LEARNING ABOUT INFORMATIVENESS

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ABSTRACT. We study whether individuals can learn the informativeness of their information technology through social learning. As in the classic sequential social learning model, rational agents arrive in order and make decisions based on the past actions of others and their private signals. There is uncertainty regarding the informativeness of the common signal-generating process. We show that learning in this setting is not guaranteed, and depends crucially on the relative tail distributions of private beliefs induced by uninformative and informative signals. We identify the phenomenon of perpetual disagreement as the cause of learning and provide a characterization of learning in the canonical Gaussian environment.

1. INTRODUCTION

Social learning plays a vital role in the dissemination and aggregation of information. The behavior of others reflects their private knowledge about an unknown state of the world, and so by observing others, individuals can acquire additional information, enabling them to make better-informed decisions. A key assumption in most existing social learning models is the presence of an informative source that provides a useful private signal to each individual. In this paper, we explore how the possibility that the source is uninformative interferes with learning, and study the conditions under which individuals can eventually distinguish an uninformative source from an informative one. This question is particularly relevant today due to the proliferation of novel information technologies, raising concerns about the accuracy and credibility of the information they provide.[1]

Formally, we incorporate the uncertainty regarding the informativeness of the source into the classic sequential social learning model (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992). A sequence of short-lived agents arrives in order, each acting once by choosing an action to match with an unknown payoff-relevant state that can be

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[1]For example, the recent surge in the popularity of ChatGPT, a generative AI language model, has led to its widespread usage by a wide range of individuals, including laypeople, artists, and college students. Despite the model's disclaimer stating that "ChatGPT may produce inaccurate information about people, places, or facts," its adoption continues to grow.

either good or bad. Before choosing an action, each agent observes the past actions of her predecessors and receives a private signal from a common source of information. We assume that this source can be either informative, generating private signals that are independent and identically distributed (i.i.d.) conditioned on the payoff-relevant state, or uninformative, producing private signals that are i.i.d. but independent of the payoff-relevant state. Both the informativeness of the source and the payoff-relevant state are realized at the beginning of time and are assumed to be fixed throughout.

If an outside observer, who aims to evaluate the informativeness of the source, were to have access to the private signals received by the agents, he would gradually accumulate empirical evidence about the source, and thus eventually learn its informativeness. However, when only the history of past actions is observable, his inference problem becomes more challenging—not only because there is less information available, but also because these past actions are correlated with each other. This correlation arises from the fact that agents base their decisions on the inferences they draw from others' actions. We say that *asymptotic learning* holds if the belief of the outside observer about the source's informativeness converges to the truth, i.e., it converges almost surely to one when the source is informative and to zero when it is uninformative. The questions we aim to address are: Can asymptotic learning be achieved and if so, under what conditions? Furthermore, what are the behavioral implications of asymptotic learning?

We focus on *unbounded signals* (Smith and Sørensen, 2000) where given an informative source, the agent's private belief induced by a signal can be arbitrarily strong, since otherwise learning can be easily precluded by agents' lack of responses to their private signals.[2] Our main result (Theorem 1) shows that even with unbounded signals, achieving asymptotic learning is far from guaranteed. In fact, the determining factor of asymptotic learning lies in the tail distributions of the private beliefs of the agents. In particular, it depends on whether the distribution induced by uninformative signals has *fatter* or *thinner* tails compared to that induced by informative signals. That is, we show that asymptotic learning holds when uninformative signals have fatter tails than informative signals, but fails when uninformative signals have thinner tails.

For example, consider an informative source that generates Gaussian signals with unit variance and mean $+1$ if the payoff-relevant state is good and mean -1 if the state is bad. If the uninformative source also generates Gaussian signals but with a variance strictly greater than one, then the uninformative signals have fatter tails. Consequently, by Theorem 1, asymptotic learning holds. In contrast, uninformative Gaussian signals with a variance strictly less than one exhibit thinner tails, and so Theorem 1 implies

[2]The phenomenon that agents follow the action of their predecessors irrespective of their private signals is known as information cascades. Once cascades are triggered, agents' actions no longer reveal any private information, and so information stops accumulating. As shown in Bikhchandani, Hirshleifer, and Welch (1992), information cascades occur almost surely when signals are bounded and the set of possible signal values is finite.

that asymptotic learning fails. Indeed, for Gaussian signals, their variances alone determine the relative tail thickness between uninformative and informative signals, so one can directly apply our main result (Corollary 1). However, there are cases where the uninformative signals exhibit neither fatter nor thinner tails, e.g., when the uninformative signals have a unit variance and a mean between -1 and $+1$. We complement our main result by showing that in this canonical environment, asymptotic learning holds if and only if uninformative signals are symmetric around zero (Theorem 2).

As another illustration of the main result, consider the case where the informative signals have the same distributions as before, but the uninformative signals are chosen uniformly from the interval $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. In this case, the distribution of private beliefs induced by these uninformative signals also has bounded support. Consequently, it can be viewed as having *extremely* thinner tails compared to those of informative Gaussian signals, and thus Theorem 1 implies that asymptotic learning fails. Nevertheless, under such an informative source, almost all agents individually learn its informativeness: Once they receive a signal outside the support $[-\varepsilon, \varepsilon]$, which is highly probable, they can infer that it must originate from the normal distribution, indicating that the source is informative. An outside observer, however, who only observes the agents' actions would not be able to determine the informativeness of the source.

A key assumption underlying our results is the uniform prior assumption regarding the payoff-relevant state when the source is uninformative. This assumption ensures that despite information uncertainty, rational agents always act as if the signals they receive are informative (Lemma 1). This is simply because in the absence of any useful information—conditioned on an uninformative source—each agent with a uniform prior is indifferent between the available actions. We make the uniform prior assumption to capture settings in which agents are not very informed a priori, thus making private signals and their informativeness a crucial determining factor of outcomes. Indeed, in many investment settings, the efficient market hypothesis (Samuelson, 1965; Fama, 1965) implies that investors should be close to indifference.[3]

The mechanism behind our main result is as follows. Since agents always act as if signals are informative, when the source is indeed informative and generates unbounded signals, agents will eventually reach a consensus on the correct action. On the one hand, suppose that the source is uninformative and generates signals with thinner tails. In this case, it is unlikely that agents will receive signals that are extreme enough to make them change their actions, so they usually mimic their predecessors. As a result, eventually agents will also reach a consensus, thus exhibiting behavior similar to that under an informative source. This makes it impossible for an outside observer who only observes agents' actions to be certain about the source's informativeness, so asymptotic learning

[3]The efficient market hypothesis states that in the financial market, asset prices should reflect all available information. Thus, if investors are not indifferent, it suggests that they possess private information that is not yet reflected in market prices, thereby challenging the hypothesis.

fails. On the other hand, suppose that the source is uninformative but generates signals with fatter tails. In this case, extreme signals are more likely, preventing agents from reaching a consensus. In fact, they will perpetually change their actions, so no consensus is ever reached. Therefore, an outside observer who observes an infinite number of action switches learns that the source is uninformative.

For general private belief distributions where the relative tail thickness is incomparable, we show that the same holds: Asymptotic learning holds if and only if, conditioned on the source being uninformative, the agents never reach a consensus (Proposition 1). In terms of behavioral implications, when the source is informative, as previously mentioned, agents eventually converge to the correct action, regardless of whether asymptotic learning is achieved. However, we show that in this case, agents can only be sure that they are taking the correct action if and only if asymptotic learning holds (Proposition 2). In contrast, when the source is uninformative, agents are clearly not guaranteed to converge to the correct action; in fact, their actions may or may not converge at all. Proposition 1 demonstrates that an outside observer eventually learns the informativeness of the source if and only if the agents' actions do not converge.

Related Literature. Our paper contributes to a rich literature on sequential social learning. The primary focus of this literature has been on determining whether agents can eventually learn to choose the correct payoff-relevant state when provided with an informative source. Various factors, such as the information structure (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992; Smith and Sørensen, 2000) and the observational networks (Çelen and Kariv, 2004; Acemoglu, Dahleh, Lobel, and Ozdaglar, 2011; Lobel and Sadler, 2015), have been extensively studied to analyze their impacts on information aggregation, including efficiency (Rosenberg and Vieille, 2019) and the speed of learning (e.g., Vives, 1993; Hann-Caruthers, Martynov, and Tamuz, 2018). However, the question of learning about the informativeness of the source that agents use to make decisions—which is the focus of this paper—remains largely unexplored.[4]

A few papers explore the idea of agents having access to multiple sources of information in the context of social learning. For example, Chen (2022) examines a sequential social learning model in which ambiguity-averse agents have access to different information sources. Instead of having uncertainty regarding the informativeness of a common source, in his model, uncertainty arises since agents are unsure about the signal precision of their predecessors. He shows that under sufficient ambiguity aversion, there can be information cascades even with unbounded signals. In a different setting, Liang and Mu (2020) consider a model in which agents endogenously choose from a set of correlated information sources, and the acquired information is then made public. They focus on the externality in the information acquisition decisions and show that information complementarity can

[4]For comprehensive surveys on recent developments in the social learning literature, see e.g., Golub and Sadler (2017); Bikhchandani, Hirshleifer, Tamuz, and Welch (2021).

result in either efficient information aggregation or "learning traps," in which society becomes stuck in choosing suboptimal information structures.

Another way of viewing our model is by considering a social learning model with four states: The source is either informative with the good or bad state, or uninformative with either the good or bad state. In such multi-state settings, a recent work by Arieli and Mueller-Frank (2021) demonstrates that pairwise unbounded signals are necessary and sufficient for learning, when the decision problem that the agents face includes, for each state, a distinct action that is uniquely optimal in that state. This is not the case in our model, because the same action is optimal in different states, and so even when agents observe a very strong signal indicating that the state is uninformative, they do not reveal it in their behavior.

More recently, Kartik, Lee, Liu, and Rappoport (2022) consider a setting with multiple states and actions on general sequential observational networks. They identify a sufficient condition for learning —"excludability" —that jointly depends on the information structure and agents' preferences. Roughly speaking, this condition ensures that agents can always displace the wrong action, which is their driving force for learning. In our model, when the source is uninformative, agents cannot displace the wrong action as all signals are pure noise.[5] Conceptually, our approach differs from theirs as we are interested in identifying the uninformative state from the informative one, instead of identifying the payoff-relevant state.

Our paper is also related to the growing literature on social learning with misspecified models. Bohren (2016) investigates a model where agents fail to account for the correlation between actions, demonstrating that different degrees of misperception can lead to distinct learning outcomes. In a broader framework, Bohren and Hauser (2021) show that learning remains robust to minor misspecifications. In contrast, Frick, Iijima, and Ishii (2020) find that an incorrect understanding of other agents' preferences or types can result in a severe breakdown of information aggregation, even with a small amount of misperception. Later, Frick, Iijima, and Ishii (2023) propose a unified approach to establish convergence results in misspecified learning environments where the standard martingale approach fails to hold. On a more positive note, Arieli, Babichenko, Müller, Pourbabaee, and Tamuz (2023) illustrate that by being mildly condescending—misperceiving others as having slightly lower-quality of information—agents may perform better in the sense that on average, only finitely many of them take incorrect actions.

2. Leading Example: Testing for a New Information Technology

As a leading economic example, consider an external evaluator tasked with assessing the informativeness of a novel information technology, such as an AI recommendation system. This system serves as a common source of information for investors interested in

[5] This observation can also be seen from Theorem 2 in Kartik, Lee, Liu, and Rappoport (2022).

buying into a start-up company of unknown quality. For concreteness, imagine that the recommendation system is either informative, providing valuable insights into the quality of the start-up, or uninformative, offering no useful information at all. We assume that the start-up's quality is either good or bad, with the corresponding optimal decision being to buy or sell. Because past investment decisions are made public, there is a positive correlation among investors' decisions: As more investors opt to buy, their decisions make later investors more convinced that the start-up is of good quality. Consequently, they are more inclined to buy upon receiving their private recommendation. Based only on these correlated investment decisions, the evaluator is then rewarded if his assessment is correct.[6]

In this scenario, the evaluator faces a hypothesis testing problem. Given a correlated sample x, which represents the history of all past investment decisions, he forms a null hypothesis $\mathcal{H}_0$ that the recommendation system is informative and an alternative hypothesis $\mathcal{H}_1$ that the system is uninformative. To test these hypotheses, he uses a log-likelihood ratio test:

$$\Lambda(x) = \log \frac{\mathbb{P}[\mathcal{H}_0 \text{ is true} \mid x]}{\mathbb{P}[\mathcal{H}_1 \text{ is true} \mid x]},$$

where $\Lambda(x)$ is the log of the likelihood ratio of $\mathcal{H}_0$ to $\mathcal{H}_1$ conditioned on the observed sample x. He sets a threshold $c > 0$, where he accepts $\mathcal{H}_0$ if $\Lambda(x) \geq c$ and rejects otherwise.

Now, the question of whether the evaluator can eventually make the correct assessment becomes: As the size of the sample x increases, can the log-likelihood ratio test $\Lambda(x)$ converge to infinity under $\mathcal{H}_0$ and to negative infinity under $\mathcal{H}_1$? In other words, can the external evaluator eventually make the correct assessment, regardless of the decision threshold used?[7] Our main result (Theorem 1) provides answers to these questions by characterizing the conditions under which it is possible.

2.1. **Why Relative Tail Thickness Matters for Learning.** Why do uninformative signals with fatter tails induce learning, whereas uninformative signals with thinner tails do not? Intuitively, the condition of having fatter tails means that it is more likely to observe a very extreme signal—say a $5 - \sigma$ signal—under an uninformative source than an informative one. As a consequence, the presence of extreme signals suggests that the source is uninformative. However, since an outside observer does not directly observe the private signals received by the agents, he can only learn by observing action switches

[6]More broadly, one can view the information source as a scientific paradigm—a set of principles that guide a specific scientific discipline. In this context, the question of whether an external evaluator can learn about the informativeness of this new technology has a similar flavor to Kuhn's question of whether society can make the right scientific paradigm shifts (Kuhn, 1962). One classic example of a paradigm shift in geology is the acceptance of the theory of plate tectonics, which only occurred in the 20th century, despite the idea of a drifted continent being proposed earlier by Abraham Ortelius in his 1596 book "Thesaurus Geographicus" (in English, "A New Body of Geography").

[7]A positive answer to this question means that the external evaluator can achieve both zero type I error (false positive) and zero type II error (false negative) asymptotically for any chosen decision rule.

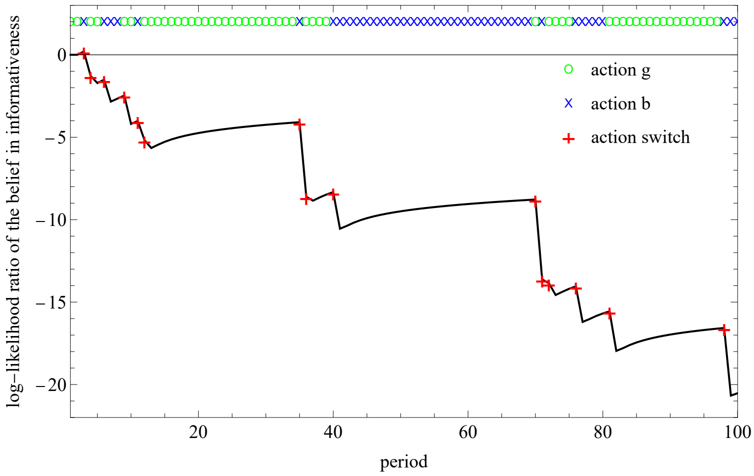

FIGURE 2.1. This simulation tracks an outside observer's belief in the informativeness of the source over 100 periods. It is generated under uninformative signals with a normal distribution with mean 0 and variance 2. The informative signals are normally distributed with a unit variace and mean -1 and $+1$, respectively. Red crosses mark the periods when action switches occur.

triggered by these extreme signals. For example, when an agent deviates from a prolonged sequence of identical actions, the observer can infer that this agent must have received an extreme signal. Since extreme signals are more likely to occur under uninformative signals with fatter tails, such a deviation makes the observer more convinced that the source is uninformative. In contrast, when the uninformative source has thinner tails, it tends to produce more moderate signals. Hence, in this case, the occurrence of extreme signals and the resulting action switches no longer suggest an uninformative source.

We illustrate this intuition in Figure 2.1. It depicts a simulation of an outside observer's belief about the source being informative (in its log-likelihood ratio) under uninformative signals with fatter tails. First, observe that every action switch, following an extended period of identical actions, leads to a significant decrease in the outside observer's belief that the source is informative. In other words, as discussed before, observing these unusual action switches makes the observer less convinced that the source is informative. Second, even though his belief gradually increases in the absence of action switches, it eventually converges to zero because it is generated under fatter-tailed uninformative signals, as suggested by Theorem 1. In contrast, if his belief had been generated under uninformative signals with thinner tails, extreme signals would have been less likely. Consequently, the action switches would eventually stop so that his belief would no longer converge to zero.

3. Model

3.1. **Setup.** There is an unknown binary state of the world $\theta \in \{\mathfrak{g}, \mathfrak{b}\}$, chosen at time 0 with equal probability. We refer to $\mathfrak{g}$ as the good state and $\mathfrak{b}$ as the bad state. A countably infinite set of agents indexed by time $t \in \mathbb{N} = \{1, 2, \ldots\}$ arrive in order, each acting once. The action of agent t is $a_t \in A = \{\mathfrak{g}, \mathfrak{b}\}$, with a payoff of one if her action matches the state θ and zero otherwise. Before agent t chooses an action, she observes the history of actions made by her predecessors $H_t = (a_1, \ldots, a_{t-1})$ and receives a private signal s_t, taking values in a measurable space (S, Σ).

A pure strategy of agent t is a measurable function $\sigma_t : A^{t-1} \times S \to A$ that selects an action for each possible pair of observed history and private signal. A pure strategy profile $\sigma = (\sigma_t)_{t \in \mathbb{N}}$ is a collection of pure strategies of all agents. A strategy profile is a Bayesian Nash equilibrium—referred to as equilibrium hereafter—if no agent can unilaterally deviate from this profile and obtain a strictly higher expected utility conditioned on their information. Given that each agent acts only once, the existence of an equilibrium is guaranteed by a simple inductive argument. In equilibrium, each agent t chooses the action a_t that maximizes her expected payoff conditional on the information available to her:

$$a_t \in \arg\max_{a \in A} \mathbb{E}[\mathbb{1}(\theta = a) | H_t, s_t].$$

Below, we make a continuity assumption which implies that agents are never indifferent, and so there is a unique equilibrium.

3.2. **The Informativeness of the Source.** So far, the above setup is the canonical setting of the sequential social learning model (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992; Smith and Sørensen, 2000). Our model builds on this classic setting and introduces another dimension of uncertainty regarding the informativeness of a common source. Specifically, at time 0, independent of θ, nature chooses an additional binary state $\omega \in \{0, 1\}$ with equal probability.[8] When $\omega = 1$, the source is informative and sends i.i.d. signals across agents conditional on the payoff-relevant state θ, with distribution μ_θ. When $\omega = 0$, the source is uninformative and still sends i.i.d. signals across agents, but independently of the state θ, with distribution μ_0. The realization of ω determines the signal-generating process for all agents. Throughout, we denote by $\mathbb{P}_0[\cdot] := \mathbb{P}[\cdot \mid \omega = 0]$ and $\mathbb{P}_1[\cdot] := \mathbb{P}[\cdot \mid \omega = 1]$ the conditional probability distributions given $\omega = 0$ and $\omega = 1$, respectively. Similarly, we use the notation $\mathbb{P}_{1,\mathfrak{g}}[\cdot] := \mathbb{P}[\cdot \mid \omega = 1, \theta = \mathfrak{g}]$ to denote the conditional probability distribution given $\omega = 1$ and $\theta = \mathfrak{g}$. We use an analogous notation for $\omega = 1$ and $\theta = \mathfrak{b}$.

We first observe that, despite the uncertainty regarding the source's informativeness, in equilibrium, each agent chooses the action that is most likely to match the state, *conditional* on the source being informative.

[8]Our results do not depend on the independence assumption between θ and ω. They hold true as long as conditioned on $\omega = 0$, both realizations of θ are equally likely.

Lemma 1. *The equilibrium action for each agent t is*

$$a_t \in \arg\max_{a \in A} \mathbb{P}_1[\theta = a | H_t, s_t]. \tag{1}$$

In other words, agents always act as if signals are informative, irrespective of the underlying signal-generating process. This is because treating signals as informative—even when they are pure noise—does not adversely affect agents' payoffs since in the absence of any useful information, each agent is indifferent between the available actions given the uniform prior assumption.

3.3. **Information Structure.** The distributions $\mu_{\mathfrak{g}}, \mu_{\mathfrak{b}}$, and μ_0 are distinct and mutually absolutely continuous, so no signal fully reveals either state θ or ω. As a consequence, conditioned on $\omega = 1$, the log-likelihood ratio of any signal

$$\ell_t = \log \frac{d\mu_{\mathfrak{g}}}{d\mu_{\mathfrak{b}}}(s_t),$$

is well-defined, and we call it the *private* log-likelihood ratio of the agents. Since agents always act as if the source is informative (Lemma 1), it captures how they update their private beliefs regarding the relative likelihood of the good state over the bad state upon receiving their signals. Therefore, regardless of the realization of ω, it is sufficient to consider ℓ_t in order to capture agents' behavior.[9] We denote by $F_{\mathfrak{g}}$ and $F_{\mathfrak{b}}$ the cumulative distribution functions of ℓ_t conditioned on the event that $\omega = 1$ and $\theta = \mathfrak{g}$ and the event that $\omega = 1$ and $\theta = \mathfrak{b}$, respectively. We denote by F_0 the cumulative distribution function of ℓ_t conditioned on $\omega = 0$. All distributions $F_{\mathfrak{g}}$, $F_{\mathfrak{b}}$, and F_0 are mutually absolutely continuous, as $\mu_{\mathfrak{g}}, \mu_{\mathfrak{b}}$ and μ_0 are. Let $f_{\mathfrak{g}}, f_{\mathfrak{b}}$ and f_0 denote the corresponding density functions of $F_{\mathfrak{g}}$, $F_{\mathfrak{b}}$ and F_0 whenever they are differentiable.

We focus on *unbounded signals* in the sense that the agent's private log-likelihood ratio can take on arbitrarily large or small values, i.e., for any $M \in \mathbb{R}$, there exists a positive probability that $\ell_t > M$ and a positive probability that $\ell_t < -M$. We informally refer to a signal s_t as *extreme* when the corresponding ℓ_t it induces has a large absolute value. A common example of unbounded private signals is the case of Gaussian signals, where s_t follows a normal distribution $\mathcal{N}(m_{(\omega,\theta)}, \sigma^2)$ with variance σ^2 and mean $m_{(\omega,\theta)}$ that depends on the pair of states (ω, θ). An extreme signal for the Gaussian case is a signal that is, for example, 5-σ away from the mean $m_{(\omega,\theta)}$.

We make two assumptions for expository simplicity. First, we assume that the pair of informative conditional CDFs $(F_{\mathfrak{g}}, F_{\mathfrak{b}})$ is symmetric around zero, i.e., $F_{\mathfrak{g}}(x) + F_{\mathfrak{b}}(-x) = 1$. This implies that our model is invariant with respect to a relabeling of the payoff-relevant state. Second, we assume that all CDFs $F_{\mathfrak{g}}$, $F_{\mathfrak{b}}$ and F_0 are continuous, so agents are never indifferent between actions.

In addition, we assume that $F_{\mathfrak{b}}$ has a differentiable left tail, i.e., is differentiable for all x negative enough and its probability density function $f_{\mathfrak{b}}$ satisfies the condition that

[9] Formally, the sequence of actions $a_1, \ldots, a_t$ is determined by $\ell_1, \ldots, \ell_t$.

$f_{\mathfrak{b}}(-x) < 1$ for all x large enough. By the symmetry assumption, this implies that $F_{\mathfrak{g}}$ also has a differentiable right tail and its density function $f_{\mathfrak{g}}$ satisfies the condition that $f_{\mathfrak{g}}(x) < 1$ for all x large enough. This is a mild technical assumption that holds for every non-atomic distribution commonly used in the literature, including the Gaussian distribution. It holds, for instance, whenever the density tends to zero at infinity.

3.4. **Asymptotic Learning.** Denote by $q_t := \mathbb{P}[\omega = 1|H_t]$ the *public belief* at time t that the source is informative given the history of actions. This is the belief that an outside observer assigned to an informative source after observing the actions of agent 1 to $t-1$. As this observer collects more information over time, his belief q_t converges almost surely since it is a bounded martingale. To ensure that he eventually learns the truth regarding the informativeness of the source, we introduce the following notion of asymptotic learning.

Definition 1. *Asymptotic learning* holds if for all $\omega \in \{0, 1\}$,

$$\lim_{t\to\infty} q_t = \omega \quad \mathbb{P}_\omega\text{-almost surely.}$$

That is, conditioned on an informative source, the public belief that the source is informative converges to one almost surely. Meanwhile, conditioned on an uninformative source, the public belief that the source is informative converges to zero almost surely. As we explain below in Section 5.1, asymptotic learning is always attainable, regardless of the underlying information structure, when all signals are publicly observable.

Another common notion of learning, concerning the convergence of actions, is called *correct herding.* That is, all, but finitely many agents take the correct action.

Definition 2. *Correct herding* occurs if $\lim_{t\to\infty} a_t = \theta$ almost surely.

In the standard model where the source is always informative, Smith and Sørensen (2000) show that correct herding occurs if and only if signals are unbounded. In our model, since agents always act as if signals are informative (Lemma 1), when the source is indeed informative and generates unbounded signals, correct herding still occurs:

$$\lim_{t\to\infty} a_t = \theta, \quad \mathbb{P}_1\text{-almost surely.}$$

That is, conditioned on an informative source, agents eventually converge to the correct action regardless of whether asymptotic learning is achieved. However, as we discuss later in Section 6, even though agents eventually herd on the correct action, they cannot be completely sure that it is the correct action unless asymptotic learning holds, and vice versa (Proposition 2). In contrast, when the source is uninformative, correct herding is clearly not guaranteed; in fact, the agents' actions may or may not converge at all. As we explain in detail later (see Section 5.2.1), it turns out that an important characterization of asymptotic learning is the non-convergence of actions in the uninformative case (Proposition 1).

4. Relative Tail Thickness

To study the conditions for asymptotic learning, it is crucial to understand the concept of relative tail thickness, i.e., the comparison between the tail distributions of agents' private log-likelihood ratios induced by different signals. This comparison is important because it captures the relative likelihood of generating extreme signals from different sources. Formally, for a fixed pair of CDFs (F_0, F_θ) where $\theta \in \{\mathfrak{g}, \mathfrak{b}\}$ and some $x \in \mathbb{R}_+$, we denote by

$$L_\theta(x) := \frac{F_0(-x)}{F_\theta(-x)} \quad \text{and} \quad R_\theta(x) := \frac{1 - F_0(x)}{1 - F_\theta(x)}$$

the left and right tail ratio of F_0 over F_θ evaluated at x, respectively. The following definition of fatter and thinner tails captures whether extreme signals are more or less likely to occur under an uninformative source versus an informative one.

Definition 3. (i) The uninformative signals have *fatter tails* than the informative signals if there exists an $\varepsilon > 0$ such that for all x large enough,

$$L_{\mathfrak{b}}(x) \geq \varepsilon \text{ and } R_{\mathfrak{g}}(x) \geq \varepsilon.$$

(ii) The uninformative signals have *thinner tails* than the informative signals if there exists an $\varepsilon > 0$ such that either

$$L_{\mathfrak{g}}(x) \leq 1/\varepsilon \quad \text{for all } x \text{ large enough,}$$

or

$$R_{\mathfrak{b}}(x) \leq 1/\varepsilon \quad \text{for all } x \text{ large enough.}$$

That is, for uninformative signals to have fatter tails, both their corresponding left and right tail ratios are eventually bounded from below. Meanwhile, for uninformative signals to have thinner tails, at least one of their corresponding left and right tail ratios is eventually bounded from above.[10,11]

Note that the first condition $L_{\mathfrak{b}}(x) \geq \varepsilon$ implies that $L_{\mathfrak{g}}(x) \geq \varepsilon$. This follows from the well-known fact that $F_{\mathfrak{g}}$ exhibits first-order stochastic dominance over $F_{\mathfrak{b}}$, i.e., $F_{\mathfrak{g}}(x) \leq F_{\mathfrak{b}}(x)$ for all $x \in \mathbb{R}$ (see, e.g., Smith and Sørensen, 2000; Chamley, 2004; Rosenberg and Vieille, 2019). Similar statements apply to the remaining three conditions. Furthermore, notice that uninformative signals cannot have fatter and thinner tails simultaneously, as $F_{\mathfrak{g}}$ and $F_{\mathfrak{b}}$ represent distributions of the private log-likelihood ratio.[12] However, there are distributions under which uninformative signals have neither fatter nor thinner tails. In

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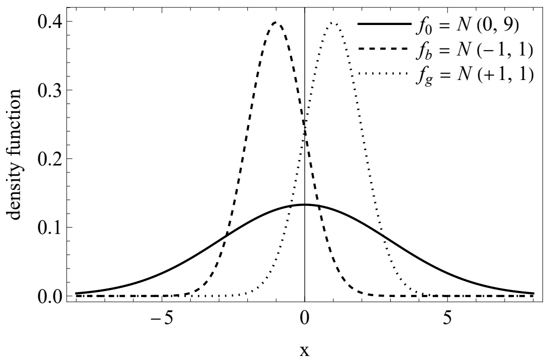
[10] For differentiable distributions, one can equivalently define the relative tail thickness in terms of the corresponding density ratios, if their limits exist.

[11] In the statistics literature, the concept of relative tail thickness has also been explored. Our definition of thinner tails is closest to that of Rojo (1992), where a CDF F is considered not more heavily tailed than G if $\limsup_{x\to\infty}(1 - F(x))/(1 - G(x)) < \infty$. Other notions of relative tail thickness represented in terms of density quantile functions can be found in Parzen (1979) and Lehmann (1988).

[12] In particular, $F_{\mathfrak{g}}$ and $F_{\mathfrak{b}}$ satisfy the following inequality: $\mathrm{e}^x F_{\mathfrak{g}}(-x) \leq F_{\mathfrak{b}}(-x)$, for all $x \in \mathbb{R}$.



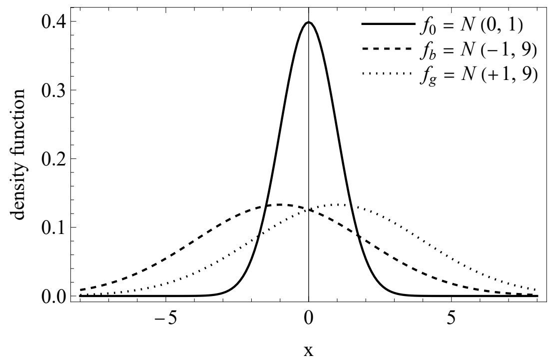


FIGURE 4.1. An example of uninformative Gaussian signals with fatter tails (on the left) and an example of uninformative Gaussian signals with thinner tails (on the right).

Section 5.3, we show that in the canonical case of Gaussian signals where the relative tail thickness is incomparable, asymptotic learning occurs only in a knife-edge case where the uninformative signals are also symmetric around zero (Theorem 2).

Intuitively, uninformative signals with fatter tails are more likely to exhibit extreme values compared to informative signals. By Bayes' Theorem, the observation of an extreme signal under fatter-tailed uninformative signals suggests that the source is uninformative. On the contrary, uninformative signals with thinner tails tend to exhibit moderate values, and thus observing an extreme signal in this case no longer suggests that the source is uninformative. Next, we provide three examples of uninformative signals with either fatter or thinner tails.

Example 1 (Gaussian Signals)**.** Consider the case where $F_{\mathfrak{g}}$ is normal with mean $+1$ and unit standard deviation and $F_{\mathfrak{b}}$ is also normal with mean -1 and unit standard deviation.

Suppose that F_0 has zero mean. If it has a standard deviation of 17, then the uninformative signals have fatter tails. In this case, if an extreme signal, such as anything above 11, is observed, it is much more likely that the source is uninformative than that an informative 10-σ signal was generated under $F_{\mathfrak{g}}$. On the other hand, if the standard deviation of F_0 is 1/17, then the uninformative signals have thinner tails, and thus an extreme signal becomes an indication that the source is informative. A graphical example of uninformative Gaussian signals with fatter and thinner tails is depicted in Figure 4.1.

Example 2 (First-Order Stochastically Dominated (FOSD) Signals)**.** Suppose that F_0 first-order stochastically dominates $F_{\mathfrak{g}}$ (or $F_{\mathfrak{b}}$ first-order stochastically dominates F_0). In this case, since the uninformative signals are much more likely to exhibit high values, F_0 has a thinner left tail than $F_{\mathfrak{g}}$, and thus by Definition 3, the uninformative signals following F_0 have thinner tails.

Now, suppose that an extreme signal of a positive value is observed. Then, it is highly unlikely that the source is informative and associated with the bad state. Instead, the source is more likely to be either uninformative or informative but associated with the good state. Therefore, while observing an extremely positive signal suggests that the state is good, it does not provide evidence for the source's informativeness, as it is likely to occur under both F_0 and $F_{\mathfrak{g}}$.

Example 3 (Mixture Signals)**.** For any fixed pair of distributions $(F_{\mathfrak{g}}, F_{\mathfrak{b}})$, let $F_0 = \alpha F_{\mathfrak{g}} + (1-\alpha) F_{\mathfrak{b}}$ for any $\alpha \in (0,1)$. Observe that in this case the uninformative signals have fatter tails than the informative signals.[13] In particular, when $\alpha = 1/2$, the corresponding mixture distribution F_0 coincides with the unconditional distribution of ℓ_t, assuming that the source is informative. In other words, a priori, this uninformative source is indistinguishable from the informative one.

Alternatively, the mixture distribution F_0 can be viewed as generating conditional i.i.d. signals, but instead of conditioning on the payoff-relevant state θ, they are generated conditioned on a different state η: In each period, the state η is randomly drawn from the same set $\{\mathfrak{g}, \mathfrak{b}\}$, independent of θ. With probability α, the event $\eta = \mathfrak{g}$ occurs and the source draws a signal from the distribution $F_{\mathfrak{g}}$. With the complementary probability, the event $\eta = \mathfrak{b}$ occurs, and the source draws a signal from $F_{\mathfrak{b}}$.

5. Main Results

5.1. **A Benchmark.** As a benchmark, we briefly discuss the case where all signals are observed by the outside observer.[14] These signals are distributed according to either $\mu_{\mathfrak{g}}$, $\mu_{\mathfrak{b}}$ or μ_0. Since these measures are distinct, as the sample size grows, this observer eventually learns which distribution is being sampled. Formally, at any time t, the empirical distribution of the signals, $\hat{\mu}_t(B)$ which assigns to a measurable set $B \subseteq S$

$$\hat{\mu}_t(B) = \frac{1}{t} \sum_{\tau=1}^{t} \mathbb{1}(s_\tau \in B).$$

Conditional on both states ω and θ, this is the empirical mean of i.i.d. Bernoulli random variables. Hence, by the strong law of large numbers, for every measurable set $B \subseteq S$,

$$\lim_{t\to\infty} \hat{\mu}_t(B) = \mu_{(\omega,\theta)}(B) \quad \text{almost surely,}$$

where $\mu_{(1,\mathfrak{g})} = \mu_{\mathfrak{g}}$, $\mu_{(1,\mathfrak{b})} = \mu_{\mathfrak{b}}$ and $\mu_{(\cdot,0)} = \mu_0$.

Thus, regardless of the underlying signal-generating process, any uncertainty concerning the informativeness of the source can eventually be resolved if all signals are public. However, if signals are private and only actions are public, then an outside observer can only rely on agents' actions to infer the type of private signals they receive. Next, we

[13]To see this, let $F_0 = \alpha F_{\mathfrak{g}} + (1-\alpha) F_{\mathfrak{b}}$ for some constant $\alpha \in (0,1)$. Since CDFs always take nonnegative values, for any $x \in \mathbb{R}$, $F_0(x) \geq (1-\alpha) F_{\mathfrak{b}}(x)$. Similarly, $1 - F_0(x) = \alpha(1 - F_{\mathfrak{g}}(x)) + (1-\alpha)(1 - F_{\mathfrak{b}}(x)) \geq \alpha(1 - F_{\mathfrak{g}}(x))$. Let $\varepsilon = \min\{\alpha, 1-\alpha\}$. Thus by definition, F_0 has fatter tails.

[14]Equivalently, one can let the outside observer observe all agents' actions in addition to their signals. Since actions contain no additional payoff relevant information, it is equivalent to observing only the signals.

turn to this setting and study the conditions under which the observer can eventually learn the truth regarding the source's informativeness.

5.2. **Public Actions.** We now present our main result (Theorem 1). In contrast to the public signal benchmark, our main result shows that the achievement of asymptotic learning is no longer guaranteed. In fact, the key determinant of asymptotic learning is the relative tail thickness between the uninformative and informative signals, as introduced in Definition 3.

Theorem 1. *When the uninformative signals have fatter tails than the informative signals, asymptotic learning holds. When the uninformative signals have thinner tails than the informative signals, asymptotic learning fails.*

For normal distributions, the relative thickness of the tails is determined solely by their variances: a higher variance corresponds to fatter tails, while a lower variance corresponds to thinner tails (see Lemma 8 in the Appendix). Thus, an immediate consequence of Theorem 1 is the following result.

Corollary 1. *Suppose all private signals are normal where the informative signals have variance σ^2, and the uninformative signals have variance τ^2. Then, asymptotic learning holds if $\tau > \sigma$ and fails if $\tau < \sigma$.*

Theorem 1 demonstrates that an outside observer can eventually learn to distinguish an uninformative source from an informative one if the former generates signals that are more widely dispersed than the latter. In contrast, when the source generates uninformative signals that are relatively concentrated compared to the informative signals, the observer can no longer differentiate between them.

To understand why relative tail thickness is crucial for achieving asymptotic learning, first consider the case where the source is informative. In this case, the probability of generating extreme signals that could overturn a long sequence of identical actions decreases rapidly. Since agents always treat signals as if they are informative (Lemma 1), they would eventually reach a consensus. Now, suppose that conditioned on an uninformative source, both actions are taken infinitely often, and so agents never reach a consensus. If that were the case, there would be distinct behavioral patterns under different sources, thereby allowing an outside observer to distinguish between them. Whether these disagreements can persist or not depends on whether the tails of the uninformative signals are thick enough to generate these extreme signals. Hence, the achievement of asymptotic learning depends on the relative tail thickness between uninformative and informative signals.

5.2.1. *Perpetual Disagreement.* The key mechanism driving our main result is the idea of *perpetual disagreement*—the event in which agents never converge to any action. We show that, in fact, achieving asymptotic learning is equivalent to having perpetual disagreement under an uninformative source. Let $S := \sum_{t=1}^{\infty} \mathbb{1}(a_t \neq a_{t+1})$ denote the total number of action switches, and so the event $\{S = \infty\}$ is the perpetual disagreement event.

Proposition 1. *Asymptotic learning holds if and only if conditioned on $\omega = 0$, the event $\{S = \infty\}$ occurs almost surely.*

This proposition characterizes asymptotic learning in terms of agents' behavior conditioned on the source being uninformative. Intuitively, since agents eventually reach a consensus under an informative source, if an outside observer never observes action convergence under an uninformative source, he can infer that the source must be uninformative. Conversely, under an uninformative source, if agents also eventually reach a consensus, the observer can no longer be sure that the source is uninformative, as such action convergence is plausible in both cases.

Building on the connection between perpetual disagreement and asymptotic learning, the proof idea behind Theorem 1 is as follows. Suppose that the source is uninformative and generates signals with fatter tails. This means that the probability of agents receiving extreme signals that would overturn a long sequence of identical actions decreases relatively slowly. As a result, the agents would never settle on any action, resulting in perpetual disagreement. Hence, in the presence of fatter-tailed uninformative signals, it follows from Proposition 1 that asymptotic learning holds. In contrast, if uninformative signals have thinner tails, the probability of agents receiving these extreme, overturning signals declines rapidly enough so that agents would eventually settle on some action. As a consequence, under thinner-tailed uninformative signals, perpetual disagreement is not guaranteed, and thus asymptotic learning fails.

5.3. **Gaussian Private Signals.** While Theorem 1 provides valuable insights into the role of relative tail thickness in determining asymptotic learning, there are situations where the uninformative signals have neither fatter nor thinner tails compared to the informative signals, rendering Theorem 1 inapplicable. For example, consider a scenario where F_0, $F_\mathfrak{g}$, and $F_\mathfrak{b}$ are normal distributions with the same variance and mean 0, 1, and -1, respectively. As x approaches infinity, both $F_0(-x)$ and $F_\mathfrak{b}(-x)$ approach zero, but the former goes to zero much faster than the latter, leading $L_\mathfrak{b}(x)$ converging to zero. As a result, F_0 does not have fatter tails. Similarly, both $L_\mathfrak{g}(x)$ and $R_\mathfrak{b}(x)$ tend to infinity as x approaches infinity, and so F_0 does not have thinner tails either.

To complement the findings of Theorem 1, we focus on the canonical Gaussian environment where all signals are normal and share the same variance σ^2. The informative signals are symmetric with mean $+1$ and -1, respectively, while the uninformative signals have mean $m_0 \in (-1, 1)$. In this setting, a simple calculation shows that the agent's private log-likelihood ratio is directly proportional to the private signal: $\ell_t = 2s_t/\sigma^2$. As a consequence, all distributions $F_\mathfrak{g}$, $F_\mathfrak{b}$, and F_0 are also Gaussian. In the following result, we show that asymptotic learning is achieved if and only if the uninformative signals are symmetric around zero.

Theorem 2. *Suppose all private signals are Gaussian with the same variance, where informative signals have means* -1 *and* $+1$*, and uninformative signals have mean* $m_0 \in (-1, 1)$*. Then, asymptotic learning holds if and only if* $m_0 = 0$*.*

Together with Corollary 1, Theorem 2 provides a complete characterization of asymptotic learning in a Gaussian environment with symmetric informative signals. In particular, it shows that even when uninformative Gaussian signals share the same variance as informative ones, and thus have neither fatter nor thinner tails, asymptotic learning still holds as long as the uninformative distribution F_0 is symmetric around zero. Intuitively, any mean shift of F_0 away from zero would move F_0 closer to either $F_{\mathfrak{g}}$ or $F_{\mathfrak{b}}$, thus making the uninformative signals more similar to the corresponding informative signals. Consequently, it becomes impossible to fully differentiate between them. Recall that in Example 2, we have observed the extreme case where the distribution F_0 shifts completely to the right of $F_{\mathfrak{g}}$. In this case, the resulting uninformative signals clearly have thinner tails, and so asymptotic learning fails.

Note that this type of learning failure is also related to the problem of non-falsifiability in hypothesis testing.[15] For example, recall the hypothesis testing problem faced by the external evaluator in Section 2, where $\mathcal{H}_0 : \omega = 1$ and $\mathcal{H}_1 : \omega = 0$. Now suppose that F_0 first-order stochastically dominates $F_{\mathfrak{g}}$ and that the realization of the payoff-relevant state is $\mathfrak{g}$. Then, based on the actions of the agents, the alternative hypothesis that the source is uninformative is non-falsifiable because agents would eventually converge to the correct action $\mathfrak{g}$, regardless of whether the null or the alternative hypothesis is true.[16] In other words, if the uninformative source is biased toward the true payoff-relevant state, it becomes impossible to recognize its lack of informativeness, so asymptotic learning fails.

5.4. **Numerical Simulation.** To further illustrate our main result, we use Monte Carlo simulations to numerically simulate an outside observer's belief process and the corresponding action switches among agents. For a fixed pair of informative Gaussian signal distributions $(\mu_{\mathfrak{g}}, \mu_{\mathfrak{b}})$ where $\mu_{\mathfrak{g}} = \mathcal{N}(+1, 2)$ and $\mu_{\mathfrak{b}} = \mathcal{N}(-1, 2)$, we simulate these processes under uninformative Gaussian signals with fatter tails where $\mu_0 = \mathcal{N}(0, 3)$ and with thinner tails where $\mu_0 = \mathcal{N}(0, 1)$.[17] We conduct these simulations 1,000 times and

[15] The idea of falsifiability, introduced by Popper (1959), highlights the importance of being able to test and potentially refute hypotheses. His famous example that "all swans are white" serves as a null hypothesis that can be easily falsified by observing a single black swan. However, the alternative hypothesis that "there exists a non-white swan" is much harder to falsify as it would require checking every swan to confirm that each of them is white.

[16] Clearly, when the realization of the payoff-relevant state is $\mathfrak{b}$, the alternative hypothesis is falsifiable, as agents would eventually herd on action $\mathfrak{b}$ if $\mathcal{H}_0$ is true and on action $\mathfrak{g}$ if $\mathcal{H}_1$ is true.

[17] Note that in this case, the agent's private log-likelihood ratio $\ell_t = s_t$, so that $F_0, F_{\mathfrak{g}}$ and $F_{\mathfrak{b}}$ have the same distribution as μ_0, $\mu_{\mathfrak{g}}$ and $\mu_{\mathfrak{b}}$, respectively.

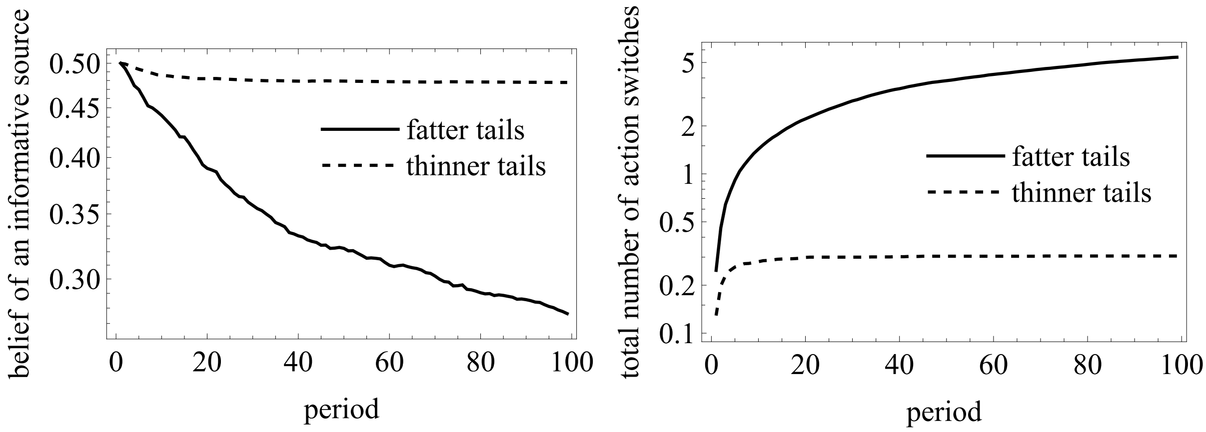


Figure 5.1. The belief that the source is informative (on the left) and the total number of action switches (on the right) under uninformative signals with fatter and thinner tails.

calculate the averages for each period. This yields approximations for the expected belief, and the expected total number of action switches under an uninformative source. Figure 5.1 displays the results of these simulations.

What immediately stands out is that under fatter-tailed uninformative signals, an outside observer's belief that the source is informative decreases much faster compared to thinner-tailed uninformative signals. By period 60, this belief is approximately 0.3 under fatter-tailed uninformative signals, which is less than two-thirds of the belief observed under thinner-tailed uninformative signals. These findings align with the predictions of Theorem 1, suggesting that in the presence of uninformative signals with fatter tails, the observer would eventually learn that the source is uninformative. In contrast, with thinner-tailed uninformative signals, the decline in the belief about the source's informativeness plateaus after period 20, suggesting that the observer would not be able to learn that the source is uninformative.

As previously discussed, the key insight driving asymptotic learning is the persistence of disagreements under an uninformative source. The intuition is that the uninformative source with fatter tails has a higher probability of generating extreme signals, which, in turn, prevents agents from converging to a consensus. This phenomenon is evident in the right plot of Figure 5.1, where the average total number of action switches under fatter-tailed uninformative signals increases over time. In contrast, under thinner-tailed uninformative signals, the total number of switches plateaus in a short amount of time, indicating that perpetual disagreement does not occur in this case.

6. Asymptotic Learning and Other Learning Notions

In this section, we discuss the connections between asymptotic learning in our model and other existing learning notions. Recall that the notion of correct herding refers to the situation in which all but finitely many agents take the correct action (see Definition

2). In our model, correct herding occurs when the source is informative, but it is not guaranteed when the source is uninformative. Indeed, as suggested by Proposition 1, when asymptotic learning fails, agents could herd on the wrong action with positive probability.

Another notion of learning, concerning the convergence of beliefs about the payoff-relevant state, is called *complete learning*. Denote by $p_t := \mathbb{P}[\theta = \mathfrak{g}|H_t]$ the *public belief* at time t that the state is good given the history of actions. We say that *complete learning* occurs if the public belief about the payoff-relevant state eventually converges to the truth. That is, conditioned on $\theta = \mathfrak{g}$, $\lim_{t\to\infty} p_t = 1$ almost surely and conditioned on $\theta = \mathfrak{b}$, $\lim_{t\to\infty} p_t = 0$ almost surely.

In the standard model where signals are always informative and unbounded (Smith and Sørensen, 2000), these two notions of learning—correct herding and complete learning—are equivalent. However, in our model, they are no longer equivalent. As shown in the following proposition, conditioned on an informative source, although agents eventually herd on the correct action, they are not completely sure that it is the correct action unless asymptotic learning holds.

Proposition 2. *Asymptotic learning holds if and only if conditioned on $\omega = 1$, complete learning occurs.*

The proof of Proposition 2 uses the idea of an outside observer who observes the same public information as agent t but has no additional information. Intuitively, as long as the observer is not certain about the source's informativeness, he cannot fully trust the public information, which consists of the agents' actions. Conversely, once the source's informativeness is confirmed, the public information becomes highly accurate, which in turn allows his belief to converge to the truth.

The next result shows that achieving asymptotic learning is equivalent to the public belief converging to the uniform prior when the source is uninformative.

Proposition 3. *Asymptotic learning holds if and only if conditioned on $\omega = 0$, $\lim_{t\to\infty} p_t = 1/2$ almost surely.*

The proof of Proposition 3 uses a similar approach to Proposition 2 and applies the result of Proposition 1. One direction is straightforward: Once an outside observer learns that the source is uninformative, his belief about the payoff-relevant state should remain uniform, since agents' actions contain no information. For the other direction, suppose that asymptotic learning fails under an uninformative source. Then, by Proposition 1, it implies that agents eventually herd on some action. Since this is also possible under an informative source, an outside observer cannot completely disregard the possibility that agents' actions are informative. As a consequence, his belief about the payoff-relevant state does not necessarily remain uniform.

7. Analysis

In this section we first examine how agents update their beliefs. We present two standard yet useful properties of their belief updating process, namely, the *overturning principle* and *stationarity*. Then, based on Proposition 1, we characterize asymptotic learning in terms of *immediate herding*—the event in which all agents immediately herd on some action—which simplifies the problem of asymptotic learning. Finally, we provide a proof sketch of our main result (Theorem 1) at the end of this section.

7.1. **Agents' Beliefs Dynamics.** Since agents always act as if signals are informative (Lemma 1), we focus on how agents update their beliefs conditioned on an informative source. Denote by π_t the *public belief* of agent t assigned to the good state conditioned on $\omega = 1$ and the history of actions H_t, i.e., $\pi_t := \mathbb{P}_1[\theta = \mathfrak{g}|H_t]$. Upon observing the history of actions, this is agent t's belief about the payoff-relevant state before receiving her private signal. Denote by r_t the log-likelihood ratio of agent t's public belief:

$$r_t := \log \frac{\pi_t}{1-\pi_t} = \log \frac{\mathbb{P}_1[\theta = \mathfrak{g}|H_t]}{\mathbb{P}_1[\theta = \mathfrak{b}|H_t]}.$$

Furthermore, we denote by L_t the log-likelihood ratio of agent t's belief regarding the good state over the bad state, conditional on $\omega = 1$ and all available information (H_t, s_t):

$$L_t := \log \frac{\mathbb{P}_1[\theta = \mathfrak{g}|H_t, s_t]}{\mathbb{P}_1[\theta = \mathfrak{b}|H_t, s_t]}.$$

Recall that ℓ_t is the log-likelihood ratio of agent t's private belief induced by signal s_t conditioned on $\omega = 1$. By Bayes' rule, we can write L_t as the sum of r_t and ℓ_t:

$$L_t = r_t + \ell_t.$$

It follows from (1) that, in equilibrium, agent t chooses action $\mathfrak{g}$ if $\ell_t \geq -r_t$ and action $\mathfrak{b}$ if $\ell_t < -r_t$.[18] Hence, conditioned on the state θ and the event that $\omega = 1$, the probability that agent t chooses action $\mathfrak{g}$ is $1 - F_\theta(-r_t)$ and the probability that she chooses action $\mathfrak{b}$ is $F_\theta(-r_t)$. As a consequence, the agents' public log-likelihood ratios (r_t) evolve as follows:

$$r_{t+1} = r_t + D_{\mathfrak{g}}(r_t) \quad \text{if } a_t = \mathfrak{g}, \tag{2}$$

$$r_{t+1} = r_t + D_{\mathfrak{b}}(r_t) \quad \text{if } a_t = \mathfrak{b}, \tag{3}$$

where

$$D_{\mathfrak{g}}(r) := \log \frac{1 - F_{\mathfrak{g}}(-r)}{1 - F_{\mathfrak{b}}(-r)} \quad \text{and} \quad D_{\mathfrak{b}}(r) := \log \frac{F_{\mathfrak{g}}(-r)}{F_{\mathfrak{b}}(-r)}.$$

Note that $D_{\mathfrak{g}}$ always takes positive values, whereas $D_{\mathfrak{b}}$ always takes negative values. Intuitively, seeing the action $\mathfrak{g}$ makes the agent more convinced that the state is good,

[18]Recall that we assumed that all distributions of ℓ_t are continuous, which rules out the possibility of indifference.

so her public belief that the state is good increases. Similarly, seeing the action $\mathfrak{b}$ makes the agent more convinced that the state is bad, which means that her public belief that the state is good decreases.

Overturning Principle and Stationarity. One important property held by the public beliefs of the agents is called the *overturning principle* (Sørensen, 1996; Smith and Sørensen, 2000). It states that a single action switch is sufficient to change the verdict of the agents' public beliefs.

Lemma 2 (Overturning Principle)**.** *For each agent t, if $a_t = \mathfrak{g}$, then $\pi_{t+1} \geq 1/2$. Similarly, if $a_t = \mathfrak{b}$, then $\pi_{t+1} \leq 1/2$.*

Another important property held by the agents' public beliefs is called *stationarity*. That is, the value of π_t captures all past information about the payoff-relevant state, independent of time. This property holds in our model because, regardless of the informativeness of the source, upon observing an action, agents always update their public log-likelihood ratios according to (2) or (3). We further write $\mathbb{P}_{\tilde{\omega},\tilde{\theta},\pi}$ to denote the conditional probability distribution given the pair of state realizations $(\tilde{\omega},\tilde{\theta})$ while highlighting the different values of the prior π.

Lemma 3 (Stationarity)**.** *For any fixed sequence $(b_\tau)_{\tau=1}^{k}$ of k actions in $A = \{\mathfrak{g}, \mathfrak{b}\}$, any prior $\pi \in (0,1)$ and any pair $(\tilde{\omega},\tilde{\theta}) \in \{0,1\} \times A$*

$$\mathbb{P}_{\tilde{\omega},\tilde{\theta}}[a_{t+1} = b_1, \ldots, a_{t+k} = b_k | \pi_t = \pi] = \mathbb{P}_{\tilde{\omega},\tilde{\theta},\pi}[a_1 = b_1, \ldots, a_k = b_k].$$

That is, regardless of the source's informativeness, if agent t's public belief equals π, then the probability of observing a sequence $(b_1, \ldots, b_k)$ of actions of length k is the same as observing this sequence starting from time 1, given that the agents' prior on the payoff-relevant state is π.

7.2. **Asymptotic Learning and Immediate Herding.** In our model, since the agent's public belief π_t evolves as in the standard model, when the source is informative, it remains a martingale. However, when the source is uninformative, π_t ceases to be a martingale under the measure $\mathbb{P}_0$. Therefore, we need to employ a different analytical approach to understand what ensures asymptotic learning.[19]

Denote the event $\{a_1 = a_2 = \ldots = a\}$ in which all agents immediately herd in an action $a \in \{\mathfrak{g}, \mathfrak{b}\}$ by $\{\bar{a} = a\}$ and refer to it as *immediate herding*. We focus on the event of immediate herding as conditioned on this event, the process of public log-likelihood ratios (r_t) evolves deterministically according to either (2) or (3).

The following lemma shows that conditioned on an informative source, immediate herding on the wrong action is impossible, whereas immediate herding on the correct

[19]A similar approach can be seen in Arieli, Babichenko, Müller, Pourbabaee, and Tamuz (2023), where the agent's public belief also ceases to be a martingale under the correct measure because overconfident agents have misspecified beliefs.

action is possible, at least for some prior. Given the symmetry of the model, we will henceforth focus on the case where $\theta = \mathfrak{g}$.

Lemma 4. *Conditioned on $\omega = 1$ and $\theta = \mathfrak{g}$, the following two conditions hold:*

(i) The event $\{\bar{a} = \mathfrak{b}\}$ occurs with probability zero.

(ii) The event $\{\bar{a} = \mathfrak{g}\}$ occurs with positive probability for some prior $\pi \in (0, 1)$.

The first part of Lemma 4 holds since, as mentioned before, conditioned on an informative source, all but finitely many agents take the correct action. This immediately implies that agents cannot immediately herd on the wrong action. Similarly, the second part of Lemma 4 holds because if agents eventually herd on the correct action, then by stationarity of the process, they can also do so immediately at least for some prior.[20]

Conditioned on an uninformative source, recall that Proposition 1 establishes an equivalence between asymptotic learning and perpetual disagreement. Building on this relationship, we now characterize asymptotic learning in terms of immediate herding. Specifically, the next proposition shows that asymptotic learning is equivalent to the absence of immediate herding on any action taken under the uniform prior when the source is uninformative.

Proposition 4. *Asymptotic learning holds if and only if conditioned on $\omega = 0$, for any action $a \in \{\mathfrak{g}, \mathfrak{b}\}$, the event $\{\bar{a} = a\}$ initiated at the uniform prior occurs with probability zero.*

To see why no immediate herding is equivalent to asymptotic learning, let us first consider a stronger condition. That is, given an uninformative source, suppose that it is impossible for agents to immediately herd on any action for *all* prior. If this were true, then again, by stationarity of the process, agents would never herd on any action for any prior. As a consequence, perpetual disagreement is guaranteed, and thus by Proposition 1, this is equivalent to asymptotic learning. The rest of the proof of Proposition 4 aims to show the equivalence between this stronger condition and the condition stated in the proposition. It utilizes the eventual monotonicity property of the belief updating process (see Lemma 6 in the Appendix).

Therefore, the problem of determining whether asymptotic learning holds is reduced to determining whether there is immediate herding, which is much easier to analyze. In particular, conditioned on the event $\{\bar{a} = \mathfrak{g}\}$, we denote the deterministic process of r_t based on (2) by $r_t^{\mathfrak{g}}$. Recall that agent t chooses $a_t = \mathfrak{g}$ if $\ell_t \geq -r_t$, and $a_t = \mathfrak{b}$ otherwise. Hence, the probability of $\{\bar{a} = \mathfrak{g}\}$ is equal to the probability that $\ell_t \geq -r_t^{\mathfrak{g}}$ for all $t \geq 1$. Moreover, conditioned on an uninformative source, since private signals are i.i.d., so are the corresponding private log-likelihood ratios. Thus, conditioned on the source being

[20]In fact, part (ii) of Lemma 4 holds not only for some prior, but also for all priors. One can see this by applying a similar argument used in the proof of Lemma 7 in the Appendix. We omit the stronger statement here, as it is not required to prove our main result.

uninformative, the probability of immediate herding on action $\mathfrak{g}$ is

$$\mathbb{P}_0[\bar{a} = \mathfrak{g}] = \prod_{t=1}^{\infty}(1 - F_0(-r_t^{\mathfrak{g}})).$$

To determine whether the above probability is positive or zero, by a standard approximation argument, it is equivalent to examining whether the sum of the probabilities of the following events is finite or infinite:

$$\mathbb{P}_0[\bar{a} = \mathfrak{g}] > 0 \ (= 0) \ \Leftrightarrow \ \sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}) < \infty \ (= \infty). \tag{4}$$

By the symmetry of the model, one has that

$$\mathbb{P}_0[\bar{a} = \mathfrak{b}] > 0 \ (= 0) \ \Leftrightarrow \ \sum_{t=1}^{\infty} \big(1 - F_0(r_t^{\mathfrak{g}})\big) < \infty \ (= \infty). \tag{5}$$

In summary, asymptotic learning holds if and only if, conditioned on the source being uninformative, as signals become more extreme, the probability of generating these extreme signals decreases slowly enough so that both sums in (4) and (5) are infinite. As we discuss below, for uninformative signals with fatter tails, these sums diverge, and for signals with thinner tails, at least one of these sums converges.

7.3. **Proof Sketch of Theorem 1.** We conclude this section by providing a sketch of the proof of Theorem 1. Suppose that the source is uninformative and generates signals with fatter tails. Then, by part (i) of Lemma 4, the probability of generating extreme signals decreases relatively slowly under informative signals. Since, under fatter-tailed uninformative signals, this probability declines even more slowly, it implies that both sums in (4) and (5) are infinite. That is, no immediate herding is possible, and thus, by Proposition 4, asymptotic learning holds. In contrast, suppose that the source is uninformative but generates signals with thinner tails. Then, by part (ii) of Lemma 4, the probability of generating extreme signals already decreases relatively fast under informative signals. Since, under thinner-tailed uninformative signals, this probability decreases even more rapidly, it implies that at least one of the sums in (4) and (5) is finite. That is, it is possible to immediately herd on some action, and hence, by Proposition 4, asymptotic learning fails.

8. Conclusion

In this paper, we study the sequential social learning problem in the presence of a potentially uninformative source, e.g., an AI recommendation system of unknown quality. We show that achieving asymptotic learning, in which an outside observer eventually discerns the informativeness of the source, is not guaranteed, and it depends on the relationship between the conditional distributions of the private signals. Specifically, it hinges on the relative tail distribution between signals: If uninformative signals have fatter tails compared to their informative counterparts, asymptotic learning holds; conversely, with thinner-tailed uninformative signals, asymptotic learning fails. We also characterize the conditions for asymptotic learning in the canonical case of Gaussian private signals.

More generally, our economic insight suggests that irregular behavior, such as an action switch following a prolonged sequence of identical actions, is the driving force behind asymptotic learning. Contrary to the public-signal benchmark case in which learning is always achieved, with private signals, an outside observer can only learn that the source is uninformative from observing these action switches. We show that when action switches accumulate indefinitely, the observer can eventually differentiate between an uninformative source and an informative one. In the context of scientific paradigms, this insight is reminiscent of Kuhn (1962)'s idea that the accumulation of anomalies may trigger scientific revolutions and paradigm shifts.

A limitation of our results is that they apply only asymptotically. Our numerical simulations suggest that in the case of uninformative Gaussian signals with fatter tails, the asymptotics can already kick in relatively early in the process. It would be interesting to understand the speed at which an outside observer learns about the informativeness of the source. Another promising extension is to explore varying degrees of informativeness, beyond the extreme cases considered in this paper. For example, instead of having either informative or completely uninformative signals, one could have a distribution over the variance of conditional i.i.d. Gaussian signals and ask whether learning about the payoff-relevant state is achievable. We leave these questions for future research.

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Appendix A. Omitted Proofs

Proof of Lemma 1. Recall that in equilibrium, each agent t chooses the action that is most likely to match the state θ conditioned on the available information (H_t, s_t). By Bayes' rule, agent t's relative likelihood between the good state and the bad state is

$$\frac{\mathbb{P}[\theta = \mathfrak{g}|H_t, s_t]}{\mathbb{P}[\theta = \mathfrak{b}|H_t, s_t]} = \frac{\sum_{\tilde{\omega}\in\{0,1\}} \mathbb{P}_{\tilde{\omega}}[\theta = \mathfrak{g}|H_t, s_t] \cdot \mathbb{P}[\omega = \tilde{\omega}|H_t, s_t]}{\sum_{\tilde{\omega}\in\{0,1\}} \mathbb{P}_{\tilde{\omega}}[\theta = \mathfrak{b}|H_t, s_t] \cdot \mathbb{P}[\omega = \tilde{\omega}|H_t, s_t]}.$$

Note that $\mathbb{P}_0[\theta = \mathfrak{g}|H_t, s_t] = \mathbb{P}_0[\theta = \mathfrak{g}]$ and $\mathbb{P}_0[\theta = \mathfrak{b}|H_t, s_t] = \mathbb{P}_0[\theta = \mathfrak{b}]$ as conditioned on $\omega = 0$, neither the public history H_t nor the private signal s_t contains information about the payoff-relevant state θ. Since the states ω and θ are independent of each other and the prior on θ is uniform, $\mathbb{P}_0[\theta = \mathfrak{g}|H_t, s_t] = \mathbb{P}_0[\theta = \mathfrak{b}|H_t, s_t]$. Thus, it follows from the above equation that

$$\frac{\mathbb{P}[\theta = \mathfrak{g}|H_t, s_t]}{\mathbb{P}[\theta = \mathfrak{b}|H_t, s_t]} \geq 1 \Leftrightarrow \frac{\mathbb{P}_1[\theta = \mathfrak{g}|H_t, s_t]}{\mathbb{P}_1[\theta = \mathfrak{b}|H_t, s_t]} \geq 1.$$

That is, in equilibrium, each agent chooses the most likely action conditioned on their information and the source being informative. $\square$

Proof of Lemma 2. Fix any $t \geq 1$. By the law of the iterated expectation,

$$\begin{aligned}\pi_{t+1} = \mathbb{P}_1[\theta = \mathfrak{g}|H_{t+1}] &= \mathbb{E}_1[\mathbb{1}(\theta = \mathfrak{g})|H_{t+1}] \\ &= \mathbb{E}_1[\mathbb{E}_1[\mathbb{1}(\theta = \mathfrak{g})|H_t, s_t]|H_{t+1}]] \\ &= \mathbb{E}_1[\mathbb{P}_1[\theta = \mathfrak{g}|H_t, s_t]|H_{t+1}]].\end{aligned}$$

Thus, if $a_t = \mathfrak{g}$, then by (1), $\mathbb{P}_1[\theta = \mathfrak{g}|H_t, s_t] \geq \mathbb{P}_1[\theta = \mathfrak{b}|H_t, s_t]$. It follows from the above equation that $\pi_{t+1} \geq 1 - \pi_{t+1}$, which implies that $\pi_{t+1} \geq 1/2$. The case where $a_t = \mathfrak{b}$ implies that $\pi_{t+1} \leq 1/2$ follows from a symmetric argument. $\square$

The following simple claim will be useful in proving Proposition 1. It employs an idea similar to the *no introspection principle* in Sørensen (1996). Recall that $q_t = \mathbb{P}[\omega = 1|H_t]$ is the belief of an outside observer assigned to an informative source based on the history of actions from time 1 to $t - 1$.

Claim 1. *For any $a \in (0, 1/2)$ and any $b \in (1/2, 1)$,*

$$\mathbb{P}_0[q_t = \tilde{q}] \leq \frac{1-a}{a} \cdot \mathbb{P}_1[q_t = \tilde{q}], \text{ for all } \tilde{q} \in [a, 1/2];$$
$$\mathbb{P}_0[q_t = \tilde{q}] \geq \frac{1-b}{b} \cdot \mathbb{P}_1[q_t = \tilde{q}], \text{ for all } \tilde{q} \in [1/2, b].$$

Proof of Claim 1. Fix any $\tilde{q} \in (0, 1)$. Let $\tilde{H}_t$ be a realization of the history of actions such that the associated belief q_t is equal to $\tilde{q}$. By the law of total expectation,

$$\mathbb{P}[\omega = 1 | q_t = \tilde{q}] = \mathbb{E}[\mathbb{E}[\mathbb{1}(\omega = 1)|\tilde{H}_t]|q_t = \tilde{q}] = \mathbb{E}[\tilde{q}|q_t = \tilde{q}] = \tilde{q}.$$

It follows from Bayes' rule that

$$\frac{\mathbb{P}_0[q_t = \tilde{q}]}{\mathbb{P}_1[q_t = \tilde{q}]} = \frac{\mathbb{P}[\omega = 0|q_t = \tilde{q}]}{\mathbb{P}[\omega = 1|q_t = \tilde{q}]} \cdot \frac{\mathbb{P}[\omega = 1]}{\mathbb{P}[\omega = 0]} = \frac{1-\tilde{q}}{\tilde{q}}.$$

Since for any $a \in (0, 1/2)$ and any $\tilde{q} \in [a, 1/2]$, $1 \leq \frac{1-\tilde{q}}{\tilde{q}} \leq \frac{1-a}{a}$, it follows from the above equation that

$$\mathbb{P}_0[q_t = \tilde{q}] = \frac{1-\tilde{q}}{\tilde{q}} \cdot \mathbb{P}_1[q_t = \tilde{q}] \leq \frac{1-a}{a} \cdot \mathbb{P}_1[q_t = \tilde{q}].$$

The second inequality follows from a symmetric argument. □

In the following proofs for Proposition 1, 2 and 3, we use the idea of an outside observer, say observer x, who observes everyone's actions. The information available to him at time t is $H_t = (a_1, \ldots, a_{t-1})$ and at time infinity is $H_\infty = \cup_t H_t$. He gets a utility of one if his action matches ω and zero otherwise. Furthermore, we denote by $q_\infty := \mathbb{P}[\omega = 1|H_\infty]$ and $p_\infty := \mathbb{P}[\theta = \mathfrak{g}|H_\infty]$ the beliefs that he holds at time infinity about the event $\omega = 1$ and the event $\theta = \mathfrak{g}$, respectively.

Proof of Proposition 1. Recall that $S = \sum_{t=1}^{\infty} \mathbb{1}(a_t \neq a_{t+1})$. We first show the if-direction, i.e., $\mathbb{P}_0[S = \infty] = 1$ implies asymptotic learning. Denote by a_t^x the action that the outside observer x would choose to maximize the probability of matching ω at time t. Fix a large positive integer $k \in \mathbb{N}$ and let $A_t(k)$ denote the event that there have been at least k action switches before time t. Let $A_t^c(k)$ denote the complementary event.

Consider the following strategy $\tilde{a}_\infty^x(k)$ for the observer x at time infinity: $\tilde{a}_\infty^x(k) = 0$ if $A_\infty(k)$ occurs and $\tilde{a}_\infty^x(k) = 1$ otherwise. That is, he would choose the action 0 if there are at least k action switches at time infinity and the action 1 otherwise. For any fixed k, the observer x's expected payoff under this strategy $\tilde{a}_\infty^x(k)$ is

$$\begin{aligned}\mathbb{P}[\tilde{a}_\infty^x(k) = \omega] &= \mathbb{P}[\omega = 0, A_\infty(k)] + \mathbb{P}[\omega = 1, A_\infty^c(k)] \\ &= \mathbb{P}_0[A_\infty(k)] \cdot \mathbb{P}[\omega = 0] + \mathbb{P}_1[A_\infty^c(k)] \cdot \mathbb{P}[\omega = 1]. \end{aligned} \tag{A.1}$$

Since conditioned on $\omega = 1$, agents eventually herd with probability one, it follows that for all k large enough,

$$\mathbb{P}_1[A_\infty^c(k)] = 1. \tag{A.2}$$

By assumption, $\mathbb{P}_0[S = \infty] = 1$, which implies that $\mathbb{P}_0[A_\infty(k)] = 1$ for all k. Thus, it follows from (A.1) and (A.2) that for all k large enough, $\mathbb{P}[\tilde{a}^x_\infty(k) = \omega] = 1$. In other words, for all k large enough, the strategy $\tilde{a}^x_\infty(k)$ achieves the maximal payoff for x.

Meanwhile, note that the optimal strategy for x at time infinity is to choose the one that is most likely to match ω conditional on H_∞: $a^x_\infty = 1$ if $q_\infty \geq 1/2$ and $a^x_\infty = 0$ otherwise. Since $\mathbb{P}[\tilde{a}^x_\infty(k) = \omega] = 1$ for all k large enough, the optimal strategy a^x_∞ must also achieve the maximal payoff of one:

$$1 = \mathbb{P}[a^x_\infty = \omega] = \mathbb{P}_1[q_\infty \geq 1/2] \cdot \mathbb{P}[\omega = 1] + \mathbb{P}_0[q_\infty < 1/2] \cdot \mathbb{P}[\omega = 0]. \tag{A.3}$$

It follows from (A.3) that $\mathbb{P}_0[q_\infty < 1/2] = \mathbb{P}_1[q_\infty \geq 1/2] = 1$. It remains to show that $\mathbb{P}_0[q_\infty = 0] = 1$ and $\mathbb{P}_1[q_\infty = 1] = 1$. To this end, first notice that $\mathbb{P}_0[q_\infty < 1/2] = 1$ implies that $\mathbb{P}_0[q_\infty \geq 1/2] = 0$. Thus, by Claim 1, it further implies that for any $b \in (1/2, 1)$ and all $\tilde{q} \in [1/2, b]$, $\mathbb{P}_1[q_\infty = \tilde{q}] = 0$. Consequently, it follows from $\mathbb{P}_1[q_\infty \geq 1/2] = 1$ that $\mathbb{P}_1[q_\infty = 1] = 1$. The case that $\mathbb{P}_0[q_\infty = 0] = 1$ follows from an identical argument.

Next, we show the only-if direction by contraposition, i.e., $\mathbb{P}_0[S < \infty] > 0$ implies the failure of asymptotic learning. Suppose that $\mathbb{P}_0[S < \infty] > 0$. Again, since conditioned on $\omega = 1$, agents eventually herd on the correct action, this implies that $\mathbb{P}_1[S < \infty] = 1$. Thus, there exists a history of actions $\tilde{H}_\infty$ at time infinity that is possible under both probability measures $\mathbb{P}_0$ and $\mathbb{P}_1$: $\mathbb{P}_0[\tilde{H}_\infty] > 0$ and $\mathbb{P}_1[\tilde{H}_\infty] > 0$. By Bayes' rule, one has $\mathbb{P}[\omega = 1|\tilde{H}_\infty] < 1$ and similarly, $\mathbb{P}[\omega = 0|\tilde{H}_\infty] < 1$.

Assume without loss of generality that under this history $\tilde{H}_\infty$, the corresponding belief $\tilde{q}_\infty \geq 1/2$. Hence, given $\tilde{H}_\infty$, the observer x should choose $a^x_\infty = 1$. As a consequence, the probability of x matching the state ω is strictly less than one:

$$\begin{aligned}
\mathbb{P}[a^x_\infty = \omega] &= \mathbb{P}[a^x_\infty = \omega, \tilde{H}_\infty] + \mathbb{P}[a^x_\infty = \omega, \tilde{H}^c_\infty] \\
&= \mathbb{P}[\omega = 1|\tilde{H}_\infty] \cdot \mathbb{P}[\tilde{H}_\infty] + \mathbb{P}[a^x_\infty = \omega, \tilde{H}^c_\infty] \\
&< \mathbb{P}[\tilde{H}_\infty] + \mathbb{P}[\tilde{H}^c_\infty] = 1.
\end{aligned}$$

But if asymptotic learning holds, then by definition and (A.3), $\mathbb{P}[a^x_\infty = \omega] = 1$, which is in contradiction with the above strict inequality. □

The following equation will be useful in proving Proposition 2 and 3. Recall that q_∞ and p_∞ are the beliefs of the outside observer assigned to the event $\omega = 1$ and to the event $\theta = \mathfrak{g}$ at time infinity, respectively. Denote by $\pi_\infty = \mathbb{P}_1[\theta = \mathfrak{g}|H_\infty]$. By the law of total probability,

$$\begin{aligned}
p_\infty &= \mathbb{P}_1[\theta = \mathfrak{g}|H_\infty] \cdot q_\infty + \mathbb{P}_0[\theta = \mathfrak{g}|H_\infty] \cdot (1 - q_\infty) \\
&= \pi_\infty \cdot q_\infty + \frac{1}{2} \cdot (1 - q_\infty),
\end{aligned} \tag{A.4}$$

where the second equality holds since conditioned on $\omega = 0$, no history of actions contains any information, and thus $\mathbb{P}_0[\theta = \mathfrak{g}|H_\infty] = \mathbb{P}_0[\theta = \mathfrak{g}] = 1/2$.

Proof of Proposition 2. Suppose that conditioned on $\omega = 1$, complete learning occurs. That is, (i) conditioned on $\omega = 1$ and $\theta = \mathfrak{g}$, $p_\infty = 1$ almost surely and (ii) conditioned on $\omega = 1$ and $\theta = \mathfrak{b}$, $p_\infty = 0$ almost surely. Since agents always act as if the signals are informative (Lemma 1), the agent's public belief $\pi_t = \mathbb{P}_1[\theta = \mathfrak{g}|H_t]$ remains a martingale under the measure $\mathbb{P}_1$. Using a standard martingale convergence argument with unbounded signals (Smith and Sørensen, 2000), we have that (i) conditioned on $\theta = \mathfrak{g}$ and $\omega = 1$, $\pi_\infty = 1$ almost surely and (ii) conditioned on $\theta = \mathfrak{b}$ and $\omega = 1$, $\pi_\infty = 0$ almost surely. Together, it follows from (A.4) that conditioned on $\omega = 1$, and any realization of θ, $q_\infty = 1$ almost surely so that asymptotic learning holds. Conversely, suppose that asymptotic learning holds. It follows from the definition and (A.4) that conditioned on $\omega = 1$, $p_\infty = \pi_\infty$ almost surely. Hence, conditioned on $\omega = 1$, complete learning occurs as π_t converges to the truth. □

Proof of Proposition 3. The only-if direction is straightforward: Suppose that asymptotic learning holds. By definition, conditioned on $\omega = 0$, $q_\infty = 0$ almost surely. It then follows from (A.4) that $p_\infty = 1/2$ almost surely.

Now, we prove the if direction. Suppose that asymptotic learning fails. Then by Proposition 1, conditioned on $\omega = 0$, there exists a history $\tilde{H}_\infty$ that is possible at time infinity in which agents eventually herd on some action. Since such history is also possible conditioned on $\omega = 1$, the corresponding belief of an outside observer $\tilde{q}_\infty \in (0, 1)$.[21] Furthermore, note that given this history $\tilde{H}_\infty$, the corresponding public belief of the agent $\tilde{\pi}_\infty$ takes values in $\{0, 1\}$. This is because conditioned on $\omega = 1$, agents eventually herd on the correct action almost surely; thus, depending on the action to which the history $\tilde{H}_\infty$ converges, the agent's public belief converges to either zero or one. Thus, it follows from (A.4) that the event $\{p_\infty \neq 1/2\}$ occurs with positive probability. □

Proof of Lemma 4. The proof idea is similar to the proof of Lemma 10 in Arieli, Babichenko, Müller, Pourbabaee, and Tamuz (2023). Recall that we use $\mathbb{P}_{1,\mathfrak{g}}$ to denote the conditional probability distribution given $\omega = 1$ and $\theta = \mathfrak{g}$. Since conditioned on $\omega = 1$, correct herding occurs, this means that $\mathbb{P}_{1,\mathfrak{g}}[\lim_t a_t = \mathfrak{g}] = 1$. As a consequence, part (i) follows directly from the fact that the events $\bar{a} = \mathfrak{b}$ and $\lim_t a_t = \mathfrak{g}$ are disjoint, and thus $\mathbb{P}_{1,\mathfrak{g}}[\bar{a} = \mathfrak{b}] = 0$.

For part (ii), define $\tau < \infty$ as the last random time at which the agent chooses the wrong action $\mathfrak{b}$. That is, $1 = \mathbb{P}_{1,\mathfrak{g}}[\lim_t a_t = \mathfrak{g}] = \sum_{k=1}^{\infty} \mathbb{P}_{1,\mathfrak{g}}[\tau = k]$. By the overturning

[21]Note that conditioned on $\omega = 0$, q_∞ has support $\subseteq [0, 1)$ as $\mathbb{P}[\omega = 0|q_\infty = 1] = 0$.

principle (Lemma 2), $a_\tau = \mathfrak{b}$ implies that $\pi_{\tau+1} \leq 1/2$. As a consequence,

$$\begin{aligned}
1 = \sum_{k=1}^{\infty} \mathbb{P}_{1,\mathfrak{g}}[\tau = k] &= \sum_{k=1}^{\infty} \mathbb{P}_{1,\mathfrak{g}}[a_{k+1} = a_{k+2} = \ldots = \mathfrak{g}, \pi_{k+1} \leq 1/2] \\
&= \sum_{k=1}^{\infty} \mathbb{E}_{1,\mathfrak{g}}\left[\mathbb{P}_{1,\mathfrak{g}}[a_{k+1} = a_{k+2} = \ldots = \mathfrak{g}, \pi_{k+1} \leq 1/2 \mid \pi_{k+1}]\right] \\
&= \sum_{k=1}^{\infty} \mathbb{E}_{1,\mathfrak{g}}\left[\mathbb{1}(\pi_{k+1} \leq 1/2) \cdot \mathbb{P}_{1,\mathfrak{g}}[a_{k+1} = a_{k+2} = \ldots = \mathfrak{g} \mid \pi_{k+1}]\right] \\
&= \sum_{k=1}^{\infty} \mathbb{E}_{1,\mathfrak{g}}\left[\mathbb{1}(\pi_{k+1} \leq 1/2) \cdot \mathbb{P}_{1,\mathfrak{g},\pi_{k+1}}[\bar{a} = \mathfrak{g}]\right],
\end{aligned}$$

where the second equality follows from the law of total expectation, and the last equality follows from the stationarity property (Lemma 3). Suppose to the contrary that for all prior $\pi \in (0,1)$, $\mathbb{P}_{1,\mathfrak{g},\pi}[\bar{a} = \mathfrak{g}] = 0$. Then the above equation equals zero, a contradiction. □

Appendix B. Proof of Theorem 1

In this section, we prove Proposition 4 and Theorem 1. To prove Proposition 4, we will first prove the following proposition (Proposition 5). Then, together with Proposition 1, they jointly imply Proposition 4. The proof of Theorem 1 is presented at the end of this section. We write $\mathbb{P}_{0,\pi}$ to denote the conditional probability distribution given $\omega = 0$ while highlighting the value of the prior π on θ.[22]

Proposition 5. *The following are equivalent.*

(i) For any action $a \in \{\mathfrak{g}, \mathfrak{b}\}$, $\mathbb{P}_{0,\pi}[\bar{a} = a] = 0$ for all prior $\pi \in (0,1)$.

(ii) $\mathbb{P}_0[S = \infty] = 1$.

(iii) For any action $a \in \{\mathfrak{g}, \mathfrak{b}\}$, $\mathbb{P}_{0,\pi}[\bar{a} = a] = 0$ for some prior $\pi \in (0,1)$.

To prove this proposition, we first establish some preliminary results on the process of the agents' public log-likelihood ratios conditioned on the event of immediate herding. These results lead to Lemma 7, a crucial part in establishing the equivalence between no immediate herding and perpetual disagreement conditioned on an uninformative source. We present the proof of Proposition 5 towards the end of this section.

Preliminaries. Recall that, conditioned on $\{\bar{a} = \mathfrak{g}\}$, the process of the agent's public log-likelihood ratio r_t evolves deterministically according to (2). We denote such a process by $r_t^{\mathfrak{g}}$, and denote the corresponding updating function by

$$\phi(x) := x + D_{\mathfrak{g}}(x).$$

That is, $r_{t+1}^{\mathfrak{g}} = \phi(r_t^{\mathfrak{g}})$ for all $t \geq 1$. Since the entire sequence $r_t^{\mathfrak{g}}$ is determined once its initial value is specified, we denote the sequence $r_t^{\mathfrak{g}}$ with an initial value of r by $r_t^{\mathfrak{g}}(r)$.

[22]We will continue to omit the prior π on the payoff-relevant state when $\pi = 1/2$.

Thus, we can write $r_t^{\mathfrak{g}}(r) = \phi^{t-1}(r)$ for all $t \geq 1$, where ϕ^t is its t-th composition and $\phi^0(r) = r$.

We remind readers of two standard properties of the sequence $r_t^{\mathfrak{g}}$, as summarized in the following lemma. The first part of this lemma states that $r_t^{\mathfrak{g}}$ tends to infinity over time, and the second part shows that it takes only some bounded time for $r_t^{\mathfrak{g}}$ to reach any positive value.

Lemma 5 (The Long-Run and Short-Run Behaviors of $r_t^{\mathfrak{g}}$)**.**

(i) $\lim_{t\to\infty} r_t^{\mathfrak{g}} = \infty$.

(ii) For any $\bar{r} \geq 0$, there exists t_0 such that $r_{t_0}^{\mathfrak{g}}(r) \geq \bar{r}$ for all $r \geq 0$.

Proof. See Lemma 6 and Lemma 12 in Rosenberg and Vieille (2019). □

Note that although the sequence $r_t^{\mathfrak{g}}$ eventually approaches infinity, it may not do so monotonically without additional assumptions on the distributions $F_{\mathfrak{g}}$ and $F_{\mathfrak{b}}$.[23] The next lemma shows that, under some mild technical assumptions on the left tail of $F_{\mathfrak{b}}$, the function $\phi(x)$ eventually increases monotonically.

Lemma 6 (Eventual Monotonicity)**.** *Suppose that $F_{\mathfrak{b}}$ has a differentiable left tail and its probability density function $f_{\mathfrak{b}}$ satisfies the condition that, for all x large enough, $f_{\mathfrak{b}}(-x) < 1$. Then, $\phi(x) := x + D_{\mathfrak{g}}(x)$ increases monotonically for all x large enough.*

Proof. By assumption, we can find a constant $\rho < 1$ such that for all x large enough, $f_{\mathfrak{b}}(-x) \leq \rho$. By definition, $D_g(x) = \log \frac{1-F_{\mathfrak{g}}(-x)}{1-F_{\mathfrak{b}}(-x)}$ and taking the derivative of $D_{\mathfrak{g}}$, one has

$$D'_{\mathfrak{g}}(x) = \frac{f_{\mathfrak{g}}(-x)}{1-F_{\mathfrak{g}}(-x)} - \frac{f_{\mathfrak{b}}(-x)}{1-F_{\mathfrak{b}}(-x)}.$$

Observe that the log-likelihood ratio of the agent's private log-likelihood ratio ℓ_t is the log-likelihood ratio itself (see, e.g., Chamley (2004)):

$$\log \frac{dF_{\mathfrak{g}}}{dF_{\mathfrak{b}}}(x) = x.$$

It follows that

$$-D'_{\mathfrak{g}}(x) = f_{\mathfrak{b}}(-x)\Big(\frac{1}{1-F_{\mathfrak{b}}(-x)} - \frac{\mathrm{e}^{-x}}{1-F_{\mathfrak{g}}(-x)}\Big) \leq \frac{f_{\mathfrak{b}}(-x)}{1-F_{\mathfrak{b}}(-x)}.$$

[23] In the case of binary states and actions, Herrera and Hörner (2012) show that the property of increasing hazard ratio is equivalent to the monotonicity of this updating function. See Smith, Sørensen, and Tian (2021) for a general treatment.

Hence, for some x large enough, there exists $\varepsilon > 0$ small enough such that $-D'_{\mathfrak{g}}(x) \leq (1+\varepsilon) f_{\mathfrak{b}}(-x)$ and $(1+\varepsilon)\rho \leq 1$. Hence, for all $x \leq x'$,

$$\begin{aligned} D_{\mathfrak{g}}(x) &= D_{\mathfrak{g}}(x') - \int_x^{x'} D'_{\mathfrak{g}}(y)dy \\ &\leq D_{\mathfrak{g}}(x') + (1+\varepsilon)\int_x^{x'} f_{\mathfrak{b}}(-x)dx \\ &= D_{\mathfrak{g}}(x') - (1+\varepsilon)(F_{\mathfrak{b}}(-x') - F_{\mathfrak{b}}(-x)). \end{aligned}$$

Rearranging the above equation, one has

$$\begin{aligned} D_{\mathfrak{g}}(x) - D_{\mathfrak{g}}(x') &\leq (1+\varepsilon)(F_{\mathfrak{b}}(-x) - F_{\mathfrak{b}}(-x')) \\ &\leq (1+\varepsilon)\rho(x'-x), \end{aligned}$$

where the second last inequality follows from the fact that $f_{\mathfrak{b}}(-x) \leq \rho < 1$. Since $(1+\varepsilon)\rho \leq 1$, it follows from the above inequality that, for some x large enough, $D_{\mathfrak{g}}(x)+x \leq D_{\mathfrak{g}}(x')+x'$ for all $x' \geq x$. That is, $\phi(x)$ eventually increases monotonically. ☐

Given these lemmas, we are ready to prove the following result. It shows that conditioned on an uninformative source, the possibility of immediate herding is independent of the prior belief.

Lemma 7. *For any action $a \in \{\mathfrak{g}, \mathfrak{b}\}$, the following statements are equivalent:*
(i) $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$ *for some prior* $\pi \in (0,1)$*;*
(ii) $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$*, for all prior* $\pi \in (0,1)$.

Proof. The second implication, namely, $(ii) \Rightarrow (i)$ is immediate. We will show the first implication, $(i) \Rightarrow (ii)$. Fix some prior $\tilde{\pi} \in (0,1)$ such that $\mathbb{P}_{0,\tilde{\pi}}[\bar{a} = \mathfrak{g}] > 0$ and let $\tilde{r} = \log \frac{\tilde{\pi}}{1-\tilde{\pi}}$. Since $r_t^{\mathfrak{g}}(\tilde{r})$ is a deterministic process, the event $\bar{a} = \mathfrak{g}$ with $\pi_1 = \tilde{\pi}$ is equivalent to the event $\{\ell_t \geq -r_t^{\mathfrak{g}}(\tilde{r}), \forall t \geq 1\}$. Conditioned on $\omega = 0$, since signals are i.i.d., so are the agents' private log-likelihood ratios. Thus, we can write

$$\mathbb{P}_{0,\tilde{\pi}}[\bar{a} = \mathfrak{g}] = \prod_{t=1}^{\infty}(1 - F_0(-r_t^{\mathfrak{g}}(\tilde{r})). \tag{B.1}$$

As a consequence, $\mathbb{P}_{0,\tilde{\pi}}[\bar{a} = \mathfrak{g}] > 0$ if and only if there exists $M < \infty$ such that

$$-\sum_{t=1}^{\infty} \log\left(1 - F_0(-r_t^{\mathfrak{g}}(\tilde{r}))\right) < M.$$

For two sequences (a_t) and (b_t), we write $a_t \approx b_t$ if $\lim_{t\to\infty}(a_t/b_t) = 1$. Since $r_t^{\mathfrak{g}}(\tilde{r}) \to \infty$ (this follows from part (i) of Lemma 5), $\log\left(1 - F_0(-r_t^{\mathfrak{g}}(\tilde{r}))\right) \approx -F_0(-r_t^{\mathfrak{g}}(\tilde{r}))$. Thus, the above sum is finite if and only if

$$\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}(\tilde{r})) < M. \tag{B.2}$$

By the overturning principle (Lemma 2), it suffices to show that (B.2) implies that

$$\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}(r)) < M, \quad \text{for any } r \geq 0.$$

By the eventual monotonicity of ϕ (Lemma 6) and the fact that $r_t^{\mathfrak{g}}(\tilde{r}) \to \infty$, we can find a large enough $\bar{t}$ such that $\bar{r} = r_{\bar{t}}^{\mathfrak{g}}(\tilde{r}) \geq 0$ and $\phi(r) \geq \phi(\bar{r})$ for all $r \geq \bar{r}$. By part (ii) of Lemma 5, there exists $t_0 \in \mathbb{N}$ such that $r_{t_0}^{\mathfrak{g}}(r) \geq \bar{r}$ for all $r \geq 0$. Since above $\bar{r}$, ϕ is monotonically increasing, one has $\phi(r_{t_0}^{\mathfrak{g}}(r)) \geq \phi(\bar{r})$ for any $r \geq 0$. Consequently, for all $\tau \geq 1$, $r_{\tau+t_0}^{\mathfrak{g}}(r) = \phi^{\tau}(r_{t_0}^{\mathfrak{g}}(r)) \geq \phi^{\tau}(\bar{r}) = r_{\tau+1}^{\mathfrak{g}}(\bar{r})$. Since $r_{\tau+1}^{\mathfrak{g}}(\bar{r}) = r_{\tau+1}^{\mathfrak{g}}(r_{\bar{t}}^{\mathfrak{g}}(\tilde{r})) = r_{\tau+\bar{t}}^{\mathfrak{g}}(\tilde{r})$, it follows that

$$F_0(-r_{\tau+t_0}^{\mathfrak{g}}(r)) \leq F_0(-r_{\tau+\bar{t}}^{\mathfrak{g}}(\tilde{r})).$$

Thus, (B.2) implies that for any $r \geq 0$, $\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}(r)) < \infty$, as required. The case of action $\mathfrak{b}$ follows from a symmetric argument. □

Now, we are ready to prove Proposition 5.

Proof of Proposition 5. We show that *(i)* ⇒ *(ii)*, *(ii)* ⇒ *(iii)*, and *(iii)* ⇒ *(i)*. To show the first implication, we prove the contrapositive statement. Suppose that $\mathbb{P}_0[S < \infty] > 0$. This implies that there exists a sequence of action realizations $(b_1, b_2, \ldots, b_{k-1}, b_k = \ldots = a)$ for some action $a \in \{\mathfrak{b}, \mathfrak{g}\}$ such that

$$\mathbb{P}_0[a_t = b_t, \forall t \geq 1] > 0.$$

By stationarity, there exists some $\pi' \in (0,1)$ such that

$$\mathbb{P}_{0,\pi'}[\bar{a} = a] > 0,$$

which contradicts (i).

To show the second implication, suppose towards a contradiction that there exists some action $a \in \{\mathfrak{g}, \mathfrak{b}\}$ such that for all prior $\pi \in (0,1)$, $\mathbb{P}_{0,\pi}[\bar{a} = a] > 0$. In particular, it holds for the uniform prior so that $\mathbb{P}_0[\bar{a} = a] > 0$. Since the event $\{\bar{a} = a\}$ is contained in the event $\{S < \infty\}$,

$$0 < \mathbb{P}_0[\bar{a} = a] \leq \mathbb{P}_0[S < \infty],$$

which implies that $\mathbb{P}_0[S = \infty] < 1$, a contradiction to (ii). Finally, by taking the negation of the statements in Lemma 7, it follows that (iii) implies (i). This concludes the proof of Proposition 5. □

Proof of Proposition 4. By the equivalence between (ii) and (iii) in Proposition 5 and Proposition 1, we have shown Proposition 4. □

Given Proposition 4, we are now ready to prove our main result.

Proof of Theorem 1. Suppose that the uninformative signals have fatter tails than the informative signals. That is, there exists an $\varepsilon > 0$ such that for all x large enough, $F_0(-x) \geq \varepsilon \cdot F_{\mathfrak{b}}(-x)$ and $1 - F_0(x) \geq \varepsilon \cdot (1 - F_{\mathfrak{g}}(x))$. By part (i) of Lemma 4, $\mathbb{P}_{1,\mathfrak{b}}[\bar{a} = \mathfrak{g}] = 0$

and $\mathbb{P}_{1,\mathfrak{g}}[\bar{a} = \mathfrak{b}] = 0$. Following a similar argument that led to (B.1), conditioned on $\omega = 1$ and $\theta = \mathfrak{b}$, one has

$$0 = \mathbb{P}_{1,\mathfrak{b}}[\bar{a} = \mathfrak{g}] = \prod_{t=1}^{\infty}(1 - F_{\mathfrak{b}}(-r_t^{\mathfrak{g}})),$$

which is equivalent to $-\sum_{t=1}^{\infty} \log(1 - F_{\mathfrak{b}}(-r_t^{\mathfrak{g}})) = \infty$. Since $r_t^{\mathfrak{g}} \to \infty$, $\log(1 - F_{\mathfrak{b}}(-r_t^{\mathfrak{g}})) \approx -F_{\mathfrak{b}}(-r_t^{\mathfrak{g}})$ and the previous sum is infinite if and only if

$$\sum_{t=1}^{\infty} F_{\mathfrak{b}}(-r_t^{\mathfrak{g}}) = \infty. \tag{B.3}$$

Similarly, applying the same logic that led to (B.3), one has

$$\mathbb{P}_{1,\mathfrak{g}}[\bar{a} = \mathfrak{b}] = 0 \Leftrightarrow \sum_{t=1}^{\infty}(1 - F_{\mathfrak{g}}(-r_t^{\mathfrak{b}})) = \infty, \tag{B.4}$$

where $r_t^{\mathfrak{b}}$ is the deterministic process of r_t that evolves according to (3). By the symmetry of the model, one has $r_t^{\mathfrak{b}} = -r_t^{\mathfrak{g}}$ for all $t \geq 1$. Since the uninformative signals have fatter tails, it follows from (B.3) and (B.4) that $\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}) = \infty$ and $\sum_{t=1}^{\infty}(1 - F_0(r_t^{\mathfrak{g}})) = \infty$. Using the same logic as we use to deduce (B.3) and (B.4), these are equivalent to $\mathbb{P}_0[\bar{a} = \mathfrak{g}] = 0$ and $\mathbb{P}_0[\bar{a} = \mathfrak{b}] = 0$. Thus, by Proposition 4, asymptotic learning holds.

Now suppose that the uninformative signals have thinner tails than the informative ones. That is, there exists an $\varepsilon > 0$ such that either $F_0(-x) \leq (1/\varepsilon) \cdot F_{\mathfrak{g}}(-x)$ for all x large enough, or $1 - F_0(x) \leq (1/\varepsilon) \cdot (1 - F_{\mathfrak{b}}(x))$ for all x large enough. By part (ii) of Lemma 4, there exist $\pi, \pi' \in (0, 1)$ such that $\mathbb{P}_{1,\mathfrak{g},\pi}[\bar{a} = \mathfrak{g}] > 0$ and $\mathbb{P}_{1,\mathfrak{b},\pi'}[\bar{a} = \mathfrak{b}] > 0$. Let $r = \log \frac{\pi}{1-\pi}$ and $r' = \log \frac{\pi'}{1-\pi'}$. Following a similar argument that led to (B.2), these are equivalent to

$$\sum_{t=1}^{\infty} F_{\mathfrak{g}}(-r_t^{\mathfrak{g}}(r)) < \infty \quad \text{and} \quad \sum_{t=1}^{\infty}(1 - F_{\mathfrak{b}}(r_t^{\mathfrak{g}}(r'))) < \infty.$$

Thus, since uninformative signals have thinner tails, the above inequalities imply that either $\sum_{t=1}^{\infty} F_0(-r_t^{\mathfrak{g}}(r)) < \infty$ or $\sum_{t=1}^{\infty}(1 - F_0(r_t^{\mathfrak{g}}(r')))$, which means that either $\mathbb{P}_{0,\pi}[\bar{a} = \mathfrak{g}] > 0$ for some $\pi \in (0, 1)$ or $\mathbb{P}_{0,\pi'}[\bar{a} = \mathfrak{b}] > 0$ for some $\pi' \in (0, 1)$. It follows from Lemma 7 that these also hold for all prior $\pi, \pi' \in (0, 1)$, including the uniform prior, so we have either $\mathbb{P}_0[\bar{a} = \mathfrak{g}] > 0$ or $\mathbb{P}_0[\bar{a} = \mathfrak{b}] > 0$. Therefore, it follows from Proposition 4 that asymptotic learning fails. $\square$

Appendix C. Gaussian Private Signals

In this section, we prove Corollary 1 and Theorem 2. We say that private signals are Gaussian when all distributions $\mu_{\mathfrak{g}}$, $\mu_{\mathfrak{b}}$ and μ_0 are normal. In particular, we assume that $\mu_{\mathfrak{g}}$ and $\mu_{\mathfrak{b}}$ share the same variance σ^2 and have mean $+1$ and -1, respectively. Meanwhile, μ_0 has mean $m_0 \in (-1, 1)$ and variance τ^2. Notice that in this case, the agent's private log-likelihood ratio induced by a signal s_t is

$$\ell_t = \log \frac{f_{\mathfrak{g}}(s_t)}{f_{\mathfrak{b}}(s_t)} = \frac{2}{\sigma^2} s_t. \tag{C.1}$$

Since ℓ_t is proportional to s_t, the corresponding CDFs F_θ and F_0 are also normal, with a variance of $4/\sigma^2$ and $4\tau^2/\sigma^4$, respectively.

Similar to Definition 3, for any pair of CDFs (F, G), we say that F has a fatter left (right) tail than G if the corresponding left (right) tail ratio is eventually bounded from below; conversely, we say that F has a thinner left (right) tail than G if the corresponding left (right) tail ratio is eventually bounded from above. The following lemma shows that for Gaussian signals, the relative thickness of the tails depends solely on their relative variances.

Lemma 8. *Suppose F and G are two Gaussian cumulative distribution functions with mean μ_1, μ_2 and variance σ_1^2 and σ_2^2, respectively. If $\sigma_1 > \sigma_2$, then F has fatter (left and right) tails than G, and conversely, G has thinner (left and right) tails than F.*

Proof. Let f and g denote the probability density functions of F and G. Then, for any $x \in \mathbb{R}$

$$\frac{f(x)}{g(x)} = \frac{\sigma_2}{\sigma_1} \exp\Big((\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}) \frac{x^2}{2} + (\frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2}) x + \frac{1}{2} (\frac{\mu_2^2}{\sigma_2^2} - \frac{\mu_1^2}{\sigma_1^2}) \Big).$$

Suppose $\sigma_1 > \sigma_2$. It follows from the above equation that $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = \infty$ and $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$. This implies that for all x large enough, $F(-x) \geq G(-x)$ and $1 - F(x) \geq 1 - G(x)$. Thus, we conclude that F has fatter (left and right) tails than G, and conversely, G has thinner (left and right) tails than F. $\square$

As a consequence, Corrollary 1 follows directly from Lemma 8 and Theorem 1.

Proof of Corollary 1. If $\tau > \sigma$, then by (C.1), F_0 has a strictly higher variance than both $F_{\mathfrak{g}}$ and $F_{\mathfrak{b}}$. Consequently, Lemma 8 implies that F_0 has fatter tails than both $F_{\mathfrak{g}}$ and $F_{\mathfrak{b}}$, and so, it follows from Theorem 1 that asymptotic learning holds. An identical argument applies to the case where $\tau < \sigma$. $\square$

We henceforth focus on the case where $\tau = \sigma$. When all private signals are Gaussian, Hann-Caruthers, Martynov, and Tamuz (2018) show that one can approximate the sequence $r_t^{\mathfrak{g}}$ by $(2\sqrt{2}/\sigma) \cdot \sqrt{\log t}$ for all t large enough (see their Theorem 4):

$$\lim_{t \to \infty} \frac{r_t^{\mathfrak{g}}}{(2\sqrt{2}/\sigma) \cdot \sqrt{\log t}} = 1. \tag{C.2}$$

Given this approximation and Proposition 4, we are ready to prove Theorem 2.

Proof of Theorem 2. In the proof, we use the Landau notation, so that $O(g(t))$ stands for some function $f : \mathbb{N} \to \mathbb{R}$ such that there exists a positive $M \in \mathbb{R}$ and $t_0 \in \mathbb{N}$ such that $|f(t)| \leq M \cdot g(t)$ for all $t \geq t_0$.

Note that by (C.1), conditioned on $\omega = 0$, we can write

$$F_0(-r_t^{\mathfrak{g}}) = \mathbb{P}_0[\ell_t \leq -r_t^{\mathfrak{g}}] = \mathbb{P}_0[s_t \leq -(\sigma^2/2) \cdot r_t^{\mathfrak{g}}].$$

Thus, it follows from (C.2) that for all t large enough,

$$F_0(-r_t^{\mathfrak{g}}) = \mathbb{P}_0[s_t \leq -\sigma\sqrt{2\log t}] =: \mu_0(-\sigma\sqrt{2\log t}),$$

where μ_0 is the CDF of s_t conditioned on $\omega = 0$. Since μ_0 is the normal distribution with mean $m_0 \in (-1,1)$ and variance σ^2, observe that $\mu_0(x) = \frac{1}{2}\operatorname{erfc}(-\frac{x-m_0}{\sigma\sqrt{2}})$, where $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-t^2}dt$ is the complementary error function.

Applying a standard asymptotic expansion of the complementary error function, i.e., $\operatorname{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} + O(\mathrm{e}^{-x^2}/x^3)$, we obtain that for all t large enough,

$$\mu_0(-\sigma\sqrt{2\log t}) = \frac{\mathrm{e}^{-\left(\frac{m_0}{\sigma}\sqrt{2\log t} + \frac{m_0^2}{2\sigma^2}\right)}}{t(\sqrt{\pi\log t} + \delta \cdot m_0)} + O\left(\frac{\mathrm{e}^{-m_0\sqrt{2\log t}}}{t(\sigma\sqrt{2\log t} + m_0)^3}\right), \tag{C.3}$$

where $\delta > 0$ is a constant.

Case (i): suppose $m_0 = 0$. Then (C.3) becomes $\frac{1}{t\cdot\sqrt{\pi\log t}} + O(\frac{1}{t\cdot(\log t)^{3/2}})$. Since the series $\frac{1}{t\log t}$ is divergent and $\frac{1}{t\log t} \leq \frac{1}{t\sqrt{\log t}}$ for all $t \geq 2$, the sum of the first term also diverges. As a consequence, one has $\sum_{t=1}^\infty F_0(-r_t^{\mathfrak{g}}) = \infty$, and by the same argument used in the proof of Theorem 1 (see, e.g., (B.3)), this is equivalent to $\mathbb{P}_0[\bar{a} = \mathfrak{g}] = 0$. By a symmetric argument, we also have $\mathbb{P}_0[\bar{a} = \mathfrak{b}] = 0$. Therefore, by Proposition 4, asymptotic learning holds.

Case (ii): suppose $m_0 \neq 0$ and $m_0 \in (-1,1)$. Let $c = \frac{m_0\sqrt{2}}{\sigma}$. By the change of variable $x = \sqrt{\log t}$,

$$\int_2^\infty \frac{\mathrm{e}^{-c\sqrt{\log t}}}{t\sqrt{\log t}}dt = 2\int_{\sqrt{\log 2}}^\infty \mathrm{e}^{-cx}dx.$$

If $m_0 > 0$, then $c = \frac{m_0\sqrt{2}}{\sigma} > 0$. By the integral test, the sum in (C.3) converges and thus $\sum_{t=1}^\infty F_0(-r_t^{\mathfrak{g}}) < \infty$. Again, by the same logic that we use to deduce (B.2), this is equivalent to $\mathbb{P}_0[\bar{a} = \mathfrak{g}] > 0$. If, on the other hand, $m_0 < 0$, then it follows from a symmetry argument that $\sum_{t=1}^\infty (1 - F_0(r_t^{\mathfrak{g}})) < \infty$, which is equivalent to $\mathbb{P}_0[\bar{a} = \mathfrak{b}] > 0$. Therefore, according to Proposition 4, asymptotic learning fails for any $m_0 \in (-1,1)$ such that $m_0 \neq 0$. □