Properties of Laplace Transform

1. $\mathcal{L}[Af(t)] = A\mathcal{L}[f(t)]$ where A is a constant.
2. $\mathcal{L}[A_1 f_1(t) + A_2 f_2(t)] = A_1\mathcal{L}[f_1(t)] + A_2\mathcal{L}[f_2(t)]$

- The transfer function of a linear, time-invariant, differential equation is defined as

$$G(s) = \frac{\mathcal{L}[output]}{\mathcal{L}[input]} = \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots a_n}$$

- In deriving a transfer function, all initial conditions are assumed zero.
- Proper Transfer Function: The transfer function $G(s)$ is called proper if $n \geq m$ (i.e., $G(s)|_{s=\infty}$ = constant).
- Strictly proper transfer function: The transfer function $G(s)$ is called strictly proper if $n > m$ (i.e., $G(s)|_{s=\infty} = 0$).
- Order of a system: The highest power of "s" in the denominator of $G(s)$ is called the order of the system.
- Poles of $G(s)$: The roots of the denominator polynomial of $G(s)$.
- Zeros of $G(s)$: The roots of the numerator polynomial of $G(s)$.

Definition: A system is called a linear system if for every t_0 and any two triplets $\{x_1(t_0), u_1(t) \text{ for } t \geq t_0, y_1(t) \text{ for } t \geq t_0\}$, $\{x_2(t_0), u_2(t) \text{ for } t \geq t_0, y_2(t) \text{ for } t \geq t_0\}$, we have for any real α the following two triplets:

$$\{x_1(t_0) + x_2(t_0), u_1(t) + u_2(t) \text{ for } t \geq t_0, y_1(t) + y_2(t) \text{ for } t \geq t_0\}(\text{additivity})$$

$$\text{and } \{\alpha x(t_0), \alpha u(t) \text{ for } t \geq t_0, \alpha y(t) \text{ for } t \geq t_0\}(\text{homogeneity})$$

The above two properties can be combined into one as

$$\{\alpha_1 x_1(t_0) + \alpha_2 x_2(t_0), \alpha_1 u_1(t) + \alpha_2 u_2(t) \text{ for } t \geq t_0, \alpha_1 y_1(t) + \alpha_2 y_2(t) \text{ for } t \geq t_0\}$$

for any real constants α_1 and α_2.

- The above property is known as superposition.
- A nonlinear system is one where the superposition property does not hold.

Norms of matrices also has the following properties

- $\|Ax\| \leq \|A\|\|x\|$
- $\|A + B\| \leq \|A\| + \|B\|$
- $\|AB\| \leq \|A\|\|B\|$

Orthonormal set of vectors

- A vector is said to be normalized if $\|x\|_2 = 1$
- Two vectors $x, y \in \mathbb{R}^n$ are orthogonal if $x^T y = 0$.
- A set of vectors, $x_1, x_2, \cdots, x_m$ is said to be orthonormal if

$$x_i^T x_j = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j. \end{cases}$$

Null Space and Nullity of a Matrix

- The null space of matrix A is

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n : \text{ such that } Ax = 0\}$$

- Nullity of A is the number of linearly independent vectors of $\mathcal{N}(A)$ and is denoted by $\nu(A)$.
- Null space of A consists of all its null vectors.
  Remark: If $\nu(A) = 0$, it means that 0 is the only element in $\mathcal{N}(A)$.

Properties of rank:

- Let $A \in \mathbb{R}^{m\times n}$ Then $\rho(A) + \nu(A) = n$.
- Let $A \in \mathbb{R}^{q\times n}$ and $B \in \mathbb{R}^{n\times p}$. Then

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\}$$

- Let $A \in \mathbb{R}^{m\times n}$. Then

$$\rho(AC) = \rho(A) \text{ and } \rho(DA) = \rho(A)$$

for any $n \times n$ and $m \times m$ non-singular matrices C and D.

- Given $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$ and $y = Ax$. There exists a vector $x \in \mathbb{R}^n$ satisfying the above equation if and only if $y \in \mathcal{R}(A)$ or equivalently,

$$\rho(A) = \rho([A \quad y])$$

- Given $A \in \mathbb{R}^{m\times n}$. For every $y \in \mathbb{R}^m$, there exists a vector $x \in \mathbb{R}^n$ such that $y = Ax$ if and only if $\rho(A) = m$.

Properties of Inverse and Determinant:

- If any two rows or columns of A are linearly dependent, then $det(A) = 0$.
- $det(A) = det(A^T)$.
- $det(AB) = det(A)det(B)$ if A and B are both square matrices.
- $det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$ where λ_is are the eigenvalues of A.
- If $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$, then

$$\det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} = det(A)det(D)$$

Theorem 2.1: Suppose $f(\lambda)$ is given and $A \in \mathbb{R}^{n\times n}$ matrix with char. polynomial

$$det(\lambda I - A) = \prod_{i=1}^{m}(\lambda - \lambda_i)^{n_i}$$

where $n = \sum_{i=1}^m n_i$. Define another $(n-1)$ degree polynomial

$$h(\lambda) := \beta_0 + \beta_1\lambda^1 + \beta_2\lambda^2 + \cdots + \beta_{n-1}\lambda^{n-1}$$

with n unknown coefficients β_i. These unknowns can be obtained by solving the following n equations:

$$\frac{d^k f(\lambda)}{d\lambda^k}\Big|_{\lambda=\lambda_i} = \frac{d^k h(\lambda)}{d\lambda^k}\Big|_{\lambda=\lambda_i} \quad \text{for } k = 0, 1, \cdots, (n_i - 1) \text{ and } i = 1, 2, \cdots, m$$

Then we have

$$f(A) = h(A)$$

and we say that $f(\lambda)$ equals to $h(\lambda)$ on the spectrum of A.

Example: Suppose $f(\lambda) = e^\lambda$ and $A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$. Choose $h(\lambda) = \beta_0 + \beta_1\lambda$.
Using Theorem 2.1 with $\lambda_i = -1, -2$ means

$$e^{-1} = \beta_0 - \beta_1$$
$$e^{-2} = \beta_0 - 2\beta_1$$

Solving $\Rightarrow \beta_0 = 2e^{-1} - e^{-2}$ and $\beta_1 = e^{-1} - e^{-2}$. Then

$$f(A) = h(A) = \beta_0 I + \beta_1 A = \begin{pmatrix} \beta_0 - \beta_1 & \beta_1 \\ 0 & \beta_0 - 2\beta_1 \end{pmatrix}$$

- A square matrix A is symmetric if $A = A^T$.
- The scalar function $x^T Ax$ where $x \in \mathbb{R}^n, A = A^T \in \mathbb{R}^{n\times n}$ is called a quadratic form.
- A real symmetric matrix A is said to be positive definite if for all $x \in \mathbb{R}^n, x \neq 0, x^T Ax > 0$.
- Similarly, a real symmetric matrix A is said to be positive semi-definite if for all $x \in \mathbb{R}^n, x \neq 0, x^T Ax \geq 0$.
- A symmetric matrix A is positive definite if all its leading minors are positive i.e.,

$$a_{11} > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0 \cdots$$

- A symmetric matrix A is positive definite if and only if its eigenvalues are positive.
- If $D \in \mathbb{R}^{n\times m}$ then $DD^T = A$ is positive definite if and only if D has full rank n.

因果性 causal/non-anticipatory → $p \geq q$ in $G(s)$
Lumped system 状态变量有限. Distributed 无限.
S.S. open loop.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad Y(s) = G(s)U(s)$$
$$y(t) = \int_{t_0}^{t} g(t-\tau)u(\tau)d\tau.$$

x: $n\times 1$, A: $n\times n$, B: $n\times p$, u: $p\times 1$; y: $q\times 1$, C: $q\times n$, D: $q\times p$

$x \in R^n$, $A \in R^{n\times n}$.

特征根: $\det(\lambda I - A) = 0$, A non-singular.

Example: find 特征根+基: $A = \begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix}$

$\det(A - \lambda I) = \begin{pmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{pmatrix} = (\lambda-2)(\lambda-5)$

$Aq = \lambda q$

for $\lambda = 2$: $(\lambda I - A)q_1 = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} q_1 = 0$, $q_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

for $\lambda = 5$: $(\lambda I - A)q_2 = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} q_2 = 0$, $q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Not all distinct 区别的 eigenvalue: Jordan Form

when λ_1 3重根, λ_2 1重根:

$$J = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_2 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & 1 \\ 0 & \lambda_1\end{matrix}} & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix}\lambda_1 & 1 \\ & \lambda_1\end{matrix}} & \\ & \boxed{\lambda_2} \end{pmatrix}$$

Homogeneous 齐次解

解: $\dot{x} = \frac{dx}{dt} = Ax$

$\frac{1}{x}dx = Adt$

$\int_{x(t_0)}^{x(t)} \frac{1}{x}dx = \int_{t_0}^{t} Adt$

$\ln x\Big|_{x(t_0)}^{x(t)} = A(t-t_0)$

$\ln\frac{x(t)}{x(t_0)} = A(t-t_0)$

$x(t) = e^{A(t-t_0)}x(t_0)$

即 $x(t) = e^{A(t-t_0)}x_0$

LTI ↓ $x(t) = e^{At}x_0$

e^{At} Properties: $e^{At} = \sum_0^\infty \frac{A^k t^k}{k!}$

$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$. A and e^{At} commute.

$e^{At} = L^{-1}[(sI-A)^{-1}]$

$I = e^{At}e^{-At}$, $\text{inv}[e^{At}] = e^{-At}$.

$[e^{At}]^T = e^{A^T t}$.

$e^{At}e^{Bt} = e^{(A+B)t}$ if & only $AB = BA$.

Method of Computing e^{At}:

① using $e^{At} = L^{-1}[(sI-A)^{-1}]$.

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, sI - A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}$$

$$\mathcal{L}^{-1}(sI-A)^{-1} = \mathcal{L}^{-1}\begin{pmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{pmatrix} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

$\frac{2}{s+1} - \frac{1}{s+2}$, $\frac{1}{s+1} - \frac{1}{s+2}$, $-\frac{2}{s+1} + \frac{2}{s+2}$, $-\frac{1}{s+1} + \frac{2}{s+2}$

$\text{adj}(sI-A) = \begin{pmatrix} s+3 & -2 \\ 1 & s \end{pmatrix}^T = \begin{pmatrix} s+3 & 1 \\ -2 & s \end{pmatrix}$

$\det(sI-A) = s(s+3) - (-1)\times 2 = s^2 + 3s + 2 = (s+1)(s+2)$

② using Cayley-Hamilton

$f(\lambda) = e^{\lambda t}$

$h(\lambda) = a_0(t) + a_1(t)\lambda$.

Example 5: $A = \begin{pmatrix} -2 & 3 \\ -\frac{1}{3} & -4 \end{pmatrix}$ ① with eigenvalues -3 and -3.
Again, let ② $h(\lambda) = \alpha_0(t) + \lambda\alpha_1(t)$ $\quad n=2$
When $\lambda = -3$, we have ③ $e^{\lambda t} = h(\lambda) = a_0(t) + \lambda a_1(t)$.
$e^{-3t} = \alpha_0(t) - 3\alpha_1(t)$
只有1st 个等式:
The second equation is obtained by differentiating $h(\lambda)$ w.r.t. λ and evaluated at $\lambda = -3$, yielding
④求导得2nd个等式: $\frac{df(\lambda)}{d\lambda}\Big|_{\lambda=-3} = \frac{h(\lambda)}{d\lambda}\Big|_{\lambda=-3} \Rightarrow te^{\lambda t} = a_1(t)$
($te^{\lambda t}$, $a_1(t)$)
$\Rightarrow te^{-3t} = \alpha_1(t)$
Solving, we get $\alpha_0(t) = e^{-3t} + 3te^{-3t}$ and
$e^{At} = \alpha_0(t)I + \alpha_1(t)A$

③ Diagonalization. $e^{At} = Qe^{\bar{A}t}Q^{-1}$. $\quad A^k = Q\bar{A}^k Q^{-1}$

Q: eigenvectors.

$\bar{A}$: eigvalue matrix. Jordan form.

$$\bar{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} e^{\bar{A}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}$$

④ Series Evaluation $\quad e^{At} = \sum_0^\infty \frac{A^k t^k}{k!}$

Full solution of S.S.:

particular 特解. $x_p(t) = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$.

$x = x_h + x_p = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$.

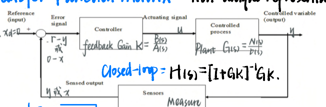
Response to Impulse:

$u(t) = \delta(t)$. $\int_0^t f(t-\tau)\delta(\tau)d\tau = f(t)$

$y_p(t) = Ce^{At}B + D\delta(t)$.

$$\begin{cases} \dot{\bar{x}} = \underbrace{\boxed{Q^{-1}AQ}}_{\bar{A}}\bar{x} + \underbrace{\boxed{Q^{-1}B}}_{\bar{B}}u \\ y = \underbrace{\boxed{CQ}}_{\bar{C}}\bar{x} + Du \end{cases}$$

Transfer Function Matrix: non-unique representation.



Closed-loop: $H(s) = [I + GK]^{-1}GK$.

Open-loop

$sX(s) - x_0 = AX(s) + BU(s)$ $\quad$ let $x_0 = 0$

$X(s) = (sI-A)^{-1}BU(s)$,

$Y(s) = [C(sI-A)^{-1}B + D]U(s) = G(s)U(s)$

$G(s) = C(sI-A)^{-1}B + D = \frac{N(s)}{D(s)}$, poles = eigvalue(A).

特征根定义 Modes of Sys.

$x(t) = e^{At}x_0 = Qe^{\bar{A}t}Q^{-1}x_0$. $\quad e^{\bar{A}t} = diag\{e^{\lambda_1 t}, \cdots, e^{\lambda_n t}\}$

$\therefore x_h(t) = \sum_{i=1}^{n} e^{\lambda_i t}q_i p_i x_0$.

q_i: eigvector, Q的列. $\quad p_i$: Q^{-1}的行

LTV System:

form $\overset{n\times n}{X} = [x_1\ x_2 \cdots x_n]$, x_i for the ith initial state $x_i(t_0)$. $\dot{X} = A(t)X(t)$. $X \leftarrow$ fundamental matrix

state-transition matrix $\varphi(t, t_0) = X(t)X^{-1}(t_0)$.

$x(t) = \varphi(t, t_0)x(t_0) + \int_{t_0}^{t}\varphi(t,\tau)B(\tau)u(\tau)d\tau$.

Controllability:

Any $x(0) = x_0$ to any $x(t_1) = x_1$ in finite t_1.

controllable 充要条件:

① C Grammian $W(0,t) = \int_0^t e^{A\tau}BB^T e^{A^T\tau}d\tau = \int_0^t e^{A(t-\tau)}BB^T e^{A^T(t-\tau)}d\tau$ is nonsingular for all $t > 0$. $\quad$ rank$[W(0,t)]$ = rank$[W_c]$. independent of t

② $W_c = \overset{n\times n}{[B\ AB \ldots A^{n-1}B]}$ is full row rank.

③ $[A - \lambda I \quad B]$ $n\times(n+r)$ is full row rank at every λ of A.

算 control $u(t)$: $u(t) = -B^T e^{A^T(t_1-t)}W^{-1}(0, t_1)(e^{At_1}x_0 - x_1)$

$\bar{W}_c = T^{-1}W_c$, controllability is not affected by T.

Lyapunov Equation of controllabity Grammian: $AW + WA^T = -BB^T$ when A is stable.

Observability:

Every unknown $x(0)$ can be from $u(t)$ & $y(t)$ in finite time. only need A, C because $u(t)$ is known, B.D useless.

O Grammian: $\overset{n\times n}{M(0,t)} = \int_0^t e^{A^T\tau}C^T C e^{A\tau}d\tau$, independent of t.

estimate $x(0)$: $x(0) = M^{-1}(0,t)\int_0^t e^{A^T\tau}C^T y(\tau)d\tau$

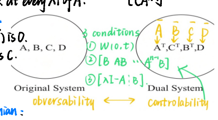
observable 充要条件:

① O Grammian is nonsingular. $\quad$ ② $O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full column rank.

③ $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ $(n+m)\times n$ has full col rank at every λ of A.

Duality 对偶.

(A, B) is C $\leftrightarrow$ (A^T, B^T) is O.
(A, C) is O $\leftrightarrow$ (A^T, C^T) is C.



Lyapunov Equation of Observation Grammian:

$A^T M + MA = -C^T C$.

Gilbert Decomposition:

rank$(W_c) = n_1 < n$. S可分解为 n_1阶可控 S_c 和 $n - n_1$阶不可控 $S_{\bar{c}}$.

$$\begin{pmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{pmatrix} = \begin{pmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{pmatrix}\begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + \begin{pmatrix} \bar{B}_c \\ 0 \end{pmatrix}u. \begin{matrix} \leftarrow \text{controllable} \\ \leftarrow \text{uncontrol} \end{matrix}$$

$$y = (\bar{C}_c \quad \bar{C}_{\bar{c}})\begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + \bar{D}u.$$

$\bar{S}_c$ has the same T.F. matrix as S.

T.F. from S to $\bar{S}$ = T.F. from S to $\bar{S}_c$:

$G(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = \bar{C}_c(sI - \bar{A}_c)^{-1}\bar{B}_c + \bar{D}$.

$n_1 = \text{rank}(W_c) = \text{rank}(\bar{W}_c) = \text{rank}(\bar{W}_c)$

Stable: if $\bar{A}_{\bar{c}}$ is stable, S is stablizable.
λ_i of $\bar{A}_{\bar{c}} < 0$

Detectable: rank$(O) = n_2 < n$. $\bar{A}_{\bar{o}}$ is stable.

$$\begin{pmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{pmatrix} = \begin{pmatrix} \bar{A}_o & 0 \\ \bar{A}_{12} & \bar{A}_{\bar{o}} \end{pmatrix}\begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + \begin{pmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{pmatrix}u \qquad \text{Duality.}$$

$$y = (\bar{C}_o \quad 0)\begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + \bar{D}u$$

Minimality: from G-K decomposition, if a LTI sys is C and O, 存在低维sys 与原系统有一样的T.F.
当 S 不存在同T.F.的低维系统时, S is minimal.

Stability:

zero-state response.

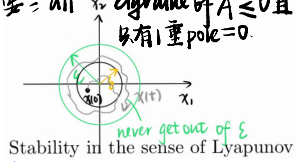
Bounded-Input-Bounded-output stable 充要条件: ① $\int_0^\infty |g(t)|dt \leq k < \infty$.
② absolute intergrable 绝对可积,
③ all poles of G(s) have 负 real.

Internal Stability: ← zero-input response.

equilibrium point x_e: $\dot{x}_e = Ax_e = 0$

Stability in the sense of Lyapunov:

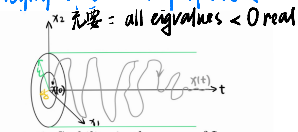
充要: all λ_i eigvalue of $A \leq 0$ 且 只有1重pole = 0.



Stability in the sense of Lyapunov

Asymptotic Stability 渐近稳定

充要: all eigvalues < 0 real



Asymptotic Stability in the sense of Lyapunov

渐近稳定 $\overset{\leftarrow}{\not\rightarrow}$ BIBO stable.
$\because Re[eig(A)] < 0 \Rightarrow$ poles of $G(s) < 0$.

Lyapunov equation: $A^T P + PA = -Q$.

当且仅当任给正定对称阵Q都有对称正定阵P满足时, 系统 $\dot{x} = Ax$, $x(0) = x_0$ 渐近稳定.

Solving the Lyapunov Equation (Grammians)

$A^T P + PA = -Q$ 可解出P, P中的各个元素 x 满足: $Mx = b$. 个数 $= \frac{n(n+1)}{2}$.

Controllable Canonical 规范化 Form:

If rank(W_c) = full, the sys controllable, sys 可转为 controllable form:

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \bar{A} & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}\bar{x} + \begin{bmatrix} 0 \\ 0 \\ \bar{B} \\ 0 \\ 1 \end{bmatrix}u$$

$$y = [\,b_0 \quad b_1 \quad \cdots \quad \bar{C}\,b_{n-2} \quad b_{n-1}\,]\,\bar{x}$$

T.F $G(s) = \frac{Y(s)}{U(s)} = \frac{\bar{C}\text{决定分子}}{\bar{A}\text{最后行决定分母}}$,

$= \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$

poles → SISO

transformation $\bar{x} = Tx$: 取 q^T 是 W_c^{-1} 的最后一行, $T = \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$. 有 $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$

T is nonsingular because $\det(TU) = 1$.

Observerable Canonical Form.

If rank(O) = n, the sys observable, 且可转为规范式:

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}\bar{x} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix}u$$

$$y = [\,0 \quad 0 \quad \cdots \quad 0 \quad 1\,]\,\bar{x}$$

Dual: O: $\bar{A}, \bar{B}, \bar{C}$
C: $\bar{A}^T, \bar{C}^T, \bar{B}^T$

T.F. $G(s) = \frac{Y(s)}{U(s)}$ 与上相同.

transform $\bar{x} = Qx$ ($\neq \bar{x} = Tx$): 取 w 是 O^{-1} 的 last col, $\bar{A} = Q^{-1}AQ$

$Q = [w \quad Aw \cdots A^{n-1}w]$. $\det(QO) = 1$

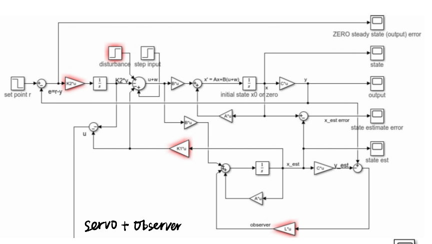
Gilbert-Kalman Canonical Decomposition:

S 分成 $C\bar{O}$, CO, $\bar{C}O$, $\bar{C}\bar{O}$ 4 subsystems.

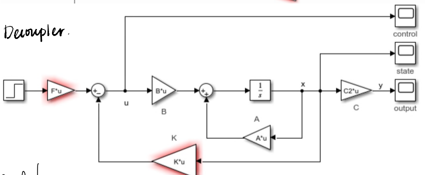
T.F. $G(s) = C_{co}(sI - \bar{A}_{co})^{-1}\bar{B}_{co} + \bar{D}$

$$\begin{pmatrix} \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \end{pmatrix} = \begin{pmatrix} \bar{A}_{c\bar{o}} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} \\ 0 & \bar{A}_{co} & 0 & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}\bar{o}} & \bar{A}_{34} \\ 0 & 0 & 0 & \bar{A}_{\bar{c}o} \end{pmatrix}\begin{pmatrix} \bar{x}_{c\bar{o}} \\ \bar{x}_{co} \\ \bar{x}_{\bar{c}\bar{o}} \\ \bar{x}_{\bar{c}o} \end{pmatrix} + \begin{pmatrix} \bar{B}_{c\bar{o}} \\ \bar{B}_{co} \\ 0 \\ 0 \end{pmatrix}u$$

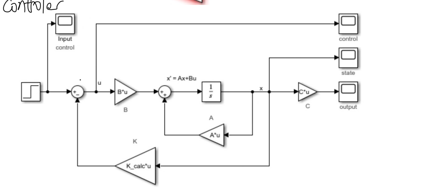
$$y = (0 \quad \bar{C}_{co} \quad 0 \quad \bar{C}_{\bar{c}o}) + \bar{D}u$$



Decoupler.



Controler



二、代数余子式计算

a_{ij}的余子式就是去除第i行和第j列剩余矩阵的行列式。
代数余子式是余子式乘以-1的i+j次方。
例如a_{11}的代数余子式就是$M_{a_{11}} = (-1)^{1+1}\left|\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}\right|$

三、矩阵的逆

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} M_{a_{11}} & M_{a_{21}} & M_{a_{31}} \\ M_{a_{12}} & M_{a_{22}} & M_{a_{32}} \\ M_{a_{13}} & M_{a_{23}} & M_{a_{33}} \end{pmatrix}$$

Example 1 Let

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}u,$$
$$y = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix}x.$$

One proceeds as follows:

$i = 1 \quad c_1^T = [1 \quad 0.5]$

$c_1^T B = [1 \quad 0.5]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [1 \quad 0.5] \neq 0 \rightarrow \sigma_1 = 1$

$i = 2 \quad c_2^T = [1 \quad 1]$

$c_2^T B = [1 \quad 1]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [1 \quad 1] \neq 0 \rightarrow \sigma_2 = 1$

We have calculated before:
$\sigma_1 = 1, \quad c_1^T B = [1 \quad 0.5]$
$\sigma_2 = 1, \quad c_2^T B = [1 \quad 1]$.

One sees that

$$B^* = \begin{pmatrix} 1 & 0.5 \\ 1 & 1 \end{pmatrix}$$

is non-singular and the system can be decoupled. One proceeds:

$$c_1^T A^{\sigma_1} = c_1^T A = [1 \quad 0.5]\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = [-0.5 \quad 0],$$
$$c_2^T A^{\sigma_2} = c_2^T A = [1 \quad 1]\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = [-1 \quad -1].$$

so that

$$C^* = \begin{pmatrix} -0.5 & 0 \\ -1 & -1 \end{pmatrix}.$$

Since $\quad \phi_{f_1} = s + 1, \qquad \phi_{f_2} = s + 2,$

$$C^{**} = \begin{pmatrix} c_1^T\phi_{f_1}(A) \\ c_2^T\phi_{f_2}(A) \end{pmatrix} = \begin{pmatrix} c_1^T(A + I) \\ c_2^T(A + 2I) \end{pmatrix} = \begin{pmatrix} -0.5 & 0 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0.5 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 1 & 1 \end{pmatrix}$$

$$K = (B^*)^{-1}C^{**} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 0.5 & 0.5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Try $g = 1$, and solve for t_1 and t_2 and check the rank of $\begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix}$.

This yields $t_1 = \frac{1}{3}$, $t_2 = -\frac{1}{9}$ and the transformation matrix will have full rank.

One then calculates $e = TB = [t_1 \quad t_2]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = t_2 = -\frac{1}{9}$

The observer is thus determined as $\dot{\xi} = -3\xi - \frac{1}{9}u + y$

One then reconstructs the state variables from the plant's output and the observer's output from (18) as

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & -\frac{1}{9} \end{bmatrix}^{-1}\begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & -9 \end{bmatrix}\begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix}$$

Let $\quad y_r(t)$ denote the output $\quad$ excited by r(t) when $w(t) = 0$. Then

$$E_r(s) = R(s) - Y_r(s)$$
$$= R(s) - \frac{G(s)K(s)}{1 + G(s)K(s)}R(s)$$
$$= \frac{R(s)}{1 + \frac{N(s)}{D(s)Q(s)}\frac{B(s)}{A(s)}} \qquad (13)$$
$$= \frac{D(s)A(s)}{D(s)Q(s)A(s) + N(s)B(s)} \cdot Q(s)\frac{N_r(s)}{D_r(s)}$$

Since all unstable roots of $D_r(s)$ are canceled by $Q(s)$, and all the poles of $P_c(s) = A(s)D(s)Q(s) + B(s)N(s)$ are stable
so we have $e_r(t) \rightarrow 0$ as $t \rightarrow \infty$.

Example 2. Find the optimal control with the cost function:

$$J = \frac{1}{2}\int_0^\infty \left[x^T\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}x + ru^2\right]dt, \qquad (18)$$

for the system:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{pmatrix}x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u, \quad x(0) = x_0. \qquad (19)$$

Let

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}.$$

The ARE $\quad A^T P + PA + Q - PBR^{-1}B^T P = 0.$

$$\begin{pmatrix} 0 & -\alpha_0 \\ 1 & -\alpha_1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\alpha_0 & -\alpha_1 \end{pmatrix} + \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}$$
$$-r^{-1}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0 \quad 1)\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = 0. \qquad (20)$$

$$-2\alpha_0 p_{12} + q_1 - p_{12}^2 r^{-1} = 0,$$
$$p_{11} - \alpha_1 p_{12} - \alpha_0 p_{22} - p_{12}p_{22}r^{-1} = 0,$$
$$2p_{12} - 2\alpha_1 p_{22} + q_2 - p_{22}^2 r^{-1} = 0,$$

There are multiple solutions

$$p_{12} = -r\alpha_0 \pm r\sqrt{\alpha_0^2 + q_1/r},$$
$$p_{22} = -r\alpha_1 \pm r\left[\alpha_1^2 + q_2/r + 2p_{12}/r\right]^{\frac{1}{2}},$$
$$p_{11} = \alpha_1 p_{12} + \alpha_0 p_{22} + p_{12}p_{22}r^{-1}$$
$$= (\alpha_1 r + p_{22})(\alpha_0 r + p_{12})r^{-1} - \alpha_1\alpha_0 r.$$

The positive definite P is

$$p_{11} = r\left\{\alpha_1^2 + q_2/r + 2\left[-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\right]\right\}^{\frac{1}{2}}\sqrt{\alpha_0^2 + q_1/r} - \alpha_0\alpha_1 r,$$
$$p_{12} = -r\alpha_0 + r\sqrt{\alpha_0^2 + q_1/r},$$
$$p_{22} = -r\alpha_1 + r\left[\alpha_1^2 + q_2/r + 2\left(-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\right)\right]^{\frac{1}{2}}.$$

The optimal control is given by

$$u = -r^{-1}(0 \quad 1)Px$$
$$= -\left[-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}, \quad -\alpha_1 + \left[\alpha_1^2 + q_2/r + 2\left(-\alpha_0 + \sqrt{\alpha_0^2 + q_1/r}\right)\right]^{1/2}\right]x.$$

$T = [t_1 \quad t_2]$ is such that the matrix $\begin{bmatrix} C \\ T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix}$ must be nonsingular.

This is the case iff $t_2 \neq 0$.

(ii) Choose $d = -3$ (or other desired decay rate).

(iii) Design T and g from $dT - TA + gC = 0$, or

$$\begin{matrix} TA & - & dT & = & gC \\ [t_1 \quad t_2]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & + & 3[t_1 \quad t_2] & = & g[1 \quad 0] \\ [0 \quad t_1] & + & [3t_1 \quad 3t_2] & = & [g \quad 0] \end{matrix}$$

$$g = 3t_1 \qquad t_1 + 3t_2 = 0$$

$H(s)=\frac{w_n^2}{s^2+2\zeta w_n s+w_n^2}$ $t_r \simeq \frac{1.8}{w_n}$ $t_s \simeq \frac{4.0}{\zeta w_n}$

$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$ 2 ~ 5 times faster

From the transient performance specifications, you can design the positions of the desired poles.

$$\phi_d(s)=\prod_{i=1}^{n}(s-\lambda_i)=s^n+\gamma_{n-1}s^{n-1}+\cdots+\gamma_1 s+\gamma_0$$

This is the desired closed loop characteristic polynomial.

Pole Placement

Given a plant $\dot{x}=Ax+Bu,$ $y=Cx.$

Build a state feedback controller: $u=-Kx+Fr,$

The closed loop: $\dot{x}=Ax+B(-Kx+Fr)=(A-BK)x+BFr$

Design K such that the characteristic polynomial of the closed loop

$$\det(sI-(A-BK))=s^n+\gamma_{n-1}s^{n-1}+\cdots+\gamma_0$$

SISO system

The open loop plant $\dot{x}=Ax+Bu,$

The state feedback controller: $u=r-k^Tx.$

The closed loop system $\dot{x}=(A-bk^T)x+br$

What is the condition on the system to place the poles to desired ones?

If the system is controllable, the poles can be placed anywhere:

We used the controllable canonical form to derive the Ackermann's formula

$$k^T=[0,\ 0,\ \cdots,\ 0,\ 1]C^{-1}\phi_d(A),$$

$$\phi_d(A)=\phi_d(s)|_{s=A}=A^n+\gamma_{n-1}A^{n-1}+\ldots+\gamma_0 I_n.$$

What if you forget Ackermann's formula?

The easiest way is directly comparing the coefficients of the two polynomials:

$$\det(sI-(A-bk^T))=s^n+\gamma_{n-1}s^{n-1}+\cdots+\gamma_0$$

How many solutions can you get? Only one!

MIMO system

The open loop plant $\dot{x}=Ax+Bu,$

The state feedback controller: $u=-Kx+Fr.$

The closed loop system $\dot{x}=(A-BK)x+BFr$

If the system is controllable, the poles can be placed anywhere:

Is the solution unique?

No. There are infinity number of solutions!

Why?

The easiest way is still comparing the coefficients of the two polynomials:

$$\det(sI-(A-BK))=s^n+\gamma_{n-1}s^{n-1}+\cdots+\gamma_0$$

Then you will find that the number of design parameters in K is greater than the number of equations. That is why you can have infinite number of solutions.

Pole Placement for MIMO system

One simple method is to force the MIMO to SISO

$$\begin{aligned}\dot{x}&=Ax-Bu\\ u=qv\ \Rightarrow\ &=Ax-Bqv\\ &=Ax+\underbrace{(Bq)}_{b}v\end{aligned}$$

Step 1. Choose the weight vector q such that the pair {A, Bq} is controllable.

Step 2. Use any single-input pole placement algorithm for the *pair* {A, Bq} to determine k^T such that $A-Bqk^T$ has the desired eigenvalues (closed-loop poles).

Overall, the required state feedback matrix is $K=qk^T.$

Algorithm For Full rank pole placement

Given controllable $\{A,B\}$ and the desired $\phi_d(s)$.

(i) Obtain the controllable canonical form in the state x via

$$\bar{x}=Tx,$$

such that

$$\dot{\bar{x}}=\bar{A}\bar{x}+\bar{B}u$$

is in the controllable canonical form.

The most time-consuming part is computing T.

Once T is obtained, then we can get the controllable canonical form:

$$\bar{A}=TAT^{-1},\qquad \bar{B}=TB$$

First compute the controllability matrix

$$W_c=\{B\ \ AB\ \ A^2B\ \ \cdots\ \ A^{n-1}B\}$$

Check whether it is full rank or not. If it is controllable, then move on to the next step.

For single input system, the controllability matrix is a square matrix, all of the vectors are independent.

For MIMO system, we need to select the n independent vectors from the controllability matrix in the strict order from left to right.

In this way, we assure that all the inputs will play a part in the placement!

After the vectors are selected, it is important to re-group them in a square matrix C in the following form

$$C=\{b_1\ \ Ab_1\ \cdots\ A^{d_1-1}b_1\ \ b_2\ \ Ab_2\ \cdots\ A^{d_2-1}b_2\cdots\ b_m\ \ Ab_m\ \cdots\ A^{d_m-1}b_m\}$$

Where the indices d_i imply the number of vectors in C related to the i-th input, u

The controllable canonical form can then be computed from this matrix C!

Compute the inverse of C, C^{-1}

For multi-input case, we need to take out m rows from C^{-1} corresponding to the m inputs, and form T as

$$T=\begin{bmatrix}q_{d_1}^T\\ q_{d_1}^TA\\ \vdots\\ q_{d_1}^TA^{d_1-1}\\ q_{d_1+d_2}^T\\ q_{d_1+d_2}^TA\\ \vdots\\ q_{d_1+d_2}^TA^{d_2-1}\\ \vdots\end{bmatrix}\Rightarrow \bar{B}=TB=\begin{array}{c}\begin{bmatrix}0&0&\cdots&0\\ \vdots&\vdots&&\vdots\\ 0&0&\cdots&0\\ 1&\times&\cdots&\times\\ 0&0&\cdots&0\\ \vdots&\vdots&&\vdots\\ 0&0&\cdots&0\\ 0&1&\cdots&\times\\ \vdots&\vdots&&\vdots\\ 0&0&\cdots&0\\ \vdots&\vdots&&\vdots\\ 0&0&\cdots&1\end{bmatrix}\end{array}\begin{array}{l}\\ \\ \\ d_1^{th}row\\ \\ \\ \\ (d_1+d_2)^{th}row\\ \\ \\ \\ n^{th}row\end{array}$$

How many T can you find? There is only one T!

(ii) Specify a desired closed-loop matrix A_d

such that $\det(sI-A_d)=\phi_d(s)$

and A_d is the same as A except non-trivial rows.

How many ways can you choose the desired closed-loop matrix A_d?

There are many ways in this step! That is why the full rank solution to the placement for MI system is still not unique!

(iii) Compare the non-trivial rows of the closed loop canonical form

$$\bar{A}-\bar{B}\bar{K}$$

with the desired one A_d, and compute $\bar{K}$.

(iv) Compute the original feedback gain K

$$K=\bar{K}T.$$

The computation is much more complicated than the unity rank method.

But it can fully exploit the freedom of all the inputs! Therefore the overall performance is usually better than the unity rank method.

For this case,

$$C=\{b_1\ \ b_2\ \ Ab_2\}=\begin{bmatrix}1&1&-1\\0&0&1\\0&1&2\end{bmatrix}$$

$$C^{-1}=\begin{bmatrix}1&3&-1\\0&-2&1\\0&1&0\end{bmatrix}$$

How many vectors in C are associated with u_1? $d_1=1$

How many vectors in C are associated with u_2? $d_2=2$

Which rows should we take out to form T?

The first (d_1) and third (d_1+d_2) rows!

$$T=\begin{bmatrix}q_1^T\\q_3^T\\q_3^TA\end{bmatrix}=\begin{bmatrix}1&3&-1\\0&1&0\\0&1&1\end{bmatrix},T^{-1}=\begin{bmatrix}1&-4&1\\0&1&0\\0&-1&1\end{bmatrix}$$

The key is to form the transformation matrix T. The rest is simple.

$$\bar{A}=TAT^{-1}=\begin{bmatrix}-1&7&0\\0&0&1\\0&-2&3\end{bmatrix},\qquad \bar{B}=TB=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}$$

There are two inputs, so there are two nontrivial rows since the inputs only affect the nontrivial row.

Design the feedback gain matrix for the controllable canonical form

$$\bar{K}=\begin{bmatrix}\bar{k}_{11}&\bar{k}_{12}&\bar{k}_{13}\\ \bar{k}_{21}&\bar{k}_{22}&\bar{k}_{23}\end{bmatrix}$$

Form the closed loop matrix:

$$\bar{A}-\bar{B}\bar{K}=\begin{bmatrix}-\bar{k}_{11}-1&-\bar{k}_{12}+7&-\bar{k}_{13}\\0&0&1\\-\bar{k}_{21}&-\bar{k}_{22}-2&-\bar{k}_{23}+3\end{bmatrix}$$

We can also construct another controllable canonical form matrix

$$\tilde{A}_d=\begin{bmatrix}0&1&0\\0&0&1\\-6&-11&-6\end{bmatrix}$$

The trivial rows must be the same.
The nontrivial rows are different.

which will lead to a different state feedback controller.

How to Solve LQR problem?

System: $\dot{x}=Ax+Bu,$

Performance Index: $J=\frac{1}{2}\int_0^\infty(\mathbf{x}^T\mathbf{Q}\mathbf{x}+\mathbf{u}^T\mathbf{R}\mathbf{u})dt.$

Controller: $u=-Kx,\quad K=R^{-1}B^TP,$

where P is the symmetric positive definite solution of the algebraic Matrix Riccati equation:

$$A^TP+PA+Q-PBR^{-1}B^TP=0.$$

The difficult part for LQR is how to solve ARE.

For first or second order system, you can first assume a matrix P with symmetric elements, and then plug it into the ARE.

It normally results in a set of nonlinear equations, with multiple solutions.

Out of the multiple solutions, there is ONLY ONE solution which is positive definite. And you have to find it out.

A systematic way to solve ARE:

Step 1: Form the 2n×2n matrix: $\Gamma=\begin{pmatrix}A&-BR^{-1}B^T\\-Q&-A^T\end{pmatrix},$

and find its *n* stable eigenvalues.

Step 2: Let the eigenvector of Γ corresponding to **stable** λ_i, *i* = 1,..., *n*, be

$$\begin{pmatrix}v_i\\ \mu_i\end{pmatrix},\ i=1,2,\cdots,n.$$

where v_i,μ_i are n-dimensional vectors.

Then, P is given by $P=[\mu_1,\ldots,\mu_n][v_1,\ldots,v_n]^{-1}.$

Decoupling by State Feedback

$$\dot{x}=Ax+Bu,\quad y=Cx,$$

where $u,y\in R^m,x\in R^n,B\in R^{n\times m},C\in R^{m\times n}$, to be decoupled by the state feedback $u=-Kx+Fr.$

We need to design both K and F!

The resultant system is

$$\dot{x}=(A-BK)x+BFr,\quad y=Cx,$$

Open loop TF: $G(s)=C(sI-A)^{-1}B.$

Closed loop TF: $H(s)=C(sI-A+BK)^{-1}BF.$

The transfer function of the closed loop, H(s), is related to the open loop G(s). To derive the expression of the closed loop transfer function in terms of the open loop is the key to the solution to decoupling problem!

$$H(s)=G(s)\left[I+K(sI-A)^{-1}B\right]^{-1}F.$$

If H(s) is diagonal, then G(s) must be nonsingular.

This is the only condition for the system to be decoupled.

First we need to find a condition to assure that $G(s)=C(sI-A)^{-1}B$ is nonsingular.

Partition C into m row vectors. $C=\begin{bmatrix}c_1^T\\c_2^T\\ \vdots\\ c_m^T\end{bmatrix}$

Define, $\sigma_i, i=1,2,\cdots,m$, *as an integer by*

$$\sigma_i=\begin{cases}\min\left(j\,|\,c_i^TA^{j-1}B\neq 0^T,\ j=1,\ 2,\ \cdots,\ n\right);\\ n,\quad if\quad c_i^TA^{j-1}B=0^T,\ j=1,\ 2,\ \cdots,\ n.\end{cases}$$

σ_i corresponds to the relative degree of the transfer function for each row.

You need to be very clear on how to compute this indicator, which is the key design parameter for decoupling.

The key to decoupling problem is to express G(s) in the form of

$$G(s)=\begin{pmatrix}s^{-\sigma_1}&0&\cdots&0\\0&s^{-\sigma_2}&\ddots&\vdots\\ \vdots&\ddots&\ddots&0\\0&\cdots&0&s^{-\sigma_m}\end{pmatrix}\left[B^*+C^*(sI-A)^{-1}B\right],$$

where B^* and C^* are defined as

$$B^*=\begin{bmatrix}c_1^TA^{\sigma_1-1}B\\c_2^TA^{\sigma_2-1}B\\ \vdots\\ c_m^TA^{\sigma_m-1}B\end{bmatrix}\qquad C^*=\begin{bmatrix}c_1^TA^{\sigma_1}\\c_2^TA^{\sigma_2}\\ \vdots\\ c_m^TA^{\sigma_m}\end{bmatrix}$$

σ_i corresponds to the relative degree of the transfer function for each row.

B^* correspond to the leading coefficients of the zero polynomials.

The key to prove this is to express $(sI-A)^{-1}$ as infinite series:

$$(sI-A)^{-1}=s^{-1}\left[I-\frac{A}{s}\right]^{-1}=s^{-1}\left[I+As^{-1}+A^2s^{-2}+\cdots\right]$$

Design the state feedback controller: $u=-Kx+Fr.$

$$F=B^{*-1},\qquad K=B^{*-1}C^*.$$

Then, the closed-loop system transfer function matrix is given by

$$H(s)=diag\left(s^{-\sigma_1},\ s^{-\sigma_2},\ \cdots,\ s^{-\sigma_m}\right),$$ unstable

which is called an integrator decoupled system.

Define $C^{**}=\begin{bmatrix}c_1^T\phi_{f1}(A)\\c_2^T\phi_{f2}(A)\\ \vdots\\ c_m^T\phi_{fm}(A)\end{bmatrix}$

where $\phi_{f_i}(A)=A^{\sigma_i}+\gamma_{i1}A^{\sigma_i-1}+\cdots+\gamma_{i\sigma_i}I,$

$\phi_{f_i}(s)=s^{\sigma_i}+\gamma_{i1}s^{\sigma_i-1}+\cdots+\gamma_{i\sigma_i}$

corresponds to the stable characteristic polynomial of the i-th input-output pair.

The state feedback controller $u=-Kx+Fr.$

$$F=B^{*-1},\qquad K=B^{*-1}C^{**}.$$

can make the resultant feedback system have the desired stable transfer functions.

Design procedure. The decoupling problem with stability by output feedback can be solved by designing $K(s)$ in two stages, i.e., $K(s)=K_d(s)K_s(s)$.

Step One: $K_d(s)$ is to make $G(s)K_d(s)$ diagonal and non-singular with no unstable pole-zero cancellations. In this step, $K_d(s)$ is not required to be proper.

Step Two: Whenever such a $K_d(s)$ is found in step one, it is always possible to design a diagonal controller $K_s(s)$ to stabilize the resultant $G(s)K_d(s)$ with SISO methods or pole placement technique loop by loop so that decoupling is not affected, and to assure that there are no unstable pole-zero cancellation between G(s) and $K(s)=K_d(s)K_s(s)$, and to make $K(s)$ proper.

Design for decoupler $K_d(s)$: Suppose that $G(s)$ is non-singular. Write

$$G(s)=\frac{N(s)}{d(s)}$$

where $d(s)$ is the least common denominator of $G(s)$, and N(s) is polynomial matrix. If $\det(N)$ and $d(s)$ have no common unstable roots, then choose

$$K_d(s)=\mathrm{adj}(N(s))$$

so that

- $G(s)K_d(s)=\frac{N(s)\bullet adj(N(s))}{d(s)}=\frac{\det(N(s))}{d(s)}I_m$ is decoupled

One problem with this simple method is that the order of the decoupled systems is large.

Refinement. One way to reduce the order of the decoupler is to elaborate the above method as follows. Express $G(s)$ as

$$G(s)=diag\left\{\frac{1}{d_1},\ \frac{1}{d_2},\ \cdots,\ \frac{1}{d_m}\right\}N_r(s)$$

where d_i is the least common denominator of i-th row of G(s) so that $N_r(s)$ is a polynomial matrix. Choose

$$K_d=adj(N_r(s))$$ Then,

$$G(s)K_d(s)=diag\left\{\frac{\det N_r(s)}{d_1},\ \frac{\det N_r(s)}{d_2},\ \cdots,\ \frac{\det N_r(s)}{d_m}\right\}$$

You still need to design diagonal controller $K_s(s)$ to stabilize the resultant $G(s)K_d(s)$ with SISO methods so that decoupling is not affected, and to assure that there are no unstable pole-zero cancellation between G(s) and $K(s)=K_d(s)K_s(s)$, and to make $K(s)$ proper.

Example 5. Let

$$G(s)=\begin{bmatrix}\frac{1}{s+2}&\frac{-1}{(s+1)(s+2)}\\ \frac{5}{s+3}&\frac{1}{s+3}\end{bmatrix}$$

What is the least common denominator?

For this case, $d(s)=(s+1)(s+2)(s+3)$, the non-refined method will give a decoupled system of order 3x2=6. With the refined method, one sees that

$$G(s)=\begin{bmatrix}d_1&0\\0&d_2\end{bmatrix}^{-1}N_r=\begin{bmatrix}(s+1)(s+2)&0\\0&s+3\end{bmatrix}^{-1}\begin{bmatrix}s+1&-1\\5&1\end{bmatrix}$$

And $\det(N_r)=s+6$ and $d_i(s)$ have no common unstable roots for *i=1,2*. We can then take

$$K_d=adj(N_r)=\begin{bmatrix}1&1\\-5&s+1\end{bmatrix}$$

$$K_d=adj(N_r)=\begin{bmatrix}1&1\\-5&s+1\end{bmatrix}\qquad GK_d=diag\left\{\frac{s+6}{(s+1)(s+2)},\ \frac{s+6}{s+3}\right\},$$

It is easy to verify that the closed loop is stable. But there is one problem.

Is K_d proper?

No. If not proper, it may not be implemented due to causality issue.

There are many solutions. One simple solution is

$$K_s=\begin{bmatrix}\frac{1}{s+6}&0\\0&\frac{1}{s+6}\end{bmatrix}$$

to stabilize GK_d and make the overall controller $K(s)=K_d(s)K_s(s)$ proper:

$$K=K_dK_s=\begin{bmatrix}\frac{1}{s+6}&\frac{1}{s+6}\\ \frac{-5}{s+6}&\frac{s+1}{s+6}\end{bmatrix}$$

$$GK=diag\left\{\frac{1}{(s+1)(s+2)},\ \frac{1}{s+3}\right\},\quad \begin{aligned}H(s)&=\left[I+G(s)K(s)\right]^{-1}G(s)K(s)\\&=diag\left\{\frac{1}{s^2+3s+3},\ \frac{1}{s+4}\right\},\end{aligned}$$

Reduced-Order Observers

Idea: We want to take advantage of the *m* state variables that are available through *y=Cx* and construct an observer of order *n − m*, lower than *n*, to estimate the remaining (n-m) state variables.

Let $\xi=Tx$, where T is $(n-m)\times n$ with $(n-m)$ rows that are linearly independent of C.

$$\begin{bmatrix}y\\ \xi\end{bmatrix}=\begin{bmatrix}C\\T\end{bmatrix}x\qquad \begin{bmatrix}C\\T\end{bmatrix}\ \text{is of } n\times n \text{ and non-singular.}$$

Using T as a design variable is one of the keys to the design of a reduced-order.

For the system: $\dot{x}=Ax+Bu,\quad y=Cx,$

construct an observer as

$$\dot{\hat{\xi}}=D\hat{\xi}+Eu+Gy$$

such that

$$\hat{\xi}\to T x\qquad \text{as } t \text{ goes to infinity.}$$

The estimation error dynamics:

$$e=\hat{\xi}-Tx$$

$$\frac{de}{dt}=De+(DT-TA+GC)x+(E-TB)u$$

If D is stable, and $DT-TA+GC=0,$ $E-TB=0,$

then $\frac{de}{dt}=De$

So, the estimation error will converge to zero.

Design: Determine *T, D, G* and *E* as follows.

(i) Find the constraints on T such that $\begin{bmatrix}C\\T\end{bmatrix}$ is non-singular; but do not choose T directly at this step.

(ii) Choose D such that its eigenvalues have negative real parts, or desired decay rates; there are multiple solution in this step.

(iii) Solve $DT-TA+GC=0$ for T and G; this is the key design equation.

The solution is also not unique in this step.

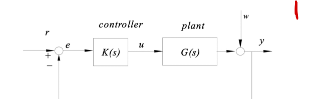
(iv) Calculate E=TB.

Finally, one needs to reconstruct the state variables from the plant's output and the observer's output

$$\hat{x}=\begin{bmatrix}C\\T\end{bmatrix}^{-1}\begin{bmatrix}y\\ \hat{\xi}\end{bmatrix}$$

Once the estimates are available, the controller is

$$u=-K\hat{x}+Fr$$



Let $G(s)=\frac{N(s)}{D(s)},\qquad K(s)=\frac{B(s)}{A(s)},$

The order of the process is n. $\deg N(s)\le\deg D(s)=n.$

The closed loop transfer function H(s) is given by

$$H(s)=\frac{G(s)K(s)}{1+G(s)K(s)}=\frac{\frac{N(s)}{D(s)}\frac{B(s)}{A(s)}}{1+\frac{N(s)}{D(s)}\frac{B(s)}{A(s)}}=\frac{N(s)B(s)}{D(s)A(s)+N(s)B(s)}.$$

Servo control design procedure for SISO system:

(i) Obtain plant coprime polynomial fraction as $G=N(s)/D(s)$.

(ii) Determine LCD, $Q(s)$ from the given types of disturbance W(s) and reference R(s).

(iii) Design $\tilde{K}$ to stabilize the generalized plant, $N/(DQ)$.

(iv) Form the servo controller as

$$K=\tilde{K}\frac{1}{Q}=\frac{B}{A}$$

From this design:
- The closed loop is stable by pole placement,
- Q(s) is inside A(s) such that the unstable factors of R(s) and W(s) disappear!

The problem with this design is that the order of the controller might be high because the order of the generalized plant is increased.

Simplified design procedure for Servo Control :

(i) Obtain plant coprime polynomial fraction as $G=N(s)/D(s)$.

(ii) Determine Q from the given types of disturbance and reference.

(iii) Skip the step of generalized plant. Directly include Q(s) as a factor of A(s) or in other words, design

$$A(s)=Q(s)A'(s)$$

Also note that if the poles of the open loop D(s) already contain some factors of Q(s), there is no need to include them again in A(s).

Cancelling out stable poles/zeros may also lead to simpler design.

(iv) Solve the pole placement problem:

$$\underbrace{D(s)Q(s)A'(s)}_{\text{分母}}+\underbrace{N(s)B(s)}_{\text{分子}}=P_c(s)$$

Always make sure that the controller B/A is proper!

Servo controller design for MIMO system

Consider an $m\times m$ plant:

$$\dot{x}=Ax+Bu+B_ww,\quad y=Cx,$$

The error is defined as $e=r-y.$

Suppose that the disturbance *w* and reference *r* are **both of step type**. We introduce one integrator to each channel:

$$v(t)=\int_0^t e(\tau)d\tau.$$

$$\dot{v}(t)=e(t)=r-y(t)=r-Cx(t).$$

Combine the integrators with the state space model to form the augmented system:

$$\begin{pmatrix}\dot{x}\\ \dot{v}\end{pmatrix}=\begin{pmatrix}A&O\\-C&O\end{pmatrix}\begin{pmatrix}x\\v\end{pmatrix}+\begin{pmatrix}B\\O\end{pmatrix}u+\begin{pmatrix}B_w\\O\end{pmatrix}w+\begin{pmatrix}0\\I\end{pmatrix}r$$

$$\dot{\bar{x}}=\bar{A}\bar{x}+\bar{B}u+\bar{B}_ww+\bar{B}_rr$$

Augmented system $\dot{\bar{x}}=\bar{A}\bar{x}+\bar{B}u+\bar{B}_ww+\bar{B}_rr$

If the augmented system is controllable, it can be stabilized by the state feedback control law:

$$u=-\bar{K}\bar{x}=-[K_1\ \ K_2]\begin{bmatrix}x\\v\end{bmatrix}$$

or

$$u(t)=-K_1x-K_2\int_0^te\,d\tau,$$
$\uparrow$ P $\uparrow$ I

which contains integral control. The feedback gain *K* may be determined by pole placement procedure or LQR.

Once the feedback system is stable, each signal in the system in response to step inputs will be constant in the steady state, and so is $v(t)$.

$$e(t)=\dot{v}(t)=0.$$

Thus, zero steady state error has been achieved.

Example 3 Let $G(s)=\frac{s+3}{s-1}:=\frac{N(s)}{D(s)}$

The reference is a sinusoid signal $r(t)=\sin(t)$, and the disturbance is of step type.

$$R(s)=\frac{1}{s^2+1},\ W(s)=\frac{1}{s}$$

What is Q(s)? $Q=s(s^2+1)$

Form the generalized plant:

$$\tilde{G}=\frac{1}{Q}G(s)=\frac{s+3}{s^4-s^3+s^2-s},$$

What is the order of the generalized plant? n=4

What is the order of the controller $\tilde{K}$? n-1=3

A controller $\tilde{K}$ of order 3 can be determined from

$$DQA+NB=P_c$$

If $P_c(s)$ is chosen as

$$P_c(s)=(s+3)^6(s+1)$$
$$=s^7+19s^6+153s^5+675s^4+1755s^3+2673s^2+2187s+7$$

then (15) is solved to get the solution as

$A(s)=A_0+A_1s+A_2s^2+A_3s^3=363+172s+20s^2+s^3$

$B(s)=B_0+B_1s+B_2s^2+B_3s^3=243+769s+571s^2+465s^3.$

Thus $\tilde{K}$ is

$$\tilde{K}=\frac{465s^3+571s^2+769s+243}{s^3+20s^2+172s+363}$$

The complete servo controller is

$$K(s)=\frac{1}{s(s^2+1)}\tilde{K}.$$

Design A(s) as

$$A(s)=Q(s)(s+A_0)=s(s^2+1)(s+A_0)$$

Since the order of A is 4, let

$$B(s)=B_0+B_1s+B_2s^2+B_3s^3$$

such that the controller is strictly proper.

$$A(s)D(s)+B(s)N(s)=s(s^2+1)(s+A_0)(s-1)+(B_0+B_1s+B_2s^2+B_3s^3)(s+3)=P_c(s)$$

Notice that (s+3) is a stable factor, which can be cancelled out. Let

$$P_c(s)=(s+1)^4(s+3)=(s^4+4s^3+6s^2+4s+1)(s+3)$$

$s+A_0=s+3$

then

$$\begin{aligned}A(s)D(s)+B(s)N(s)&=(s+3)s(s^2+1)(s-1)+(B_0+B_1s+B_2s^2+B_3s^3)(s+3)\\&=(s+3)(B_0+(B_1-1)s+(B_2+1)s^2+(B_3-1)s^3+s^4)\\&=(s^4+4s^3+6s^2+4s+1)(s+3)\end{aligned}$$

$$B_0+(B_1-1)s+(B_2+1)s^2+(B_3-1)s^3+s^4=1+4s+6s^2+4s^3+s^4$$

Comparing the coefficients, we have

$$B(s)=1+5s+5s^2+5s^3$$

$$A(s)=s(s^2+1)(s+3)$$