7

ELASTICITY

7.1 INTRODUCTION

If the distance between any two points in a body is invariable, the body is said to be a rigid body. In practice it is not possible to have a perfectly rigid body. All bodies get deformed under the action of force–some more some less. When the shape or size of a body has been altered by the application of a force or a system of forces, there is usually some tendency for the body to recover its original shape or size on the removal of the force. This property of the body by virtue of which it tends to regain its original shape or size on the removal of deforming forces is called *elasticity*.

If a body resists deformation and recovers its original shape, size or volume completely and immediately on the removal of the deforming forces it is called *perfectly elastic body* and if it completely retains its altered size and shape, it is said to be *perfectly plastic*. In general bodies are in between these two extreme limits. There is no perfectly elastic or perfectly plastic body.

Fluids resist a change of volume, but not of shape and possess only volume elasticity. Solids, on the other hand, resist change of shape as well as change of volume and possess *rigidity or shear elasticity* as well as volume elasticity.

7.2 STRESS AND STRAIN

When a body is subjected to a force or a system of forces it undergoes a change in size or shape or both. Elastic bodies offer appreciable resistance to the deforming forces; as a result, work has to be done to deform them. This amount of work done is stored in body as elastic potential energy. When the deforming force is removed, its increased elastic potential energy produces a tendency in the body to restore the body to its original state of zero energy or stable equilibrium. This tendency is due to the internal forces called into play by the deformation to restore the body to its original position.

The change per unit amount is called *strain*. *Strain is the fractional deformation produced in a body when it is subjected to a set of deforming forces.* Strain being ratio has no units. There are following three types of strains.

(*i*) Longitudinal or Tensile Strain

It is defined as the change in length per unit original length, without any change in shape. When the forces act along the length of the body, a change in length is produced. The ratio of change in length to the original length is called the longitudinal strain.

(*ii*) Shear or Shearing Strain

It is defined as the ratio of the relative displacement between two layers, under the action of a tangential force to the distance between them, the distance being measured at right angles to the direction of stress. When tangential forces act on a body, it undergoes a change in shape. The angular deformation produced is called shearing strain or shear.

(*iii*) Volumetric Strain and Tensile Strain

It is defined as the change in volume per unit original volume, without any change in shape. When the forces are applied normal to the surface of a body in all the directions it undergoes a change in volume. The ratio of change in volume to the original volume is called volume strain and stress producing it is called normal stress. However in case of an extension, the strain is called tensile strain and the stress is called tensile stress.

When a body is deformed the internal force called into play per unit area to restore it to its original state is called *stress*. Since the deformed body is at rest, the stress must be equal and opposite to the deforming force per unit area. Hence stress is measured in terms of the load or the force applied per unit area and its units are N/m^2.

When the force is applied normal to the surface of a body the stress is called the normal stress. If the force is applied along the surface of the body, *i.e.* tangential force per unit area is called shearing stress. A normal stress is called compressive or expansive (tensile) according as the body is subjected to thrust *i.e.* compression or pull *i.e.* tension.

It may be noted that wherever there is stress, strain is also there or where there is strain stress is also there. In fact stress and strain always accompany each other.

The maximum stress after an application of which the material is able to regain its original shape or size is called elastic limit.

7.3 HOOKE'S LAW

There is a simple relationship between stress and strain discovered by Hooke and is called Hooke's law. *According to this law strain is proportional to the stress producing it within elastic limits. i.e.*

Stress $\propto$ Strain

or $$\frac{\text{Stress}}{\text{Strain}} = \text{a constant} = E \qquad ...(7.1)$$

The constant E is called the coefficient of Elasticity or Modulus of Elasticity. The value of this constant depends upon the nature of the material and also the conditions to which it has been subjected after manufacturing.

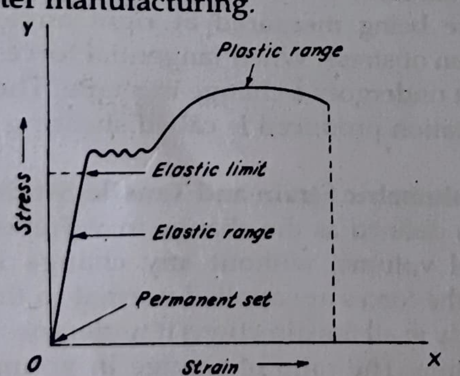

Fig. 7.1. Stress-strain diagram.

If we plot a graph between stress and strain we get a curve as shown in Fig. (7.1) and it is called stress-strain diagram. It is clear from this graph that Hooke's law holds good only for the straight line portion of the curve. As soon as the elastic limit is crossed, the strain increases more rapidly than stress and corresponds to partly elastic and partly plastic region. After this the body does not come back to its position of rest and acquires a permanent residual strain or what is called permanent set acquired by the body. Beyond this plastic region starts.

Corresponding to the three types of strain we have three elastic modulus namely :

(*i*) Young's modulus or linear elasticity corresponding to linear to tensile strain.

(*ii*) Bulk modulus or elasticity of volume corresponding to volume strain.

(*iii*) Shear modulus or elasticity of shape or modulus of rigidity corresponding to shearing strain.

7.4 YOUNG'S MODULUS

When the applied force is along one direction only the strain produced in that direction is called longitudinal strain and the corresponding stress is called longitudinal stress. According to Hooke's law the ratio of longitudinal stress to the corresponding longitudinal strain within the elastic limit is constant and this constant is called Young's modulus of the material of the body. Consider a wire of length L and area of cross-section a. If F is the force applied along its length and normal to its area of cross section it produces a change in the length of the wire and let this change be l, then

Longitudinal stress = F/a, and

Longitudinal strain = l/L

The ratio of the longitudinal stress to the longitudinal strain within elastic limit is called Young's modulus of elasticity, denoted by a letter Y and is given by

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

or $$Y = \frac{F/a}{l/L} = \frac{FL}{l\,a} \qquad ...(7.2)$$

This relation holds good for compressive as well as extensive stresses.

The units of Young's modulus of elasticity is same as that of stress as strain has no units. Hence unit of Y is N/m^2.

7.5 BULK MODULUS

If the force is applied normally and uniformly to the surface of the body it undergoes a change in volume, its shape remaining unchanged. Such a force per unit area is called normal stress and the change in volume per unit volume is called volume strain. The ratio of normal stress to volume strain is called bulk modulus of elasticity and denoted by K.

If a body having volume V is subjected to a force F applied uniformly and normally on a total surface area A of body, and its volume is changed by v, then

$$\text{Bulk modulus } K = \frac{F/A}{v/V} = \frac{F\,V}{A\,v} = \frac{PV}{v} \qquad ...(7.3)$$

where P is bulk pressure.

The unit of bulk modulus is same as that of pressure i.e. N/m^2. The reciprocal of Bulk modulus is called *compressibility*.

7.6 MODULUS OF RIGIDITY

When a body is subjected to a force in such a way so as to change the shape of the body only without changing the size of the body, the body is said to be sheared for example when tangential force is applied on the upper surface of a pack of cards, each card is displaced relative to the other. The displacement of each card with respect to that on the table is proportional to its distance from latter. The pack of card is said to be sheared as the change of shape is without any change in the volume.

The ratio of tangential force per unit area to the angular deformation produced is called modulus of rigidity.

Consider a cube as shown in Fig. 7.2 (*a*) and (*b*). Let the face $ABCD$ be such that lower face CD is clamped. If a tangential force F is applied on the

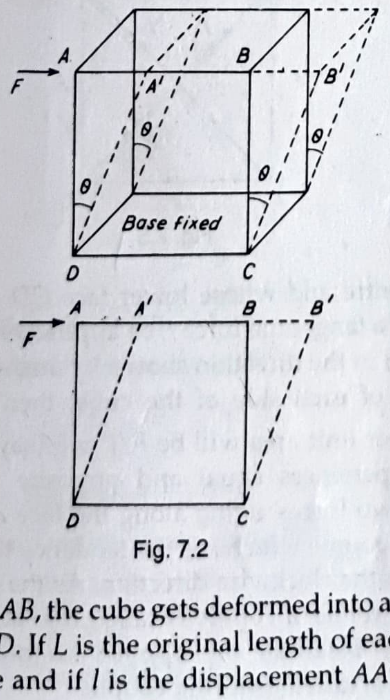

Fig. 7.2

upper face AB, the cube gets deformed into a rhombus $A'B'\,CD$. If L is the original length of each side of the cube and if l is the displacement $AA' = BB'$, then shearing strain is θ such that $\theta = l/L$ as shear strain or angle of shear is the angle in radians through which each line, ordinary perpendicular to the fixed face, is turned when the cube is sheared.

$\therefore$ Modulus of rigidity, η

$$= \frac{\text{Tangential Stress}}{\text{Shear Strain}} = \frac{T}{\theta}$$

In general the value of θ is very small [Fig. 7.2(b)]

$$\therefore \qquad \theta = \tan\theta = AA'/AD$$

Hence we can define shear strain as the ratio of relative displacement between two layers in the direction of the stress to the distance between them, distance being measured perpendicular to the stress,

$$\because \qquad \theta = \frac{AA'}{AD} = \frac{l}{L}$$

$$\therefore \qquad \eta = \frac{F/a}{l/L} = \frac{F\,L}{a\,l} \qquad ...(7.4)$$

The unit of modulus of rigidity is same as that of pressure *i.e.* N/m^2.

7.7 POISSON'S RATIO

When a body is subjected to a force or a system of force the deformation is not only in one direction but all along. For example if a wire is stretched, besides undergoing extension along the direction of force, it also undergoes contraction in the perpendicular directions. The strain produced along

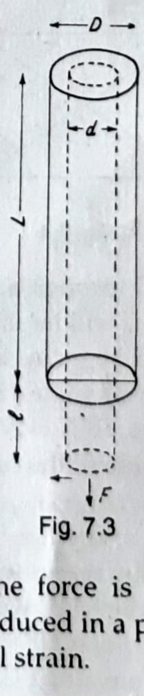

Fig. 7.3

the direction of the force is called longitudinal strain and that produced in a perpendicular direction is called lateral strain.

Within the elastic limits the ratio of the lateral strain to the longitudinal strain is constant for the material of the body and is known as Poisson's ratio and is denoted by σ.

The longitudinal strain per unit stress is denoted by α and the lateral strain per unit stress by β. The Poisson's ratio is denoted by σ, and

$$\sigma = \beta/\alpha \qquad ...(7.3)$$

For example if a wire of original length L and diameter D is subjected to a tensile stress in the direction shown in Fig. 7.3. The length of the wire increases from L to $L + l$ and the diameter decreases from D to d.

Then, longitudinal strain = l/L

$$\therefore\ \sigma = \frac{\text{Lateral strain}}{\text{Longitudinal strain}}$$

$$= -\frac{(D-d)/D}{l/L} = -\frac{L\,(D-d)}{D\,l} \qquad ...(7.6)$$

The negative sign indicates that longitudinal and lateral strains are in opposite sense.

7.8 EQUIVALENCE OF SHEAR TO COMPRESSION AND EXTENSION

Consider a cube such that its lower surface CD is fixed and let $ABCD$ be its front face (Fig. 7.4). If a tangential force F is applied at the upper surface AB in the direction shown, its cube will be sheared

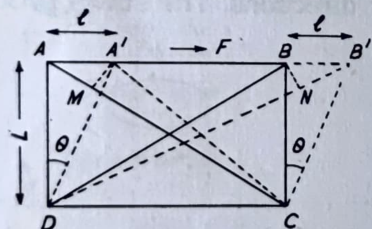

Fig. 7.4

to the shape $A'B'CD$ through a small angle. In this way the diagonal AC will be shortened to $A'C$ and the diagonal DB will be increased in length to DB'. As in practice the shear angle θ is very small, therefore $\angle BB'C = 90°$ and $BB'N = 45°$. Let

Length of each side of the cube = L

$AA' = BB' = l$.

As θ is small, $\theta = \tan\theta = l/L$

As diagonal DB increases to DB' and diagonal CA is compressed to AC', we have

Extension strain along $DB = NB'/DB$

Compression strain along $CA = MA/CA$

Since $\angle BB'C = 90°$ and $\angle BB'N = 45°$,

we have, $NB' = BB' \cos 45° = \dfrac{1}{\sqrt{2}}$

and $AM = AA' \cos 45° = \dfrac{1}{\sqrt{2}}$

Also $OB = CA = \sqrt{L^2 + L^2} = L\sqrt{2}$

$\therefore$ Extension strain

$$= \frac{NB'}{DB} = \frac{l/\sqrt{2}}{L/\sqrt{2}} = \frac{l}{2L} = \frac{\theta}{2} \quad \text{along } DB \qquad ...(7.7)$$

and compression strain

$$= \frac{MA'}{CA} = \frac{l/\sqrt{2}}{L/\sqrt{2}} = \frac{l}{2\,L} = \frac{\theta}{2} \quad \text{along } CA \qquad ...(7.8)$$

From above relations it is clear that a simple shear θ is equivalent to an extension strain and compression strain at right angles to each other and each of value $\theta/2$.

The converse of this is also true *i.e.* simultaneous equal compression and extension at right angles to each other are equivalent to shear.

7.9 EQUIVALENCE OF SHEARING STRESS TO A COMPRESSIVE STRESS AND A TENSILE STRESS

Consider a section of cube $ABCD$ (Fig. 7.5) having

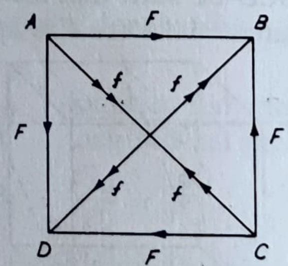

Fig. 7.5

0 as centre and whose lower face CD is clamped and let a tangential force F be applied on the upper face AB in the direction shown by arrow. If L is the length of each side of the cube, then tangential force per unit area will be $F/l^2 = T$ (say). The face CD experiences equal and opposite forces and these two forces acting along the face AB and CD form a couple which has the tendency to rotate the cube in the clockwise direction. As the cube is unable to rotate, a couple is called into action which counter balances the applied couple and this couple is called restoring couple.

The restoring couple consists of forces acting tangentially on the faces CB and AD and each of magnitude F. Under the action of these four forces (along AB, CD, BC and AD) each of F the cube is deformed as these four forces are in equilibrium the cube does not move. The force acting along AB can be resolved along AO and OB *i.e.*, into two equal components of magnitude f. Similarly the remaining three forces can be resolved into three pairs of components each of f. Thus we have four forces of value F each acting tangentially on the faces of the cube which are resolved into eight forces each of value f acting along the diagonals as shown in Fig. 7.5. The force f acting along AC tends to compress the diagonal and those acting along DB tend to produce extension along that diagonal. Therefore the area of the triangular face ABC cut parallel to AC and perpendicular to the plane of the paper $= DB \times AO = L\sqrt{2} \times L = L^2\sqrt{2}$

as $DB = \sqrt{AB^2 + AD^2} = \sqrt{L^2 + L^2} = L\sqrt{2}$.

Since the force acting on either side of the section normal to it is $2f$, (Fig. 7.6), therefore compressive stress along AC

$$= \frac{\text{Force}}{\text{Area}} = \frac{2f}{L^2\sqrt{2}}$$

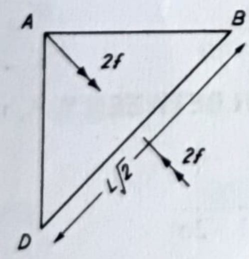

Fig. 7.6

But $f = F\cos 45° = F/\sqrt{2}$

$\therefore$ Compressive stress along AC

$$= \frac{2F/\sqrt{2}}{L^2 \times \sqrt{2}} = \frac{F}{L^2} = T \quad \text{...(7.9)}$$

= Tangential stress

Similarly tensile stress along BD

$$= \frac{2F \times 1/\sqrt{2}}{L^2 \times \sqrt{2}} = \frac{F}{L^2} = T \quad \text{...(7.10)}$$

= Tangential stress

From relation (7.9) and (7.10) it is clear that a shearing stress is equivalent to an equal linear tensile stress and an equal linear compressive stress at right angles to each other.

7.10 RELATION BETWEEN Y AND α

Consider a cube of unit side and let a unit tension act along one side, the extension produced will be equal to α. Now we know, that

$$Y = \frac{\text{Stress}}{\text{Longitudinal stress}}$$

As stress = 1 and longitudinal stress = α, we have

$$\therefore \quad Y = 1/\alpha \quad \text{...(7.11)}$$

7.11 RELATION BETWEEN K, α AND β

Consider a cube of unit length such that its one vertex coincides with the origin and its three sides are parallel to three mutually perpendicular axes OX, OY and OZ as shown in Fig. 7.7. Let the pairs of oppositely directed tensile forces T_x, T_x; T_y T_y, and T_z T_z be applied. Then if α is the increase per unit length per unit tension along the direction of the force and β, is the contraction per unit tension,

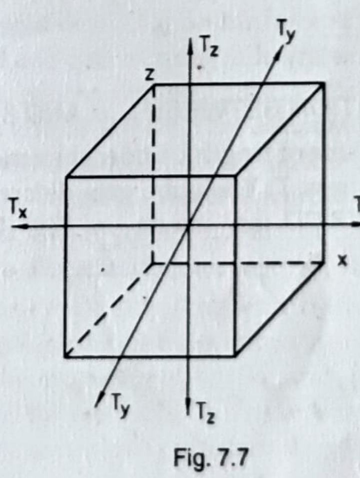

Fig. 7.7

in a direction perpendicular to the force, the length of side OX increases while along OY and OZ there is contraction. Hence, after the application of the forces the length of the side will be

$$OX = 1 + \alpha T_x - \beta T_y - \beta T_z$$

Similarly for the length of the other sides

$$OY = 1 + \alpha T_y - \beta T_x - \beta T_z$$

and

$$OZ = 1 + \alpha T_z + \beta T_x - \beta T_y$$

$\therefore$ Volume of the cube = $OX \times OY \times OZ$

$$= (1 + \alpha T_x - \beta T_y - \beta T_z)(1 + \alpha T_y - \beta T_x - \beta T_z)(1 + \alpha T_z - \beta T_x - \beta T_y)$$

$$= 1 + (\alpha - 2\beta)(T_x + T_y + T_z)$$

The higher power of α and β are neglected being small quantities. If $T_x = T_y = T_z = T$

The volume of the cube $= 1 + (\alpha - 2\beta)\, 3T$

The volume of the cube $= 1$ (being unit cube).

$\therefore$ Change in volume $= 3T(\alpha - 2\beta)$

Hence, Volume strain $\dfrac{3T(\alpha - 2\beta)}{1}$

$$\text{Bulk modulus } K = \frac{\text{Stress}}{\text{Volume Strain}} = \frac{T}{3T(\alpha - 2\beta)}$$

or $$K = \frac{1}{3(\alpha - 2\beta)} \quad \text{....(7.12)}$$

7.12 RELATION BETWEEN Y, K AND α

From relation (7.5), (7.11) and (7.12), we have

$$\sigma = \frac{\beta}{\alpha}, \; Y = \frac{1}{\alpha} \text{ and } K = \frac{1}{3(\alpha - 2\beta)}$$

or $$K = \frac{1/\alpha}{3\left(1 - \dfrac{2\beta}{\alpha}\right)}$$

or $$K = \frac{Y}{3(1 - 2\sigma)} \quad \text{...(7.13)}$$

7.13 RELATION BETWEEN η, α AND β

Consider a cube of length L under the action of the tangential stress T. The cube gets deformed to a rhombus $A'B'CD$ as shown in Fig. 7.8. The

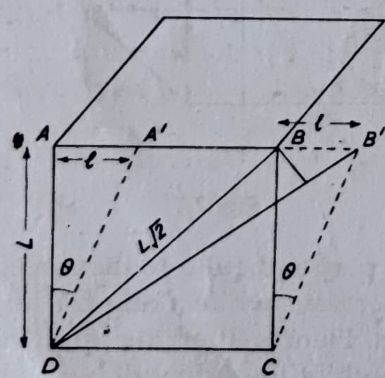

Fig. 7.8

diagonal AC undergoes a contraction and the diagonal DB' undergoes an elongation of equal amount.

As shown in article 7.9 the tangential stress T is equivalent to a compressive stress T along the diagonal CB and tensile stress T along the diagonal DB. These compressive and tensile stresses together produce extension along the diagonal DB. If α is longitudinal strain per unit stress and β the lateral strain per unit stress, then the strain produced along the diagonal is equal to $T(\alpha + \beta)$. As BM is normal on DB'.

$$\therefore \text{ Strain } = \frac{MB'}{DB} = T(\alpha + \beta) \quad \text{...(7.14)}$$

$$MB' = BB' \cos(BB'M) = BB' \cos 45^\circ$$

$$= \frac{l}{\sqrt{2}} \quad (\text{where } AA' = BB' = l)$$

and diagonal $DB = \sqrt{L^2 + L^2} = L\sqrt{2}$

Substituting the values of DB and MB' in equation (7.14), we have

$$T(\alpha + \beta) = \frac{l/\sqrt{2}}{L\sqrt{2}} = \frac{l}{2L}$$

or $T(\alpha + \beta) = \theta/2$ (as $\theta = l/L$ for small values)

or $T/\theta = 1/2\,(\alpha + \beta)$

or $$\eta = \frac{1}{2(\alpha + \beta)} \quad \text{...(7.15)}$$

(as $\eta = T/\theta$). The above equation can be written as

$$\eta = \frac{1/\alpha}{2(1 + \beta/\alpha)}$$

$$\therefore \quad \eta = \frac{Y}{2(1 + \sigma)} \quad \text{...(7.16)}$$

7.14 RELATION BETWEEN Y, K, η AND σ

We know that

$$K = \frac{Y}{3(1 - 2\sigma)} \quad \text{...}(i)$$

and $$\eta = \frac{Y}{2(1 + \sigma)} \quad \text{...}(ii)$$

from relations 7.13 and 7.16. From equation (*i*), we have

$$\frac{Y}{3K} = 1 - 2\sigma \quad \text{...(7.17)}$$

and from equation (*ii*), we get

$$\frac{Y}{\eta} = 2(1 + \sigma) \quad \text{...(7.18)}$$

From relations (7.17) and (7.18), we get

$$\frac{Y}{3K} + \frac{Y}{\eta} = 3 \text{ or } \frac{1}{3K} + \frac{1}{\eta} = \frac{3}{Y}$$

$$Y = \frac{9\eta K}{3K+\eta} \quad \text{...(7.19)}$$
or

By dividing equation 7.17 and 7.18, we get

$$\frac{\eta}{3K} = \frac{1-2\sigma}{2(1+\sigma)} \quad \text{...(7.20)}$$

or $\quad 2\eta(1+\sigma) = 3K(1-2\sigma)$

or $\quad 2\eta + 2\sigma\eta = 3K - 6K\sigma$

or $\quad 2\eta\sigma + 6K\sigma = 3K - 2\eta$

or $$\sigma = \frac{3K-2n}{2\eta+6K}$$

or $$\sigma = \frac{3K-2\eta}{2(\eta+3K)} \quad \text{...(7.21)}$$

7.15 DETERMINATION OF YOUNG'S MODULUS

Searle's Method

In this method a sensitive spirit level and a micrometer screw is used for measuring the increase in length. Searle's apparatus is shown in Fig. 7.9 (*a*) and (*b*).

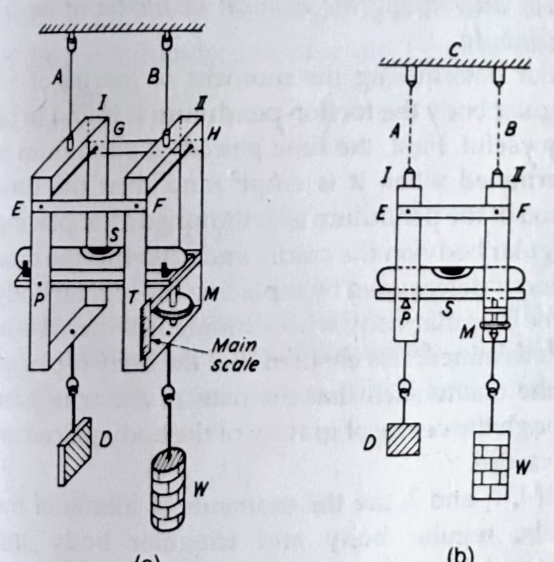

Fig. 7.9

Searle's apparatus consists of a pair of frames, hinged together by cross pieces *EF* and *GH* in such a way, that they are free to move up and down with respect to each other. A strip mounted in the frame *I* rests on the tip of micrometer screw *M* which passes through the cross piece of the frame *H*. A sensitive spirit level *S* is mounted on a strip which is pivoted at *P* in frame *I*.

The wires *A* and *B* are identical in length, material and diameter and are suspended from a rigid support *C*. At the lower end of these wires two frames are connected. The wire *A* which is used for comparison is kept tight by suspending from its frame *I* a constant load *D*. From the frame *II* of the experimental wire *B*, is suspended by a hook to carry a load which can be changed at will.

To remove the kinks if any in the wire load and unload the wire *B*, a number of times and then stretch the wire by placing a load W_0. Measure its length from the top of the frame to the point of suspension. Let this original length be L_m.

Measure the diameter of the wire at different places along its length, taking two readings at right angles to each other at each position. Determine the mean diameter of the wire from these measurements and calculate with this value the area of cross-section πr^2 in m^2.

With the dead load W_0, adjust the micrometer screw, until the air bubble of the spirit level is in its centre. Record the reading of the linear millimetre scale and the corresponding reading of the circular scale of the micro meter screw against the former. Take it as the zero reading.

Increase load on the experimental wire by placing a weight of 1/2 kg on the hook. The wire *B* is stretched and gets increase in length and the frame *II* moves relative to the frame I. Now raise the lowered side of the strip by adjusting the micrometer screw upward till the air bubble again comes back in the central position. Record the reading of micrometer with respect to the linear scale. The difference of the reading for load W_0 and this gives the increase in length of the wire due to the weight of 1/2 kg. Repeat the observations by increasing the load in the multiples of 1/2 kg and adjusting the spirit level for each load. In this way determine the successive extensions corresponding to each additional load.

Now the weight on the hanger is decreased in stages removing every time 1/2 kg weight and the readings of the micrometer screw are recorded. The mean of the two readings corresponding to equal weight while loading and unloading is found out for each step. Plot a graph between the load in kg and extension in m. A straight line graph is obtained. From this graph the ratio F/l N/m is found out.

If *L* is the length of the wire, then

$$\text{Young's modulus } Y = \frac{F/A}{l/L} = \frac{F \times L}{A \times l}$$

If M is the mass of the weight and r mean radius, then

$$Y = \frac{M g L}{\pi r^2 L} \text{ N/m}^2 \qquad ...(7.22)$$

In this method the error due to yielding of support and change of temperature during the experiment effect both the wires, thus these errors are eliminated in this method.

7.16 TORSION PENDULUM

A torsion pendulum is shown in Fig. 7.10 and consists of a cradle D suspended from a fixed torsion

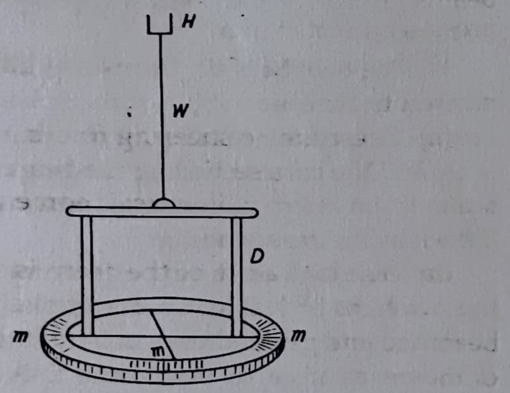

Fig. 7.10 Torsion Pendulum.

head H by means of a wire W. The cradle is in the form of a horizontal circular disc of alluminium fixed to a rectangular metallic frame. The disc is levelled with the help of three small lead masses m, m and m placed in a concentric circular grove without altering the moment of inertia of the cradle.

When the disc is rotated in a horizontal plane so as to twist the wire, the various elements of the wire undergo shearing strains. Restoring couples, which tend to restore the unstrained conditions, are called into action. Now when the cradle or disc is released, it starts executing torsional vibrations.

If the angle of twist at the lower end of the wire is θ, then the restoring couple is $C\,\theta$, where C is the torsional rigidity of the wire. This couple acting on the disc produces in it an angular acceleration given by

$$C\,\theta = -I \frac{d^2\theta}{dt^2} \qquad ...(7.23)$$

Where I is the moment of inertia of the disc about the axis of the wire. The minus sign indicates that the couple $C\,\theta$ tends to decrease the twist. Equation (7.23) can be rewritten as

$$\frac{d^2\theta}{dt^2} = -\frac{C}{I}\theta$$

The above relation shows that the angular acceleration is proportional to the angular displacement θ and is always directed towards the mean position. Hence the motion of the disc is simple harmonic motion and the time period of the vibration will be given by

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{\theta}{\left(\frac{C}{I}\times\theta\right)}}$$

or

$$T = 2\pi\sqrt{I/C} \qquad ...(7.24)$$

7.17 USES OF TORSION PENDULUM

(i) *For determining the moment of inertia of an irregular body*

For determining the moment of inertia of an irregular body the torsion pendulum is found to be very useful. First, the time period of pendulum is determined when it is empty and then the time period of the pendulum is determined after placing a regular body on the cradle and after this the time period is determined by replacing the regular body by the irregular body whose moment of inertia is to be determined. It is ensured that the body is placed on the cradle such that the axes of the wire pass through the centre of gravity of the body placed on the cradle.

If I, I_1 and I_2 are the moments of inertia of the cradle, regular body and irregular body and T, T_1 and T_2 are the time periods in the three cases respectively, then

$$T = 2\pi\sqrt{\frac{I}{C}} \qquad ...(7.25)$$

$$T_1 = 2\pi\sqrt{\frac{I+I_1}{C}} \qquad ...(7.26)$$

$$T_2 = 2\pi\sqrt{\frac{I+I_2}{C}} \qquad ...(7.27)$$

From relations (7.26) and (7.25), we have

$$T_1^2 - T^2 = \frac{4\pi^2 I_1}{C}$$

and from relations (7.26) and (7.27), we have

$$T_2^2 - T^2 = \frac{4\pi^2 I_2}{C}$$

$$\therefore \quad \frac{T_1^2 - T^2}{T_2^2 - T^2} = \frac{4\pi^2 I_1/C}{4\pi^2 I_2/C} = \frac{I_1}{I_2}$$

or $$I_2 = I_1 \times \frac{T_2^2 - T^2}{T_1^2 - T^2} \quad ...(7.28)$$

The moment of inertia of the regular body I_1 is determined with the help of the dimensions of the body, thus the moment of inertia of the irregular body is calculated.

(ii) **Determination of Torsional Rigidity, η**

For determining the modulus of rigidity η the time period of the pendulum is found (i) when the cradle is empty, (ii) when a regular body is placed on the cradle with axis of wire passing through the centre of gravity of the body. If T_1 is the time period of the pendulum in first case and T_2 in the second case, then we have

$$T = 2\pi\sqrt{\frac{I}{C}} \quad ...(7.29)$$

and $$T_1 = 2\pi\sqrt{\frac{I+I_1}{C}} \quad ...(7.30)$$

where I is the moment of inertia of the cradle and I_1 the moment of inertia of the regular body placed on the cradle. From relations (7.29) and (7.30), we have

$$T_1^2 - T^2 = \frac{4\pi^2 I_1}{C}$$

or $$C = \frac{4\pi^2 I_1}{T_1^2 - T^2} \quad ...(7.31)$$

For a wire of modulus of rigidity η length l and radius r we have

$$C = \frac{\pi \eta r^4}{2l} \quad ...(7.32)$$

Equating (7.31) and (7.32)

$$\frac{4\pi^2 I_1}{T_1^2 - T^2} = \frac{\pi \eta r^4}{2l}$$

or $$\eta = \frac{8\pi l I_1}{(T_1^2 - T^2) r^4} \quad ...(7.33)$$

Thus the value of η can be determined.

7.18 BENDING OF BEAM

A beam is a rod of uniform cross section whose length is much greater as compared to its other dimensions, so that the shearing stresses over any section of it are negligibly small. They are usually set in horizontal position and are designed to support heavy loads. They are used in buildings to support roofs and in bridges to support the load of vehicles passing over them. A beam supported at one end loaded at the other end is called cantilever.

A simple theory concerning the bending of beams making the following assumptions is developed.

(i) The weight of the beam is negligible in comparison to the load.

(ii) There are no shearing forces.

(iii) The cross section of the beam remains unaltered so that the geometrical moment of inertia of the beam remains same.

(iv) The curvature of the beam is small.

In whatever way couple is applied to bend the beam the longitudinal filaments on the convex side of the beam are extended while those on the concave side are compressed. In between these filaments, which is neither lengthened nor shortened, but remains constant in length. This filament is called neutral filament and the axis of the beam lying on the neutral filament is called the neutral axis.

7.19 BENDING MOMENT OF A BEAM

Consider the section $PBCP'$ (Fig. 7.11), the extended filaments lying above the neutral axis EF are in a state of tension and exert an inward pull on

Fig. 7.11

the filament adjacent to them towards the fixed end of the beam. In the same way the shortened filaments lying below the neutral axis EF are in a state of compression and exert an outward push on the filaments adjacent to them towards the loaded end of the beam. As a result tensile and compressive stresses develop in the upper and lower halves of the beam respectively and form a couple which opposes to bending of the beam. The moment of this couple is called the moment of the resistance. When the beam is in equilibrium position the bending moment and restoring moment or moment of resistance should be equal.

To find an expression for the moment of the restoring couple consider a fibre DE at a distance r from the neutral axis EF as shown in Fig. 7.12. Let the radius of curvature be R of the part PB and Φ be the angle subtended by it at the centre of curvature. In unstrained position of the beam, the length of the fibre $DC = EF = R\,\Phi$. In the strained position the length of the fibre $DC = (R + r)\,\Phi$.

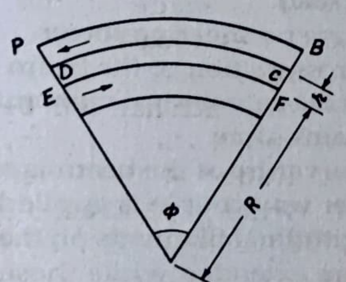

Fig. 7.12

Strain in the fibre DC

$$= \frac{\text{Change in the length}}{\text{Original length}}$$

$$\text{or Strain} = \frac{(R+r)\,\Phi - R\,\Phi}{R\,\Phi} = \frac{r}{R} \qquad \text{...(7.34)}$$

i.e., strain is proportional to the distance from the neutral axis.

Let the area of the fibre be A and its neutral axis be at a distance r from neutral axis of the beam and the strain produced be r/R. We have

$$\text{Stress} = Y \times \text{Strain} = Y\,r/R$$

Hence force on the area A

$$F = Y\,(r/R) \times A \qquad \text{...(7.35)}$$

Therefore the moment of this force about EF

$$= Y\,(r/R) \times A \times r = Y A r^2/R \qquad \text{...(7.36)}$$

As the moment of the forces acting on both the upper and lower halves of the section are in the same direction, the total moment of the forces acting on the filaments due to straining

$$= \Sigma Y \frac{a r^2}{R} = \frac{Y}{R}\,\Sigma a r^2 = \frac{Y}{R} I_g \qquad \text{...(7.37)}$$

where I_g is the geometrical moment of inertia and is equal to AK^2, A being the total area of the section and K being the radius of gyration of the beam

$$\therefore \text{ moment of the forces} = \frac{Y}{R} I_g$$

In equilibrium bending moment of the beam is equal and opposite to the moment of bending couple due to the load on one end.

$$\therefore \text{ Bending moment of the beam} = \frac{Y}{R} I_g$$

The quantity YI_g ($= Y A K^2$) is called the flexural rigidity of the beam. Flexural rigidity is defined as the bending moment required to produce a unit radius of curvature.

7.20 CANTILEVER LOADED AT THE FREE END

A beam loaded at one end and fixed at the other end is shown in Fig. 7.11 and 7.13. Consider a transverse section of the beam PP' and consider the forces which keep the portion $PBCP'$ in equilibrium. The force W acting at B vertically downward produces an equal reaction at A acting vertically upwards. These two forces form a couple, which bend the beam. As the length of the rod is along OX in its unstrained state and the depression is produced along the axis OY, the moment of the

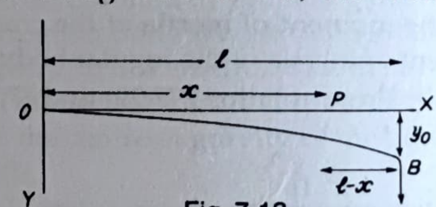

Fig. 7.13

bending couple or bending moment is $W \times (l - x)$ as shown in Fig. 7.13. Where l is the length of the beam and $x = OP$. This bending couple is balanced by a restoring couple formed by the forces acting on this part PB.

As discussed earlier in equilibrium bending moment is equal to the restoring moment, hence we have

$$\frac{Y}{R} I_g = W\,(l - x)$$

If y is the depression of the beam at the sectoin at P, the radius of curvature R of this section is given by

$$\frac{1}{R} = \frac{(d^2y/dx^2)}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$

where dy/dx is the slope of the tangent at the point (x, y). As the slope is very small because depression is small, the $\left(\frac{dy}{dx}\right)^2$ will be negligible hence the above expression approximates

$$\frac{1}{R} = \frac{d^2 y}{dx^2} \quad ...(7.39)$$

Comparing equations (7.38) and (7.39), we get

$$\frac{d^2y}{dx^2} = \frac{W(l-x)}{Y I_g}$$

by integrating above relation, we hance

$$\frac{dy}{dx} = \frac{W}{Y I_g}\left(l x - \frac{x^2}{2}\right) + C_1 \quad ....(7.40)$$

C_1 being the constant of integration. When $x = 0$ *i.e.*, (at the fixed point A the tangent is horizontal and hence the slope of the curve is zero) $\frac{dy}{dx} = 0$.

Substituting this value in (7.40), we get $C_1 = 0$

$$\therefore \quad \frac{dy}{dx} = \frac{W}{YI_g}\left(lx - \frac{x^2}{2}\right)$$

Again integrating the above equation, we get

$$Y = \frac{W}{YI_g}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_2 \quad ...(7.41)$$

where C_2 is the constant of integration. Now when $x = 0$ (*i.e.*, at A) $y = 0$. Substituting these values of x and y in (7.41), we get $C_2 = 0$.

$$\therefore \quad y = \frac{W}{YI_g}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \quad ...(7.42)$$

The above relation gives the depression of point B at a distance x from the fixed end. At the free end $x = l$, hence the depression produced at the free end is

$$y_0 = \frac{W}{YI_g}\left(\frac{l^3}{2} - \frac{l^3}{6}\right) = \frac{W l^3}{3YI_g} \quad ...(7.43)$$

Now if the beam is of rectangular cross-section of breadth b and thickness d

$$I_g = \frac{b\,d^3}{12}$$

and hence from (7.43) we have

$$y_0 = \frac{4\,W\,l^3}{Y\,b\,d^3} \quad ...(7.44)$$

If beam is of circular cross-section $I_g = \pi r^4/4$, r being radius of the beam, then

$$y_0 = \frac{4W\,l^3}{3Y\,\pi\,r^4} \quad ...(7.45)$$

7.21 CANTILEVER SUPPORTED AT ITS ENDS LOADED IN THE MIDDLE

Let AB be a beam supported on two knife edges at a distance L apart in the same horizontal line. Let the beam be loaded at the middle point O with a weight W. The reaction at each knife edge will be $W/2$ acting vertially upward. The beam bends as shown in Fig. 7.14, the maximum displacement being at the point the beam is loaded. The beam may be regarded as made of two cantilevers, whose free end carries a load $W/2$ each of length $L/2$ and fixed at the point O.

The depression in each half will be equal to the depression produced in a beam of length $L/2$ fixed at one end and loaded at the other end with a weight of $W/2$. Writing $L/2$ instead of l and $W/2$ instead of W in equation (7.43), we have for the elevation of the ends with respect to the middle point

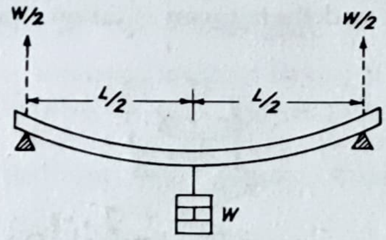

Fig. 7.14

$$y_0 = \frac{W}{2}\,\frac{L^3}{8 \times 3\,YI_g} \quad \text{or} \quad y_0 = \frac{WL^3}{48\,YI_g} \quad ...(7.46)$$

For a beam of rectangular cross-section

$$I_g = b\,d^3/12$$

(b, being the breadth and d the depth of the rod)

$$\therefore \quad y_0 = \frac{W L^3}{4\,Y\,b\,d^3}$$

For a beam of circular cross-section of radius r.

$\therefore \quad I_g = \frac{\pi r^4}{4}$ and hence $y_0 = \frac{W L^3}{12 \pi Y r^4}$

7.22 DETERMINATION OF Y BY BENDING OF BEAM

It has been shown above that if we measure the depression y_0 for a beam of known dimensions, supported at both ends and loaded at the centre, the value of Y can be determined with the help of relation (7.46). However, it is more convenient to take the beam in the form of a rectangular shape as its moment of inertia can be easily determined by measuring its dimensions, thus knowing W, L, b, d and y the value of Y can be determined.

Necessary arrangement for determining Y by bending is shown in Fig. 7.15. The beam is placed symmetrically on two strong knife edges K_1 and K_2. Exactly midway between the two knife edges at a point O a hanger with hook is hung. The central screw of the spherometer is made to touch. Now a known load is applied by placing weights on the hanger. The depression y of the mid point thus produced is noted directly with the help of micrometer screw or more accurately with the help of a microscope having cyepiece fitted with cross wires.

First the readings are taken by increasing the load in steps and then by decreasing the load in the same steps and their mean is taken. This gives the

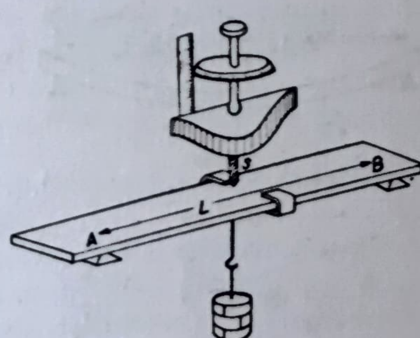

Fig. 7.15

mean depression y_0. A graph between the depression and load $W (= mg)$ is plotted which is a straight line. The slope of this graph gives the value of W/y_0. The distance L between the knife edges is measured with the help of a metre scale. The breadth b and thickness d of the beam is measured by a screw gauge or vernier callipers and their mean value is determined. Now substituting the value of $W/y_0, L, b$ and d in the following relation

$$Y = \frac{W L^3}{4 y_0 b d^3} \quad ...(7.47)$$

the value of Y is calculated.

EXAMPLE 1 *A load of 2 kg produces an extension of 1 mm in a wire of 3 metres in length and 1 mm in diameter. Calculate the Young's modulus of the wire.*

Solution : Here, load applied

$$W = 2 \times 9.8 = 19.6 \text{ N}.$$

Increase in length, $l = 1$ mm $= 0.001$ m
Original length, $L = 3$ m
Diameter of the wire, $r = 1$ mm $= 0.001$ m.

$$Y = \frac{FL}{Al} = \frac{FL4}{l \pi r^2} = \frac{19.6 \times 3 \times 4}{0.001 \times \pi \times (0.001)^2}$$

$$= 7.48 \times 10^{10} \text{ N/m}^2.$$

EXAMPLE 2 *What force is required to stretch a steel wire to double its length when its area of cross-section is 1 sq. cm and Young's modulus 2×10^{11} N/m^2.*

Solution : Let the original length $= L$

$\therefore$ Increase in length $= L$

Area of cross section $A = 10^{-4}$ m^2

Youn'g modulus $Y = 2 \times 10^{11}$ N/m^2

$$\therefore \; Y = \frac{F/A}{L/l} \text{ or } 2 \times 10^{11} = \frac{F}{10^{-4}}$$

or $F = 2 \times 10^7$ N.

EXAMPLE 3 *A steel wire 2 mm in diameter is just stretched between two fixed points at a temperature of 20°C. Determine its tension when temperature falls to 10°C. Coefficient of linear expansion of steel $= 11 \times 10^{-6}$/°C and Y for steel is 2×10^{11} N/m^2.*

Solution : Let its length at 20° C $= L_2$, its length at 10° C $= L_1$ ...(i)

$\therefore$ Change in length $= L_2 - L_1 = L_1 \alpha t$

α, being the coefficient of thermal expansion and t rise in temperature. We know

$$\text{Strain} = \frac{\text{Change in length}}{\text{Original length}} = \frac{L_2 - L_1}{L_1}, \text{ and}$$

$$\text{Stress} = \frac{\text{Force}}{\text{Area}} = \frac{F}{\pi r^2}$$

$$= \frac{F}{\pi\,(1\times 10^{-3})^2} = \frac{F\times 10^6}{\pi}\ \text{N/m}^2.$$

$$\therefore\ Y = \frac{\text{Stress}}{\text{Strain}} = \frac{L_1}{L_2 - L_1}\times\frac{F\times 10^6}{\pi}$$

$$\text{or}\ (L_2 - L_1) = \frac{L_1}{\pi Y}\times F\times 10^6 \qquad \text{...(ii)}$$

From relations (*i*) and (*ii*), we have

$$L_2 - L_1 = \frac{L_1\times F\times 10^6}{\pi Y} = L_1\,\alpha\, t$$

$$\text{or}\qquad F = \alpha\, t\, Y\,\pi\times 10^{-6}$$

$$= 11\times 10^{-6}\times 10\times 2\times 10^{11}\times\frac{22}{7}\times 10^{-6}$$

$$= \mathbf{69.14\ N.}$$

EXAMPLE 4 *A steel wire of length 2.0 m and cross section 1×10^{-6} m^2 is held between rigid supports with a tension 200 N. If the middle of the wire is pulled 5 mm side ways, calculate change in tension. Also calculate the change in tension if temperature changes by 5°C. [For steel $Y = 2.2\times 10^{11}$ N/m^2 and $\alpha = 8\times 10^{-6}$ K^{-1}]*

Solution : Let the change in tension

$$F = \frac{YAl}{L} \qquad \text{...(i)}$$

when the wire is pulled 5×10^{-3} m sideways at the middle, then the new length of the wire will be

$$L' = 2\sqrt{[(1)^2 + (5\times 10^{-3})^2]}$$

Hence increase in length of the wire

$$l = 2\sqrt{[(1)^2 + (5\times 10^{-3})^2]} - 2$$

$$= 25\times 10^{-6}\ \text{m} \qquad \text{...(ii)}$$

Substituting the value of l in relation (*i*) we get

$$F = \frac{2.2\times 10^{11}\times 10^{-6}\times 25\times 10^{-6}}{2} = 2.75\ \text{N.}$$

Now when temperature changes by 5°C, the change in length of the wire is given by

$$l = L\alpha t = 2\times 8\times 10\times^{-6}\times 5 = 8\times 10^{-5}\ \text{m.}$$

$\therefore$ Change in tension

$$F = 2.2\times 10^{11}\times 10^{-6}\times 25\times 10^{-5}/2 = \mathbf{8.8\ N}$$

EXAMPLE 5 *A fixed beam of 50 cm is depressed by 15.0 mm at the loaded end. Calculate the depression at a distance 0.3 m from the fixed end.*

Solution : From relation (7.42) we have for the depression at a point x distance apart from the fixed end.

$$y = \frac{W}{YI_g}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) \qquad \text{...(i)}$$

and at the free end $x = l$ and $y = 15\times 10^{-3}$ m.

$$\therefore\ 15\times 10^{-3} = \frac{W}{YI_g}\left(\frac{l^3}{2} - \frac{l^3}{6}\right) = \frac{W}{YI_g}\times\frac{l^3}{3} \qquad \text{...(ii)}$$

From (*i*) and (*ii*)

$$\frac{y}{15\times 10^{-3}} = \frac{\dfrac{W}{YI_g}\left(\dfrac{lx^2}{2} - \dfrac{x^3}{6}\right)}{\dfrac{W}{YI_g}\dfrac{l^3}{3}}$$

$$\text{or}\ y = \frac{15\times 10^{-3}\left(\dfrac{lx^2}{2} - \dfrac{x^3}{6}\right)}{l^3/3}$$

In this problem $x = 0.3$ m and $l = 0.5$ m.

$$\therefore\ y = \frac{15\times 10^{-3}}{(1/3)\times(0.5)^3}\times\left[\frac{0.5\times(0.3)^2}{2} - \frac{(0.003)^2}{6}\right]$$

$$= 8.05\times 10^{-3}\ \text{m} = \mathbf{8.05\ mm.}$$

EXAMPLE 6 *A wire of radius r stretched without tension, along a straight line is lightly fixed at A and B (Fig. 7.16). What is the tension in the wire when it is pulled into the shape ACB? Assume Young's Modulus of the material of wire to be Y.*

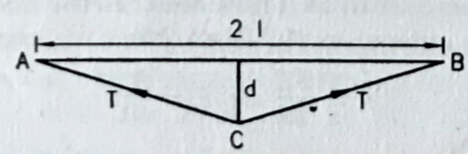

Fig. 7.16

Solution : As final length is $AB + CD$ and initial length was $2l$, we have

$$\text{Increase in length} = AC + BD - 2l$$

$$= 2\sqrt{(l^2 + d^2)} - 2l$$

Longitudinal stress $= \dfrac{T}{\pi (r)^2}$ (r, radius of wire)

Longitudinal strain $= \dfrac{2(\sqrt{(l^2+d^2)}-l)}{2l}$

$\therefore\ Y = \dfrac{\text{Longitudinal stress}}{\text{Longitudinal strain}}$

or $Y = \dfrac{T/\pi r^2}{(\sqrt{l^2+d^2}-l)/l} = Y = \dfrac{Tl}{\pi r^2[(\sqrt{l^2+d^2}-l]}$

or $T = \dfrac{Y\pi r^2[(\sqrt{l^2+d^2}-l]}{l}$

If $l >> d$, then $= \sqrt{l^2+d^2} = l\left(1+\dfrac{d^2}{2l^2}\right)$

$\therefore\ T = \dfrac{Y\pi r^2 d^2}{2l^2}$

EXAMPLE 7 *A sphere of radius 10 cm and mass 25 kg is attached to the lower end of steel wire which is suspended from the ceiling of a room. the point of support is 521 cm above floor. When the sphere is set swinging as a simple pendulum, its lowest point just grazes floor. Calculate the velocity of the ball at its lower position. Young's modulus of steel 2×10^{11} N/m². Unstretched length of wire = 5.2 m and radius of the steel wire = 0.5 cm.*

Solution : When the sphere swings, then at its lower position the force $F = mg + mv^2/r$

or $$F = 25\times9.8+\frac{25v^2}{5.11} \quad ...(i)$$

(as $r = 5.21 - 0.10 = 5.11$)

Due to this force the original length of wire 5.2 m becomes 5.21 m as it now touches the floor *i.e.*, it causes an extension of 0.01 m. Now we know that

$$Y = \frac{FL}{Al} \text{ or } F = \frac{YAl}{L}.$$

Given $Y = 2\times10^{11}$ N/m², $L = 5.2$ m, $l = 0.01$ m.

$$A = \pi r^2 = 3.14\times\left(\frac{0.05}{100}\right)^2 = 7.85\times10^{-7}\text{ m}^2.$$

$$\therefore\ F = 2\times10^{11}\times7.85\times10^{-7}\times0.01/5.2 = 301.92\text{ N}.$$

Substituting in equation (*i*), we have

$$301.92 = 25\times9.8+\frac{25\times v^2}{5.11}$$

$$v^2 = \frac{(301.92-25\times9.8)\times5.11}{25} = 11.69 \text{ or } v = 3.41\text{ m/s}.$$

EXAMPLE 8 *A light rod of length 200 cm is suspended from the ceiling horizontally by means of two vertical wires of equal length tied to its ends. One of the wires is made of steel and is of cross section 0.1 sq. cm. and other is of brass of cross section 0.2 sq. cm. Find the position along the rod at which a weight may be hung to produce (i) Equal stresses in both wires. (ii) Equal strain in both wires. Young's modulus of elasticity of brass and steel are 10^{11} N/m² and 2×10^{11} N/m² respectively.*

Solution : Suspended rod is shown in Fig. 7.17. Consider the rod *AB* be of 200 cm and a weight *W*

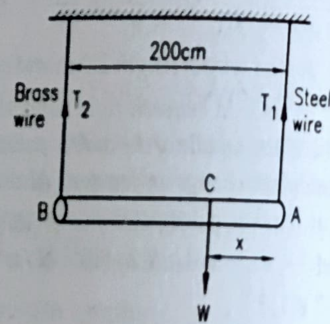

Fig. 7.17

be hung at C at a distance, *x* from *A* and T_1 and T_2 be tensions in steel and brass wires respectively.

(*i*) Stress in steel wire (=*F*/*A*)

$$= \frac{T_1}{A_1} = \frac{T_1}{0.1\times10^{-4}}$$

Stress in brass wire $= \dfrac{T_2}{A_2} = \dfrac{T_2}{0.2\times10^{-4}}$

where T_1 and T_2 are in Newtons.

According to the problem

$$\frac{T_1}{0.1\times10^{-4}} = \frac{T_2}{0.2\times10^{-4}} \text{ or } \frac{T_1}{T_2} = 0.5 \quad ...(i)$$

As the rod is in equilibrium taking moment of forces about C, we have

$$T_1 x = T_2(2-x)$$

or $$\frac{T_1}{T_2} = \frac{2-x}{x} = \frac{2}{x}-1 \quad ...(ii)$$

Equating (*i*) and (*ii*), we have

$$(2/x) - 1 = 0.5$$

or $\quad x = 4/3 = 1.333$ m or **133.33 cm**

Hence to produce equal stresses in two wires the load should be hung at a distance at **133.33 cm** from end *A*.

(*ii*) As $Y = \dfrac{T/A}{\text{Strain}}$ or $\text{Strain} = \dfrac{T}{AY}$

For steel wire strain $= \dfrac{T_1}{A_1 Y_1}$

For brass wire strain $= \dfrac{T_2}{A_2 Y_2}$

As both these strains are equal, we have on equating

$$\frac{T_1}{T_2} = \frac{A_1 Y_1}{A_2 Y_2} = \frac{0.1 \times 10^{-4} \times 2 \times 10^{11}}{0.2 \times 10^{-4} \times 10^{11}} = 1$$

or $T_1 = T_2$.

Again taking moments about C, we have

$$T_1/T_2 = (2 - x)/x$$

$\therefore$ $(2-x)/x = 1$ or $x = 1$ m or 100 cm.

Thus the weight should be hung at the centre of the rod so that strains produced are same.

EXAMPLE 9 *The end of a given strip, cantilever depresses* 10 mm *under a certain load. Calculate the depression under same load for another cantilever of same material,* 2 times *in length,* 2 times *in width and* 3 times *in thickness (vertical).*

Solution : For a free end cantilever of length *l*, the depression is given by

$$Y = \frac{M g l^3}{3 Y I} = \frac{W l^3}{3 Y I} = \frac{4 W l^3}{Y b d^3}$$

(as for beam $I = b\,d^3/12$)

Now if y' is the depression for a beam of length $2l$, width $2b$ and thickness, $3d$ then

$$y' = \frac{4W\,(2l)^3}{Y\,(2b)\,(3d)^3} = \frac{16 W l^3}{27\,Ybd^3} = \frac{4}{27}\left(\frac{4 W l^3}{Ybd^3}\right) = \frac{4}{27} y$$

Since $y = 10 \times 10^{-3}$ m $= 10$ mm

$\therefore$ $y' = (4/27) \times 10 =$ **1.48 mm.**

EXAMPLE 10 *Calculate Poisson's ratio for silver. Given its Young's modulus* $= 7.25 \times 10^{10}$ N/m^2 *and bulk modulus* 11×10^{10} N/m^2.

Solution : We know $Y = 3\,K\,(1 - 2\sigma)$

or $\quad \sigma = \dfrac{1}{2}\left(1 - \dfrac{Y}{3K}\right)$

Given $Y = 7.25 \times 10^{10}$, $K = 11 \times 10^{10}$ N/m^2

$$\therefore\ \sigma = \frac{1}{2}\left(1 - \frac{7.25 \times 10^{10}}{3 \times 11 \times 10^{10}}\right) = 0.39.$$

EXAMPLE 11 *A material has Poisson's ratio 0.2. If a uniform rod of it suffers longitudinal strain* 4.0×10^{-3}, *calculate the percentage change in its volume.*

Solution : We know

$$\sigma = \frac{\Delta r/r}{\Delta l/l} = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

Given $\Delta l/l = 4.0 \times 10^{-3}$ and $\sigma = 0.2$

$\therefore$ $\quad \Delta r/r = 0.20 \times 4.0 \times 10^{-3} = 0.80 \times 10^{-3}$

If V is the volume initially and $V + \Delta V$ is final volume then $V = \pi r^2 l$

and $\quad V + \Delta V = \pi\,(r - \Delta r)^2\,(l + \Delta l)$

From above two equations neglecting higher power terms, we have

$$\Delta V = \pi r^2 \Delta l - 2\pi r l \Delta r$$

$$\therefore\ \frac{\Delta V}{V} = \frac{\Delta l}{l} - 2\frac{\Delta r}{r}$$

Substituting the value of $\Delta l/l$ and $\Delta r/r$, we have

$$\frac{\Delta V}{V} = 4 \times 10^{-3} - 2\,(0.8 \times 10^{-3}) = 2.4 \times 10^{-3}$$

$\therefore$ percentage change in volume

$= (2.4 \times 10^{-3}) \times 100 =$ **0.24 %.**

EXAMPLE 12 *A wire of length l* m *and diameter* 1 mm *is clamped at one of its ends. Calculate the couple required to twist the other end by* 90°. *Given* $\eta = 2.8 \times 10^{10}$ N/m^2.

Solution : The torque required to twist the free end of a clamped wire of length l through ϕ radian will be

$$\tau = \frac{\pi \eta r^4}{2l}\phi$$

For $\phi = 90° = \dfrac{\pi}{2}$ radian $\tau = \dfrac{\pi^2 \eta r^4}{4l}$

Given $\eta = 2.8 \times 10^{-10}$ N/m^2, $l = 1$ m

$r = 5\text{ mm} = 0.0005\text{ m}$

$$\therefore\ \tau = \frac{\pi^2 \times 2.8 \times 10^{10} \times (5 \times 10^{-4})^4}{4}$$

$$= 4.32 \times 10^{-3}\ \text{N-m.}$$

EXAMPLE 13 *An iron wire of length 1 m and radius 0.5 mm elongates by 0.32 mm when stretched by a force of 49 N, and twists through 0.4 radian when equal and opposite torques of 3×10^{-3} N-m are applied at its ends. Calculate elastic constant for iron.*

Solution : $Y = \dfrac{\text{Stress}}{\text{Strain}} = \dfrac{F/\pi r^2}{x/l} = \dfrac{F\,l}{\pi r^4 x}$

Here force, $F = 49$ N, radius $r = 5 \times 10^{-4}$ m and elongation $x = 32 \times 10^{-5}$ m and $l = 1$.

$$\therefore\ Y = \frac{49 \times 1}{\pi \times (5 \times 10^{-4})^2\ (32 \times 10^{-5})}$$

$$= 19.5 \times 10^{10}\ \text{N/m}^2.$$

EXAMPLE 14 *Find the amount of work done in twisting a steel wire of radius 1 mm and length 25 cm through an angle of 45°, the modulus of rigidity of steel being 8×10^{10} N/m².*

Solution : The work done by a torque τ in twisting through θ is $\tau . \theta$. Hence the work done for producing a total twist θ_0 will be

$$W = \int_0^{\theta_0} \tau\,\theta\,d\theta = \frac{1}{2}\tau\,\theta_0^2$$

Now we also know $\tau = \pi\,\eta\,r^4/2\,l$

$$\therefore\ W = \frac{1}{2}\frac{\pi\,\eta\,r^4}{2\,l}\theta_0^2$$

Given $\eta = 8 \times 10^{10}$ N/m², $r = 10^{-3}$ and $l = 0.25$ m and $\theta_0 = \pi/4$.

$$\therefore\ W = \frac{1}{2}\ \frac{\pi \times 8 \times 10^{10} \times (10^{-3})^4}{2 \times 0.25} \times \left(\frac{\pi}{4}\right)^2$$

$$= 15.5 \times 10^{-2}\ \text{J.}$$

EXAMPLE 15 *A wire of radius 1 mm and length 2 m is twisted through 90°. Calculate the angle of shear at the surface, at the axis of the wire and at a point mid way, if the modulus of rigidity of material is 5×10^{10} N/m², what is the torsional couple?*

Solution :) The angle of shear at a radial distance x from the axis of the wire of length l and radius r for a twist ϕ at the free end will be $\theta = (x/l)\,\phi$ at the axis of the wire $x = 0$ so that $\theta = 0$. At the surface of the wire $x = r$, so that $\theta = r\,\phi/l$,

As $r = 10^{-3}$ m, $l = 2$ m and $\phi = \pi/2 = 90°$

$\therefore\ \theta = 10^{-3} \times 90/2 = 0.045°$

Now at a point midway, $x = r/2$

$\therefore\ \theta = 0.045/2 = 0.0225°$

The torsional couple $\tau = \dfrac{\pi\,\eta\,r^4}{2\,l}\,\phi$

For $\phi = \dfrac{\pi}{2}$ radian we have $\tau = \dfrac{\pi^2\,\eta\,r^4}{4\,l}$

Substituting the values of η, r and l, we have

$$\tau = \frac{\pi^2 \times 5 \times 10^{10} \times (10^{-3})^4}{4 \times 2} = 6.168 \times 10^{-2}\ \text{N.m.}$$

EXAMPLE 16 *Two solid cylinders of the same material having length l and 2l, and radii r and 2r are joined coaxially. Under a couple applied between the free ends, the shorter cylinder shows a twist of 30°. calculate the twist of the longer cylinder.*

Solution : If τ is the couple, and it produces twist θ in the shorter cylinder and twist θ' in the larger cylinder. Then

$$\tau = \frac{\pi\,\eta\,r^4}{2\,l}\,\theta = \frac{\pi\,\eta\,(2r)^4}{2\,(2\,l)}\,\theta'$$

$\therefore\ \theta' = \theta/8 = 30°/8 = 3.75°.$

EXAMPLE 17 *One end of a wire of 4 mm radius and 100 cm in length is twisted through 60°. Calculate the angle of shear on its surface.*

Solution : If ϕ is the twist at the free end and θ is the angle of shear, then $\theta = r\,\phi/l$

As $l = 1$ m, $r = 4 \times 10^{-3}$ m and $\phi = 60°$

$\therefore\ \theta = 4 \times 10^{-3} \times 60 = 0.24°.$

EXAMPLE 18 *The restoring couple per unit twist in a solid cylinder of radius 5.0 cm is 10^{-1} N-m. Find the restoring couple per unit twist in a hollow cylinder of the same material, mass and length but the internal radius 12 cm.*

Solution : The restoring couple per unit twist in a solid cylinder of length of l is given by

$$C = \pi\,\eta\,r^4/2\,l$$

Here $r = 0.05$ m and $C = 10^{-1}$ N-m.

$$\therefore\ \frac{\pi\,\eta}{2\,l} = \frac{C}{r^4} = \frac{10^{-1}}{(0.5)^4}$$

If r_1 and r_2 are the internal and external radii of the hollow cylinder of the same length and mass and of same material, then

$$m = \pi (r_2^2 - r_1^2)\, l\rho = \pi r^2 l\rho \text{ or } r_2^2 - r_1^2 = r^2$$

Here $r_1 = 0.12$ m and $r = 0.05$ m

$$\therefore\ r_2^2 = r^2 + r_1^2 = (0.05)^2 + (0.12)^2 = (0.13)^2$$

$$\therefore\ r_2 = 0.13 \text{ m.}$$

The restoring couple per unit twist for this hollow cylinder is

$$C' = \frac{\pi \eta (r_2^4 - r_1^4)}{2l} = \frac{10^{-1}}{(0.5)^4}[(0.13)^4 - (0.12)^4]$$

$$= 1.25 \times 10^{-4} \text{ N.m.}$$

EXAMPLE 19 *A cantilever of length of 0.5 m has a depression of 15 mm at its free end. Calculate the depression at a distance of 0.3 metre from the fixed end.*

Solution : We know the depression of the cantilever at a distance x from its fixed end is given by

$$y = \frac{W}{YI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) = \frac{Wx^2}{2YI}\left(l - \frac{x}{3}\right) \quad ...(i)$$

and at the free end ($x = l$) is $\delta = \dfrac{Wl^3}{3YI}$ ...(ii)

Here l is the length of cantilever, W is the load, Y is the Young's modulus of the material and I is the geometrical moment of inertia.

From above two equations (i) and (ii), we have

$$\frac{y}{\delta} = \frac{\frac{x^2}{2}\left(l - \frac{x}{3}\right)}{l^3/3} \text{ or } y = \frac{3x^2\left(l - \frac{x}{3}\right)}{2l^2}\delta$$

Here $l = 0.5$ m, $x = 0.3$ m and $\delta = 0.015$ m

$$\therefore\ y = \frac{3 \times (0.3)^2 \times \left(0.5 - \frac{0.3}{3}\right)}{2 \times (0.5)^3} \times 15 \times 10^{-3}$$

$$= 6.48 \times 10^{-3} \text{ m} = \mathbf{6.48\ mm.}$$

EXAMPLE 20 *Calculate the maximum power which can be transmitted by means of a steel shaft rotating about its axis with an angular velocity ω. If its length is L, radius = r and the permissible torsion angle ϕ.*

Solution : Torque on the shaft (Eq. 7.32)

$\tau = 4\pi^2 r^4 \eta / 2l$. As work done = $\tau\theta$ and power $P = \tau\theta/t = \tau\omega$

$$\therefore\ P = \frac{4\pi^2 r^4 \eta}{2l} \times \omega.$$

OBJECTIVE TYPE QUESTIONS

1. Four wires of the same material are stretched by the same load. The dimensions are given below. Which of them will elongate the most?
   (a) diameter 1 mm, length 100 cm
   (b) diameter 2 mm, length 200 cm
   (c) diameter 0.5 mm, length 50 cm

2. The extension of a wire by the application of a load is 3 mm. The extension in a wire of the same material and length but half the radius by the same load will be
   (a) 12.0 mm (b) 0.75 mm
   (c) 6.0 mm (d) 1.5 mm

3. A wire of length L and r is fixed at one end and a force F applied to the other end produces an extension l. The extension produced in another wire of the same material of length $2L$ and radius $2r$ by a force $2F$ is
   (a) l (b) $2l$ (c) $l/2$ (d) $4l$

4. Young's modulus of material of a wire is defined as
   (a) ratio of linear strain to normal stress
   (b) ratio of normal stress to linear strain
   (c) product of linear strain and normal stress
   (d) square root of the ratio between normal stress and linear strain

5. Young's modulus (Y), modulus of ragidity (η) and Poisson ratio are related as
   (a) $Y = 2\eta/(1+\sigma)$ (b) $\sigma = \dfrac{2Y}{(1+\eta)}$
   (c) $\dfrac{Y}{\eta} = 2(1+\sigma)$ (d) $2Y = \eta(1+\sigma)$

6. Young's modulus (Y), modulus of rigidity (η) and Bulk modulus are related as
   (a) $\dfrac{1}{K} = \dfrac{1}{3\eta} + \dfrac{1}{9Y}$ (b) $\dfrac{1}{\eta} = \dfrac{1}{3Y} + \dfrac{1}{9K}$
   (c) $\dfrac{Y}{3} = \dfrac{1}{\eta} + \dfrac{1}{9K}$ (d) $\dfrac{1}{Y} = \dfrac{1}{3\eta} + \dfrac{1}{9K}$

7. A spiral spring is stretched by a weight attached to it. The strain will be
   (a) elastic (b) bulk
   (c) shear (d) tensile

8. The effect of temperature on the value of modulus of elasticity for various substances in general is
(a) it increases with increase in temperature
(b) remains constant
(c) decreases with rise in temperature

9. The ratio of lateral contraction to longitudinal strain, when a body undergoes a linear tensile strain is known as
(a) modulus of elasticity (b) Young's modulus
(c) Bulk modulus (d) Poisson's ratio

10. If the diameter of the suspension wire is doubled without changing the length in case of a torsional pendulum, the time period
(a) will increase (b) will not be affected
(c) will decrease (d) will double

11. A simple shear θ is equivalent to an extension strain and compression strain at right angles to each other of value
(a) 2θ (b) θ (c) $\frac{\theta}{2}$ (d) $\frac{3\theta}{2}$

12. A metal ball immersed in alcohol weighs W_1 at 0°C and W_2 at 50°C. The coefficient of cubical expansion of the metal is less than that of the alcohol. Assuming that the density of the metal is large compared to that of alcohol, it can be shown that
(a) $W_1 > W_2$ (b) $W_1 = W_2$
(c) $W_1 < W_2$

Answers

1. (d)	2. (a)	3. (a)	4. (b)	5. (c)
6. (d)	7. (c)	8. (c)	9. (d)	10. (c)
11. (c)	12. (b)			

EXERCISES AND PROBLEMS

1. Explain the terms stress, strain, Young's modulus, Bulk modulus, Modulus of rigidity and Yield point.

2. State and explain Hooke's law. Describe an experiment to determine the Young's modulus of a wire.

3. Define Young's modulus Y, Bulk modulus K, and Poisson's ratio σ, and derive the relation between them.

4. Show that a shear θ is equivalent to a compression strain $\theta/2$ and an extension strain $\theta/2$ in two mutually perpendicular directions.

5. Define modulus of rigidity η. Show that a tangential stress T is equivalent to a tensile stress T and a compressive stress T at right angles to each other and also prove the relation

$$\eta = \frac{Y}{2(1+\sigma)}$$

6. Show that the bulk modulus K, Young's modulus Y and Poisson's ratio σ are connected by the relation

$$K = \frac{Y}{3(1-2\sigma)}$$

7. Prove the following relations
(a) $\frac{Y}{\eta} = 2(1+\sigma)$ (b) $\frac{1}{K} = \frac{9}{Y} - \frac{3}{\eta}$
(c) $Y = \frac{9\eta K}{3K+\eta}$

8. Show that for a light cantilever of length l and carrying a load W at the free end, the depression of a point x distance apart from fixed end is given by

$$y = \frac{W}{Y I_g} \cdot \left(\frac{lx^2}{2} - \frac{x^3}{6}\right)$$

9. If one end of a bar is fixed and a load is applied to the other end, calculate depression at the free end.

10. A rectangular bar of iron is supported at its two ends on knife edges and a load is applied at the middle point. Calculate the depression at the middle point. How can this be utilized to determine Young's modulus of iron?

11. Describe Torsion Pendulum. How it can be used to measure the moment of inertia of an irregular body and torsional rigidity?

12. A steel wire, length 2.75 m and diameter 1.0 mm, has its upper end attached to a beam. If a load of 1.0 kg is suspended from the lower end, by how much will the wire be extended?

13. A wire 0.32 mm in diameter elongates by 1 mm when stretched by a force of 0.33 kg wt. and twists through 1 radian when equal and opposite torques of 145×10^{-7} N-m are applied at its ends. Find Poissons's ratio for the wire. [Ans. 0.429]

14. A uniform rigid rod 1.2 m long is clamped horizontally at one end. A weight of 1 kg is attached to the free end. Calculate the depression of a point 0.9 m distance apart from the clamped end. The diameter of the rod is 2 cm. Y for the material of the rod is 1.013×10^{10} N/m^2. [Ans. 0.48 mm]

15. A rectangular bar 2 cm in breadth and 1 cm in depth and 1 metre in length is supported at its ends and a load of 2 kg is applied at its middle. Calculate the depression if the Young's modulus of the material of the beam is 2×10^{11} N/m^2.

[Ans. 1.2 mm]

16. In an experiment a rod of diameter 1.26 cm was supported on two knife edges, placed 0.7 m apart. On applying a load of 9 kg exactly midway between the knife edges, the depression on the middle point was observed to be 0.025 cm. Calculate the Young's modulus of the material of the rod.

[Ans. 20.42×10^{10} N/m^2]

17. How much work is done in stretching a wire 1 mm by a force of 1 kN. How far would a wire of the same material and length but of four times that diameter be stretched ? [Ans. 0.5 J and 1/16 mm]

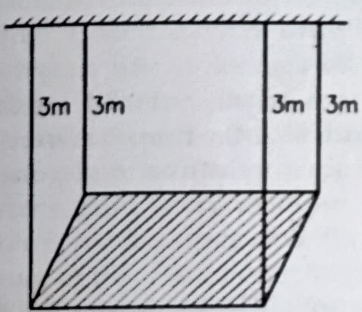

Fig. 7.18

18. Four wires each of 3 m length and diameter 2 mm support a platform as shown in Fig. 7.18. The Young's modulus of the material of the wire is 1.8×10^{11} N/m^2. How far will the platform drop due to elongation of the wire if an 80 kg. load is placed at the centre of the platform.

19. Show that the density of stored elastic energy in the stretched wire is independent of the wire dimension as is equal to $\tau^2/2Y$ where τ is the stress and Y is the Young's modulus of elasticity.

20. If the certain volume of water is to be compressed by 0.2 percent how large pressure must be applied to water. The bulk modulus of water is 21×10^9 N/m^2.

[Ans. 1.05×10^6 N/m^2]

21. An unstretched block of gelatine has the dimensions $6 \times 6 \times 2$ cm. If a force of 0.3 N is applied tangentially to the upper surface as shown in Fig. 7.19 causes a displacement of 5 mm relative to the lower surface. Calculate the shearing strain and stress and the shear modulus.

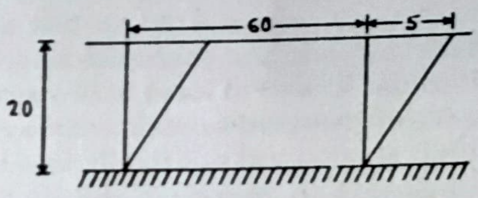

Fig. 7.19

[Ans. 0.25; 83.33 N/m^2 and 333.33 N/m^2.]

22. Two metallic sheets on an aircraft wing are to be held together by the rivets of the same material of cross sectional area 1.6 cm^2. The shearing stress on each rivet must not exceed one-tenth of the elastic limit for the metal. How many rivets are needed if each rivet supports the same fraction of a total shearing force of 3.5 kN? Assume that the elastic limit stress is 4×10^6 N/m^2.

[Ans. 53 rivets]