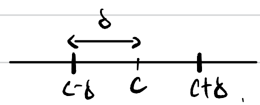
Lecture 15, 16. Extreme values + shapes of functions.

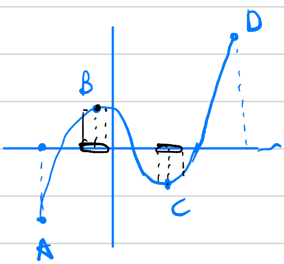
* $f(x)$ is defined on D.

$$\begin{cases} \text{if } f(x) \geq f(c) \text{ for } \forall x \in D, & f(c) \text{ is the global minimum} \\ \text{if } f(x) \leq f(c) \text{ for } \forall x \in D, & f(c) \text{ is the global maximum} \end{cases}$$

$$\begin{cases} \text{if } f(x) \geq f(c) \text{ for } \forall x \in U(c,\delta), & f(c) \text{ is the local minimum} \\ \text{if } f(x) \leq f(c) \text{ for } \forall x \in U(c,\delta), & f(c) \text{ is the local maximum} \end{cases}$$

$U(c,\delta) = (c-\delta, c) \cup (c, c+\delta)$





* Find extreme values? → the existence can be guaranteed by continuity.

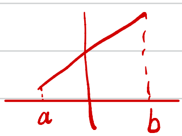
★ check ① two ends and ② $f'(x)=0$

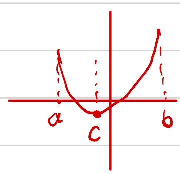
continuity + close set

⇓

existence of extremes

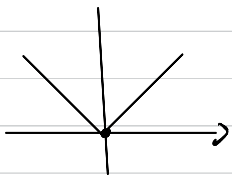
①





Fermat's theorem: If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c)=0$.

Remark: It is possible that $f(c)$ is local extreme and $f'(c)$ does not exist.



* Mean Value Theorem (by Michel Rolle)

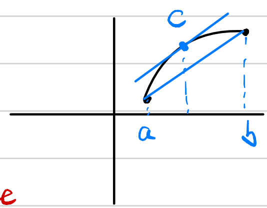
Rolle's Theorem: ① $f(x)$ is continuous on the closed intervals $[a,b]$

② $f(x)$ is differentiable on the open interval (a,b).

Then there must exist $c \in (a,b)$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

trend at c — trend on the whole $[a,b]$.



* Monotone and Extreme test by Derivatives.

① If $f'(x)>0$ on I, the f is increasing

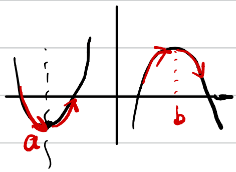
② If $f'(x)<0$ on I, the f is decreasing.

③ If $f'(x) > 0$ for $x > c$ and $f'(x) < 0$ for $x < c$ near c.

then $f(c)$ is a local minimum.

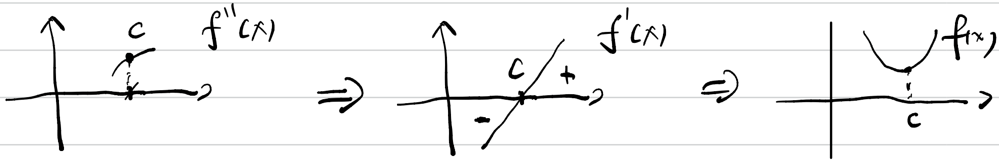
④ If $f'(x) < 0$ for $x > c$ and $f'(x) > 0$ for $x < c$ near c.

then $f(c)$ is a local maximum.



⑤ If $f'(c) = 0$ and $f''(c) > 0$. then $f(c)$ is a local minimum.

⑥ If $f'(c) = 0$ and $f''(c) < 0$. then $f(c)$ is a local maximum



Example 1. Find the absolute maximum and minimum values of

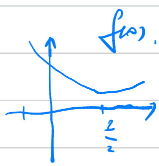
1.1. $f(x) = x^4 - 2x^3$ on $[-2, 2]$

1.2 $f(x) = x^{\frac{2}{3}}(2-x)$ on $[-1, 2]$.

1.1. $f'(x) = 4x^3 - 6x^2 = 4x^2(x - \frac{3}{2})$

$x \in [-2, \frac{3}{2}]$ $f'(x) \le 0$. $f(x) \searrow$.

$x \in (\frac{3}{2}, 2]$ $f'(x) > 0$ $f(x) \nearrow$.



We only need to check $f(2)$, $f(-2)$, $f(\frac{3}{2})$.

$f(2)=0$, $f(-2)=32$ ← maximum.

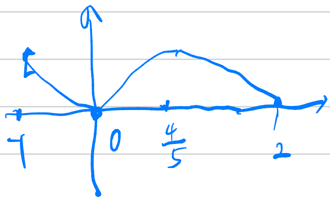
$f(\frac{3}{2}) = -\frac{27}{16}$ ← minimum,

1.2. $f'(x) = \frac{1}{3} x^{-\frac{1}{3}} (4-5x)$.

$x \in [-1, 0)$. $f'(x) < 0$ $f(x) \searrow$

$x \in (0, \frac{4}{5})$ $f'(x) > 0$ $f(x) \nearrow$

$x \in (\frac{4}{5}, 2)$ $f'(x) < 0$ $f(x) \searrow$.



$f(0)=0$, $f(2)=0$, ← min

$f(-1)=3$, $f(\frac{4}{5}) = 1.03$

↑

max.

Exercise 1. prove that $f(x) = x^{101} + x^{51} + x + 1$ has no local extremes on $\mathbb{R}$.

Solution. $f'(x) = 101x^{100} + 51 \cdot x^{50} + 1 \geq 1 > 0$.

$f(x)$ is monotonicly increasing on $\mathbb{R}$.

Exercise 2. Find the Range of functions on given intervals.

2.1 $f(x) = 3x^2 - 12x + 5$. $x \in [0,3]$

2.2 $f(x) = \frac{x}{x^2+1}$ $x \in [0,2]$.

2.3 $f(x) = x e^{-\frac{x^2}{8}}$, $x \in [-1,4]$

2.4 $f(x) = \ln(x^2+x+1)$ $x \in [-1,1]$.

Solution = 2.1 $f(x) = 3x^2 - 12x + 5$. $x \in [0,3]$

$f'(x) = \underline{6x - 12}$ $x \in [0,2]$, $f'(x) < 0$. $f(x) \searrow$

$x \in (2,3]$ $f'(x) > 0$. $f(x) \uparrow$

max: $f(0)$ or $f(3)$, $\underline{f(0) = 5} > f(3) = -4$

min $= \underline{f(2) = -7}$

2.2 $f(x) = \frac{x}{x^2+1}$ $x \in [0,2]$.

$f'(x) = \frac{(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2} = \frac{(1+x)(1-x)}{(x^2+1)^2}$.

$x \in [0,1]$ $f'(x) > 0$ $f(x) \uparrow$.

$x \in (1,2]$ $f'(x) < 0$ $f(x) \searrow$.

max $= f(1) = \frac{1}{2}$

min $= f(0)$ or $f(2)$.

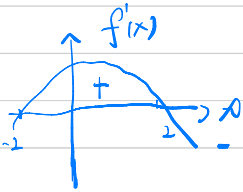
$f(0)=0$. $f(2)=\frac{2}{5}$

$\uparrow$ min.

2.3 $f(x)= x e^{-\frac{x^2}{8}}$, $x\in[-1,1]$

$f'(x)= e^{-\frac{x^2}{8}} + x\cdot(-\frac{x}{4})\cdot e^{-\frac{x^2}{8}}$

$= (1-\frac{x^2}{4})\, e^{-\frac{x^2}{8}}$. $x\in[-1,4]$.



$x\in[-1,2]$, $f'(x)>0$. $f(x)\uparrow$.

$x\in(2,4]$, $f'(x)<0$. $f(x)\downarrow$.

max $= f(2) = 2e^{-\frac{1}{2}}$

min $f(-1)$ or $f(4)$. but $f(4)>0>f(-1)$

Thus, min $= f(-1) = -e^{-\frac{1}{8}}$.

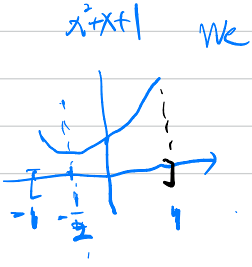
2.4 $f(x)= \ln(x^2+x+1)$ $x\in[-1,1]$.

Notice that $\ln x$ is monotonic increasing.

max $f(x)= \ln(\max[x^2+x+1])$

min $f(x)= \ln(\min[x^2+x+1])$.

We tend to study the extreme values of x^2+x+1.



$x^2+x+1 \to$ $[-1,-\frac{1}{2}]$ $\downarrow$.

$[-\frac{1}{2},1]$ $\uparrow$.

$$\min \ x^2+x+1 = \left(-\tfrac{1}{2}\right)^2 - \tfrac{1}{2} + 1 = \frac{3}{4}.$$

$$\max \ x^2+x+1 = 1+1+1 = 3.$$

$$\min f(x) = \ln\left(\tfrac{3}{4}\right)$$

$$\max f(x) = \underline{\ln(3)}.$$