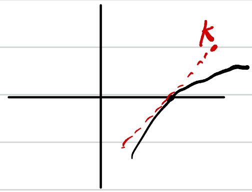
Tutorial 8 Rate of change & linear approximation.

$\star$ $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ $\to$ rate of change



f' has very practical means in broad areas:

- Physics. velocity, acceleration.
- Biology. birth rate, rate of growth.
- Economics. marginal cost, marginal revenue.
- Chemistry. reaction rate.

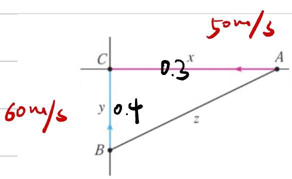
To solve questions regarding rate of changes,

① Introduce notations

② Find relations of quantities.

③ Compute derivatives.

Example 6: Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?



solution: ① Denote time → t. At this moment, $t=0$ h.

② $y = 0.4 - 60t$.

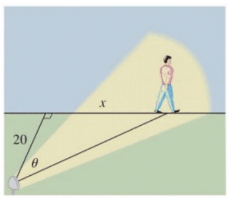
$x = 0.3 - 50t$.

$$z = \sqrt{y^2+x^2} = \sqrt{(0.4-60t)^2+(0.3-50t)^2}.$$

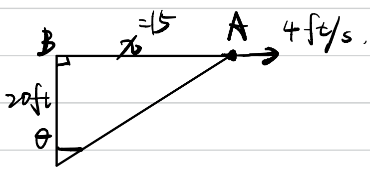
③ $$v = z'(0) = \left.\frac{60\times 2(60t-0.4)+50\times 2(50t-0.3)}{2\sqrt{(0.4-60t)^2+(0.3-50t)^2}}\right|_{t=0}$$

$$= \frac{-48-30}{2\times 0.5} = -78 \text{ mi/h}$$ unit of measurement

Example 7: A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



Solution =



① Denote x as dist between A and B. t as time and start with 0s.

② $$\begin{cases} x = \tan\theta \cdot 20 \\ x = 15 + 4t. \end{cases} \Rightarrow 20\cdot\tan\theta = 15+4t.$$

$$\tan\theta = \frac{3}{4} + \frac{1}{5}t.$$

③ we want to find $\frac{d\theta}{dt}$ at $t=0$.

method 1 $\theta = \arctan\left(\frac{3}{4} + \frac{1}{5}t\right)$, $(\arctan z)' = \frac{1}{1+z^2}$.

$$\frac{d\theta}{dt} = \frac{\frac{1}{5}}{1+\left(\frac{3}{4}+\frac{1}{5}t\right)^2} \to t=0$$

$$\left.\frac{d\theta}{dt}\right|_{t=0} = \frac{\frac{1}{5}}{1+\frac{9}{16}} = \frac{16}{125}.$$

method 2 = $\tan\theta = \frac{3}{4} + \frac{1}{5}t$.

$$\Downarrow$$

$$\tan'\theta \cdot \frac{d\theta}{dt} = \frac{1}{5}. \qquad (\tan\theta)' = \frac{1}{\cos^2\theta}.$$

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{5}.$$

$$\frac{d\theta}{dt} = \frac{1}{5}\cos^2\theta$$

when $t=0$, $\tan\theta = \frac{3}{4} \Rightarrow \cos\theta = \frac{4}{5}$.

i.e., $\left.\frac{d\theta}{dt}\right|_{\theta=0} = \frac{1}{5}\times\left(\frac{4}{5}\right)^2 = \frac{16}{125}$.

* linear approximation and differentials.

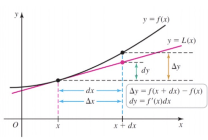
$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

2) Differentials

DEFINITION Differentials

Let f be differentiable on an interval containing x. A small change in x is denoted by the **differential** dx. The corresponding change in f is approximated by the **differential** $dy = f'(x)\,dx$; that is,

$$\Delta y = f(x+dx) - f(x) \approx dy = f'(x)\,dx.$$



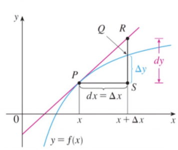
→ convex

Relationship Between Δx and Δy

Suppose f is differentiable on an interval I containing the point a. The change in the value of f between two points a and $a + \Delta x$ is approximately

$$\Delta y \approx f'(a)\,\Delta x,$$

where $a + \Delta x$ is in I.



→ concave.

Newton's method.

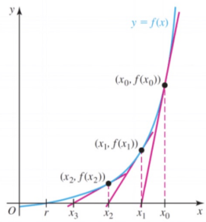
PROCEDURE **Newton's Method for Approximating Roots of $f(x) = 0$**

1. Choose an initial approximation x_0 as close to a root as possible.

2. For $n = 0, 1, 2, \ldots$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{provided } f'(x_n) \neq 0.$$

3. End the calculations when a termination condition is met.



Solving for x and calling it x_{n+1}, we find that

$$\underbrace{x_{n+1}}_{\text{new approximation}} = \underbrace{x_n}_{\text{current approximation}} - \frac{f(x_n)}{f'(x_n)}, \text{ provided } f'(x_n) \neq 0.$$

$\Rightarrow$ $f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0.$ intersect between tangent line and $y=0$.

$\Downarrow$

We are solving the linear approximation of $f(x)$ at x_n.

Newton's method may fail:

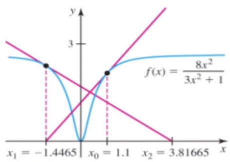


FIGURE 4.85

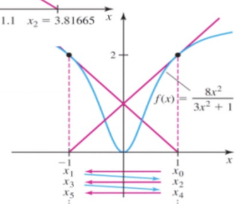
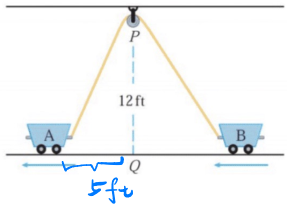


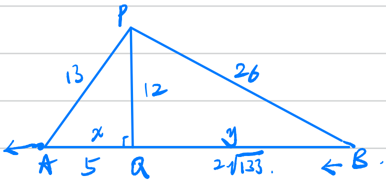
FIGURE 4.83

Exercise 1.

Two carts, A and B, are connected by a rope 39 ft long that passes over a pulley P (see the figure). The point Q is on the floor 12 ft directly beneath P and between the carts. Cart A is being pulled away from Q at a speed of 2 ft/s. How fast is cart B moving toward Q at the instant when cart A is 5 ft from Q?



Solution:



Denote time t, 0 s.

$x = AQ = 5 + 2t$.

$y = BQ$. $\rightarrow$ goal $\left(\frac{dy}{dt}\right)$.

$AP + BP = 39$ ☆

$AP = \sqrt{x^2+12^2}$, $BP = \sqrt{y^2+12^2}$.

Relationship: $\sqrt{x^2+12^2} + \sqrt{y^2+12^2} = 39$.

method 1. $y = \sqrt{\left(39 - \sqrt{(5+2t)^2+12^2}\right)^2 - 12^2}$.

Find $y'(0)$ w.r.t time t.

method 2. Taking derivative on both sides w.r.t t.

$$\frac{1}{2}\frac{2x}{\sqrt{x^2+12^2}}\cdot\frac{dx}{dt}\Big|_{t=0} + \frac{2y}{2\sqrt{y^2+12^2}}\cdot\frac{dy}{dt}\Big|_{t=0} = 0.$$

$$\left(\frac{x}{\sqrt{x^2+12^2}}\cdot 2\right)\Big|_{t=0} + \frac{y}{\sqrt{y^2+12^2}}\cdot\frac{dy}{dt}\Big|_{t=0} = 0.$$

$t=0$, $x=5$, $y = 2\sqrt{133}$.

$$\frac{10}{13} + \frac{\sqrt{133}}{13}\cdot\frac{dy}{dt}\Big|_{t=0} = 0$$

$$\frac{dy}{dt}\Big|_{t=0} = -\frac{10}{\sqrt{133}} \approx -0.87 \text{ ft/s}.$$

A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.

(a) Find an expression for the number of bacteria after t hours.

(b) Find the number of bacteria after 3 hours.

(c) Find the rate of growth after 3 hours.

(d) When will the population reach 10,000?

Solution: growth rate is proportional to size $\Rightarrow$ exponential function.

number of cells y, time t. (h)

$$y = 100 \cdot a^t.$$

$t = 1$ h, $y = 420 \Rightarrow a = 4.2$.

(a) $y = 100 \cdot 4.2^t$

(b) $y(3) = 100 \cdot 4.2^3 = 7408.8 \approx 7409$.

(c) $y'(t) = 100 \cdot \ln 4.2 \cdot 4.2^t$.

$$\left.\frac{dy}{dt}\right|_{t=3} = y'(3) = 100 \times \ln 4.2 \cdot 4.2^3 \approx 10{,}632 \text{ cells/h}.$$

(d) solve. $y(t) > 10{,}000$ $=$ $100 \cdot 4.2^t > 10{,}000 \rightarrow t > \frac{\ln 100}{\ln 4.2} \approx 3.2$ h.

Exercise 3. $f(x) = \sqrt{1-x}$.

Find an approximation of $\sqrt{0.9}$ and $\sqrt{0.99}$ using $f(x)$.

Solution: $f'(x) = -\frac{1}{2\sqrt{1-x}}$ $f'(0) = -\frac{1}{2}$.

$$\sqrt{0.9} = f(0.1) \approx f(0) + f'(0) \times 0.1$$

$$= 1 - 0.05$$

$$= 0.95.$$

Input

$\sqrt{0.9}$

Result

0.9486832980505137995996680

Fewer digits More digits

$$\sqrt{0.99} = f(0.01) \approx f(0) + f'(0) \times 0.01$$

$$= 1 - 0.005$$

$$= 0.995.$$

Input

$\sqrt{0.99}$

Result

0.99498743710661995473447

Fewer digits More digits