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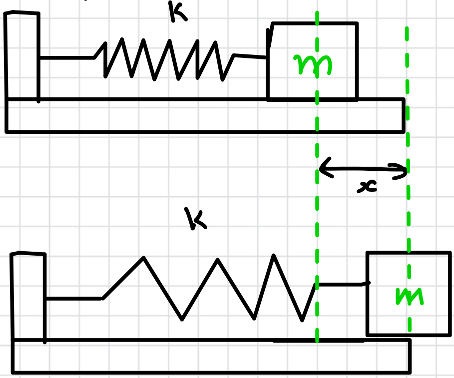
UNIT-1

1) Simple Harmonic Oscillators



$$m\frac{d^2x}{dt^2} = -Kx$$
↳ equation of motion

$$\ddot{x} + \omega_0^2 x = 0$$

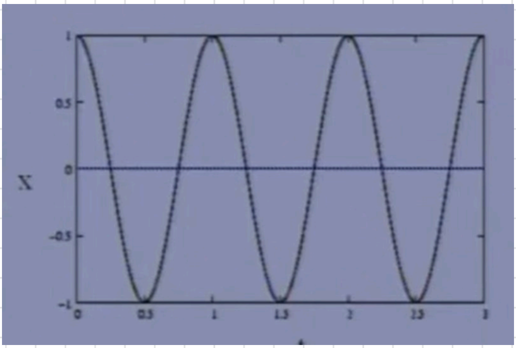
$$\Rightarrow \omega_0 = \sqrt{\frac{K}{m}}$$

solution is as follows :-

$$x(t) = A\cos(\omega_0 t + \phi) \longrightarrow ①$$

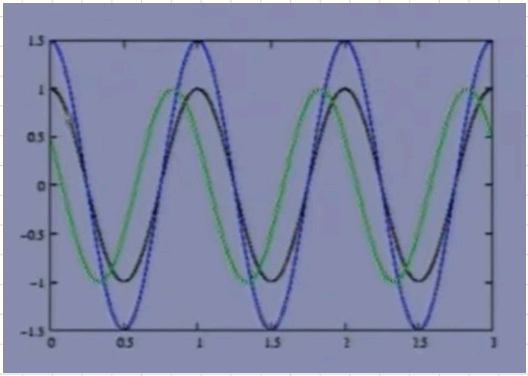
Discussing each of the constants in the equation ①

Angular Frequency



$$T = \frac{2\pi}{\omega_0}$$

$$x(t) = A\cos(\omega_0 t + \phi)$$



Complex numbers

$$\tilde{x} = Ae^{i\phi}$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

Complex representation of an oscillating quantity.

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$\tilde{x}(t) = Ae^{i(\omega_0 t + \phi)}$$
$$\Rightarrow A[\cos(\omega_0 t + \phi) + i(\sin(\omega_0 t + \phi)]$$

The real part of the complex number

$$\Rightarrow \tilde{x}(t) = Ae^{i(\omega_0 t + \phi)}$$

represents the oscillating quantity

$$x(t) = A\cos(\omega_0 t + \phi)$$

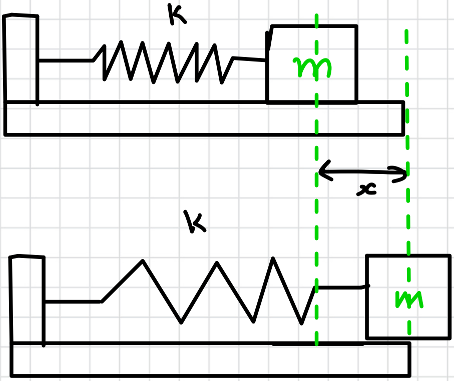
Complex velocity

$$\tilde{x}(t) = Ae^{i(\omega_0 t + \phi)}$$

$$\Rightarrow \tilde{v} = \dot{\tilde{x}} = i\omega_0 \tilde{x}$$

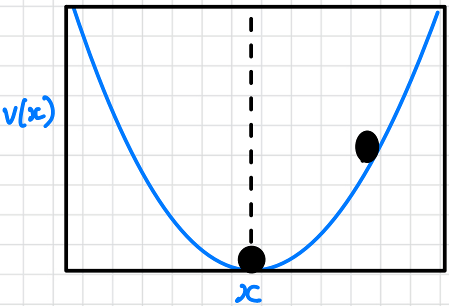
$$\Rightarrow -\omega_0 A[\sin(\omega_0 t + \phi) - i\cos(\omega_0 t + \phi)]$$

$v(t) = -\omega_0 A \sin(\omega_0 t + \phi)$



Potential $V(x)$

$$V(x) = \frac{Kx^2}{2}$$



Potential energy

$$U = \frac{1}{2} K A^2 \cos^2(\omega_0 t + \phi)$$
$$\Rightarrow \frac{1}{4} m\omega_0^2 A^2 \{1 + \cos[2\omega_0 t + \phi)]\}$$

Kinetic energy

$$T = \frac{1}{2} m\omega_0 A^2 \sin^2(\omega_0 t + \phi)$$
$$\Rightarrow \frac{1}{4} m\omega_0^2 A^2 \{1 - \cos[2(\omega_0 t + \phi)]\}$$

Energy

$$U = \frac{1}{4} m\omega_0^2 A^2 \{1 + \cos[2(\omega_0 t + \phi)]\}$$
$$T = \frac{1}{4} m\omega_0^2 A^2 \{1 - \cos[2(\omega_0 t + \phi)]\}$$
$$\Rightarrow E = T + U = \frac{m\omega_0^2 A^2}{2}$$

Time average

$$\langle Q \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} Q(t)\, dt$$

Average oscillations

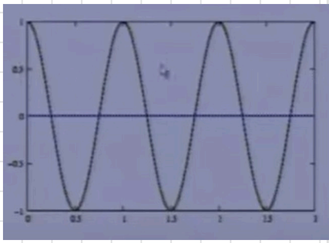
$$\langle \cos(\omega_0 t + \phi) \rangle = 0$$

Average energy

$$\langle U \rangle = \langle T \rangle = \frac{1}{4} m\omega_0^2 A^2$$
$$= \frac{1}{4} m\tilde{v}\tilde{v}^* = \frac{1}{4} K\tilde{x}\tilde{x}^*$$

complex conjugate

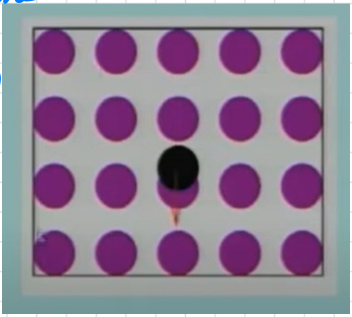
Root Mean Square



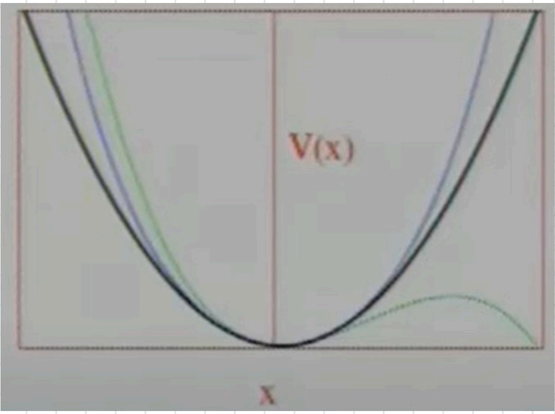
$$\sqrt{\langle x^2 \rangle} = \sqrt{\frac{\tilde{x}\tilde{x}^*}{2}}$$

Atomic Vibrations

Particle distributed from stable equilibrium



Stable Equilibrium



$V(x) = x^2$

$V(x) = \exp\left(\frac{x^2}{2}\right) - 1$

$V(x) = x^2 - x^3$

The Potential near Equilibrium

$$V(x) \approx V(x)_{x=0} + \left(\frac{dV(x)}{dx}\right) x + \frac{1}{2}\left(\frac{d^2V(x)}{dx^2}\right)_{x=0}$$

$$F = \frac{-dV(x)}{dx} = 0$$

$$K = \left(\frac{d^2V(x)}{dx^2}\right)_{x=0} > 0$$

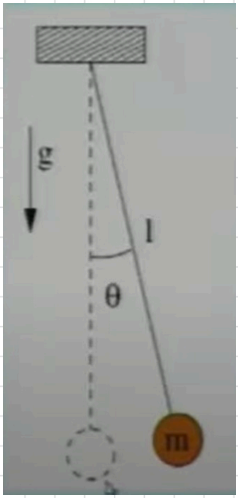
$$V(x) \approx V(x)_{x=0} + \frac{1}{2} Kx^2$$

Simple Pendulum

$V(\theta) = mgl[1-\cos\theta]$

$\cos\theta \approx 1 - \frac{\theta^2}{2}$

$V(\theta) = \frac{1}{2} mgl\theta^2$



Solution

$I\ddot{\theta} = -mgl\theta$

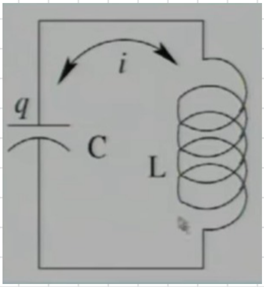
$\omega_0 = \sqrt{\frac{g}{l}}$

LC Oscillator

$L\dot{I} + \frac{Q}{C} = 0$

$\ddot{Q} + \frac{1}{LC} Q = 0$

$\omega_0 = \sqrt{\frac{1}{LC}}$



2) Damped Oscillator

$x(t) = A\cos(\omega_0 t + \phi)$

$\omega_0 = \sqrt{\frac{k}{m}}$

$\tilde{x}(t) = Ae^{i(\omega t + \phi)}$

$\longrightarrow$ x tilda t

$\tilde{x}(t) = \tilde{A}e^{i\omega t}$

$\tilde{A} = Ae^{i\phi}$

Q) consider a simple harmonic oscillator with

and $\omega_0 = 2s^{-1}$
$x_0 = 0.3m$, $V_0 = 0.7ms^{-1}$
$t = 0$

$\tilde{x}(t) = \tilde{A}e^{i\omega t}$

$\tilde{A} = a + ib$

$\tilde{A} = Ae^{i\phi}$

$\tilde{x}(t) = (a + ib)e^{i\omega t}$

$\boxed{x_0 = a = 0.3m}$

now let consider velocity

$\tilde{v}(t) = \dot{\tilde{x}}(t) = i\omega\tilde{x}$

$\Rightarrow i\omega(a + ib)e^{i\omega t}$

$\Rightarrow i\omega(a + ib)$

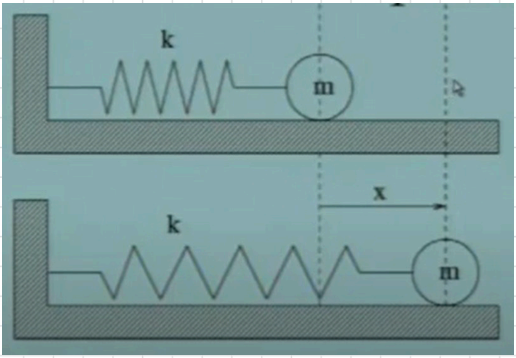
as $V_0 = 0.7m/s = \omega_0 b$

$\Rightarrow \boxed{b = -0.35m}$

$A = \sqrt{a^2 + b^2} = 0.46m$

$\phi = \tan^{-1}\left(\frac{b}{a}\right) = -49.4°$

Damped Oscillator



$F = -kx = -c\dot{x}$

$\Rightarrow m\ddot{x} = -kx - c\dot{x}$

$\Rightarrow \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

Equation for Damped Oscillator

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ ——— ①

Trial solution $x(t) = Ae^{\alpha t}$

$\Rightarrow x(t) = Ae^{\alpha t}$

$\Rightarrow \dot{x}(t) = A\alpha e^{\alpha t}$

$\Rightarrow \ddot{x}(t) = A\alpha^2 e^{\alpha t}$ ——— ②

putting ② in ①, we get

$\Rightarrow \alpha^2 + 2\beta\alpha + \omega_0^2 = 0$ ——— ③

Roots of ③ are as follows :-

$\alpha_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$

$\alpha_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$

Underdamped

$\beta < \omega_0$

$\omega = \sqrt{\omega_0^2 - \beta^2}$

$\alpha_1 = -\beta + i\omega$ and $\alpha_2 = -\beta - i\omega$

putting the values in the trial solution

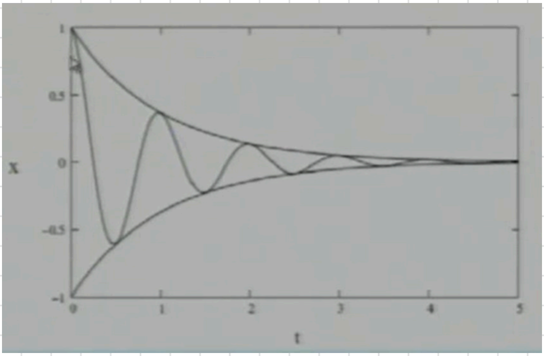
$x(t) = e^{-\beta t}\left[A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right]$

$\Rightarrow x(t) = Ae^{-\beta t}\cos(\omega t + \phi)$

$\Rightarrow \tilde{x}(t) = \tilde{A}e^{(i\omega - \beta)t}$

example :-

$e^{-t}\cos(2\pi t)$



Logarithmic Decrement

$\lambda = \ln\left[\frac{x(t)}{x(t+T)}\right] = \frac{2\pi\beta}{\omega}$

$\Rightarrow x(t) = e^{-\beta t}$

$\Rightarrow$ at $t = 0$, $x(0) = 1$

at $t = \frac{2\pi}{\omega}$, $x(t) = e^{-\beta\frac{2\pi}{\omega}}$

$\Rightarrow \ln e^{2\pi\beta/\omega} = \frac{2\pi\beta}{\omega}$

Q2) An underdamped oscillation

$x(t) = \tilde{A}e^{(i\omega - \beta)t}$

x_0 V_0 at $t = 0$

now find $\tilde{A}$ in terms of x_0 & V_0

$\tilde{x}(t)$ in terms of x_0 and V_0

ans) $\tilde{x}(t) = \tilde{A}e^{-\beta t}e^{i\omega t}$

$\Rightarrow x_0\ V_0 \quad t = 0$

$\Rightarrow \tilde{V}(t) = (i\omega - \beta)\tilde{x}(t)$

$\hookrightarrow \sqrt{\omega^2 + \beta^2} = \sqrt{\omega_0^2 - \beta^2 + \beta^2}$

$\Rightarrow \omega_0$

$\Rightarrow i = e^{i\pi/2}$

$\Rightarrow \tilde{V}(t) = \omega_0 e^{i\phi}\tilde{x}(t)$

$\Rightarrow \phi = \tan^{-1}\left(-\frac{\sqrt{\omega_0^2 - \beta^2}}{\beta}\right)$

$\Rightarrow \tilde{A} = a + ib$

$\Rightarrow a = x_0$

now determining b

$V_0 = Re\left[(i\omega - \beta)(a + ib)\right]$

$\Rightarrow -\beta x_0 - \omega b$

$b = -\frac{V_0 + \beta x_0}{\omega}$

$\Rightarrow \tilde{x}(t) = \left[x_0 - i\frac{V_0 + \beta x}{\omega}\right]e^{-\beta t}e^{i\omega t}$

$\Rightarrow x(t) = e^{-\beta t}\left[x_0\cos\omega t + \frac{V_0 + \beta x}{\omega} + \sin(\omega t)\right]$

Over damped ($\beta > \omega_0$)

$\alpha_1 = -\beta + \sqrt{\beta^2-\omega_0^2} = -\gamma_1$

$\alpha_2 = -\beta - \sqrt{\beta^2-\omega_0^2} = -\gamma_2$

$\gamma_1 , \gamma_2 > 0$ and $\gamma_2 > \gamma_1$

we have 2 solutions

$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$

$$x(t) = \frac{V_0 + \gamma_2 x_0}{\gamma_2-\gamma_1} e^{-\gamma_1 t} - \frac{V_0 + \gamma_1 x_0}{\gamma_2 - \gamma_1} e^{-\gamma_2 t}$$

High Damping ($\beta \gg \omega_0$)

$\sqrt{\beta^2-\omega_0^2}$

$$\Rightarrow \beta = \sqrt{1-\frac{\omega^2}{\beta^2}} \approx \beta\left[1-\frac{1}{2}\frac{\omega^2}{\beta^2}\right]$$

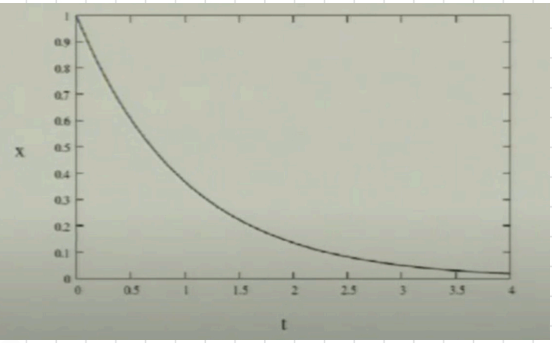
$\Rightarrow \gamma_1 = \frac{\omega_0^2}{2\beta}$ and $\gamma_2 = 2\beta$

the solutions when the damping is very large

$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t}$

$$x(t) = \frac{V_0 + \gamma_2 x_0}{\gamma_2-\gamma_1} e^{-\gamma_1 t} - \frac{V_0 + \gamma_1 x_0}{\gamma_2 - \gamma_1} e^{-\gamma_2 t}$$

Example:-



ω_0 $\quad T_0 \sim \frac{1}{\omega_0}$

β $\quad T_0 \sim \frac{1}{\beta}$

$T_0 > T_D$ $\quad$ $T_D > T_0$

Critical Damping [$\beta = \omega_0$]

general solution:-

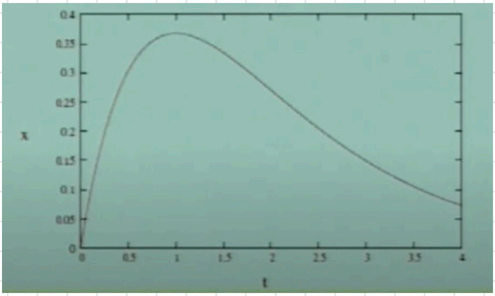
$x(t) = e^{-\beta t}[A_1 + A_2 t]$

at rest x_0

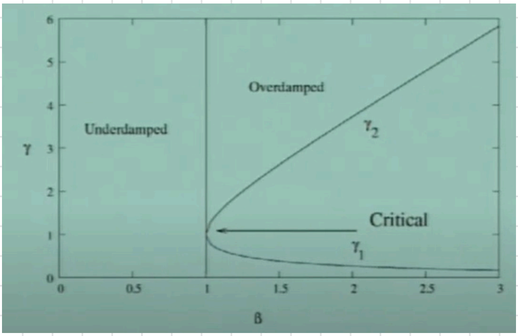
$x(t) = x_0 e^{-\beta t}[1 + \beta t]$

solutions:-

$x = 0$ with speed V_0 $\quad x(t) = V_0 e^{-\beta t} t$



Summary ($\omega_0 = 1$)



3) Oscillator with external forcing

External forcing

SHO with an additional external force

$F = \cos(\omega t + \psi)$

Why this particular type of force

Reason:- Fourier series for any arbitrary time varying force

$$F(t) = \sum_{\omega} F_{\omega} \cos(\omega t + \psi_{\omega})$$

Solution using Fourier series

Find solution for a single frequency

$$m\ddot{x}_{\omega} + kx_{\omega} = F_{\omega}\cos(\omega t + \psi_{\omega})$$

Superpose solutions

$$x(t) = \sum_{\omega} x_{\omega}(t)$$

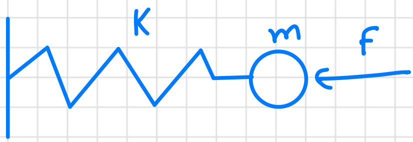
The equation

$$\ddot{\tilde{x}} + \omega_0^2 \tilde{x} = \tilde{f} e^{i\omega t} \quad \text{where,}$$

$$\tilde{f} = \frac{Fe^{i\psi}}{m}$$

solution = complimentary function + particular integral

$$m\ddot{x} - kx = F\cos(\omega t + \varphi)$$



$$\ddot{x} - \omega_0^2 x = \frac{F}{m}\cos(\omega t + \varphi)$$

$$\frac{F}{m} e^{i(\omega t + \psi)}$$

$$\ddot{\tilde{x}} + \omega_0^2 \tilde{x} = \tilde{f} e^{i\omega t} \quad \text{where,}$$

$$\tilde{f} = \frac{Fe^{i\psi}}{m}$$

$$\ddot{\tilde{x}} + \omega_0^2 \tilde{x} = 0$$

$$\Rightarrow A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t}$$

Particular Integral

$$\ddot{\tilde{x}} + \omega_0^2 \tilde{x} = \tilde{f} e^{i\omega t} \qquad \tilde{x}(t) = \tilde{B} e^{i\omega t}$$

$$\Rightarrow \ddot{\tilde{x}} + \omega_0^2 \tilde{x} = \tilde{f} e^{i\omega t}$$

$$\tilde{x} = Be^{i\omega t}$$

$$\Rightarrow -\omega^2 Be^{i\omega t} + \omega_0^2 \tilde{B} e^{i\omega t}$$

$$\Rightarrow \tilde{f} e^{i\omega t}$$

$$\Rightarrow \tilde{B} = \frac{\tilde{f}}{\omega_0^2 - \omega^2}$$

$$\Rightarrow [-\omega^2 + \omega_0^2]\tilde{B} = \tilde{f}$$

$$\Rightarrow \tilde{x}(t) = \frac{\tilde{f}}{\omega_0^2 - \omega^2} e^{i\omega t}$$

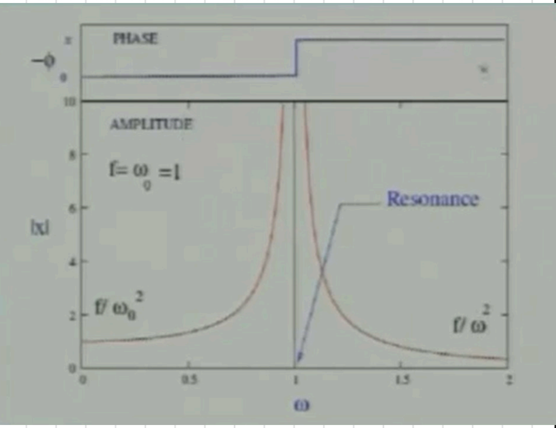
Amplitude And Phase

$$|\tilde{x}| = \frac{f}{|\omega_0^2 - \omega^2|}$$

$\phi = 0$ for $\omega < \omega_0$ and

$\phi = -\pi$ for $\omega > \omega_0$

$$\tilde{f}(-1) = \tilde{f} e^{i\pi} = \tilde{f} e^{-i\pi}$$



low Frequency Response $\omega \ll \omega_0$

$$\tilde{x}(t) = \frac{\tilde{f}}{\omega_0^2} e^{i\omega t} = \frac{F}{K} e^{i(\omega t + \phi)}$$

stiffness controlled Regime

$$m\ddot{x} + Kx = F$$

$$\boxed{x = \frac{F}{K}} \qquad \omega = 0$$

$$x(t) = \frac{F}{K} \cos(\omega t + \varphi)$$

High Frequency Response

$$\tilde{x}(t) = -\frac{\tilde{f}}{\omega^2} e^{i\omega t} = -\frac{F}{m\omega^2} e^{i(\omega t + \phi)}$$

mass controlled Regime solution of

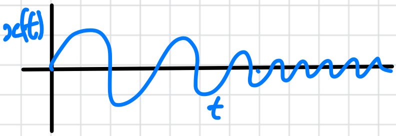
$$m\ddot{x} = F\cos(\omega t + \phi)$$

$$m\ddot{x} = F\cos(\omega t + \varphi)$$

$$\Rightarrow x(t) = A\cos(\omega t + \varphi)$$

$$\Rightarrow A = \frac{-F}{m}$$

$$\Rightarrow x(t) = \frac{-F}{m} \cos(\omega t + \varphi)$$



Solutions

Complimentary functions are transients

Steady state Behavior is decided by the particular integral

Solutions with Damping

$$\tilde{x}(t) = \frac{\tilde{f}}{(\omega_0^2 - \omega^2) + 2i\beta\omega} e^{i\omega t}$$

$$\tilde{x}(t) = Ce^{i\phi} \tilde{f} e^{i\omega t}$$

$$\ddot{\tilde{x}} + 2\beta\dot{\tilde{x}} + \omega_0^2 \tilde{x} = \tilde{f} e^{i\omega t}$$

$$\Rightarrow \tilde{x}(t) = \tilde{B} e^{i\omega t}$$

$$\Rightarrow \left[-\omega^2 + 2\beta i\omega + \omega_0^2\right] \tilde{B} e^{i\omega t} = \tilde{f} e^{i\omega t}$$

$$\Rightarrow \tilde{B} = \frac{\tilde{f}}{\left[(\omega_0^2 - \omega^2) + 2i\beta\omega\right]}$$

Amplitude and Phase

$$|\tilde{x}| = \frac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

$$\phi = \tan^{-1}\left(\frac{-2\beta\omega}{\omega_0^2-\omega^2}\right)$$

Summary

Some key points are as follows:-

High frequency and low frequency behaviour unchanged by damping

Amplitude is finite throughout

Maximum amplitude at

$$\omega = \sqrt{\omega_0^2 - 2\beta^2}$$

Average Energy

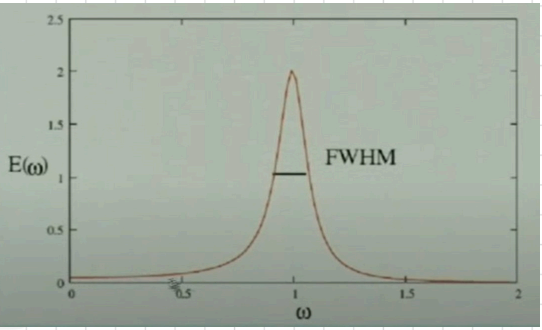
$$E = \frac{k\tilde{x}\tilde{x}^*}{2}$$

$$E(\omega) = \frac{k}{2}\,\frac{f^2}{(\omega_0^2-\omega^2)^2+4\beta^2+\omega^2}$$

solution with damping

$$\tilde{x}(t) = \frac{\tilde{f}}{(\omega_0^2-\omega^2)+2i\beta\omega}\,e^{i\omega t}$$

$$\tilde{x}(t) = Ce^{i\phi}\tilde{f}e^{i\omega t}$$



Mild Damping ($\beta \ll \omega_0$)

Maxima at $\omega = \omega_0$

$$(\omega_0^2-\omega^2)^2 = (\omega_0+\omega)^2(\omega_0-\omega)^2$$
$$\approx 4\omega_0^2(\omega_0-\omega)^2$$

$$E(\omega) \approx \frac{1}{8}\,\frac{f^2}{\omega_0^2[(\omega_0-\omega)^2+\beta^2]}$$

FWHM

$$E_{max} = \frac{f^2}{8\omega_0^2\beta^2}$$

$$\Rightarrow E(\omega_0+\Delta\omega) = \frac{E_{max}}{2}$$

$$\Rightarrow \text{FWHM} = 2\Delta\omega = 2\beta$$

Power

$$P(t) = F(t)\dot{x}(t)$$

$$P(t) = [F\cos(\omega t)] - |\tilde{x}|\omega\sin[\omega t+\phi]]$$

$$F(t) = F\cos(\omega t)$$

$$x(t) = [\tilde{x}|\cos(\omega t+\phi)$$

$$P(t) = F\cos(\omega t)[-\omega|\tilde{x}|\sin(\omega t+\phi)]$$

$$\langle P\rangle(\omega) = \frac{-F\omega|\tilde{x}|\sin\phi}{2}$$

$\sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi$

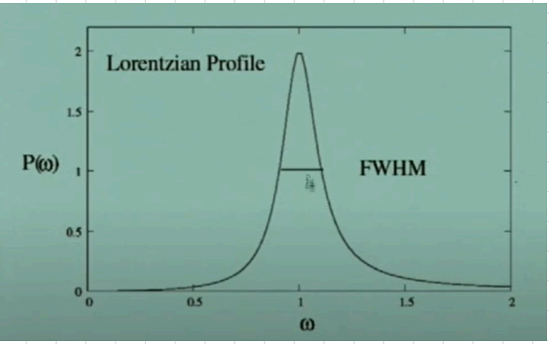
$$\tilde{x} = \frac{f}{(\omega_0^2-\omega^2)+2i\beta\omega} \times \frac{(\omega_0^2-\omega^2)-2i\beta\omega}{[\quad '' \quad]}$$

$$\Rightarrow \frac{f[(\omega_0^2-\omega^2)-2i\beta\omega]}{(\omega_0^2-\omega^2)+2\beta^2\omega^2}$$

$$\Rightarrow |\tilde{x}|\sin\phi = \frac{-2\beta\omega f}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

average power as

$$\langle P\rangle(\omega) = \frac{\beta\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\left(\frac{F^2}{m}\right)$$



$$\frac{d}{d\omega}\frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

$$\Rightarrow \frac{2\omega}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2} - \frac{\omega^2[2(\omega_0^2-\omega^2)(-2\omega)+4\beta^2\omega_2^2]}{[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2]^2}$$

$$\Rightarrow \frac{2\omega\{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2+2\omega^2(\omega_0^2-\omega^2) - 4\omega^2\beta\}}{[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2]^2}$$

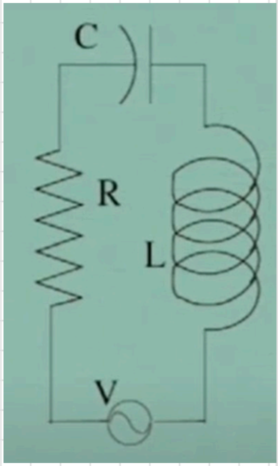
$$\Rightarrow (\omega_0^2-\omega^2)[\omega_0^2+\omega^2]=0$$

$$\Rightarrow \omega=\omega_0$$

$$\Rightarrow \langle P\rangle(\omega) = \frac{\beta\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\left(\frac{F^2}{m}\right)$$

3) Resonance

Electrical Circuits



from kirchoff's law

$$L\frac{dI}{dt} + \frac{q}{C} + RI = V\cos(\omega t+\phi)$$

$$\Rightarrow L\ddot{q} + R\dot{q} + \frac{q}{C} = V\cos(\omega t+\phi)$$

$$\omega_0^2 = \frac{1}{LC},\ \beta = \frac{R}{2L}\ \ \&\ \tilde{V} = \left(\frac{V}{L}\right)e^{i\phi}$$

$$\Rightarrow \ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \tilde{V}e^{i\omega t}$$

Impedance

$$\tilde{Z}(\omega) = i\omega L - \frac{i}{\omega C} + R$$

$$\Rightarrow \tilde{V} = \tilde{I}\tilde{Z}$$

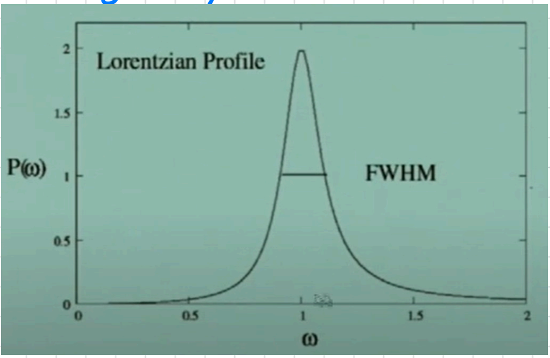
Average Power

$$\langle P(\omega)\rangle = \frac{R\tilde{I}\tilde{I}^*}{2}$$

$$\tilde{I} = \frac{\tilde{V}}{i(\omega L - 1/\omega C)+R}$$

$$\langle P(\omega)\rangle = \frac{\omega^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}\left(\frac{RV^2}{2L^2}\right)$$

Lorentizian Profile



Q3) $L = 10\,mH$ and $C = 1\,\mu F$ ω_0?
Choose R for critical Damping
What is the maximum power from a 10V source for $R = 2\,\Omega$. FWHM?, Quality factor

ans)

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{10^{-2} \times 10^{-6}}}$$

$$\Rightarrow 10^4$$

$$\Rightarrow 10\,kHz = \omega_0$$

now $\beta = \omega_0 = 10^4 = \frac{R}{2L}$ | $R = 2 \times 10^4 \times 10^{-2}$

$$\Rightarrow R = 200\,\Omega$$

$$\langle P(\omega) \rangle = \frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \left(\frac{RV^2}{2L^2} \right)$$

$$\Rightarrow 25\,W$$

FWHM $\Rightarrow$ 200 Hz

$$\Rightarrow Q = \frac{\omega_0}{2\beta} = 50$$

Time period T? $2\pi\,10^{-4}$ sec

Log Decrement ? λ

$$\tilde{x}(t) = [\tilde{A} e^{-\beta t}] e^{i\omega t}$$

$$\lambda' = \ln\left(\frac{x_n}{x_{n+1}}\right) = \beta T$$

$$\Rightarrow 2\pi\,10^{-2}$$

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