Lecture 17, 18.

L'Hopital's rule

$\lim_{x\to a} \frac{f(x)}{g(x)}$

① $\frac{\infty}{\infty}$ $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = -\infty$.

② $\frac{0}{0}$ $\lim_{x\to a} f(x)=0$, $\lim_{x\to a} g(x)=0$.

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ is still $\frac{\infty}{\infty}$ or $\frac{0}{0}$, we can use L'Hopital's rule again and compute $\lim_{x\to a} \frac{f''(x)}{g''(x)}$, ... and so on.

Example 1. $\lim_{x\to\infty} (1-\frac{1}{x})^x = L$.

$$\ln L = \lim_{x\to\infty} x \ln(1-\frac{1}{x})$$

$$= \lim_{x\to\infty} \frac{\ln(1-\frac{1}{x})}{\frac{1}{x}} \overset{\text{L'Hopital}}{\Longrightarrow} \frac{\frac{1}{1-\frac{1}{x}} \cdot \frac{1}{x^2}}{-\frac{1}{x^2}} \Rightarrow -1$$

Thus, we have $\ln L = -1$

$L = \frac{1}{e}$.

How about: $\lim_{x\to\infty} (1-\frac{1}{x^2})^x$?

① $\lim_{x\to\infty} (1-\frac{1}{x})^x \cdot (1+\frac{1}{x})^x = \left[\lim_{x\to\infty} (1-\frac{1}{x})^x\right]\left[\lim_{x\to\infty} (1+\frac{1}{x})^x\right]$

$= e \cdot \frac{1}{e} = 1.$

② $\lim_{x\to\infty} \left[(1-\frac{1}{x^2})^{x^2}\right]^{\frac{1}{x}}$

since $(1-\frac{1}{x^2})^{x^2} \longrightarrow \frac{1}{e}$, when x is large,

$\left|(1-\frac{1}{x^2})^{x^2} - \frac{1}{e}\right| \leq c$. we can take c as, e.g. $\frac{1}{2e}$.

$$\frac{1}{2e} \leq (1-\frac{1}{x^2})^{x^2} \leq \frac{3}{2e}.$$

$$\begin{array}{ccccc} \lim_{x\to\infty} (\frac{1}{2e})^{\frac{1}{x}} & \leq & \lim_{x\to\infty} \left[(1-\frac{1}{2x})^{x^2}\right]^{\frac{1}{x}} & \leq & \lim_{x\to\infty} (\frac{3}{2e})^{\frac{1}{x}} \\ \Downarrow & & \| & & \Downarrow \\ 1 & & 1 & & 1 \end{array}$$

Example 2. $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} = L$

$$\ln L = \lim_{x\to 0} \frac{1}{x^2} \ln \cos x$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x} \sin x}{2x} = \lim_{x\to 0} \frac{1}{2\cos x} \cdot \lim_{x\to 0} \frac{\sin x}{x}$$

$$= \frac{1}{2}.$$

$$L = e^{\frac{1}{2}}.$$

Exercise 1. Compute the following limits.

1.1. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ 1.2 $\lim_{x\to 0} \frac{\sin^{-1} x}{x}$.

1.3. $\lim_{x\to 0^+} \frac{x^x - 1}{\ln x + x - 1}$ 1.4 $\lim_{x\to 1} \frac{\cos x \ln(x-1)}{\ln(e^x - e)}$

Solution :

1.1. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \to \frac{\infty}{\infty}$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}\cdot\frac{1}{\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{2\sqrt{x}} = 0.$$

1.2. $\frac{\sin^{-1}x}{x} \to \frac{0}{0}$

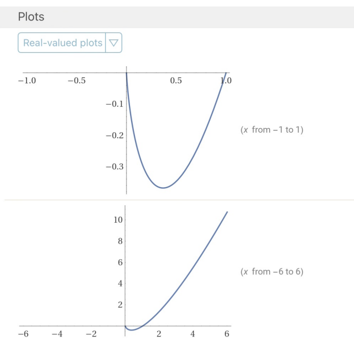
$$= \lim_{x\to 0} \frac{(\sin^{-1}x)'}{x'} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1.$$

1.3. $\lim_{x\to 0^+} \frac{x^x - 1}{\ln x + x - 1}$

$\lim_{x\to 0^+} x^x$? Denote $\lim_{x\to 0^+} x^x = L$.

$$\lim_{x\to 0^+} \underline{x\ln x} = \ln L.$$

What is the shape of $x\ln x$?



$g(x) = x\ln x$. $g'(x) = \ln x + 1$.

$(0, \frac{1}{e}]\ \downarrow$. $(\frac{1}{e}, +\infty)\ \uparrow$.

$$\lim_{x\to 0^+} x\ln x$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

$$\ln L = 0 \Rightarrow \lim_{x\to 0^+} x^x = L = 1.$$

$$\lim_{x\to 0^+} \frac{x^x - 1}{\ln x + x - 1} \to \frac{0}{-\infty} = 0.$$

1.4 $\lim_{x\to 1^+} \frac{\cos x \ln(x-1)}{\ln(e^x - e)} = \lim_{x\to 1^+} \cos x \lim_{x\to 1^+} \frac{\ln(x-1)}{\ln(e^x - e)}$

$$= \cos 1 \cdot \lim_{x\to 1^+} \frac{\frac{1}{x-1}}{\frac{e^x}{e^x - e}}$$

$$= \cos 1 \cdot \lim_{x\to 1^+} \frac{e^x - e}{x-1} \frac{1}{e^x}$$

$$= \cos 1 \cdot \lim_{x\to 1^+} \frac{1}{e^x} \lim_{x\to 1^+} \frac{e^x - e}{x-1}$$

① split the limit

$$= \cos 1 \cdot \frac{1}{e} \cdot (e^x)'(1)$$

② $\lim_{x\to 1^+} \frac{e^x - e}{x-1} = (e^x)'(1)$

$= e^1.$

$$= \cos 1 \cdot \frac{1}{e} \cdot e = \cos 1.$$

Exercise 2. Can the following limits be solved by L'Hopital's rule? If no, please find the limits by other methods.

2.1 $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$.

2.2 $\lim_{x\to(\frac{\pi}{2})^-} \frac{\sec x}{\tan x}$. $\quad (\sec x = \frac{1}{\cos x})$.

solution:

2.1. $\lim_{x\to\infty} \frac{\overset{\to\infty}{x}}{\underset{\to\infty}{\sqrt{x^2+1}}} = \lim_{x\to\infty} \frac{1}{\frac{1\cdot 2x}{2\sqrt{x^2+1}}} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$ ⊗.

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{1} = 1.$$

2.2. $\lim_{x\to\frac{\pi}{2}^-} \frac{\overset{\to\infty}{\sec x}}{\underset{\to\infty}{\tan x}} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\tan x}{\sec x}$ ⊗.

$$\lim_{x\to(\frac{\pi}{2})^-} \frac{\sec x}{\tan x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}} = \lim_{x\to(\frac{\pi}{2})^-} \frac{1}{\sin x} = 1.$$

* Curve sketching.

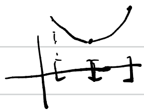
① Domain.

② Intercepts. $x=0$, $y=0$.

③ symmetricity.

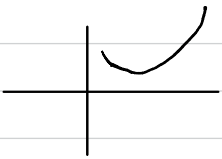
④ asymptotes. $\lim_{x\to\infty} f(x) = L$

⑤ Increasing / decreasing in I

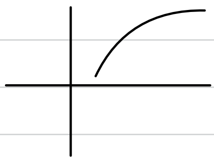


⑥ local maximum / minimum.

⑦ Convex $[f''(x) > 0]$ or concave $[f''(x) < 0]$



x^2, e^x

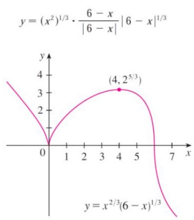


$\sqrt{x}, \ln x$.

Example 2 : **Sketch the graph of the function** $f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$.

Calculation of the first two derivatives gives

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}} \qquad f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$



Since $f'(x) = 0$ when $x = 4$ and $f'(x)$ does not exist when $x = 0$ or $x = 6$, the critical numbers are 0, 4, and 6.

Interval	$4-x$	$x^{1/3}$	$(6-x)^{2/3}$	$f'(x)$	f
$x < 0$	+	−	+	−	decreasing on $(-\infty, 0)$
$0 < x < 4$	+	+	+	+	increasing on $(0, 4)$
$4 < x < 6$	−	+	+	−	decreasing on $(4, 6)$
$x > 6$	−	+	+	−	decreasing on $(6, \infty)$

$f''(x)$ also determine the shape, $x < 6 \to f''(x) < 0$. concave.

$x > 6 \to f''(x) > 0$. convex.

Exercise. 1. sketch the graph $y = \frac{x}{x^3-1}$.

$y = f(x) = \dfrac{x}{x^3 - 1}$ **A.** $D = (-\infty, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \iff x = 0$

C. No symmetry **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x}{x^3 - 1} = 0$, so $y = 0$ is a HA. $\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^+} f(x) = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \dfrac{(x^3 - 1)(1) - x(3x^2)}{(x^3 - 1)^2} = \dfrac{-2x^3 - 1}{(x^3 - 1)^2}$. $f'(x) = 0 \Rightarrow x = -\sqrt[3]{1/2}$. $f'(x) > 0 \iff x < -\sqrt[3]{1/2}$ and $f'(x) < 0 \iff -\sqrt[3]{1/2} < x < 1$ and $x > 1$, so f is increasing on $\left(-\infty, -\sqrt[3]{1/2}\right)$ and decreasing on $\left(-\sqrt[3]{1/2}, 1\right)$ and $(1, \infty)$. **F.** Local maximum value $f\left(-\sqrt[3]{1/2}\right) = \frac{2}{3}\sqrt[3]{1/2}$; no local minimum

G. $f''(x) = \dfrac{(x^3 - 1)^2(-6x^2) - (-2x^3 - 1)2(x^3 - 1)(3x^2)}{[(x^3 - 1)^2]^2}$

$$= \frac{-6x^2(x^3 - 1)[(x^3 - 1) - (2x^3 + 1)]}{(x^3 - 1)^4} = \frac{6x^2(x^3 + 2)}{(x^3 - 1)^3}.$$

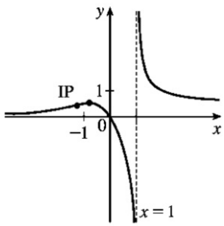
$f''(x) > 0 \iff x < -\sqrt[3]{2}$ and $x > 1$, $f''(x) < 0 \iff -\sqrt[3]{2} < x < 0$ and $0 < x < 1$, so f is CU on $(-\infty, -\sqrt[3]{2})$ and $(1, \infty)$ and CD on $(-\sqrt[3]{2}, 1)$.

IP at $\left(-\sqrt[3]{2}, \frac{1}{3}\sqrt[3]{2}\right)$

convex

concave.

H.

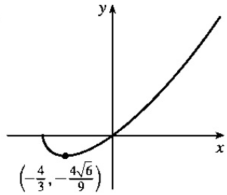


Exercise 2. sketch $y = x\sqrt{2+x}$.

$y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: -2 and 0 **C.** No symmetry

D. No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so f is decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum value $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$, no local maximum

H.



G. $$f''(x) = \frac{2\sqrt{2+x}\cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \frac{6(2+x) - (3x+4)}{4(2+x)^{3/2}}$$
$$= \frac{3x+8}{4(2+x)^{3/2}}$$

$f''(x) > 0$ for $x > -2$, so f is CU on $(-2, \infty)$. No IP