THE EMERGENCE OF FADS IN A CHANGING WORLD

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Abstract. We study how fads emerge from social learning in a changing environment. We consider a sequential learning model in which rational agents arrive in order, each acting only once, and the underlying unknown state is constantly evolving. Each agent receives a private signal, observes all past actions of others, and chooses an action to match the current state. Since the state changes over time, cascades cannot last forever, and actions fluctuate too. We show that in the long run, actions change more often than the state. This describes many real-life faddish behaviors in which people often change their actions more frequently than what is necessary.

1. Introduction

The term "fads" describes transient behaviors or objects that rise and fade quickly in popularity. From daily matters such as diet and parenting to important economic and environmental policy designs, we often encounter fads in many situations (Naim, 2000; Myers, 2011). While sociologists have studied some characteristics of fads, the question of how and why fads emerge has not been completely resolved in the economic literature.[1] In this paper, we show how fads can arise from social learning in an ever-changing environment and provide rationales for the development of fads.

A pioneering study by Bikhchandani, Hirshleifer, and Welch (1992) presents an early theory of fads where the authors briefly discuss the idea that a one-time shock to the underlying environment can dislodge a trend in action and thus trigger whimsical shifts in behavior. They mainly focus on learning in a fixed environment where fads cannot recur indefinitely. In contrast, the recurrence of fads is possible in a changing environment, a setting that has recently attracted some attention in the literature (see, e.g., Dasaratha, Golub, and Hak, 2020; Lévy, Pęski, and Vieille, 2022)[2]. Indeed, this setting is important to study because many applications concerning social learning, such as investment, employment, cultural norms, and technological advancement, often operate in a dynamic environment.

We study the long-term behavior of agents, who arrive sequentially and learn from observing the past actions of others as well as their private signals, in a changing state.

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[1]See, e.g., a sociology study on a popular streaking fad in the 1970s (Aguirre, Quarantelli, and Mendoza, 1988).
[2]See below for a detailed literature review on social learning studies in a changing environment.

As the underlying state evolves, the best action to take changes over time. The question we ask is: how frequently do actions change compared to state changes? Because of information cascades, agents sometimes ignore their private signals, and thus their actions do not change even when the state changes. On the other hand, because signals are noisy, actions sometimes change unnecessarily. We say *fads* emerge if there are more action changes than state changes. Under some mild conditions, our main result shows that fads do emerge in the long run. We stress that in our model, fads arise from agents' desire to match the ever-changing state rather than any heuristics or irrationality of the agents.

Following Moscarini, Ottaviani, and Smith (1998), we consider a sequential learning model of a binary state and action in a Markovian environment with a small and symmetric transition probability. In every period, a newly-born rational agent arrives and acts once. Before her decision, she receives a private signal that is informative about the current state and all past actions made by her predecessors. Each agent obtains a positive payoff if her action matches the current state. Unlike a standard fixed state learning model, each agent here also considers the possibility that the state may have changed after observing all available information.

The slowly evolving state in our model allows temporary cascades to arise, and as a result, agents cannot herd on a single action forever (Moscarini, Ottaviani, and Smith, 1998). Intuitively, older information is depreciated over time as it becomes less relevant to the current agent so that eventually, agents will return to utilize their private signals and change the action. Thus, we focus on how often agents change their actions. If agents are perfectly informed about the current state, their actions should change as often as the state. However, agents only receive imperfectly informative signals and observe the past actions of others, which are also noisy indicators of their private signals. Moreover, because agents in a temporary cascade would act independently of the private signals, there may be cases where the binary state has changed an even number of times, e.g., say twice, while agents mistakenly treat the state as unchanged, thus not changing their actions.

Perhaps surprisingly, our main result (Theorem 1) shows that even with a reasonably small probability of changing states, where temporary cascades last longer, actions change more often than the state in the long run. In other words, fads arise in a changing environment even though the probability of a state change is small. For example, consider a private signal that matches the state with a probability of 0.8. On average, when the state changes once every a hundred periods, we show that agents take less than sixty-one periods to change the action. Thus, the long-term frequency of action changes must be higher than state changes, resulting in fads. This relatively high frequency of action changes is also in line with the fragility of fads, where small shocks to the environment could cause rapid shifts in actions.[3]

The mechanism behind the long-run emergence of fads is as follows. First, for any fixed signal precision and probability of changing state, there exists a maximum length of any cascade. As a result, even though the rise of temporary cascades prolongs action inertia, such an effect on actions is limited by its bounded length. Meanwhile, agents only require one opposing signal to switch their actions whenever they have an opportunity, i.e., whenever the public belief exits a cascade. We lower bound the probability of agents switching actions and thus establish an upper bound for the expected time between action changes. This upper bound turns out to be less than the expected time of state changes, implying that it takes less time on average for actions to change than the state. Finally, we translate the expected time of changes in both the state and action into their long-term relative frequency of changes and conclude that fads emerge in the long run.

It is worth mentioning that the small probability of changing states is not the driving force behind our main result. When the state is unlikely to change, it also slows down action changes. Intuitively, actions change slowly in this case because past actions are more informative about the current state, and as a result, the public information consisting of all past actions becomes stronger. Consequently, information cascades last longer, prolonging action inertia. Nevertheless, we show that actions change more often than the state in the long run.

1.1. **Related Literature.** This paper is closely related to a small stream of studies on social learning in a changing state. As mentioned before, in the original paper of Bikhchandani, Hirshleifer, and Welch (1992), the authors show that a one-time shock to the state could break the cascade, even though that shock may never be realized. Thus, information cascades are not robust to changes in the underlying environment, and the fragility of information cascades often leads to volatile behaviors, such as fads.[4]

Later, Moscarini, Ottaviani, and Smith (1998) further explore this idea and show that if the underlying environment is evolving in every period and the state is relatively persistent, an information cascade must arise, but it can only be temporary, i.e., it must end in finite time. Intuitively, older information becomes less relevant to the current agent as the best action fluctuates with the ever-evolving state. As a result, everyone would eventually return to relying on their own and most current signals, thus ending the cascades. Our work builds on their model but with a different focus. While their main focus is on the short-term behavior of information cascades, e.g., under what conditions do they end or arise, we ask: in the long run, should one expect to see more action changes than state changes or vice versa?

[3]See a detailed discussion of the fragility of fads and many other examples of fads in Bikhchandani, Hirshleifer, and Welch (1992, 1998).

[4]They also provide a numeric example where the probability of an action change is at least 87% higher than the probability of a state change (See Result 4) which is in line with our main result.

Hirshleifer and Welch (2002) also consider a changing environment in their stylized model (see their §3), but their focus is on examining the effect of *memory loss* on the continuity of behavior.[5] They show that in a relatively stable environment, memory loss causes the agent to exhibit excess action inertia relative to a full-recall regime, whereas in a violate environment, memory loss leads to excess action impulsiveness.[6] One can think of our question as a natural next step from theirs – since there is excess action inertia or impulsiveness when agents only observe past actions in a changing environment, how does the frequency of action changes compare with state changes?

Among a few more recent studies that consider a dynamic state, the efficiency of learning has been a primary focus of study. For example, Frongillo, Schoenebeck, and Tamuz (2011) consider a specific dynamic environment in which the underlying state follows a random walk with non-Bayesian agents who use different linear rules when updating. Their main result is that the equilibrium weights may not be Pareto optimal, causing inefficiency in learning.[7] In a similar but more general environment, Dasaratha, Golub, and Hak (2020) show that having sufficiently diverse network neighbors with different signal distributions improves learning. Intuitively, diverse signals allow agents to decipher the most relevant information from the old and confounded information, thus achieving higher efficiency in information aggregation.[8]

A more recent study by Lévy, Pęski, and Vieille (2022) considers a setup similar to ours with a focus on equilibrium welfare. In their model, agents observe a random subsample drawn from all past behaviors and then decide whether to acquire private signals that are potentially costly. These model generalizations allow them to highlight the trade-off between the learning efficiency and the need to be responsive to environmental changes, which results in a reduction in equilibrium welfare. In contrast, we assume that all past actions are observable and the private signals are free of charge, all of which are typical features of the classic sequential learning model (Banerjee, 1992; Bikhchandani, Hirshleifer, and Welch, 1992). We consider this simple learning model without further complications as our focus is on comparing the long-term relative frequency of action and state changes – a question turns out to be non-trivial even in this simple setup.

[5]The term "memory loss" refers to the case where the agent only recalls past actions but not past signals.

[6]Intuitively, as the volatility of the environment increases, past actions become less relevant to the current state. At some point, this information weakens enough so that the amnesiac agent would always follow her latest signal, but the full-recall agent may not do so at this point. Hence there is an increase in the probability of an action change due to amnesia.

[7]See more studies in the computer science literature, e.g., Acemoglu, Nedic, and Ozdaglar (2008); Shahrampour, Rakhlin, and Jadbabaie (2013) that consider a dynamic environment with non-Bayesian agents.

[8]They further show that this efficiency in information aggregation is no longer achievable if one substitutes diverse network structures for diverse signals, stressing the importance of signal asymmetry.

2. Model

We follow the setup from Moscarini, Ottaviani, and Smith (1998) closely. Time is discrete and the horizon is infinite, i.e., $t \in \mathbb{N}^+ = \{1, 2, \ldots\}$. There is a constantly evolving binary state $\theta_t \in \{-1, +1\}$ that follows a Markov chain. For simplicity, assume that both states are equally likely at the beginning of time, and the transition between states is symmetric: $\mathbb{P}[\theta_{t+1} \neq i | \theta_t = i] = \varepsilon$, for $i \in \{-1, +1\}$, where ε is the probability of switching states. This assumption implies that the stationary distribution of θ_t is uniform over $\{-1, +1\}$. If $\varepsilon = 1/2$, states are i.i.d., and one can simply think of the evolution of states as a sequence of fair coin flips. For our interest, we assume that $\varepsilon \in (0, 1/2)$ throughout. Equivalently, one can think of this assumption as follows: in every period, with probability $2\varepsilon \in (0, 1)$ the state will be redrawn from $\{-1, +1\}$ with equal probability. Thus, the probability of switching states is equal to $\varepsilon \in (0, 1/2)$.

A sequence of agents indexed by time t arrive in order, each acting once by choosing an action $a_t \in \{-1, +1\}$. The payoff function of agent t is the indicator function $\mathbb{1}(a_t = \theta_t)$ so that agent t would like to choose the action that is more likely to match the current state. As the state evolves, the best action to take also fluctuates over time. We say that a *fad* emerges by time $n+1$ if the fraction of time periods $t \leq n$ for which $a_t \neq a_{t+1}$ is larger than the fraction of those for which $\theta_t \neq \theta_{t+1}$, i.e.,

$$\mathcal{Q}_a(n) := \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(a_t \neq a_{t+1}) > \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}(\theta_t \neq \theta_{t+1}) =: \mathcal{Q}_\theta(n). \tag{1}$$

By multiplying both sides of (1) by n, the emergence of fads by time $n+1$ implies that actions would have changed more often than the state by time $n+1$.

Before choosing an action, agent t receives a private signal s_t and observes the history of all past actions made by her predecessors, $h^{t-1} = (a_1, a_2, \ldots, a_{t-1})$. Conditional on the entire sequence of states, the private signals s_t are independent, and each s_t has a Bernoulli distribution $B_{\theta_t}(\alpha)$ where α is the symmetric probability of matching the current state: $\mathbb{P}[s_t = i | \theta_t = i] = \alpha \in (1/2, 1)$, for $i \in \{-1, +1\}$. We often refer to signal $s = +1$ as an up-signal and to $s = -1$ as a down-signal. Let $\mathcal{I}^t = \{-1, +1\}^{t-1} \times \{-1, +1\}$ be the space of information available to agent t prior to her decision so that $I_t = (h^{t-1}, s_t)$ is an element of $\mathcal{I}^t$.

At any time t, the timing of the events is as follows. First, agent t arrives and observes the history h^{t-1} of all past actions. Second, the state θ_{t-1} transitions to θ_t with probability ε of switching. Agent t then receives a private signal s_t that matches the current state θ_t with probability α. Finally, she chooses an action a_t that maximizes the probability of matching θ_t conditional on I_t, the information available to her.

In this Markovian environment, we can view the agent's private signal as a direct piece of information about the current state and the history of past actions as an indirect piece of information. These histories of actions inform the agent about the private signals received by her predecessors and thus are informative about the previous states. However, as the state switches in every period with probability ε, the current state may differ from the previous states. In the extreme case where $\varepsilon = 1/2$, as states are i.i.d. over time, the history of past actions ceases to contain any information about the current state, and thus the agent should only act according to her own private signal.

2.1. **Agents' Beliefs.** Let $q_t := \mathbb{P}[\theta_t = +1|h^{t-1}]$ denote the *public belief* assigned to state $\theta_t = +1$ at time t after observing the history of actions h^{t-1}. Let $p_t := \mathbb{P}[\theta_t = +1|I_t]$ denote the *private belief* assigned to state $\theta_t = +1$ after observing $I_t = (h^{t-1}, s_t)$. Denote the log-likelihood ratio (LLR) of the private belief of agent t by

$$L_t := \log \frac{p_t}{1-p_t} = \log \frac{\mathbb{P}[\theta_t = +1|I_t]}{\mathbb{P}[\theta_t = -1|I_t]},$$

and call it the *posterior likelihood* at time t. It follows from Bayes' rule that the posterior likelihood at time t satisfies

$$L_t = \log \frac{\mathbb{P}[s_t|\theta_t = +1, h^{t-1}]}{\mathbb{P}[s_t|\theta_t = -1, h^{t-1}]} + \log \frac{\mathbb{P}[\theta_t = +1|h^{t-1}]}{\mathbb{P}[\theta_t = -1|h^{t-1}]}. \tag{2}$$

As the private signal is independent of the history conditional on the current state, the first term in (2) reduces to the LLR induced by the signal and it is equal to $c_\alpha := \log \frac{\alpha}{1-\alpha}$ if $s_t = +1$ and $-c_\alpha$ if $s_t = -1$. Denote the second term in (2) by

$$l_t := \log \frac{q_t}{1-q_t} = \log \frac{\mathbb{P}[\theta_t = +1|h^{t-1}]}{\mathbb{P}[\theta_t = -1|h^{t-1}]},$$

the *public likelihood* at time t. Intuitively, anyone who observes all past actions until time $t-1$ can calculate this log-likelihood ratio.

Thus, depending on the realization of the private signals, the posterior likelihood L_t is the sum of the public likelihood l_t and the LLR induced by the private signal at time t:

$$L_t = \begin{cases} l_t - c_\alpha & \text{if } s_t = -1, \\ l_t + c_\alpha & \text{if } s_t = +1. \end{cases} \tag{3}$$

2.2. **Agents' Actions.** The optimal action for agent t is the action that maximizes her expected payoff conditional on the information available to her:

$$a_t \in \operatorname*{arg\,max}_{a \in \{-1,+1\}} \mathbb{E}[\mathbb{1}(\theta_t = a)|I_t] = \operatorname*{arg\,max}_{a \in \{-1,+1\}} \mathbb{P}[\theta_t = a|I_t].$$

Thus $a_t = +1$ if $L_t > 0$ and $a_t = -1$ if $L_t < 0$. When agent t is indifferent, i.e., $L_t = 0$, we assume that she would follow what her immediate predecessor did in the previous period, i.e., $a_t = a_{t-1}$.[9] This assumption ensures that action changes are not due to the specification of the tie-breaking rule but rather due to her strict preference for one action over another.

[9]Our tie-breaking rule is different from the one in Moscarini et al. (1998) where they assume that agents would follow their private signals when indifferent. Under this assumption, it is likely for an indifferent agent to choose a different action from her predecessor's.

2.3. **Information Cascades and Regions.** An *information cascade* is the event in which the past actions of others form an overwhelming influence on agents so that they act independently of the private signals. Specifically, it follows from (3) that the sign of the posterior likelihood L_t is purely determined by the sign of the public likelihood l_t once the absolute value of l_t exceeds c_α. Since the sign of L_t determines the optimal action of agent t, in this case, a_t would also be purely determined by the sign of l_t, independent of the private signal s_t. That is, $a_t = +1$ if $l_t > c_\alpha$ and $a_t = -1$ if $l_t < -c_\alpha$. When $|l_t| < c_\alpha$, agent t chooses the action according to her private signal so that $a_t = s_t$.

When $|l_t| = c_\alpha$, by the tie-breaking rule at indifference, regardless of the private signal that agent t receives, she would choose $a_t = a_{t-1} = \text{sign}(l_t)$.[10] Thus, we call the region of the public likelihood in which $|l_t| \geq c_\alpha$ the *cascade region* and the region in which $|l_t| < c_\alpha$ the *learning region*. We refer to the cascade in which $a = +1$ as an up-cascade and to the cascade in which $a = -1$ as a down-cascade.

3. Results

We now state our main result. Recall that in (1) we define the emergence of fads by some time $n+1$. For reasonably small probability of state change (see how we obtain the bound $\bar{\varepsilon}(\alpha)$ in Lemma 2 in §4)), we show that the long-term relative frequency of action changes is higher than that of state changes, suggesting the emergence of fads in the long run.

Theorem 1. *For any signal precision $\alpha \in (1/2, 1)$ and probability of state change $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$, a fad emerges in the long run almost surely, i.e.,*

$$\lim_{n\to\infty} \mathcal{Q}_a(n) > \lim_{n\to\infty} \mathcal{Q}_\theta(n) \quad a.s.$$

Perhaps surprisingly, Theorem 1 shows that when the underlying state evolves slowly, information cascades last longer, but agents who learn from observing others' past actions change their actions more often than the state in the long run. In other words, fads emerge from social learning in a changing environment even though the probability of a state change is small. For example, consider a private signal with a precision of 0.8. When the probability of changing states is equal to 0.01, the state switches once every a hundred periods on average, and meanwhile, the average time for an action to switch is strictly less than six-one periods.[11] Thus, in the long run, actions would switch strictly more frequently than the state, resulting in faddish behaviors.

The idea behind the proof of Theorem 1 is as follows. Intuitively, one might think that actions would change less often than the state due to the effect of cascades on action

[10] Without loss of generality, consider $l_t = c_\alpha$. To see why $a_{t-1} = \text{sign}(l_t) = +1$, suppose to the contrary that $a_{t-1} = -1$. Given that $l_t = c_\alpha$ and $a_{t-1} = -1$, it must be that $l_{t-1} > c_\alpha$, which implies that $a_{t-1} = +1$. A contradiction.

[11] This follows from Proposition 1 in §5 by substituting $\alpha = 0.8$ and $\varepsilon = 0.01$ into $M(\alpha, \varepsilon)$, and we have $M(0.8, 0.01) \approx 60.7$.

inertia. However, for any fixed signal precision and probability of switching state, there exists a maximum length of any cascade (Moscarini, Ottaviani, and Smith, 1998). We show that once the public belief exits a cascade region, the action either switches, or else the public belief enters the same cascade region again. We upper bound the probability of the latter event and thus establish an upper bound to the expected time between action changes, which turns out to be less than the expected time between state changes (by Proposition 1 in §5). Finally, our main result (Theorem 1) translates the expected time between both action and state switches into their long-run relative frequency of switches.[12] Building on this, we conclude that the long-run relative frequency of action changes is higher than that of state changes, meaning that a fad emerges in the long run.

4. The Public Belief Dynamics

In this section, we study the dynamics of the public likelihood in different regions for both fixed and changing states. We then discuss the connection between the public likelihood and the agent's action and regulate the public likelihood's behavior by focusing on small probabilities of switching states. The regulated public likelihood exhibits distinct patterns, which will facilitate our analysis later.

4.1. **Learning region.** When the state is fixed ($\varepsilon = 0$), as the agent's action is informative about her private signal in the learning region, i.e., $a_t = s_t$, the public belief at time $t+1$ coincides with the private belief at time t:

$$q_{t+1} := \mathbb{P}[\theta_{t+1} = 1 | h^{t-1}, a_t] = \mathbb{P}[\theta_t = 1 | h^{t-1}, s_t] = p_t.$$

Hence, $l_{t+1} = L_t$ and l_{t+1} evolves according to (3). When the state changes with probability $\varepsilon > 0$ in every period, upon observing the latest history, each agent also needs to consider the possibility that the state may have changed after the latest action was taken. However, neither the learning region nor the cascade region is affected by a changing state as the state only transitions after the history of past actions was observed. By Bayes' rule, the public likelihood at time $t+1$ in the learning region is

$$\begin{aligned} l_{t+1} &:= \log \frac{q_{t+1}}{1 - q_{t+1}} \\ &= \log \frac{\mathbb{P}[\theta_{t+1} = +1 | h^{t-1}, a_t]}{\mathbb{P}[\theta_{t+1} = -1 | h^{t-1}, a_t]} \\ &= \log \frac{\sum_{\theta_t \in \{-1,+1\}} \mathbb{P}[\theta_{t+1} = +1, a_t | h^{t-1}, \theta_t] \mathbb{P}[\theta_t | h^{t-1}]}{\sum_{\theta_t \in \{-1,+1\}} \mathbb{P}[\theta_{t+1} = -1, a_t | h^{t-1}, \theta_t] \mathbb{P}[\theta_t | h^{t-1}]}. \end{aligned} \tag{4}$$

Since the process $(\theta_t)_t$ is a Markov chain and $a_t = s_t$ in the learning region, conditioned on θ_t, both θ_{t+1} and a_t are independent of h^{t-1} and independent of each other. Hence,

[12] Note that since these action changes are not independent events, this connection for actions does not directly follow from the law of large numbers.

we can write (4) as

$$f_1(l_t) := \log \frac{(1-\varepsilon)\alpha \mathrm{e}^{l_t} + \varepsilon(1-\alpha)}{\varepsilon\alpha \mathrm{e}^{l_t} + (1-\varepsilon)(1-\alpha)} \quad \text{if } s_t = +1,$$

and

$$f_0(l_t) := \log \frac{(1-\varepsilon)(1-\alpha)\mathrm{e}^{l_t} + \varepsilon\alpha}{\varepsilon(1-\alpha)\mathrm{e}^{l_t} + (1-\varepsilon)\alpha} \quad \text{if } s_t = -1.$$

In sum, with a changing state, the public likelihood in the learning region evolves as follows:

$$l_{t+1} = \begin{cases} f_1(l_t) & \text{if } s_t = +1, \\ f_0(l_t) & \text{if } s_t = -1. \end{cases} \tag{5}$$

It is straightforward to check that both $f_1(l)$ and $f_0(l)$ are strictly increasing in l. Intuitively, when an agent starts with a higher prior belief, her posterior belief would be higher upon receiving either an up or down signal. The following lemma shows that it is relatively easy for the public likelihood to switch its sign if it is in the learning region where the agent's action is the same as her private signal.

Lemma 1. *When the public likelihood is in the learning region, one opposing action is sufficient to overturn its sign. I.e., for all $|l_t| < c_\alpha$, $a_t \neq sign(l_t)$ implies that $sign(l_{t+1}) = a_t$.*

The proof of Lemma 1 relies on the idea that for any non-uniform public belief, the effect of observing different actions on the public likelihood is asymmetric. In particular, an opposing action has a bigger impact on the current belief than a supporting one. Intuitively, as an opposing action is somewhat unexpected, the public belief would act as if it overweighs the opposing evidence. In addition, the public belief about either state is not too extreme, given that its corresponding log-likelihood ratio is in the learning region, and as a result, it can be easily persuaded by the most current action. On the other hand, as we shall see in §4.4, observing one supporting action may not be enough evidence to push the public likelihood into the cascade region.

4.2. **Cascade region.** When the state is fixed ($\varepsilon = 0$), the public likelihood stays forever at the value at which it first entered the cascade region.[13] When the state is changing ($\varepsilon > 0$), however, the behavior of the public likelihood changes significantly. To see this, suppose that t is a time at which the public likelihood enters the cascade region from the learning region. Although no agent's actions reveal more information about the state after time t, the state still evolves and switches with probability ε in every period. Since the process $(\theta_t)_t$ is a Markov chain, it follows from the law of total probability that the

[13]This is the classical result from the information cascade literature introduced independently by Bikhchandani, Hirshleifer, and Welch (1992) and Banerjee (1992).

public belief updates deterministically as follows:

$$q_{t+1} := \mathbb{P}[\theta_{t+1} = +1 | h^t] = \sum_{\theta_t \in \{-1,+1\}} \mathbb{P}[\theta_{t+1} = +1 | h^t, \theta_t] \mathbb{P}[\theta_t | h^t]$$
$$= (1-\varepsilon) q_t + \varepsilon(1-q_t),$$

so that

$$q_{t+1} = \ (1-2\varepsilon) q_t + (2\varepsilon)\frac{1}{2}. \tag{6}$$

Equivalently, we can write it recursively in terms of the public likelihood:

$$l_{t+1} := \log \frac{q_{t+1}}{1-q_{t+1}} = \log \frac{(1-\varepsilon)\mathrm{e}^{l_t} + \varepsilon}{1-\varepsilon+\varepsilon \mathrm{e}^{l_t}}. \tag{7}$$

From (6), we see that q_{t+1} tends to $1/2$, and by (7), l_{t+1} moves towards zero over time so that eventually it will exit the cascade region. Intuitively, having a changing state depreciates the value of older information as actions observed in earlier periods become less relevant to the current state. Consequently, even a strong belief about the state will become weaker after some finite number of periods so that the information cascade will not last forever. Indeed, this is the main insight from Moscarini, Ottaviani, and Smith (1998) (see their Proposition 1), where they show that information cascades must end in finite time.

Another important observation is that regardless of whether the state is fixed or changing, the process of the public likelihoods $(l_t)_t$ in either case forms a Markov chain.[14] In the case of a fixed state, the state space of $(l_t)_t$ is finite as the size of steps between states is equal to c_α, which is a constant for any fixed signal precision α. However, in the case of a changing state, the state space of $(l_t)_t$ is infinite. This is because from (5), one can see that the step size between l_t and l_{t+1} depends on the current value of l_t, and in almost all cases, two consecutive opposing signals would not exactly offset each other, i.e., $f_i(f_j(l)) \neq l$, for $i \neq j = 0, 1$. As a result, there are many more possible states for l_t in a changing state relative to a fixed state.

We demonstrate these public likelihood dynamics by simulations with specific values of signal precision α and probability of changing states ε. Figure 1 displays the results of two simulations in the first hundred periods with the same signal precision $\alpha = 0.8$ and different state-changing probabilities, i.e., $\varepsilon = 0.05$ and 0.03, respectively.

[14] To see this, note that conditioned on the state θ_t, the private signal s_t is independent of $l_{t'}$, for any $t' < t$. So its distribution conditioned on l_t is the same as its distribution conditioned on $(l_1, \ldots, l_{t-1})$ which implies that $\mathbb{P}[l_{t+1} = l | l_1, \ldots, l_t] = \mathbb{P}[l_{t+1} = l | l_t]$.

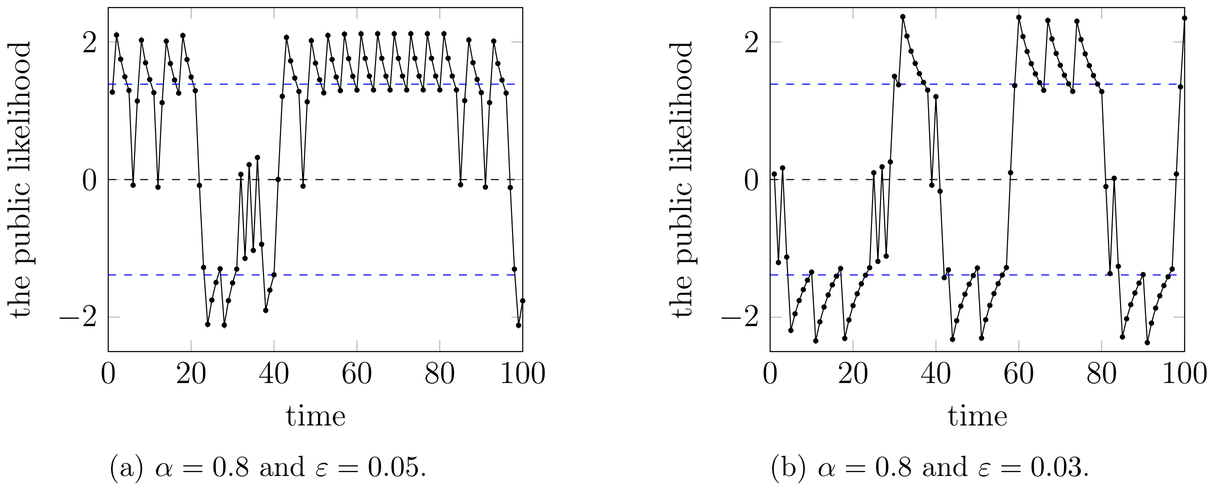


FIGURE 1. Two numerical simulations depict the evolution of the public likelihood in the first 100 periods. The learning region is strictly between the two blue dash lines.

In line with the above discussion, Figure 1 shows that in the cascade region (above or below the blue dash lines), the public likelihood moves towards zero over time and exits the cascade region after some finite number of periods. In the learning region (between the two blue dash lines), the public likelihood changes its sign with each opposing signal, and thus, with consecutively opposing signals, the public likelihood oscillates around zero. By comparing Figure 1a with 1b, it seems that the number of periods the public likelihood spends in a cascade may depend on the probability of switching states – when ε decreases from 0.05 to 0.03, the time in which the public likelihood spends in a cascade generally increases from three to six periods. Indeed, we will see in §5 that the maximum length of any cascades not only depends on the probability of switching states but also on the precision of the private signal.

4.3. **The Public Likelihood and Action.** There is a one-to-one mapping between the public likelihood and the agent's action, namely $\text{sign}(l_{t+1}) = a_t$ for all $t > 1$. To see this, consider first the case where l_t is in the learning region. If $a_t = \text{sign}(l_t)$, then it follows from (5) that $\text{sign}(l_{t+1}) = a_t$. If $a_t \neq \text{sign}(l_t)$, then by Lemma 1, we have $\text{sign}(l_{t+1}) = a_t$. Second, if l_t is in the cascade region, there are two possible cases: (i) suppose that t is a time at which the public likelihood enters the cascade region. In this case, $a_{t-1} = \text{sign}(l_t)$ so that l_t can enter the cascade region; and (ii) suppose that the public likelihood entered the cascade region before time t. From (7), we know that the public likelihood tends to zero, and thus l_{t-1} must also be in the cascade region, which implies that $\text{sign}(l_{t-1}) = \text{sign}(l_t)$. Since l_{t-1} is in the cascade region, it follows that $a_{t-1} = \text{sign}(l_{t-1}) = \text{sign}(l_t)$.

4.4. **The Regulated Public Likelihood.** To facilitate our analysis in the next section, we regulate the behavior of the public likelihood by restricting to small ε and focus on up-cascades henceforth as the case of down-cascades is symmetric. This restriction guarantees that upon exiting the cascade region, the public likelihood will land in the learning region where only one up-signal is required to initiate a cascade. We call such a region the *one up-signal away region*, and denote it by the interval $[c_u, c_\alpha)$, where $c_u := f_1^{-1}(c_\alpha)$. Note that for any public likelihood in the rest of the non-negative learning region, it will jump into the one up-signal away region upon receiving an up-signal. In summary, we have the following lemma.

Lemma 2. *For any signal precision $\alpha \in (1/2, 1)$, there exists an upper bound $\bar{\varepsilon}(\alpha)$ to the probability of state change such that for all $0 < \varepsilon \leq \bar{\varepsilon}(\alpha)$,*

(i) when exiting an up-cascade, the public likelihood always lands in the one up-signal away region;

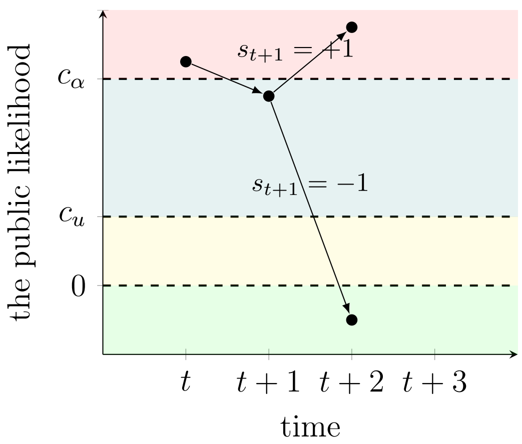
(ii) for any non-negative public likelihood, at most two up-signals are required to initiate a cascade.

Lemma 1 and 2 together imply that for the parameter space of our interest, there are two distinct behavioral patterns of the public likelihood:

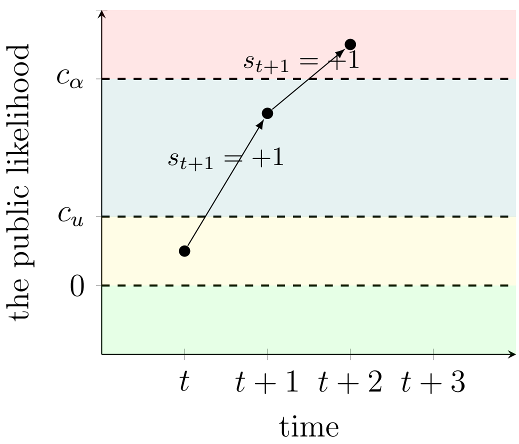
(i) when exiting a cascade, the public likelihood always lands in the one up-signal away region, and in the next period, it either jumps back to the same cascade region or switches its sign;

(ii) for any public likelihood in $[0, c_u)$, it requires two up-signals to initiate a cascade.

We illustrate these behavioral patterns of the public likelihood in Figure 2 below.



(a) Behavioral Pattern (i)



(b) Behavioral Pattern (ii)

FIGURE 2. The region in red, including the boundary at c_α, is the up-cascade region. The region in blue, only including the boundary at c_u, is the one up-signal away region, and for any public likelihood in the yellow region (including zero), it needs two up-signals to initiate a cascade.

5. Expected Time Between State and Action Changes

This section provides a detailed analysis of the comparison between the expected time between state changes and action changes, which is crucial in proving our main result.

We first calculate the expected time between state changes. Since the process $(\theta_t)_t$ is a Markov chain with two states and a symmetric transition probability ε, the expected time between state changes is inversely proportional to the probability of state change, which is equal to $1/\varepsilon$.[15] Intuitively, if the state becomes more likely to change in every period, it takes less time to change on average.

In contrast to the state, the question of how long it takes on average for the action to change is more difficult as the process $(a_t)_t$ is not a Markov chain. Nevertheless, since a_t is a function of l_{t+1} (namely, the sign of l_{t+1}), which is a Markov chain, the expected time between action changes equals that between the sign switches of the public likelihood. However, as we have seen in §4, this Markov chain $(l_t)_t$ is complicated: it has infinitely many states and different transition probabilities between states. Thus, instead of directly calculating the expected time between the sign switches of l_t, we bound it from above and compare this upper bound with the expected time between state changes.

To do so, let us consider the maximum length of any cascade. Based on the dynamics of the public likelihood in §4, we know that the supremum of the public likelihood is equal to $f_1(c_\alpha)$. For any fixed state-changing probability, since no cascade can last longer than the cascade starting at $f_1(c_\alpha)$, one can calculate a tight upper bound to the length of any cascade from (7). We follow Moscarini, Ottaviani, and Smith (1998) and denote this bound by[16]

$$K(\alpha, \varepsilon) = \frac{\log(1-2\alpha(1-\alpha))}{\log|1-2\varepsilon|}.$$

In other words, if $l_t = f_1(c_\alpha)$, then $l_{t+\lfloor K(\alpha,\varepsilon)\rfloor}$ would have just exited the cascade region, so the length of any cascade is less than $\lfloor K(\alpha, \varepsilon)\rfloor$, which is by definition less or equal to $K(\alpha, \varepsilon)$. For instance, when $\alpha = 0.8$ and $\varepsilon = 0.05$, the greatest integer less than or equal to this upper bound is three. As seen in Figure 1a, the duration of any up-cascades never exceeds three periods.

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[15] To see this, suppose that the expected time between state changes is equal to x. As the state follows a two-state Markov chain with symmetric transition probability ε, x satisfies:

$$x = \varepsilon + (1-\varepsilon)(1+x),$$

which implies that $x = 1/\varepsilon$.

[16] For completeness, we provide a similar calculation of this bound to the one in §3.B of Moscarini et al. (1998). Fix α and ε. Let m denote the supremum of the public belief and note that $m = \frac{(1-\varepsilon)\alpha^2+\varepsilon(1-\alpha)^2}{\alpha^2+(1-\alpha)^2}$. After h periods in a cascade, by (6), the public belief starting at m becomes

$$g(h) := \varepsilon \sum_{i=1}^{h-1} (1-2\varepsilon)^i + (1-2\varepsilon)^h m.$$

Since m is the supremum, any public belief after h periods in a cascade would have a value that is strictly less than $g(h)$. Thus, whenever $g(h) \leq \alpha$, or equivalently, $(1-2\varepsilon)^{h+1} \leq 1-2\alpha(1-\alpha)$, the public likelihood would have exited the cascade. Hence, the maximum length of any cascade is $K(\alpha, \varepsilon) = \frac{\log(1-2\alpha(1-\alpha))}{\log|1-2\varepsilon|}$.

Given this tight upper bound to the length of any cascade and the regulated public likelihood, we next establish an upper bound to the expected time between the sign switches of the public likelihood. For all $i = 1, 2, \ldots$, we denote the random time at which the public likelihood switches its sign for the i-th time by $\mathcal{T}_i$ and let $\mathcal{T}_0 = 0$. Denote the random time elapsed between the $i-1$ and i-th sign switch by $\mathcal{D}_i := \mathcal{T}_i - \mathcal{T}_{i-1}$.

Proposition 1. *For any signal precision $\alpha \in (1/2, 1)$, probability of state change $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$ and positive integers $i \geq 2$, conditional on the public likelihood that just had its $i-1$-th sign switch, the expected time to the next sign switch is strictly bounded above by $M(\alpha, \varepsilon) := 1 + \frac{K(\alpha,\varepsilon)}{2\alpha(1-\alpha)}$, which in turn is strictly less than the expected time between state switches, i.e.,*

$$E[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < M(\alpha, \varepsilon) < 1/\varepsilon.$$

Proposition 1 implies that the expected time between the sign switches of the public likelihood is less than that between state changes. Since there is a one-to-one mapping between the sign of the public likelihood and the agent's action, this implies that, on average, actions take less time to switch than the state. Note that $M(\alpha, \varepsilon)$ is decreasing in α, and thus $M(1/2, \varepsilon)$ is the greatest upper bound to the expected time between action switches. Intuitively, as the signal becomes less informative, agents rely more on public information than private information, and as a result, information cascades are more likely to arise, and so is action inertia.

We thus illustrate the proof idea of Proposition 1 using a weakly informative signal, that is, $\alpha = 1/2 + \delta$ where δ is strictly positive and close to 0. Denote the maximum length of any cascades as δ approaches zero by $K(\frac{1}{2}, \varepsilon)$. In this case, upon exiting a cascade, the probability of the public likelihood switching its sign is about $1/2$ as it is almost equally likely that the agent, who follows her private signal, will receive an up or down signal. Thus, we can bound the expected time between the sign switches from above by

$$1 + \sum_{i=1}^{\infty} \frac{i}{2^i} K(\frac{1}{2}, \varepsilon) = 1 + 2K(\frac{1}{2}, \varepsilon) = M(1/2, \varepsilon),$$

which turns out to be strictly less than $1/\varepsilon$ for all $\varepsilon \in (0, \bar{\varepsilon}(1/2 + \delta)]$.

Note that our restriction to small ε is not the driving force behind Proposition 1. We argue that even for small ε, it remains unclear whether on average, actions or the state take less time to change. When ε is smaller, the state changes more slowly, but actions also change more slowly because past actions become more informative about the current state, resulting in longer cascades. For example, when ε decreases from 0.05 to 0.03 in Figure 1, the maximum length of information cascades increases from three to six periods and more generally, $\partial K(\alpha, \varepsilon)/\partial \varepsilon < 0$ for $\varepsilon \in (0, 1/2)$.[17]

[17]In fact, for $\varepsilon \geq \alpha(1-\alpha)$, information cascades never arise (see Proposition 2 in Moscarini, Ottaviani, and Smith (1998)) and note that in Lemma 2, the upper bound $\bar{\varepsilon}(\alpha) < \alpha(1-\alpha)$ for all $\alpha \in (1/2, 1)$, so temporary information cascades arise for $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$.

Finally, we provide a lemma that allows us to have well-defined moments when working with the process $(\mathcal{D}_i)_i$ – a collection of the random time elapsed between the sign switches of the public likelihood.

Lemma 3. *Fix any signal precision* $\alpha \in (1/2, 1)$ *and probability of state change* $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$. *Then, for every* $r \in \{1, 2, \dots\}$ *there is a constant* c_r *that depends on* α *and* ε *such that for all* i, $\mathbb{E}[|\mathcal{D}_i|^r] < c_r$. *I.e., each moment of* $\mathcal{D}_i$ *is uniformly bounded, independently of* i.

6. Conclusion

We study the long-term behavior of agents who receive a private signal and observe the past actions of their predecessors in a changing environment. As the state evolves, agents adjust accordingly so that their actions fluctuate over time. We show that in the long run, the relative frequency of action changes is higher than that of state changes, suggesting fads can emerge from social learning in a changing environment.

One can ask what happens if a single long-lived agent receives a private signal about a changing state in every period. We conjecture that in this case, the agent's actions would change less often than the state, in contrast to our main result. If this were the case, such a stark contrast would highlight the importance of observational learning in accelerating action fluctuations, especially when the underlying environment is slowly evolving.

There are a number of possible avenues for future research. Recall that Proposition 1 implies that $M(\alpha, \varepsilon)$ is an upper bound to the expected time between action changes. One could ask whether this upper bound $M(\alpha, \varepsilon)$ is tight, and if so, for any given finite time N, the number of action changes would be close to $N/M(\alpha, \varepsilon)$. Based on the simulation results, we conjecture that it is not a tight bound. E.g., we let $\alpha = 0.8$ and $\varepsilon = 0.01$, and $N = 100,000$. Since $M(0.8, 0.01) \approx 61$, it implies that within these ten thousand periods, the action should at least change about $1,640$ times. However, our numerical simulation shows that the action changes about $3,500$ times, which is more than double the number suggested by $M(0.8, 0.01)$.

Furthermore, our simulations suggest that as the private signal becomes less informative ($\alpha \to 1/2$) and the state changes more slowly ($\varepsilon \to 0$), the ratio between the frequency of action changes and state changes approaches a constant that is close to 4. This suggests that it might be possible to achieve a very accurate understanding of fads in this regime.

Appendix A. Proofs

Proof of Lemma 1. Fix $\alpha \in (1/2, 1)$ and $\varepsilon \in (0, 1/2)$. Denote an upward step size by $\Delta_1(l_t) := f_1(l_t) - l_t > 0$ and similarly, a downward step size by $\Delta_0(l_t) := l_t - f_0(l_t) > 0$. We show that the upward step size $\Delta_1(l_t)$ decreases in l_t and the downward step size

$\Delta_0(l_t)$ increases in l_t, and in particular,

$$\sup_{|l|<c_\alpha} \Delta_1(l) = \Delta_1(-c_\alpha) = c_\alpha = \Delta_0(c_\alpha) = \sup_{|l|<c_\alpha} \Delta_0(l).$$

To see this, note that by symmetry, $\Delta_0(l_t) = \Delta_1(-l_t)$, and in particular, $\Delta_0(c_\alpha) = \Delta_1(-c_\alpha) = c_\alpha$. Thus, it suffices to show that $\Delta_1(l_t)$ decreases in l_t. Let $l_t \in (-c_\alpha, c_\alpha)$ and suppose that $s_t = +1$. To show that $\Delta_1(l_t)$ decreases in l_t, it is equivalent to show that $c_\alpha - \Delta_1(l_t)$ increases in l_t. By (5),

$$\begin{aligned}
c_\alpha - \Delta_1(l_t) &= c_\alpha - (f_1(l_t) - l_t) \\
&= c_\alpha - (\log \frac{(1-\varepsilon)\alpha e^{l_t} + \varepsilon(1-\alpha)}{\varepsilon\alpha e^{l_t} + (1-\varepsilon)(1-\alpha)} - l_t) \\
&= \log \frac{\varepsilon e^{l_t + c_\alpha} + 1 - \varepsilon}{1 - \varepsilon + \frac{\varepsilon}{e^{l_t + c_\alpha}}}.
\end{aligned}$$

Since $l_t \in (-c_\alpha, c_\alpha)$, $l_t + c_\alpha > 0$, and since $\log(\cdot)$ is a strictly increasing function, $\partial(c_\alpha - \Delta_1(l_t))/\partial l_t > 0$. Thus, it follows that $\sup_{|l|<c_\alpha} \Delta_1(l) = \Delta_1(-c_\alpha) = c_\alpha = \Delta_0(c_\alpha) = \sup_{|l|<c_\alpha} \Delta_0(l)$.

Finally, since $\Delta_0(c_\alpha) = c_\alpha - f_0(c_\alpha) = c_\alpha$, $f_0(c_\alpha) = 0$ and thus for all $0 < l_t < c_\alpha$, $f_0(l_t) < 0$ as $f_0(\cdot)$ is strictly increasing. By an analogous argument, for all $-c_\alpha < l_t < 0$, $f_1(l_t) > 0$. Since agents follow their signals in the learning region, for all $|l_t| < c_\alpha$, one opposing action $a_t = s_t \neq \operatorname{sign}(l_t)$ implies that $\operatorname{sign}(l_{t+1}) = a_t$. $\square$

Proof of Lemma 2. Fix $\alpha \in (1/2, 1)$ and let $\bar{\varepsilon}(\alpha) := \frac{1}{2}(1+2\alpha-2\alpha^2-\sqrt{1+4\alpha^2-8\alpha^3+4\alpha^4})$.

(i) To see how we obtain this upper bound $\bar{\varepsilon}(\alpha)$, note that from (6), $\partial q_{t+1}/\partial\varepsilon < 0$ for any $q_t > 1/2$ and $\partial q_{t+1}/\partial q_t = 1 - 2\varepsilon > 0$ for all $\varepsilon \in (0, 1/2)$. It follows from (7) that for any fixed $l_t > 0$ and $\varepsilon \in (0, 1/2)$, $\partial l_{t+1}/\partial\varepsilon < 0$ and $\partial l_{t+1}/\partial l_t > 0$ since $\log(\cdot)$ is a strictly increasing function. Thus, there exists a maximum of ε such that for any $l_t \geq c_\alpha$, $c_u \leq l_{t+1} < c_\alpha$, and we obtain $\bar{\varepsilon}(\alpha)$ by substituting $l_t = c_\alpha$ and $l_{t+1} = c_u$ in (7). Furthermore, for all $0 < \varepsilon < \alpha(1-\alpha)$,

$$c_u := f^{-1}(c_\alpha) = \log \frac{(1-\alpha)(\alpha-\varepsilon)}{\alpha(1-\alpha-\varepsilon)} \in (0, c_\alpha).$$

Since $\bar{\varepsilon}(\alpha) < \alpha(1-\alpha)$ for all $\alpha \in (1/2, 1)$, we have $0 < c_u < c_\alpha$, and so the one up-signal away region is well-defined.

(ii) Since $f_1(\cdot)$ is strictly increasing, it suffices to show that $f_1(f_1(0)) \geq c_\alpha$. Note that for all $0 < \varepsilon < \alpha(1-\alpha)$, $f_1(0) > c_u$, and thus it also holds for $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$ since $\bar{\varepsilon}(\alpha) < \alpha(1-\alpha)$. It follows from the definition of c_u that $f_1(f_1(0)) > f_1(c_u) = f_1(f^{-1}(c_\alpha)) = c_\alpha$.

$\square$

Proof of Proposition 1. Fix $\alpha \in (1/2, 1)$, $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$ and some $i \geq 2$. Denote the greatest integer than is less than or equal to $K(\alpha, \varepsilon)$ by K. We will first provide an upper bound to $\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}]$ and show that this upper bound is less than $1/\varepsilon$.

Suppose first that $l_{\mathcal{T}_{i-1}} \geq 0$. Denote the probability of receiving an up-signal conditional on the public likelihood being l by $\pi(l)$. By Bayes' rule and the fact that the signal is independent of q conditional on the (current) state,

$$\begin{aligned}\pi(l) &= \mathbb{P}[s = +1 | q = \frac{\mathrm{e}^l}{1+\mathrm{e}^l}] \\ &= \sum_{i \in \{-1,+1\}} \mathbb{P}[s = +1 | q = \frac{\mathrm{e}^l}{1+\mathrm{e}^l}, \theta = i] \cdot \mathbb{P}[\theta = i | q = \frac{\mathrm{e}^l}{1+\mathrm{e}^l}] \\ &= \alpha \frac{\mathrm{e}^l}{1+\mathrm{e}^l} + (1-\alpha)\frac{1}{1+\mathrm{e}^l} = \frac{1+\alpha(\mathrm{e}^l - 1)}{1+\mathrm{e}^l}.\end{aligned}$$

Since $\partial \pi(l)/\partial l = \frac{\mathrm{e}^l(2\alpha-1)}{(1+\mathrm{e}^l)^2} > 0$, the probability of receiving an up-signal increases as the public likelihood increases. Thus, denote the supremum of $\pi(l)$ over all non-negative public likelihood in the learning region by

$$\bar{\pi} := \sup_{l \in [0, c_\alpha)} \pi(l) = \pi(c_\alpha) = 1 - 2\alpha(1-\alpha). \tag{8}$$

Let $\kappa(l)$ denote the length of the cascade initiated by l and let $\mathcal{L}(l)$ denote the new value of l after it first exits the cascade region. There are three possible cases for any non-negative $l_{\mathcal{T}_{i-1}}$.

Case (i). The one up-signal away region where $l_{\mathcal{T}_{i-1}} \in [c_u, c_\alpha)$. By part (i) of Lemma 2, upon exiting the cascade region, $\mathcal{L}(l_{\mathcal{T}_{i-1}}) \in [c_u, c_\alpha)$ since $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$. Thus, for all $c_u \leq l_{\mathcal{T}_{i-1}} < c_\alpha$, it follows from Lemma 1 that

$$\begin{aligned}\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] &= 1 - \pi(l_{\mathcal{T}_{i-1}}) + \pi(l_{\mathcal{T}_{i-1}})\Big(\kappa(l_{\mathcal{T}_{i-1}}) + \mathbb{E}[\mathcal{D}_i | \mathcal{L}(l_{\mathcal{T}_{i-1}})]\Big) \\ &< 1 - \bar{\pi} + \bar{\pi}\Big(\kappa(l_{\mathcal{T}_{i-1}}) + \mathbb{E}[\mathcal{D}_i | \mathcal{L}(l_{\mathcal{T}_{i-1}})]\Big) \\ &\leq 1 - \bar{\pi} + \bar{\pi}\Big(K + \mathbb{E}[\mathcal{D}_i | \mathcal{L}(l_{\mathcal{T}_{i-1}})]\Big),\end{aligned}$$

where the second strict inequality follows from (8) and the last inequality follows from the definition of K. Taking the supremum on both sides and rearranging,

$$\sup_{c_u \leq l_{\mathcal{T}_{i-1}} < c_\alpha} \mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] \leq 1 + \frac{K\bar{\pi}}{1-\bar{\pi}}. \tag{9}$$

Case (ii). The up-cascade region where $l_{\mathcal{T}_{i-1}} \geq c_\alpha$. After at most K steps, the public likelihood starting at $l_{\mathcal{T}_{i-1}}$ would have exited the cascade region, and by part (i) of Lemma 2 again, $\mathcal{L}(l_{\mathcal{T}_{i-1}}) \in [c_u, c_\alpha)$. Hence,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] \leq K + \mathbb{E}[\mathcal{D}_i | \mathcal{L}(l_{\mathcal{T}_{i-1}})].$$

It then follows from (9) that for all $l_{\mathcal{T}_{i-1}} \geq c_\alpha$,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < K + 1 + \frac{K\bar{\pi}}{1-\bar{\pi}} = 1 + \frac{K}{1-\bar{\pi}}. \tag{10}$$

Case (iii). The rest of the non-negative learning region where $l_{\mathcal{T}_{i-1}} \in [0, c_u)$. As $\pi(l)$ increases in l, it follows from Lemma 1 and (8) that for all $0 \leq l_{\mathcal{T}_{i-1}} < c_u$,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < (1-\bar{\pi}) + \bar{\pi}(1 + \mathbb{E}[\mathcal{D}_i | f_1(l_{\mathcal{T}_{i-1}})]).$$

By part (ii) of Lemma 2, for any l in $[0, c_u)$, $f_1(l) \in [c_u, c_\alpha)$. Thus, it follows from (9) that for all $0 \leq l_{\mathcal{T}_{i-1}} < c_u$,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < (1-\bar{\pi}) + \bar{\pi}(1 + 1 + \frac{K\bar{\pi}}{1-\bar{\pi}}) = \frac{(K-1)(\bar{\pi})^2 + 1}{1-\bar{\pi}}. \tag{11}$$

The maximum of all three upper bounds from (9) to (11) is

$$\max\{1 + \frac{\bar{\pi}K}{1-\bar{\pi}}, \frac{(K-1)(\bar{\pi})^2 + 1}{1-\bar{\pi}}, 1 + \frac{K}{1-\bar{\pi}}\} = 1 + \frac{K}{1-\bar{\pi}}.$$

Thus, for any $l_{\mathcal{T}_{i-1}} \geq 0$,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < 1 + \frac{K}{1-\bar{\pi}} \leq 1 + \frac{K(\alpha, \varepsilon)}{2\alpha(1-\alpha)},$$

where the second inequality follows from the definition of K that $K \leq K(\alpha, \varepsilon)$. The case where $l_{\mathcal{T}_{i-1}} \leq 0$ follows by a symmetric argument. Hence, for all $l_{\mathcal{T}_{i-1}}$,

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < 1 + \frac{K(\alpha, \varepsilon)}{2\alpha(1-\alpha)} := M(\alpha, \varepsilon).$$

Since $M(\alpha, \varepsilon)$ is decreasing in α,

$$\limsup_{\alpha \to 1/2} M(\alpha, \varepsilon) = 1 + \frac{2 \log 2}{-\log |1 - 2\varepsilon|}.$$

Moreover, for any $\varepsilon \in (0, 1/2)$,

$$\limsup_{\alpha \to 1/2} M(\alpha, \varepsilon) < 1/\varepsilon,$$

since

$$2 \log 2 < 2 = \liminf_{\varepsilon \to 0} -(\frac{1}{\varepsilon} - 1) \log |1 - 2\varepsilon|.$$

Thus, for any $\alpha \in (1/2, 1)$, $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$ and $l_{\mathcal{T}_{i-1}}$, we have

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < M(\alpha, \varepsilon) < 1/\varepsilon.$$

$\square$

Proof of Lemma 3. Fix $\alpha \in (1/2, 1)$, $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$ and some $i = 1, 2, \ldots$. Recall that $\mathcal{D}_i$ is the random time elapsed between the $i-1$ and i-th sign switches of the public likelihood. By Lemma 1 and part (ii) of Lemma 2, we know that one opposing signal is sufficient to overturn the sign of the public likelihood and two consecutive signals of the same sign would lead it into the corresponding cascade region. For any positive integer $n \geq 2$, denote by

$$k(n) := \frac{n-2}{\lfloor K(\alpha, \varepsilon) \rfloor}$$

the minimum number (may not be an integer) of cascades required for $\mathcal{D}_i > n$. Hence, for any $n \geq 2$, we can bound the probability that $\mathcal{D}_i > n$ by

$$\mathbb{P}[\mathcal{D}_i > n] < \bar{\pi}^{2+(\lfloor k(n) \rfloor - 1)},$$

where $\bar{\pi} \in (1/2, 1)$ is the maximum probability of receiving a private signal that has the same sign as the public likelihood (Recall that we define $\bar{\pi}$ in (8) in the proof of Proposition 1, and we write it as $\bar{\pi}$ here for the ease of notation). Since $\mathcal{D}_i$ is a positive random variable, it then follows that for any $p > 0$,

$$\begin{aligned} \lim_{n\to\infty} n^p \mathbb{P}[|\mathcal{D}_i| > n] &= \lim_{n\to\infty} \frac{n^p}{1/\mathbb{P}[\mathcal{D}_i > n]} \\ &< \lim_{n\to\infty} \frac{n^p}{(1/\bar{\pi})^{1+\lfloor k(n) \rfloor}} = 0. \end{aligned} \tag{12}$$

For any $r \geq 1$, the r-th moment of $|\mathcal{D}_i|$ satisfies

$$\begin{aligned} \mathbb{E}[|\mathcal{D}_i|^r] &= \int_0^\infty \mathbb{P}[|\mathcal{D}_i|^r > t] dt \\ &< 1 + \int_1^\infty \mathbb{P}[\mathcal{D}_i > y] r y^{r-1} dy \\ &= 1 + \sum_{n=1}^{\infty} \int_n^{n+1} \mathbb{P}[\mathcal{D}_i > y] r y^{r-1} dy \\ &< 1 + \sum_{n=1}^{\infty} \mathbb{P}[\mathcal{D}_i > n] r (n+1)^{r-1}, \end{aligned}$$

where the second inequality follows from a change of variable $y = t^{1/r}$. Since (12) implies that $\mathbb{P}[\mathcal{D}_i > n] < Cn^{-p}$ for some nonnegative constant C, it follows that for any $p > r$,

$$\begin{aligned} \mathbb{E}[|\mathcal{D}_i|^r] &< 1 + rC \sum_{n=1}^{\infty} \frac{(n+1)^{r-1}}{n^p} \\ &< 1 + r2^{r-1} C \sum_{n=1}^{\infty} \frac{1}{n^{p-r+1}} < \infty, \end{aligned}$$

which holds for all i. Hence, for every $r \in \{1, 2, \ldots\}$, there exists a constant $c_r = 1 + r2^{r-1} C \sum_{n=1}^{\infty} \frac{1}{n^{p-r+1}}$, independently of i, that uniformly bounds $\mathbb{E}[|\mathcal{D}_i|^r]$. $\square$

Proof of Theorem 1. Fix $\alpha \in (1/2, 1)$ and $\varepsilon \in (0, \bar{\varepsilon}(\alpha)]$. Let S_t denote the event that the state switches at time t: $S_t = \{\theta_{t-1} \neq \theta_t\}$. Since the process $(\theta_t)_t$ follows a Markov chain with two states and a symmetric transition probability ε, $(\mathbb{1}(S_2), \mathbb{1}(S_3), \ldots)$ is a sequence of i.i.d. random variables. By the strong law of large numbers,

$$\lim_{n\to\infty} \mathcal{Q}_\theta(n) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=2}^{n+1} \mathbb{1}(S_t) = \mathbb{E}[\mathbb{1}(S_t)] = \mathbb{P}[\theta_{t-1} \neq \theta_t] = \varepsilon \quad \text{a.s.}$$

Let E_t denote the event that the action changes at time t: $E_t = \{a_{t-1} \neq a_t\}$. For all $t > 1$, since $a_t = \text{sign}(l_{t+1})$, $E_t = \{\text{sign}(l_{t+1}) \neq \text{sign}(l_t)\}$. Recall that $\mathcal{T}_i$ is the random time at which the public likelihood switches for the i-th time and $\mathcal{D}_i = \mathcal{T}_i - \mathcal{T}_{i-1}$ is the time elapsed between the $i-1$ and i-th sign switch.

Let $\Phi = (\mathcal{F}_1, \mathcal{F}_2, \ldots)$ be the filtration where each $\mathcal{F}_i = \sigma(\mathcal{D}_1, \ldots, \mathcal{D}_i)$ and thus $\mathcal{F}_j \subseteq \mathcal{F}_i$ for any $j \leq i$. Hence, the sequence of random variables $(\mathcal{D}_1, \mathcal{D}_2, \ldots)$ is adapted to Φ so that each $\mathcal{D}_i$ is $\mathcal{F}_i$-measurable. By Proposition 1, there exists $\delta = 1/\varepsilon - M(\alpha, \varepsilon) > 0$ such that

$$\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}] < 1/\varepsilon - \delta,$$

and thus by the law of total expectation and the Markov property of l,

$$\mathbb{E}[\mathcal{D}_i|\mathcal{F}_{i-1}] = \mathbb{E}[\mathbb{E}[\mathcal{D}_i | l_{\mathcal{T}_{i-1}}, \mathcal{F}_{i-1}]|\mathcal{F}_{i-1}] < 1/\varepsilon - \delta. \tag{13}$$

Let $X_i = \mathcal{D}_i - \mathbb{E}[\mathcal{D}_i|\mathcal{F}_{i-1}]$ and denote the partial sum of the process $(X_i)_i$ by

$$S_n = X_1 + \cdots + X_n.$$

By the definition of X_i, $\mathbb{E}[X_i|\mathcal{F}_{i-1}] = 0$. Since each S_{n-1} is $\mathcal{F}_{n-1}$ measurable, it follows that the process $(S_n)_n$ forms a martingale:

$$\mathbb{E}[S_n|\mathcal{F}_{n-1}] = \mathbb{E}[\sum_{i=1}^n X_i|\mathcal{F}_{n-1}] = S_{n-1} + \mathbb{E}[X_n|\mathcal{F}_{n-1}] = S_{n-1}.$$

Note that $\mathbb{E}[X_i^2]$ is uniformly bounded since both $\mathbb{E}[\mathcal{D}_i^2]$ and $\mathbb{E}[\mathcal{D}_i|\mathcal{F}_{i-1}]$ are uniformly bounded by Lemma 3 and (13). Hence, $\sum_{i=1}^\infty \frac{1}{i^2}\mathbb{E}[X_i^2] < \infty$. By the strong law for martingales (See p.238, Theorem 2 in Feller (1966)),[18]

$$\lim_{n\to\infty} \frac{1}{n} S_n = 0 \quad \text{a.s.}$$

By the definition of S_n, we can write

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i}^n \mathcal{D}_i = \lim_{n\to\infty} \frac{1}{n}\sum_{i}^n \mathbb{E}[\mathcal{D}_i|\mathcal{F}_{i-1}] \quad \text{a.s.}$$

It then follows from (13) that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i}^n \mathcal{D}_i \leq 1/\varepsilon - \delta < 1/\varepsilon \quad \text{a.s.}$$

[18] In fact, we do not need to refer to the strong law for martingales. Let $Y_n = \sum_{i=1}^n \frac{1}{i} X_i$. It is easy to see that $(Y_n)_n$ is a martingale and it is bounded in $\mathcal{L}^2$ since $\mathbb{E}[|Y_n|^2] = \sum_{i=1}^n \frac{1}{i^2}\mathbb{E}[X_i^2] < \infty$. By the martingale convergence theorem, Y_n converges a.s. It then follows from Kronecker's Lemma that $\frac{1}{n}S_n \to 0$ a.s.

Hence,

$$\lim_{n\to\infty} \mathcal{Q}_a(n) = \lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \mathbb{1}(E_t) = \lim_{n\to\infty} \frac{n}{\sum_i^n \mathcal{D}_i} > \varepsilon \quad \text{a.s.}$$

□

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