CS-C3100 Computer Graphics

Bézier Curves and Splines

Majority of slides from Frédo Durand

Before We Begin

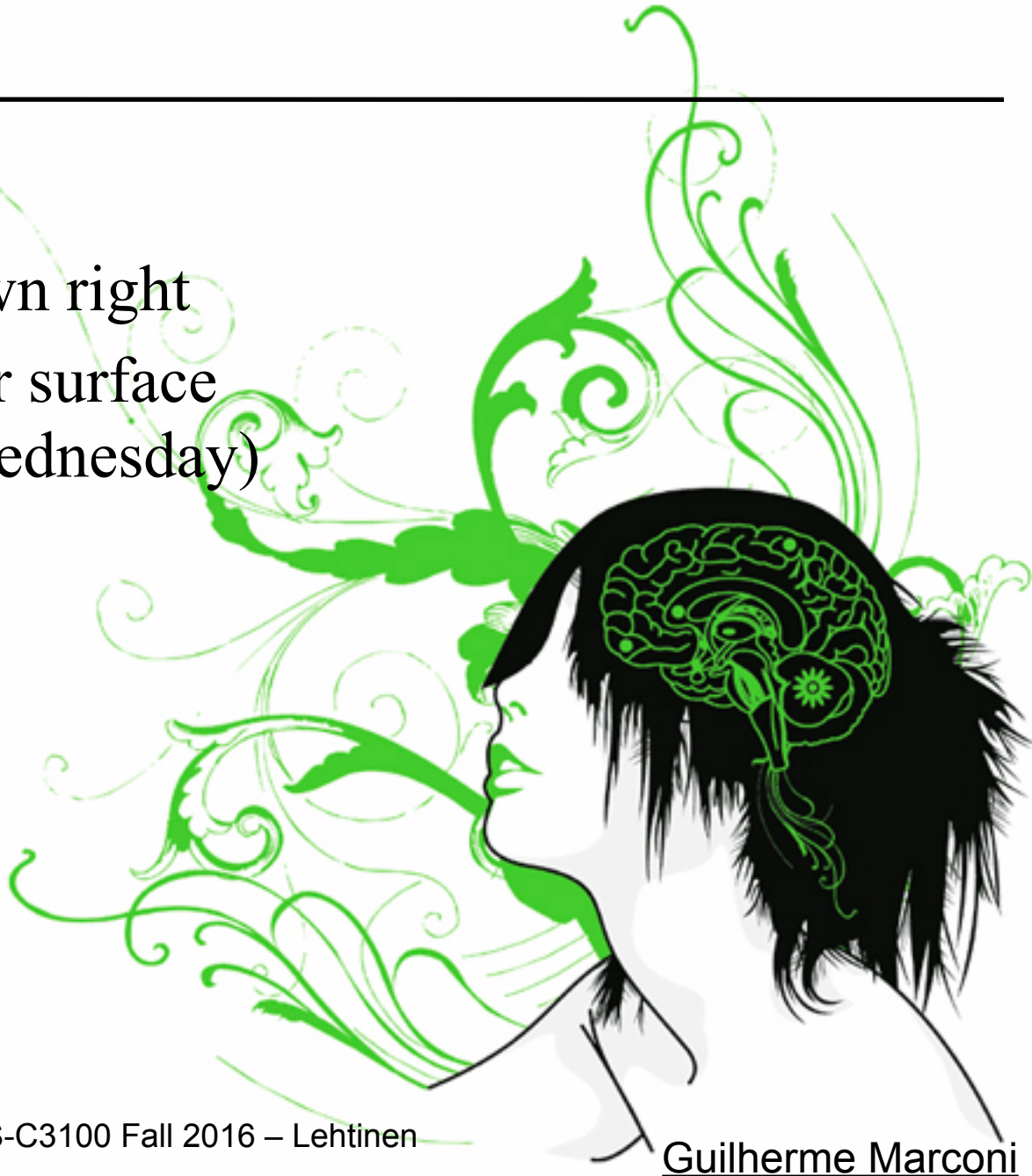
- Anything on your mind concerning Assignment 1?

- Linux makefiles released on MyCourses
  - Unsupported! For you to play with

- Assignment 2 (Curves & Surfaces, stuff covered today!) will be posted before Sunday's deadline

Today

- Curves in 2D
  - Useful in their own right
  - Provides basis for surface modeling (this Wednesday)



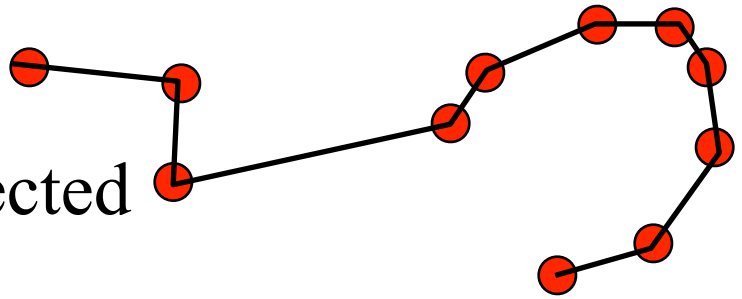
Guilherme Marconi

Modeling Curves in 2D

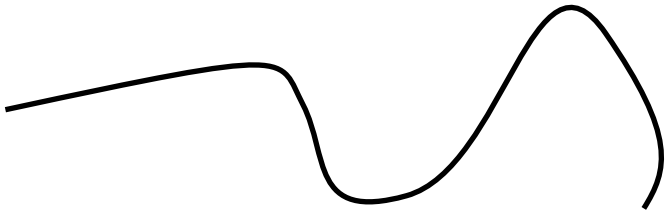
- How?

Modeling Curves in 2D

- Polylines
  - Sequence of vertices connected by straight line segments
  - Useful, but not for smooth curves
  - Very easy!
  - This is the representation that usually gets drawn in the end (smooth curves converted into these)

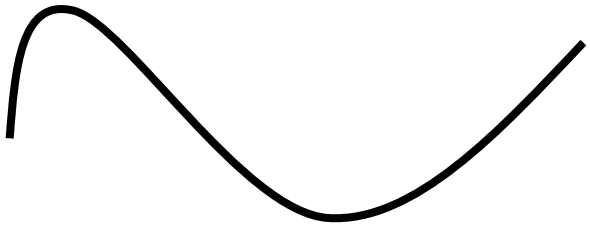


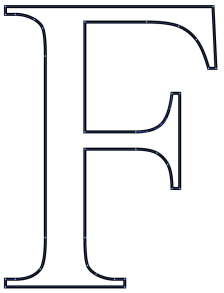
- **Smooth curves**
  - How do we specify them?
  - A little harder (but not too much)

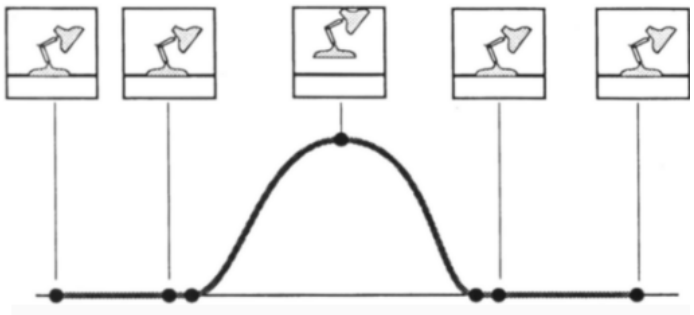


Splines

- A type of smooth curve in the plane or in 3D
- Many uses
  - 2D Illustration (e.g. Adobe Illustrator)
  - Fonts
  - 3D Modeling
  - Animation: Trajectories
- In general: Interpolation and approximation

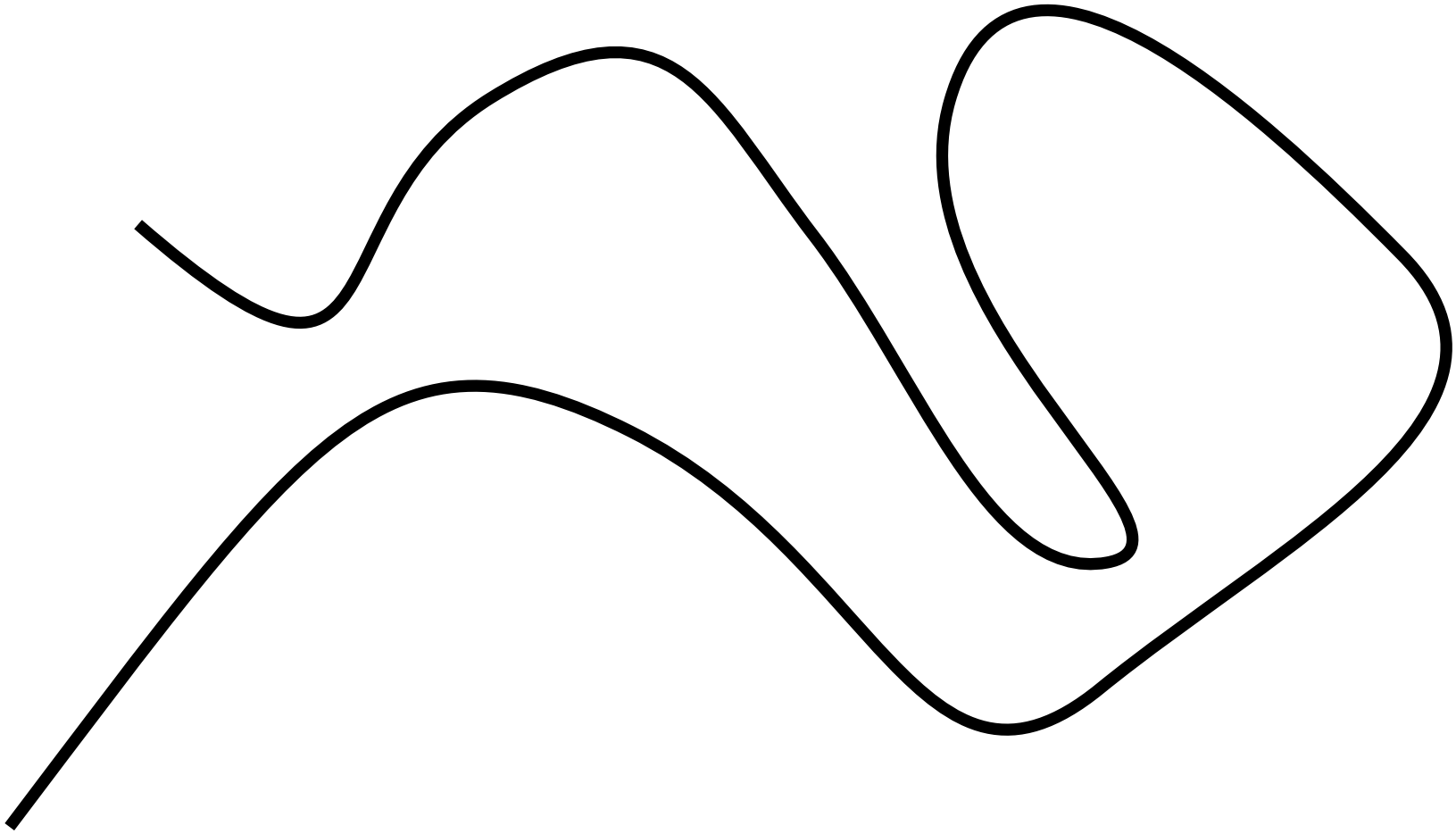




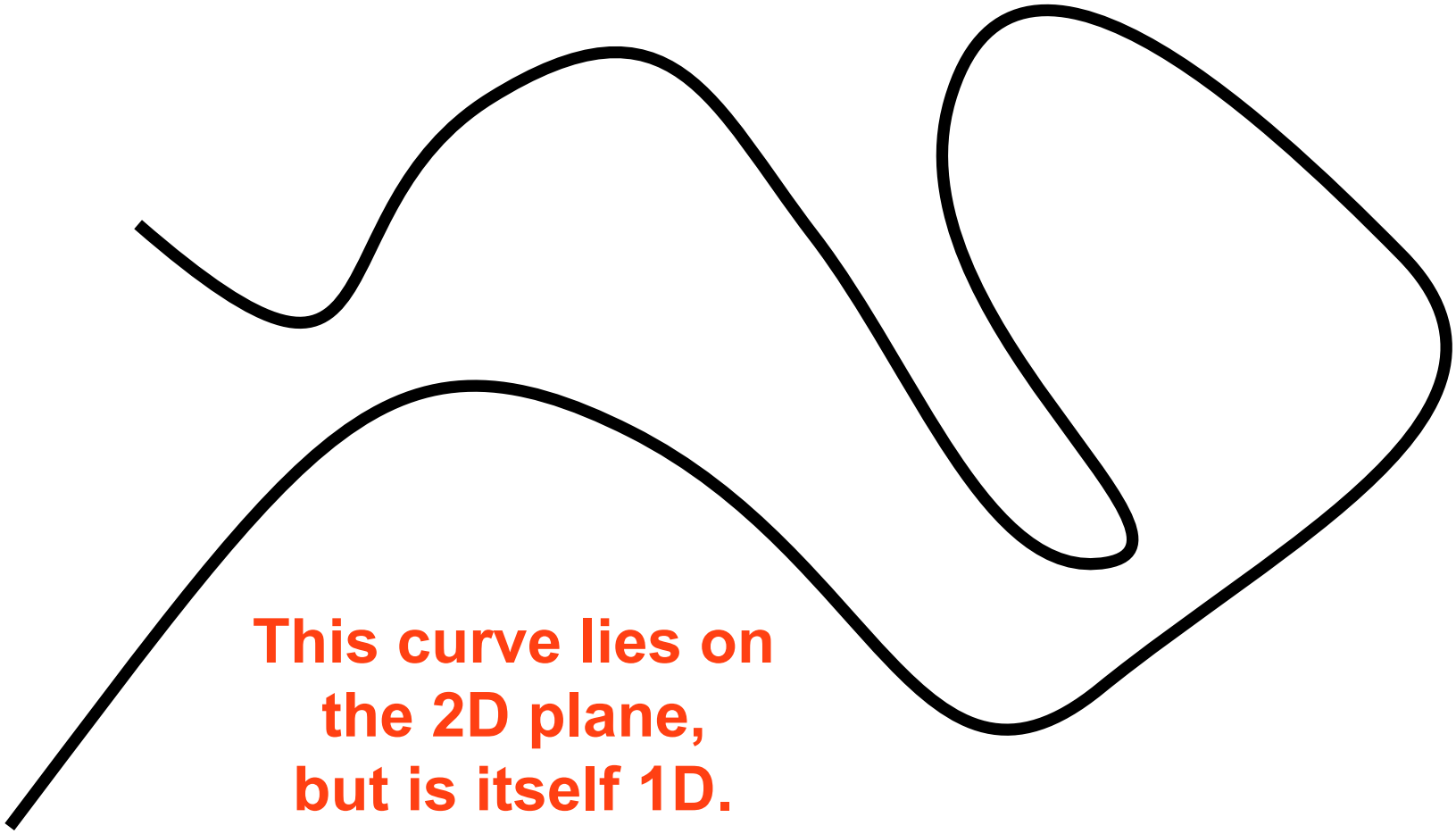


ACM © 1987 "Principles of traditional animation applied to 3D computer animation"

How Many Dimensions?



How Many Dimensions?



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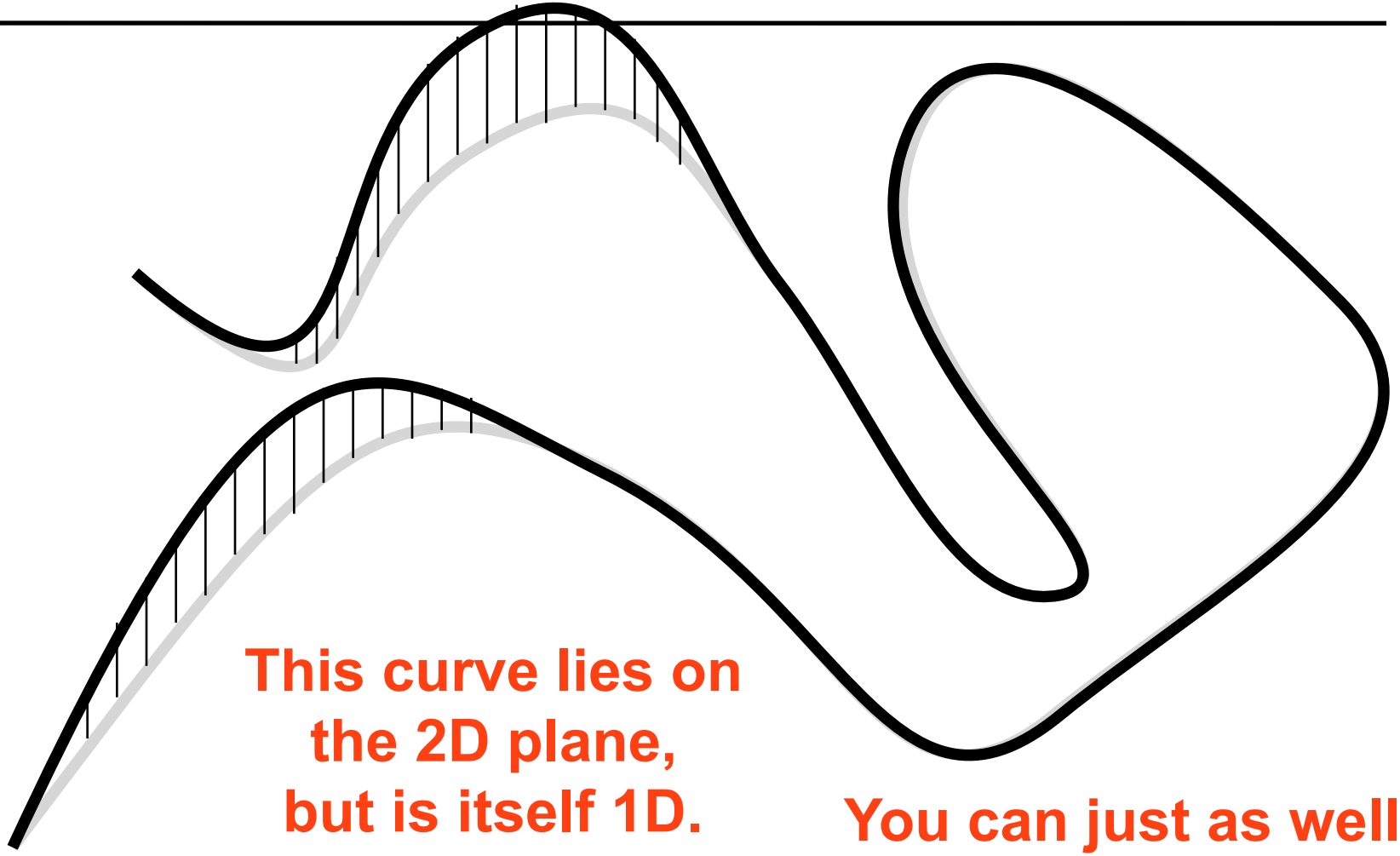

This curve lies on the 2D plane, but is itself 1D.

You can just as well define 1D curves in 3D space.

Two Definitions of a Curve

1. A continuous 1D point set on the plane or space
2. A mapping from an interval S onto the plane
   - That is, P(t) is the point of the curve at parameter t

$$P : \mathbb{R} \ni S \mapsto \mathbb{R}^2, \quad P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

- Big differences
  - It's easy to generate points on the curve from the 2nd
  - The second definition can describe trajectories, the speed at which we move on the curve

Example of the First Definition?

- **A continuous 1D point set on the plane or space**
- A mapping from an interval S onto the plane
  - That is, P(t) is the point of the curve at parameter t

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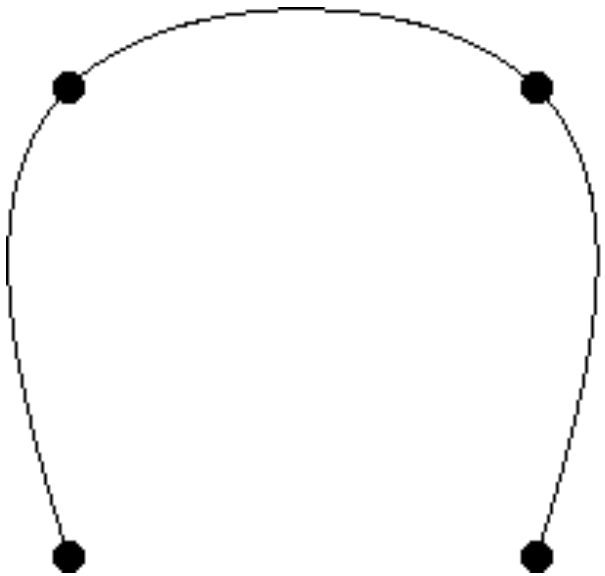
- An "algebraic curve", e.g. intersection of

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad z = 0$$

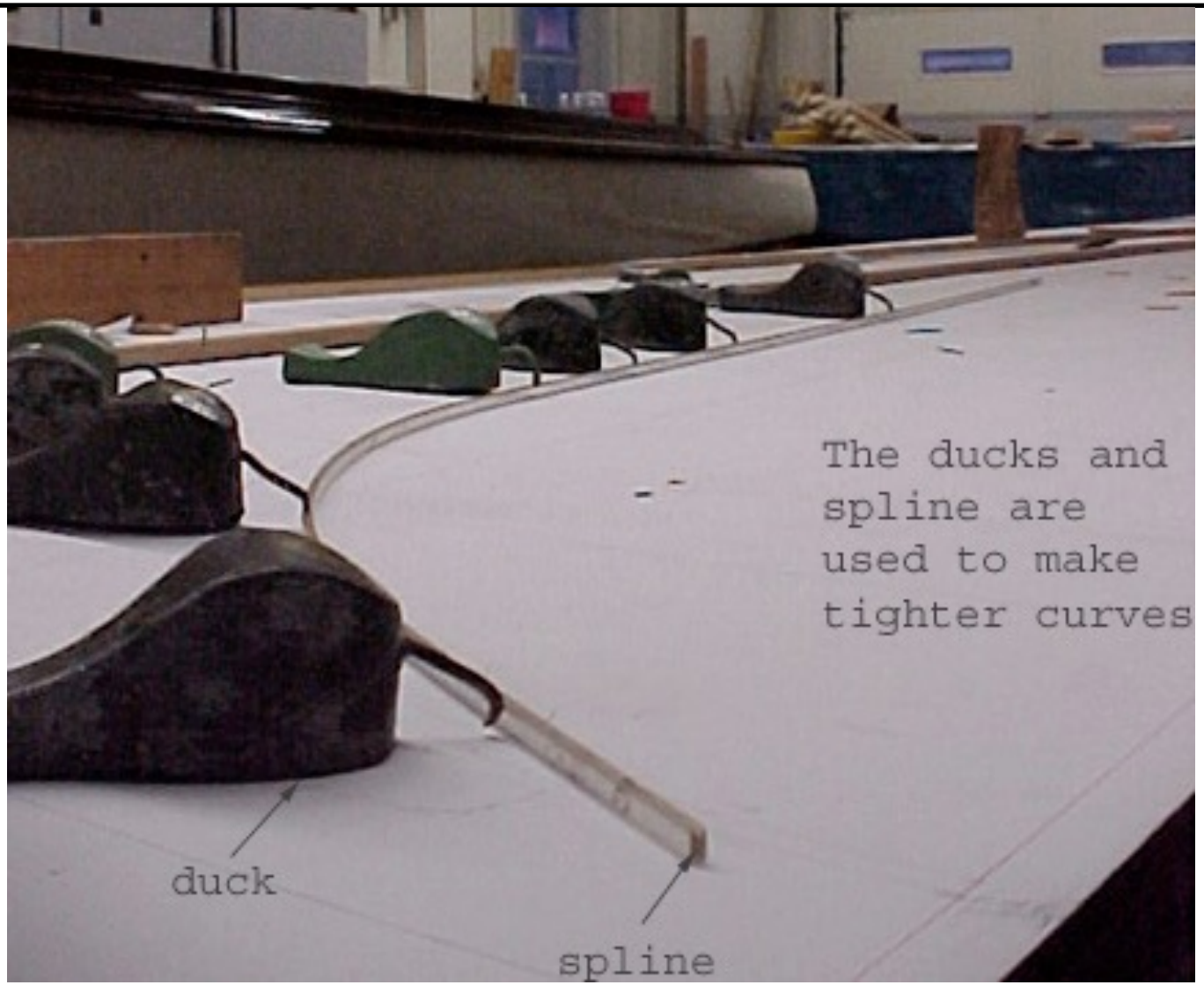
It's a circle on the xy plane :)

General Principle for Modeling

- User specifies "**control points**"
- We'll interpolate the control points by a smooth curve
  - The curve is completely determined by the control points.
  - I.e., we need an unambiguous rule that gives us the curve points given the control points.



Physical Splines

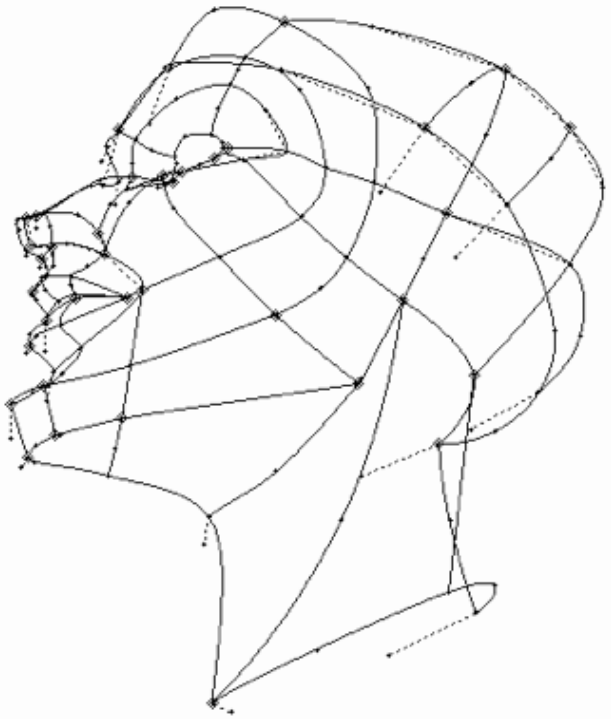


www.abm.org

See http://en.wikipedia.org/wiki/Flat_spline

Two Points of View

- Approximation/interpolation
  - We have “data points”, how can we interpolate?
  - Important in lots of applications, both graphics and non-graphics
- User interface/modeling
  - What is an easy way to specify a smooth curve?
  - Our main perspective today.



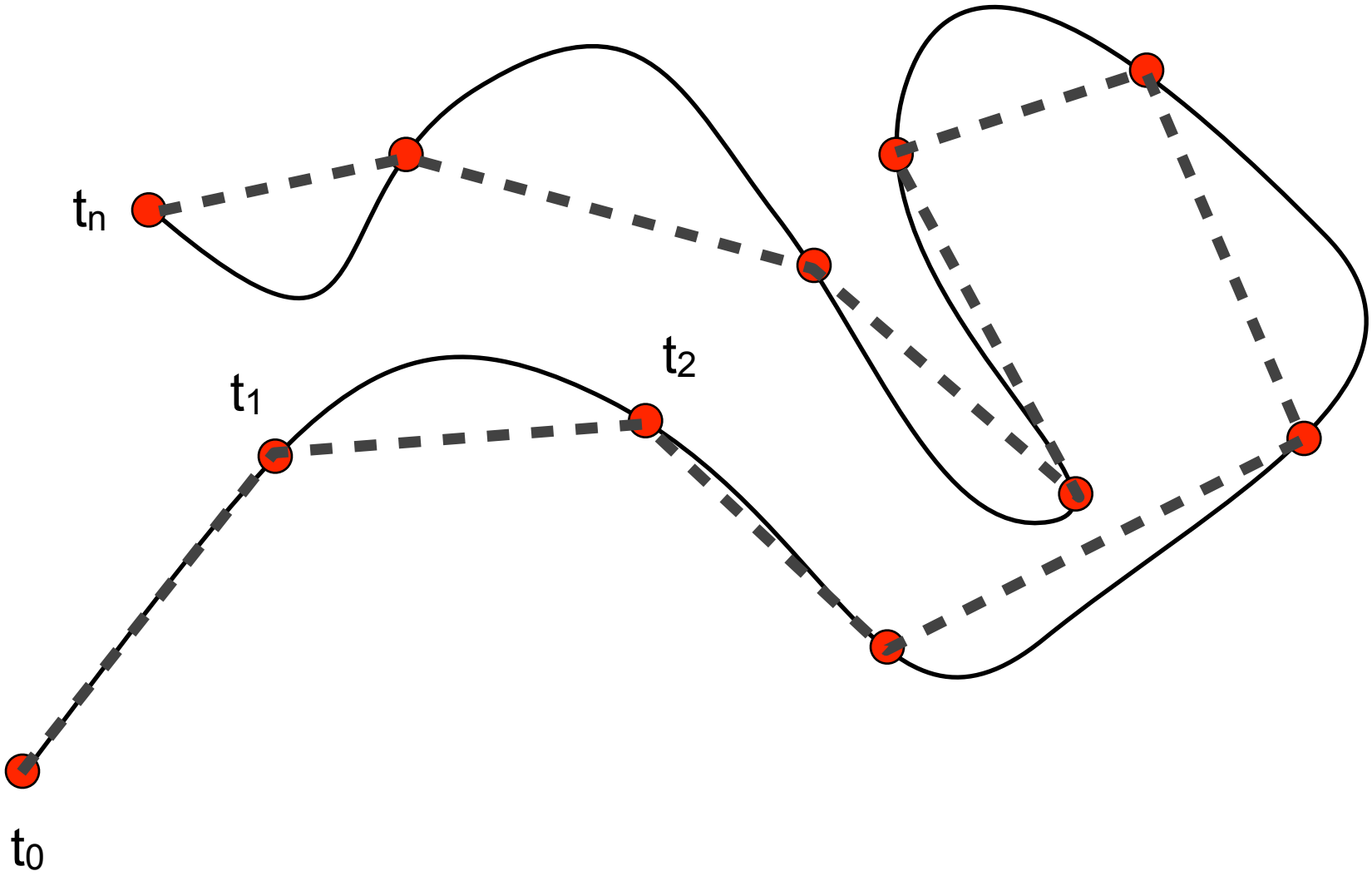
celshader.com

Splines

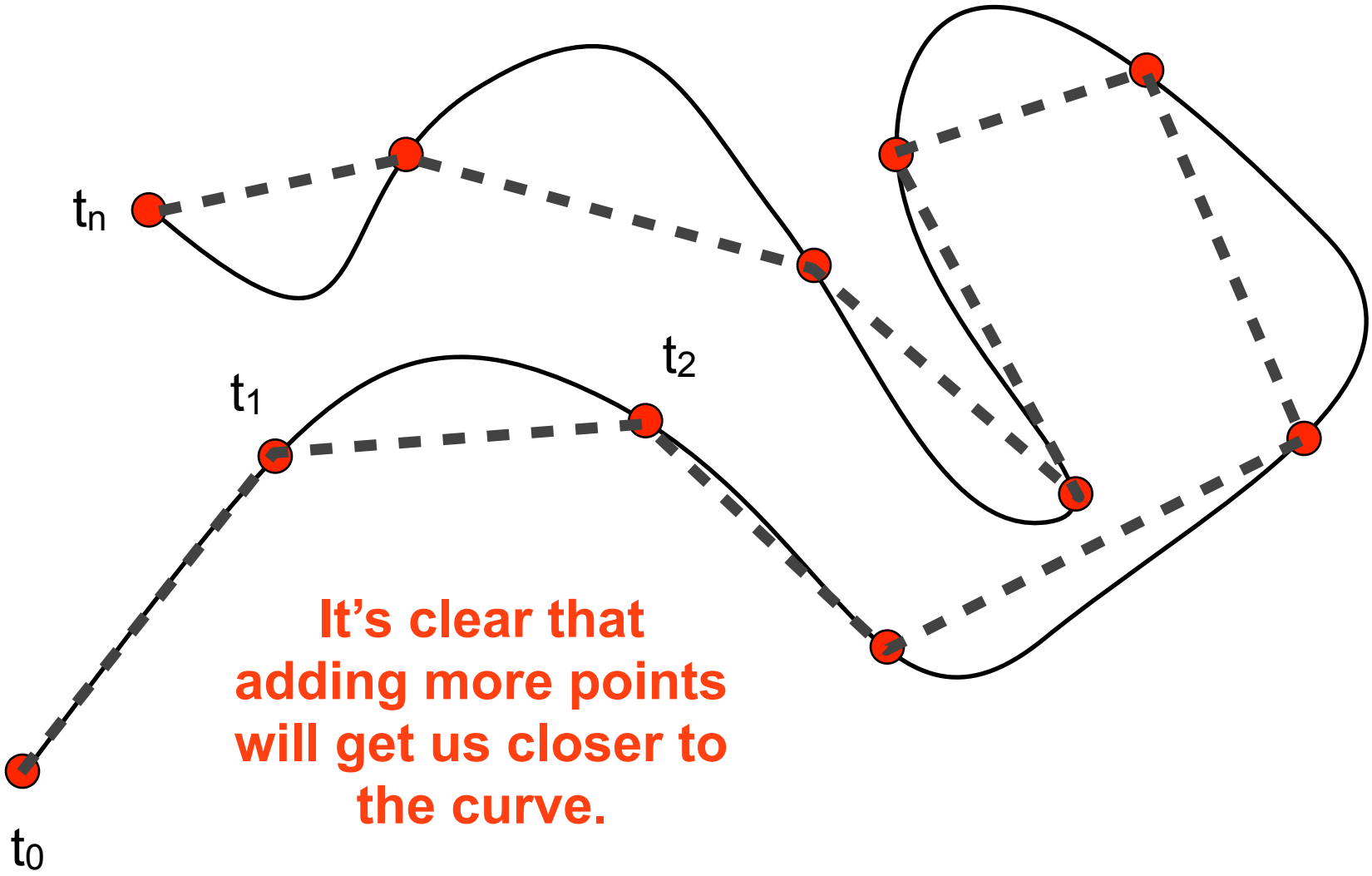
- Specified by a few control points
  - Good for UI
  - Good for storage

- Results in a smooth parametric curve P(t)
  - Just means that we specify $x(t)$ and $y(t)$
  - In practice **low-order polynomials chained together**
  - Convenient for animation, where t is time
  - Convenient for *tessellation* because **we can discretize t and approximate the curve with a polyline**

Tessellation



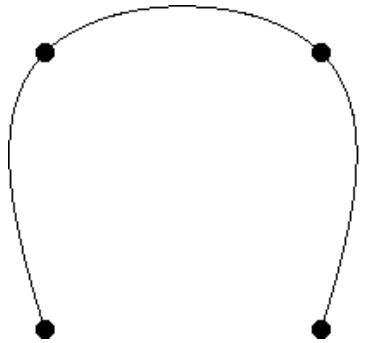
Tessellation



It's clear that adding more points will get us closer to the curve.

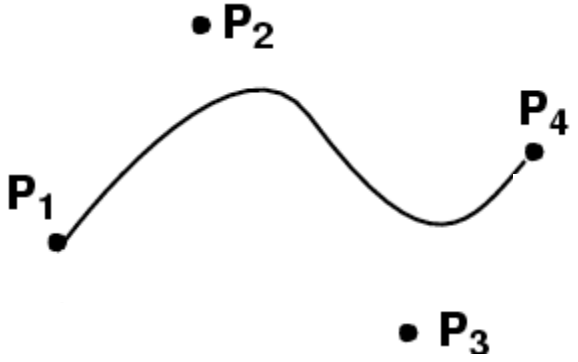
Interpolation vs. Approximation

- Interpolation
  - Goes through all specified points
  - Sounds more logical



Interpolation

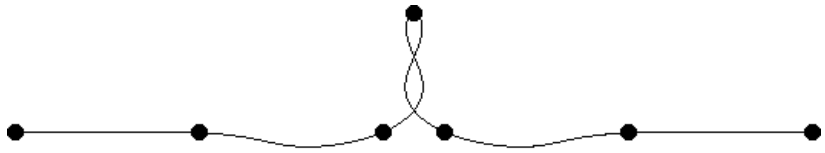
- Approximation
  - Does not go through all points



Approximation

Interpolation vs. Approximation

- Interpolation
  - Goes through all specified points
  - Sounds more logical
  - But can be more unstable, “ringing”



Interpolation

- Approximation
  - Does not go through all points
  - Turns out to be convenient

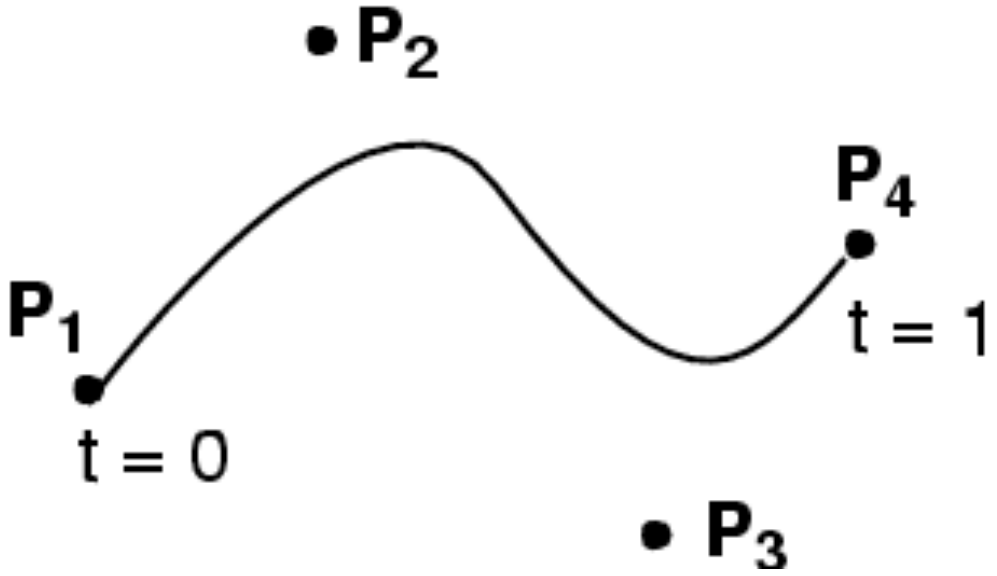
Approximation

- In practice, we’ll do something in between.

Questions?

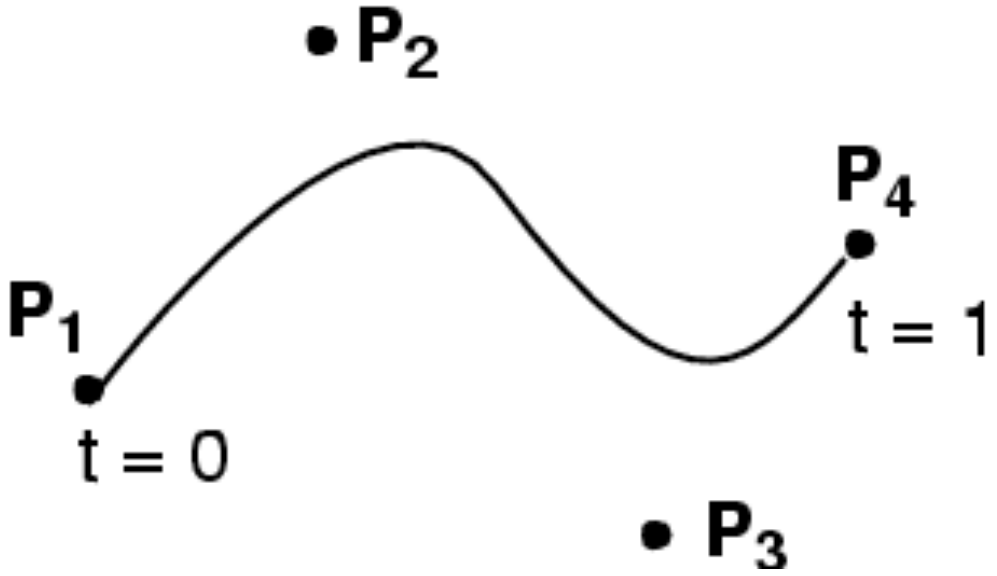
Cubic Bézier Curve

- User specifies 4 control points $\mathbf{P}_1$ ... $\mathbf{P}_4$
- Curve goes through (interpolates) the ends $\mathbf{P}_1$, $\mathbf{P}_4$
- Approximates the two other ones
- Cubic polynomial



Cubic Bézier Curve

- $P(t) = (1-t)^3 \mathbf{P}_1 + 3t(1-t)^2 \mathbf{P}_2 + 3t^2(1-t) \mathbf{P}_3 + t^3 \mathbf{P}_4$



That is,

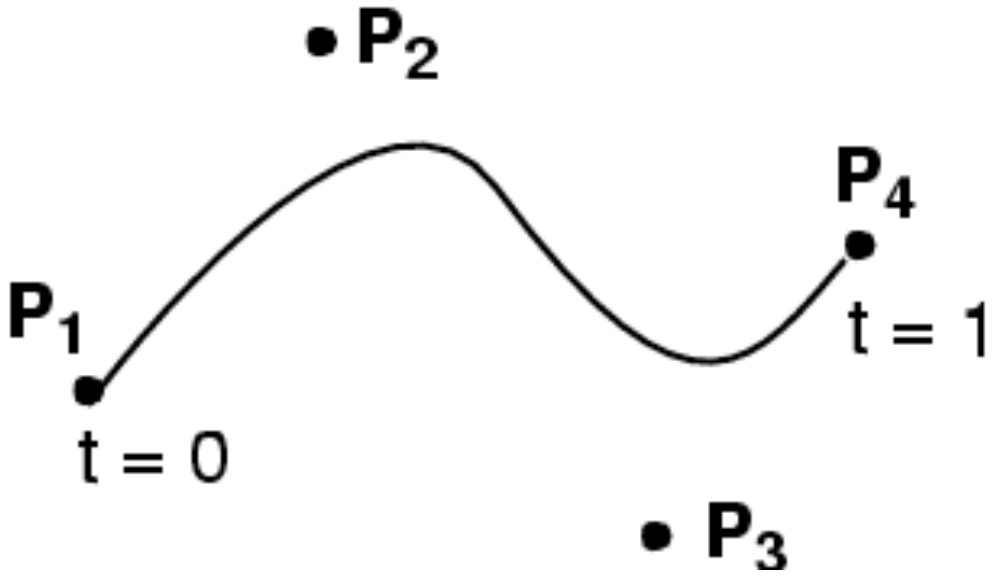
$$x(t) = (1-t)^3 x_1 + 3t(1-t)^2 x_2 + 3t^2(1-t) x_3 + t^3 x_4$$

$$y(t) = (1-t)^3 y_1 + 3t(1-t)^2 y_2 + 3t^2(1-t) y_3 + t^3 y_4$$

Cubic Bézier Curve

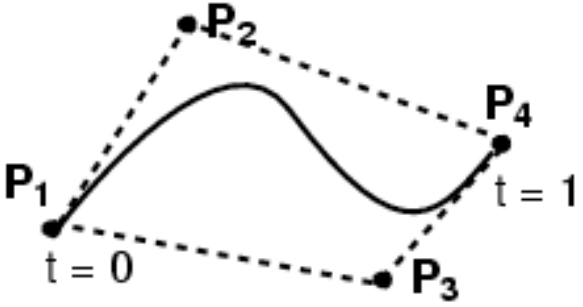
- $P(t) = (1-t)^3 \mathbf{P}_1 + 3t(1-t)^2 \mathbf{P}_2 + 3t^2(1-t) \mathbf{P}_3 + t^3 \mathbf{P}_4$

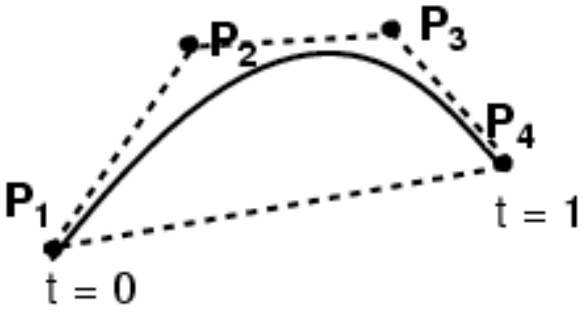
Verify what happens for t=0 and t=1

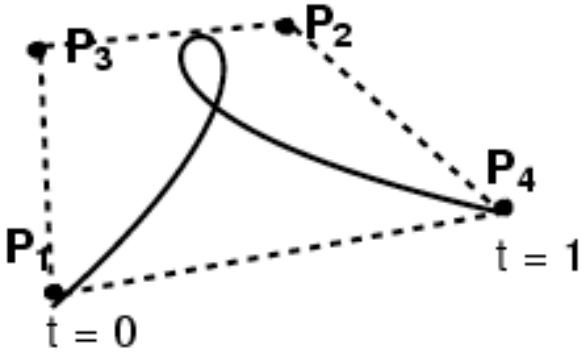


Cubic Bézier Curve

- 4 control points
- Curve passes through first & last control point
- Curve is tangent at $\mathbf{P}_1$ to $(\mathbf{P}_1\text{-}\mathbf{P}_2)$ and at $\mathbf{P}_4$ to $(\mathbf{P}_4\text{-}\mathbf{P}_3)$







A Bézier curve is bounded by the **convex hull** of its control points.

Cubic Bézier Curve

- 4 control points
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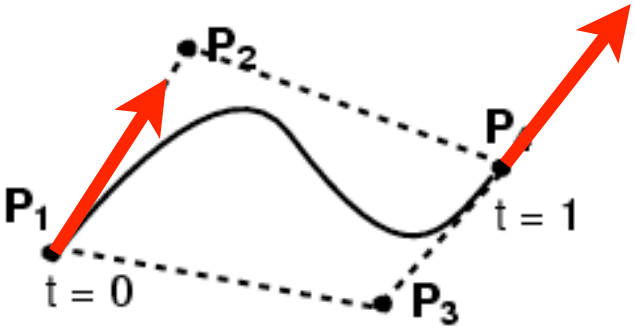
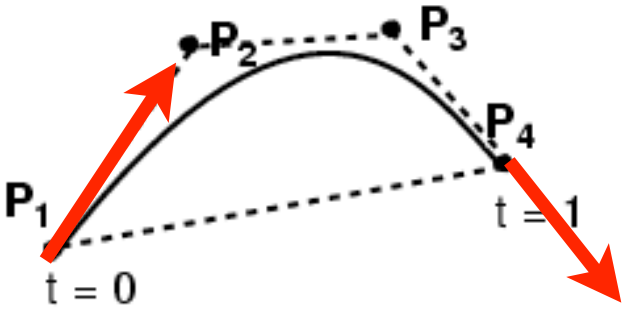
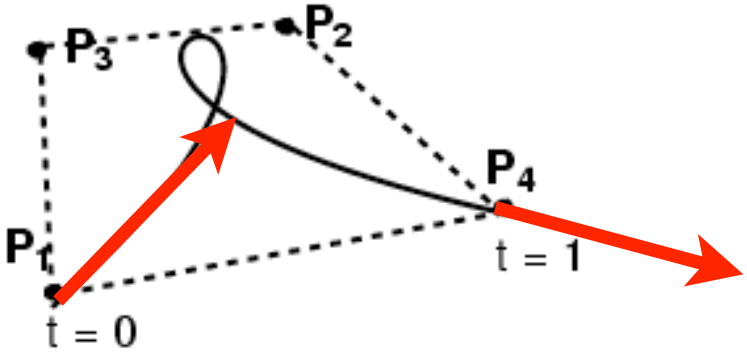






A Bézier curve is bounded by the **convex hull** of its control points.

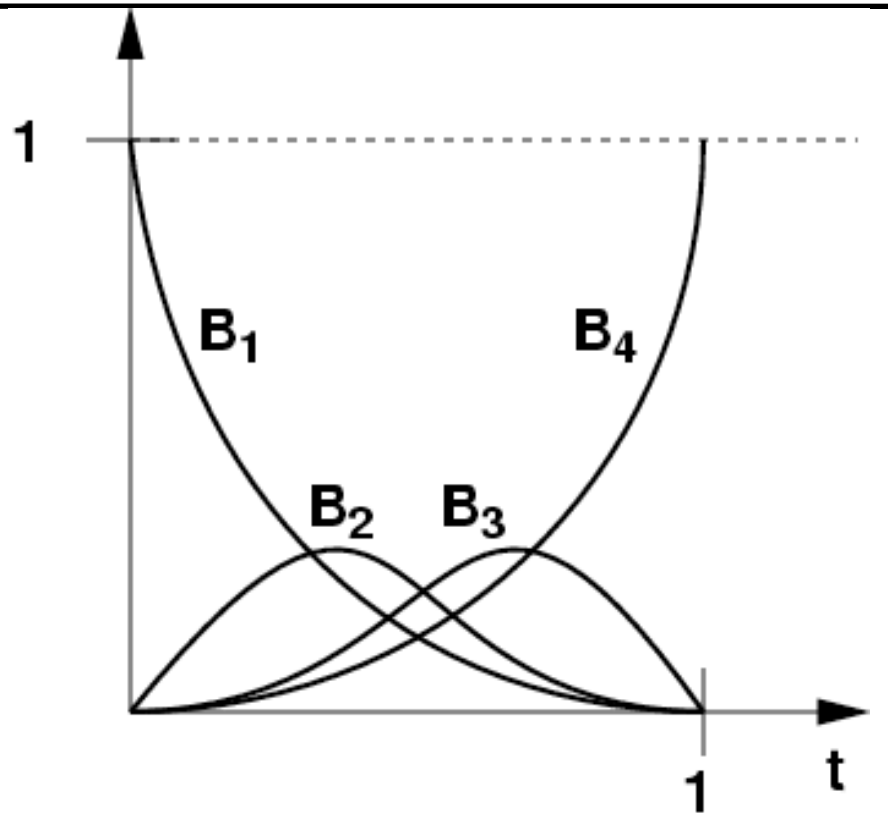
“Bernstein Polynomials”

For cubic:

- $B_1(t)=(1-t)^3$
- $B_2(t)=3t(1-t)^2$
- $B_3(t)=3t^2(1-t)$
- $B_4(t)=t^3$

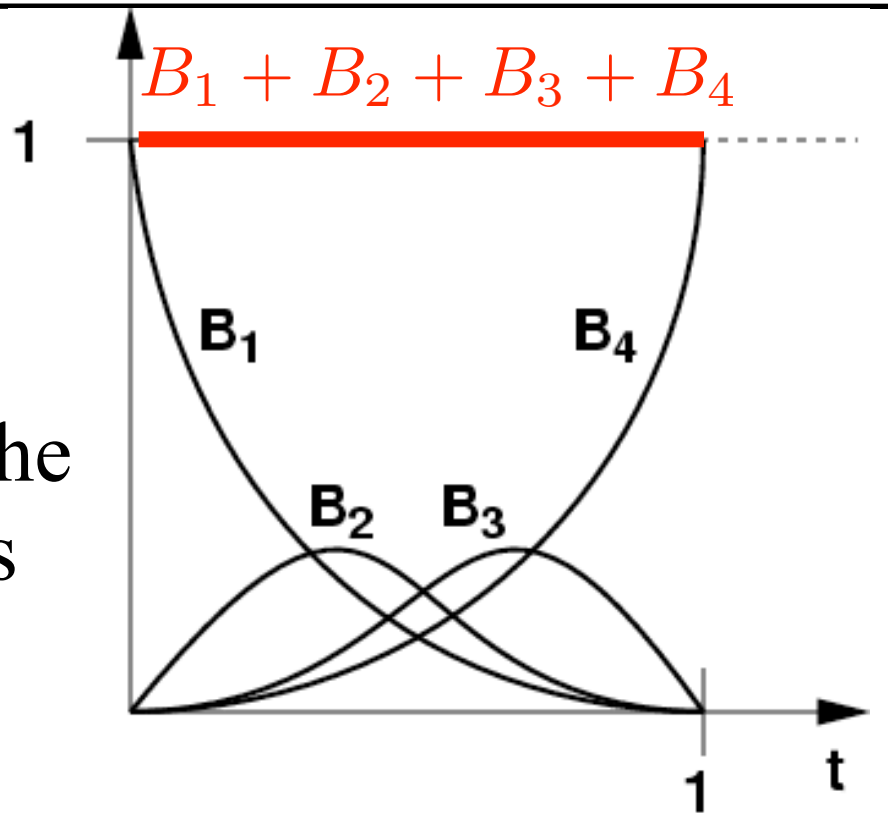
  – (careful with indices, many authors start at 0)

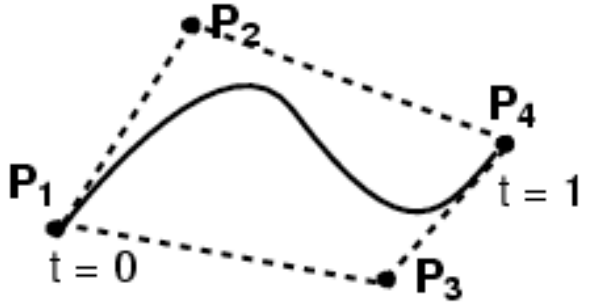
- But defined for any degree



Properties of Bernstein polynomials

- ≥ 0 for all $0 \leq t \leq 1$
- Sum to 1 for every t
  - called *partition of unity*
- (These two together are the reason why Bézier curves lie within convex hull)
- Only B_1 is non-zero at 0
  - Bezier interpolates P_1
  - Same for B_4 and P_4 for t=1



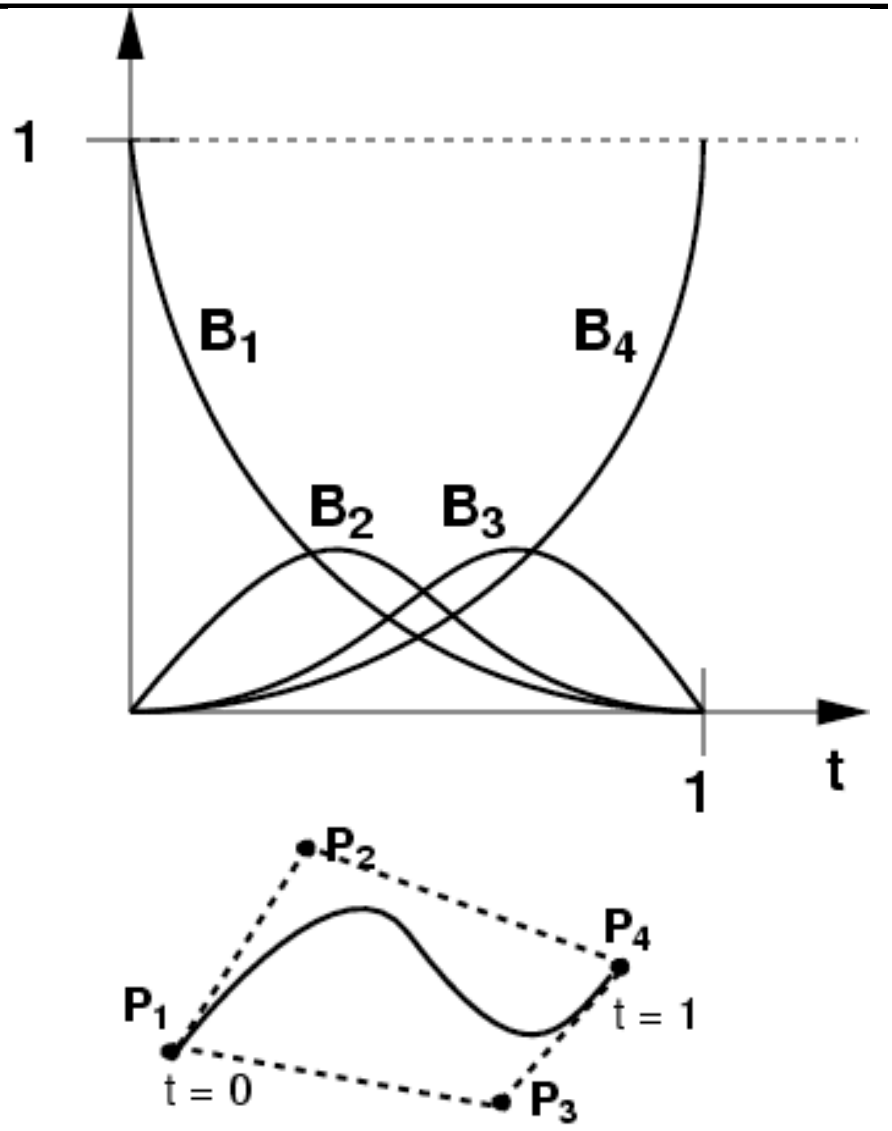


Bézier Curves

- $\mathbf{P}(t) = \mathbf{P}_1 B_1(t) + \mathbf{P}_2 B_2(t) + \mathbf{P}_3 B_3(t) + \mathbf{P}_4 B_4(t)$
  - $\mathbf{P}_i$ are 2D control points (x_i, y_i)
  - For each t, the point $\mathbf{P}(t)$ on a Bézier curve is a linear combination of the control points with weights given by the Bernstein polynomials at t

Interpretation as "Influence"

- Each B_i specifies the influence of P_i
- First, P_1 is the most influential point, then P_2, P_3, and P_4
- P_2 and P_3 never have full influence
  - Not interpolated!



Questions?

What's with the Formula?

- Explanation 1:
  - Magic!

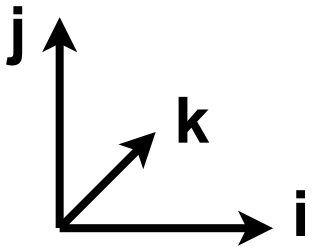
- Explanation 2:
  - It's a linear combination of ***basis polynomials***

  - Let's study this in 1D using curves $y=f(t)$

Usual Vector Spaces

- In 3D, each vector has three components x, y, z
- But geometrically, each vector is actually the sum

$$v = x\,\vec{\boldsymbol{i}} + y\,\vec{\boldsymbol{j}} + z\,\vec{\boldsymbol{k}}$$



- **i**, **j**, **k** are basis vectors

- Vector addition: just add components
- Scalar multiplication: just multiply components

Polynomials as a Vector Space

- Polynomials $y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$
- Can be added: just add the coefficients

$$(y + z)(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \ldots + (a_n + b_n)t^n$$

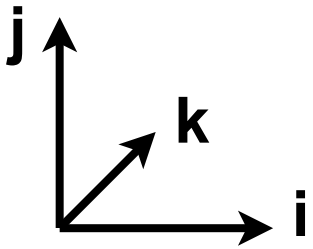
- Can be multiplied by a scalar: multiply the coefficients

$$s \cdot y(t) = (s \cdot a_0) + (s \cdot a_1)t + (s \cdot a_2)t^2 + \ldots + (s \cdot a_n)t^n$$

Polynomials as a Vector Space

- In 3D, each vector has three components x, y, z
- But geometrically, each vector is actually the sum

$$\vec{v} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$



- **i**, **j**, **k** are basis vectors

- In the polynomial vector space, $\{1, t, ..., t^n\}$ are the basis vectors, $a_0, a_1, ..., a_n$ are the components

$$y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

Polynomials as a Vector Space

- Polynomials are like vectors; you can add them and scale them, and the result is still a polynomial.

- Connections
  - The flavor of math that is concerned with such things is called *functional analysis*.
    - Many familiar concepts, such as orthogonality, carry over from "ordinary" *n*-D vectors to functions!
    - Trivializes many "difficult" things like Fourier series by giving geometric analogue.
    - Many other uses in graphics, e.g. finite element global illumination algorithms such as radiosity.

Questions?

Subset of Polynomials: Cubic

$$y(t) = a_0 + a_1\, t + a_2\, t^2 + a_3\, t^3$$

- Closed under addition & *scalar* multiplication
  - Means the result is still a cubic polynomial (verify!)
  - Not under general multiplication: $t^3 * t^3 = t^6$!
- This means it is also a vector space, a 4D **subspace** of the full space of polynomials
  - How many dimensions does the full space have?
- The *x* and *y* coordinates of cubic Bézier curves belong to this subspace as functions of *t*.

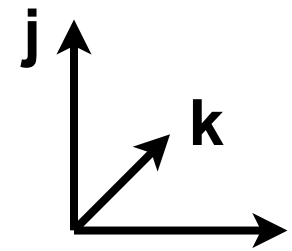
Basis for Cubic Polynomials

More precisely:

What's a basis?

$$\vec{v} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

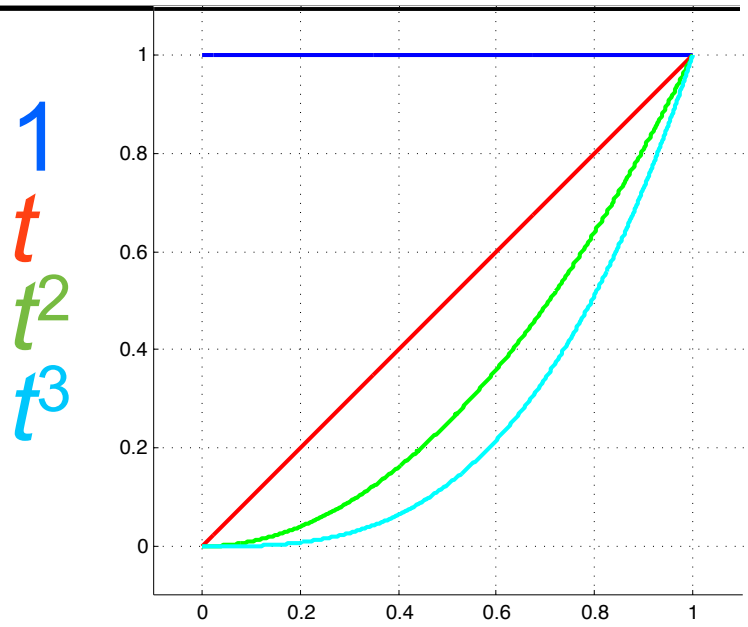
In 3D



- A set of "atomic" vectors
  - Called **basis vectors**
  - Linear combinations of basis vectors span the space
- Linearly independent
  - Means that no basis vector can be obtained from the others by linear combination
    - Example: **i**, **j**, **i**+**j** is not a basis (missing **k** direction!)

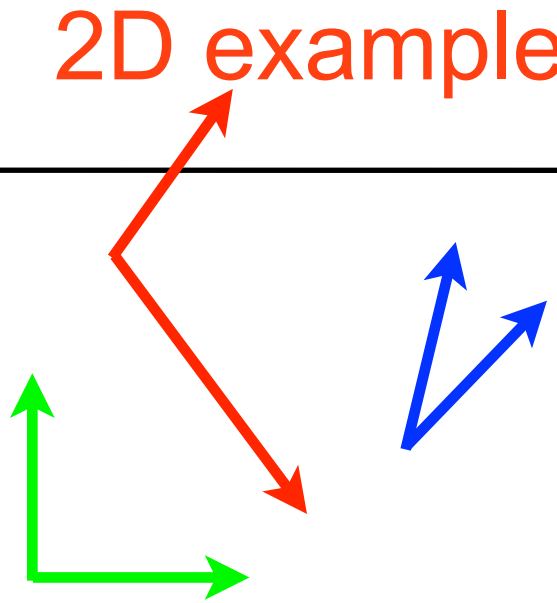
Canonical Basis for Cubics

$$\{1, t, t^2, t^3\}$$



- *Any* cubic polynomial is a linear combination of these:
  - $a_0 + a_1\, t + a_2\, t^2 + a_3\, t^3 = a_0 * 1 + a_1 * t + a_2 * t^2 + a_3 * t^3$
- They are linearly independent
  - Means you can't write any of the four monomials as a a linear combination of the others. (You can try.)

Different basis



- For example:
  - $\{1,\ 1+t,\ 1+t+t^2,\ 1+t-t^2+t^3\}$
  - $\{t^3,\ t^3+t^2,\ t^3+t,\ t^3+1\}$

- These can all be obtained from $1, t, t^2, t^3$ by linear combination
- Infinite number of possibilities, just like you have an infinite number of bases to span R^2

Why we bother:

Matrix-Vector Notation For Polynomials

- For example:
  - $1,\ 1+t,\ 1+t+t^2,\ 1+t-t^2+t^3$
  - $t^3,\ t^3+t^2,\ t^3+t,\ t^3+1$

Change-of-basis matrix

"Canonical" monomial basis

These relationships hold for each value of *t*

$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ 1+t-t^2+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} t^3 \\ t^3+t^2 \\ t^3+t \\ t^3+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Matrix-Vector Notation For Polynomials

- For example:
  - $1,\ 1+t,\ 1+t+t^2,\ 1+t-t^2+t^3$
  - $t^3,\ t^3+t^2,\ t^3+t,\ t^3+1$

Change-of-basis matrix

"Canonical" monomial basis

$$\begin{pmatrix} 1 \\ 1+t \\ 1+t+t^2 \\ 1+t-t^2+t^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

$$\begin{pmatrix} t^3 \\ t^3+t^2 \\ t^3+t \\ t^3+1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Not any matrix will do! If it's singular, the basis set will be linearly dependent, i.e., redundant.

Matrix Form of Bernstein

Cubic Bernstein:

- $B_1(t)=(1-t)^3$
- $B_2(t)=3t(1-t)^2$
- $B_3(t)=3t^2(1-t)$
- $B_4(t)=t^3$

Expand these out and collect powers of *t*. The coefficients are the entries in the matrix B!

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

More Matrix-Vector Notation

- Remember:
  $\mathbf{P}(t) = \mathbf{P}_1 \mathrm{B}_1(t) + \mathbf{P}_2 \mathrm{B}_2(t) + \mathbf{P}_3 \mathrm{B}_3(t) + \mathbf{P}_4 \mathrm{B}_4(t)$
  is a linear combination of control points

- or, in matrix-vector notation

Bernstein polynomials (4x1 vector)

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

point on curve (2x1 vector)

matrix of control points (2 x 4)

More Matrix-Vector Notation

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

<= Flasback from two slides ago, let's combine with below:

Bernstein polynomials (4x1 vector)

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

point on curve (2x1 vector)

matrix of control points (2 x 4)

Phase 3: Profit

- Combined, we get cubic Bézier in matrix notation

point on curve (2x1 vector)

$$\mathrm{P(t)} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} =$$

Canonical monomial basis

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

"Geometry matrix" of control points $P_1..P_4$ (2 x 4)

"Spline matrix" (Bernstein)

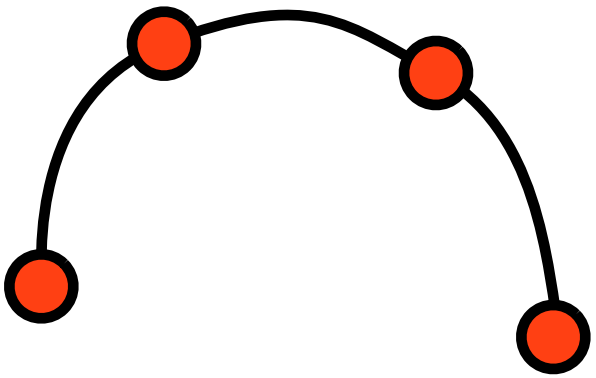
General Spline Formulation

$$Q(t) = \mathbf{GBT(t)} = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

- Geometry: control points coordinates assembled into a matrix (P_1, P_2, …, P_{n+1})
- Spline matrix: defines the type of spline
  - Bernstein for Bézier
- Power basis: the monomials (1, t, ..., t^n)
- Advantage of general formulation
  - Compact expression
  - Easy to convert between types of splines
  - Dimensionality (plane or space) doesn't really matter

What can we do with this?

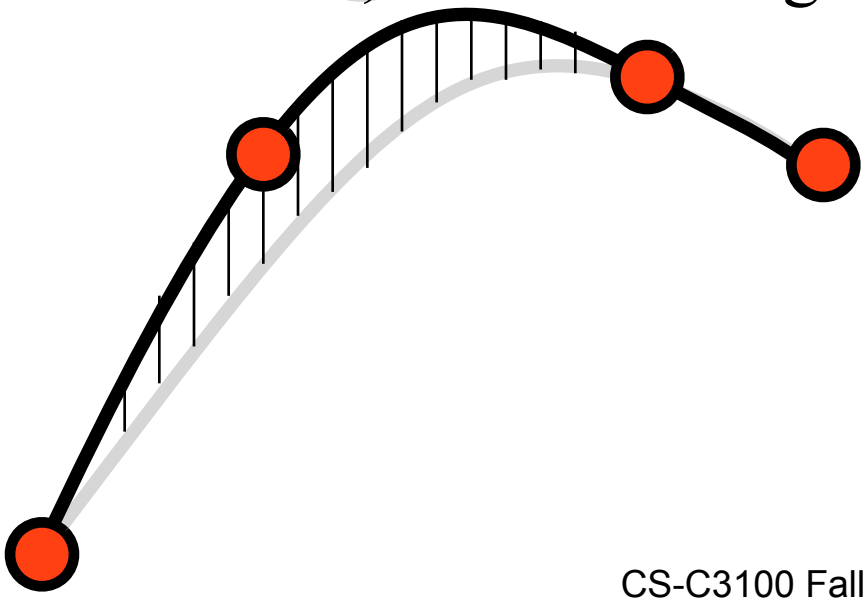
- Cubic polynomials form a vector space.
- Bernstein basis is canonical for Bézier.
- Can do other bases as well: Catmull-Rom splines interpolate all the control points, for instance



$$\mathbf{B}_{\mathrm{CR}} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 2 & -1 \\ 2 & 0 & -5 & 3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

3D Spline has 3D control points

- The 2D control points can be replaced by 3D points – this yields space curves.
  - All the above math stays the same except with the addition of 3rd row to control point matrix (z coords)
  - In fact, can do homogeneous coordinates as well!



Easy Brain Teaser for Next Time

- Can you derive formulas for conversion between two different cubic spline matrices, say Bézier and Catmull-Rom?
  - Both are cubics, so you can represent the other curve exactly with new control points $\mathbf{G}_2$

$$\mathbf{G}_1\mathbf{B}_1\mathbf{T}(t) \equiv \mathbf{G}_2\mathbf{B}_2\mathbf{T}(t)$$

$$\mathbf{G}_2 = ???$$

Change of Basis, Other Direction

- Given B1...B4, how to get back to canonical 1, t, t^2, t^3 ?

$$\begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Change of Basis, Other Direction

That's right, with the inverse matrix!

- Given B1...B4, how to get back to canonical 1, t, t^2, t^3 ?

$$\begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}^{\boldsymbol{B}^{-1}} \begin{pmatrix} B_1(t) \\ B_2(t) \\ B_3(t) \\ B_4(t) \end{pmatrix}$$

Recap

- Cubic polynomials form a vector space.
- Bernstein basis is canonical for Bézier.
  - Can be seen as influence function of data points
  - Or data points are coordinates of the curve in the Bernstein basis
- We can change between basis with matrices.

Questions?

A Cubic Only Gets You So Far

- What if you want more control?

Higher-Order Bézier Curves

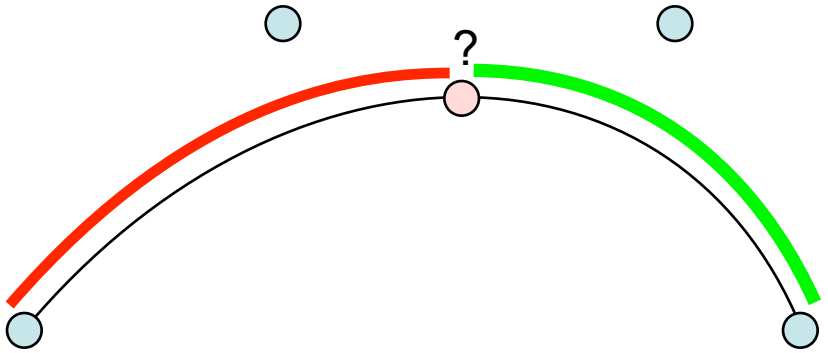
- > 4 control points
- Bernstein Polynomials as basis functions
  - For polynomial order n, i^{th} basis function is

$$B_i^n(t) = \frac{n!}{i!(n-i)!} t^i (1-t)^{n-i}$$

- Every control point affects entire curve
  - Not simple local effect
  - More difficult to control for modeling
- You will not need this in this class

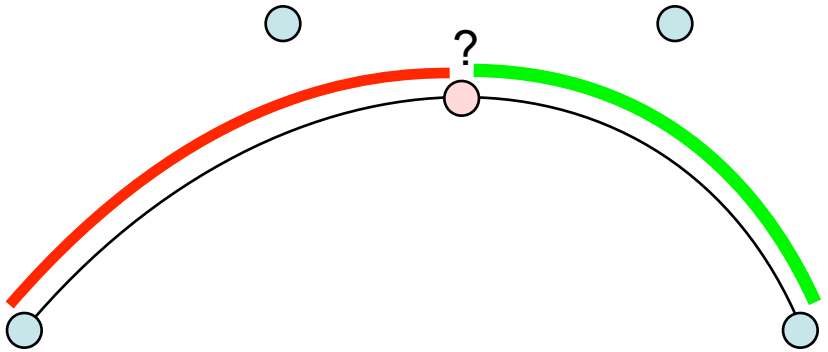
Subdivision of a Bézier curve

- Can we split a Bezier curve into two in the middle, using two new Bézier curves?
  - Would be useful for adding detail, as a single cubic doesn't get you very far, and higher-order curves are nasty.



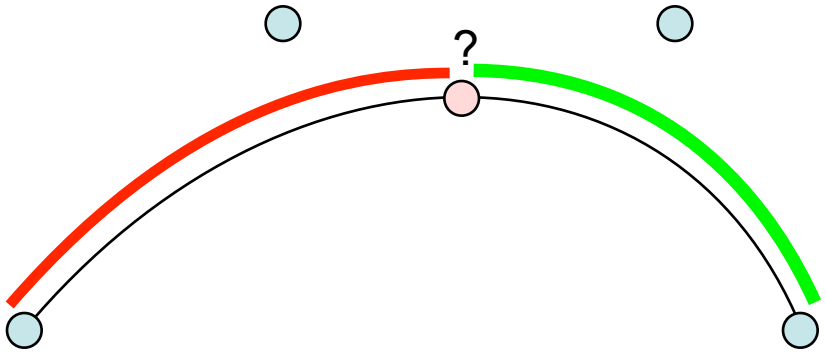
Subdivision of a Bezier curve

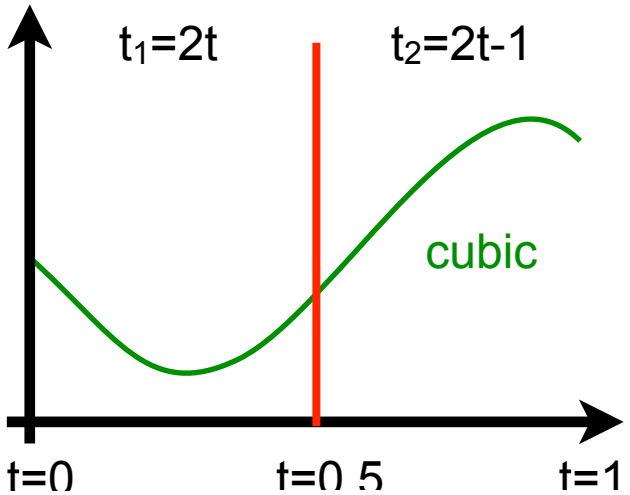
- Can we split a Bezier curve into two in the middle, using two new Bézier curves?
  - The resulting curves are again a cubic (Why?)



Subdivision of a Bezier curve

- Can we split a Bezier curve into two in the middle, using two new Bézier curves?
  - The resulting curves are again a cubic (Why? A cubic in t is also a cubic in $2t$)
  - Hence it must be representable using the Bernstein basis. So **yes, we can!**





Subdivision of a Bezier curve

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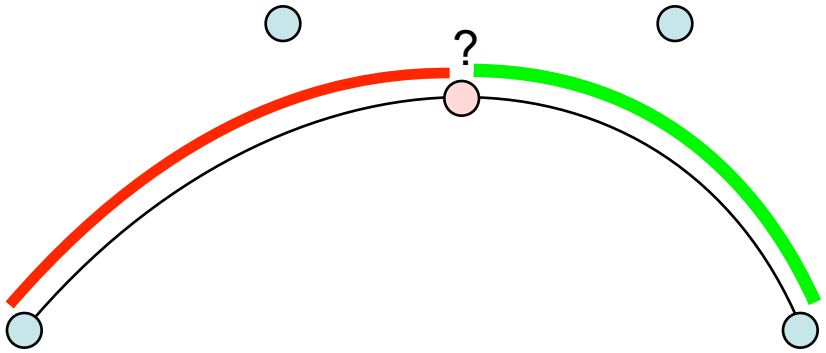
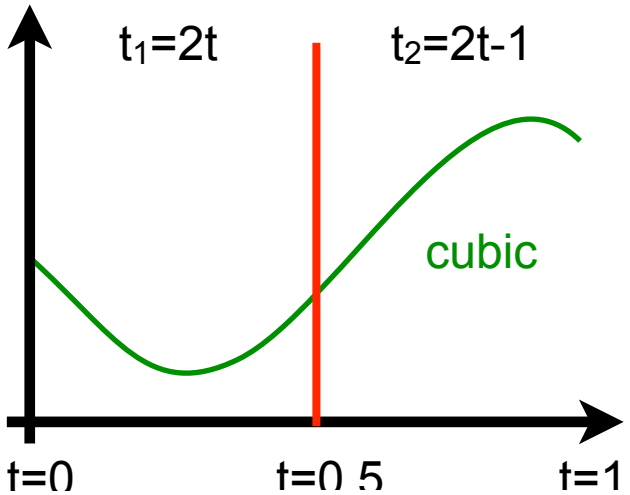
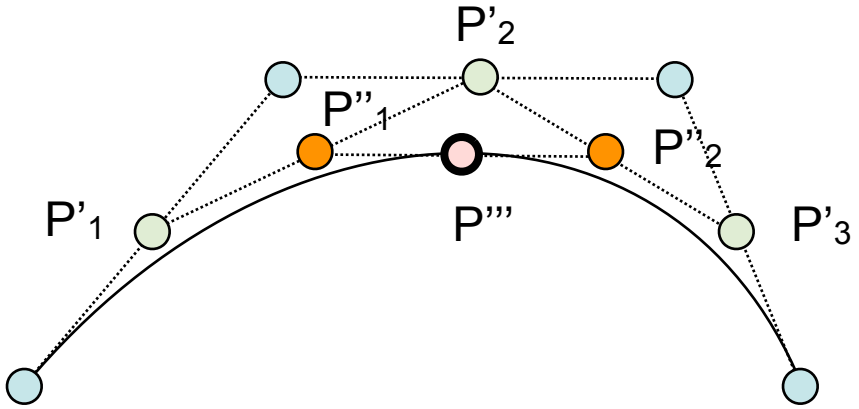




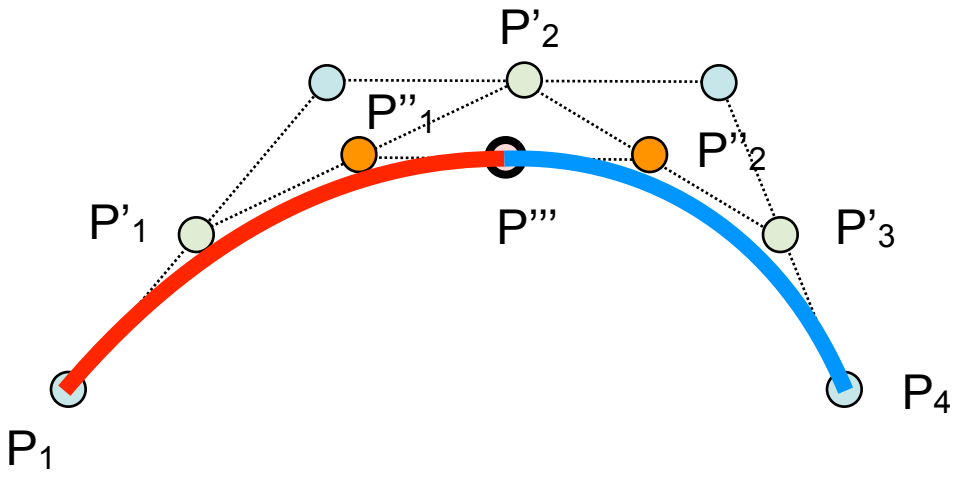
“De Casteljau Construction”

- Take the middle point of each of the 3 segments
- Construct the two segments joining them
- Take the middle of those two new segments
- Join them
- Take the middle point P’’’



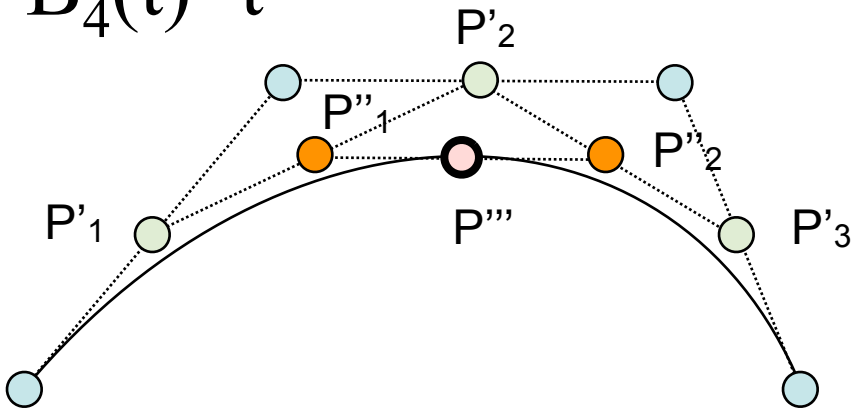
Result of Split in Middle

- The two new curves are defined by
  - P_1, P'_1, P''_1, and P'''
  - P''', P''_2, P'_3, and P_4
- Together they exactly replicate the original curve!
  - Originally 4 control points, now 7 (more control)



Sanity Check

- Do we get the middle point?
- $B_1(t)=(1-t)^3$
- $B_2(t)=3t(1-t)^2$
- $B_3(t)=3t^2(1-t)$
- $B_4(t)=t^3$



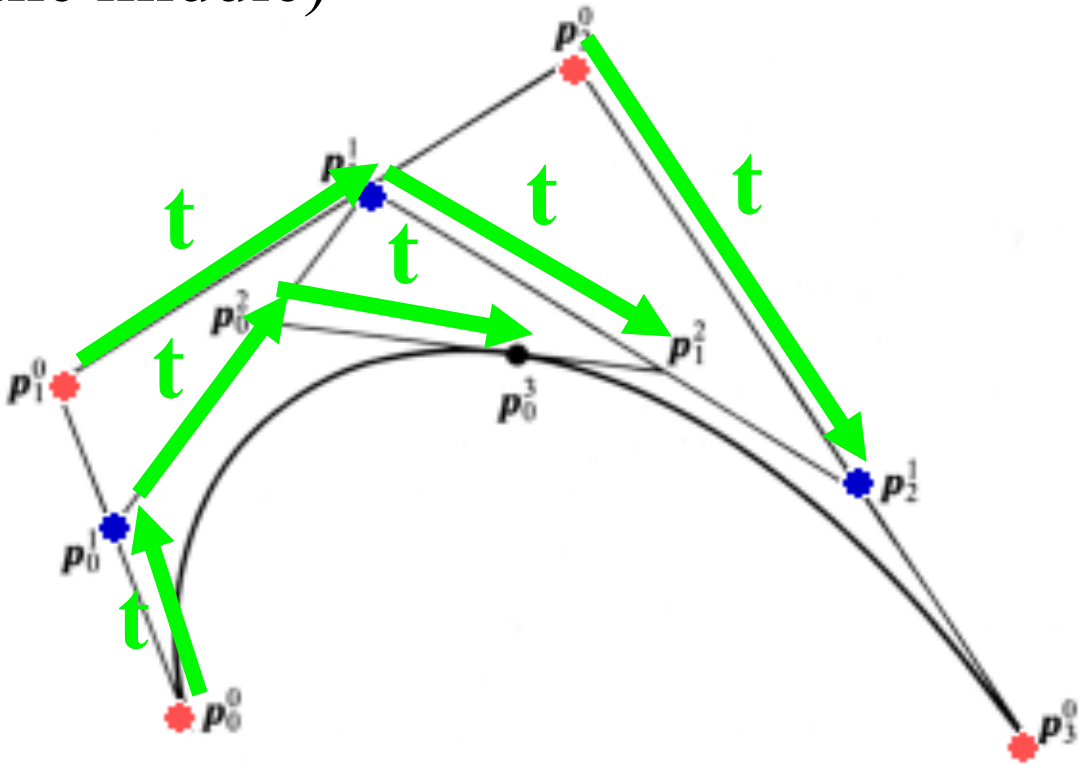
$$P_1' = 0.5(P_1 + P_2)$$
$$P_2' = 0.5(P_2 + P_3)$$
$$P_3' = 0.5(P_3 + P_4)$$

$$P_1'' = 0.5(P_1' + P_2')$$
$$P_2'' = 0.5(P_2' + P_3')$$

$$\begin{aligned} P''' &= 0.5(P_1'' + P_2'') \\ &= 0.5\,(0.5(P_1' + P_2') + 0.5(P_2' + P_3')) \\ &= 0.5\,(0.5\,[0.5(P_1 + P_2) + 0.5(P_2 + P_3)] + \\ &\quad 0.5\,[0.5(P_2 + P_3) + 0.5(P_3 + P_4)] \\ &= 1/8P_1 + 3/8P_2 + 3/8P_3 + 1/8P_4 \end{aligned}$$

De Casteljau Construction

- Actually works to construct a point at any *t*, not just 0.5
- Just subdivide the segments with ratio (1-*t*), *t* (not in the middle)



Questions?

Aside: Extra Credit Suggestion

- TrueType font glyphs (the images of the characters) consist of *quadratic* Bézier curves chained together.
- Why don't you write code that loads the glyphs using the Win32 API, and either
  - Convert them to cubic Bézier and display them using the drawing code you already wrote
  - Or write code for drawing quadratic curves directly.
- See this MSDN documentation page for a starting point!

Linear Transformations & Cubics

- What if we want to transform each point on the curve with a linear transformation **M**?

$$P'(t) = \mathbf{M}\left(\begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} \right)$$

Linear Transformations & Cubics

- What if we want to transform each point on the curve with a linear transformation **M**?
  - Because everything is linear, it's the same as transforming the only the control points

$$P'(t) = \mathbf{M}\left(\begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} \right)$$

$$= \left(\mathbf{M} \begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \end{pmatrix} \right) \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Linear Transformations & Cubics

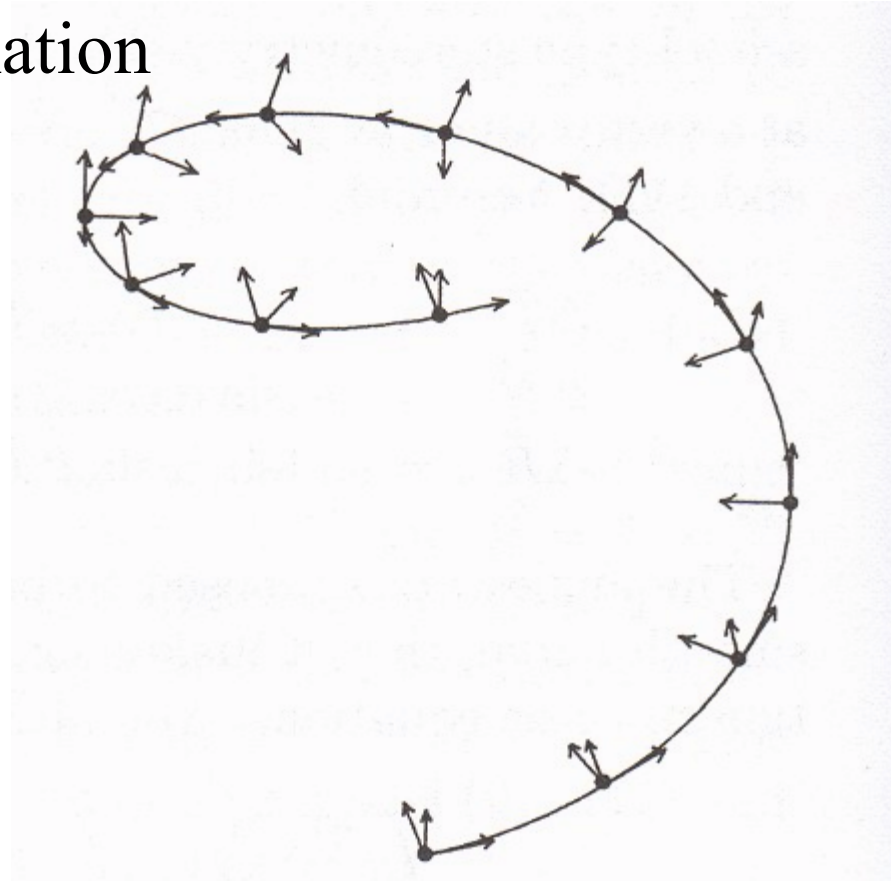
- Homogeneous coordinates also work
  - Means you can translate, rotate, shear, etc.
  - Also, changing *w* gives a "tension" parameter
    - Note though that you need to normalize P' by $1/w$

$$P'(t) = \mathbf{M}\left[\begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix} \right]$$

$$= \left[\mathbf{M} \begin{pmatrix} P_{1,x} & P_{2,x} & P_{3,x} & P_{4,x} \\ P_{1,y} & P_{2,y} & P_{3,y} & P_{4,y} \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}$$

Differential properties of curves

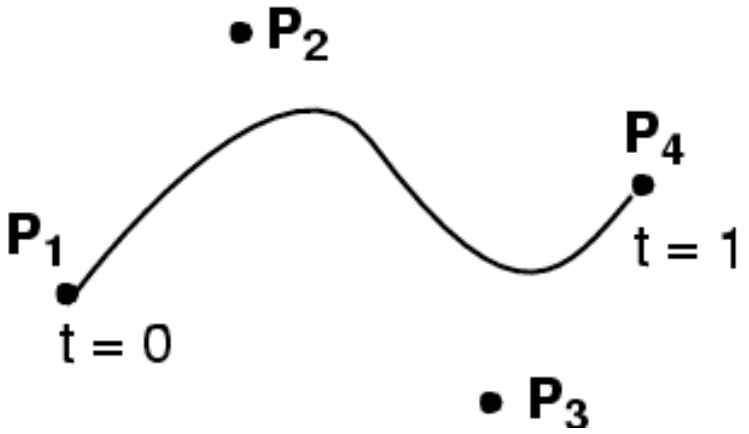
- Motivation
  - Compute normal for surfaces
  - Compute velocity for animation
  - Analyze smoothness



Velocity

- First derivative w.r.t. *t*
- Can you compute this for Bezier curves?

$$P(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

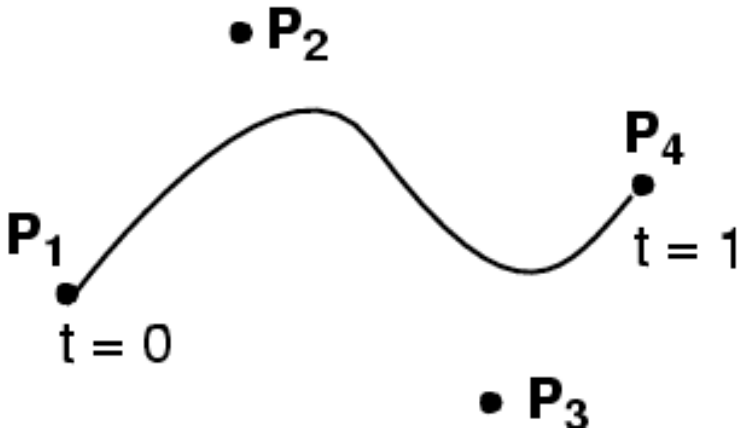


- You know how to differentiate polynomials...

Velocity

- First derivative w.r.t. *t*
- Can you compute this for Bezier curves?

$$P(t) = (1-t)^3 P_1 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$



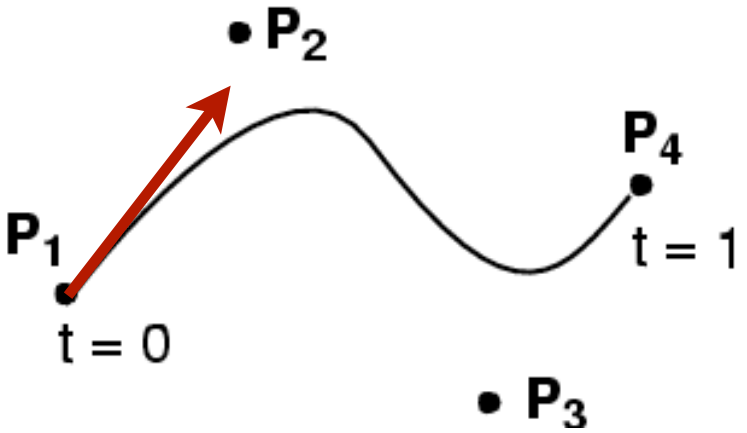
- **$P'(t) = -3(1-t)^2 P_1 + [3(1-t)^2 - 6t(1-t)] P_2 + [6t(1-t) - 3t^2] P_3 + 3t^2 P_4$**

Sanity check: t=0; t=1

Can also write this using a matrix **B'** – try it out!

Tangent

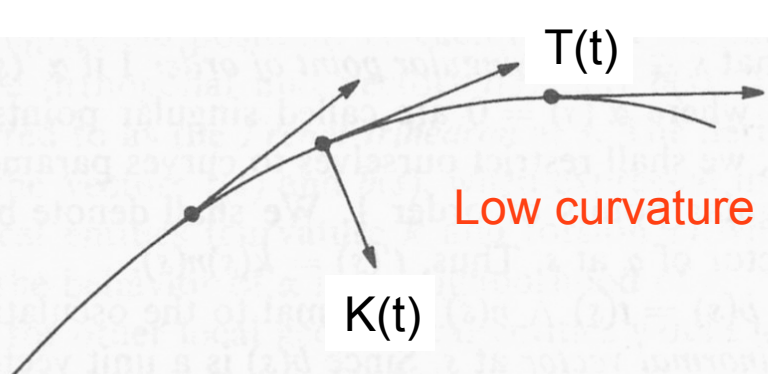
- The tangent to the curve P(t) can be defined as $T(t)=P'(t)/||P'(t)||$
  - normalized velocity, $||T(t)|| = 1$
- This provides us with one orientation for swept surfaces in a little while

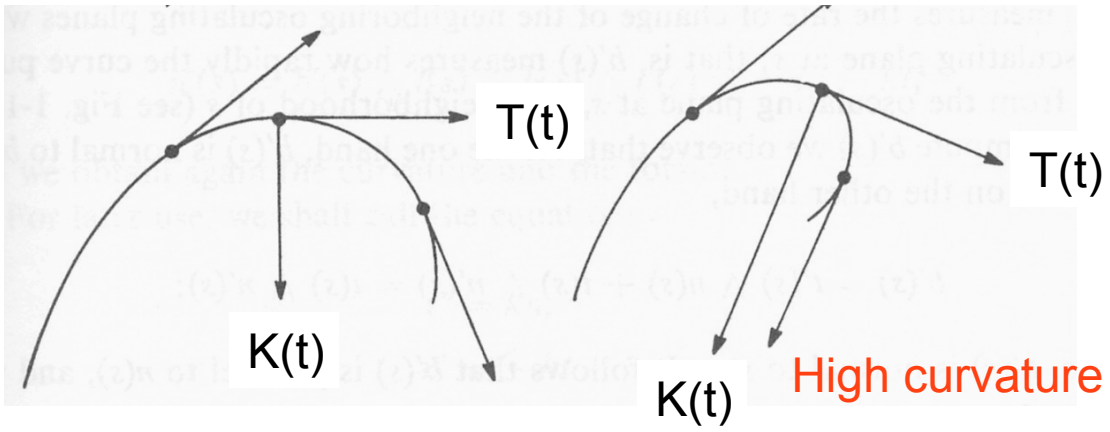


Questions?

Curvature

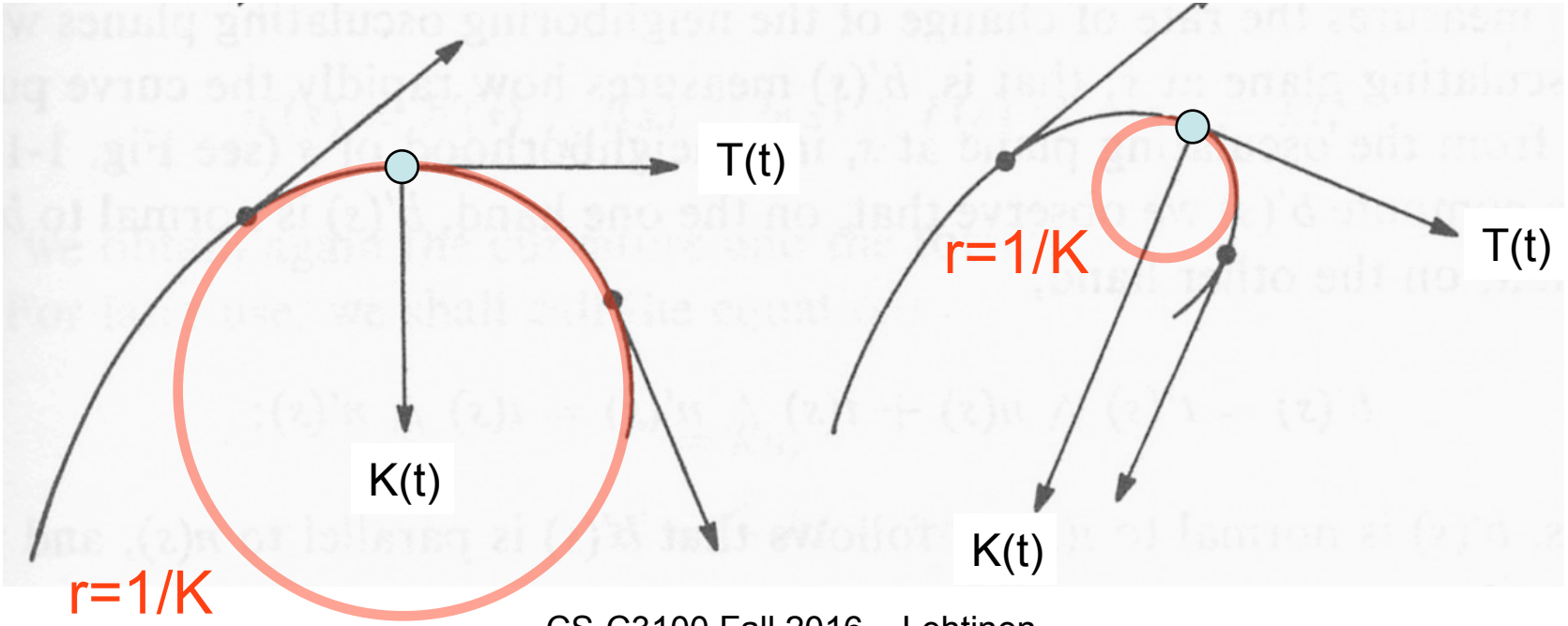
- Derivative of unit tangent
  - K(t)=T'(t)
  - Magnitude ||K(t)|| is constant for a circle
  - Zero for a straight line
- Always orthogonal to tangent, ie. $K \cdot T = 0$
  - Can you prove this? (Hints: ||T(t)||=1, (x(t)y(t))'=?)





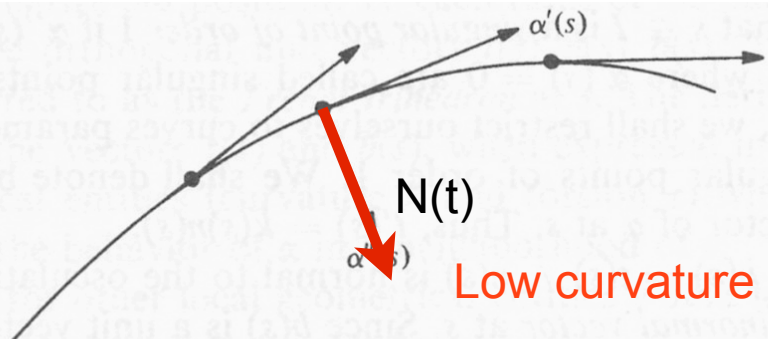
Geometric Interpretation

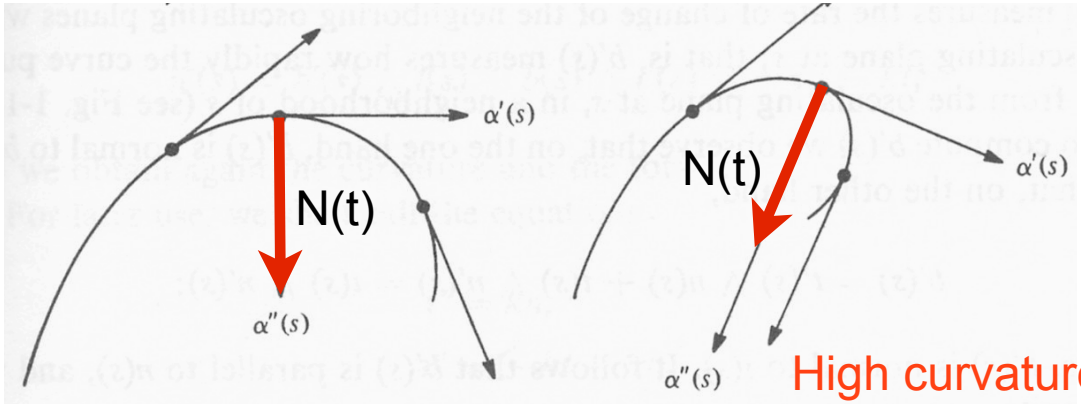
- K is zero for a line, constant for circle
  - What constant? 1/r
- 1/||K(t)|| is the radius of the circle that touches P(t) at *t* and has the same curvature as the curve



Curve Normal

- Normalized curvature: T’(t)/||T’(t)||

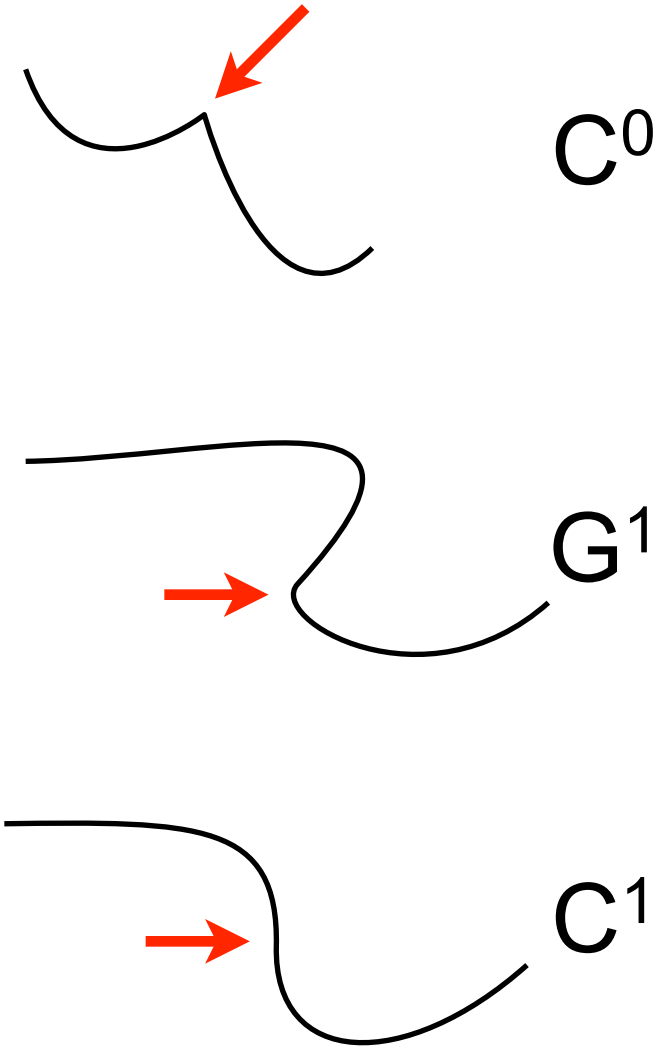




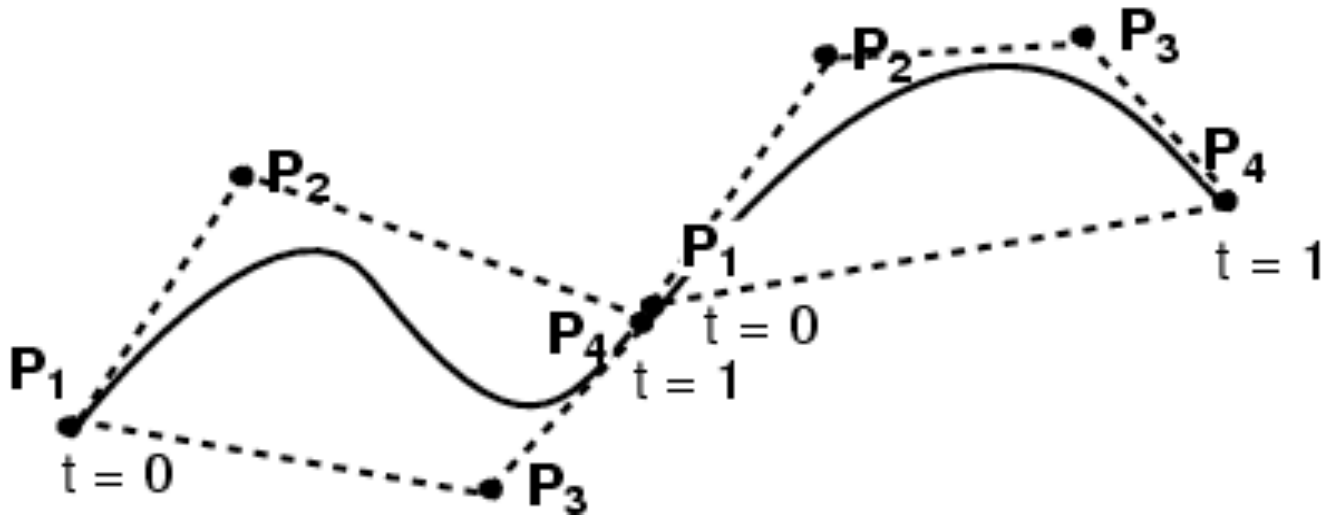
Questions?

Orders of Continuity

- C^0 = continuous
  - The seam can be a sharp kink
- G^1 = geometric continuity
  - Tangents **point to the same direction** at the seam
- C^1 = parametric continuity
  - Tangents **are the same** at the seam, implies G^1
- C^2 = curvature continuity
  - Tangents and their derivatives are the same

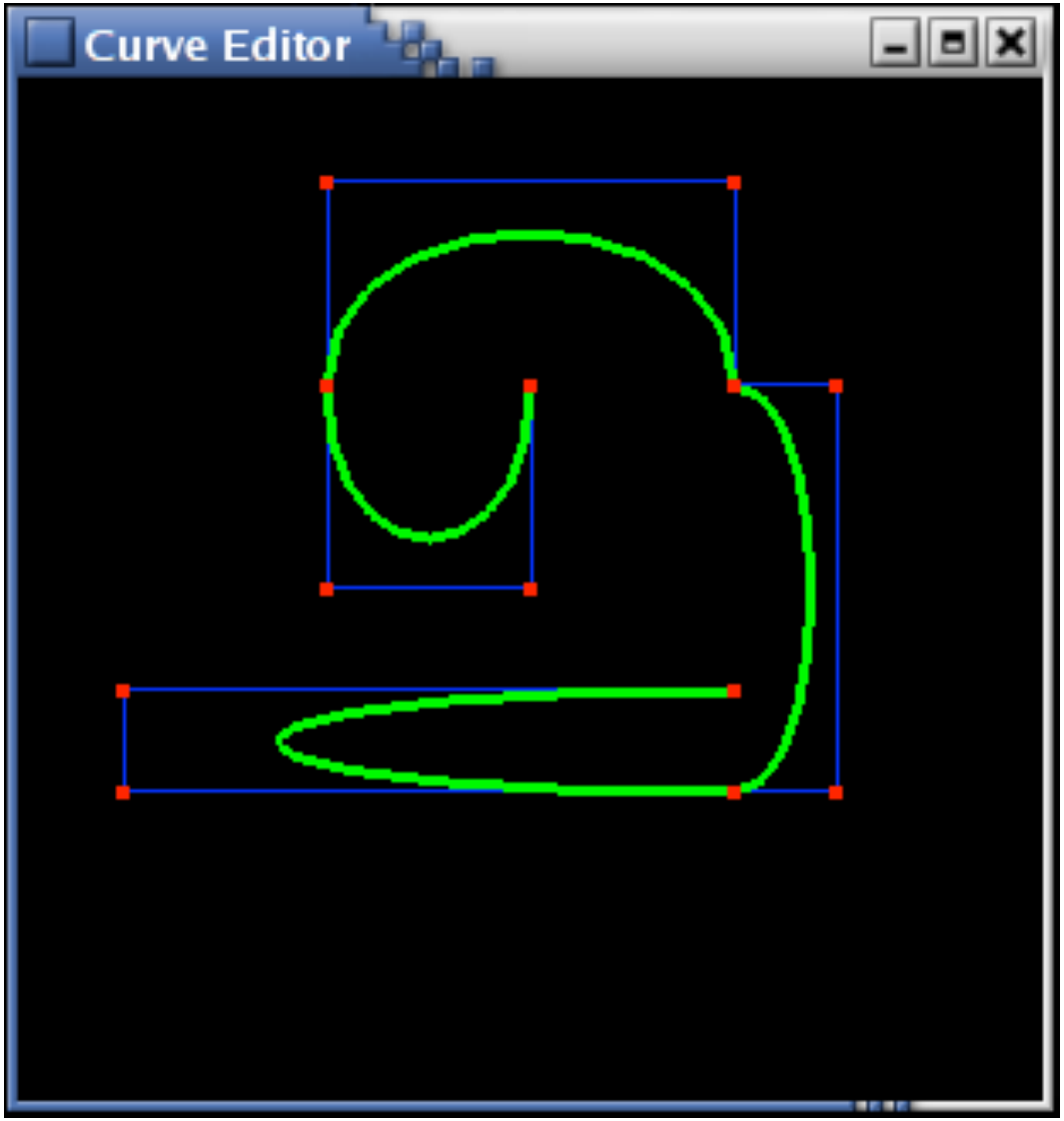


Connecting Cubic Bézier Curves



- How can we guarantee C^0 continuity?
- How can we guarantee G^1 continuity?
- How can we guarantee C^1 continuity?
- C^2 and above gets difficult

Connecting Cubic Bézier Curves

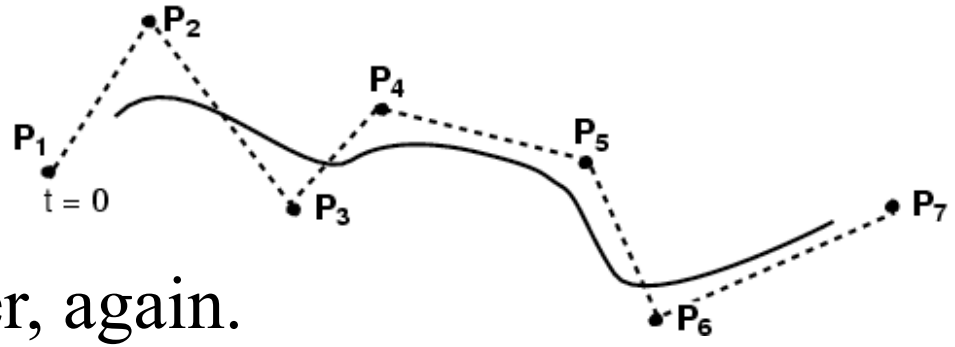


- Where is this curve
  - C^0 continuous?
  - G^1 continuous?
  - C^1 continuous?
- What's the relationship between:
  - the # of control points, and the # of cubic Bézier subcurves?

Questions?

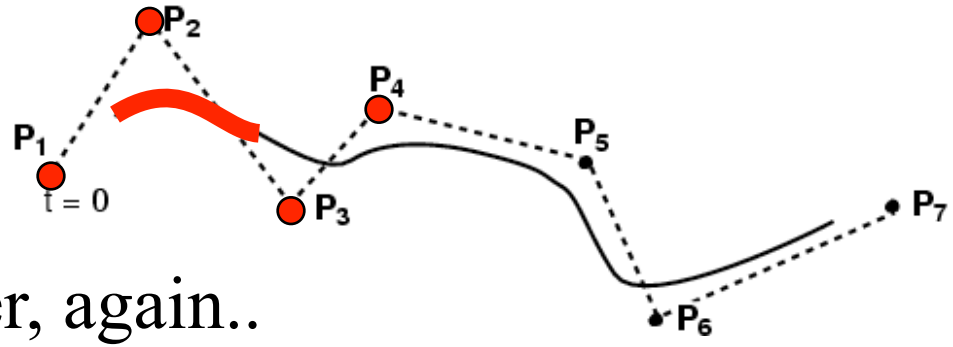
Cubic B-Splines

- ≥ 4 control points
- Locally cubic
  - Cubics chained together, again.



Cubic B-Splines

- ≥ 4 control points
- Locally cubic
  - Cubics chained together, again..
  - BUT with a sliding window of 4 control points!



Cubic B-Splines

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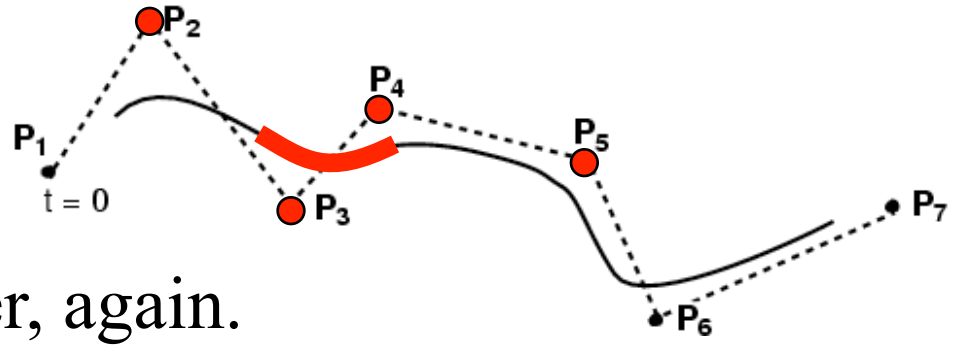

Cubic B-Splines

- ≥ 4 control points
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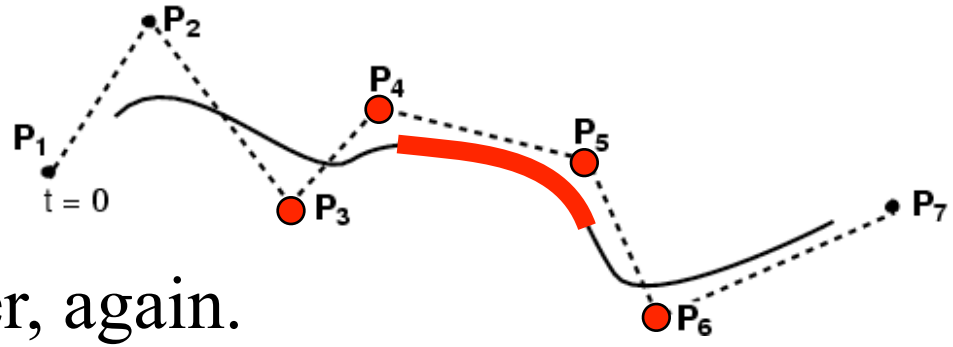
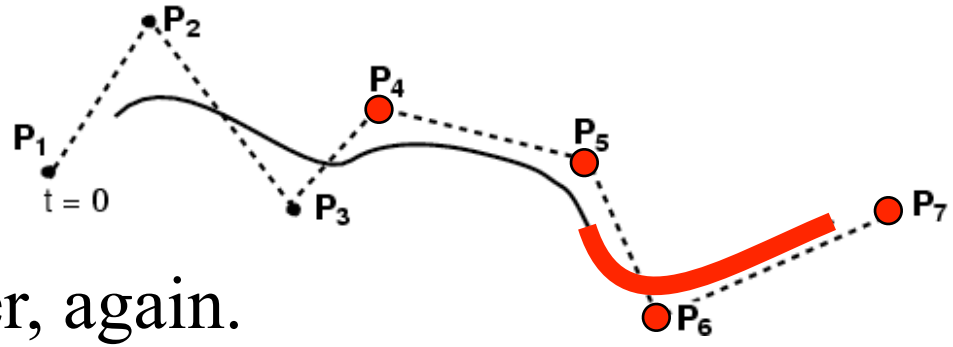
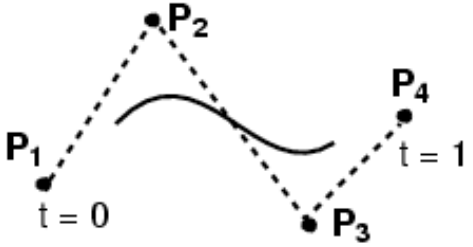
  - BUT with a sliding window of 4 control points!

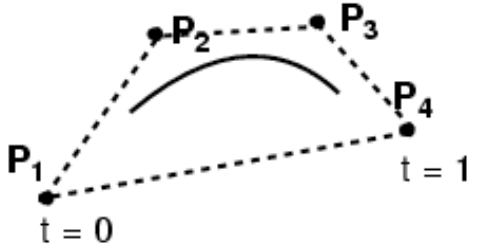


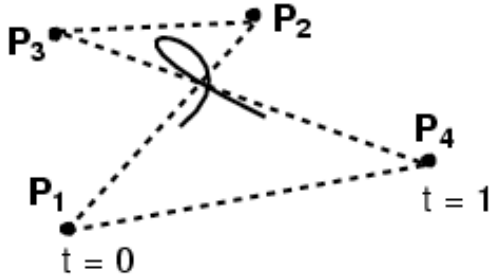
Cubic B-Splines

- ≥ 4 control points
- Locally cubic
  - Cubics chained together, again.
- Curve is not constrained to pass through any control points



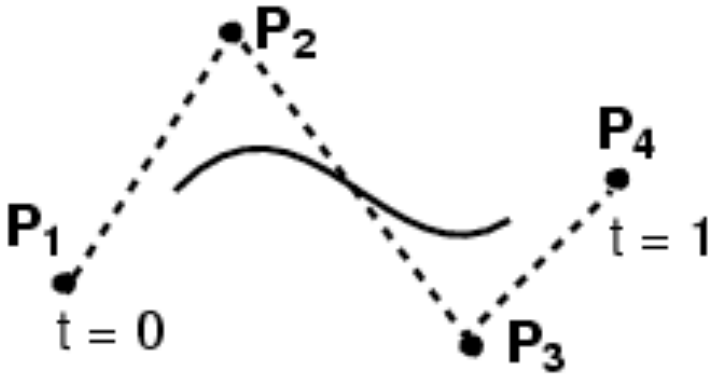


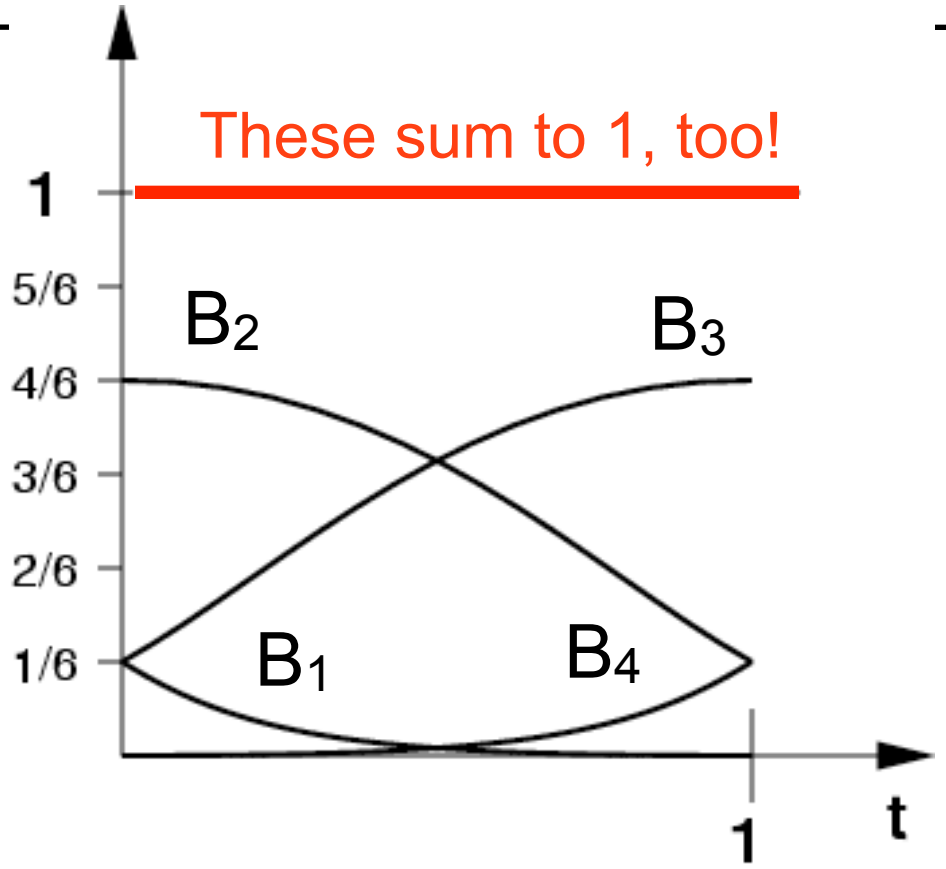




A BSpline curve is also bounded by the convex hull of its control points.

Cubic B-Splines: Basis





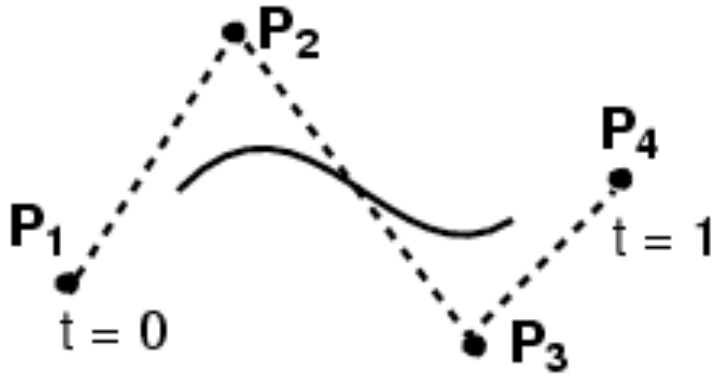
$$B_1(t) = \frac{1}{6}(1-t)^3$$

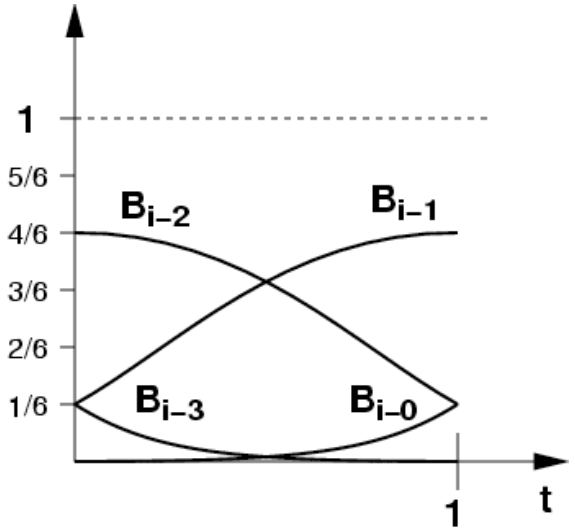
$$B_2(t) = \frac{1}{6}(3t^3 - 6t^2 + 4)$$

$$B_3(t) = \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1)$$

$$B_4(t) = \frac{1}{6}t^3$$

Cubic B-Splines: Basis





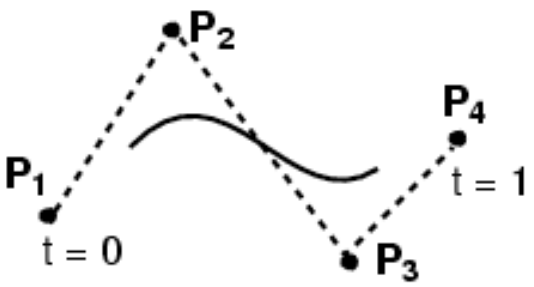
$$Q(t) = \frac{(1-t)^3}{6}P_{i-3} + \frac{3t^3 - 6t^2 + 4}{6}P_{i-2} + \frac{-3t^3 + 3t^2 + 3t + 1}{6}P_{i-1} + \frac{t^3}{6}P_i$$

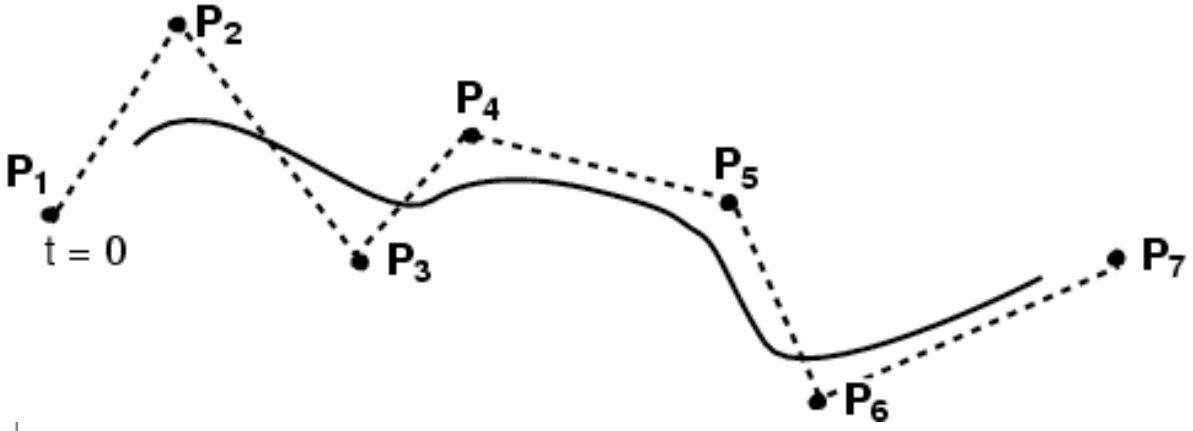
$$Q(t) = \mathbf{GBT(t)}$$

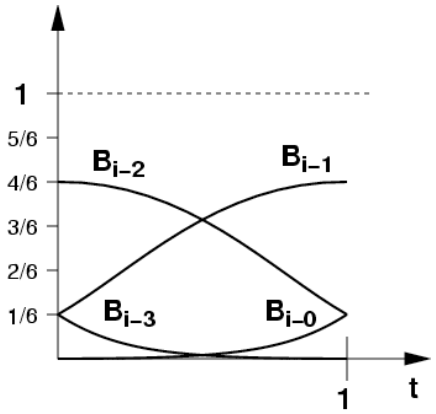
$$B_{B-Spline} = \frac{1}{6}\begin{pmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

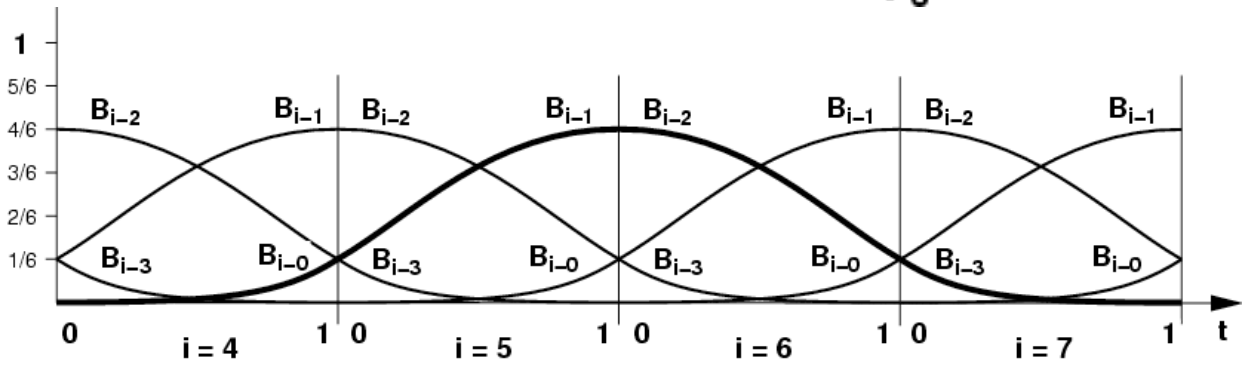
Cubic B-Splines

- Local control (windowing)
- Automatically C^2, and no need to match tangents!

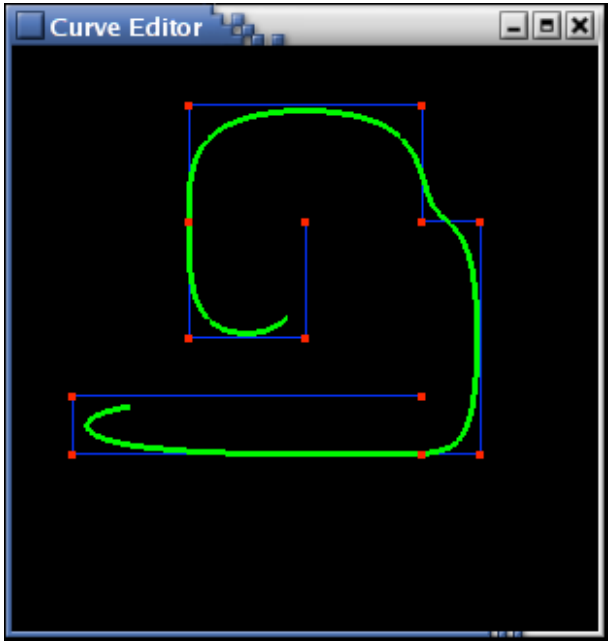




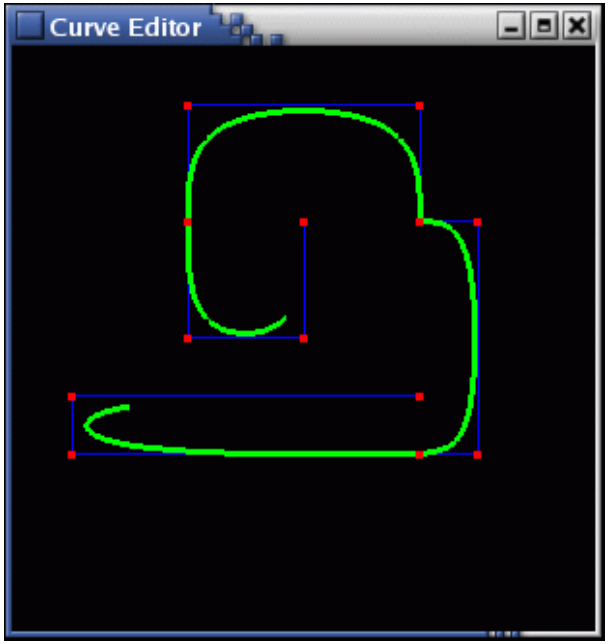




B-Spline Curve Control Points

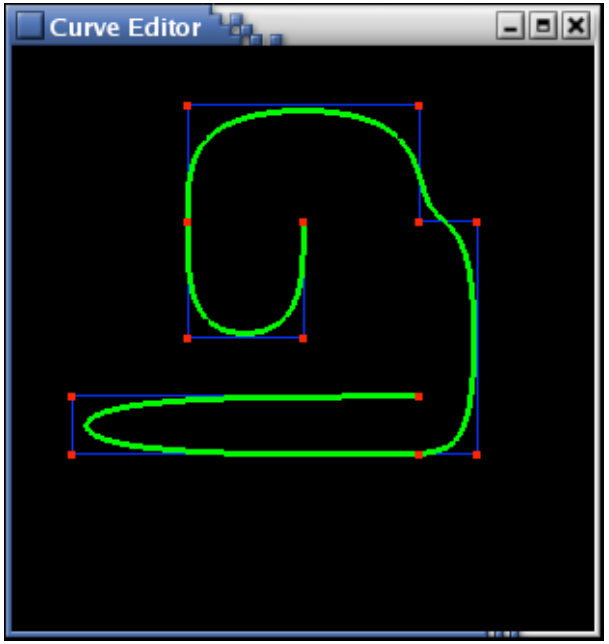


Default BSpline



BSpline with derivative discontinuity

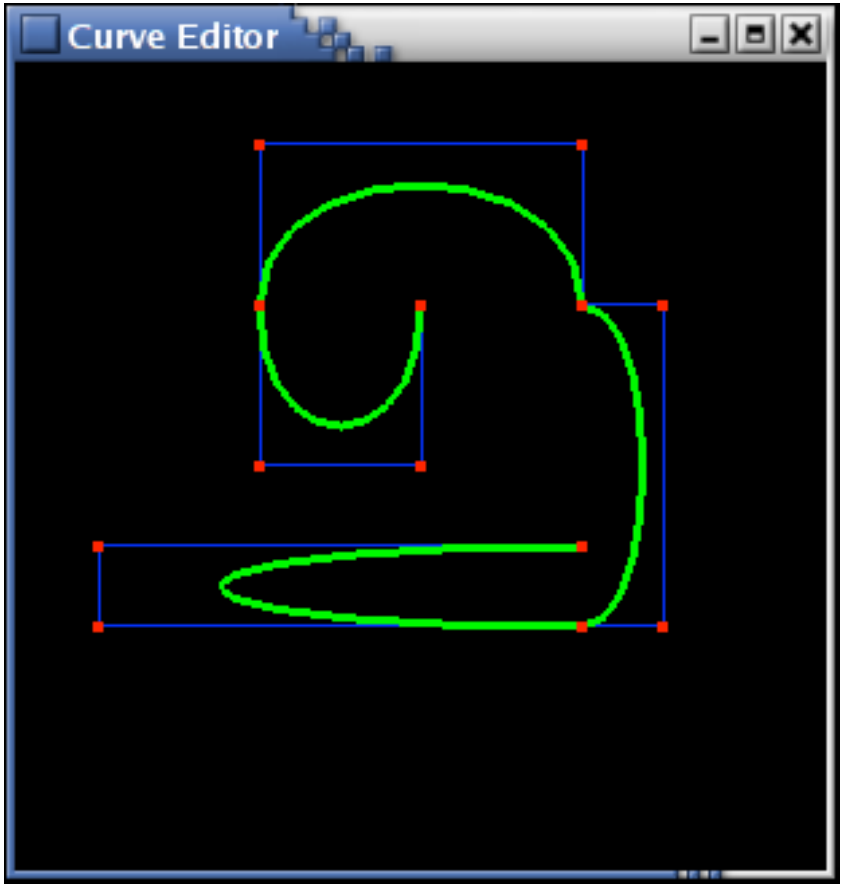
Repeat interior control point



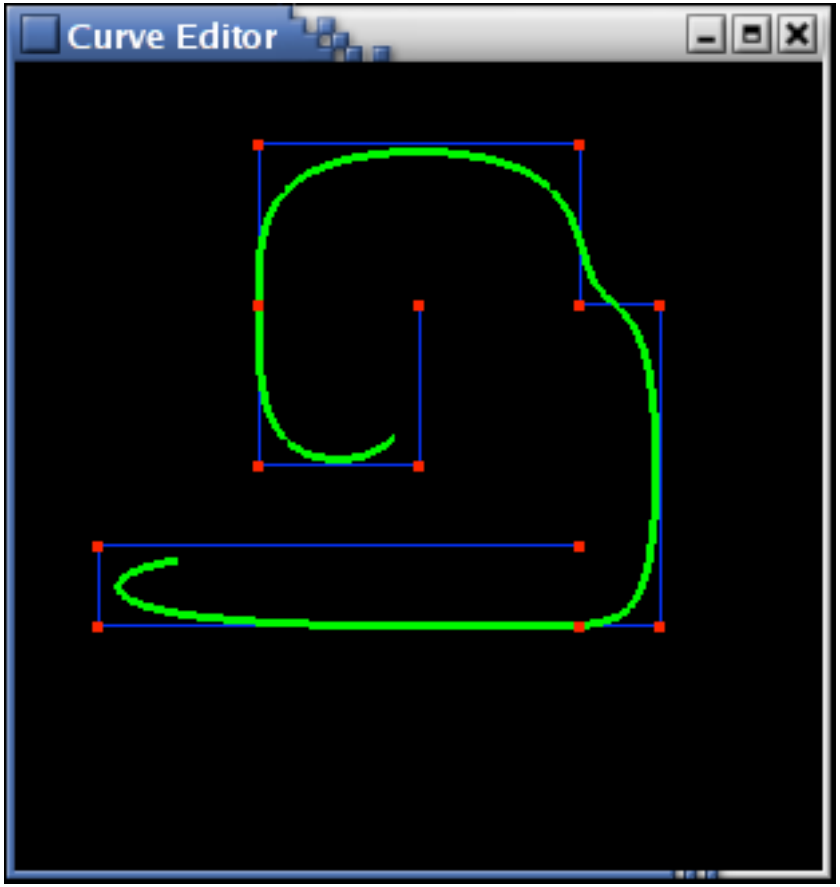
BSpline which passes through end points

Repeat end points

Bézier ≠ B-Spline



Bézier



BSpline

But both are cubics, so one can be converted into the other!

Easy Brain Teaser

- Can you derive formulas for conversion between two different cubic spline matrices, say Bézier and Catmull-Rom?
  - Both are cubics, so you can represent the other curve exactly with new control points $\mathbf{G}_2$

$$\mathbf{G}_1\mathbf{B}_1\mathbf{T}(t) \equiv \mathbf{G}_2\mathbf{B}_2\mathbf{T}(t)$$

$$\mathbf{G}_2 = ???$$

Solution

$$\mathbf{G}_1\mathbf{B}_1\mathbf{T}(t) \equiv \mathbf{G}_2\mathbf{B}_2\mathbf{T}(t)$$

$$\Leftrightarrow \mathbf{G}_1\mathbf{B}_1 = \mathbf{G}_2\mathbf{B}_2$$ **(This must hold for all t!)**

$$B_{B-Spline} = \frac{1}{6}\begin{pmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_{Bezier} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution

$$\mathbf{G}_1\mathbf{B}_1\mathbf{T}(t) \equiv \mathbf{G}_2\mathbf{B}_2\mathbf{T}(t)$$

$$\Leftrightarrow \mathbf{G}_1\mathbf{B}_1 = \mathbf{G}_2\mathbf{B}_2$$ **(This must hold for all t!)**

$$\Rightarrow \boxed{\mathbf{G}_2 = \mathbf{G}_1\mathbf{B}_1\mathbf{B}_2^{-1}}$$

$$B_{B-Spline} = \frac{1}{6}\begin{pmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_{Bezier} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Converting between Bézier & BSpline

- Simple with the basis matrices!
  - Note that this only works for a single segment of 4 control points

$$B_{Bezier} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $P(t) = \mathbf{G}\,\mathbf{B}_1\,\mathbf{T}(t) = \mathbf{G}\,\mathbf{B}_1\,(\mathbf{B}_2^{-1}\mathbf{B}_2)\,\mathbf{T}(t) = (\mathbf{G}\,\mathbf{B}_1\,\mathbf{B}_2^{-1})\,\mathbf{B}_2\,\mathbf{T}(t)$

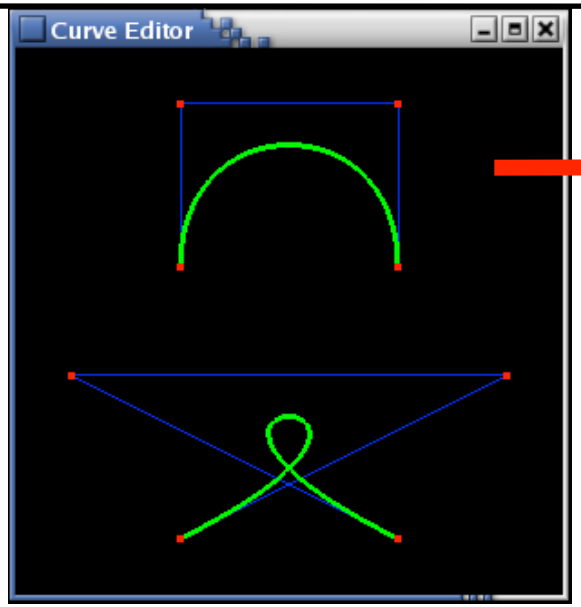
$$B_{B-Spline} = \frac{1}{6}\begin{pmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

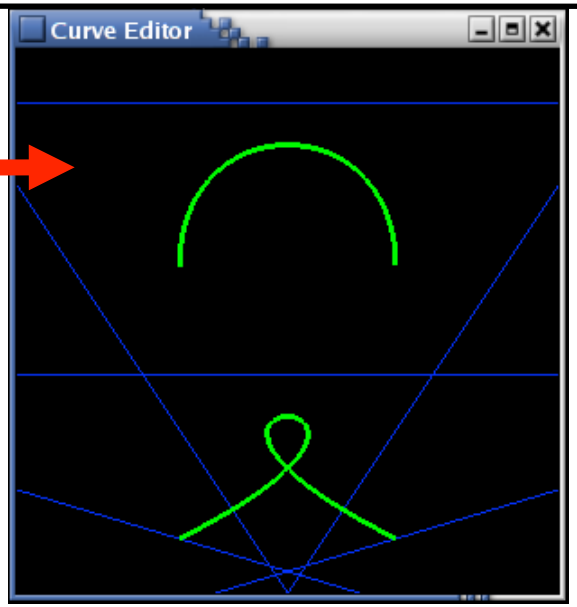
- $\mathbf{G}\,\mathbf{B}_1\,\mathbf{B}_2^{-1}$ are the control points for the segment in new basis.

$$Q(t) = \mathbf{GBT(t)} \quad = \text{Geometry } \mathbf{G} \cdot \text{Spline Basis } \mathbf{B} \cdot \text{Power Basis } \mathbf{T(t)}$$

Converting between Bézier & BSpline

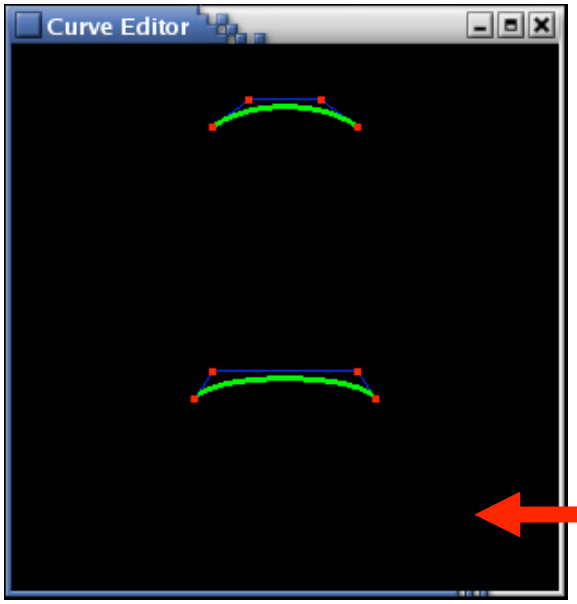
original control points as Bézier

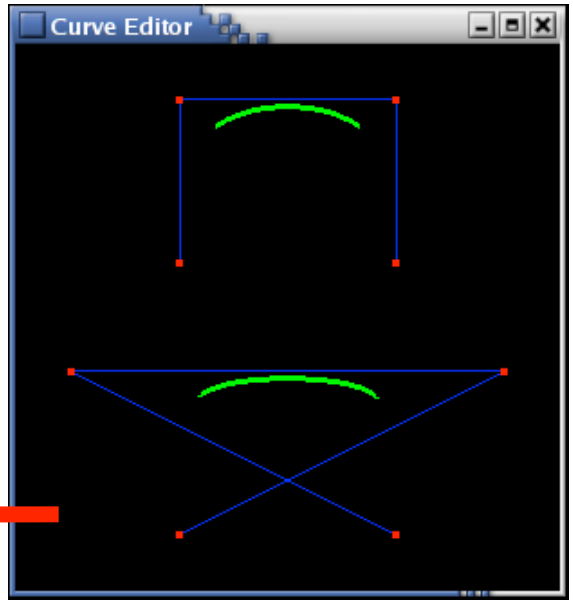




new BSpline control points to match Bézier

new Bézier control points to match BSpline

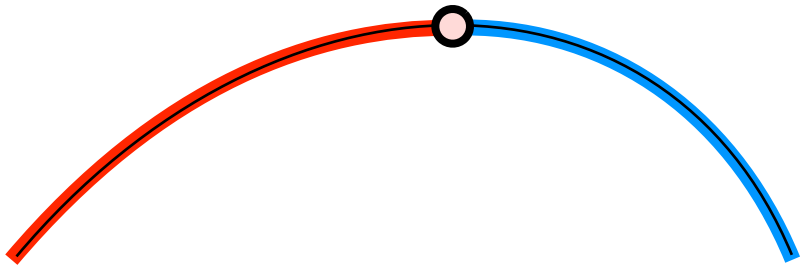




original control points as BSpline

Why Bother with B-Splines?

- Automatic C^2 is nice!
- Also, B-Splines can be split into segments of non-uniform length without affecting the global parametrization.
  - "Non-uniform B-Splines"
  - We'll not do this, but just so you know.



NURBS (Generalized B-Splines)

- Rational cubics
  - Use homogeneous coordinates, just add *w* !
    - Provides a "tension" parameter to control points

- NURBS: Non-Uniform Rational B-Spline
  - **non-uniform** = different spacing between the blending functions, a.k.a. "knots"
  - **rational** = ratio of cubic polynomials (instead of just cubic)
    - implemented by adding the homogeneous coordinate *w* into the control points.

Questions?

That's All for Today, Folks

- Fun stuff to know about function/vector spaces
  - http://en.wikipedia.org/wiki/Vector_space
  - http://en.wikipedia.org/wiki/Functional_analysis
  - http://en.wikipedia.org/wiki/Function_space

- Inkscape is an open source vector drawing program for Mac/Windows. Try it out!