

Learning about Informativeness

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Abstract

We study a sequential social learning model in which there is uncertainty about the informativeness of a common signal-generating process. Rational agents arrive in order and make decisions based on the past actions of others and their private signals. We show that, in this setting, asymptotic learning about informativeness is not guaranteed and depends crucially on the relative tail distributions of the private beliefs induced by uninformative and informative signals. We identify the phenomenon of perpetual disagreement as the cause of learning and characterize learning in the canonical Gaussian environment.

1 Introduction

Social learning plays a vital role in the dissemination and aggregation of information. The behavior of others reflects their private knowledge about an unknown state of the world, and so by observing others, individuals can acquire additional information, enabling them to make better-informed decisions. A key assumption in most existing social learning models is the presence of an informative source that provides a useful private signal to each individual. In this paper, we explore how the possibility that the source is uninformative interferes with learning, and study the conditions under which individuals can eventually distinguish an uninformative source from an informative one. This question is particularly relevant today due to the proliferation of novel information technologies, raising concerns about the accuracy and credibility of the information they provide.

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Formally, we introduce uncertainty regarding the informativeness of the source into the classic sequential social learning model (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992; Smith and Sørensen, 2000). As usual, a sequence of short-lived agents arrives in order, each acting once by choosing an action to match an unknown payoff-relevant state that can be either good or bad. Before making their decisions, each agent observes the past actions of her predecessors and receives a private signal from a common source of information. However, unlike in the usual setting, there is uncertainty surrounding this common information source. In particular, we assume that this source can be either informative, generating private signals that are independent and identically distributed (i.i.d.) conditioned on the payoff-relevant state, or uninformative, producing private signals that are i.i.d. but independent of the payoff-relevant state. Both the payoff-relevant state and the informativeness of the source are realized independently at the outset and are assumed to be fixed throughout.

If an outside observer, who aims to evaluate the informativeness of the source, were to have access to the private signals received by the agents, he would gradually accumulate empirical evidence about the source and eventually learn its informativeness. However, when only the history of past actions is observable, his inference problem becomes more challenging—not only because there is less information available, but also because these past actions are correlated with each other. This correlation arises from social learning behavior, where agents’ decisions are influenced by the inferences they draw from observing others’ actions. We say that *asymptotic learning of informativeness* holds if the outside observer’s belief about the source’s informativeness converges to the truth, i.e., it converges almost surely to one when the source is informative and to zero when it is uninformative. The questions we aim to address are: Can learning about informativeness be achieved asymptotically, and if so, under what conditions? Furthermore, what are the behavioral implications of such learning?

We consider *unbounded signals* under which the agent’s private belief induced by an informative signal can be arbitrarily strong. We focus on this setting since otherwise learning is precluded by agents’ lack of response to their private signals.¹ Our main result (Theorem 1) shows that even with unbounded signals, achieving asymptotic learning of informativeness is far from guaranteed. The key factor in determining such learning lies in the tail distributions of agents’ private beliefs. Specifically, it depends on whether the belief distribution induced by uninformative signals has *fatter* or *thinner* tails compared to that induced by informative signals. We show that asymptotic learning of informativeness holds when uninformative signals have fatter tails than informative signals, but

¹As shown by Smith and Sørensen (2000), with bounded signals, an information cascade—where agents stop responding to their private signals—would be triggered, blocking further information aggregation. In contrast, cascades do not form with unbounded signals, thus allowing information to continue accumulating through agents’ actions.

fails when uninformative signals have thinner tails.

For example, consider an informative source that generates Gaussian signals with unit variance and a mean of $+1$ if the payoff-relevant state is good and a mean of -1 if the state is bad. Meanwhile, the uninformative source generates Gaussian signals with mean 0 , independent of the state. If the uninformative source generates signals with a variance strictly greater than one, then the uninformative signals have fatter tails, and thus asymptotic learning of informativeness holds. In contrast, when the uninformative signals have a variance strictly less than one, they exhibit thinner tails, and so asymptotic learning of informativeness fails.

The mechanism behind the main result is as follows. First, in our model, despite information uncertainty, agents always act as if the signals are informative. Therefore, when the source is indeed informative and generates unbounded signals, agents will eventually take the correct action. Now, suppose the source is uninformative and generates signals with thinner tails. In this case, it is unlikely that agents will receive signals extreme enough to break the consensus on actions, so they usually mimic their predecessors. As a result, an outside observer who only observes agents' actions cannot discern that the source is uninformative, as an action consensus will be reached under both uninformative and informative sources. In contrast, suppose the source is uninformative but generates signals with fatter tails. In this scenario, extreme signals are more likely, allowing agents to break the consensus; in fact, both actions will be taken infinitely often, so agents will never settle on an action consensus. Hence, an outside observer who observes an infinite number of action switches learns that the source is uninformative.

For some private belief distributions, their relative tail thickness is neither thinner nor fatter. For these, we show that the same holds: Asymptotic learning of informativeness is achieved if and only if conditioned on the source being uninformative, agents never settle on an action consensus (Proposition 1). In terms of behavior, as mentioned before, when the source is informative, agents eventually choose the correct action, regardless of information uncertainty. In contrast, when the source is uninformative, agents are clearly not guaranteed to settle on the correct action; in fact, their actions may or may not converge at all. As demonstrated by Proposition 1, an outside observer eventually learns the informativeness of the source if and only if the agents' actions do not converge when the source is uninformative.

Related Literature

Our paper contributes to a rich literature on sequential social learning. Assuming that the common source of information is always informative, the primary focus of this literature has been on determining whether agents can eventually learn to choose the correct

action. Various factors, such as the information structure (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992; Smith and Sørensen, 2000) and the observational networks (Celen and Kariv, 2004; Acemoglu, Dahleh, Lobel, and Ozdaglar, 2011; Lobel and Sadler, 2015), have been extensively studied to analyze their impact on information aggregation, including its efficiency (Rosenberg and Vieille, 2019) and the speed of learning (e.g., Vives, 1993; Hann-Caruthers, Martynov, and Tamuz, 2018). However, the question of learning about the informativeness of the source—which is the focus of this paper—remains largely unexplored.²

A few papers explore the idea of agents having access to multiple sources of information in the context of social learning. For example, Liang and Mu (2020) consider a model in which agents endogenously choose from a set of correlated information sources, and the acquired information is then made public and learned by other agents. They focus on the externality in agents’ information acquisition decisions and show that information complementarity can result in either efficient information aggregation or “learning traps,” in which society gets stuck in choosing suboptimal information structures. In a different setting, Chen (2022) examines a sequential social learning model in which ambiguity-averse agents have access to different sources of information. Consequently, information uncertainty arises in his model because agents are unsure about the signal precision of their predecessors. He shows that under sufficient ambiguity aversion, there can be information cascades even with unbounded signals. Our paper differs from these prior works as we focus on rational agents with access to a common source of information of unknown informativeness.

Another way of viewing our model is by considering a social learning model with four states: The source is either informative with the good or bad state, or uninformative with either the good or bad state. In such multi-state settings, recent work by Arieli and Mueller-Frank (2021) demonstrates that pairwise unbounded signals are necessary and sufficient for learning, when the decision problem that agents face includes a distinct action that is uniquely optimal for each state. This is not the case in our model, because the same action is optimal in different states, e.g., when the source is uninformative, and so even when agents observe a very strong signal indicating that the state is uninformative, they do not reveal it in their behavior.

More recently, Kartik, Lee, Liu, and Rappoport (2022) consider a setting with multiple states and actions on general sequential observational networks. They identify a sufficient condition for learning —“excludability” —that jointly depends on the information structure and agents’ preferences. Roughly speaking, this condition ensures that agents can always displace the wrong action, which is their driving force for learning.

²For comprehensive surveys on recent developments in the social learning literature, see e.g., Golub and Sadler (2017); Bikhchandani, Hirshleifer, Tamuz, and Welch (2021).

In our model, when the source is uninformative, agents cannot displace the wrong action as all signals are pure noise.³ Conceptually, our approach differs from theirs as we are interested in identifying the uninformative state from the informative one, instead of identifying the payoff-relevant state.

Our paper is also related to the growing literature on social learning with misspecified models. [Bohren \(2016\)](#) investigates a model where agents fail to account for the correlation between actions, demonstrating that different degrees of misperception can lead to distinct learning outcomes. In a broader framework, [Bohren and Hauser \(2021\)](#) show that learning remains robust to minor misspecifications. In contrast, [Frick, Iijima, and Ishii \(2020\)](#) find that an incorrect understanding of other agents’ preferences or types can result in a severe breakdown of information aggregation, even with a small amount of misperception. Later, [Frick, Iijima, and Ishii \(2023\)](#) propose a unified approach to establish convergence results in misspecified learning environments where the standard martingale approach fails to hold. On a more positive note, [Arieli, Babichenko, Müller, Pourbabaee, and Tamuz \(2023\)](#) illustrate that by being mildly condescending—misperceiving others as having slightly lower-quality of information—agents may perform better in the sense that on average, only finitely many of them take incorrect actions.

2 Model

There is an unknown binary state of the world $\theta \in \{\mathbf{g}, \mathbf{b}\}$, chosen at time 0 with equal probability. We refer to \mathbf{g} as the good state and \mathbf{b} as the bad state. Concurrently, nature chooses an additional binary state $\omega \in \{0, 1\}$, independent of θ , where $\omega = 1$ with probability $\gamma \in (0, 1)$.⁴ A countably infinite set of agents indexed by time $t \in \mathbb{N} = \{1, 2, \dots\}$ arrive in order, each acting once. The action of agent t is $a_t \in A = \{\mathbf{g}, \mathbf{b}\}$, with a payoff of one if her action matches the state θ and zero otherwise. Before agent t chooses an action, she observes the history of her predecessors’ actions $H_t = (a_1, \dots, a_{t-1})$, and receives a private signal s_t , taking values in a measurable space (S, Σ) .

The informativeness of private signals depends on ω , which determines the signal-generating process for all agents. If $\omega = 1$, the source is informative, and conditional on θ , signals are drawn i.i.d. across agents from a distribution μ_θ . If $\omega = 0$, the source is uninformative, and signals are drawn i.i.d. from a distribution μ_0 that is independent of θ . We denote by $\mathbb{P}_0[\cdot] := \mathbb{P}[\cdot \mid \omega = 0]$ and $\mathbb{P}_1[\cdot] := \mathbb{P}[\cdot \mid \omega = 1]$ the conditional probability distributions given $\omega = 0$ and $\omega = 1$, respectively. Similarly, we use the notation $\mathbb{P}_{1,\mathbf{g}}[\cdot] := \mathbb{P}[\cdot \mid \omega = 1, \theta = \mathbf{g}]$ to denote the conditional probability distribution given $(\omega, \theta) = (1, \mathbf{g})$.

³This observation can also be seen from Theorem 2 in [Kartik, Lee, Liu, and Rappoport \(2022\)](#).

⁴Our results do not rely on the independence between θ and ω . They hold true as long as conditioned on $\omega = 0$, both realizations of θ are equally likely.

We use an analogous notation for $(\omega, \theta) = (1, \mathfrak{b})$.

Agents' Behavior. A pure strategy of agent t is a measurable function $\sigma_t : A^{t-1} \times S \rightarrow A$ that selects an action for each possible pair of observed history and private signal. A pure strategy profile $\sigma = (\sigma_t)_{t \in \mathbb{N}}$ is a collection of pure strategies of all agents. A strategy profile is a Bayesian Nash equilibrium—referred to as equilibrium hereafter—if no agent can unilaterally deviate from this profile and obtain a strictly higher expected payoff conditioned on their information. Given that each agent acts only once, the existence of an equilibrium is guaranteed by a simple inductive argument. In equilibrium, each agent t chooses the action a_t that maximizes her expected payoff given the available information:

$$a_t \in \arg \max_{a \in A} \mathbb{E}[\mathbb{1}(\theta = a) | H_t, s_t]. \quad (1)$$

Below, we make a continuity assumption which implies that agents are never indifferent, and so there is a unique equilibrium.

We first observe that, despite the uncertainty regarding the informativeness of the source, in equilibrium, each agent chooses the action that is most likely to match the state, *conditional* on the source being informative.

Lemma 1. *The equilibrium action for each agent t is*

$$a_t \in \arg \max_{a \in A} \mathbb{P}_1[\theta = a | H_t, s_t]. \quad (2)$$

In essence, agents always respond to their private signals, acting as if they are informative irrespective of the underlying signal-generating process. Intuitively, treating signals as informative—even when they are pure noise—does not adversely affect agents' payoffs, since in the absence of any useful information, each agent with a uniform prior is indifferent between the available actions. As we discuss in §5, this responsiveness breaks down under a non-uniform prior; consequently, the informativeness of the source can never be fully revealed.

Information Structure. We assume the distributions $\mu_{\mathfrak{g}}$, $\mu_{\mathfrak{b}}$, and μ_0 are distinct⁵ and mutually absolutely continuous, so no signal fully reveals either state θ or ω . As a

⁵Formally, two distributions are distinct if there exists a positive measure set to which they assign a different measure. From an economic perspective, requiring μ_0 to differ from both $\mu_{\mathfrak{g}}$ and $\mu_{\mathfrak{b}}$ rules out “fake” information technologies that can perfectly mimic the informative source at one realization of θ while being uninformative. Likewise, $\mu_{\mathfrak{g}} \neq \mu_{\mathfrak{b}}$ ensures that signals are informative in the informative state.

consequence, conditioned on $\omega = 1$, the log-likelihood ratio of any signal s

$$\ell = \log \frac{d\mu_{\mathbf{g}}}{d\mu_{\mathbf{b}}}(s),$$

is well-defined. We refer to ℓ as the *agent's private log-likelihood ratio (LLR)*. Since regardless of ω , agents always act as if the signals are informative, it is sufficient to consider ℓ to capture their behavior.⁶ Let $F_{\mathbf{g}}$ and $F_{\mathbf{b}}$ be the CDFs of ℓ under an informative source ($\omega = 1$) when the state θ is \mathbf{g} and \mathbf{b} , respectively. Let F_0 denote the CDF of ℓ under an uninformative source ($\omega = 0$). Note that these CDFs are mutually absolutely continuous, as $\mu_{\mathbf{g}}, \mu_{\mathbf{b}}$ and μ_0 are. Let $f_{\mathbf{g}}, f_{\mathbf{b}}$ and f_0 denote the corresponding density functions of $F_{\mathbf{g}}, F_{\mathbf{b}}$ and F_0 whenever they are differentiable.

We focus on *unbounded signals*, where ℓ can take on arbitrarily large or small values: for any $M \in \mathbb{R}$, there exists a positive probability that $\ell > M$ and a positive probability that $\ell < -M$. We informally refer to a signal s as *extreme* when the corresponding ℓ it induces has a large absolute value. A common example of unbounded private signals is the case of Gaussian signals, where s follows a normal distribution $\mathcal{N}(m_{(\omega, \theta)}, \sigma^2)$ with variance σ^2 and mean $m_{(\omega, \theta)}$ that depends on the pair of states (ω, θ) . An extreme Gaussian signal is a signal that is, for example, $5 - \sigma$ away from the mean $m_{(\omega, \theta)}$.

We make two assumptions for expository simplicity. First, we assume that the pair $(F_{\mathbf{g}}, F_{\mathbf{b}})$ of informative conditional CDFs is symmetric around zero, i.e., $F_{\mathbf{g}}(x) + F_{\mathbf{b}}(-x) = 1$. Given that the prior on θ is uniform, this assumption makes our model invariant to a relabeling of θ .⁷ Second, we assume that all conditional CDFs— $F_{\mathbf{g}}, F_{\mathbf{b}}$, and F_0 —are continuous, so agents are never indifferent between actions.

In addition, we assume that $F_{\mathbf{b}}$ has a differentiable left tail, i.e., it is differentiable for all sufficiently negative x , and its density function $f_{\mathbf{b}}$ satisfies the condition that $f_{\mathbf{b}}(-x) < 1$ for all x large enough. By symmetry, this implies that $F_{\mathbf{g}}$ also has a differentiable right tail and its density function $f_{\mathbf{g}}$ satisfies the condition that $f_{\mathbf{g}}(x) < 1$ for all x large enough. This is a mild technical assumption that holds for every non-atomic distribution commonly used in the literature, including the Gaussian distribution. It holds, for instance, whenever the density tends to zero at infinity.

Asymptotic Learning of Informativeness. We now define our notion of asymptotic learning. Let $q_t := \mathbb{P}[\omega = 1 | H_t]$ be the belief that an outside observer assigns to the source being informative after observing the history of agents' actions from time 1 to $t - 1$. As this observer collects more information over time, his belief q_t converges almost surely since it is a bounded martingale. Formally, we say that *asymptotic learning of*

⁶Formally, the sequence of actions a_1, \dots, a_t is determined by ℓ_1, \dots, ℓ_t .

⁷Our main result can be easily extended to a non-symmetric case since our notion of relative tail thickness does not require symmetry.

informativeness holds if for all $\omega \in \{0, 1\}$,

$$\lim_{t \rightarrow \infty} q_t = \omega \quad \mathbb{P}_\omega\text{-almost surely.}$$

That is, the outside observer's belief eventually converges to the truth. In §5, we discuss the relationship between asymptotic learning of informativeness and other notions of learning.

3 Main Results

Before stating the results, we introduce the concept of relative tail thickness, which captures the relative likelihood of generating extreme signals from different sources.

Relative Tail Thickness. For any pair of CDFs (F_0, F_θ) where $\theta \in \{\mathfrak{g}, \mathfrak{b}\}$, we denote their corresponding ratios for any $x \in \mathbb{R}_+$ by

$$L_\theta(x) := \frac{F_0(-x)}{F_\theta(-x)} \quad \text{and} \quad R_\theta(x) := \frac{1 - F_0(x)}{1 - F_\theta(x)}.$$

For large x , these represent the left and right tail ratios of F_0 over F_θ . Formally, we say the uninformative signals have *fatter tails* than the informative signals if there exists an $\varepsilon > 0$ such that

$$\begin{aligned} L_{\mathfrak{b}}(x) &\geq \varepsilon \quad \text{for all } x \text{ large enough,} \\ \text{and } R_{\mathfrak{g}}(x) &\geq \varepsilon \quad \text{for all } x \text{ large enough.} \end{aligned}$$

We say the uninformative signals have *thinner tails* than the informative signals if there exists an $\varepsilon > 0$ such that

$$\begin{aligned} \text{either } L_{\mathfrak{g}}(x) &\leq 1/\varepsilon \quad \text{for all } x \text{ large enough,} \\ \text{or } R_{\mathfrak{b}}(x) &\leq 1/\varepsilon \quad \text{for all } x \text{ large enough.} \end{aligned}$$

Intuitively, compared to informative signals, uninformative signals with fatter tails are more likely to exhibit extreme values. Thus, by Bayes' Theorem, observing an extreme signal suggests that the source is uninformative. In contrast, uninformative signals with thinner tails tend to exhibit moderate values, so observing an extreme signal in this case suggests that the source is informative.

As an example of uninformative signals with fatter tails, consider those represented by $F_0 = \frac{1}{2}(F_{\mathfrak{g}} + F_{\mathfrak{b}})$, a mixture distribution between $F_{\mathfrak{g}}$ and $F_{\mathfrak{b}}$.⁸ Note that this mixture

⁸More generally, it is straightforward to see that any positive mixture distribution $F_0 = \alpha F_{\mathfrak{g}} + (1-\alpha)F_{\mathfrak{b}}$

distribution F_0 coincides with the unconditional distribution of ℓ under an informative source. Thus, one can view it as representing an uninformative source that is a priori indistinguishable from the informative one. By contrast, consider an uninformative distribution F_0 that first-order stochastically dominates F_g , i.e., $F_0(x) \leq F_g(x)$ for all $x \in \mathbb{R}$. In other words, the uninformative signals are more likely to exhibit high values than the informative signals associated with the good state. It follows immediately from the resulting inequality, i.e., $L_g(x) \leq 1$ for all $x \in \mathbb{R}$, that the uninformative source has thinner tails.

Now, we are ready to state our main result (Theorem 1). It shows that the key determinant of asymptotic learning of informativeness is the relative tail thickness between the uninformative and informative signals.

Theorem 1. *When the uninformative signals have fatter (thinner) tails than the informative signals, asymptotic learning of informativeness holds (fails).*

This result demonstrates that an observer learning from agents' actions can eventually distinguish between informative and uninformative sources, but only if the uninformative source generates signals with greater dispersion. In contrast, when the uninformative signals are relatively concentrated, this differentiation becomes impossible.

The idea behind our proof of Theorem 1 is as follows. First consider the case where the source is informative. In this case, the likelihood of generating extreme signals that overturn a long streak of correct action consensus decreases rapidly. Consequently, agents will eventually choose the correct action since they always treat signals as informative. Now, suppose that the source is uninformative, and instead of reaching a consensus, agents continue to disagree indefinitely, leading to both actions being taken infinitely often. If this were the case, an outside observer would eventually be able to distinguish between informative and uninformative sources, as they induce distinct behavioral patterns among agents. Whether these disagreements persist or not depends on whether the tails of the uninformative signals are thick enough to generate these overturning extreme signals.

In summary, when the tails of uninformative signals are sufficiently thick, overturning extreme signals occur frequently enough so that disagreements persist. Conversely, when the tails are relatively thin, these signals are less likely to occur and disagreements eventually cease. Hence, we conclude that the relative tail thickness between uninformative and informative signals plays an important role in determining the achievement of asymptotic learning of informativeness.

In the case of Gaussian signals, the relative tail thickness of normal distributions is determined solely by their variances. An immediate corollary is the following:

where $\alpha \in (0, 1)$ has fatter tails.

Corollary 1. *For normal private signals with informative and uninformative variances σ^2 and τ^2 , respectively, asymptotic learning of informativeness holds if $\tau > \sigma$ and fails if $\tau < \sigma$.*

4 Analysis

In this section, we first analyze how agents update their beliefs. We then use two important phenomena to characterize asymptotic learning of informativeness. This leads to a complete proof of Theorem 1 at the end of this section.

Agents' Belief Dynamics

Since agents act as if signals are informative, to understand their behavior, it suffices to focus on their belief updating process conditioned on $\omega = 1$. We denote the public belief of agent t , based on the history of actions H_t , by $\pi_t := \mathbb{P}_1[\theta = \mathfrak{g}|H_t]$. The corresponding LLR of π_t is

$$r_t := \log \frac{\pi_t}{1 - \pi_t} = \log \frac{\mathbb{P}_1[\theta = \mathfrak{g}|H_t]}{\mathbb{P}_1[\theta = \mathfrak{b}|H_t]}.$$

Similarly, we denote the LLR of agent t 's posterior belief, based on both H_t and her private signal s_t by

$$L_t := \log \frac{\mathbb{P}_1[\theta = \mathfrak{g}|H_t, s_t]}{\mathbb{P}_1[\theta = \mathfrak{b}|H_t, s_t]}.$$

Recall that ℓ_t is the private LLR of agent t ' conditioned on $\omega = 1$. By Bayes' rule, we can write

$$L_t = r_t + \ell_t.$$

From (2) we know that in equilibrium, agent t chooses action \mathfrak{g} if $\ell_t \geq -r_t$ and action \mathfrak{b} if $\ell_t < -r_t$. Hence, conditioned on θ and the event $\omega = 1$, the probability that $a_t = \mathfrak{g}$ is $1 - F_\theta(-r_t)$ and the probability that $a_t = \mathfrak{b}$ is $F_\theta(-r_t)$. Consequently, the agents' public LLRs evolve as follows:

$$r_{t+1} = r_t + D_{\mathfrak{g}}(r_t) \quad \text{if } a_t = \mathfrak{g}, \tag{3}$$

$$r_{t+1} = r_t + D_{\mathfrak{b}}(r_t) \quad \text{if } a_t = \mathfrak{b}, \tag{4}$$

where

$$D_{\mathfrak{g}}(r) := \log \frac{1 - F_{\mathfrak{g}}(-r)}{1 - F_{\mathfrak{b}}(-r)} \quad \text{and} \quad D_{\mathfrak{b}}(r) := \log \frac{F_{\mathfrak{g}}(-r)}{F_{\mathfrak{b}}(-r)}.$$

As this is a standard belief updating process, it satisfies well-known properties such as the *overturning principle* and *stationarity*. For completeness, we state them here.

Claim 1 (Overturning Principle). *For each agent t , if $a_t = \mathbf{g}$, then $\pi_{t+1} \geq 1/2$. Similarly, if $a_t = \mathbf{b}$, then $\pi_{t+1} \leq 1/2$.*

This property asserts that a single action is sufficient to change the verdict of π_t . We further write $\mathbb{P}_{\tilde{\omega}, \tilde{\theta}, \pi}$ to denote the conditional probability distribution given the pair of state realizations $(\tilde{\omega}, \tilde{\theta})$ while highlighting the different values of the prior π .

Claim 2 (Stationarity). *For any fixed sequence $(b_\tau)_{\tau=1}^k$ of k actions in $\{\mathbf{g}, \mathbf{b}\}$, any prior $\pi \in (0, 1)$ and any pair $(\tilde{\omega}, \tilde{\theta}) \in \{0, 1\} \times \{\mathbf{g}, \mathbf{b}\}$*

$$\mathbb{P}_{\tilde{\omega}, \tilde{\theta}}[a_{t+1} = b_1, \dots, a_{t+k} = b_k | \pi_t = \pi] = \mathbb{P}_{\tilde{\omega}, \tilde{\theta}, \pi}[a_1 = b_1, \dots, a_k = b_k].$$

This claim states that the value of π_t captures all past information about the payoff-relevant state, independent of time. This holds in our model because, regardless of the informativeness of the source, agents always update their public LLRs according to either (3) or (4).

Perpetual Disagreement and Immediate Agreement

We introduce two events that are important for characterizing asymptotic learning of informativeness. All proofs for the results in this section are provided in the Appendix.

The first is *perpetual disagreement*—the event in which agents’ actions never converge, which we denote by $\{S = \infty\}$, where $S = \sum_{t=1}^{\infty} \mathbb{1}(a_t \neq a_{t+1})$ is the total number of action switches. The following proposition establishes the key mechanism underlying Theorem 1: it shows that learning about informativeness is equivalent to the almost sure occurrence of perpetual disagreement when the source is uninformative.

Proposition 1. *Asymptotic learning of informativeness holds if and only if conditioned on $\omega = 0$, the event $\{S = \infty\}$ occurs almost surely.*

Intuitively, if agents never reach a consensus under an uninformative source, then the outside observer infers that the source must be uninformative since agents would have reached a consensus under an informative source. Conversely, suppose agents could also reach a consensus when the source is uninformative. This implies that action convergence is plausible under both informative and uninformative sources. Consequently, the observer can no longer be sure of the source’s informativeness.

The second important event is *immediate agreement*, where agents reach a consensus from the outset. We denote such an immediate agreement on action a by $\{\bar{a} = a\} = \{a_1 = a_2 = \dots = a\}$. The next lemma shows that when the source is informative, immediate agreement on the wrong action is impossible, whereas immediate agreement on the correct action is possible. For brevity, we state this result only for the case where $\theta = \mathbf{g}$, as the analogous statements hold for $\theta = \mathbf{b}$ by symmetry.

Lemma 2. *Conditioned on $\omega = 1$ and $\theta = \mathbf{g}$, the following two conditions hold:*

- (i) *The event $\{\bar{a} = \mathbf{b}\}$ occurs with probability zero.*
- (ii) *The event $\{\bar{a} = \mathbf{g}\}$ occurs with positive probability for some prior $\pi \in (0, 1)$.*

Next, we focus on the immediate agreement event under an uninformative source. Building on Proposition 1, we can now characterize asymptotic learning of informativeness in terms of this phenomenon. This is a crucial step in proving our main result, as conditioned on this event, the belief process is much easier to analyze.

Proposition 2. *Asymptotic learning of informativeness holds if and only if conditioned on $\omega = 0$, both events $\{\bar{a} = \mathbf{g}\}$ and $\{\bar{a} = \mathbf{b}\}$ occur with zero probability.*

This proposition shows that learning is equivalent to the absence of immediate agreement starting from a uniform prior given an uninformative source. Given this, we are now ready to prove our main result.

Proof of Theorem 1

By part (i) of Lemma 2, we know that $\mathbb{P}_{1,\mathbf{b}}[\bar{a} = \mathbf{g}] = 0$ and $\mathbb{P}_{1,\mathbf{g}}[\bar{a} = \mathbf{b}] = 0$. Moreover, conditioned on $\{\bar{a} = \mathbf{g}\}$, the public LLR r_t evolves deterministically according to (3), and we denote such sequence by $r_t^{\mathbf{g}}$. Recall that agent t chooses $a_t = \mathbf{g}$ if $\ell \geq -r_t$ and $a_t = \mathbf{b}$ otherwise. Consequently, the event $\{\bar{a} = \mathbf{g}\}$ is equivalent to $\{\ell_t \geq -r_t^{\mathbf{g}}, \forall t \geq 1\}$. Conditioned on $\omega = 0$, since signals are i.i.d., so are the agents' private LLRs. Thus, we can write

$$0 = \mathbb{P}_{1,\mathbf{b}}[\bar{a} = \mathbf{g}] = \prod_{t=1}^{\infty} (1 - F_{\mathbf{b}}(-r_t^{\mathbf{g}})).$$

By taking the logarithm on both sides, this infinite product equals zero if and only if the sum of its logarithms diverges: $-\sum_{t=1}^{\infty} \log(1 - F_{\mathbf{b}}(-r_t^{\mathbf{g}})) = \infty$. For two sequences (a_t) and (b_t) , we write $a_t \approx b_t$ if $\lim_{t \rightarrow \infty} (a_t/b_t) = 1$. Since $r_t^{\mathbf{g}} \rightarrow \infty$, $\log(1 - F_{\mathbf{b}}(-r_t^{\mathbf{g}})) \approx -F_{\mathbf{b}}(-r_t^{\mathbf{g}})$ and the previous sum is infinite if and only if

$$\sum_{t=1}^{\infty} F_{\mathbf{b}}(-r_t^{\mathbf{g}}) = \infty. \tag{5}$$

Similarly, we have $\mathbb{P}_{1,\mathbf{g}}[\bar{a} = \mathbf{b}] = 0$ if and only if $\sum_{t=1}^{\infty} (1 - F_{\mathbf{g}}(-r_t^{\mathbf{b}})) = \infty$, where $r_t^{\mathbf{b}}$ denotes the deterministic process of r_t conditioned on $\{\bar{a} = \mathbf{b}\}$. By symmetry, $r_t^{\mathbf{b}} = -r_t^{\mathbf{g}}$ for all $t \geq 1$. Hence, $\mathbb{P}_{1,\mathbf{g}}[\bar{a} = \mathbf{b}] = 0$ if and only if

$$\sum_{t=1}^{\infty} (1 - F_{\mathbf{g}}(r_t^{\mathbf{g}})) = \infty, \tag{6}$$

Now, suppose the uninformative signals have fatter tails than the informative signals. By definition, there exists an $\varepsilon > 0$ such that for all x large enough, $F_0(-x) \geq \varepsilon \cdot F_{\mathbf{b}}(-x)$ and $1 - F_0(x) \geq \varepsilon \cdot (1 - F_{\mathbf{g}}(x))$. It then follows from (5) and (6) that

$$\sum_{t=1}^{\infty} F_0(-r_t^{\mathbf{g}}) = \infty \quad \text{and} \quad \sum_{t=1}^{\infty} (1 - F_0(r_t^{\mathbf{g}})) = \infty.$$

Applying the same logic we used to deduce (5) and (6), having these two divergent sums is equivalent to $\mathbb{P}_0[\bar{a} = \mathbf{g}] = 0$ and $\mathbb{P}_0[\bar{a} = \mathbf{b}] = 0$. Thus, by Proposition 2, asymptotic learning of informativeness holds.

By part (ii) of Lemma 2, there exist priors $\pi', \pi'' \in (0, 1)$ such that $\mathbb{P}_{1,\mathbf{g},\pi'}[\bar{a} = \mathbf{g}] > 0$ and $\mathbb{P}_{1,\mathbf{b},\pi''}[\bar{a} = \mathbf{b}] > 0$. Let $r' = \log \frac{\pi'}{1-\pi'}$ and $r'' = \log \frac{\pi''}{1-\pi''}$. Denote by $r_t^{\mathbf{g}}(r)$ the deterministic process with an initial value r . Following a similar argument that led to (5) and (6), these are equivalent to

$$\sum_{t=1}^{\infty} F_{\mathbf{g}}(-r_t^{\mathbf{g}}(r')) < \infty \quad \text{and} \quad \sum_{t=1}^{\infty} (1 - F_{\mathbf{b}}(r_t^{\mathbf{g}}(r''))) < \infty.$$

Now, suppose the uninformative signals have thinner tails than the informative signals. By definition, there exists an $\varepsilon > 0$ such that either (i) $F_0(-x) \leq (1/\varepsilon) \cdot F_{\mathbf{g}}(-x)$ for all x large enough, or (ii) $1 - F_0(x) \leq (1/\varepsilon) \cdot (1 - F_{\mathbf{b}}(x))$ for all x large enough. It then follows from the above inequalities that

$$\text{either} \quad \sum_{t=1}^{\infty} F_0(-r_t^{\mathbf{g}}(r')) < \infty, \quad \text{or} \quad \sum_{t=1}^{\infty} (1 - F_0(r_t^{\mathbf{g}}(r''))) < \infty.$$

Following a similar argument, we have either (i) $\mathbb{P}_{0,\pi'}[\bar{a} = \mathbf{g}] > 0$ for some $\pi' \in (0, 1)$ or (ii) $\mathbb{P}_{0,\pi''}[\bar{a} = \mathbf{b}] > 0$ for some $\pi'' \in (0, 1)$. By a technical lemma (Lemma 4) in the Appendix, if these hold for some prior, these also hold for all prior $\pi', \pi'' \in (0, 1)$, including the uniform prior. Hence, either (i) $\mathbb{P}_0[\bar{a} = \mathbf{g}] > 0$, or (ii) $\mathbb{P}_0[\bar{a} = \mathbf{b}] > 0$. It follows from Proposition 2 that asymptotic learning of informativeness fails. This concludes the proof of Theorem 1.

5 Discussion

Other Learning Notions. Our model highlights a crucial distinction between two common notions of learning: *correct herding*, where agents' actions converge to the truth, and *complete learning*, where their beliefs do. As is well-known, when the source is always informative and generates unbounded signals, both correct herding and complete learning hold. Our model with information uncertainty, however, decouples these two outcomes.

When the source is informative, correct herding is still achieved, as agents’ behavior mimics that of a standard social learning model. Yet, as we show in the Online Appendix, this no longer implies complete learning: even when agents settle on the correct action, they remain uncertain about its correctness if asymptotic learning of informativeness fails.

The Role of the Uniform Prior. The uniform prior is central to our analysis because it ensures that agents always respond to their private signals. In the Online Appendix, we prove that when such responsiveness fails—as is the case with a non-uniform prior—no signal structure can guarantee asymptotic learning of informativeness. Intuitively, this is because a non-uniform prior introduces a default action. Consequently, an observer who sees a consensus on this action cannot distinguish between two possibilities: (i) the source is informative and the default is the correct action, or (ii) the source is uninformative and agents, having learned this, have stopped responding to their signals and reverted to the default action. This observational equivalence makes learning impossible.

Gaussian Signals with the Same Variance. As shown by Corollary 1, our main result applies directly to Gaussian private signals with different variances. When Gaussian signals share the same variance, this turns out to be a special case where the uninformative signals have neither fatter nor thinner tails. To complement our main result, in the Online Appendix, we show that in this case, asymptotic learning of informativeness occurs if and only if the uninformative signals are also symmetric around zero. The intuition here is similar to that used in establishing Proposition 1: any deviation of the uninformative mean from zero shifts its distribution closer to one of the informative distributions, leading agents to form an action consensus, just as they would under an informative source. This in turn creates an observational equivalence that prevents learning.

A Proofs Omitted from the Main Text

Proof of Lemma 1. From (1), we know $a_t = \mathbf{g}$ if

$$1 \leq \frac{\mathbb{P}[\theta = \mathbf{g}|H_t, s_t]}{\mathbb{P}[\theta = \mathbf{b}|H_t, s_t]} = \frac{\sum_{\tilde{\omega} \in \{0,1\}} \mathbb{P}_{\tilde{\omega}}[\theta = \mathbf{g}|H_t, s_t] \cdot \mathbb{P}[\omega = \tilde{\omega}|H_t, s_t]}{\sum_{\tilde{\omega} \in \{0,1\}} \mathbb{P}_{\tilde{\omega}}[\theta = \mathbf{b}|H_t, s_t] \cdot \mathbb{P}[\omega = \tilde{\omega}|H_t, s_t]},$$

where the second equality follows from the law of total probability. Since conditioned on $\omega = 0$, neither the history H_t nor the signal s_t contains any information about θ , we have $\mathbb{P}_0[\theta = \mathbf{g}|H_t, s_t] = \mathbb{P}_0[\theta = \mathbf{g}]$ and $\mathbb{P}_0[\theta = \mathbf{b}|H_t, s_t] = \mathbb{P}_0[\theta = \mathbf{b}]$. As long as conditioned on $\omega = 0$, the prior on θ is uniform, $\mathbb{P}_0[\theta = \mathbf{g}|H_t, s_t] = \mathbb{P}_0[\theta = \mathbf{b}|H_t, s_t] = 1/2$. Thus, the above inequality is equivalent to

$$1 \leq \frac{\mathbb{P}_1[\theta = \mathbf{g}|H_t, s_t]}{\mathbb{P}_1[\theta = \mathbf{b}|H_t, s_t]}.$$

That is, in equilibrium, each agent chooses the most likely action conditioned on the available information and the source being informative. \square

The following simple claim will be useful in proving Proposition 1. Recall that $q_t = \mathbb{P}[\omega = 1|H_t]$ is the belief that an outside observer assigns to an informative source based on the history of actions from time 1 to $t - 1$ and $\gamma = \mathbb{P}[\omega = 1] \in (0, 1)$ is the prior that the source is informative.

Claim 3. *For any $a \in (0, 1/2)$ and any $b \in (1/2, 1)$,*

$$\begin{aligned} \mathbb{P}_0[q_t = \tilde{q}] &\leq \frac{1-a}{a} \frac{\gamma}{1-\gamma} \cdot \mathbb{P}_1[q_t = \tilde{q}], \text{ for all } \tilde{q} \in [a, 1/2]; \\ \mathbb{P}_0[q_t = \tilde{q}] &\geq \frac{1-b}{b} \frac{\gamma}{1-\gamma} \cdot \mathbb{P}_1[q_t = \tilde{q}], \text{ for all } \tilde{q} \in [1/2, b]. \end{aligned}$$

Proof. Fix a prior $\gamma \in (0, 1)$. For any $\tilde{q} \in (0, 1)$, let \tilde{H}_t be a history of actions such that the associated belief q_t is equal to \tilde{q} . By the law of total expectation,

$$\mathbb{P}[\omega = 1|q_t = \tilde{q}] = \mathbb{E}[\mathbb{E}[\mathbb{1}(\omega = 1)|\tilde{H}_t]|q_t = \tilde{q}] = \mathbb{E}[\tilde{q}|q_t = \tilde{q}] = \tilde{q}.$$

It follows from Bayes' rule that

$$\frac{\mathbb{P}_0[q_t = \tilde{q}]}{\mathbb{P}_1[q_t = \tilde{q}]} = \frac{\mathbb{P}[\omega = 0|q_t = \tilde{q}]}{\mathbb{P}[\omega = 1|q_t = \tilde{q}]} \cdot \frac{\mathbb{P}[\omega = 1]}{\mathbb{P}[\omega = 0]} = \frac{1-\tilde{q}}{\tilde{q}} \frac{\gamma}{1-\gamma}.$$

Since for any $a \in (0, 1/2)$ and any $\tilde{q} \in [a, 1/2]$, $1 \leq \frac{1-\tilde{q}}{\tilde{q}} \leq \frac{1-a}{a}$, it thus follows from the

above equation that

$$\mathbb{P}_0[q_t = \tilde{q}] = \frac{1 - \tilde{q}}{\tilde{q}} \frac{\gamma}{1 - \gamma} \cdot \mathbb{P}_1[q_t = \tilde{q}] \leq \frac{1 - a}{a} \frac{\gamma}{1 - \gamma} \cdot \mathbb{P}_1[q_t = \tilde{q}].$$

The second inequality follows from an identical argument. \square

Proof of Proposition 1. (If direction) Suppose $\mathbb{P}_0[S = \infty] = 1$. Let a_t^x be the guess that the outside observer x would make to maximize his probability of guessing ω correctly at time t . The information available to x at time t is $H_t = (a_1, \dots, a_{t-1})$ and at time infinity is $H_\infty = \cup_t H_t$. Denote by $q_\infty = \mathbb{P}[\omega = 1 | H_\infty]$ the belief that x has at time infinity that $\omega = 1$.

Fix a large positive integer $k \in \mathbb{N}$. Let $A_t(k)$ denote the event that there have been at least k action switches before time t and denote its complementary event by $A_t^c(k)$. Consider the following strategy $\tilde{a}_\infty^x(k)$ for x at time infinity: $\tilde{a}_\infty^x(k) = 0$ if $A_\infty(k)$ occurs and $\tilde{a}_\infty^x(k) = 1$ otherwise. The probability of x guessing it correctly under this strategy is

$$\mathbb{P}[\tilde{a}_\infty^x(k) = \omega] = \mathbb{P}_0[A_\infty(k)] \cdot \mathbb{P}[\omega = 0] + \mathbb{P}_1[A_\infty^c(k)] \cdot \mathbb{P}[\omega = 1]. \quad (\text{A.1})$$

Since $\mathbb{P}_{1,\theta}[a_t \rightarrow \theta] = 1$, it follows that for all k large enough,

$$\mathbb{P}_1[A_\infty^c(k)] = 1. \quad (\text{A.2})$$

By assumption, $\mathbb{P}_0[S = \infty] = 1$, and thus $\mathbb{P}_0[A_\infty(k)] = 1$ for all k . Together, (A.1) and (A.2) imply that for all k large enough, $\mathbb{P}[\tilde{a}_\infty^x(k) = \omega] = 1$.

Meanwhile, the optimal strategy for x at time infinity is the following: $a_\infty^x = 1$ if $q_\infty \geq 1/2$ and $a_\infty^x = 0$ otherwise. Since $\mathbb{P}[\tilde{a}_\infty^x(k) = \omega] = 1$ for all k large enough, the probability of guessing it correctly under a_∞^x must also be one:

$$1 = \mathbb{P}[a_\infty^x = \omega] = \mathbb{P}_1[q_\infty \geq 1/2] \cdot \mathbb{P}[\omega = 1] + \mathbb{P}_0[q_\infty < 1/2] \cdot \mathbb{P}[\omega = 0]. \quad (\text{A.3})$$

From (A.3), $\mathbb{P}_0[q_\infty < 1/2] = \mathbb{P}_1[q_\infty \geq 1/2] = 1$. To establish asymptotic learning of informativeness, by definition, we need to show $\mathbb{P}_0[q_\infty = 0] = 1$ and $\mathbb{P}_1[q_\infty = 1] = 1$. To this end, note that $\mathbb{P}_0[q_\infty < 1/2] = 1$ implies that $\mathbb{P}_0[q_\infty \geq 1/2] = 0$. Thus, by Claim 3, it implies that for any $b \in (1/2, 1)$ and all $\tilde{q} \in [1/2, b]$, $\mathbb{P}_1[q_\infty = \tilde{q}] = 0$. Consequently, it follows from $\mathbb{P}_1[q_\infty \geq 1/2] = 1$ that $\mathbb{P}_1[q_\infty = 1] = 1$. The case that $\mathbb{P}_0[q_\infty = 0] = 1$ follows from an identical argument.

(Only-if direction) Suppose by contraposition that $\mathbb{P}_0[S < \infty] > 0$. Again, since $\mathbb{P}_{1,\theta}[a_t \rightarrow \theta] = 1$, we have $\mathbb{P}_1[S < \infty] = 1$. Thus, there exists a history of actions at time infinity \tilde{H}_∞ that is possible under both probability measures \mathbb{P}_0 and \mathbb{P}_1 : $\mathbb{P}_0[\tilde{H}_\infty] > 0$ and

$\mathbb{P}_1[\tilde{H}_\infty] > 0$. By Bayes' rule, $\mathbb{P}[\omega = 1 | \tilde{H}_\infty] < 1$ and $\mathbb{P}[\omega = 0 | \tilde{H}_\infty] < 1$. Assume without loss of generality that under \tilde{H}_∞ , the corresponding belief $\tilde{q}_\infty \geq 1/2$, and so the observer x would guess $a_\infty^x = 1$. Hence, the probability of x guessing correctly about ω is strictly less than one:

$$\begin{aligned}\mathbb{P}[a_\infty^x = \omega] &= \mathbb{P}[a_\infty^x = \omega, \tilde{H}_\infty] + \mathbb{P}[a_\infty^x = \omega, \tilde{H}_\infty^c] \\ &= \mathbb{P}[\omega = 1 | \tilde{H}_\infty] \cdot \mathbb{P}[\tilde{H}_\infty] + \mathbb{P}[a_\infty^x = \omega, \tilde{H}_\infty^c] < \mathbb{P}[\tilde{H}_\infty] + \mathbb{P}[\tilde{H}_\infty^c] = 1.\end{aligned}$$

This contradicts $\mathbb{P}[a_\infty^x = \omega] = 1$, which is required for asymptotic learning of informativeness to hold. \square

Recall that $\mathbb{P}_{\omega, \theta, \pi}$ is the conditional probability distribution given ω and θ while emphasizing the prior value π .

Proof of Lemma 2. Since $\mathbb{P}_{1, \mathbf{g}}[a_t \rightarrow \mathbf{g}] = 1$, part (i) follows directly from the fact that $\{\bar{a} = \mathbf{b}\}$ and $\{a_t \rightarrow \mathbf{g}\}$ are two disjoint events. For part (ii), let $\tau < \infty$ denote the last (random) time at which the agent chooses the wrong action \mathbf{b} . It is well-defined since $a_t \rightarrow \mathbf{g}$. Hence, $1 = \mathbb{P}_{1, \mathbf{g}}[a_t \rightarrow \mathbf{g}] = \sum_{k=1}^{\infty} \mathbb{P}_{1, \mathbf{g}}[\tau = k]$. By the overturning principle, $a_\tau = \mathbf{b}$ implies that $\pi_{\tau+1} \leq 1/2$. Hence,

$$\begin{aligned}1 = \sum_{k=1}^{\infty} \mathbb{P}_{1, \mathbf{g}}[\tau = k] &= \sum_{k=1}^{\infty} \mathbb{E}_{1, \mathbf{g}}[\mathbb{P}_{1, \mathbf{g}}[a_{k+1} = a_{k+2} = \dots = \mathbf{g}, \pi_{k+1} \leq 1/2 \mid \pi_{k+1}]] \\ &= \sum_{k=1}^{\infty} \mathbb{E}_{1, \mathbf{g}}[\mathbb{1}(\pi_{k+1} \leq 1/2) \cdot \mathbb{P}_{1, \mathbf{g}}[a_{k+1} = a_{k+2} = \dots = \mathbf{g} \mid \pi_{k+1}]] \\ &= \sum_{k=1}^{\infty} \mathbb{E}_{1, \mathbf{g}}[\mathbb{1}(\pi_{k+1} \leq 1/2) \cdot \mathbb{P}_{1, \mathbf{g}, \pi_{k+1}}[\bar{a} = \mathbf{g}]],\end{aligned}$$

where the second equality follows from the law of total expectation, and the last equality follows from the stationarity property. Suppose $\mathbb{P}_{1, \mathbf{g}, \pi}[\bar{a} = \mathbf{g}] = 0$ for all prior $\pi \in (0, 1)$. This implies that the above equation equals zero, a contradiction. \square

To prove Proposition 2, we first prove the following proposition (Proposition 3). Together with Proposition 1, they jointly imply Proposition 2.

Proposition 3. *The following are equivalent.*

- (i) For any action $a \in \{\mathbf{g}, \mathbf{b}\}$, $\mathbb{P}_{0, \pi}[\bar{a} = a] = 0$ for all prior $\pi \in (0, 1)$.
- (ii) $\mathbb{P}_0[S = \infty] = 1$.
- (iii) For any action $a \in \{\mathbf{g}, \mathbf{b}\}$, $\mathbb{P}_{0, \pi}[\bar{a} = a] = 0$ for some prior $\pi \in (0, 1)$.

We will first prove the following technical lemma.

Lemma 3 (Eventual Monotonicity). *Suppose that $F_{\mathbf{b}}$ has a differentiable left tail and its density function $f_{\mathbf{b}}$ satisfies the condition that, for all x large enough, $f_{\mathbf{b}}(-x) < 1$. Then, $\phi(x) := x + D_{\mathbf{g}}(x)$ increases monotonically for all x large enough.*

Proof. By assumption, we can find a constant $\rho < 1$ such that for all x large enough, $f_{\mathbf{b}}(-x) \leq \rho$. By definition, $D_{\mathbf{g}}(x) = \log \frac{1-F_{\mathbf{g}}(-x)}{1-F_{\mathbf{b}}(-x)}$. Taking the derivative of $D_{\mathbf{g}}$,

$$D'_{\mathbf{g}}(x) = \frac{f_{\mathbf{g}}(-x)}{1-F_{\mathbf{g}}(-x)} - \frac{f_{\mathbf{b}}(-x)}{1-F_{\mathbf{b}}(-x)}.$$

Observe that the LLR of ℓ is the LLR itself (see, e.g., [Chamley \(2004\)](#)): $\log \frac{dF_{\mathbf{g}}}{dF_{\mathbf{b}}}(x) = x$. It follows that

$$-D'_{\mathbf{g}}(x) = f_{\mathbf{b}}(-x) \left(\frac{1}{1-F_{\mathbf{b}}(-x)} - \frac{e^{-x}}{1-F_{\mathbf{g}}(-x)} \right) \leq \frac{f_{\mathbf{b}}(-x)}{1-F_{\mathbf{b}}(-x)}.$$

Fix some $\varepsilon > 0$ small enough so that $(1+\varepsilon)\rho \leq 1$. It follows from the above inequality that there exists some x large enough such that $-D'_{\mathbf{g}}(x) \leq (1+\varepsilon)f_{\mathbf{b}}(-x)$. Furthermore, for all $x' \geq x$,

$$\begin{aligned} D_{\mathbf{g}}(x) &= D_{\mathbf{g}}(x') - \int_x^{x'} D'_{\mathbf{g}}(y) dy \\ &\leq D_{\mathbf{g}}(x') + (1+\varepsilon) \int_x^{x'} f_{\mathbf{b}}(-x) dx \\ &= D_{\mathbf{g}}(x') - (1+\varepsilon)(F_{\mathbf{b}}(-x') - F_{\mathbf{b}}(-x)). \end{aligned}$$

Rearranging the above equation,

$$\begin{aligned} D_{\mathbf{g}}(x) - D_{\mathbf{g}}(x') &\leq (1+\varepsilon)(F_{\mathbf{b}}(-x) - F_{\mathbf{b}}(-x')) \\ &\leq (1+\varepsilon)\rho(x' - x), \end{aligned}$$

where the second last inequality follows from the fact that $f_{\mathbf{b}}(-x) \leq \rho < 1$. Since $(1+\varepsilon)\rho \leq 1$, the above inequality implies that there exists some x large enough such that $D_{\mathbf{g}}(x) + x \leq D_{\mathbf{g}}(x') + x'$ for all $x' \geq x$. That is, $\phi(x)$ eventually increases monotonically. \square

Applying the above lemma, we show that conditioned on an uninformative source, the possibility of immediate agreement is independent of the prior belief.

Recall that $r_t^{\mathbf{g}}$ is the sequence that evolves according to (3). Since $\phi(x) = x + D_{\mathbf{g}}(x)$, we can write $r_{t+1}^{\mathbf{g}} = \phi(r_t^{\mathbf{g}})$ for all $t \geq 1$. Furthermore, we write $r_t^{\mathbf{g}}(r) = \phi^{t-1}(r)$ for all $t \geq 1$, where ϕ^t is its t -th composition and $\phi^0(r) = r$.

Lemma 4. *For any action $a \in \{\mathbf{g}, \mathbf{b}\}$, the following statements are equivalent:*

- (i) $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$, for some prior $\pi \in (0, 1)$;
- (ii) $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$, for all prior $\pi \in (0, 1)$.

Proof. The implication from (ii) \Rightarrow (i) is immediate. We will show (i) \Rightarrow (ii). Fix some prior $\tilde{\pi} \in (0, 1)$ such that $\mathbb{P}_{0,\tilde{\pi}}[\bar{a} = \mathbf{g}] > 0$ and let $\tilde{r} = \log \frac{\tilde{\pi}}{1-\tilde{\pi}}$. Since $r_t^{\mathbf{g}}(\tilde{r})$ is a deterministic process with an initial value \tilde{r} , the event $\{\bar{a} = \mathbf{g}\}$ initiated at prior $\tilde{\pi}$ is equivalent to $\{\ell_t \geq -r_t^{\mathbf{g}}(\tilde{r}), \forall t \geq 1\}$. Following a similar argument that led to (5),

$$\mathbb{P}_{0,\tilde{\pi}}[\bar{a} = \mathbf{g}] > 0 \Leftrightarrow \sum_{t=1}^{\infty} F_0(-r_t^{\mathbf{g}}(\tilde{r})) < \infty. \quad (\text{A.4})$$

By the overturning principle, it suffices to show the above implies that

$$\sum_{t=1}^{\infty} F_0(-r_t^{\mathbf{g}}(r)) < \infty, \quad \text{for any } r \geq 0.$$

Since $r_t^{\mathbf{g}}(\tilde{r}) \rightarrow \infty$, it follows from Lemma 3 that there exists \bar{t} large enough such that $r_{\bar{t}}^{\mathbf{g}}(\tilde{r}) := \bar{r} \geq 0$ and $\phi(r) \geq \phi(\bar{r})$ for all $r \geq \bar{r}$. Note that there exists $t_0 \in \mathbb{N}$ such that $r_{t_0}^{\mathbf{g}}(r) \geq \bar{r}$ for all $r \geq 0$.⁹ Since above \bar{r} , ϕ is monotonically increasing, we have that $\phi(r_{t_0}^{\mathbf{g}}(r)) \geq \phi(\bar{r})$ for any $r \geq 0$. Hence, for all $\tau \geq 1$, $r_{\tau+t_0}^{\mathbf{g}}(r) = \phi^{\tau}(r_{t_0}^{\mathbf{g}}(r)) \geq \phi^{\tau}(\bar{r}) = r_{\tau+1}^{\mathbf{g}}(\bar{r})$ where ϕ^{τ} is its τ -th composition and $\phi^0(r) = r$. Since $r_{\tau+1}^{\mathbf{g}}(\bar{r}) = r_{\tau+1}^{\mathbf{g}}(r_{\bar{t}}^{\mathbf{g}}(\tilde{r})) = r_{\tau+\bar{t}}^{\mathbf{g}}(\tilde{r})$, it follows that

$$F_0(-r_{\tau+t_0}^{\mathbf{g}}(r)) \leq F_0(-r_{\tau+\bar{t}}^{\mathbf{g}}(\tilde{r})).$$

Thus, it follows from (A.4) that for any $r \geq 0$, $\sum_{t=1}^{\infty} F_0(-r_t^{\mathbf{g}}(r)) < \infty$, as required. The case for action \mathbf{b} follows from a symmetric argument. \square

Now, we are ready to prove Proposition 3.

Proof of Proposition 3. We show that (i) \Rightarrow (ii), (ii) \Rightarrow (iii), and (iii) \Rightarrow (i). To show the first implication, we prove the contrapositive statement. Suppose $\mathbb{P}_0[S < \infty] > 0$. This implies that there exists a sequence of action realizations $(b_1, b_2, \dots, b_{k-1}, b_k = \dots = a)$ for some action $a \in \{\mathbf{b}, \mathbf{g}\}$ such that

$$\mathbb{P}_0[a_t = b_t, \forall t \geq 1] > 0.$$

By stationarity, there exists some $\pi' \in (0, 1)$ such that

$$\mathbb{P}_{0,\pi'}[\bar{a} = a] > 0,$$

which contradicts (i).

⁹See Lemma 12 in Rosenberg and Vieille (2019).

To show the second implication, suppose towards a contradiction that there exists some $a \in \{\mathbf{g}, \mathbf{b}\}$ such that $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$ for all prior $\pi \in (0, 1)$. In particular, it holds for the uniform prior. Since the event $\{\bar{a} = a\}$ is contained in the event $\{S < \infty\}$,

$$0 < \mathbb{P}_0[\bar{a} = a] \leq \mathbb{P}_0[S < \infty].$$

This implies that $\mathbb{P}_0[S = \infty] < 1$, a contradiction to (ii).

Finally, we show the last implication by contraposition. Suppose that there exists some $a \in \{\mathbf{g}, \mathbf{b}\}$ such that $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$ for some prior $\pi \in (0, 1)$. By Lemma 4, it also holds for all prior $\pi \in (0, 1)$, which is a contradiction to (iii). This concludes the proof of Proposition 3. \square

Proof of Proposition 2. By the equivalence between (ii) and (iii) in Proposition 3 and Proposition 1, we have shown Proposition 2. \square

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