# Lecture 10. Derivative.

## Derivatives of Elementary Functions:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$, for $x > 0$

For real numbers $p$ and for $x > 0$, $\frac{d}{dx}(x^p) = px^{p-1}$.

If $b > 0$, then for all $x$, $\frac{d}{dx}(b^x) = b^x \ln b$.

## Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

# Important differentiation Rules.

### Table of differentiation Formulas:

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

### Product Rule and Quotient Rule

If $f$ and $g$ are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

If $f$ and $g$ are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

### Chain Rule

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at $x$. The composite function $y = f(g(x))$ is differentiable at $x$, and its derivative can be expressed in two equivalent ways:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \qquad \text{Version 1}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x) \qquad \text{Version 2}$$

E.g. Prove $\sin' x = \cos x$.

$$\lim_{\Delta\to 0}\frac{\sin(x+\Delta)-\sin x}{\Delta} = \lim_{\Delta\to 0} \sin\left(x+\tfrac{1}{2}\Delta+\tfrac{1}{2}\Delta\right) - \sin\left(x+\tfrac{1}{2}\Delta-\tfrac{1}{2}\Delta\right)$$

Recall that $\left.\begin{array}{l}\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.\end{array}\right\}\Rightarrow \begin{array}{l}\sin(\alpha+\beta)-\sin(\alpha-\beta)\\ =2\cos\alpha\sin\beta.\end{array}$

$$= \lim_{\Delta\to 0}\frac{2\cos(x+\frac{1}{2}\Delta)\sin\frac{1}{2}\Delta}{\Delta}$$

$$= \underbrace{\left[\lim_{\Delta\to 0}\cos(x+\tfrac{1}{2}\Delta)\right]}_{\text{continuity of } \cos x} \cdot \left[\lim_{\Delta\to 0}\frac{\sin\frac{1}{2}\Delta}{\frac{1}{2}\Delta}\right].$$

$$= \cos x \cdot 1$$

E.g. $f(x)=\dfrac{(x^3-1)^4\sqrt{3x-1}}{x^2+4}$. Find $f'(x)$.

Solution 1. by produce rule,

Denote $f_1(x) = (x^3-1)^4\cdot\sqrt{3x-1}$ $\quad f_1'(x) = 4(x^3-1)^3\cdot 3x^2\cdot\sqrt{3x-1} + (x^3-1)^4\cdot\frac{3}{2}(3x-1)^{-\frac{1}{2}}$.

$f_2(x) = x^2+4$. $\quad f_2'(x) = 2x$.

$$\left(\frac{f_1(x)}{f_2(x)}\right)' = \frac{f_1'(x)f_2(x) - f_1(x)f_2'(x)}{f_2^2(x)} = \frac{12x^2(x^3-1)^3\sqrt{3x-1}+\frac{3}{2}(3x-1)^{-\frac{1}{2}}(x^3-1)^4}{x^2+4} - \frac{2(x^3-1)^4\sqrt{3x-1}}{x^2+4}x$$

Solution 2. by taking ln on both sides:

$$\ln f(x) = 4\ln(x^3-1) + \frac{1}{2}\ln(3x-1) - \ln(x^2+4)$$

$$\Downarrow$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{4}{x^3-1} \cdot 3x^2 + \frac{3}{2}\,\frac{1}{3x-1} - \frac{2x}{x^2+4}$$

$$f'(x) = f(x) \cdot \left[ \frac{12x^2}{x^3-1} + \frac{3}{2(3x-1)} - \frac{2x}{x^2+4} \right].$$

Exercise 1. Given that $\lim\limits_{x\to 0} \frac{\sin x}{x} \to 1$, find the following limits.

1. $\lim\limits_{x\to 0} \frac{\tan 2x}{\sin x}$

2. $\lim\limits_{x\to 0} \frac{\tan nx}{\sin x}$

3. $\lim\limits_{x\to 0} \frac{\cos^2 x - 1}{x}$

4. $\lim\limits_{x\to 2} \frac{\sin(x-2)}{x^2-4}$

5. $\lim\limits_{x\to 0} \frac{6x - \sin 2x}{2x - 3\sin 4x}$.

Solution :

1. $\lim\limits_{x\to 0} \frac{\tan 2x}{\sin x} = \lim\limits_{x\to 0} \frac{\sin 2x}{\sin x} \frac{1}{\cos 2x} = \lim\limits_{x\to 0} \frac{\sin 2x}{\sin x} \cdot \lim\limits_{x\to 0} \frac{1}{\cos 2x} = 2.$

2. $\lim\limits_{x\to 0} \frac{\tan nx}{\sin x} = \lim\limits_{x\to 0} \frac{\sin(nx)}{\sin x} \frac{1}{\cos nx} = n.$

3. $\lim\limits_{x\to 0} \frac{\cos^2 x - 1}{x} = \lim\limits_{x\to 0} -\frac{\sin^2 x}{x} = -\lim\limits_{x\to 0} \frac{\sin x}{x} \cdot \lim\limits_{x\to 0} \sin x = 0.$

4. $\lim_{x\to 2} \frac{\sin(x-2)}{x^2-4} = \lim_{x\to 2} \frac{\sin(x-2)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{\sin(x-2)}{x-2} \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}.$

5. $\lim_{x\to 0} \frac{6x - \sin 2x}{2x - 3\sin 4x}. = \lim_{x\to 0} \frac{6 - \frac{\sin 2x}{x}}{2 - \frac{3\sin 4x}{x}} = \frac{\lim_{x\to 0} 6 - \frac{\sin 2x}{x}}{\lim_{x\to 0} 2 - 3\frac{\sin 4x}{x}}$

$= \frac{6-2}{2-12} = -\frac{4}{10} = -\frac{2}{5}.$

Exercise 2. Use the definition of derivative to prove that

$$\frac{d(x^4)}{dx} = 4 \cdot x^3.$$

Solution: $\frac{d(x^4)}{dx} = \lim_{\Delta\to 0} \frac{(x+\Delta)^4 - x^4}{\Delta} = \lim_{\Delta\to 0} \frac{4\Delta x^3 + 6\Delta^2 x^2 + 4\Delta^3 x + \Delta^4}{\Delta}.$

$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} \cdot a^i b^{n-i}$

$n=4.$ $\binom{4}{0}=1,$ $\binom{4}{1}=4,$ $\binom{4}{2}=6,$ $\binom{4}{3}=4,$ $\binom{4}{4}=1.$

$(a+b)^4 = b^4 + 4 \cdot a \cdot b^3 + 6a^2b^2 + 4 \cdot a^3 b + a^4.$

$\lim_{\Delta\to 0} \frac{4\Delta x^3 + 6\Delta^2 x^2 + 4\Delta^3 x + \Delta^4}{\Delta}$

$= 4x^3 + \lim_{\Delta\to 0} 6\Delta x^2 + 4\Delta^2 x + \Delta^3 = 4x^3.$

Exercise 3. Differentiate the function.

1. $y = x^{\frac{5}{3}} - x^{\frac{2}{3}}.$

2. $y = \frac{x^2+4x+3}{\sqrt{x}}$

3. $y = (4x - x^2)^{100}$

4. $y = x^4 e^x$.

5. $y = e^{-\frac{1}{2}x^2}$.

6. $y = \sqrt{1 + xe^{-2x}}$.

solution :

1. $y = x^{\frac{5}{3}} - x^{\frac{2}{3}}$.

$y' = \frac{5}{3}x^{\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}}$.

2. $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$

$y = x^{\frac{3}{2}} + 4 \cdot x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$

$y' = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$.

3. $y = (4x - x^2)^{100}$

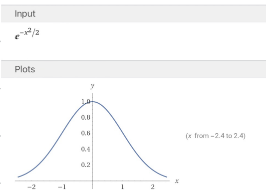
$y' = 100\,(4x - x^2)^{99} \cdot (4x - 2x)' = 100 \cdot (4x - x^2)^{99} \cdot (4 - 2x)$

4. $y = x^4 e^x$.

$y' = x^4 \cdot (e^x)' + (x^4)' e^x = \underline{x^4} e^x + 4x^3 e^x$

5. $y = e^{-\frac{1}{2}x^2}$.

$y' = e^{-\frac{1}{2}x^2} \cdot (-\frac{1}{2}x^2)' = -x \cdot e^{-\frac{1}{2}x^2}$




bell curve.

6. $y = \sqrt{1 + xe^{-2x}}$.

$y' = \frac{1}{2}(1 + xe^{-2x})^{-\frac{1}{2}} \cdot (1 + xe^{-2x})'$

$$= \frac{1}{2}\left(1+xe^{-2x}\right)^{-\frac{1}{2}}(1-2x)\,e^{-2x}.$$

Exercise 4. Find the derivative of $y = \dfrac{x^{\frac{3}{4}} \cdot \sqrt{x^2+1}}{(3x+2)^5}$. (hint: take ln).

Solution: $\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$.

$$\Downarrow$$

$$\frac{1}{y}\cdot y' = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - \frac{5\cdot 3}{3x+2}.$$

$$y' = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$= \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).$$