EE5801 Tutorial 3 Solutions

Q1

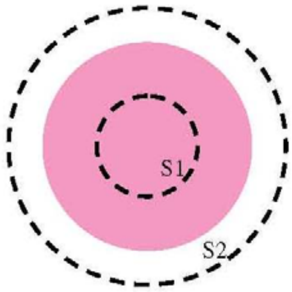
check for spherical symmetry $\rightarrow$ can apply Gauss's Law

$$\text{LHS} = \oint\!\!\oint \vec{E} \bullet d\vec{A} = E_r \oint\!\!\oint dA = E_r 4\pi r^2$$

$$\text{RHS} = \frac{1}{\varepsilon}\iiint \rho(r)\, dV = \frac{1}{\varepsilon}\int_0^r \frac{\rho_0 r}{a} r^2 dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{\pi\rho_0 r^4}{\varepsilon a}$$

equate LHS and RHS expressions for S1 within sphere

$$\vec{E}\,(r<a) = \frac{\rho_0 r^2}{4\varepsilon a}\,\hat{u}_r$$



change upper bound of RHS for S2 outside sphere to obtain $Q = 4\pi \int_0^a \frac{\rho_0 r}{a} r^2 dr = \pi\rho_0 a^3$

$$\vec{E}\,(r>a) = \frac{\rho_0 a^3}{4\varepsilon_0 r^2}\,\hat{u}_r$$

add metallic spherical shell (to shield charged sphere)

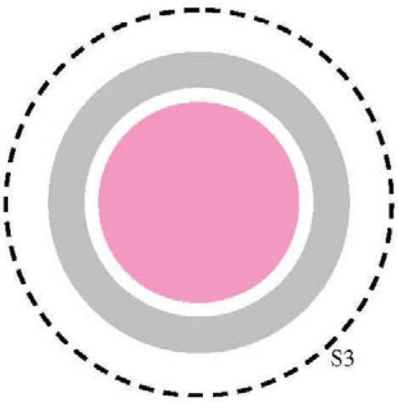
$-Q$ induced on $r = b$ surface of spherical shell

(i) <u>without earthing</u>

$+Q$ residing on $r = c$ surface of spherical shell

no change in total charge for RHS expression

$$\vec{E}\,(r>c) = \frac{\rho_0 a^3}{4\varepsilon_0 r^2}\,\hat{u}_r \quad \text{before earthing}$$



(ii) <u>with earthing</u>

zero charge on $r = c$ surface of spherical shell

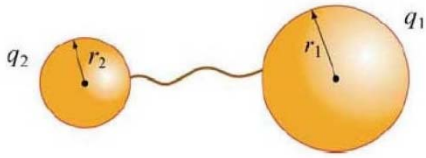
need to include in RHS expression addition of charge $-Q$ from earth

$$\vec{E}\,(r>c) = 0 \qquad \text{after earthing}$$

Q2

(a)

common potential because of wire connection

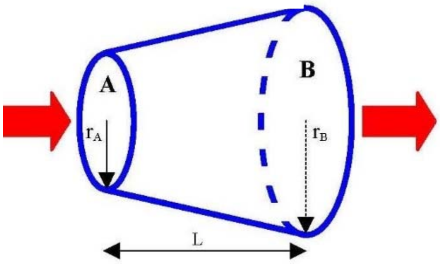


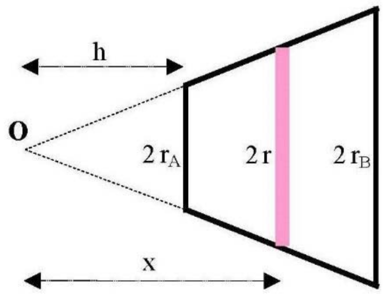
$$\frac{q_1}{4\pi\varepsilon_o r_1} = \frac{q_2}{4\pi\varepsilon_o r_2} \qquad \frac{q_1}{q_2} = \frac{r_1}{r_2}$$

$$\therefore \quad \frac{E_1}{E_2} = \frac{\dfrac{q_1}{4\pi\varepsilon_o r_1^2}}{\dfrac{q_2}{4\pi\varepsilon_o r_2^2}} = \frac{q_1}{q_2}\frac{r_2^2}{r_1^2} = \frac{r_2}{r_1}$$

implication: stronger electric field expected at sharper corners since $E \propto \frac{1}{r}$

(b)





elemental resistance of dx strip at x from O: $\quad dR = \frac{dx}{\pi\sigma r^2}$

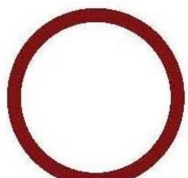
need to integrate from $x = h$ to $x = h + L$: $\quad R = \frac{1}{\pi\sigma}\int_h^{h+L}\frac{dx}{r^2} = \frac{L}{\pi\sigma r_A r_B}$

based on geometrical relationships $\frac{r}{x} = \frac{r_A}{h} = \frac{r_B}{h+L}$

(c)

$$I = \iint\limits_{\text{circle}} J_0 \exp\left(\frac{r^2 \quad a^2}{\delta^2}\right) r\,dr\,d\phi$$

$$= J_0 \int_0^a \exp\left(\frac{r^2 \quad a^2}{\delta^2}\right) r\,dr \int_0^{2\pi} d\phi$$

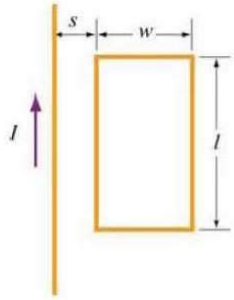


current flowing within thin layer

$$= J_0 \pi \delta^2 \left\{1 - \exp\left(-\frac{a^2}{\delta^2}\right)\right\}$$ (using substitution $u = \frac{r^2 \quad a^2}{\delta^2}$ as integration aid)

$$\rightarrow J_0 \pi \delta^2 \quad \text{when } \delta << a$$

Q3



choose to place (fictitious) current I on straight wire

can thus apply Ampere's Law

$$B_\phi = \frac{\mu_0 I}{2\pi r} \qquad \text{(valid only if wire is sufficiently long)}$$

gives rise to flux linkage with loop

$$d\Phi = B_\phi dA = \frac{\mu_0 I}{2\pi r}(l\,dr) = \frac{\mu_0 I l}{2\pi}\frac{dr}{r}$$

obtain mutual inductance by dividing total flux to current

$$M = \frac{\Phi}{I} = \frac{\mu_0 l}{2\pi}\int_s^{s+w}\frac{dr}{r} = \frac{\mu_0 l}{2\pi}\ln\left(1+\frac{w}{s}\right)$$

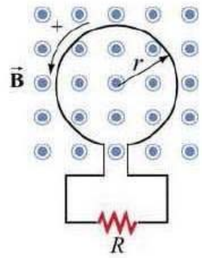
alternate approach: place (fictitious) current on wire loop

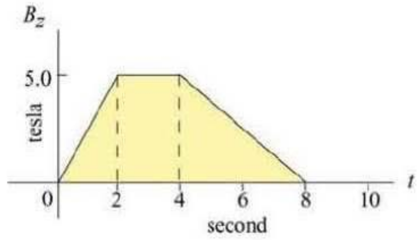
will obtain same numerical answer for M

prefer not to pursue this option:

- more cumbersome to derive B of loop
- also need to consider question of area enclosed by straight wire.

Q4





current in R due to EMF given by Faraday's Law

$$I = \frac{V_{EMF}}{R} = -\frac{1}{R}\frac{d}{dt}\iint \vec{B} \bullet d\vec{A} = -\frac{\pi r^2}{R}\frac{dB_z}{dt}$$

Lenz's Law: negative sign $\rightarrow$ current in clockwise sense if $\frac{dB_z}{dt} > 0$

for $0 < t < 2$, EMF due to (linear) increase in B

$$\therefore\ I = -\frac{\pi\, 0.5^2}{100}\frac{5}{2} = -19.6 \text{ mA } (\textit{i.e.} \text{ clockwise})$$

for $2 < t < 4$, no EMF due to constant B

$$\therefore\ I = -\frac{\pi\, 0.5^2}{100}\frac{0}{2} = 0$$

for $4 < t < 8$, EMF due to (linear) decrease in B

$$\therefore\ I = -\frac{\pi\, 0.5^2}{100}\left(-\frac{5}{4}\right) = +9.8 \text{ mA } (\textit{i.e.} \text{ anti-clockwise})$$

for $t > 8$, no EMF due to constant B and hence back to zero current