Lecture 9

The derivative as a function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

* We say $f(x)$ is differentiable on an open set C if for any $x \in C$, $f'(x)$ exists.

☆ $\boxed{\text{Differentible} \Rightarrow \text{Continuous}}$.

* Differentiation rules:

1. $\dfrac{d(x^n)}{dx} = n x^{n-1}$. (for any real number).

   $\Downarrow$ (when $n=0 \Rightarrow$ constant function)

   any polynomial f.

2. $\dfrac{d(e^x)}{dx} = e^x$.

How about $y = \alpha^x$? $(\alpha > 0)$.

$y = \alpha^x$ $\quad \ln y = x \cdot \ln a$. $\quad f(g(x))' = f'(g(x)) \cdot g'(x)$.

$y' \frac{1}{y} = \ln a$.

$y' = y \ln a$. , i.e., $\dfrac{d(\alpha^x)}{dx} = \alpha^x . \ln a$

Derivative $\Rightarrow$ Limit $\Rightarrow$ follows the rule of limit.

$$(f(x)+g(x))' = f'(x) + g'(x)$$

$$(c\cdot f(x))' = c\cdot f'(x).$$

Exercise 1. Find $\lim_{x\to\frac{\pi}{2}}\left(\frac{\frac{1}{\sqrt{\sin x}}-1}{x-\frac{\pi}{2}}\right)$, and point out the meaning of this limit in terms of derivative.

Solution : $f'(\frac{\pi}{2})$ for $f(x)=\frac{1}{\sqrt{\sin x}}$.

denote $x-\frac{\pi}{2}=y$, $\Rightarrow x=\frac{\pi}{2}+y$.

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{\frac{1}{\sqrt{\sin x}}-1}{x-\frac{\pi}{2}}\right) = \lim_{y\to 0}\frac{\frac{1}{\sqrt{\sin(y+\frac{\pi}{2})}}-1}{y}$$

$$\underline{\sin(y+\tfrac{\pi}{2})} \overset{\star}{=} \sin(\tfrac{\pi}{2}-(-y)) = \cos(-y) = \underline{\cos y} \ \star$$

$$\lim_{y\to 0}\frac{\frac{1}{\sqrt{\sin(y+\frac{\pi}{2})}}-1}{y} = \lim_{y\to 0}\frac{\frac{1}{\sqrt{\cos y}}-1}{y} = \lim_{y\to 0}\frac{1-\sqrt{\cos y}}{y\sqrt{\cos y}}.$$

$\star$ rationalization.

$$= \lim_{y\to 0}\frac{(1-\sqrt{\cos y})(1+\sqrt{\cos y}) = 1-\cos y}{y\sqrt{\cos y}(1+\sqrt{\cos y})}$$

To transform $\underline{\cos}$ into $\underline{\sin}$.

$$= \lim_{y\to 0}\frac{(1-\cos y)(1+\cos y) = 1-\cos^2 y = \sin^2 y}{y\sqrt{\cos y}(1+\sqrt{\cos y})(1+\cos y)}$$

$$= \lim_{y\to 0}\frac{\sin^2 y}{y\sqrt{\cos y}(1+\sqrt{\cos y})(1+\cos y)}.$$

$$= \left[\lim_{y\to 0}\sin y\right]\cdot\lim_{y\to 0}\left[\frac{\sin y}{y\sqrt{\cos y}(1+\sqrt{\cos y})(1+\cos y)}\right]$$

$$= 0. \ \triangle$$

Exercise 2. Find the tangent line of the following functions:

1. $f(x) = x^2 - 4$ at $P(2,0)$

2. $f(x) = \sqrt{x+3}$ at $P(1,2)$.

Solution: 2.1. $f'(x) = 2x$,

$f'(2) = 4 \Rightarrow$ line with slope 4.

$y = 4(x-2)$

$= 4x - 8$.

2.2. $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x+3}}$

$f'(1) = \frac{1}{4}$.

$y - 2 = \frac{1}{4}(x-1) \Rightarrow y = \frac{1}{4}x + \frac{7}{4}$.

A line goes across (x_0, y_0) with slop k:

$y - y_0 = k(x - x_0)$.

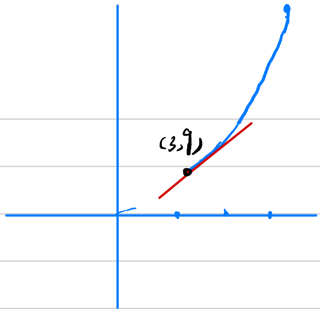
Exercise 3. $f(x) = \begin{cases} x^2 & x \geq 3 \\ ax+b & x < 3 \end{cases}$. Find a, b such that f is differentiable on $\mathbb{R}$.

solution: $x = 3$: tangent line of x^2?

$\frac{d(x^2)}{dx} = 2x$. slope $= 2x = 6$. + $(3, 9)$

$$y-9=6(x-3)$$

$$y=6x-9.$$



since when $x\geq 3$, $f(x)=x^2$

$$\lim_{h\to 0^+}\frac{f(3+h)-f(3)}{h}=6.$$

In order to ensure the existence of $f'(3)$ $\Rightarrow$ $\lim_{h\to 0^-}\frac{f(3+h)-f(3)}{h}=6$.

$$(ax+b)'\big|_{x=3}=6.$$

That is, $ax+b$ must be the tangent line of x^2 at $x=3$.

$$ax+b=\underset{a}{6}x-\underset{b}{9}.$$

Exercise 4. $f(x)=\begin{cases} x^m\sin\frac{1}{x}. & x\neq 0 \\ 0 & x=0 \end{cases}$ $m\in N_+$ (positive integer).

Find the range of m s.t.: 1. f is continuous at $x=0$.

2. f is differentiable at $x=0$.

$$(x^m\sin\tfrac{1}{x})'=mx^{m-1}\sin\tfrac{1}{x}-x^{m-2}\cos\tfrac{1}{x}$$

3. f' is continuous at $x=0$.

Solution: 1. $\lim_{x\to 0}f(x)=0$. $x^m\sin\frac{1}{x}\to 0$. for any $m\geq 1$,

$$|x^m\sin\tfrac{1}{x}|\leq|x^m|\to 0,$$

Thus, $m\geq 1$.

2. $f'(0)$ exists $\Rightarrow$ $\lim\limits_{x\to 0} \dfrac{x^m \sin\frac{1}{x}}{x}$ exists.

$\lim\limits_{x\to 0} \dfrac{x^m \sin\frac{1}{x}}{x} = \lim\limits_{x\to 0} x^{m-1}\sin\frac{1}{x}$. if $m=1$ $\otimes$

if $m \geq 2$ $\checkmark$. $|x^{m-1}| \to 0$.

Thus $m \geq 2$.

3. $f'(x)$ is continuous at $x=0$. $\Rightarrow$ $f'(0)$ exists. $\Rightarrow$ at least $m \geq 2$.

If $m \geq 2$. $f'(0) = \lim\limits_{x\to 0} \dfrac{x^m \sin\frac{1}{x}}{x} = 0$. , as $|x^{m-1}\sin\frac{1}{x}| \leq |x^{m-1}|$.

$f'(x)$ is continuous $\Rightarrow$ $\underline{\lim\limits_{x\to 0} f'(x)} = f'(0) = 0$.

$\lim\limits_{x\to 0} \left[(x^m \sin\frac{1}{x})' = m x^{m-1}\sin\frac{1}{x} - x^{m-2}\cos\frac{1}{x} \right] = 0.$

If $m=2$, $\cos\frac{1}{x} \to$ no limit,

If $m \geq 3$. $|m \cdot x^{m-1}\sin\frac{1}{x} - x^{m-2}\cos\frac{1}{x}| \leq m|x^{m-1}| + |x^{m-2}| \to 0$.

Thus, $m \geq 3$.

Exercise 5. If $g(0) = g'(0) = 0$, and $f(x) = \begin{cases} g(x)\sin\frac{1}{x} & x \neq 0 \\ 0 & x=0 \end{cases}$.

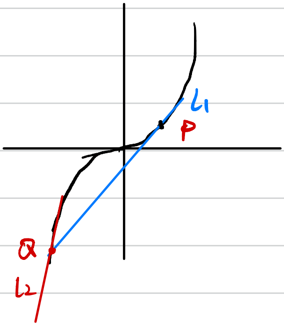
Find $f'(0)$.

Solution: $f'(0) = \lim\limits_{h\to 0} \dfrac{f(h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{g(h)\sin\frac{1}{h} - 0}{h}$

However, $0 \leq \left|\dfrac{g(h)\sin\frac{1}{h} - 0}{h}\right| \leq \underline{\left|\dfrac{g(h)}{h}\right|}$, where $\lim\limits_{h\to 0} \dfrac{g(h)-0}{h} = g'(h) = 0$.

By sandwich Thm, we have $f'(0) = 0$.

Exercise 6. Take a point P at curve $y=x^3$. The tangent line of the curve at P intersects with the curve at another point Q. Prove that the slope of tangent line at Q is 4 times as the slope of tangent line at P.



Solution. suppose $P(x_1, x_1^3)$.

$$\frac{d(x^3)}{dx} = 3x^2 \rightarrow \text{Slope of } l_1 = 3x_1^2$$

$$l_1:\ y - x_1^3 = 3x_1^2(x - x_1)$$

$$l_1:\ y = 3x_1^2 \cdot x - 2x_1^3.$$

$$\begin{cases} y = 3x_1^2 x - 2x_1^3 \\ y = x^3 \end{cases} \Rightarrow x^3 = 3x_1^2 \cdot x - 2x_1^3.$$

$$x^3 - 3x_1^2 x + 2x_1^3 = 0$$

$$\|$$

$$(x-x_1)(x^2 + x_1 x - 2x_1^2)$$

$$= (x-x_1)(x-x_1)(x+2x_1)$$

$$= (x-x_1)^2(x+2x_1)$$

$$\begin{array}{r} x^2 + x_1 x - 2x_1^2 \\ (x-x_1)\overline{)\,x^3 + \boxed{} - 3x_1^2 x + 2x_1^3} \\ \underline{x^3 - x_1 x^2 \qquad\qquad} \\ x_1 x^2 - 3x_1^2 x + 2x_1^3 \\ x_1 x^2 - x_1^2 x \qquad \end{array}$$

$$-2x_1^2x+2x_1^3$$

$$-2x_1^2x+2x_1^3$$

$$0.$$

Another solution $= \underline{\underline{x=-2x_1}}$

Thus, $Q=(-2x_1,-8x_1^3).$

with the slope $3\cdot(-2x_1)^2$

$$=\underline{12\cdot x_1^2}.$$

Another path:

$$x^3-3x_1^2x+2x_1^3=0$$

$$\Downarrow$$

$$\left(\frac{x}{x_1}\right)^3-3\left(\frac{x}{x_1}\right)+2=0$$

solve $y^3-3y+2=0$

$$\Downarrow$$

$$(y-1)^2(y+2)=0.$$

$l_1: 3x_1^2,\ l_2: 12x_1^2.$ Q.E.D.