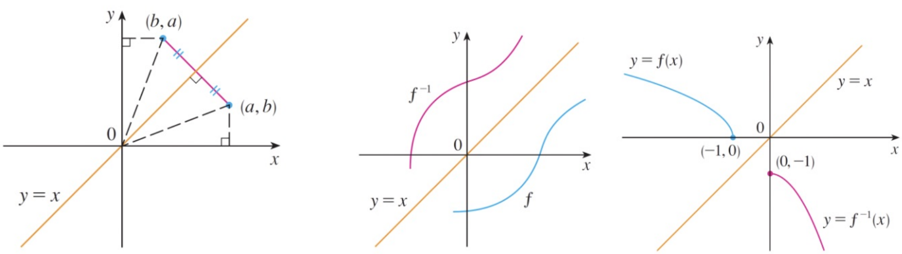
Lecture 4.

Inverse function :

(a) **If a function $f(x)$ is bijective, the function $f(x)$ has an inverse function $f^{-1}(x)$. i.e. It's inverse function is well defined.**

(b) **The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $= y$.**



① inverse of exponential / logarithmic functions

$y = \log_a x \Rightarrow x = a^y \Rightarrow g(z) = a^z$ is the inverse function

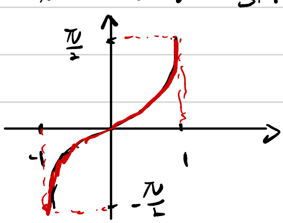
$y = a^x \Rightarrow x = \log_a y \Rightarrow g(z) = \log_a z$ is the inverse function.

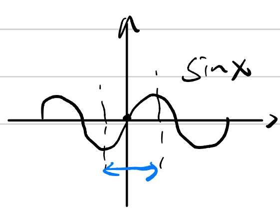
Rmk: $\log_a x = \dfrac{\ln x}{\ln a}$

② Inverse of trigonometric functions :

* $\arcsin x = \sin^{-1} x$, $x \in [-1, 1]$.

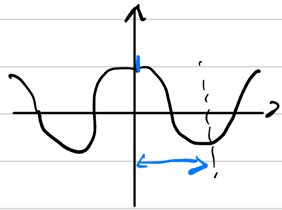
domain.



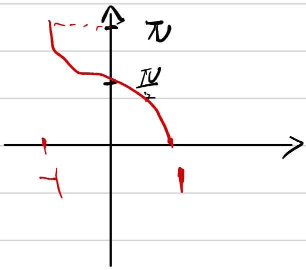


$\arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\rightarrow$ range.

$\sin(\arcsin x) = x$ for $x \in [-1, 1]$.



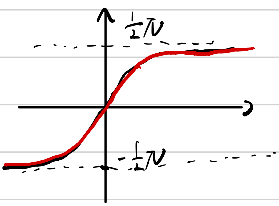
* $\arccos x = \cos^{-1} x$ $\quad x \in [-1, 1]$.



$\arccos x \in [0, \pi]$.

$\cos(\arccos x) = x$ for $x \in [-1, 1]$.

* $\arctan x = \tan^{-1} x$ $\quad x \in \mathbb{R}$.



$\arctan x \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

Exercise 1. (a) Find $\tan(\sin^{-1} x)$.

(b) Find $\sin(\tan^{-1} x)$

(c) Find $\cos(2\tan^{-1} x)$.

Solution: (a) Since $\sin^{-1}x$ has the domain $[-1,1]$,

we have $x\in[-1,1]$.

Then, we denote $\sin^{-1}x = z$, $\sin(z)=x$.

$$y=\tan(\sin^{-1}x)=\frac{\sin z}{\cos z}=\frac{x}{\cos z}.$$

Recall that $\cos(\sin^{-1}x)=\sqrt{1-x^2}$.

$\uparrow$ tri $\uparrow$ inv-tri

We have $y=\tan(\sin^{-1}x)=\dfrac{x}{\sqrt{1-x^2}}$, $x\in[-1,1]$.

(b) ... $\tan^{-1}x$ ... domain $\mathbb{R}$.

Denote $\tan^{-1}x=z \rightarrow \tan(z)=x$, what about $\sin z$?

$\in[-\frac{\pi}{2},\frac{\pi}{2}]$ ☆

Rmk: $\tan x=\dfrac{\sin x}{\cos x} \rightarrow \tan^2 x=\dfrac{\sin^2 x}{\cos^2 x}$

We apply $\sin^2x+\cos^2x=1$ to derive the relationship between $\sin x$ and $\tan x$.

$$\tan^2 x=\frac{\sin^2 x}{1-\sin^2 x} \Rightarrow \tan^2 x-\tan^2 x\sin^2 x=\sin^2 x$$

$$\sin^2 x=\frac{\tan^2 x}{1+\tan^2 x}.$$

$\Downarrow$

☆ if $x\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $\sin x=\dfrac{\tan x}{\sqrt{1+\tan^2 x}}$. (be carefull about the sign!).

Since $z\in(-\frac{\pi}{2},\frac{\pi}{2})$.

we always have $\sin z=\dfrac{\tan z}{\sqrt{1+\tan^2 z}}$.

Thus, $\sin(\tan^{-1}x) = \dfrac{x}{\sqrt{1+x^2}}$, $x \in \mathbb{R}$.

(c). For $\cos(2\tan^{-1}x)$. $\to$ $\underline{x \in \mathbb{R}}$.

Denote $\tan^{-1}x = z$.

$y = \cos(2\tan^{-1}x) = \cos(2z) = 1 - 2\sin^2 z$.

Rmk: $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$.

By (b), we have $\sin z = \dfrac{x}{\sqrt{1+x^2}}$.

Thus, $y = \cos(2\tan^{-1}x) = 1 - 2\sin^2 z = 1 - \dfrac{2x^2}{1+x^2} = \dfrac{1-x^2}{1+x^2}$.

Lecture 5.

☆ Limit.

Limit

DEFINITION Limit of a Function (Preliminary)

Suppose the function f is defined for all x near a except possibly at a. If $f(x)$ is arbitrarily close to L (as close to L as we like) for all x sufficiently close (but not equal) to a, we write

$$\lim_{x \to a} f(x) = L$$

and say the limit of $f(x)$ as x approaches a equals L.

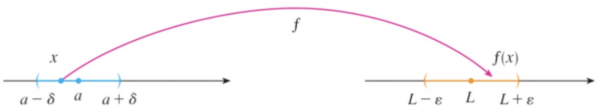
1) The Precise Definition of Limit

2 Definition Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$



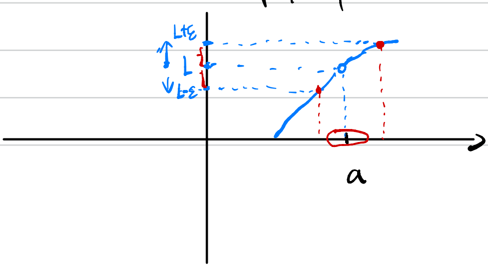
Mathematically, we say $\lim_{x \to a} f(x) = L$ iff:

for any $\varepsilon > 0$, $\exists\ \delta > 0$, such that when $x \in U(a,\delta)/\{a\}$.

$$|f(x) - L| < \varepsilon$$

Here $U(a,\varepsilon)$ means the neighbourhood of a with radius ε.

$U(a,\varepsilon) = \{x \mid |x-a| < \varepsilon\}$. $\longrightarrow$ interval $(a-\varepsilon,\ a+\varepsilon)$.



① $\frac{\sin x}{x}$

⇓

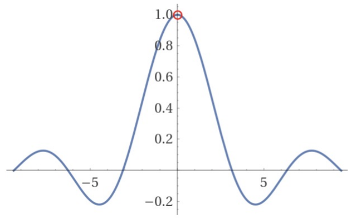
$\cos(x)$, $x=0$

Limit

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Step-by-step solution

Plot

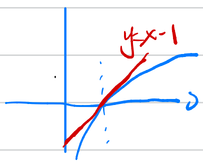


(x from −9.4 to 9.4)

② $\frac{\ln(x+1)}{x}$

⇓

$\frac{1}{x+1}$, $x=0$

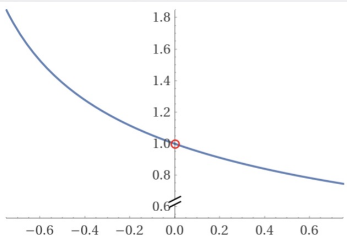


Limit

$$\lim_{x\to 0} \frac{\log(x+1)}{x} = 1$$

Step-by-step solution

Plot



(x from −0.8 to 0.8)

③ $\frac{e^x - 1}{x}$

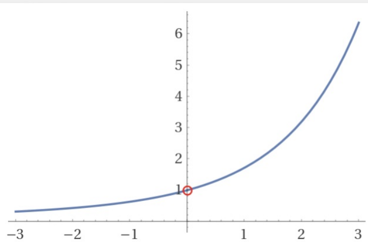
e^x, $x=0$

Limit

$$\lim_{x\to 0} \frac{\exp(x) - 1}{x} = 1$$

Step-by-step solution

Plot



(x from −3 to 3)

One-sided limit:

DEFINITION One-Sided Limits

1. **Right-sided limit** Suppose f is defined for all x near a with $x > a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x > a$, we write

$$\lim_{x \to a^+} f(x) = L$$

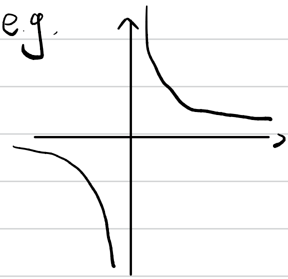
and say the limit of $f(x)$ as x approaches a from the right equals L.

2. **Left-sided limit** Suppose f is defined for all x near a with $x < a$. If $f(x)$ is arbitrarily close to L for all x sufficiently close to a with $x < a$, we write

$$\lim_{x \to a^-} f(x) = L$$

and say the limit of $f(x)$ as x approaches a from the left equals L.

e.g.



$y = \frac{1}{x}$.

$$\lim_{x \to 0^+} \frac{1}{x} \to +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x} \to -\infty.$$

* The limit exists only when R-side limit = L-side limit.

* If the limit exists, then it is unique.

Exercise 2: Do the following limits exist?

2.1: $\lim_{x \to (-3)^+} \frac{x+2}{x+3}$, 2.2. $\lim_{x \to (-3)^-} \frac{x+2}{x+3}$.

2.3. $\lim_{x \to 1} \frac{2-x}{(x-1)^2}$ 2.4 $\lim_{x \to 5^-} \frac{e^x}{(x-5)^3}$.

2.5 $\lim_{x \to 3^+} \ln(x^2-9)$.

Solution : 2.1 → no. $x+2 \to -1$

$x+3 \to 0^+$

$\lim_{x \to (-3)^+} \frac{x+2}{x+3} \to -\infty$.

2.2 no. $\lim_{x \to (-3)^-} \frac{x+2}{x+3} \to +\infty$.

2.3. no. $2-x \to 1$

$(x-1)^2 \to 0^+$

$\lim_{x \to 1} \frac{2-x}{(x-1)^2} \to +\infty$.

2.4. no. $e^x \to e^5$

$(x-5)^3 \to 0^-$

$\lim_{x \to 5^-} \frac{e^x}{(x-5)^2} -\infty$.

2.5 no $x^2-9 \to 0^+$

$\lim \ln(x^2-9) \to -\infty$.