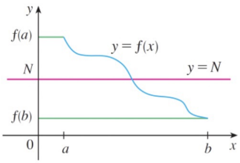
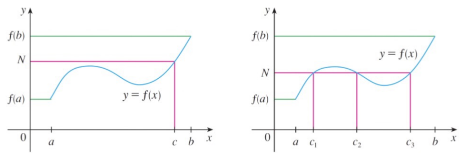
Lecture 7

Continuity.

10 **The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].





(a) (b)

Exercise 1: Check the values of the following function at, e.g $x=0,1,2$ to find at least one solution of $f(x)=0$.

1.1. $f(x) = x^4 + x - 3$. 1.2 $f(x) = \sqrt[3]{x} + x - 1$.

1.3 $f(x) = e^x + 2x - 3$ 1.4 $f(x) = x^2 - x - \sin(x+1)$.

Solution: 1.1 $f(1) = -1$, $f(2) = 15$. $\exists\, x_0 \in (1,2)$, such that $f(x_0) = 0$

1.2. $f(0) = -1$ $f(1) = 1$ $\exists\, x_0 \in (0,1)$. such that $f(x_0) = 0$

1.3 $f(0) = -2$ $f(1) = e - 1 > 0$ $\exists\, x_0 \in (0,1)$, such that $f(x_0) = 0$.

1.4 $f(1) = -\sin 2 < 0$, $f(2) = 2 - \sin(3) > 0$, $\exists\, x_0 \in (1,2)$, such that $f(x_0) = 0$

Exercise 2. Extend the domain and definition of $f(x) = x \cos\frac{1}{x}$ to make it continuous in $\mathbb{R}$.

solution: Define $g(x) = \begin{cases} 0 \\ x\cos\frac{1}{x} \end{cases}$. $g(x)$ is continuous.

$|g(x)| \leq |x| \cdot |\cos\frac{1}{x}| \leq |x|$. $\Rightarrow$ $-|x| \leq g(x) \leq |x|$.

when $x \to 0$, $-|x| = |x| = 0$. $\Rightarrow$ $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} x\cos\frac{1}{x} = 0$.

Exercise 3: For the following 2 statements:

A: f is continuous at point x_0.

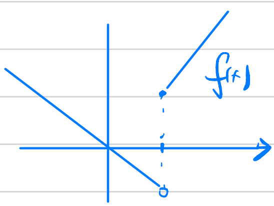
B: f^2 and $|f|$ are continuous at point x_0.

Can $A \Rightarrow B$? Can $B \Rightarrow A$?

solution: $A \Rightarrow B$ ✓ Because $g(z) = z^2$, $g(z) = |z|$ are continuous.

$B \Rightarrow A$ ✗ Counterexample:

$f(x) = \begin{cases} x & x \geq 1 \\ -x & x < 1 \end{cases}$ $\boxed{x_0 = 1}$.



$f^2(x) = x^2$, $|f(x)| = |x|$ $\Rightarrow$ continuous.

Lecture 8.

Limit of a function $f(x)$ for x tends to infinity

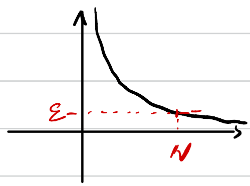
7 Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

Example: 1. $\lim_{x \to +\infty} \frac{1}{x} = 0$

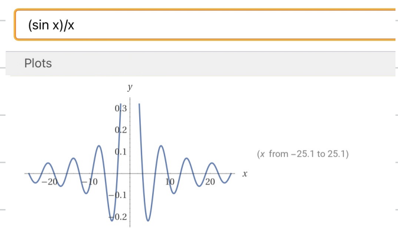


proof: For any $\varepsilon > 0$,

take $N = \frac{2}{\varepsilon}$. We then have:

$\forall\, x > N = \frac{2}{\varepsilon}$, $|f(x) - 0| = \left|\frac{1}{x}\right| < \frac{\varepsilon}{2} < \varepsilon$.

2. $\lim_{x \to +\infty} \frac{\sin x}{x} = 0$.



proof: by sandwich thm,

$\left|\frac{\sin x}{x}\right| \leq \frac{1}{x}$. $(x > 0)$.

$-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$

☆ limit laws, sandwith thm and other rules of limts can also by applied to limits at infinity.

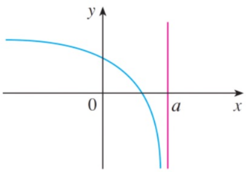
Asymptote

(a) Vertical Asymptote

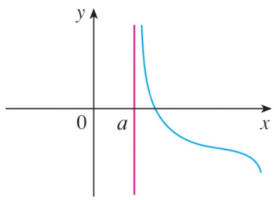
6 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$

$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$



(c) $\lim_{x\to a^-} f(x) = -\infty$

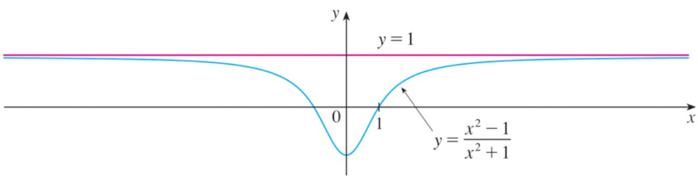


(b) $\lim_{x\to a^+} f(x) = \infty$

(b) Horizontal Asymptote

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to -\infty} f(x) = L$$



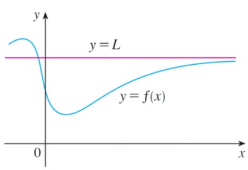


FIGURE 2
Examples illustrating $\lim_{x\to\infty} f(x) = L$

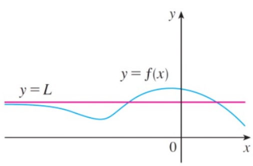


FIGURE 3
Examples illustrating $\lim_{x\to -\infty} f(x) = L$

Exercise 1. Find all the horizontal and vertical asymptotes of

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}.$$

Solution: Vertical asymptotes: at point $x = \frac{5}{3}$ ☆.

$\lim_{x \to \frac{5}{3}^+} f(x) = +\infty$,

$\lim_{x \to \frac{5}{3}^-} f(x) = -\infty$.

$\Rightarrow x = \frac{5}{3}$

horizontal asymptotes: $\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \to \frac{\sqrt{2}}{3}$,

☆ change limit at inf to limit at 0.

$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \boxed{\frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}} \to \frac{-\sqrt{2}}{3}$.

here $f(x) = \frac{\sqrt{2x^2+1}\,/x}{(3x-5)/x} = \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}$.

$y = \frac{\sqrt{2}}{3}$, $y = -\frac{\sqrt{2}}{3}$

Derivatives and rate of changes:

2) The Derivative of a Function

DEFINITION The Derivative

The **derivative** of f is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

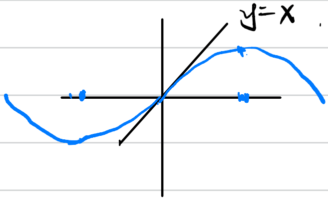
provided the limit exists. If $f'(x)$ exists, we say f is **differentiable** at x. If f is differentiable at every point of an open interval I, we say that f is differentiable on I.

$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

= **The slope of tangent of the curve ($y = f(x)$) at the point $x = a$.**

= **The instantaneous rate of change of $f(x)$ at $x = a$.**

Example: Find the slope of the tangent line of $y=\sin x$ at $x=0$.



$$f'(0)=\lim_{h\to 0}\frac{\sin(0+h)-\sin(0)}{h-0}$$

$$=\lim_{h\to 0}\frac{\sin h}{h}=1.$$

Exercise 2. Find: 1. $\lim_{x\to 0}\frac{\sin 4x}{x}$

2. $\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}$

Solution: 1. $\lim_{x\to 0}\frac{\sin 4x}{x}=\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot 4$

$$=\lim_{y\to 0}\frac{\sin y}{y}\cdot 4 \quad (y=4x)$$

$$=4.$$

2. $\lim_{x\to 0}\frac{\sin 3x}{\sin 5x}=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{5x}{\sin 5x}\cdot\frac{3}{5}$

$$=\left[\lim_{x\to 0}\frac{\sin 3x}{3x}\right]\cdot\left[\lim_{x\to 0}\frac{5x}{\sin 5x}\right]\cdot\frac{3}{5}$$

$$=\lim_{z_1\to 0}\frac{\sin z_1}{z_1}\cdot\lim_{z_2\to 0}\frac{z_2}{\sin z_2}\cdot\frac{3}{5}$$

$$=\frac{3}{5}.$$

Exercise 3: Find all the asymptotes of

1. $f(x) = e^{\frac{1}{x}}$

2. $f(x) = \frac{|1-x^2|}{x(x+1)}$

3. $f(x) = \frac{x^2+\sqrt{16x^4+16x^2}}{2x^2-4}$.

Solution: 1. $f(x)=e^{\frac{1}{x}}$, Vertical: $x=0$.

$x \to 0^+$. $f(x) = e^{+\infty} = +\infty$

$x \to 0^-$. $f(x) = e^{-\infty} = 0$,

Horizontal: $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = e^0 = 1$.

$y=1$

2. $f(x) = \frac{|1-x^2|}{x(x+1)}$

$= \frac{|(1+x)(1-x)|}{x(x+1)}$. ← $x=0$, $x=-1$.

☆ $x=-1$ is not asymptote. $\lim_{x\to-1} f(x) = -2$.

Vertical $x=0$.

$f(x) = \frac{|1-x^2|/x^2}{(x^2+x)/x^2} = \frac{|(\frac{1}{x^2}-1)|}{1+\frac{1}{x}}$, $\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = 1$.

Horizontal: $y=1$.

3. $f(x) = \dfrac{x^2+\sqrt{16x^4+16x^2}}{2x^2-4}$

$= \dfrac{x^2+\sqrt{16x^4+16x^2}}{2(x+\sqrt{2})(x-\sqrt{2})}$

Vertical : $\underline{x=\sqrt{2}}$, $\underline{x=-\sqrt{2}}$.

$f(x) = \dfrac{(x^2+\sqrt{16x^4+16x^2})/x^2}{(2x^2-4)/x^2} = \dfrac{1+\sqrt{16+16/x^2}}{2-\frac{4}{x^2}}$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \dfrac{1+\sqrt{16}}{2-0} = \dfrac{5}{2}$.

Horizontal : $y=\frac{5}{2}$.