Lecture 6.

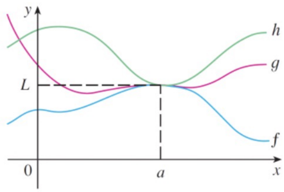
Sandwich Theorem:

3 The Squeeze Theorem If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

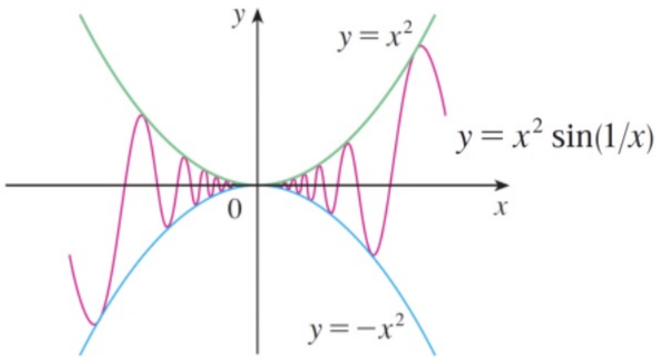


e.g.1 $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

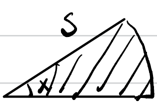
$-1 \le \sin\frac{1}{x} \le 1$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\begin{array}{ccc} \downarrow & \| & \downarrow \\ 0 & 0. & 0 \end{array}$$



e.g. 2. Show that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

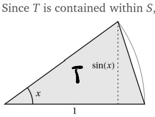


Area $S = \frac{x}{2}$.

The triangle T with the same corners as S has base 1 and height $\sin(x)$.

This means the area of T is $A = \frac{\sin(x)}{2}$.

Since T is contained within S, its area must be smaller:

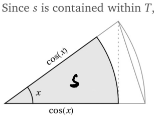


Area $S = \frac{x}{2}$

Area $T = \frac{\sin(x)}{2}$

The sector s with radius $\cos(x)$ has area $A = \frac{x\cos^2(x)}{2}$.

Since s is contained within T, its area must be smaller:



Area $S = \frac{x}{2}$

Area $T = \frac{\sin(x)}{2}$

Area $s = \frac{x\cos^2(x)}{2}$

Area S ≤ Area T ≤ Area S.

$$\frac{x\cos^2 x}{2} \le \frac{\sin x}{2} \le \frac{x}{2}, \quad (x>0)$$

$$\Downarrow$$

$$\cos^2 x \le \frac{\sin x}{x} \le 1 \quad (x>0).$$

$$\begin{array}{ccc} \downarrow & \downarrow & \| \\ 1 & 1 & 1 \end{array}$$

By sandwich theorem.

Limit laws.

suppose c, d are constant, and $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist.

$*$ $\lim_{x\to a}[cf(x)+dg(x)] = c\lim_{x\to a} f(x) + d\lim_{x\to a} f(x)$. (add, subtract, scale.)

How about: $\lim_{x\to a}\sum_{i=1}^{n} f_i(x) \overset{?}{=} \sum_{i=1}^{n}\lim_{x\to a} f_i(x)$. n is finite ✓

n approaches $+\infty$ ✗

$*$ $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$.

$*$ $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (if $\lim_{x\to a} g(x) \neq 0$).

$*$ limit for composite functions.

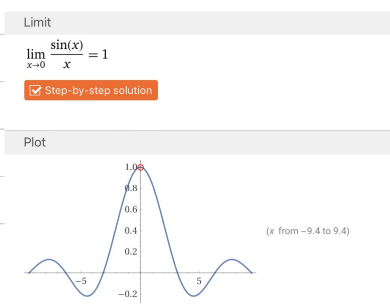
$\lim_{x\to a} g\circ f(x) = \lim_{x\to a} g(f(x))$.

① $g(z) = z^n$. $\Rightarrow$ $\lim_{x\to a}(f(x))^n = (\lim_{x\to a} f(x))^n$.

② $g(z) = z^{\frac{1}{n}}$. $\lim_{x\to a}[f(x)]^{\frac{1}{n}} = (\lim_{x\to a} f(x))^{\frac{1}{n}}$.

As long as $g(z)$ is $\boxed{\text{continuous}}$ at the point $L = \lim_{x\to a} f(x)$,
we have $\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x)) = g(L)$.

e.g: $\lim_{x\to 0}\sin\left(\frac{\sin x}{x}\right) = \sin 1$.



How about $\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}$?

$$\lim_{\underset{\triangle}{x\to 0}} \frac{\sin(\sin x)}{\sin x} = \lim_{\underset{\triangle}{\underline{z\to 0}}} \frac{\sin z}{z} = 1.$$

Extra exercises:

$$\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$$

solution $: \lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} = \frac{(\sqrt{6-x}-2)(\sqrt{3-x}+1)(\sqrt{6-x}+2)}{(\sqrt{3-x}-1)(\sqrt{3-x}+1)(\sqrt{6-x}+2)}$ (rationalize it)

$$= \lim_{x\to 2} \frac{(2-x)(\sqrt{3-x}+1)}{(2-x)(\sqrt{6-x}+2)}$$

$$= \lim_{x\to 2} \frac{\sqrt{3-x}+1}{\sqrt{6-x}+2} = \frac{2}{4} = \frac{1}{2}.$$

Exercise 1.1: find $\lim_{x\to 4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$.

1.2: find $\lim_{x\to 0} \frac{\frac{1}{x^2}}{\sqrt{\frac{1}{x^4}+1}}$

1.3: if $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x\to 1} f(x)$.

1.4: if $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$, find (a). $\lim_{x\to 0} f(x)$.

(b) $\lim_{x\to 0} \frac{f(x)}{x}$.

1.1: find $\lim_{x\to4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}$.

solution = $\lim_{x\to4} \frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}} = \lim_{x\to4} \frac{3(x-4)\sqrt{x+5}(3+\sqrt{x+5})}{(3-\sqrt{x+5})(3+\sqrt{x+5})}$

$= \lim_{x\to4} \frac{3(x-4)\sqrt{x+5}(3+\sqrt{x+5})}{4-x}$

$= \lim_{x\to4} -3\sqrt{x+5}(3+\sqrt{x+5})$

$= -3\times3\times6 = -54$.

1.2: find $\lim_{x\to0} \frac{\frac{1}{x^2}}{\sqrt{\frac{1}{x^4}+1}}$

$= \lim_{x\to0} \frac{1}{\sqrt{1+x^4}} = 1$.

1.3: if $\lim_{x\to1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x\to1} f(x)$.

$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{f(x)-8}{(x-1)} \times (x-1) + 8$ what if $\lim_{x\to1} f(x) \neq 8$?

$= \lim_{x\to1} \frac{f(x)-8}{x-1} \cdot \lim_{x\to1}(x-1) + 8 = 8$.

1.4: if $\lim_{x\to0} \frac{f(x)}{x^2} = 5$, find (a). $\lim_{x\to0} f(x)$.

(b) $\lim_{x\to0} \frac{f(x)}{x}$.

(a): $\lim_{x\to0} f(x) = \lim_{x\to0} \frac{f(x)}{x^2} \cdot x^2 = \lim_{x\to0} \frac{f(x)}{x^2} \cdot \lim_{x\to0} x^2 = 0$.

(b): $\lim_{x\to0} \frac{f(x)}{x} = \lim_{x\to0} \frac{f(x)}{x^2} \cdot x = \lim_{x\to0} \frac{f(x)}{x^2} \cdot \lim_{x\to0} x = 0$.

Lecture 7

continuity:

Continuity

Definition: A Function $f(x)$ is continuous at a number a if

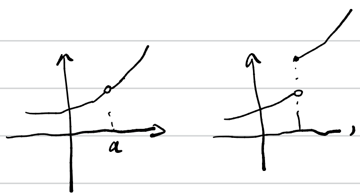
$$\lim_{x \to a} f(x) = f(a)$$

Such that $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

☆ Can pass the limit:

$$\lim_{x \to a} f(x) = f(\lim_{x \to a} x) = f(a).$$

If f is continuous at a, then $\begin{cases} f(a) \text{ exists.} \\ \lim_{x \to a} f(x) \text{ exists.} \\ f(a) = \lim_{x \to a} f(x). \end{cases}$



Thm: If f, g are continuous at a, and c, d are constants, then we have:

$$\left.\begin{cases} c \cdot f + d \cdot g \\ f \cdot g \\ \frac{f}{g} \quad (\text{if } g(a) \neq 0) \end{cases}\right\} \Rightarrow \text{continuous at } a.$$

Many functions are continuous in their domains

* polynomials: $a_1x^1 + a_2x^2 + \cdots$ $\Rightarrow$ rational

* root: $\sqrt{x}$

* trigonometric: $\sin x$, $\cos x$, $\tan x$.

* exp, log.

☆ If f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

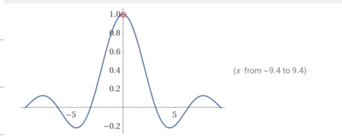
Define $f = \begin{cases} 1, & x = 0 \\ \frac{\sin x}{x}, & x \neq 0 \end{cases}$ $\Rightarrow$

Limit

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Step-by-step solution

Plot



Is f continuous? $\checkmark$

Thus: $\lim_{x\to 0} \frac{\sin(\sin x)}{\sin x}$

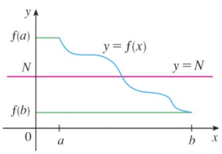
$= \lim_{x\to 0} f(\sin x)$

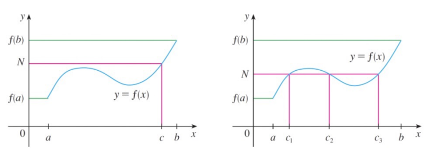
$= f(\lim_{x\to 0} \sin x) = f(0) = 1.$

☆ Intermediate value theorem.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

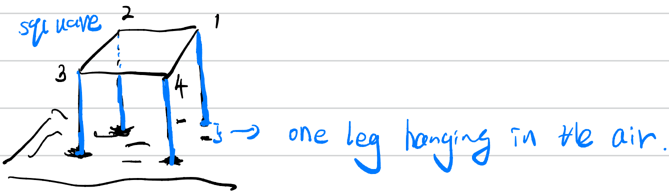




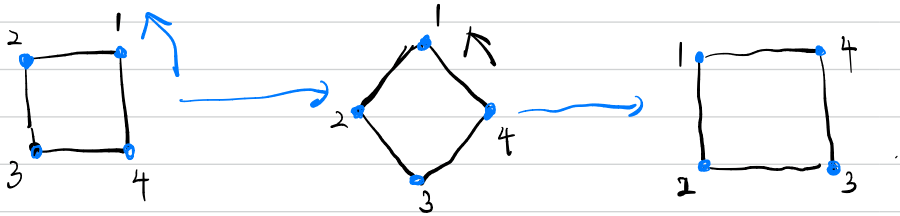
(a) (b)

One interesting Application:

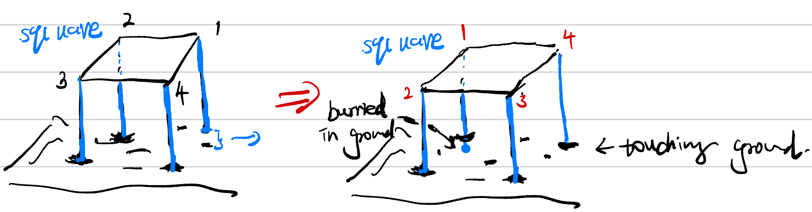
Table rotation: suppose on a continuous ground. We have a table with 4 Legs:

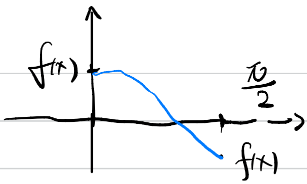


Claim: rotate the table to any direction up to at most 90°. we can stabilize the table with 4 legs touching the ground.



Reason: denote the distance between leg 1 and ground as $f(x)$. $f(0)>0$. Suppose we allow $f(x)$ to be $\underline{\leq 0}$ (i.e. we allow the leg to be inserted in the ground). If we always force leg 2, 3, 4. to touch the ground, then after a 90° rotation, $f(x)$ will be negative.





By intermidiate value thm,

we can always find a angle such that

$f(x) = 0 \quad x \in (0, \frac{\pi}{2})$.

Exercise 2. define $f(2)$ so that $f(x)$ is always continuous:

$$f(x) = \frac{x^2 - x - 2}{x - 2}$$

solution: $f(x) = \frac{(x-2)(x+1)}{x-2}$, $\lim_{x \to 2} f(x) = \lim_{x \to 2} x+1 = 3$.

$$f(x) = \begin{cases} 3 & \text{if } x = 2. \\ \frac{x^2 - x - 2}{x - 2} = x+1 & \text{if } x \neq 2. \end{cases}$$