Index

CIE

Tests
3 Tests
50 marks each
(Reduced 40m)

Quiz
10 marks
3 Quiz (best 2)
(20m)

EL
Project
MATLAB
Quiz
Grader

20 marks

40M/100

24/60 a 16/40 minimum

for, SEE for 100 marks

for CIE	45	40
for SEE	35	40

Pattern of SEE (3 hours)

PART A (10 marks)
MCQ's (2 marks from each unit)
PART B (90 marks)
unit 1 and (18 marks from each unit)
2 are compulsory questions and
unit 3, 4 and 5 can be skipped.
form of marks 6m + 6m + 6m
6m + 7m + 5m
10m + 8m

fundamentals of linear algebra, calculus, differential equations

unit 1 :- Rank, Consistency, sloving of system of equation, Gauss elimination Gauss Jorden, Gauss seidel Rayleigh's power method

unit 2 :- Differential Calculus

Differential Calculus: Basics of polar coordinates, polar curves, angle between the radius vector and tange Curvature, radius of curvature-Cartesian, polar & parametric forms (without proof), centre and circle of curvat (formulae only) and problems. Taylor's and Maclaurin's series for a function of a single variable (statements on and problems.

unit 3 :- multivariable functions and partial differentiation

Unit – III | 09 Hrs

Multivariable Functions and Partial Differentiation: Functions of several variables, Partial derivatives definition and notations, higher order partial derivatives-problems, total differentials, total derivatives, composi functions and chain rule-problems. Extreme values for a function of two variables-method of Lagrange multipliers Jacobians - properties and problems.

unit 4 :- Multiple Integrals

Unit – IV | 09 Hrs

Multiple Integrals: Double integrals–Introduction and method of evaluation-problems. Change of order integration and change of variables to polar coordinates-problems. Applications–area, volume and center gravity. Triple integrals-introduction and method of evaluation and problems. Applications-volume of a solid center of gravity.

Unit – V | 09 Hrs

Linear Ordinary Differenti...

unit 5 :- Linear ordinary differential equations of higher order.

Unit – V | 09 Hrs

Linear Ordinary Differential Equations of Higher Order: Standard form of a higher-order linear different equation with constant coefficients. Solution of homogeneous equations–complementary functio Nonhomogeneous equations-concept of inverse differential operator, methods of finding particular integral bas on input function (force function), method of variation of parameters. Equations with functional coefficient Cauchy equation. Applications-simple harmonic motion, LRC circuits.

Course Outco...

Reference Books

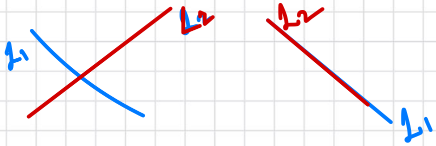
✓ 1	Advanced Engineering Mathematics, E. Kreyszig, 10th Edition (Reprint), 2016, John Wiley & Sc ISBN: 978-04-70458-36-5.
2	Calculus, Saturinino L. Salas, Einar Hille and Garret J. Etgen, 10th Edition, 2022, Wiley India, ISBN: 978-93-90421-96-1.
3	Schaum's Outline of Advanced Calculus, Robert Wrede and Murray Spiegel, 3rd Edition, 2010, McGr Hill Education, ISBN -10: 0071623663, ISBN -13: 978-00-71623-66-7.
4	Calculus, James Stewart, 8th Edition, 2016, Cengage Learning, ISBN: 978-12-85740-62-1.
✓ 5	Higher Engineering Mathematics, B. S. Grewal, 44th Edition, 2015, Khanna Publishers, ISBN: 978-81-93328-49-1.

UNIT-1

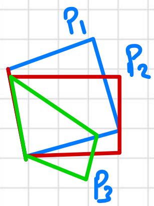
I Linear Algebra

$a_1x + b_1y = c_1 \leftarrow L_1$

$a_2x + b_2y = c_2 \leftarrow L_2$



$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \rightarrow AX = B$$



now, $AX = B$

$$\Rightarrow A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

1) Elementary Transformations

These are 3 elementary row/coloumn transformations

(i) Interchanging of any 2 rows or columns.

(ii) Multiplication of a non zero scalar with any row or column.

$$R_i \longrightarrow KR_i \;/\; C_i \longrightarrow KC_i$$

(iii) Adding any row with K times any other row.

$$R_i + KR_j \;/\; C_i + KC_j$$

Ex:- $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 3 & 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 6 & 9 \\ 1 & 2 & 4 \end{bmatrix}$

$R_2 \longleftrightarrow R_3$ $\qquad$ $R_2 \longrightarrow \frac{1}{3} R_2$

$C = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$ $\qquad$ $R_3 \longrightarrow 3R_3 - R_1$

$D = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 3 \\ 1 & 7 & 9 \end{bmatrix}$

Note:- By applying the elementry row or coloumn transmission we get a matrix equivalent to the given matrix. i.e., A is said to be equivalent to B denoted by $A \sim B$ if one can be obtained from other by applying finite no. of row or coloumn transmissions.

2) Echelon form of a matrix

A non zero matrix A is said to be in the echelon form if

(i) all the zero rows are below non zero rows.

(ii) Number of zeros before the leading entry in any row exceeds number of such zeros in previous row.

(iii) leading entry in each row is non zero.

Examples:-

$$A = \begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{Echelon form}$$

$$B = \begin{bmatrix} 3 & 5 & -7 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

B after $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 3 & 5 & -7 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{Echelon form}$$

C after $R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We are applying only row transmissons to echelon form.

Reading entry in each row is the frist non zero element

Row reduced form :-

A matrix in the echelon form is said to be a row reduced echelon form if the leading entry of each row is one

Reduse the matrices to echelon form

Q1) $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

ans) $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

now, $R_1 \leftrightarrow R_2$ (If in leading row zero is present use this)

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 2R_1$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$A \sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form

The row reduced echelon form is obtained by using the transformation

$R_1 \rightarrow \frac{1}{2} R$ and $R_2 \rightarrow \frac{1}{2} R_2$

$$A \sim \begin{bmatrix} 1 & 3/2 & 5/2 & 2 \\ 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q2) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

ans) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

now, $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form

The row reduced echelon form is obtained by using the transformation

$R_1 \to \frac{1}{2} R$ and $R_2 \to \frac{1}{2} R_2$

$$A \sim \begin{bmatrix} 1/2 & 1 & 3/2 & 1 \\ 0 & -1/2 & -1/2 & -3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q3) $\begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$

ans)

$$A \sim \begin{bmatrix} 3 & 4 & -1 & -6 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 1 & 3 & 13 & 3 \end{bmatrix}$$

$R_1 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -9 \\ 3 & 4 & -1 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -15 \\ 0 & -5 & -40 & -15 \end{bmatrix}$$

$R_2 \to -1/3 R_2$

$R_3 \to -1/5 R_3$

$R_4 \to -1/5 R_4$

$$A \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 1 & 8 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the echelon form

The row reduced echelon form is obtained by using the transformation

$R_1 \to \frac{1}{2} R$ and $R_2 \to \frac{1}{2} R_2$

$$A \sim \begin{bmatrix} 1/2 & 3/2 & 13/2 & 3/2 \\ 0 & 1/2 & 4 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Minor of a matrix

Minor of a matrix is the determinant obtained by deleting few rows and coloums of the matrix

Ex:- $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 3 & 1 & 4 \\ 3 & 1 & 4 & 2 \end{bmatrix}_{3 \times 4}$

minors of order 3 are

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 3 & 1 \\ 3 & 1 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 3 & 7 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 5 & 7 \\ 2 & 1 & 4 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 3 & 5 & 7 \\ 3 & 1 & 4 \\ 1 & 4 & 2 \end{vmatrix}$$

minors of order 2 are

$$\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 5 \\ 3 & 1 \end{vmatrix}$$

3) Rank of a matrix

Let A be non zero $m \times n$ matrix a positive integer r is said to be the rank of A if

(i) there exists at least one non zero minor of order r

(ii) All the minors of order greater than r are zero.

In other words the rank of matrix is the order of the largest non zero minor.

Ex :- $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 0 & -1 & 2 & -4 \end{bmatrix}_{3\times 4}$

minors of order 3 are

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & -1 & 2 \end{vmatrix} = 0, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & -1 & -4 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ 0 & -1 & -4 \end{vmatrix} = 0, \begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ -1 & 2 & -4 \end{vmatrix} = 0$$

minors of order 2

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0, \begin{vmatrix} 4 & 2 \\ -1 & 2 \end{vmatrix} = 10$$

$$\rho(A) = 2$$

$\rho(A)$ → Rank of a matrix denotion

The rank of a matrix in the echelon form is the number of non zero rows.

The rank of a matrix = the equivalent matrix

Note :- If A is a non zero matrix $n \times n$ then rank of A $\rho(A)$ cannot exceed one.

$$\rho(A) \le \min(m,n)$$

If I is an identity matrix of order n then Rank of I is n

$$\rho(I) = n$$

Rank of A and rank of its transpose are the same

$$\rho(A) = \rho(A^T)$$

Next Page

Find the rank of the following

Q4) $A = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$

ans)

$$A = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

$\Rightarrow R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + 3R_1$
$R_4 \to R_4 + 5R_1$

$$\Rightarrow A \sim \begin{bmatrix} -1 & 2 & 3 & 2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix}$$

$R_3 \to R_4 - 2R_2$ & $R_4 \to R_4 - 2R_2$

$$\Rightarrow A \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q5) $A = \begin{bmatrix} 0 & 6 & -3 & -1 \\ -1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

ans)

$$A = \begin{bmatrix} 0 & 6 & -3 & -1 \\ -1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_4 \leftrightarrow R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ -1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 6 & -3 & -1 \end{bmatrix}$$

$\Rightarrow R_2 \to R_2 + R_1$
and $R_3 \to R_3 - 3R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 6 & 2 \\ 0 & 6 & -3 & -1 \end{bmatrix}$$

$\Rightarrow R_3 \to R_3 + 2R_2$
$R_4 \to R_4 - 6R_2$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 3 & -7 \end{bmatrix}$$

$\Rightarrow R_4 \to 4R_4 - 3R_3$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & -14 \end{bmatrix}$$

Q6) For what value of x, will the matrix

$$A = \begin{bmatrix} 3-x & 2 & 2 \\ 1 & 4-x & 0 \\ -2 & -4 & 1-x \end{bmatrix}$$

be of the rank (i) equal to 3
(ii) less than 3

ans) given matrix A is 3×3 matrix

hence the rank of $A = 3$ if $|A| \neq 0$

the equation is

$\Rightarrow -x^3 + 8x^2 - 22x + 18$

$\Rightarrow x^3 - 8x^2 + 22x + 18$

$x = +2, \; x = -3$

for $x = 2$ or 3, $|A| = 0$
$\Rightarrow \rho(A) < 3$

for $x \neq 2$ and $x \neq 3$

$|A| \neq 0 \Rightarrow \rho(A) = 3$

Q7) find the value of P for which the following matrix will be of (i) rank one, 2 & 3

given:- $A = \begin{bmatrix} 3 & p & p \\ p & 3 & p \\ p & p & 3 \end{bmatrix}$

ans) $\rho(A) = 3$ if $|A| \neq 0$

& $\rho(A) < 3$ if $|A| = 0$

$|A| = 0 \Rightarrow 2p^3 - 9p^2 + 27 = 0$

$p = 3$ & $\frac{-3}{2}$

for $P \neq 3$ & $P \neq \frac{-3}{2}$. $|A| \neq 0 \Rightarrow \rho(A) = 3$

for $P = 3$, $A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \rho(A) = 1$

for $P = \frac{3}{2}$, $A = \begin{bmatrix} 3/2 & 3/2 & 3/2 \\ 3/2 & 3/2 & 2/2 \\ 3/2 & 3/2 & 3/2 \end{bmatrix}$

$\sim \begin{bmatrix} 3/2 & 3/2 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \rho(A) = 2$

Q) The value of b such that rank of A is 3

$\rho(A) = 3$

where $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$

ans) $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$

$\Rightarrow R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - bR_1$

$R_4 \to R_4 - 9R_1$

$A \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & b+9 & 3 \end{bmatrix}$

$\Rightarrow R_2 \longleftrightarrow R_3$

$A \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & b+9 & 3 \end{bmatrix}$

$\Rightarrow R_4 \to R_4 - (b+9)R_3$

$A \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-b & 2+b & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -b-6 \end{bmatrix}$

$\Rightarrow \rho(A) = 3$ if $|A| = 0$

$1(2-b).1.(-b-6) = 0$

$b = 2$ & $b = -6$

Solution of system of Linear equations

Consider a system of m linear equations in n unknowns

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x + a_{13}x_3 + \dots\dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots a_{mn}x_n = b_m \end{array}\right\} \quad (1)$$

Equation (1) can be written in the form of matrices which results in the following matrix equation.

$$AX = B \text{ —— } (2)$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If all the elements of matrix B i.e., all b_i's are zero then equation (2) is called homogeneous system of linear equations.

If atleast one B_i is not equal to zero then equation (2) is called a non homogeneous system of linear equations.

Solution of a system of equations is the set of values of the unknows which satisfy all the equations.

Consistency of a system of linear equations

A system of linear equations may processes

(i) unique solution

(ii) Infinite no. of solutions

(iii) No solution.

If the system processes a solution then we say that the system of linear equations is consistent otherwise the system is said to be inconsistent.

The matrix which consists of the elements of the coffeient matrix A and the elements of the matrix of constants B denoted by $[A:B]$ is called the augmented matrix.

Consistency of non-homogenous system of equations

A non homogeneous system of linear equation $AX=B$ is said to be consistent if the rank of

$$\rho(A:B) = \rho(A)$$

otherwise the system is inconsistent

(i) If $\rho(A:B) = \rho(A) = n$ (no. of unknowns) then the system has unique solution

(ii) If rank of $\rho(A:B) = \rho(A) < n$ then the system has infinitely many solutions

(iii) $\rho(A:B) \neq \rho(A)$ then the system has no solution

Consistency of a homogeneous system $AX=0$

It is clear that the homogenous system of linear equations awalys posses a solution it may be a trivial solution (0 solution) or non trivial solution

(i) If $\rho(A) = n$ (no. of unknown) then the system has trivial solution or zero solution

(ii) If $\rho(A) < n$ then the system has non trivial or infintely many solutions.

Test for consistency and hence solve

(i)
$$\begin{aligned} x+2y+2z &= 5 \\ 2x+y+3z &= 6 \\ 3x-y+2z &= 4 \\ x+y+z &= -1 \end{aligned}$$

ans)

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 2 & 5 \\ 2 & 1 & 3 & 6 \\ 3 & -1 & 2 & 4 \\ 1 & 1 & 1 & -1 \end{array}\right]$$

Reduce the augmented matric to echelon form.

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & 2 & 5 \\ 0 & -3 & -1 & -4 \\ 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -60 \end{array}\right]$$

$\rho(A:B) = 4 \neq \rho(A) = 3$

Thus the system is inconsistent

If the system of linear equations is consistent that is if the rank

$$\rho(A:B) = \rho(A)$$

then we solve the system of equations from the echelon form of a matrix $[A:B]$ back substitution method.

Test for consistency and hence solve

Q10)
$$\begin{aligned} x+y+z &= 3 \\ x+3z+2y &= 4 \\ x+4y &= 6-9z \end{aligned}$$

ans)

$x+y+z=3$

$x+3z+2y=4 \longrightarrow x+2y+3z=4$

$x+4y=6-9z \longrightarrow x+4y+9z=6$

The corresponding augmented matrix is as follows:-

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array}\right]$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

$\rho(A:B) = 3$, $\rho(A) = 3$, $n = 3$

$\Rightarrow \rho(A:B) = \rho(A) = n = 3$

$\Rightarrow$ Thus the system of equations is consistent and has unique solution

The system of equations corespoinding to the above echelon form is

$x+y+z=3$ —— ①
$y+2z=1$ —— ②
$z=0$ —— ③

we obtain the solution of the given system by back substitution i.e., $z=0$

from ③, $z=0$

put $z=0$ in ② $\Rightarrow y=1$
put $z=0$ & $y=1$ in ① $\Rightarrow x=2$

$(x,y,z)=(2,1,0)$

Q11) Test for the consistency to solve

$$2x_1+x_2+2x_3+x_4=6$$
$$6x_1-6x_2+6x_3+12x_4=36$$
$$4x_1+3x_2+3x_3-3x_4=-1$$
$$2x_1+2x_2-x_3+x_4=10$$

ans)

Consistency Check and Solution

To check for consistency and solve the system, we can use Gaussian elimination on the augmented matrix.

The given system of equations is:

1. $2x_1+x_2+2x_3+x_4=6$
2. $6x_1-6x_2+6x_3+12x_4=36$
3. $4x_1+3x_2+3x_3-3x_4=-1$
4. $2x_1+2x_2-x_3+x_4=10$

Step 1: Form the Augmented Matrix

First, we write the system as an augmented matrix $[A|B]$.

$$\begin{pmatrix} 2 & 1 & 2 & 1 & | & 6 \\ 6 & -6 & 6 & 12 & | & 36 \\ 4 & 3 & 3 & -3 & | & -1 \\ 2 & 2 & -1 & 1 & | & 10 \end{pmatrix} To simplify, we can divide the second row($$

Step 2: Apply Row Operations

We will now perform row operations to get the matrix into row echelon form. Let's start by swapping R_1 and R_2 to get a leading 1.

$$R_1 \leftrightarrow R_2$$

$$\begin{pmatrix} 1 & -1 & 1 & 2 & | & 6 \\ 2 & 1 & 2 & 1 & | & 6 \\ 4 & 3 & 3 & -3 & | & -1 \\ 2 & 2 & -1 & 1 & | & 10 \end{pmatrix}$$

Now, let's create zeros in the first column below the pivot.

- $R_2 \to R_2 - 2R_1$
- $R_3 \to R_3 - 4R_1$
- $R_4 \to R_4 - 2R_1$

This gives us:

$$\begin{pmatrix} 1 & -1 & 1 & 2 & | & 6 \\ 0 & 3 & 0 & -3 & | & -6 \\ 0 & 7 & -1 & -11 & | & -25 \\ 0 & 4 & -3 & -3 & | & -2 \end{pmatrix} Simplify R_2 by dividing by 3 (R_2 \to \frac{R_2}{3}). \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Next, create zeros in the second column below the pivot.

- $R_3 \to R_3 - 7R_2$
- $R_4 \to R_4 - 4R_2$

This gives us:

$$\begin{pmatrix} 1 & -1 & 1 & 2 & | & 6 \\ 0 & 1 & 0 & -1 & | & -2 \\ 0 & 0 & -1 & -4 & | & -11 \\ 0 & 0 & -3 & 1 & | & 6 \end{pmatrix}$$

Finally, let's work on the third and fourth columns.

- $R_3 \to -R_3$
- $R_4 \to R_4 + 3R_3$ (using the new R_3)
- $R_4 \to \frac{R_4}{13}$

This results in the row echelon form:

$$\begin{pmatrix} 1 & -1 & 1 & 2 & | & 6 \\ 0 & 1 & 0 & -1 & | & -2 \\ 0 & 0 & 1 & 4 & | & 11 \\ 0 & 0 & 0 & 1 & | & 3 \end{pmatrix}$$

The rank of the coefficient matrix A is 4, and the rank of the augmented matrix $[A|B]$ is also 4. Since the rank equals the number of variables, the system is **consistent** and has a **unique solution**. ✨

Step 3: Solve Using Back Substitution

We can now write the system of equations from the row echelon form:

1. $x_1-x_2+x_3+2x_4=6$
2. $x_2-x_4=-2$
3. $x_3+4x_4=11$
4. $x_4=3$

From equation (4), we have $x_4=3$.

Substitute $x_4=3$ into equation (3):

$x_3+4(3)=11 \Longrightarrow x_3+12=11 \Longrightarrow x_3=-1$

Substitute $x_4=3$ into equation (2):

$x_2-3=-2 \Longrightarrow x_2=1$

Substitute $x_2=1$, $x_3=-1$, and $x_4=3$ into equation (1):

$x_1-(1)+(-1)+2(3)=6 \Longrightarrow x_1-2+6=6 \Longrightarrow x_1+4=6 \Longrightarrow x_1=2$

The final unique solution is $(2,1,-1,3)$.

Q12) $4x - 2y + 6z = 8$
$x + y - 3z = -1$
$15x - 3y + 9z = 21$

ans)
$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21$$

now

$$[A:B] = \left[\begin{array}{ccc:c} 4 & -2 & 6 & 8 \\ 1 & 1 & -1 & -1 \\ 15 & -3 & 9 & 21 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 2 & -1 & 3 & 4 \\ 1 & -1 & 3 & -1 \\ 5 & -1 & 3 & 7 \end{array}\right]$$

$\Rightarrow R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 3 & 7 \end{array}\right]$$

$\Rightarrow R_2 \to R_2 - 2R_1$ & $R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & -3 & 9 & 2 \\ 0 & -1 & 3 & 2 \end{array}\right]$$

$\Rightarrow R_3 \to R_1 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -3 & -1 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad \rho(A:B) = 2 = \rho(A) < 3$

The system is consistent & has infinite number of solution.

The equations corresponding to the above echelon form is

$$x + y - 3z = -1$$
$$-y + 3z = 2$$

In the above echelon form the column 3 is non-pivotal form

z is the variable corresponding to the 3^{rd} coloumn which is called the free variable

let $z = k$

$\Rightarrow y = 3k - 2$

$\Rightarrow x = 1$

solution is $(x, y, z) = [1, 3k-2, k]$

Q12) Test for consistency and solve
$x_1 + x_2 - x_3 = 0$
$2x_1 - x_2 + x_3 = 3$
$4x_1 + 2x_2 - 2x_3 = 2$

ans)
$$x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 3$$
$$4x_1 + 2x_2 - 2x_3 = 2$$

$$\Rightarrow [A:B] = \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 2 & 1 & -1 & 1 \end{array}\right]$$

$\Rightarrow R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$\Rightarrow R_3 \to 3R_3 - R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow R_2 \to 1/3 R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A:B) = \rho(A) = 2 < 3$

system had infinite number of solutions

$x_1 + x_2 - x_3 = 0$

$-x_2 + x_3 = 1$

let $x_3 = k$

$x_2 = k - 1$

$x_1 = 1$

$\therefore (x_1, x_2, x_3) = (1, k-1, k)$

Q13) $2x - y + 3z = 0$
$3x - 2y + z = 0$
$x - 4y + 5z = 0$

ans)

$2x - y + 3z = 0$
$3x - 2y + z = 0$
$x - 4y + 5z = 0$

The corresponding coefficient matrix

$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2 < 3$

$\Rightarrow$ The system has non-trivial / infinite no. of solutions.

$x - 4y + 5z = 0$

$y - z = 0$

put $z = k$

$\Rightarrow y = k, x = k, z = k$

Q14) Solve and check consistency

$x + y - z + u = 0$
$x - y + 2z - u = 0$
$3x + y + u = 0$

ans)

Consistency Check

The given system of equations is:

1. $x + y - z + u = 0$
2. $x - y + 2z - u = 0$
3. $3x + y + 0z + u = 0$

This is a **homogeneous system** because all constant terms are zero. A homogeneous system is **always consistent**. It will always have at least the trivial solution ($x = y = z = u = 0$).

To determine if there are non-trivial solutions, we analyze the rank of the coefficient matrix.

Solving the System

We can solve the system using Gaussian elimination.

Step 1: Form the Augmented Matrix

Write the coefficients of the variables into an augmented matrix.

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & -1 & 0 \\ 3 & 1 & 0 & 1 & 0 \end{array}\right)$$

Step 2: Apply Row Operations

Our goal is to reduce the matrix to row echelon form.

- $R_2 \to R_2 - R_1$
- $R_3 \to R_3 - 3R_1$

This gives us:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 \\ 0 & -2 & 3 & -2 & 0 \end{array}\right)$$

Now, we can eliminate the third row.

- $R_3 \to R_3 - R_2$

This results in the row echelon form:

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -2 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Step 3: Back Substitution

The rank of the matrix is 2, while the number of variables is 4. This means we will have $4 - 2 = 2$ free variables. Let's choose z and u as our free variables.

Let $z = k_1$ and $u = k_2$, where k_1 and k_2 are any real numbers.

Now, we write the equations from the row echelon form:

1. $x + y - z + u = 0$
2. $-2y + 3z - 2u = 0$

From equation (2), solve for y:

$-2y + 3(k_1) - 2(k_2) = 0$

$-2y = -3k_1 + 2k_2$

$y = \frac{3}{2}k_1 - k_2$

Now, substitute y, z, and u into equation (1) to solve for x:

$x + (\frac{3}{2}k_1 - k_2) - k_1 + k_2 = 0$

$x + \frac{1}{2}k_1 = 0$

$x = -\frac{1}{2}k_1$

This confirms the system has infinitely many solutions, defined by the two parameters k_1 and k_2. 🤓

Q15) Find the values of μ and λ for which the systems

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

has a (i) unique solution
(ii) infinitely many solutions
(iii) no solution

ans) The given system of equations are as follows:-

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\Rightarrow \quad R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$$\Rightarrow \quad R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

now
(i) for unique solution

$$\rho(A) = 3$$

Step-by-Step Analysis

To find the conditions for λ and μ, we'll use Gaussian elimination to convert the system's augmented matrix into row echelon form.

The system of equations is:

1. $x + y + z = 6$
2. $x + 2y + 3z = 10$
3. $x + 2y + \lambda z = \mu$

1. Form the Augmented Matrix

First, we write the system as an augmented matrix $[A|B]$.

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 1 & 2 & 3 & | & 10 \\ 1 & 2 & \lambda & | & \mu \end{pmatrix}$$

2. Apply Row Operations

We perform row operations to simplify the matrix.

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to create zeros in the first column.

$$\begin{pmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 2 & | & 4 \\ 0 & 1 & \lambda-1 & | & \mu-6 \end{pmatrix} Next, apply R_3 \to R_3 - R_2 to create a zero in th$$

The nature of the solution now depends entirely on the last row, which represents the equation:

$(\lambda - 3)z = \mu - 10$

3. Analyze the Conditions 🤔

(i) Unique Solution

For a unique solution, the number of non-zero rows in the coefficient matrix (the rank) must be equal to the number of variables. Here, we need the rank to be 3. This requires that the term $\lambda - 3$ is **not zero**.

- $\lambda - 3 \neq 0 \implies \lambda \neq 3$
- If $\lambda \neq 3$, the value of μ doesn't matter, as we can always solve for z.

(ii) Infinitely Many Solutions

For **infinitely many solutions**, the rank must be less than the number of variables, and the system must be consistent. This happens when the last row is entirely zeros, representing the identity $0z = 0$.

- This requires both terms in the last row to be zero.
- $\lambda - 3 = 0 \implies \lambda = 3$
- $\mu - 10 = 0 \implies \mu = 10$

(iii) No Solution

For **no solution**, the system must be inconsistent. This occurs when the last row represents a contradiction, like $0z = k$, where k is a non-zero number.

- This requires the coefficient of z to be zero, but the constant on the right side to be non-zero.
- $\lambda - 3 = 0 \implies \lambda = 3$
- $\mu - 10 \neq 0 \implies \mu \neq 10$

Q16) Find the values of λ for which the system

$$x+y+z=1, \quad x+2y+4z=\lambda$$

$$x+4y+10z=\lambda^2$$

has a solution in each case.

ans) The corresponding argument matrix is

$$A = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$$

$\Rightarrow R_2 \to R_2 - R_1, \; R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{array}\right]$$

$\Rightarrow R_3 \to R_3 - 3R_2$

$$\sim \left(\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right)$$

$\Rightarrow \rho(A) = 2$ & the system has a sol^n if $\rho(A:B) = 2$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$\lambda = 1, \; \lambda = 2$

The equations corresponding to the alone echelon form is

$$x+y+z=1$$

$$y+3z=\lambda-1$$

For $\lambda = 1$, we get $x+y+z=1$
$y+3z=0$

Let $z = k$, $y = -3k$, $x = 2k+1$, $(x,y,z) = (2k+1, -3k, k)$

for $\lambda = 2$, we get $x+y+z=1$
$y+3z=1$

Let $z = l$, $y = 1-3l$, $x = 2l$

$$(x,y,z) = (2l, 1-3l, l)$$

3

Q17) Determine the values of λ for the set of equations

$$3x - y + \lambda z = 0$$
$$2x + y + z = 2$$
$$x - 2y - \lambda z = -1$$

will fail to have a unique solution and for what values of λ are the equation inconsistent?

ans) The matrix would be as follows:-

$$[A:B] = \begin{bmatrix} 3 & -1 & \lambda & : & 0 \\ 2 & 1 & 1 & : & 2 \\ 1 & -2 & -\lambda & : & -1 \end{bmatrix}$$

$\Rightarrow R_3 \longleftrightarrow R_1$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$[A:B] \sim \begin{bmatrix} 1 & -2 & -\lambda & : & -1 \\ 0 & 5 & 1+2\lambda & : & 4 \\ 0 & 0 & 2\lambda-1 & : & 1 \end{bmatrix}$$

$\Rightarrow \det(A:B) = \frac{1}{2} = \lambda$

Q18) Find the values of b for which the system has non-trivial solution to find them

$$2x + 3by + (3b+4)z = 0$$

$$x + (b+4)y + (4b+2)z = 0$$

$$x + 2(b+1)y + (3b+4)z = 0$$

ans)

$$[A:B] = \begin{bmatrix} 2 & 3b & 3b+4 & : & 0 \\ x & b+4 & 4b+2 & : & 0 \\ x & 2b+2 & 3b+4 & : & 0 \end{bmatrix}$$

$\Rightarrow R_1 \to R_1 - R_3$

$$\sim \begin{bmatrix} 2-x & 3b-2b+2 & 3b+4-3b+4 & : & 0 \\ x & b+4 & 4b+2 & : & 0 \\ x & 2b+2 & 3b+4 & : & 0 \end{bmatrix}$$

$\Rightarrow \det(A) = b = 2, -2$

$\Rightarrow$ now case 1, $b = 2$

$y = \frac{-5}{3}(3k) \Rightarrow -5k$

, for $z = 3k$, $x = 0$

so, $(x, y, z) = (0, -5k, 3k)$

case 2:- $b = -2$

$y = k$, $x = 4k$, $z = k$

$(x, y, z) = (4k, k, k)$

Q19) for what values of λ does the following system of equation posses a national solution solution solve for real values of λ

$$3x + y - \lambda z = 0$$
$$4x - 2y - 3z = 0$$
$$2\lambda x + 4y - \lambda z = 0$$

Gauss Elimination method

The augmented matrix corresponding to given system is reduced to echelon form using intern reduces matrix A to upper triangular matrix and the sol^n is obtained by back substitution

Q20) $2x - 7y + 4z = 9$
$x + 9y - 6z = 1$
$-3x + 8y + 5z = 6$

ans)
$$\begin{bmatrix} 2 & -7 & 4 & : & 9 \\ 1 & 9 & -6 & : & 1 \\ -3 & 8 & 5 & : & 6 \end{bmatrix}$$

$\Rightarrow \quad R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 9 & -6 & : & 1 \\ 2 & -7 & 4 & : & 9 \\ -3 & 8 & 5 & : & 6 \end{bmatrix}$$

$\Rightarrow R_2 \longrightarrow R_2 - 2R_1, \; R_3 \longrightarrow R_3 + 3R_1$

$$\sim \begin{bmatrix} 1 & 9 & -6 & : & 1 \\ 0 & -25 & 16 & : & 7 \\ 0 & 35 & -13 & : & 9 \end{bmatrix}$$

$\Rightarrow R_3 \longrightarrow 25R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 9 & -6 & : & 1 \\ 0 & -25 & 16 & : & 7 \\ 0 & 0 & 235 & : & 470 \end{bmatrix}$$

$x + 9y - 6z = 1$

$-2y + 16z = 7$

$235z = 470$

$\Rightarrow x = 4, \; y = 1, \; z = 2$

Q21) $2x_1 + 2x_2 + 4x_3 = 18$
$x_1 + 3x_2 + 2x_3 = 13$
$3x_1 + x_2 + 3x_3 = 14$

ans)
$$\begin{bmatrix} 2 & 2 & 4 & : & 18 \\ 1 & 3 & 2 & : & 13 \\ 3 & 1 & 3 & : & 14 \end{bmatrix}$$

$\Rightarrow R_1 \longleftrightarrow R_2$

$$\begin{bmatrix} 1 & 3 & 2 & : & 13 \\ 2 & 2 & 4 & : & 18 \\ 3 & 1 & 3 & : & 14 \end{bmatrix}$$

$\Rightarrow R_2 \longrightarrow R_2 - 2R_1, \; R_3 \longrightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 3 & 2 & : & 13 \\ 0 & -4 & 0 & : & -8 \\ 0 & -8 & -3 & : & -25 \end{bmatrix}$$

$\Rightarrow R_2 \longleftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 3 & 2 & : & 13 \\ 0 & -8 & -3 & : & 25 \\ 0 & -4 & 0 & : & -8 \end{bmatrix}$$

$\Rightarrow x + 3y + 2z = 13$

$-8y - 3z = 25$

$-y = -8$

$x = 1, \; y = 2, \; z = 3$

Q3) $x_1 + 2x_2 - x_3 = 3$
$2x_1 - 2x_2 + 3x_3 = 2$
$3x_1 - x_2 + 2x_3 = 1$
$x_1 - x_2 + x_3 = -1$

Gauss Jordan method

This is another method of solving a non homogeneous of equations

In this method the augmented matrix

$$[A:B] = \begin{bmatrix} a_1 & b_1 & c_1 & : d_1 \\ a_2 & b_2 & c_2 & : d_2 \\ a_3 & b_3 & c_3 & : d_3 \end{bmatrix}$$

is recused to

$$[A:B] \sim \begin{bmatrix} a_1'' & 0 & 0 & : d_1'' \\ 0 & b_2'' & 0 & : d_2'' \\ 0 & 0 & c_3'' & : d_3'' \end{bmatrix}$$

Diagonal matrix using elementry row transmisions and the solution is obtained

$$a_1'' x = d_1''$$

$$\Rightarrow x = \frac{d_1''}{a_1''}$$

$$y = \frac{d_2''}{b_2''}$$

$$z = \frac{d_3''}{c_3''}$$

Q24) Solve by Gauss Jorden method

$$2x + y + 3z = 1$$

$$4x + 4y + 7z = 1$$

$$2x + 5y + 9z = 3$$

ans) The corresponding augmented matrix is

$$[A:B] = \begin{bmatrix} 2 & 1 & 3 & : 1 \\ 4 & 4 & 7 & : 1 \\ 2 & 5 & 9 & : 3 \end{bmatrix}$$

$\Rightarrow R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 1$

$$\sim \begin{bmatrix} 2 & 1 & 3 & : 1 \\ 0 & 2 & 1 & : -1 \\ 0 & 4 & 6 & : 2 \end{bmatrix}$$

$\Rightarrow R_1 \to 2R_1 - R_2,\ R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 4 & 0 & 5 & : 3 \\ 0 & 2 & 1 & : -1 \\ 0 & 0 & 4 & : 4 \end{bmatrix}$$

$\Rightarrow R_3 \to \frac{1}{4} R_3$

$$\sim \begin{bmatrix} 4 & 0 & 5 & : 3 \\ 0 & 2 & 1 & : -1 \\ 0 & 0 & 1 & : 1 \end{bmatrix}$$

$\Rightarrow R_1 \to R_1 - 5R_3,\ R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 4 & 0 & 0 & : -2 \\ 0 & 2 & 0 & : -2 \\ 0 & 0 & 1 & : 1 \end{bmatrix}$$

$4x = -2,\ x = -\frac{1}{2}$

$2y = -2 \Rightarrow y = -1$

$z = 1$

25) $x + y + z = 8$
$x - y + 2z = 6$
$3x + 5y - 7z = 14$

ans) $$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : 8 \\ 1 & -1 & 2 & : 6 \\ 3 & 5 & -7 & : 14 \end{bmatrix}$$

$\Rightarrow R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : 8 \\ 1 & -1 & 2 & : 6 \\ 0 & 2 & -4 & : -7 \end{bmatrix}$$

$\Rightarrow \quad R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & -1 & 2 & 6 \\ 0 & 2 & -4 & -7 \end{array}\right]$$

$\Rightarrow$

Q26) Find inverse using gauss jordan matrix. Let A be a non-singular matrix consider the matrix [A|I] where I is the identity matrix of size same as A. Using elementary road transformations reduse matrix A of [A | I] to identity matrix which results in the change of the identity matrix I. The resultant matrix is the inverse of A.

Q27) Use Gauss Jorden method to find inverse of A

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

ans)

consider,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow R_2 \to R_2 - R_1, R_3 \to R_3 + 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$\Rightarrow R_1 \to 2R_1 - R_2, R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 12 & 3 & -1 & 0 \\ 0 & 2 & 6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

$\Rightarrow R_1 \to R_1 + 3R_3, R_2 \to 2R_2 - 3R_3$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 6 & 2 & 3 \\ 0 & 4 & 0 & -5 & -1 & -3 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

$\Rightarrow R_1 \to \frac{1}{2} R_1, R_2 \to \frac{1}{4} R_2, R_3 \to \frac{-1}{4} R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

$$\Rightarrow A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Q28) $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

ans)

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$\Rightarrow$ consider,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$\Rightarrow R_1 \to R_1 - 3R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$\Rightarrow R_1 \to R_1 - 3R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Diagonally dominant system:-

consider $a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$
$a_3x + b_2y + c_3z = d_3$

$|a_1| \geq |b_1| + |c_1|$,

$|b_1| \geq |a_2| + |c_2|$,

$|c_3| \geq |a_3| + |b_3|$

The above system is said to be diagonally dominant if it satisfies

$|a_1| \geq |b_1| + |c_1|$,

$|b_1| \geq |a_2| + |c_2|$,

$|c_3| \geq |a_3| + |b_3|$

Gauss Siden method:-

This method is applicable only if the system is diagonally dominant. This is an iterative method where an initial solution is assumed and in the successive process we attain at the required solution.

The working procedure of this method is discussed in the following example:-

Q29) Solve the following system of equations using gauss sidel method

$6x + 15y + 2z = 72$
$x + y + 54z = 110$
$27x + 6y - z = 85$

ans)

Here $|15| \geq |6| + |2|$
$|54| > |1| + |1|$
$|27| > |6| + |1|$

The system in the given form is not diagonally dominant by rearranging the equation it can be reduced to diagonally dominant

$27x + 6y - z = 85$ —— ①
$6x + 15y + 2z = 72$ —— ②
$x + y + 54z = 110$ —— ③

from ①, we get

$x = \frac{85 - 6y + z}{27}$ —— ④

from ②, we get

$y = \frac{72 - 6x - 2z}{15}$ —— ⑤

from ③, we get

$z = \frac{110 - x - y}{54}$ —— ⑥

let the initial solution be
$(x^{(0)}, y^{(0)}, z^{(0)}) = (0, 0, 0)$

first approximation:-

put $y = 0$ & $z = 0$ in ④

we get, $x^{(1)} = \frac{85}{27} = 3.1481$

put $x = 3.1481$, $z = 0$

$y = \frac{72 - 6(3.1481) - 2(0)}{15}$
$\Rightarrow 3.5408$

put $x = 3.1481$, $y = 3.5408$

$\Rightarrow z^{(1)} = 1.9132$

$(x^{(1)}, y^{(1)}, z^{(1)}) = (3.1481, 3.5408, 1.9132)$

second approximation

$y = 3.5408$, $z = 1.9132$ in ④
$x^{(2)} = 2.4322$

put $x = 2.4322$, $z = 1.9132$, in (5)

$y^{(2)} = 3.5720$

put $x = 2.4322$, $y = 3.5720$ in (6)

$z^{(2)} = 1.9258$

3^{rd} approximation :-

$x^{(3)} = 2.4257$, $y^{(3)} = 3.5729$, $z^{(3)} = 1.9260$

4^{th} approximation:-

$x^{(4)} = 2.4257$, $y^{(4)} = 3.5730$, $z^{(4)} = 1.9259$

The 3^{rd} and 4^{th} approximated values are coinciding

hence the approximate value is as given above

Q30) Solve by gaussian method

$2x + 2y + 10z = 14$

$10x + y + z = 12$

$2x + 10y + z = 13$

ans) diagonally dominant system is

$10x + y + z = 12$

$2x + 10y + z = 13$

$2x + 2y + 10z = 14$

$\Rightarrow x = \frac{12 - y - z}{10}$ $\quad z = \frac{14 - 2x - 2y}{10}$

$y = \frac{14 - 2x - 2y}{10}$

Let the initial solution be

$(x^{(0)}, y^{(0)}, z^{(0)}) = (0, 0, 0)$

	x	y	z
I^{st}	1.2	1.06	1.16
II^{nd}	0.978	0.988	1.0884
III^{rd}	0.99	0.99	1.09
IV^{th}	0.99	0.99	1.09

The 3^{rd} and 4^{th} approximated values are coinciding

hence the approximate value is as given above

Q31) $5x_1 - x_2 + x_3 = 10$
$x_1 + x_2 + 5x_3 = -1$
$2x_1 + 4x_2 = 12$

ans) diagonal dominant system is

$5x_1 - x_2 + x_3 = 10$
$2x_1 + 4x_2 = 12$
$x_1 + x_2 + 5x_3 = -1$

$x_1 = \frac{10 + x_2 - x_3}{5}$

$x_2 = \frac{12 - 2x_1}{4}$

$x_3 = \frac{-1 - x_1 - x_2}{5}$

	x	y	z
I^{st}	2	2	-1
II^{nd}	2.6	1.7	-1.06
III^{rd}	2.55	1.7	-1.055
IV^{th}	2.55	1.7	-1.0556

The 3rd and 4th approximated values are coinciding

hence the approximate value is as given above

Q32) $10x - 2y - z - u = 3$

$-2x + 10y - z - u = 15$

$-x - y + 10z - 2u = 27$

$-x - y - 2z + 10u = -9$

ans) $x = \dfrac{3 + 2y + z + u}{10}$

$y = \dfrac{15 + 2x + z + u}{10}$

$z = \dfrac{27 + x + y + 2u}{10}$

$u = \dfrac{-9 + x + y + 2z}{10}$

	x	y	z	u
I^{st}	0.3	1.56	2.88	-0.13
II^{nd}	0.88	1.95	2.95	-0.024
III^{rd}	0.98	1.98	2.99	-0.0062
IV^{th}	0.99	1.99	2.99	-0.0008
V^{th}	0.99	1.99	2.99	-0.0001

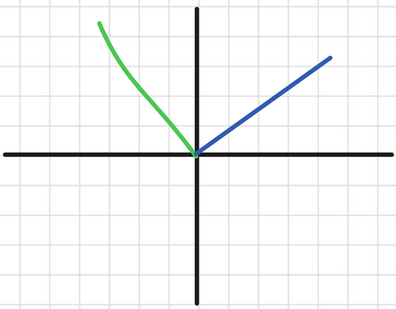
The 4th and 5th approximated values are coinciding

hence the approximate value is as given above

Eigen values and eigen vectors

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\theta = \frac{\pi}{2}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$



$X = \lambda$

$$T(x) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T.u. = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$T.u. = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$T.u. = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Let A be a square matrix of order n a real number λ is said to be the eigen value of A if there exists a column matrix X such that $AX = \lambda X$, λ is called the eigen value and X is the corresponding eigen vector.

consider

$$AX = \lambda X \text{ ——— } ①$$
$$AX = \lambda I X$$
$$AX - \lambda I X = 0$$
$$(A - \lambda I)X = 0 \text{ ——— } ②$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$\Rightarrow$ Equation ② represents a homogeneous system of linear equations & this system possesses a non-trivial solⁿ if

$$|A - \lambda I| = 0 \text{ ——— } ③$$

Expanding the determinant in ③ we get, an n^{th} degree equation in λ. solving which we get n roots for λ. These roots are called the eigen values or characteristic roots or latent roots. and equation ③ is called characteristic equation.

substituting each value of λ in ② & solving the homogeneous system of equations we get the vectors X corresponding to each λ. Which are called eigen vectors.

Find the eigen values & the corresponding eigen vectors

Q1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

ans) Let λ be the Eigen value & the X be the corresponding Eigen vector

$$\Rightarrow AX = \lambda X$$

$$\Rightarrow [A - \lambda I]X = 0 \text{ ——— } ①$$

The corresponding characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 = 0$$

$$\lambda = 4 \text{ \& } -1$$

put $\lambda = -1$ in ① which is

$$\Rightarrow (1-\lambda)x + 2y = 0$$

$$\Rightarrow 3x + (2-\lambda)y = 0$$

$$\Rightarrow \begin{array}{l} 2x + 2y = 0 \\ 3x + 3y = 0 \end{array} \Rightarrow x + y = 0 \Rightarrow y = -x$$
$$\& \; x = K, \; y = -K$$

$$X_1 = \begin{bmatrix} K \\ -K \end{bmatrix}$$

put $\lambda = 4$ in ①

$\Rightarrow$ $-3x + 2y = 0$ $\Rightarrow y = \frac{3x}{2}$
$3x - 2y = 0$

$y = \frac{3l}{2}$

$X_2 = \begin{bmatrix} l \\ 3l/2 \end{bmatrix}$

Properties of Eigen values & vectors

A square matrix of order n will posses n eigen values which maybe distinct or not.

If all the n eigen values of A are distinct then there exists exactly one eigen vector corresponding to each of them.

If A is a non singular matrix then all the eigen values are non-zero.

If λ is the eigen value of A then the eigen value of A^{-1} is $\frac{1}{\lambda}$.

Any matrix A and it's transpose will have the same eigen values.

The eigen values of a triangular matrix diagonal elements of the given matrix.

Sum of the eigen values of the matrix is its trace.

The product of the eigen values of a matrix A is equal to it's determinant.

If $\lambda_1, \lambda_2 \dots \lambda_n$ are the eigen values of a matrix X then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \dots \lambda_n^m$ provided m is a positive integer.

Q34) check weather

$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigen vector

given $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

ans)

$AX = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-1)X$

eigen value $= -1$

Q35) Find the sum of product of eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

ans)

sum of eigen value = Trace of A
$= 3+2+5 = 10$

product of eigen values $= |A| = 3.2.5 = 30$

Q36) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are 1 each

ans) Eigen value of $A^{-1} = 1, 1, \frac{1}{5}$

Q37) If the eigen values of

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

are $-2, 3$ & 6. Then find the eigen values of A^3

ans)

$-8, 27, 216$

Rayleigh's Power method

In many problems we need to calculate only the numerically largest eigen value which can be obtained by a numerical technique called rayleigh's power method. This method is itarative in nature

Working Procedure

Let A be a given non zero square matrice of order 3. Let X_0 be the initial vector which can be taken in any of the forms $[1\,0\,0]'$, $[0\,1\,0]'$, $[0\,0\,1]'$ or sometimes $[1, 1, 1]'$

Find the product AX_0 and express it in the form

$$AX_0 = \lambda_1 X_1 \quad \text{where,}$$

$\lambda_1 \rightarrow$ the numerically largest value of AX_0

and this process is called normalization.

Step 3:-

Find the product AX_1 and express it has $\lambda_2 X_2$ using normalization.

Step 4:-

This process is continued till 2 consecutive iterations have the same eigen value & the eigen vector upto the desicreed degree of accuracy.

Note:- To find the numerically smallest eigen value and Corresponding eigen vector we use the inverse power method. In this method we first find A^{-1} then applying Rayleigh's power method we get the numerically largest value of A^{-1} which is required smallest eigen value of A.

Q38) Using Power method find an approximate value of largest eigen value and corresponding eigen vector

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

ans)

Let us take the initial vector X_0 as

$$X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ consider

$$AX_0 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow 4\begin{bmatrix} 1 \\ 0 \\ 0.25 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1.5 \end{bmatrix}$$

$$\Rightarrow 4\begin{bmatrix} 1 \\ 0 \\ 0.375 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.375 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1.75 \end{bmatrix}$$

$$= 4\begin{bmatrix} 1 \\ 0 \\ 0.4375 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 1.875 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 0.4688 \end{bmatrix}$$

$$= \lambda_4 X_4$$

$$\begin{bmatrix} 4 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Q39) Using Power method find an approximate value of largest eigen value and corresponding eigen vector

$$A = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$$

initial vector $X_0 = (1\ 0\ 0)^T$

ans)

Consider:- $AX_0 = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$

Q40) Using inverse power method find the smallest eigen value & vector by taking

$[1, 1, 1]^T$

given, $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

ans)

now A^{-1} using scientific calculator

$$A^{-1} = \begin{bmatrix} 0.4166 & 0.25 & 0.0833 \\ 0.25 & 0.75 & 0.25 \\ 0.0833 & 0.25 & 0.4166 \end{bmatrix}$$

$$A^{-1}X_0 = \begin{bmatrix} 0.7499 \\ 1.25 \\ 0.7499 \end{bmatrix} \Rightarrow 1.25 \begin{bmatrix} 0.59992 \\ 1 \\ 0.59992 \end{bmatrix} = \lambda_1 X_1$$

$$X_6 \approx x_7 = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$$

Q41) Find corresponding eigen value & vetor

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \text{ taking } [1\ 0\ 0]^T$$

ans)

$$A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^{-1}X_0 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0.33 \\ 0.33 \end{bmatrix} = \lambda_1 X_1$$

$$A^{-1}X_1 = \begin{bmatrix} 1.666 \\ 1.999 \\ 1.999 \end{bmatrix} = 1.999 \begin{bmatrix} 0.833 \\ 1 \\ 1 \end{bmatrix} = \lambda_2 X_2$$

largest E.V. of $A^{-1} = 3.732$, $x = \begin{bmatrix} 1 \\ -0.366 \\ -0.366 \end{bmatrix}$

smallest E.V. of $A = 0.268$ $x = \begin{bmatrix} 1 \\ -0.366 \\ -0.366 \end{bmatrix}$

Q42) Find the Eigen values & eigen vectors of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

ans)

Q43) Find the rank of the matrix

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

ans)

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$\Rightarrow R_1 \longleftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$\Rightarrow R_2 = R_2 + R_1, \ R_3 = R_3 + 2R_1$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$\Rightarrow R_2 = \frac{1}{2}R_2, \ R_3 = \frac{1}{5}R_3$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 2 & -3 & -3 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$\Rightarrow R_4 + 3R_2 = R_4, \ R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

$\Rightarrow R_3 \longleftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \rho(A) = 3$$

Q44) Solve by gauss elimination method

$$6x - 2y + 2z + 4u = 16$$
$$12x - 8y + 6z + 10u = 26$$
$$3x - 13y + 9z + 3u = -19$$
$$-6x + 4y + z - 18u = -34$$

ans)

$6x - 2y + 2z + 4u = 16$

$\Rightarrow 6x = \frac{16 + 2y - 2z + 4u}{1}$

$\Rightarrow x = \frac{16 + 2y - 2z + 4u}{16}$

$12x - 8y + 6z + 10u = 26$

$\Rightarrow -8y = \frac{26 - 12x - 6z - 10u}{1}$

$\Rightarrow y = \frac{26 - 12x - 6z - 10u}{-8}$

$3x - 13y + 9z + 3u = -19$

$\Rightarrow 9z = -19 - 3x + 13y - 3u$

$\Rightarrow z = \frac{-19 - 3x + 13y - 3u}{9}$

$-6x + 4y + z - 18u = -34$

$\Rightarrow -18u = -34 + 6x - 4y + z$

$\Rightarrow u = \frac{-34 + 6x - 4y + z}{-18}$

	x	y	z	u
I	2.22	20.25	22.52	0.0849
II	12.10	31.69	39.06	2.69
III	10.83	46.12	59.99	5.19
	9.02	61.7	82.36	8.03
	6.92	78.95	106.95	11.18
	4.75	98.0680	134.23	14.64
	2.23	119.08	164.269	18.479
	−0.45	142.3644	197.51	22.704
	−3	168.03	334.201	27.3
	−6.8	196.42	274.75	32.54
	−10.48	227.46	319.52	38.24
	−14			

Q45) Solve by gauss Seidel method

$$9x + 2y + 4z = 20$$

$$x + 10y + 4z = 6$$

$$2x - 4y + 10y = -15$$

ans)

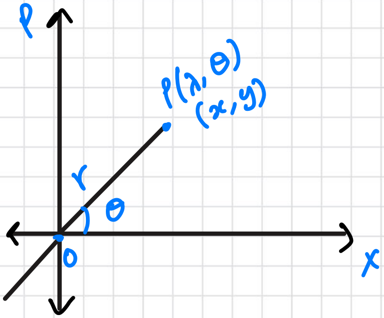
Q46) Check for consistency and hence solve

$$x_1 - 2x_2 - x_3 + 3x_4 = 0$$
$$-2x_1 + 4x + 8x_3 - 5x_4 = 3$$
$$3x_1 - 6x_2 - 6x_3 + 8x_4 = 2$$

UNIT-2

Differential Calculus

polar coordinates



Let P be any point in the plane and O be a fixed point. let OX be a line through O. The fixed point is called pole and OX is called the initial line. Join OP. Let R be the distance between O and P which is called the radius vector. The line OP makes an angle θ with the initial line which is called the vectorian angle. The point P can be described by a new set of coordinates P(r, θ) known as the polar coordinates. If the point B is a variable point then it traces a curve which is called the polar curve represented by $r = f(\theta)$

Note:- The radius vector r is always positive and the -ve sign indicates the direction of r

θ is measured in the anti clockwise direction.

Relation between the Cartesian and polar coordinates

from the ΔOPQ, we have

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$ —— ①

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$ —— ②

squaring and adding ① & ②, we get

$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$ —— ③

now ② ÷ ①

$\tan\theta = \frac{y}{x}$

$\Rightarrow \theta = \tan^{-1}\frac{y}{x}$ —— ④

Find the equation in the polar form for the given Cartesian curves.

Q1) $x^2 + y^2 = 4$

ans) put $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow r^2 = 4$ or $r = 2$

Q2) $(x-a)^2 + y^2 = a^2$

ans) $(r\cos\theta - a)^2 + (r\sin\theta)^2 = a^2$

$\Rightarrow r^2\cos^2\theta - 2ar\cos\theta + a^2 + r^2\sin^2\theta = a^2$

$\Rightarrow r^2 - 2ar\cos\theta = 0$

$\Rightarrow r = 2a\cos\theta$

Q3) $4r\cos\theta + r\sin\theta = 8$
convert to cartisian form
ans)
$$4x + y = 8$$

Q4) Find the polar cordinates corresponding to cartsian coordinates

(i) $(x, y) = (1, 1)$
ans)
$$r = \sqrt{x^2+y^2} \ \& \ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$r = \sqrt{2} \qquad \theta = \tan^{-1}(1) = \frac{\pi}{4}$$

$(1, 1)$ can be written as $\left(\sqrt{2}, \frac{\pi}{4}\right)$

(ii) $(x, y) = (-1, 1)$
ans)
$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$
$$\Rightarrow \theta = \tan^{-1}\left(\frac{3}{-1}\right)$$
$$\Rightarrow \tan^{-1}(-1) = \frac{3\pi}{4}$$
$$\Rightarrow \pi + \tan^{-1}\left(\frac{y}{x}\right) = \pi - \frac{\pi}{4}$$
$$\Rightarrow \frac{3\pi}{4}$$

$(-1, 1)$ can be written as $\left(\sqrt{2}, \frac{3\pi}{4}\right)$

(iii) $(x, y) = (-1, 1)$
ans)
$$r = \sqrt{2}, \theta = \pi + \tan^{-1}\left(\frac{y}{x}\right)$$
$$\Rightarrow \frac{5\pi}{4}$$

(iv) $(x, y) = (1, -1)$
ans)
$$r = \sqrt{2}, \theta = 2\pi + \tan^{-1}\left(\frac{y}{x}\right)$$
$$\Rightarrow 2\pi - \frac{\pi}{4}$$
$$\Rightarrow \frac{7\pi}{4}$$

(i) $(x, y) = (2, -6)$
ans)
$$r = \sqrt{40} \text{ and } \theta = 2\pi + \tan^{-1}\left(\frac{-6}{2}\right)$$
$$\Rightarrow 2\pi - \tan^{-1}(3)$$

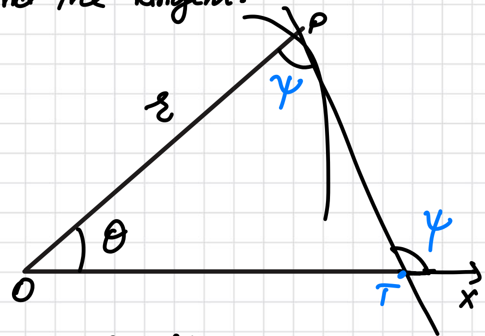
(ii) $\left(8, \frac{2\pi}{3}\right) = (r, \theta)$

ans)
$$x = r\cos\theta = 8\cos\left(\frac{2\pi}{3}\right) = -4$$
$$y = r\sin\theta = 8\sin\left(\frac{2\pi}{3}\right) = 4\sqrt{3}$$

(iii) $\left(3, \frac{\pi}{4}\right)$
ans) $x = 3\cos\frac{\pi}{4} = 3 \times \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$
$y = 3\sin\frac{\pi}{4} = 3 \times \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

$$\Rightarrow \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$$

Angle between the radius vector and the tangent.



$C = f(\theta)$

Let C be a polar curve represented by $r = f(\theta)$ and P be any point on it. OP be the radius vector and OX be the initial line. Draw a tangent to the curve at point P and extend it to meet the initial line at T. Let ψ be the angle made by the tangent PT with OX. Let ϕ be the angle b/w radius vector OP and tangent PT.

$$\tan\phi = r.\frac{d\theta}{dr}$$

(or)

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\psi = \theta + \phi$$

$$\Rightarrow \tan\psi = \tan(\theta+\phi)$$

$$\Rightarrow \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} \text{ —— } ①$$

w.k.t. $x = r\cos\theta$ and $y = r\sin\theta$

slope of tangent PT is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} =$$

$$\Rightarrow \frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta}$$

$$\Rightarrow \frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\, dr/d\theta + r\cos\theta}{\cos\theta\, dr/d\theta \quad r\sin\theta}$$

$\div$ A $\cos\theta\frac{dr}{d\theta}$

$$\Rightarrow \frac{dy}{d\theta} = \frac{\tan\theta + r\,d\theta/dr}{1 - r\tan\theta\, d\theta/dr} \text{ —— } ②$$

from ① & ②

the angle b/w the radius vector & the tangent ψ is obtained by

$$\tan\phi = r\frac{d\theta}{dr}$$

or

$$\cot\phi = \frac{1}{r}\frac{d\theta}{r}$$

Q5) Find the angle b/w the radius vector and the tangent for the following polar curves

(i) $r = a(1+\cos\theta)$

ans) taking log on both sides

$$\log r = \log[a(1+\cos\theta)]$$

$$\Rightarrow \log a + \log(1+\cos\theta)$$

$\Rightarrow$ Diff w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+\cos\theta} - \sin\theta$$

$\Rightarrow$ If ϕ is the angle b/w radius vector and the tangen, then

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\Rightarrow \cot\phi = \frac{-\sin\theta}{1+\cos\theta} = \frac{-\cancel{2}\sin\theta/2\,\cancel{2\cos\theta/2}}{2\cos^{\cancel{2}}\theta/2}$$

$$\Rightarrow \tan\frac{\theta}{2}$$

$$\Rightarrow \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\Rightarrow \phi = \frac{\pi}{2} + \frac{\theta}{2}$$

(ii) $r^2\cos 2\theta = a^2$

ans)

$$r^2\cos 2\theta = a^2$$

$\Rightarrow$ taking log on both sides

$$\log(r^2\cos 2\theta) = \log a^2$$

$$\Rightarrow 2\log r + \log\cos 2\theta = 2\log a$$

$\Rightarrow$ diff w.r.t. θ we get

2. $\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\cos 2\theta} \cdot 2\sin 2\theta = 0$

$\Rightarrow$ Divide by 2

$\frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$

$\Rightarrow \cot\phi = \tan 2\theta = \cot\left(\frac{\pi}{2} - 2\theta\right)$

$\Rightarrow \phi = \frac{\pi}{2} - 2\theta$

(iii) $r^n = a^n \sec(n\theta + \alpha)$

ans

$r^n = a^n \sec(n\theta + \alpha)$

$\Rightarrow$ taking log both sides

$n\log r = n\log a + \log(\sec(n\theta + \alpha))$

$\Rightarrow n.\frac{1}{r} = \frac{dr}{d\theta}$

$\Rightarrow \frac{1}{\sec(n\theta+\alpha)} n.\sec(n\theta+\alpha)\tan(n\theta+\alpha)$

$\Rightarrow \cot\phi = \tan(n\theta + \alpha)$

$\Rightarrow \cot\left(\frac{\pi}{2} - (n\theta + \alpha)\right)$

$\Rightarrow \phi = \frac{\pi}{2} - n\theta - \alpha$

(iv) $r^m = a^m(\cos m\theta + \sin m\theta)$

ans)

$r^m = a^m(\cos m\theta + \sin m\theta)$

$\Rightarrow \cot\phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$

$\Rightarrow \frac{1 - \tan m\theta}{1 + \tan m\theta}$

$\Rightarrow \frac{\tan\pi/4 - \tan m\theta}{1 + \tan\pi/4 \tan m\theta} = \tan\left(\frac{\pi}{4} - m\theta\right)$

$\Rightarrow \cot\left(\frac{\pi}{2} - \left(\frac{\pi}{4} - m\theta\right)\right)$

$\Rightarrow \phi = \frac{\pi}{2} + m\theta$

(v) $r = ae^{\theta\cot\alpha}$

ans)

$r = ae^{\theta\cot\alpha}$

$\Rightarrow$ using log on both sides

$\log r = \log ae^{\theta\cot\alpha}$

$\Rightarrow \log r = \theta\cot\alpha \log ae$

$\Rightarrow$ Diff w.r.t.

$\frac{1}{r}\frac{dr}{d\theta} =$

Q6) Find the angle ϕ for the angle $a(1-\cos\theta)$ and hence determine the slope of the curve at $\theta = \frac{\pi}{6}$.

ans) slope of a tangent

$\tan\psi = (\tan\theta + \phi)$ —— ①

$\Rightarrow \log r = \log a + \log(1-\cos\theta)$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}\sin\theta$

$\Rightarrow \cot\phi = \cot\frac{\theta}{2}$

$\Rightarrow \phi = \frac{\theta}{2}$ —— ②

$\Rightarrow$ ② in ①

$\tan\psi = \tan\left(\frac{\theta}{2} + \theta\right)$

$\Rightarrow$ at $\theta = \frac{\pi}{6}$

$\tan\psi = \tan\left(\frac{\pi}{4}\right) = 1$

Q) Find the slope of the tangent for the curve

$\frac{2a}{r} = 1-\cos\theta$ and $\theta = \frac{2\pi}{3}$

ans)

$\frac{2a}{r} = 1-\cos\theta$

$\Rightarrow \frac{2a}{1-\cos\theta} = r$

$\Rightarrow \log r = \log 2a - \log(1-\cos\theta)$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-\sin\theta}{1-\cos\theta} \Rightarrow -\cot\frac{\theta}{2}$

$\cos\phi = -\cot\left(\frac{\theta}{2}\right) = \cot\left(\frac{-\theta}{2}\right)$

$\Rightarrow \phi = -\frac{\theta}{2}$

$\Rightarrow \phi = \left(\frac{-2\pi}{3}\right) \times \frac{1}{2} = \frac{-2\pi}{6}$

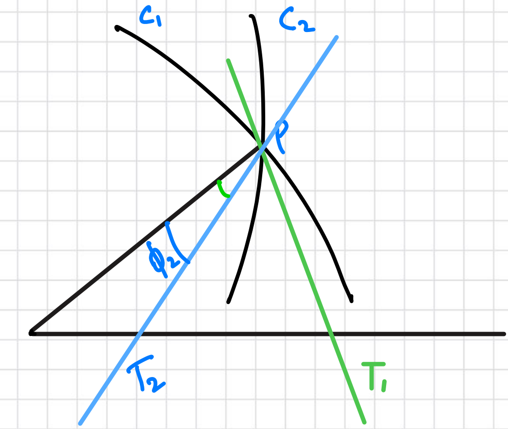
$\Rightarrow \tan t = \tan\left(\frac{4\pi}{3}\right)$

$\Rightarrow \tan\left(\frac{3\pi}{2} - \frac{\pi}{6}\right)$

$\Rightarrow \cot\frac{\pi}{6}$

$\Rightarrow \sqrt{3}$

Angle between 2 polar curves



The angle between 2 polar curves is the angle between the tangents to the curves at the point of intersection.

From the above figure we have the angle b/w the radius OP and the tangent PT_1 is ϕ_1 & the angle b/w the radius vector b/w OP & T_2 is ϕ_2, hence angle intersection b/w 2 polar curves is

$$|\phi_1 - \phi_2|$$

$$\Rightarrow \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$$

$$\Rightarrow \tan\phi_1 . \tan\phi_2 = -1$$

If the 2 curves intersect at 90°

i.e., $\phi_1 - \phi_2 = \frac{\pi}{2}$, then we get

$\tan\phi_1 . \tan\phi_2 = -1$

$\therefore$ the condition for 2 polar curves to be orthoganol is

$$\tan\phi_1 . \tan\phi_2 = -1$$

OR

$$\cot\phi_1 . \cot\phi_2 = -1$$