Lecture 11 & 12 Derivatives + Reviews for Midterm

Derivatives of Elementary Functions:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$, for $x > 0$

For real numbers p and for $x > 0$, $\frac{d}{dx}(x^p) = px^{p-1}$.

If $b > 0$, then for all x, $\frac{d}{dx}(b^x) = b^x \ln b$.

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \text{ for } -\infty < x < \infty$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \text{ for } |x| > 1$$

Important differentiation Rules.

Table of differentiation Formulas:

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1} \qquad \frac{d}{dx}(e^x) = e^x$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Product Rule and Quotient Rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Chain Rule

Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x. The composite function $y = f(g(x))$ is differentiable at x, and its derivative can be expressed in two equivalent ways:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \qquad \text{Version 1}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x) \qquad \text{Version 2}$$

Derivatives of inverse functions:

$$\frac{d f^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))}.$$

e.g. Use this rule to show $\frac{d \sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}$.

$$\sin' x = \cos x.$$

$$\frac{d \sin^{-1}x}{dx} = \frac{1}{\cos(\sin^{-1}x)}$$

☆ Denote $\sin^{-1}x = z$. $\sin(\sin^{-1}x) = \sin z = x$.

$\sin^{-1}x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos(\sin^{-1}x) \geq 0$.

$$\sin^2 z + \cos^2 z = 1 \rightarrow \cos z = \sqrt{1-\sin^2 z}$$

$$= \sqrt{1-x^2}.$$

Thus, $\frac{d \sin^{-1}x}{dx} = \frac{1}{\cos(\sin^{-1}x)} = \frac{1}{\sqrt{1-x^2}}$.

Review Exercises

1. Evaluate the following composite functions:

$f(x) = \sqrt{x}$, $g(x) = x^3 - 2$, find $f\circ g$, $g \circ f$, $f \circ f$, and $g \circ g$.

solution: $f\circ g = \sqrt{x^3+2}$

$g\circ f = x^{\frac{3}{2}} + 2$.

$f \circ f = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$

$$g\circ g = (x^3-2)^3 - 2 = x^9 - 6x^6 + 12x^3 - 10.$$

2. Find the domain and range of

(1). $g(x) = (x^2-60)\sqrt{x+5}$.

(2). $f(x) = (9-x^2)^{\frac{3}{2}}$.

solution (1). $\sqrt{x+5} \Rightarrow x \geq -5$. domain.

To find the range of $g(x)$, we do the substitution:

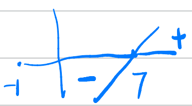
$z = x+5 \Rightarrow x = z-5$.

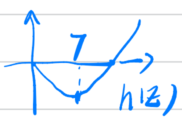
$g(x) = x^2\sqrt{x+5} = ((z-5)^2-60)\sqrt{z} = z^{\frac{5}{2}} - 10z^{\frac{3}{2}} - 35z^{\frac{1}{2}} = h(z)$.

$z \geq 0$

$h'(z) = \frac{5}{2}z^{\frac{3}{2}} - 15z^{\frac{1}{2}} - \frac{35}{2}z^{-\frac{1}{2}}$

$h'(z)$ $= \frac{5}{2}z^{-\frac{1}{2}}(z^2-6z-7) = \frac{5}{2}z^{-\frac{1}{2}}(z-7)(z+1)$





$\min\limits_{z} h(z) = h(7) = 7^{\frac{5}{2}} - 10\cdot 7^{\frac{3}{2}} - 35\cdot 7^{\frac{1}{2}}$

$= \sqrt{7}(49-70-35)$

$= -56\sqrt{7}$.

Range of $g(x)$ = Range of $h(z)$ $(z\geq 0)$ $= [-56\sqrt{7}, +\infty)$.

(2). $9-x^2 \geq 0$ $\quad x\in[-3,3]$.

$9-x^2 \in [0,9]$, $(9-x^2)^{\frac{3}{2}} \in [0,27]$.

3. $g(x) = \dfrac{x-100}{\sqrt{x}-10}$,

(1). find $\lim_{x\to 100^-} g(x)$. and $\lim_{x\to 100^+} g(x)$

(2) extend the function such that it is continuous at $x=100$.

Solution (1). $g(x) = \dfrac{(\sqrt{x}-10)(\sqrt{x}+10)}{\sqrt{x}-10}$.

$\lim_{x\to 100} g(x) = \lim_{x\to 100^+} g(x) = \lim_{x\to 100^-} g(x) = \sqrt{100}+10 = 20$

(2). $g(x) = 20$ when $x=100$.

4. Evaluate $\lim_{x\to +\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$ for

(1). $f(x) = \dfrac{(x^6+8)^{\frac{1}{3}}}{4x^2+\sqrt{3x^4+1}}$. (2). $f(x) = 4x(3x-\sqrt{9x^2+1})$

Solution : (1) $f(x) = \dfrac{(x^6+8)^{\frac{1}{3}}}{4x^2+\sqrt{3x^4+1}}$. $\to$ divide by x^2.

$= \dfrac{(1+\frac{8}{x^6})^{\frac{1}{3}}}{4+\sqrt{3+\frac{1}{x^4}}}$

$x\to+\infty$, $x\to-\infty$ $\gg$ $x^2\to+\infty$ $\quad \lim_{x\to\infty} f(x) = \dfrac{1}{4+\sqrt{3}} = \dfrac{4-\sqrt{3}}{13}$

(2) $f(x) = 4x(3x - \sqrt{9x^2+1})$

$$= \frac{4x\cdot(-1)}{3x+\sqrt{9x^2+1}}$$

if $x \to +\infty$. $f(x) = \frac{-1\times 4}{3+\sqrt{9+\frac{1}{x^2}}} \to \frac{-4}{3+3} = -\frac{2}{3}$.

if $x \to -\infty$. $x \to -\infty$. $3x - \sqrt{9x^2+1} \to -\infty$.

$f(x) = 4x(3x-\sqrt{9x^2+1}) \to +\infty$.

5. $g(x) = \begin{cases} 5x-2 & \text{if } x<1 \\ a & \text{if } x=1 \\ ax^2+bx & \text{if } x>1. \end{cases}$ Determine a, b such that g is continuous at $x=1$.

$g(x)$ is continuous $= \lim_{x\to 1^+} g(x) = \lim_{x\to 1^-} g(x) = g(1) = a$.

$a = \lim_{x\to 1^-} 5x-2 = 3$,

$\lim_{x\to 1^+} ax^2+bx = a = 3$, $\Rightarrow b=0$.

$\begin{cases} a=3 \\ b=0. \end{cases}$

6. (1). Suppose the slope of curve $y=f^{-1}(x)$ at $(4,7)$ is $\frac{4}{5}$, Find $f'(7)$.

(2). Find $(f^{-1})'(3)$ if $f(x)=x^3+x+1$. $(x \geq 0)$.

(1) $(f^{-1})'(4) = \frac{4}{5}$.

Derivatives of inverse functions =

$$\frac{df^{-1}}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{4}{5}$$

$$f'(f^{-1}(4)) = \frac{5}{4} \Rightarrow f'(7) = \frac{5}{4}.$$

(2). $f^{-1}(3)$? $\Rightarrow$ $x^3 + x + 1 = 3$

$x^3 + x - 2 = 0$.

$(x-1)(x^2 + x + 2) = 0$.

$x^3 + x^2 + 2x$

$- x^2 - x - 2$

$(x-1)(x+2)(x-1) = 0$

$x = -2$, $\underline{x = 1}$.

$f^{-1}(3) = 1$.

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{3 \cdot 1^2 + 1} = \frac{1}{4}.$$