Review

* Definition of functions. $y=f(x)$, $\boxed{x \in D}$

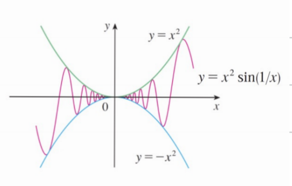
Domain, Range.

* Linear functions. $y = ax+b$
  - representation
  - intercept
  - slope (direction).

* Transformations of functions.
  - $y = a f(c(x+d))$ a, c, d.
  - $y = f^{-1}(x) \Rightarrow \sin^{-1}x, \cos^{-1}x, \tan^{-1}x$

* Limits.
  - Definition $\longrightarrow$ when does the limit exist?
  - $\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\lim_{x\to 0} \frac{\ln(x+1)}{x} = 1$
  - One-sided limits.

* First way to compute the limit: Sandwich Theorem



$g(x) \le f(x) \le h(x)$
↑
Can be constants.

* Limit laws:

- $\lim_{x \to a} f(x) \cdot g(x)$, $f(x) + g(x)$, $\frac{f(x)}{g(x)}$
- Important trick: rationalization. $\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$

* Continuity (based on the limit).

$\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x) = f(x_0)$.

- Continuous $\Rightarrow$ $\boxed{\lim}$ operator can pass through the function.

  $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$

- law of preserving continuity $\to$ limit laws.
- If $f(x)$ is continuous $\Rightarrow$ Intermediate Value Theorem. (condition)

  $[a, b]$ closed. $f(a) \neq f(b)$.

- Extend a funtion to make it continuous. in $\mathbb{R}$.

* Limit at infinity & Asymptotes.

- $\lim_{x \to +\infty} f(x) = \lim_{x \to 0^+} f(\frac{1}{x})$.
- Compute asymptotes.

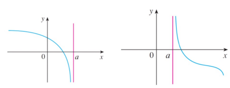
  ☆ check the limit.

(a) Vertical Asymptote

6 **Definition** The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$



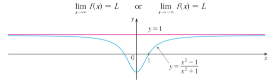
(c) $\lim_{x \to a^-} f(x) = -\infty$

(b) $\lim_{x \to a^+} f(x) = \infty$

(b) Horizontal Asymptote

3 **Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$



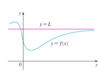


FIGURE 2
Examples illustrating $\lim_{x \to \infty} f(x) = L$

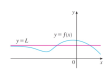


FIGURE 3
Examples illustrating $\lim_{x \to -\infty} f(x) = L$

* Derivatives. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

- Differentiable $\Longrightarrow$ continuous.
- Two-sided
- Differentiation rules: x^n, e^x, $\sin x$, etc
- Extend a funtion to make it Differentiable

two-sided

e.g. $x^n \sin\frac{1}{x}$.

* . **Derivatives of Elementary Functions:**

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$, for $x > 0$

For real numbers p and for $x > 0$, $\frac{d}{dx}(x^p) = px^{p-1}$.

If $b > 0$, then for all x, $\frac{d}{dx}(b^x) = b^x \ln b$.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

Table of differentiation Formulas:

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Product Rule and Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Chain Rule

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{Version 1}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \qquad \text{Version 2}$$

* Derivatives of inverse functions =

$$\frac{d f^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))}$$

$\frac{d(\sin^{-1}x)}{dx} \rightarrow$ what is $\underline{\cos(\sin^{-1}x)}$?

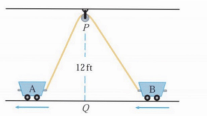
* Rate of changes + solving practical problems.

① Introduce notations

② Find relations of quantities.

③ Compute derivatives.

Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P (see the figure). The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



* Linear approximation

$y = f(x) \Rightarrow y = f(x_0) + f'(x_0)(x - x_0)$. accurate when $x \rightarrow x_0$.

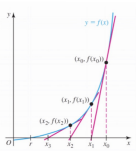
- Newton's method. $\Rightarrow$ solving $f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0$

PROCEDURE **Newton's Method for Approximating Roots of $f(x) = 0$**

1. Choose an initial approximation x_0 as close to a root as possible.
2. For $n = 0, 1, 2, \ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{provided } f'(x_n) \neq 0.$$

3. End the calculations when a termination condition is met.



Solving for x and calling it x_{n+1}, we find that

$$\underbrace{x_{n+1}}_{\text{new approximation}} = \underbrace{x_n}_{\text{current approximation}} - \frac{f(x_n)}{f'(x_n)}, \text{ provided } f'(x_n) \neq 0.$$

* Extreme Values

- Global maximum / minimum. $\Rightarrow$ $\boxed{\text{Range}}$ of functions.

- local maximum/minimum

- Mean Value Theorem = $f'(c) = \frac{f(a)-f(b)}{a-b}$. (Recall IVT).

- Monotonic and Extreme value tests.

- Convexity $f''(x)>0$ convex 

  $f''(x)<0$ concave 

- Curve sketching

☆ L'Hopital's rule

$\lim\limits_{x\to a} \frac{f(x)}{g(x)}$

① $\frac{\infty}{\infty}$ $\lim\limits_{x\to a} f(x) = \infty$

$\lim\limits_{x\to a} g(x) = -\infty$.

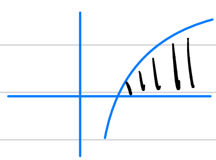
② $\frac{0}{0}$ $\lim\limits_{x\to a} f(x)=0$

$\lim\limits_{x\to a} g(x)=0$.

$\Rightarrow \lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$.

☆. Integral

- . Inverse of derivative. original function → rate of change of integral.

- Area problem.  ☆ positive or negative.

- Definite Integral. Computation, comparasion.