Financial Engineering

Homework 1

Due at 07:00 pm (Korea Standard Time) on Saturday, February 11.

Submit one file: written solutions with executable Python code

Problem 1. An investor deposits \$15,000 in a bank today at 6% annual interest. Assuming that no additional deposits are made, and that no money is withdrawn, how large will the account balance be 23 years from today if

(a) the interest is computed using the simple interest convention?
(b) the interest is compounded annually?
(c) the interest is compounded quarterly?
(d) the interest is compounded monthly?
(e) the interest is compounded continuously?

(You should assume that the 6% annual interest rate applies to deposits of any length.)

Problem 2. If you buy a lottery ticket in 50 lotteries, in each of which your chances of winning a prize of 1/100, what is the probability that you will win a prize:

(a) at least once?
(b) exactly twice?
(c) at least twice?

Calculate both the exact probabilities (using the binomial distribution) and the approximate probabilities (using the Poisson distribution).

Problem 3. Consider the following situation:

$$Y \sim N(\mu, \sigma^2)$$
$$(X \mid Y = y) \sim N(y, \nu^2)$$

We want to find the distribution of X.

(a) Find $f_{XY}(x, y)$ and then use a Law of Total Probability to determine the distribution of X

$$f_X(x) = \int f_{X|Y}(x \mid y) f_Y(y)\, dy$$

(b) Find the MGF $M_X(t)$ and determine the distribution of X

$$M_X(t) = E(e^{tX}) = E[E(e^{tX} \mid Y)]$$

where the "inner" expected value, $E(e^{tX} \mid Y)$ is the moment generating function of the random variable $(X \mid Y = y)$

Problem 4. Many people believe that the daily change of price of a company's stock on the stock market is a random variable with mean 0 and variance σ^2. That is, if S_t represents the price of the stock on the t-th day, then

$$S_t = S_{t-1} + W_t, \quad t \geq 1$$

where $W_1, W_2, \cdots,$ are independent and identically distributed random variables with mean 0 and variance σ^2. Suppose that the stock's price today is 100 and $\sigma^2 = 1$.

(a) What can you say about the probability that the stock's price will exceed 105 after 10 days? (Hint: Chebyshev inequality)

(b) Suppose $W_1, W_2, \cdots,$ are independent and identically distributed Normal random variables with mean 0 and variance σ^2. What is the probability that the stock's price will exceed 105 after 10 days?

(c) Simulate $M = 10{,}000$ trajectories of the stock price S_t for $t = 0, 1, \ldots, 10$. Plot 10 pairs of trajectories of the stock and calculate the probability that the stock's price exceeds 105 after 10 days.

Problem 5. The IRR is generally calculated using an iterative procedure. Suppose that we define $f(\lambda) = -a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$, where all a_i are positive and $n > 1$. Here is an iterative technique that generates a sequence $\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_k, \dots$ of estimates that converges to the root $\bar{\lambda} > 0$, solving $f(\bar{\lambda}) = 0$. Start with any $\lambda_0 > 0$ close to the solution. Assuming λ_k has been calculated, evaluate

$$f'(\lambda_k) = a_1 + 2a_2\lambda_k + 3a_3\lambda_k^2 + \cdots + na_n\lambda_k^{n-1}$$

and define

$$\lambda_{k+1} = \lambda_k - \frac{f(\lambda_k)}{f'(\lambda_k)}$$

Try the procedure on $f(\lambda) = -1 + \lambda + \lambda^2$ starting with $\lambda_0 = 1$.

(a) Write a Python function to calculate $\bar{\lambda}$ accurate up to 0.000001 and compute the computation time

(b) Use the Bisection method to calculate $\bar{\lambda}$ accurate up to 0.000001 and compare the computation time with the result from (a)

Problem 6. Given N points drawn randomly on the circumference of a circle, what is the probability that they are all within a semicircle?