THE EMERGENCE OF FADS IN A CHANGING WORLD

WANYING HUANG†

Abstract. We study how fads emerge from social learning in a changing environment. We consider a sequential learning model in which rational agents arrive in order, each acting only once, and the underlying unknown state is constantly evolving. Each agent receives a private signal, observes all past actions of others, and chooses an action to match the current state. Since the state changes over time, cascades cannot last forever, and actions fluctuate too. We show that in the long run, actions change more often than the state. This describes many real-life faddish behaviors in which people often change their actions more frequently than what is necessary.

1. Introduction

The term "fad" describes transient behavior that rises and fades quickly in popularity. From daily matters such as diet and parenting to important economic and environmental policy designs, we often encounter fads in many situations (Naim, 2000; Myers, 2011). In finance, it has been documented that fads are more likely to appear when there is uncertainty regarding the asset's intrinsic value, such as during the initial public offering (IPO) market (Camerer, 1989; Aggarwal and Rivoli, 1990). While sociologists have studied some characteristics of fads, the question of how and why fads emerge has not been completely resolved in the economic literature.[1] In this paper, we show how fads can arise from social learning in an ever-changing environment and provide rationales for the development of fads.

The pioneering work in social learning by Banerjee (1992) and Bikhchandani et al. (1992) (BHW, hereafter) shows that under appropriate conditions, *information cascades* always occur – this is the event in which agents find it optimal to follow others regardless of their private signals. However, as discussed in BHW, these information cascades are also fragile to small shocks such as the possibility of a one-time change in the underlying state, thus causing "seemingly whimsical swings in mass behavior without obvious external stimulus". They refer to these behavioral changes as fads and suggest that behavior may change more frequently than the state since a fad can change even though the state

Date: August 30, 2022.

†California Institute of Technology. Email: whhuang@caltech.edu.

I am indebted to my advisor Omer Tamuz for his continued support and encouragement. I am grateful (in alphabetical order) to Wade Hann-Caruthers, Pawel Janas, Aldo Lucia, Thomas Palfrey, Luciano Pomatto and Jean-Laurent Rosenthal for their helpful comments. All errors are my own.

[1]See, e.g., a sociology study on the streaking fad in U.S. college campuses in the 1970s, where the authors discuss many descriptive characteristics of fads (Aguirre et al., 1988).

remains unchanged. Inspired by this original idea of BHW's, we introduce a formal concept of fad and study its long-term behavior.

While BHW present an early theory of fads, they mainly focus on learning in a fixed environment where fads cannot recur indefinitely. In contrast, the recurrence of fads is possible in a changing environment, a setting that has recently attracted some attention in the literature (see, e.g., Dasaratha et al., 2020; Lévy et al., 2022). Indeed, this setting is important to study because many applications concerning social learning, such as investment, employment, cultural norms, and technological advancement, often operate in a dynamic environment. In this paper, we adopt a simple Markovian environment where there is a binary state with symmetric and small transition probability.[2]

We study the long-term behavior of agents, who arrive sequentially and learn from observing the past actions of others as well as their own private signals. Each agent acts once and obtains a positive payoff if her action matches the current state. As the underlying state evolves, the best action to take also fluctuates over time. The question we ask is: how frequently do actions change compared to state changes? On the one hand, agents sometimes ignore their private signals because of information cascades, and thus they do not change their actions even when the state changes. On the other hand, because signals are noisy, agents sometimes change actions unnecessarily. We say *fads* emerge if there are more action changes than state changes, and our main result shows that fads do emerge in the long run. We stress that in our model, fads arise from rational agents' desire to match the ever-changing state rather than any heuristics or irrationality of the agents.

More specifically, the slowly evolving state in our model allows cascades to arise, but they can only be temporary so that agents cannot herd on a single action forever (Moscarini et al., 1998). Intuitively, older information is depreciated over time as it becomes less relevant to the current agent, and eventually, agents will return to utilize their private signals and change their actions. Nevertheless, the question of how often agents change their actions compared to state changes remains unclear. At first glance, one may expect that agents would change their actions less often than the state as temporary cascades prevent agents from following their private signals and thus prevent action changes. This effect of cascades on action inertia is magnified by the possibility that the binary state has changed an even number of times, e.g., say twice, while agents in a cascade mistakenly treat the state as unchanged, thus not changing their actions.

Perhaps surprisingly, our main result (Theorem 1) shows that even if there are temporary cascades, actions change more often than the state in the long run. In other words, fads emerge in a changing environment even though it is unlikely that the state

[2]See other studies of social learning in a changing environment that also consider a Markovian environment, e.g., Moscarini et al. (1998); Hirshleifer and Welch (2002) and Lévy et al. (2022). Our model is mostly close to Moscarini et al. (1998) except for the tie-breaking rule. We provide a detailed literature review on these studies next.

has changed at all. For example, consider a private signal that matches the state with a probability of 0.8. On average, when the state changes once every one hundred periods, we show that agents take less than sixty-one periods to change their actions. Thus, the long-term frequency of action changes must be higher than state changes, resulting in fads. This relatively high frequency of action changes is also in line with the fragility of fads, where small shocks to the system could cause rapid shifts in agents' behavior.[3]

The mechanism behind the emergence of fads is as follows. First, for any fixed signal precision and probability of a state change, there exists a maximum length of any cascade. As a result, even though the rise of temporary cascades prolongs action inertia, such an effect on actions is limited by its bounded length. Meanwhile, agents only require one opposing signal to switch their actions whenever they have an opportunity, i.e., whenever the public belief exits a cascade. We bound the probability of agents switching actions from below and thus establish an upper bound for the expected time between action changes. This upper bound turns out to be less than the expected time of state changes, implying that it takes less time on average for actions to change than the state. Finally, we translate the expected time of changes for both the state and the action into their long-term relative frequency of changes and conclude that fads emerge in the long run.

We restrict our model to a persistent state so temporary cascades can arise (see Proposition 2, Moscarini et al. (1998)). The reason for this restriction is twofold. First, if, on the other hand, cascades never arise, agents would follow their signals and change their actions accordingly. Here, since action changes are purely driven by the volatility in the state and the noise from the private signals, social learning plays no role in agents' behavior. In contrast, in our model, even with the intervention of social learning, which slows down action changes, actions still change more often than the state, suggesting that the emergence of fads is robust to observational learning. Second, even for a persistent state, it is not straightforward to see whether actions or the state changes more often – when the state is unlikely to change, it also slows down action changes. Intuitively, actions change slowly in this case because past actions become more informative about the current state, and as a result, temporary cascades last longer, prolonging action inertia. Nevertheless, we show that in the long run, actions change more often than the state.

1.1. **Related Literature.** This paper is closely related to a small stream of studies on social learning in a changing state. As mentioned before, BHW show that a one-time shock to the state could break the cascade, even though that shock may never be realized. They provide a numeric example where the probability of an action change is at least 87% higher than the probability of a state change (see their Result 4) which is in line with our main result. Later, Moscarini et al. (1998) further explore this idea and show

[3] As discussed in Bikhchandani et al. (1992, 1998), fads are fragile precisely because they are typically formed based on little information. Thus, different kinds of shocks, such as uncertainty in the underlying state as in our model or the arrival of a better informed agent, etc., would dislodge the previous trend and cause drastic behavioral changes.

that if the underlying environment is evolving in every period and the state is sufficiently persistent, an information cascade must arise, but it can only be temporary, i.e., it must end in finite time. Our work builds on their model but with a different focus. While their focus on analyzing the short-term behavior of information cascades, e.g., under what conditions do they end or arise, we ask: in the long run, should one expect to see more action changes than state changes or vice versa?

Hirshleifer and Welch (2002) also consider a changing environment in their stylized model (see their §3), but their focus is on examining the effect of *memory loss* on the continuity of behavior of a single agent.[4] They show that in a relatively stable environment, memory loss causes the agent to exhibit excess action inertia relative to a full-recall regime, whereas in a violate environment, memory loss leads to excess action impulsiveness.[5] One can think of our question as a natural next step from theirs – since there is excess action inertia or impulsiveness when agents only observe past actions in a changing environment, how does the frequency of action changes compare with state changes?

Among a few more recent studies that consider a dynamic state, the efficiency of learning has been a primary focus of study. For example, Frongillo et al. (2011) consider a specific dynamic environment in which the underlying state follows a random walk with non-Bayesian agents who use different linear rules when updating. Their main result is that the equilibrium weights may not be Pareto optimal, causing inefficiency in learning.[6] In a similar but more general environment, Dasaratha et al. (2020) show that having sufficiently diverse network neighbors with different signal distributions improves learning. Intuitively, diverse signals allow agents to decipher the most relevant information from the old and confounded information, thus achieving higher efficiency in information aggregation.

A more recent study by Lévy et al. (2022) considers a similar setup where they focus on studying the implication of a dynamic state on equilibrium welfare. In their model, agents observe a random subsample drawn from all past behaviors and then decide whether to acquire private signals that are potentially costly. These model generalizations allow them to highlight the trade-off between learning efficiency and the need to be responsive to environmental changes, which results in a reduction in equilibrium welfare. In contrast, we assume that all past actions are observable and the private signals are free of charge. We consider this simple sequential learning model without further complications as our focus is on comparing the long-term relative frequency of action and state changes – a question turns out to be non-trivial even in this simple setup.

[4]The term "memory loss" refers to the case where the agent only recalls past actions but not past signals.

[5]Intuitively, as the volatility of the environment increases, past actions become less relevant to the current state. At some point, this information weakens enough so that the amnesiac agent would always follow her latest signal, but the full-recall agent may not do so at this point. Hence there is an increase in the probability of an action change due to amnesia.

[6]See more studies in the computer science literature, e.g., Acemoglu et al. (2008); Shahrampour et al. (2013) that consider a dynamic environment with non-Bayesian agents.

2. Model

We follow the setup from Moscarini et al. (1998) closely. Time is discrete and the horizon is infinite, i.e., $t \in \mathbb{N}^+ = \{1, 2, \ldots\}$. There is a binary state $\theta_t \in \{-1, +1\}$ that constantly evolves over time. A sequence of agents indexed by time t arrive in order, each acting once by choosing an action $a_t \in \{-1, +1\}$ with a payoff function that depends on the unknown state at time t: $\mathbb{1}(a_t = \theta_t)$, i.e., a positive payoff of one if the action matches the current state and zero otherwise.

Before choosing an action, agent t receives a private signal s_t and observes the history of all past actions made by her predecessors, $h^{t-1} = (a_1, a_2, \ldots, a_{t-1})$. Conditional on the entire sequence of states, the private signals s_t are independent, and each s_t has a Bernoulli distribution $B_{\theta_t}(\alpha)$ where α is the symmetric probability of matching the current state: $\mathbb{P}[s_t = i|\theta_t = i] = \alpha \in (1/2, 1)$, for $i \in \{-1, +1\}$. We often refer to signal $s = +1$ as an up-signal and to $s = -1$ as a down-signal. Let $\mathcal{I}^t = \{-1, +1\}^{t-1} \times \{-1, +1\}$ be the space of information available to agent t prior to her decision so that $I_t = (h^{t-1}, s_t)$ is an element of $\mathcal{I}^t$.

For simplicity, we assume that both states are equally likely at the beginning of time, and the state evolves according to a symmetric Markov chain with transition probability ε, i.e., $\mathbb{P}[\theta_{t+1} \neq i|\theta_t = i] = \varepsilon$, for $i \in \{-1, +1\}$. This assumption implies that the stationary distribution of θ_t is uniform. We assume throughout that the state is sufficiently persistent and in particular, $\varepsilon \in (0, \alpha(1 - \alpha))$ so that temporary cascades can arise (Proposition 2, Moscarini et al. (1998)). Equivalently, one can think of this assumption as follows: in every period, with probability $2\varepsilon \in (0, 2\alpha(1 - \alpha))$ the state will be redrawn from $\{-1, +1\}$ with equal probability. Thus, the probability of switching states is equal to $\varepsilon \in (0, \alpha(1 - \alpha))$.

At any time t, the timing of the events is as follows. First, agent t arrives and observes the history h^{t-1} of all past actions. Second, the state θ_{t-1} transitions to θ_t with probability ε of switching. Agent t then receives a private signal s_t that matches the current state θ_t with probability α. Finally, she chooses an action a_t that maximizes the probability of matching θ_t conditional on I_t, the information available to her.

2.1. **Fads.** As the state evolves, the best action to take also fluctuates over time. BHW informally discuss the idea that agents exhibit faddish behavior if they change their actions more often than the state as a fad can change while the state remains unchanged. Formally, we say that *fads* emerge by time $n + 1$ if the fraction of time periods $t \leq n$ for which $a_t \neq a_{t+1}$ is larger than the fraction of those for which $\theta_t \neq \theta_{t+1}$, i.e.,

$$\mathcal{Q}_a(n) := \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(a_t \neq a_{t+1}) > \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(\theta_t \neq \theta_{t+1}) =: \mathcal{Q}_\theta(n). \tag{1}$$

Multiplying both sides of (1) by n, the emergence of fads by time $n + 1$ implies that actions would have changed more often than the state by time $n + 1$.

2.2. **Agents' Beliefs.** Let $q_t := \mathbb{P}[\theta_t = +1 | h^{t-1}]$ denote the *public belief* assigned to state $\theta_t = +1$ at time t after observing the history of actions h^{t-1}. Let $p_t := \mathbb{P}[\theta_t = +1 | I_t]$ denote the *private belief* assigned to state $\theta_t = +1$ after observing $I_t = (h^{t-1}, s_t)$. Denote the log-likelihood ratio (LLR) of the private belief of agent t by

$$L_t := \log \frac{p_t}{1-p_t} = \log \frac{\mathbb{P}[\theta_t = +1 | I_t]}{\mathbb{P}[\theta_t = -1 | I_t]},$$

and call it the *posterior likelihood* at time t. It follows from Bayes' rule that the posterior likelihood at time t satisfies

$$L_t = \log \frac{\mathbb{P}[s_t | \theta_t = +1, h^{t-1}]}{\mathbb{P}[s_t | \theta_t = -1, h^{t-1}]} + \log \frac{\mathbb{P}[\theta_t = +1 | h^{t-1}]}{\mathbb{P}[\theta_t = -1 | h^{t-1}]}. \tag{2}$$

As the private signal is independent of the history conditional on the current state, the first term in (2) reduces to the LLR induced by the signal and it is equal to $c_\alpha := \log \frac{\alpha}{1-\alpha}$ if $s_t = +1$ and $-c_\alpha$ if $s_t = -1$. Denote the second term in (2) by

$$l_t := \log \frac{q_t}{1-q_t} = \log \frac{\mathbb{P}[\theta_t = +1 | h^{t-1}]}{\mathbb{P}[\theta_t = -1 | h^{t-1}]},$$

the *public likelihood* at time t. Intuitively, anyone who observes all past actions until time $t-1$ can calculate this log-likelihood ratio.

Thus, depending on the realization of the private signals, the posterior likelihood L_t is the sum of the public likelihood l_t and the LLR induced by the private signal at time t:

$$L_t = \begin{cases} l_t - c_\alpha & \text{if } s_t = -1, \\ l_t + c_\alpha & \text{if } s_t = +1. \end{cases} \tag{3}$$

2.3. **Agents' Behavior.** The optimal action for agent t is the action that maximizes her expected payoff conditional on the information available to her:

$$a_t \in \operatorname*{arg\,max}_{a \in \{-1,+1\}} \mathbb{E}[\mathbb{1}(\theta_t = a) | I_t] = \operatorname*{arg\,max}_{a \in \{-1,+1\}} \mathbb{P}[\theta_t = a | I_t].$$

Thus $a_t = +1$ if $L_t > 0$ and $a_t = -1$ if $L_t < 0$. When agent t is indifferent, i.e., $L_t = 0$, we assume that she would follow what her immediate predecessor did in the previous period, i.e., $a_t = a_{t-1}$.[7] This assumption ensures that action changes are not due to the specification of the tie-breaking rule but rather due to her strict preference for one action over another.

2.4. **Information Cascades and Regions.** An *information cascade* is the event in which the past actions of others form an overwhelming influence on agents so that they act independently of the private signals. Specifically, it follows from (3) that the sign of the posterior likelihood L_t is purely determined by the sign of the public likelihood l_t once the absolute value of l_t exceeds c_α. Since the sign of L_t determines the optimal action of

[7]This tie-breaking rule is different from the one in Moscarini et al. (1998) where they assume that agents would follow their private signals when indifferent. Under their assumption, it is likely for an indifferent agent to choose a different action from her predecessor's.

agent t, in this case, a_t will also be purely determined by the sign of l_t, independent of the private signal s_t. That is, $a_t = +1$ if $l_t > c_\alpha$ and $a_t = -1$ if $l_t < -c_\alpha$. When $|l_t| < c_\alpha$, agent t chooses the action according to her private signal so that $a_t = s_t$.

When $|l_t| = c_\alpha$, by the tie-breaking rule at indifference, regardless of the private signal that agent t receives, she chooses $a_t = a_{t-1} = \text{sign}(l_t)$.[8] Thus, we call the region of the public likelihood in which $|l_t| \geq c_\alpha$ the *cascade region* and the region in which $|l_t| < c_\alpha$ the *learning region*. We refer to the cascade in which $a = +1$ as an up-cascade and to the cascade in which $a = -1$ as a down-cascade.

3. Results

We now state our main result. Recall that in (1) we define the emergence of fads by some time $n+1$ as a higher relative frequency of action changes compared to state changes.

Theorem 1. *For any signal precision $\alpha \in (1/2, 1)$ and probability of a state change $\varepsilon \in (0, \alpha(1-\alpha))$, fads emerge in the long run almost surely, i.e.,*

$$\lim_{n\to\infty} \mathcal{Q}_a(n) > \lim_{n\to\infty} \mathcal{Q}_\theta(n) \quad a.s.$$

Perhaps surprisingly, Theorem 1 shows that even there are times in which agents stop responding to their private signals, i.e., when cascades arise temporarily, agents who observe their predecessors' past actions still change their actions more often than the state in the long run. In other words, fads can emerge from social learning even though the underlying environment evolves very slowly. For example, consider a private signal with a precision of 0.8. When the probability of a state change is equal to 0.01, the state switches once every one hundred periods on average. Meanwhile, the average time for the action to switch is strictly less than six-one periods.[9] Thus, in the long run, actions would switch strictly more often than the state, resulting in faddish behavior.

The idea behind the proof of Theorem 1 is as follows. At first glimpse, one might expect that actions would change less often than the state due to the effect of cascades on action inertia. However, for any fixed signal precision and probability of a state change, there exists a maximum length of any cascade. We show that once the public belief exits the cascade region, the action either switches, or else the public belief enters the same cascade region again. We upper bound the probability of the latter event and thus establish an upper bound to the expected time between action changes, which turns out to be less than the expected time between state changes (by Proposition 1 in §5). Finally, our main result (Theorem 1) translates the expected time between both action and state switches

[8] Without loss of generality, consider $l_t = c_\alpha$. To see why $a_{t-1} = \text{sign}(l_t) = +1$, suppose to the contrary that $a_{t-1} = -1$. Given that $l_t = c_\alpha$ and $a_{t-1} = -1$, it must be that $l_{t-1} > c_\alpha$, which implies that $a_{t-1} = +1$. A contradiction.

[9] This follows from Proposition 1 in §5 by substituting $\alpha = 0.8$ and $\varepsilon = 0.01$ into $M(\alpha, \varepsilon)$, and we have $M(0.8, 0.01) \approx 60.7$.

into their long-run relative frequency of switches.[10] Building on this, we conclude that the long-run relative frequency of action changes is higher than that of state changes.

4. The Public Belief Dynamics

This section provides a detailed analysis of the dynamics of the public likelihood in different regions, which will facilitate our analysis in the next section.

4.1. **Cascade region.** When the state is fixed ($\varepsilon = 0$), it is well-known that the public likelihood stays forever at the value at which it first entered the cascade region, and an incorrect cascade can occur forever with positive probability (Bikhchandani et al., 1992; Banerjee, 1992). When the state is changing ($\varepsilon > 0$), however, the behavior of the public likelihood changes significantly. To see this, suppose that t is a time at which the public likelihood enters the cascade region from the learning region. Although no agent's actions reveal more information about the state after time t, the state still evolves and switches with probability ε in every period. Since the process $(\theta_t)_t$ is a Markov chain, it follows from the law of total probability that the public belief updates deterministically as follows:

$$\begin{aligned} q_{t+1} := \mathbb{P}[\theta_{t+1} = +1|h^t] &= \sum_{\theta_t \in \{-1,+1\}} \mathbb{P}[\theta_{t+1} = +1|h^t, \theta_t]\mathbb{P}[\theta_t|h^t] \\ &= (1-\varepsilon)q_t + \varepsilon(1-q_t), \end{aligned}$$

so that

$$q_{t+1} = \ (1-2\varepsilon)q_t + (2\varepsilon)\frac{1}{2}. \tag{4}$$

Equivalently, we can write it recursively in terms of the public likelihood:

$$l_{t+1} := \log \frac{q_{t+1}}{1-q_{t+1}} = \log \frac{(1-\varepsilon)\mathrm{e}^{l_t} + \varepsilon}{1-\varepsilon+\varepsilon \mathrm{e}^{l_t}}. \tag{5}$$

From (4), we see that q_{t+1} tends to $1/2$, and by (5), l_{t+1} moves towards zero over time so that eventually it will exit the cascade region. Intuitively, having a changing state depreciates the value of older information as actions observed in earlier periods become less relevant to the current state. Consequently, after some finite number of periods, the public belief will slowly converge towards uniform, and thus information cascades built upon this public belief cannot last forever. Indeed, this is the main insight from Moscarini et al. (1998) (see their Proposition 1), where they show that information cascades (if they arise) must end in finite time.

4.2. **Learning region.** When the state is fixed ($\varepsilon = 0$), as the agent's action is informative about her private signal in the learning region, i.e., $a_t = s_t$, the public belief at time

[10] Notice that since these action changes are not independent events, the connection between the expected time of action changes and its long-run relative frequency of changes does not directly follow from the standard result of the law of large numbers.

$t+1$ coincides with the private belief at time t:

$$q_{t+1} := \mathbb{P}[\theta_{t+1} = 1 | h^{t-1}, a_t] = \mathbb{P}[\theta_t = 1 | h^{t-1}, s_t] = p_t.$$

Hence, $l_{t+1} = L_t$ and l_{t+1} evolves according to (3). When the state changes with probability $\varepsilon > 0$ in every period, upon observing the latest history, each agent also needs to consider the possibility that the state may have changed after the latest action was taken. However, neither the learning region nor the cascade region is affected by a changing state as the state only transitions after the history of past actions is observed. By Bayes' rule, the public likelihood at time $t+1$ in the learning region is

$$\begin{aligned} l_{t+1} &:= \log \frac{q_{t+1}}{1 - q_{t+1}} \\ &= \log \frac{\mathbb{P}[\theta_{t+1} = +1 | h^{t-1}, a_t]}{\mathbb{P}[\theta_{t+1} = -1 | h^{t-1}, a_t]} \\ &= \log \frac{\sum_{\theta_t \in \{-1,+1\}} \mathbb{P}[\theta_{t+1} = +1, a_t | h^{t-1}, \theta_t] \mathbb{P}[\theta_t | h^{t-1}]}{\sum_{\theta_t \in \{-1,+1\}} \mathbb{P}[\theta_{t+1} = -1, a_t | h^{t-1}, \theta_t] \mathbb{P}[\theta_t | h^{t-1}]}. \end{aligned} \tag{6}$$

Since the process $(\theta_t)_t$ is a Markov chain and $a_t = s_t$ in the learning region, conditioned on θ_t, both θ_{t+1} and a_t are independent of h^{t-1} and independent of each other. Hence, we can write (6) as

$$f_1(l_t) := \log \frac{(1-\varepsilon)\alpha \mathrm{e}^{l_t} + \varepsilon(1-\alpha)}{\varepsilon\alpha \mathrm{e}^{l_t} + (1-\varepsilon)(1-\alpha)} \quad \text{if } s_t = +1,$$

and

$$f_0(l_t) := \log \frac{(1-\varepsilon)(1-\alpha) \mathrm{e}^{l_t} + \varepsilon\alpha}{\varepsilon(1-\alpha) \mathrm{e}^{l_t} + (1-\varepsilon)\alpha} \quad \text{if } s_t = -1.$$

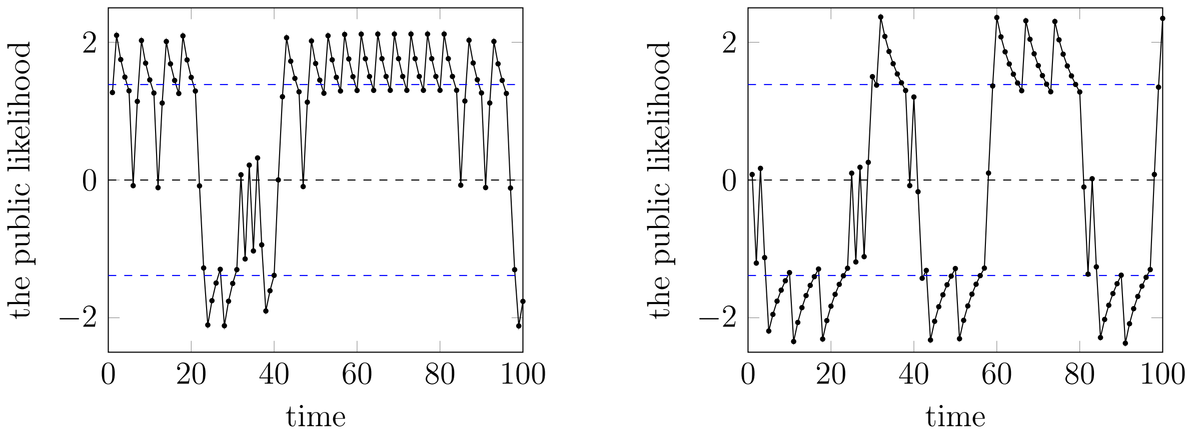
In summary, with a changing state, the public likelihood in the learning region evolves as follows:

$$l_{t+1} = \begin{cases} f_1(l_t) & \text{if } s_t = +1, \\ f_0(l_t) & \text{if } s_t = -1. \end{cases} \tag{7}$$

Note that in our model, both $f_1(l)$ and $f_0(l)$ are strictly increasing in l, and moreover, $f_1(l) > l$ and $f_0(l) < l$. Intuitively, when an agent starts with a higher prior belief, her posterior belief will be higher upon receiving either an up or down signal. Similarly, an agent's posterior belief will also be higher (lower) than her prior belief upon receiving an up-signal (down-signal).

From (7), one can see that the magnitude difference between l_t and l_{t+1} depends on both the realization of the private signal s_t and the current value of l_t. The following lemma summarizes two distinct patterns of the transition of the public likelihood in the learning region. At any time t, we say an action is *opposing* to the current public belief if $a_t \neq \operatorname{sign}(l_t)$ and *supporting* if $a_t = \operatorname{sign}(l_t)$.

Figure 1. Simulated Evolution of the Public Likelihood



(a) $\alpha = 0.8$ and $\varepsilon = 0.05$.

(b) $\alpha = 0.8$ and $\varepsilon = 0.03$.

Lemma 1. *For any signal precision $\alpha \in (1/2, 1)$ and probability of a state change $\varepsilon \in (0, \alpha(1-\alpha))$, if the public likelihood is in the learning region, then*

(i) observing one opposing action is sufficient to overturn its sign;

(ii) at most two supporting actions are required to initiate a cascade.

Another important observation is that regardless of whether the state is fixed or changing, the process of the public likelihoods $(l_t)_t$ in either case forms a Markov chain.[11] In the case of a fixed state, the state space of $(l_t)_t$ is finite as the magnitude difference between l_t and l_{t+1} is constant for any fixed signal precision, which is equal to c_α. However, in the case of a changing state, the state space of $(l_t)_t$ is infinite as such magnitude differences also depend on the current value of l_t. Note that in almost all cases, two consecutive opposing signals would not exactly offset each other, i.e., $f_i(f_j(l)) \neq l$, for$i \neq j = 0, 1$. As a result, there are many more possible values for l_t in a changing state compared to a fixed state.

We demonstrate these public likelihood dynamics by running simulations under specific values of signal precision α and probability of a state change ε. Figure 1 displays the results of two simulations in the first hundred periods. In line with the above discussion, Figure 1 shows that in the cascade region (above or below the blue dash lines), the public likelihood moves towards zero over time and exits the cascade region after some finite number of periods. In the learning region (strictly between the two blue dash lines), the public likelihood changes its sign with each opposing signal, and thus, with consecutively opposing signals, the public likelihood oscillates around zero. By comparing Figure 1a with 1b, it seems that the number of periods the public likelihood spends in a cascade may depend on the probability of a state change. For example, when ε decreases from 0.05 to 0.03, the time the public likelihood spends in a cascade generally increases from

[11] To see that the process $(l_t)_t$ is a Markov chain, notice that conditioned on the state θ_t, the private signal s_t is independent of $l_{t'}$, for any $t' < t$. So its distribution conditioned on l_t is the same as its distribution conditioned on $(l_1, \ldots, l_{t-1})$ which implies that $\mathbb{P}[l_{t+1} = l | l_1, \ldots, l_t] = \mathbb{P}[l_{t+1} = l | l_t]$.

three to six periods. Indeed, we will see in the next section that the maximum length of any cascades depends on both the probability of a state change and the precision of the private signal.

5. Analysis – Expected Time Between State and Action Changes

This section provides a detailed analysis of the comparison between the expected time between state changes and action changes, which is crucial in proving our main result.

We first calculate the expected time between state changes. Since the process $(\theta_t)_t$ is a simple Markov chain, the expected time between state changes is inversely proportional to the probability of a state change, which is equal to $1/\varepsilon$. To see this, suppose that the expected time between state changes is equal to x. As $(\theta_t)_t$ follows a two-state Markov chain with a symmetric transition probability of ε, x satisfies

$$x = \varepsilon + (1-\varepsilon)(1+x).$$

This implies that $x = 1/\varepsilon$. Intuitively, if the state becomes more likely to change in every period, it takes less time to change on average.

In contrast, the question of how long it takes on average for the action to change is more difficult as the process $(a_t)_t$ is not a Markov chain. Nevertheless, a_t is a function of l_{t+1}, namely, the sign of l_{t+1}, which is a Markov chain. However, as discussed in §4, this Markov chain is complicated as it has infinitely many states (values) and different transition probabilities between states. The complexity of $(l_t)_t$ raises difficulty in directly analyzing the expected time between the sign switches of the public likelihood. To overcome this, we provide an upper bound instead and thus obtain an upper bound to the expected time between action changes.

To do so, let us consider the maximum length of any cascade. For any signal precision and probability of a state change, since no cascade can last longer than the cascade starting at the supremum of the public likelihood $f_1(c_\alpha)$, one can calculate a tight upper bound to the length of any cascade from (5). Following Moscarini et al. (1998), we denote this bound by[12]

$$K(\alpha,\varepsilon) = \frac{\log(1-2\alpha(1-\alpha))}{\log|1-2\varepsilon|}.$$

[12] For completeness, we provide a similar calculation of this bound to the one in §3.B of Moscarini et al. (1998). Fix α and ε. Let m denote the supremum of the public belief and note that $m = \frac{(1-\varepsilon)\alpha^2+\varepsilon(1-\alpha)^2}{\alpha^2+(1-\alpha)^2}$. After h periods in a cascade, by (4), the public belief starting at m becomes

$$g(h) := \varepsilon \sum_{i=1}^{h-1}(1-2\varepsilon)^i + (1-2\varepsilon)^h m.$$

Since m is the supremum, any public belief after h periods in the cascade region has a value that is strictly less than $g(h)$. Thus, whenever $g(h) \le \alpha$, or equivalently, $(1-2\varepsilon)^{h+1} \le 1-2\alpha(1-\alpha)$, the public likelihood exits the cascade region. Hence, the public likelihood can never be in the cascade region for more than $K(\alpha,\varepsilon) := \frac{\log(1-2\alpha(1-\alpha))}{\log|1-2\varepsilon|}$ periods.

In other words, if $l_t = f_1(c_\alpha)$, then $l_{t+\lfloor K(\alpha,\varepsilon)\rfloor}$ would just have exited the cascade region, so the length of any cascade is less than $\lfloor K(\alpha, \varepsilon)\rfloor$, the greatest integer that is less than or equal to $K(\alpha, \varepsilon)$. For instance, when $\alpha = 0.8$ and $\varepsilon = 0.05$, $\lfloor K(0.8, 0.05)\rfloor = 3$, and as seen in Figure 1a, the duration of any up-cascades never exceeds three periods.

For all $i = 1, 2, \ldots$, denote the random time at which the public likelihood switches its sign for the i-th time by $\mathcal{T}_i$ and let $\mathcal{T}_0 = 0$. Denote the random time elapsed between the $i-1$-th and i-th sign switch by $\mathcal{D}_i := \mathcal{T}_i - \mathcal{T}_{i-1}$.

Proposition 1. *For any signal precision $\alpha \in (1/2, 1)$, probability of state change $\varepsilon \in (0, \alpha(1-\alpha))$ and positive integers $i \geq 2$, conditional on the public likelihood that just switched its sign for the $i-1$-th time, the expected time to the next sign switch is strictly bounded above by $M(\alpha, \varepsilon) := 1 + \frac{K(\alpha,\varepsilon)}{2\alpha(1-\alpha)}$, which in turn is strictly less than the expected time between state switches, i.e.,*

$$E[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < M(\alpha, \varepsilon) < 1/\varepsilon.$$

Since there is a one-to-one mapping between the public likelihood and the agent's action, Proposition 1 implies that on average, actions take less time to switch than the state. For any fixed ε, note that $M(\alpha, \varepsilon)$ is decreasing in α, and thus $M(1/2, \varepsilon)$ is the greatest upper bound to the expected time between action switches. Intuitively, as the signal becomes less informative, agents rely more on public information than private information, and as a result, information cascades are more likely to arise, and so is action inertia. Hence, the upper bound on the expected time between action switches grows as the private signal becomes less precise.

We thus illustrate the proof idea of Proposition 1 using a weakly informative signal, that is, $\alpha = 1/2 + \delta$ where δ is strictly positive and close to 0. Denote the maximum length of any cascades as δ approaches zero by $K(\frac{1}{2}, \varepsilon)$. In this case, upon exiting a cascade, the probability of the public likelihood switching its sign is about $1/2$ as it is almost equally likely that the agent, who follows her private signal, will receive an up or down signal. Thus, we can bound the expected time between the sign switches from above by

$$1 + \sum_{i=1}^{\infty} \frac{i}{2^i} K(\frac{1}{2}, \varepsilon) = 1 + 2K(\frac{1}{2}, \varepsilon) = M(1/2, \varepsilon),$$

which turns out to be strictly less than $1/\varepsilon$ for all $\varepsilon \in (0, 1/4)$.

Note that our restriction to a persistent state is not the driving force behind Proposition 1. This is because even for small probability of a state change ε, it remains unclear whether on average, actions or the state take less time to change. When ε is smaller, the state changes more slowly, but actions also change more slowly because cascades last longer as past actions become more informative about the current state. For example, when ε decreases from 0.05 to 0.03 in Figure 1, the maximum length of information cascades increases from three to six periods and more generally, $\partial K(\alpha, \varepsilon)/\partial \varepsilon < 0$ for $\varepsilon \in (0, 1/2)$.

Finally, we end this section by providing a lemma that allows us to have well-defined moments when working with the process $(\mathcal{D}_i)_i$.

Lemma 2. *Fix any signal precision $\alpha \in (1/2, 1)$ and probability of a state change $\varepsilon \in (0, \alpha(1-\alpha))$. Then, for every $r \in \{1, 2, \ldots\}$ there is a constant c_r that depends on α and ε such that for all i, $\mathbb{E}[|\mathcal{D}_i|^r] < c_r$. I.e., each moment of $\mathcal{D}_i$ is uniformly bounded, independently of i.*

6. Conclusion

We study the long-term behavior of agents who receive a private signal and observe the past actions of their predecessors in a changing environment. As the state evolves, agents adjust accordingly so that their actions fluctuate over time. We show that in the long run, the relative frequency of action changes is higher than that of state changes, suggesting fads can emerge from social learning in a changing environment.

One could study the frequency of action changes for a single long-lived agent who receives a private signal about a changing state in every period. We conjecture that action changes would be less frequent in this case than in our model, where each agent only observes her predecessors' past actions rather than signals. Intuitively, by shutting down the channel of noisy observations of others' private signals, the frequency of unnecessary action changes would reduce. If this were the case, it would highlight the importance of observational learning in accelerating action fluctuations, especially when the underlying environment is slowly evolving.

One may wonder if the driving force behind our main result is due to the high frequency of action changes when the posterior likelihood is around zero. Accordingly, we can further restrict the definition of fads to action changes that do not have consecutive switches, i.e., $a_t \neq a_{t-1}$ and $a_{t-1} \neq a_{t-2}$. Simulation results show that actions still change more frequently than the state, even under this more restricted definition of fads. For example, for $\alpha = 0.8$, $\varepsilon = 0.05$ and a total of $100,000$ periods, the action changes about 8,200 times which is more frequent than the number of state changes, that is about 5,000 times.

There are a number of possible avenues for future research. Recall that Proposition 1 implies that $M(\alpha, \varepsilon)$ is an upper bound to the expected time between action changes. One could ask whether this upper bound $M(\alpha, \varepsilon)$ is tight, and if so, for any given finite time N, the number of action changes would be close to $N/M(\alpha, \varepsilon)$. Based on the simulation results, we conjecture that it is not a tight bound. E.g., we let $\alpha = 0.8$ and $\varepsilon = 0.01$, and $N = 100,000$. Since $M(0.8, 0.01) \approx 61$, it implies that within these hundred thousand periods, the action should at least change about $1,640$ times. However, our numerical simulation shows that the action changes about $3,500$ times, which is more than double the number suggested by $M(0.8, 0.01)$.

Furthermore, our simulations suggest that as the private signal becomes less informative ($\alpha \to 1/2$) and the state changes more slowly ($\varepsilon \to 0$), the ratio between the frequency of action changes and state changes approaches a constant that is close to 4. This suggests that it might be possible to achieve a very accurate understanding of fads in this regime.

Appendix A. Proofs

Proof of Lemma 1. Fix $\alpha \in (1/2, 1)$, $\varepsilon \in (0, \alpha(1-\alpha))$. By symmetry, it is without loss to consider the case where $0 < l_t < c_\alpha$.

(i) First notice that $f_0(c_\alpha) = 0$, and since $f_0(\cdot)$ in (7) is strictly increasing, $f_0(l_t) < 0$ for all $0 < l_t < c_\alpha$. Since l_t is in the learning region, $a_t = s_t$. Hence, $-1 = a_t \neq \mathrm{sign}(l_t) = +1$ implies that $\mathrm{sign}(l_{t+1}) = \mathrm{sign}(f_0(l_t)) = -1$.

(ii) As $f_1(\cdot)$ is strictly increasing, it suffices to show that $f_1(f_1(0)) \geq c_\alpha$. Note that for all $0 < \varepsilon < \alpha(1-\alpha)$, $f_1(0) > c_u$ where

$$c_u := f_1^{-1}(c_\alpha) = \log \frac{(1-\alpha)(\alpha-\varepsilon)}{\alpha(1-\alpha-\varepsilon)} \in (0, c_\alpha)$$

is the threshold at which exactly one up-signal is required to push the public likelihood into the up-cascade region. Thus, $f_1(f_1(0)) > f_1(c_u)$ and by the definition of c_u, we have $f_1(f_1(0)) > f_1(c_u) = f_1(f_1^{-1}(c_\alpha)) = c_\alpha$, as required.

□

Proof of Proposition 1. Fix $\alpha \in (1/2, 1)$, $\varepsilon \in (0, \alpha(1-\alpha))$ and some positive integer $i \geq 2$. For the ease of notation, we denote by K the greatest integer that is less than or equal to $K(\alpha, \varepsilon)$ and note that $K \geq 1$ since $K(\alpha, \varepsilon) \geq 1$. Consider $l_{\mathcal{T}_{i-1}} > 0$ (without loss of generality). We will first provide an upper bound to the expected time of the public likelihood of switching its sign to negative and then show that it is strictly less than $1/\varepsilon$.

Conditioned on the public likelihood being l, or equivalently, the public belief being $\mathrm{e}^l/(1+\mathrm{e}^l)$, denoting the probability of receiving an up-signal by

$$\pi(l) := \mathbb{P}[s = +1 | q = \frac{\mathrm{e}^l}{1+\mathrm{e}^l}] = \frac{1 + \alpha(\mathrm{e}^l - 1)}{1+\mathrm{e}^l}.$$

Since $\partial \pi(l)/\partial l > 0$, the supremum of $\pi(l)$ over all positive learning region is equal to $\pi(c_\alpha) = 1 - 2\alpha(1-\alpha)$, and we denote it as $\bar{\pi}$.

Let $\kappa(l)$ denote the length of the cascade initiated by receiving an up-signal conditioned on the public likelihood being l and let $\mathcal{L}(l)$ denote the value of this public likelihood after it first exits the cascade region. We further divide the region $(0, f_1(c_\alpha))$ into three sub-regions: (i) $[c_u, c_\alpha)$ – the one up-signal away from the cascade region, where $c_u := f^{-1}(c_\alpha)$; (ii) $(0, c_u)$ – the two up-signal away from the cascade region, and (iii) $[c_\alpha, f_1(c_\alpha))$ – the up-cascade region. Next, we obtain an upper bound for each sub-region.

First, consider $l_{\mathcal{T}_{i-1}} \in [c_u, c_\alpha)$. By part (i) of Lemma 1, since $l_{\mathcal{T}_{i-1}}$ is in the learning region, one opposing signal is sufficient to change the sign of $l_{\mathcal{T}_{i-1}}$. Thus, the expected time to the next sign switch satisfies

$$\begin{aligned}
\mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}] &= 1-\pi(l_{\mathcal{T}_{i-1}})+\pi(l_{\mathcal{T}_{i-1}})\Big(\kappa(l_{\mathcal{T}_{i-1}})+\mathbb{E}[\mathcal{D}_i|\mathcal{L}(l_{\mathcal{T}_{i-1}})]\Big)\\
&< 1-\bar{\pi}+\bar{\pi}\Big(\kappa(l_{\mathcal{T}_{i-1}})+\mathbb{E}[\mathcal{D}_i|\mathcal{L}(l_{\mathcal{T}_{i-1}})]\Big)\\
&\leq 1-\bar{\pi}+\bar{\pi}\Big(K+\mathbb{E}[\mathcal{D}_i|\mathcal{L}(l_{\mathcal{T}_{i-1}})]\Big), \qquad (8)
\end{aligned}$$

where the second last strict inequality follows from the definition $\bar{\pi}$ and the last inequality follows from the definition of K. Note that there are two possible cases for $\mathcal{L}(l_{\mathcal{T}_{i-1}})$.
Case (1): $\mathcal{L}(l_{\mathcal{T}_{i-1}}) \in [c_u, c_\alpha)$. Taking the supremum on both sides of (8) and rearranging,

$$\sup_{c_u \leq l_{\mathcal{T}_{i-1}} < c_\alpha} \mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}] \leq 1+\frac{K\bar{\pi}}{1-\bar{\pi}}.$$

Case (2): $\mathcal{L}(l_{\mathcal{T}_{i-1}}) \in [0, c_u)$. Similarly, we can write

$$\mathbb{E}[\mathcal{D}_i|\mathcal{L}(l_{\mathcal{T}_{i-1}})] < 1-\bar{\pi}+\bar{\pi}\Big(1+\mathbb{E}[\mathcal{D}_i|f_1(\mathcal{L}(l_{\mathcal{T}_{i-1}}))]\Big).$$

Substituting the above inequality into (8), we obtain

$$\mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}] < 1-\bar{\pi}+\bar{\pi}\Big(K+1-\bar{\pi}+\bar{\pi}\big(1+\mathbb{E}[\mathcal{D}_i|f_1(\mathcal{L}(l_{\mathcal{T}_{i-1}}))]\big)\Big).$$

By part (ii) of Lemma 1, $f_1(\mathcal{L}(l_{\mathcal{T}_{i-1}})) \in [c_u, c_\alpha)$, and thus by taking the supremum on both sides and arranging

$$\sup_{c_u \leq l_{\mathcal{T}_{i-1}} < c_\alpha} \mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}] \leq \frac{1-\bar{\pi}+(K+1)\bar{\pi}}{1-\bar{\pi}^2} \leq 1+\frac{K\bar{\pi}}{1-\bar{\pi}}.$$

Together, we have

$$\sup_{c_u \leq l_{\mathcal{T}_{i-1}} < c_\alpha} \mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}] \leq 1+\frac{K\bar{\pi}}{1-\bar{\pi}}. \qquad (9)$$

Next, consider $l_{\mathcal{T}_{i-1}} \in (0, c_u)$. By part (i) of Lemma (1) and the definition of $\bar{\pi}$, the expected time to the next sign switch is bounded above by

$$\mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}] < (1-\bar{\pi})+\bar{\pi}(1+\mathbb{E}[\mathcal{D}_i|f_1(l_{\mathcal{T}_{i-1}})]).$$

It follows from part (ii) of Lemma 1 that $f_1(l_{\mathcal{T}_{i-1}}) \in [c_u, c_\alpha)$, and thus by (9), we have that for all $l_{\mathcal{T}_{i-1}} \in (0, c_u)$,

$$\begin{aligned}
\mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}] &< (1-\bar{\pi})+\bar{\pi}(1+1+\frac{K\bar{\pi}}{1-\bar{\pi}})\\
&= \frac{(K-1)(\bar{\pi})^2+1}{1-\bar{\pi}}. \qquad (10)
\end{aligned}$$

Finally, consider $l_{\mathcal{T}_{i-1}} \in [c_\alpha, f_1(c_\alpha))$. In this case, after at most K periods, the public likelihood starting at $l_{\mathcal{T}_{i-1}}$ would have exited the cascade region. Hence, the expected time to the next sign switch is bounded above by

$$\mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}] \leq K+\mathbb{E}[\mathcal{D}_i|\mathcal{L}(l_{\mathcal{T}_{i-1}})].$$

Again, there are two possible cases for $\mathcal{L}(l_{\mathcal{T}_{i-1}})$.
Case (1): $\mathcal{L}(l_{\mathcal{T}_{i-1}}) \in [c_u, c_\alpha)$. It follows from (9) that

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < K + 1 + \frac{K\bar{\pi}}{1-\bar{\pi}} = 1 + \frac{K}{1-\bar{\pi}}.$$

Case (2): $\mathcal{L}(l_{\mathcal{T}_{i-1}}) \in [0, c_u)$. It then follows from (10) that

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < K + \frac{(K-1)(\bar{\pi})^2 + 1}{1-\bar{\pi}} = 1 + K + \bar{\pi} + \frac{K\bar{\pi}^2}{1-\bar{\pi}} \leq 1 + \frac{K}{1-\bar{\pi}}.$$

Together, for all $l_{\mathcal{T}_{i-1}} \in [c_\alpha, f_1(c_\alpha))$,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < 1 + \frac{K}{1-\bar{\pi}}. \tag{11}$$

Since the maximum over all three upper bounds from (9) to (11) is $1 + \frac{K}{1-\bar{\pi}}$,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < 1 + \frac{K}{1-\bar{\pi}} \leq 1 + \frac{K(\alpha, \varepsilon)}{2\alpha(1-\alpha)}, \text{ for all } l_{\mathcal{T}_{i-1}} > 0.$$

We show next that this upper bound $1 + \frac{K(\alpha,\varepsilon)}{2\alpha(1-\alpha)}$ is strictly less than $1/\varepsilon$. As $\frac{K(\alpha,\varepsilon)}{2\alpha(1-\alpha)}$ decreases in α,

$$\limsup_{\alpha \to 1/2} 1 + \frac{K(\alpha, \varepsilon)}{2\alpha(1-\alpha)} = 1 + \frac{2 \log 2}{-\log|1-2\varepsilon|}.$$

Since

$$2 \log 2 < 2 = \liminf_{\varepsilon \to 0} -(\frac{1}{\varepsilon} - 1) \log|1-2\varepsilon|,$$

we conclude that

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < 1 + \frac{K(\alpha, \varepsilon)}{2\alpha(1-\alpha)} < 1/\varepsilon, \text{ for all } l_{\mathcal{T}_{i-1}} > 0.$$

□

Proof of Lemma 2. Fix $\alpha \in (1/2, 1)$, $\varepsilon \in (0, \alpha(1-\alpha))$ and some positive integer $i \geq 2$. Consider the case where $\mathcal{D}_i$ is the (random) time elapsed from a positive public likelihood to a negative one (without loss of generality). Denote by $\bar{\pi} \in (1/2, 1)$ the supremum of the probability of receiving an up-signal conditional on the public likelihood being in the positive learning region.

By Lemma 1, we know that one opposing signal is sufficient to overturn the sign of the public likelihood and at most two consecutive supporting signals is required to initiate a cascade. For any positive integer $n \geq 2$, denote by

$$k(n) := \max\left\{\frac{n-1}{\lfloor K(\alpha, \varepsilon) \rfloor}, 1\right\}$$

the minimum number (may not be an integer) of cascades required for $\mathcal{D}_i > n$. Hence, for any $n \geq 2$, we can bound the probability of event $\{\mathcal{D}_i > n\}$ by

$$\mathbb{P}[\mathcal{D}_i > n] < \bar{\pi}^{2+(\lfloor k(n) \rfloor - 1)}.$$

Since $\mathcal{D}_i$ is a positive random variable, it then follows that for any $p > 0$,

$$\lim_{n\to\infty} n^p \mathbb{P}[|\mathcal{D}_i| > n] = \lim_{n\to\infty} \frac{n^p}{1/\mathbb{P}[\mathcal{D}_i > n]}$$
$$< \lim_{n\to\infty} \frac{n^p}{(1/\bar{\pi})^{1+\lfloor k(n)\rfloor}} = 0. \tag{12}$$

For any $r \geq 1$, the r-th moment of $|\mathcal{D}_i|$ satisfies

$$\begin{aligned}\mathbb{E}[|\mathcal{D}_i|^r] &= \int_0^\infty \mathbb{P}[|\mathcal{D}_i|^r > t]dt \\ &< 1 + \int_1^\infty \mathbb{P}[\mathcal{D}_i > y]ry^{r-1}dy \\ &= 1 + \sum_{n=1}^\infty \int_n^{n+1} \mathbb{P}[\mathcal{D}_i > y]ry^{r-1}dy \\ &< 1 + \sum_{n=1}^\infty \mathbb{P}[\mathcal{D}_i > n]r(n+1)^{r-1},\end{aligned}$$

where the second inequality follows from a change of variable $y = t^{1/r}$. Since (12) implies that $\mathbb{P}[\mathcal{D}_i > n] < Cn^{-p}$ for some nonnegative constant C, it follows that for any $p > r$,

$$\begin{aligned}\mathbb{E}[|\mathcal{D}_i|^r] &< 1 + rC\sum_{n=1}^\infty \frac{(n+1)^{r-1}}{n^p} \\ &< 1 + r2^{r-1}C\sum_{n=1}^\infty \frac{1}{n^{p-r+1}} < \infty,\end{aligned}$$

which holds for all i. Hence, for every $r \in \{1, 2, \ldots\}$, there exists a constant $c_r = 1 + r2^{r-1}C\sum_{n=1}^\infty \frac{1}{n^{p-r+1}}$, independently of i, that uniformly bounds $\mathbb{E}[|\mathcal{D}_i|^r]$. $\square$

Proof of Theorem 1. Fix $\alpha \in (1/2, 1)$ and $\varepsilon \in (0, \alpha(1-\alpha))$. Since the process $(\theta_t)_t$ follows a Markov chain with two states and a symmetric transition probability ε, $(\mathbb{1}(\theta_1 \neq \theta_2), \mathbb{1}(\theta_2 \neq \theta_3), \ldots)$ is a sequence of i.i.d. random variables. It follows from the strong law of large numbers and that

$$\lim_{n\to\infty} \mathcal{Q}_\theta(n) := \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^n \mathbb{1}(\theta_t \neq \theta_{t+1}) = \mathbb{P}[\theta_t \neq \theta_{t+1}] = \varepsilon \quad \text{a.s.}$$

Let $\Phi = (\mathcal{F}_1, \mathcal{F}_2, \ldots)$ be the filtration where each $\mathcal{F}_i = \sigma(\mathcal{D}_1, \ldots, \mathcal{D}_i)$ and thus $\mathcal{F}_j \subseteq \mathcal{F}_i$ for any $j \leq i$. Hence, the sequence of random variables $(\mathcal{D}_1, \mathcal{D}_2, \ldots)$ is adapted to Φ so that each $\mathcal{D}_i$ is $\mathcal{F}_i$-measurable. By Proposition 1, there exists $\delta = 1/\varepsilon - M(\alpha, \varepsilon) > 0$ such that for all $i \geq 2$,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < 1/\varepsilon - \delta,$$

and thus by the law of total expectation and the Markov property of l,

$$\mathbb{E}[\mathcal{D}_i|\mathcal{F}_{i-1}] = \mathbb{E}[\mathbb{E}[\mathcal{D}_i|l_{\mathcal{T}_{i-1}}, \mathcal{F}_{i-1}]|\mathcal{F}_{i-1}] < 1/\varepsilon - \delta. \tag{13}$$

Let $X_i = \mathcal{D}_i - \mathbb{E}[\mathcal{D}_i|\mathcal{F}_{i-1}]$ for all $i \geq 2$ and denote a partial sum of the process $(X_i)_i$ by

$$Y_n = X_2 + \frac{1}{2}X_3 + \cdots + \frac{1}{n-1}X_n.$$

By the definition of X_i, $\mathbb{E}[X_i|\mathcal{F}_{i-1}] = 0$ for all $i \geq 2$. Since each Y_{n-1} is $\mathcal{F}_{n-1}$-measurable, it follows that the process $(Y_n)_n$ forms a martingale:

$$\mathbb{E}[Y_n|\mathcal{F}_{n-1}] = \mathbb{E}[\sum_{i=2}^{n} \frac{1}{i-1}X_i|\mathcal{F}_{n-1}] = Y_{n-1} + \frac{1}{n-1}\mathbb{E}[X_n|\mathcal{F}_{n-1}] = Y_{n-1}.$$

Note that $\mathbb{E}[X_i^2]$ is uniformly bounded for all $i \geq 2$ since both $\mathbb{E}[\mathcal{D}_i^2]$ and $\mathbb{E}[\mathcal{D}_i|\mathcal{F}_{i-1}]$ are uniformly bounded by Lemma 2 and (13). Hence, $\mathbb{E}[Y_n^2] = \sum_{i=2}^n \frac{1}{(i-1)^2}\mathbb{E}[X_i^2] < \infty$ for all n. By the martingale convergence theorem, Y_n converges a.s. It then follows from Kronecker's lemma that

$$\lim_{n\to\infty} \frac{1}{n-1}(X_2 + \cdots X_n) = 0 \quad \text{a.s.}^{13}$$

By the definition of X_i, we ca write

$$\lim_{n\to\infty} \frac{1}{n-1}\sum_{i=2}^{n} \mathcal{D}_i = \lim_{n\to\infty} \frac{1}{n-1}\sum_{i=2}^{n} \mathbb{E}[\mathcal{D}_i|\mathcal{F}_{i-1}] \quad \text{a.s.}$$

It then follows from (13) that

$$\lim_{n\to\infty} \frac{1}{n-1}\sum_{i=2}^{n} \mathcal{D}_i \leq 1/\varepsilon - \delta < 1/\varepsilon \quad \text{a.s.} \tag{14}$$

Since $a_t = \text{sign}(l_{t+1})$ for all $t \geq 2$, $\mathbb{1}(a_t \neq a_{t+1}) = \mathbb{1}(\text{sign}(l_{t+1}) \neq \text{sign}(l_{t+2}))$. Hence,

$$\begin{aligned}\lim_{n\to\infty} \mathcal{Q}_a(n) &:= \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \mathbb{1}(a_t \neq a_{t+1}) \\ &= \lim_{n\to\infty} \frac{1}{n-1}\sum_{t=2}^{n} \mathbb{1}(\text{sign}(l_{t+1}) \neq \text{sign}(l_{t+2}))\end{aligned}$$

It follows from the definition of $\mathcal{T}_i$ and $\mathcal{D}_i$ that

$$\lim_{n\to\infty} \mathcal{Q}_a(n) = \lim_{n\to\infty} \frac{n-1}{\mathcal{T}_n - \mathcal{T}_1} = \lim_{n\to\infty} \frac{n-1}{\sum_{i=2}^{n} \mathcal{D}_i}.$$

Finally, by (14), we conclude that

$$\lim_{n\to\infty} \mathcal{Q}_a(n) > \varepsilon = \lim_{n\to\infty} \mathcal{Q}_\theta(n) \quad \text{a.s.}$$

□

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