

# Linear Algebra : Determinants, Matrices

1. Introduction. 2. Determinants, Cofactors, Laplace's expansion. 3. Properties of determinants. 4. Matrices, Special matrices. 5. Matrices operations. 6. Related matrices. 7. Rank of a matrix, Elementary transformations, Elementary matrices, Inverse from elementary matrices, Normal form of a matrix. 8. Partition method of finding the inverse. 9. Solution of linear system of equations. 10. Consistency of linear system of equations. 11. Linear and orthogonal transformations. 12. Vectors ; Linear dependence. 13. Eigen values and eigen vectors. 14. Properties of eigen values. 15. Cayley-Hamilton theorem. 16. Reduction to diagonal form. 17. Reduction of quadratic form to canonical form. 18. Nature of quadratic form. 19. Complex matrices. 20. Objective Types of Questions.

## 2.1 INTRODUCTION

*Linear algebra* comprises of the theory and applications of linear system of equation, linear transformations and eigen value problems. In linear algebra, we make a systematic use of matrices and to a lesser extent determinants and their properties.

Determinants were first introduced for solving linear systems and have important engineering applications in systems of differential equations, electrical networks, eigen-value problems and so on. Many complicated expressions occurring in electrical and mechanical systems can be elegantly simplified by expressing them in the form of determinants.

Cayley\* discovered matrices in the year 1860. But it was not until the twentieth century was well-advanced that engineers heard of them. These days, however, matrices have been found to be of great utility in many branches of applied mathematics such as algebraic and differential equations, mechanics theory of electrical circuits, nuclear physics, aerodynamics and astronomy. With the advent of computers, the usage of matrix methods has been greatly facilitated.

## 2.2 DETERMINANTS

(1) **Definition.** The expression  $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$  is called a *determinant of the second order* and stands for ' $a_1b_2 - a_2b_1$ '. It contains 4 numbers  $a_1, b_1, a_2, b_2$  (called *elements*) which are arranged along two horizontal lines (called *rows*) and two vertical lines (called *columns*).

Similarly,  $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  is called a *determinant of the third order*. It consists of 9 elements which are arranged in 3 rows and 3 columns.

\*Arthur Cayley (1821–1895) was a professor at Cambridge and is known for his important contributions to algebra, matrices and differential equations.

In general, a determinant of the  $n$ th order is denoted by

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \dots l_1 \\ a_2 & b_2 & c_2 & d_2 \dots l_2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & d_n \dots l_n \end{vmatrix}$$

which is a block of  $n^2$  elements arranged in the form of a square along  $n$ -rows and  $n$ -columns. The diagonal through the left hand top corner which contains the elements  $a_1, b_2, c_3, \dots, l_n$  is called the leading or principal diagonal.

## (2) Cofactors

The cofactor of any element in a determinant is obtained by deleting the row and column which intersect in that element with the proper sign. The sign of an element in the  $i$ th row and  $j$ th column is  $(-1)^{i+j}$ . The cofactor of an element is usually denoted by the corresponding capital letter.

For instance, in  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ , the cofactor of  $b_3$  i.e.,  $B_3 = (-1)^{3+2} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$  and  $C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$ .

## (3) Laplace's expansion.\* A determinant can be expanded in terms of any row (or column) as follows:

Multiply each element of the row (or column) in terms of which we intend expanding the determinant, by its cofactor and then add up all these terms.

∴ Expanding by  $R_1$  (i.e., 1st row),

$$\begin{aligned} \Delta &= a_1 A_1 + b_1 B_1 + c_1 C_1 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2) \end{aligned}$$

Similarly, expanding by  $C_2$  (i.e., 2nd column)

$$\begin{aligned} \Delta &= b_1 B_1 + b_2 B_2 + b_3 B_3 = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ &= -b_1(a_2 c_3 - a_3 c_2) + b_2(a_1 c_3 - a_3 c_1) - b_3(a_1 c_2 - a_2 c_1) \end{aligned}$$

and expanding by  $R_3$  (i.e., 3rd row),  $\Delta = a_3 A_3 + b_3 B_3 + c_3 C_3$ .

Thus  $\Delta$  is the sum of the products of the elements of any row (or column) by the corresponding cofactors.

If, however, the sum of the products of the elements of any row (or column) by the cofactors of another row (or column) be taken, the result is zero.

e.g., in  $\Delta$ ,  $a_3 A_2 + b_3 B_2 + c_3 C_2 = -a_3 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_3 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

$$= -a_3(b_1 c_3 - b_3 c_1) + b_3(a_1 c_3 - a_3 c_1) - c_3(a_1 b_3 - a_3 b_1) = 0$$

In general,  $a_i A_j + b_i B_j + c_i C_j = \Delta$  when  $i = j = 0$  when  $i \neq j$

**Example 2.1.** Expand  $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

**Solution:** Expanding by  $R_1$ ,  $\Delta = a \begin{vmatrix} b & f \\ f & c \end{vmatrix} - h \begin{vmatrix} h & f \\ g & c \end{vmatrix} + g \begin{vmatrix} h & b \\ g & f \end{vmatrix}$

$$= a(bc - f^2) - h(hc - gf) + g(hf - gb) = abc + 2fg - af^2 - bg^2 - ch^2$$

\*Named after a great French mathematician Pierre Simon Marquis De Laplace (1749–1827). He made important contributions to probability theory, special functions, potential theory and astronomy. While a professor in Paris, he taught Napoleen Bonapart for a year.

**Example 2.2.** Find the value of  $\Delta = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{vmatrix}$ .

**Solution:** Since there are two zeros in the second row, therefore, expanding by  $R_2$ , we get

$$\begin{aligned}\Delta &= -\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} + 0 - 3 \begin{vmatrix} 0 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} + 0 \\ &\quad (\text{Expand by } C_1) \quad (\text{Expand by } R_1) \\ &= -[1(0 \times 2 - 1 \times 1) - 3(2 \times 2 - 1 \times 3) + 0] - 3[0 - (2 \times 2 - 3 \times 1) + 3(2 \times 0 - 3 \times 3)] \\ &= -(-1 - 3) - 3(-1 - 27) = 4 + 84 = 88.\end{aligned}$$

## 2.3 PROPERTIES OF DETERMINANTS

The following properties, are proved for determinants of the third order, but these hold good for determinants of any order. These properties enable us to simplify a given determinant and evaluate it without expanding the given determinant.

**I. A determinant remains unaltered by changing its rows into columns and columns into rows.**

Let  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  [Expand by  $R_1$ ]

$$\begin{aligned}&= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \\ \text{Then } \Delta' &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad [\text{Expand by } R_1] \\ &= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \\ &= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = \Delta.\end{aligned}$$

**Obs. 1.** Any theorem concerning the rows of a determinant, therefore, applies equally to its columns and vice-versa.  
**2.** When a row or a column is referred to in a general manner, it is called a *line*.

**II. If two parallel lines of a determinant are interchanged, the determinant retains its numerical value but changes in sign.**

Let  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  [Expand by  $R_1$ ]

$$\begin{aligned}&= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\end{aligned}$$

Interchanging  $C_2$  and  $C_3$ , we have

$$\begin{aligned}\Delta' &= \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} \quad [\text{Expand by } R_1] \\ &= a_1(c_2b_3 - c_3b_2) - c_1(a_2b_3 - a_3b_2) + b_1(a_2c_3 - a_3c_2) \\ &= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] = -\Delta.\end{aligned}$$

**Cor. If a line of  $\Delta$  be passed over two parallel lines, i.e., if the resulting determinant is like**

$$\Delta' = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}, \quad \text{then } \Delta' = (-1)^m \Delta.$$

In general, if any line of a determinant be passed over  $m$  parallel lines, the resulting determinant

$$\Delta' = (-1)^m \Delta.$$

**III. A determinant vanishes if two parallel lines are identical.**

Consider a determinant  $\Delta$  in which two parallel lines are identical.

Interchange of the identical lines leaves the determinant unaltered yet by the previous property, the interchanges of two parallel lines changes the sign of the determinant.

Hence

$$\Delta = \Delta' = -\Delta \quad \text{or} \quad 2\Delta = 0, \quad \text{or} \quad \Delta = 0.$$

**IV. If each element of a line be multiplied by the same factor, the whole determinant is multiplied by that factor.**

i.e., 
$$\begin{vmatrix} a_1 & pb_1 & v_1 \\ a_2 & pb_2 & v_2 \\ a_3 & pb_3 & v_3 \end{vmatrix} = p \begin{vmatrix} a_1 & b_1 & v_1 \\ a_2 & b_2 & v_2 \\ a_3 & b_3 & v_3 \end{vmatrix}$$

For on expanding by  $C_2$ ,

$$\begin{aligned} \text{L.H.S.} &= -pb_1(a_2v_3 - a_3v_2) + pb_2(a_1v_3 - a_3v_1) - pb_3(a_1v_2 - a_2v_1) \\ &= p(-b_1B_1 + b_2B_2 - b_3B_3) = \text{R.H.S.} \end{aligned}$$

Similarly, 
$$\begin{vmatrix} a_1 & b_1 & v_1 \\ ka_1 & kb_2 & kv_2 \\ a_3 & b_3 & v_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & v_1 \\ a_2 & b_2 & v_2 \\ a_3 & b_3 & v_3 \end{vmatrix}$$

**Cor. If two parallel lines be such that the elements of one are equi-multiples of the elements of the other, the determinant vanishes.**

i.e., 
$$\begin{vmatrix} a_1 & b_1 & pb_1 \\ a_2 & b_2 & pb_2 \\ a_3 & b_3 & pb_3 \end{vmatrix} = p \begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = p(0) = 0$$

**V. If each element of a line consists of  $m$  terms, the determinant can be expressed as the sum of  $m$  determinants.**

Consider the determinant 
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 + d_1 - e_1 \\ a_2 & b_2 & c_2 + d_2 - e_2 \\ a_3 & b_3 & c_3 + d_3 - e_3 \end{vmatrix}$$

end of whose third column elements consists of three terms.

Expanding  $\Delta$  by  $C_3$ , we have

$$\begin{aligned} \Delta &= (c_1 + d_1 - e_1)(a_2b_3 - a_3b_2) - (c_2 + d_2 - e_2)(a_1b_3 - a_3b_1) + (c_3 + d_3 - e_3)(a_1b_2 - a_2b_1) \\ &= [c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)] + [d_1(a_2b_3 - a_3b_2) - d_2(a_1b_3 - a_3b_1) \\ &\quad + d_3(a_1b_2 - a_2b_1)] - [e_1(a_2b_3 - a_3b_2) - e_2(a_1b_3 - a_3b_1) + e_3(a_1b_2 - a_2b_1)] \\ &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & e_3 \end{vmatrix} \end{aligned}$$

Further, if the elements of three parallel lines consist of  $m, n$  and  $p$  terms respectively, the determinants can be expressed as the sum of  $m \times n \times p$  determinants.

**Example 2.3.** If 
$$\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0$$
 in which  $a, b, c$  are different, show that  $abc = 1$ .

**Solution:** As each term of  $C_3$  in the given determinant consists of two terms, we express it as a sum of two determinants.

$$\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & -1 \\ b & b^2 & -1 \\ c & c^2 & -1 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

[Taking common  $a, b, c$  from  $R_1, R_2, R_3$  respectively of the first determinant and  $-1$  from  $C_3$  of the second determinant.]

$$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

(Passing  $C_3$  over  $C_2$  and  $C_1$  in the second determinant)

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} \text{ (take } \det = 1\text{)} = 0. \text{ Hence } abc = 1, \text{ since } \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} \neq 0 \text{ as } a, b, c \text{ are all different.}$$

VI. If to each elements of a line be added equi-multiples of the corresponding elements of one or more parallel lines, the determinants remains unaltered.

Let  $A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then  $\Delta' = \begin{vmatrix} a_1 + pb_1 & b_1 & c_1 \\ a_2 + pb_2 & b_2 & c_2 \\ a_3 + pb_3 & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta + 0 + 0 = \Delta.$$

[by IV-Cor.]

This property is very useful for simplifying determinants. To add equi-multiples of parallel lines, we shall employ the following notation :

Suppose to the elements of the second row, we add  $p$  times the elements of the first row and  $q$  times the element of the third row ; then we say :

Operate  $R_2 + pR_1 + qR_3$ .

Similarly Operate  $C_2 + mC_1 + nC_3 = nC_2$ ,

means that to the elements of the third column add  $m$  times the elements of the first column and  $-n$  times the elements of the second column.

**Example 2.1** Evaluate  $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 6 & 7 & 1 & 2 \end{vmatrix}$ .

**Solution:** Operating  $R_1 - R_2 - R_4$ ,  $R_2 - 3R_3$ ,  $R_3 - 2R_4$ , the given determinant becomes

$$\Delta = \begin{vmatrix} -8 & -12 & 0 & -2 \\ 0 & -2 & 0 & 1 \\ -4 & -6 & 0 & -1 \\ 6 & 7 & 1 & 2 \end{vmatrix} \quad [\text{Expand by } C_1]$$

$$= \begin{vmatrix} -8 & -12 & -2 \\ 0 & -2 & 1 \\ -4 & -6 & -1 \end{vmatrix} = 0 \quad [\because R_1 = 2R_2]$$

**Example 2.2** Solve the equations  $\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 3x+8 & 5x+17 \end{vmatrix} = 0$ .

**Solution:** Operating  $R_3 - (R_1 + R_2)$ , we get

$$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0 \quad (\text{Operate } R_3 - R_1 \text{ and } R_3 + R_2)$$

$$\begin{vmatrix} x+2 & 2x+4 & 3x+12 \\ x+1 & x+1 & x+1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0 \quad \text{or} \quad (x+1)(x+2) \begin{vmatrix} 1 & 2 & 6 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

To bring one more zero in  $C_3$ , operate  $R_1 - R_2$ .

$$\therefore (x+1)(x+2) \begin{vmatrix} 0 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & 1 & 3x+8 \end{vmatrix} = 0$$

Now expand by  $C_1$ .  $\therefore -(x+1)(x+2)(3x+8-5) = 0$  or  $-3(x+1)(x+2)(x+1) = 0$

Thus,  $x = -1, -1, -2$ .

**Example 2.6.** Prove that  $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$ .

**Solution:** Let  $\Delta$  be the given determinant. Taking  $a, b, c, d$  common from  $R_1, R_2, R_3, R_4$  respectively, we get

$$\begin{aligned} \Delta &= abcd \begin{vmatrix} a^{-1}+1 & a^{-1} & a^{-1} & a^{-1} \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix} \\ &\quad [\text{Operate } R_1 + (R_2 + R_3 + R_4) \text{ and take out the common factor from } R_1] \\ &= abcd (1 + a^{-1} + b^{-1} + c^{-1} + d^{-1}) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix} \\ &\quad [\text{Operate } C_2 - C_1, C_3 - C_1, C_4 - C_1] \\ &= abcd \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \begin{vmatrix} 1 & 0 & 0 & 0 \\ b^{-1} & 1 & 0 & 0 \\ c^{-1} & 0 & 1 & 0 \\ d^{-1} & 0 & 0 & 1 \end{vmatrix} = abcd \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right) \end{aligned}$$

**Obs.** If all elements on one side of the leading diagonal are zero, then the determinant is equal to the product of leading diagonal elements and such a determinants is called a *triangular determinant*.

**VII. Factor Theorem.** If the elements of a determinant  $\Delta$  are functions of  $x$  and two parallel lines become identical when  $x = a$ , then  $x - a$  is a factor of  $\Delta$ .

Let  $\Delta = f(x)$

Since  $\Delta = 0$  when  $x = a$ ,  $\therefore f(a) = 0$ .

i.e.,  $(x - a)$  is a factor of  $f(x)$ .

Hence  $x - a$  is a factor of  $\Delta$ .

**Obs.** If  $k$  parallel lines of a determinant  $\Delta$  become identical when  $x = a$ , then  $(x - a)^{k-1}$  is a factor of  $\Delta$ .

**Example 2.7. Factorize**  $\Delta = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$ .

**Solution:** Putting  $a = b$ ,  $R_1 = R_2$  and hence  $\Delta = 0$ .  $\therefore a - b$  is a factor of  $\Delta$ .

Similarly,  $a - c$  and  $a - d$  are also factors of  $\Delta$ .

Again putting  $b = c$ ,  $R_2 = R_3$  and hence  $\Delta = 0$ .

Similarly  $b - d$  and  $c - d$  are also factors of  $\Delta$ .

Also  $\Delta$  is of the sixth degree in  $a, b, c, d$  and therefore, there cannot be any other algebraic factor of  $\Delta$ .

$\therefore$  Suppose  $\Delta = k(a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$ , where  $k$  is a numerical constant.

The leading term in  $\Delta = a^3b^2c$ . The corresponding term on R.H.S. =  $ka^3b^2c$ .

$\therefore k = 1$ .

Hence,  $\Delta = (a - b)(a - c)(a - d)(b - c)(b - d)(c - d)$ .

**Example 2.8.** Prove that  $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$ .

**Solution:** Let the given determinant be  $\Delta$ . If we put  $a = 0$ ,

$$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ 0 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = 0$$

$\therefore a$  is a factor of  $\Delta$ . Similarly  $b$  and  $c$  are its factors.

Again if we put  $a + b + c = 0$ ,

$$\Delta = \begin{vmatrix} (-a)^2 & a^2 & a^2 \\ b^2 & (-b)^2 & b^2 \\ c^2 & c^2 & (-c)^2 \end{vmatrix} = 0$$

In this, three columns being identical,  $(a + b + c)^2$  is a factor of  $\Delta$ .

As  $\Delta$  is of the sixth degree and is symmetrical in  $a, b, c$  the remaining factor must therefore, be of the first degree and of the form  $k(a + b + c)$ .

Thus  $\Delta = kabc(a + b + c)^3$

To determine  $k$ , put  $a = b = c = 1$ , then

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 27k \quad \text{or} \quad 54 = 27k \quad \text{i.e., } k = 2$$

Hence  $\Delta = 2abc(a + b + c)^3$ .

**Otherwise :** Operating  $C_1 - C_3$  and  $C_2 - C_3$ , we have

$$\begin{aligned} \Delta &= \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix} \quad [\text{Take } (a+b+c) \text{ common from } C_1 \text{ and } C_2] \\ &= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \quad [\text{Operate } R_3 - R_1 - R_2] \\ &= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad \left[ \text{Operate } C_1 + \frac{1}{a} C_3, C_2 + \frac{1}{b} C_3 \right] \\ &= (a+b+c)^2 \begin{vmatrix} b+c & a^2/b & a^2 \\ b^2/a & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \quad [\text{Expand by } R_3] \\ &= 2ab(a+b+c)^2 [(b+c)(c+a) - ab] = 2abc(a+b+c)^3. \end{aligned}$$

**VIII. Multiplication of Determinants.** The product of two determinants of the same order is itself a determinant of that order.

Let  $\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  and  $\Delta_2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$

then their product is defined as

$$\Delta_1 \Delta_2 = \begin{vmatrix} a_1 l_1 + b_1 m_1 + c_1 n_1, & a_1 l_2 + b_1 m_2 + c_1 n_2, & a_1 l_3 + b_1 m_3 + c_1 n_3 \\ a_2 l_1 + b_2 m_1 + c_2 n_1, & a_2 l_2 + b_2 m_2 + c_2 n_2, & a_2 l_3 + b_2 m_3 + c_2 n_3 \\ a_3 l_1 + b_3 m_1 + c_3 n_1, & a_3 l_2 + b_3 m_2 + c_3 n_2, & a_3 l_3 + b_3 m_3 + c_3 n_3 \end{vmatrix}$$

Similarly, the product of two determinants of the  $n$ th order is a determinant of the  $n$ th order.

**Example 2.9.** Evaluate  $\begin{vmatrix} a^2 + \lambda^2 & ab + c\lambda & ca - b\lambda \\ ab - c\lambda & b^2 + \lambda^2 & bc + a\lambda \\ ca + b\lambda & bc - a\lambda & c^2 + \lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}$

**Solution:** By the rule of multiplication of determinants, the resulting determinant

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$$

where  $d_{11} = (a^2 + \lambda^2)\lambda + (ab + c\lambda)b + (ca - b\lambda)(-b) = \lambda(a^2 + b^2 + c^2 + \lambda^2)$

$d_{12} = (a^2 + \lambda^2)(-c) + (ab + c\lambda)a + (ca - b\lambda)c = 0$

$d_{13} = 0,$

$d_{21} = 0, d_{22} = \lambda(a^2 + b^2 + c^2 + \lambda^2), d_{23} = 0,$

$d_{31} = 0, d_{32} = 0, d_{33} = \lambda(a^2 + b^2 + c^2 + \lambda^2),$

Hence  $\Delta = \begin{vmatrix} \lambda(a^2 + b^2 + c^2 + \lambda^2) & 0 & 0 \\ 0 & \lambda(a^2 + b^2 + c^2 + \lambda^2) & 0 \\ 0 & 0 & \lambda(a^2 + b^2 + c^2 + \lambda^2) \end{vmatrix} = \lambda^3(a^2 + b^2 + c^2 + \lambda^2)^3,$

**Example 2.10.** Show that  $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$  where  $A, B$  etc. are the co-factors of  $a, b$  etc.

respectively in the determinant  $(a_1 b_2 c_3).$

**Solution:** Let  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  and  $\Delta' = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

Then  $\Delta \Delta' = \begin{vmatrix} a_1 A_1 + b_1 B_1 + c_1 C_1, & a_1 A_2 + b_1 B_2 + c_1 C_2, & a_1 A_3 + b_1 B_3 + c_1 C_3 \\ a_2 A_1 + b_2 B_1 + c_2 C_1, & a_2 A_2 + b_2 B_2 + c_2 C_2, & a_2 A_3 + b_2 B_3 + c_2 C_3 \\ a_3 A_1 + b_3 B_1 + c_3 C_1, & a_3 A_2 + b_3 B_2 + c_3 C_2, & a_3 A_3 + b_3 B_3 + c_3 C_3 \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$

Hence  $\Delta' = \Delta^2.$

**Obs.**  $\Delta'$  is called the reciprocal or adjugate determinant of  $\Delta.$

**Example 2.11.** Express  $\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$

as the square of a determinant, and hence find its value.

**Solution:** Given determinant

$$= \begin{vmatrix} a, (-a) + b, c + b, b, a, (-b) + b, a + c, c, a, (-c) + b, b + c, a \\ b, (-a) + c, c + a, b, b, (-b) + c, a + a, c, b, (-c) + c, b + a, a \\ c, (-a) + a, c + b, b, c, (-b) + a, a + b, c, c, (-c) + a, b + b, a \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

[Taking out  $(-1)$  common from  $C_1$  and interchange  $C_2, C_3]$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1)^2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \Delta^2$$

where  $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

Hence the given determinant  $= \Delta^2 = (a^3 + b^3 + c^3 - 3abc)^2.$

## PROBLEMS 2.1

1. Prove, without expanding, that  $\begin{vmatrix} 1 & \alpha & \alpha^2 & -\beta\gamma \\ 1 & \beta & \beta^2 & -\gamma\alpha \\ 1 & \gamma & \gamma^2 & -\alpha\beta \end{vmatrix}$  vanishes. (W.B.T.U., 2016)

2. If  $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$ , then prove, without expansion, that  $xyz = -1$  where  $x, y, z$  are unequal.

3. Show that (i)  $\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)$ .

(ii)  $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ a^2 & b^2 & c^2 \end{vmatrix} = -(a-b)(b-c)(c-a)(a+b+c)$ .

4. If  $a, b, c$  are all different and  $\begin{vmatrix} a & a^3 & a^4 - 1 \\ b & b^3 & b^4 - 1 \\ c & c^3 & c^4 - 1 \end{vmatrix} = 0$ , then show that  $abc(bc+ca+ab) = a+b+c$ .

5. Evaluate (i)  $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \\ 1 & 2 & 4 & 4 \\ 1 & 2 & 3 & 5 \end{vmatrix}$  (ii)  $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

Prove the following results : (6 to 13)

6.  $\begin{vmatrix} a+b & b+c & c+a \\ l+m & m+n & n+l \\ p+q & q+r & r+p \end{vmatrix} \div \begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix} = 2$  7.  $\begin{vmatrix} a-b-c & 2b & 2c \\ 2a & b-c-a & 2c \\ 2a & 2b & c-a-b \end{vmatrix} = (a+b+c)^3$

8.  $\begin{vmatrix} 1+a^2-b^2 & 2b & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a^2 & 1-a^2-b^2 \end{vmatrix}$  is a perfect cube.

9.  $\begin{vmatrix} 1 & \cos A & \sin A \\ 1 & \cos B & \sin B \\ 1 & \cos C & \sin C \end{vmatrix} = 4 \sin \frac{B-C}{2} \sin \frac{C-A}{2} \sin \frac{A-B}{2}$ .

10.  $\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$  is a perfect square. 11.  $\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af-be+cd)^2$  (W.B.T.U., 2016)

12.  $\begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + dab \\ 1 & d & d^2 & d^3 + abc \end{vmatrix}$  vanishes.

13.  $\begin{vmatrix} a^2 + \lambda & ab & ac & ad \\ ab & b^2 + \lambda & bc & bd \\ ac & bc & c^2 + \lambda & cd \\ ad & bd & cd & d^2 + \lambda \end{vmatrix} = \lambda^3(a^2 + b^2 + c^2 + d^2 + \lambda)$

Factorize each of the following determinants : (14 to 17)

14.  $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

15.  $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ .

$$16. \begin{vmatrix} b^2c^3 + c^2 & bc + a & 1 \\ c^2a^2 + b^2 & ca + b & 1 \\ a^2b^2 + c^2 & ab + c & 1 \end{vmatrix}$$

$$17. \begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ a & b & c & d \\ 1 & 1 & 1 & 1 \\ bcd & cda & dab & abc \end{vmatrix}$$

$$18. \text{ If } a + b + c = 0, \text{ solve } \begin{vmatrix} a - x & c & b \\ c & b - x & a \\ b & a & c - x \end{vmatrix} = 0$$

$$19. \text{ Solve the equation } \begin{vmatrix} x + 1 & 2x + 1 & 3x + 1 \\ 2x & 4x + 3 & 6x + 3 \\ 4x - 1 & 6x + 4 & 8x + 4 \end{vmatrix} = 0.$$

$$20. \text{ Show that } \begin{vmatrix} b^2 + c^2 & a^2 & a^2 \\ b^2 & c^2 + a^2 & b^2 \\ c^2 & c^2 & a^2 + b^2 \end{vmatrix} = 4a^2b^2c^2. \quad (\text{W.B.T.U., 2012})$$

## 2.4 MATRICES

**(1) Definition.** A system of  $mn$  numbers arranged in a rectangular formation along  $m$  rows and  $n$  columns and bounded by the brackets [ ] is called an  $m$  by  $n$  matrix ; which is written as  $m \times n$  matrix. A matrix is also denoted by a single capital letter.

Thus

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1j} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2j} & \dots a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots a_{ij} & \dots a_{in} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{mj} & \dots a_{mn} \end{bmatrix}$$

is a matrix of order  $mn$ . It has  $m$  rows and  $n$  columns. Each of the  $mn$  numbers is called an element of the matrix.

To locate any particular element of a matrix, the elements are denoted by a letter followed by two suffixes which respectively specify the rows and columns. Thus  $a_{ij}$  is the element in the  $i$ -th row and  $j$ -th column of  $A$ . In this notation, the matrix  $A$  is denoted by  $[a_{ij}]$ .

A matrix should be treated as a single entity with a number of components, rather than a collection of numbers. For example, the coordinates of a point in solid geometry, are given by a set of three numbers which can be represented by the matrix  $[x, y, z]$ . Unlike a determinant, a matrix cannot reduce to a single number and the question of finding the value of a matrix never arises. The difference between a determinant and a matrix is brought out by the fact that an interchange of rows and columns does not alter the determinant but gives an entirely different matrix.

### (2) Special matrices

**Row and column matrices.** A matrix having a single row is called a row matrix, e.g.,  $[1 \ 3 \ 5 \ 7]$ .

A matrix having a single column is called a column matrix, e.g.,  $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Row and column matrices are sometimes called row vectors and column vectors.

**Square matrix.** A matrix having  $n$  rows and  $n$  columns is called a square matrix of order  $n$ .

The determinant having the same elements as the square matrix  $A$  is called the determinant of the matrix and is denoted by the symbol  $|A|$ . For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

The diagonal of this matrix containing the elements 1, 3, 5 is called the leading or principal diagonal. The sum of the diagonal elements of a square matrix  $A$  is called the trace of  $A$ .

A square matrix is said to be singular if its determinant is zero otherwise non-singular.

**Diagonal matrix.** A square matrix all of whose elements except those in the leading diagonal, are zero is called a diagonal matrix.

A diagonal matrix whose all the leading diagonal elements are equal is called a scalar matrix. For example,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

are the diagonal and scalar matrices respectively.

**Unit matrix.** A diagonal matrix of order  $n$  which has unity for all its diagonal elements, is called a unit matrix or an identity matrix of order  $n$  and is denoted by  $I_n$ . For example, unit matrix of order 3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Null matrix.** If all the elements of a matrix are zero, it is called a null or zero matrix and is denoted by '0'; e.g.,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ is a null matrix.}$$

**Symmetric and skew-symmetric matrices.** A square matrix  $A = [a_{ij}]$  is said to be symmetric when  $a_{ij} = a_{ji}$  for all  $i$  and  $j$ .

If  $a_{ij} = -a_{ji}$  for all  $i$  and  $j$  so that all the leading diagonal elements are zero, then the matrix is called a skew-symmetric matrix. Examples of symmetric and skew-symmetric matrices are

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ and } \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix} \text{ respectively.}$$

**Triangular matrix.** A square matrix all of whose elements below the leading diagonal are zero, is called an upper triangular matrix. A square matrix all of whose elements above the leading diagonal are zero, is called a lower triangular matrix. Thus

$$\begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -5 & 4 \end{bmatrix}$$

are upper and lower triangular matrices respectively.

**Transposed matrix:** This matrix obtained from any given matrix  $A$  by changing rows into columns and columns into rows is called transposed matrix or transpose of  $A$  and it is denoted by  $A'$  or  $A^t$ .

Thus if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}; \text{ then } A^t = \begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

It follows from the definition that  $(A^t)^t = A$

**Adjoint matrix:** Let  $A$  be the square matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Now consider the determinant  $|A|$ . Let  $A_{11}, A_{12}, A_{13}, A_{21}, \dots$  be the cofactors of the elements  $a_{11}, a_{12}, a_{13}, a_{21}, \dots$

Then the matrix formed by these cofactors is given by

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

The transpose of this co-factor matrix is called adjoint of the matrix  $A$ . It is denoted by  $\text{adj. } A$ .

$$\text{Thus, } \text{adj. } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

**Equal matrixes:** Two matrices  $A$  and  $B$  are said to be equal when they are of the same order and the elements in the corresponding positions are equal. In other words the two matrices are identical. We write then  $A = B$ .

$$\text{Thus if } \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 4 & 0 & 7 \\ 3 & 1 & 6 \end{bmatrix}$$

then  $a = 4, b = 0, c = 7, d = 3, e = 1$  and  $f = 6$ .

Also note that the matrices are of the same order i.e.,  $2 \times 3$ .

## 2.5 MATRICES OPERATIONS

### (1) Equality of matrices

Two matrices  $A$  and  $B$  are said to equal if and only if

(i) they are of the same order

and (ii) each element of  $A$  is equal to the corresponding element of  $B$ .

**(2) Addition and subtraction of matrices.** If  $A, B$  be two matrices of the same order, then their sum  $A + B$  is defined as the matrix each element of which is the sum of the corresponding elements of  $A$  and  $B$ .

$$\text{Thus, } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \\ c_3 & d_3 \end{bmatrix} = \begin{bmatrix} a_1 + c_1 & b_1 + d_1 \\ a_2 + c_2 & b_2 + d_2 \\ a_3 + c_3 & b_3 + d_3 \end{bmatrix}$$

Similarly,  $A - B$  is defined as a matrix whose elements are obtained by subtracting the elements of  $B$  from the corresponding elements of  $A$ .

$$\text{Thus, } \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} - \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 - c_1 & b_1 - d_1 \\ a_2 - c_2 & b_2 - d_2 \end{bmatrix}$$

**Obs.** 1. Only matrices of the same order can be added or subtracted.

2. Addition of matrices is commutative,

i.e.,  $A + B = B + A$ .

3. Addition and subtraction of matrices is associative.

i.e.  $(A + B) - C = A + (B - C) = B + (A - C)$ .

**(3) Multiplication of matrix by a scalar.** The product of a matrix  $A$  by a scalar  $k$  is a matrix whose each element is  $k$  times the corresponding elements of  $A$ .

$$\text{Thus, } k \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}$$

The distributive law holds for such products, i.e.,  $k(A + B) = kA + kB$ .

**Obs.** All the laws of ordinary algebra hold for the addition or subtraction of matrices and their multiplication by scalars.

**Example 2.12.** Find  $x, y, z$  and  $w$  given that

$$3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 5 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 6 & x+y \\ z+w & 5 \end{bmatrix}$$

**Solution:** We have  $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+6 & 5+x+y \\ -1+z+w & 2w+5 \end{bmatrix}$

Equating the corresponding elements, we get

$$3x = x + 6, 3y = 5 + x + y, 3z = -1 + z + w, 3w = 2w + 5.$$

$$\text{or } 2x = 6, 2y = 5 + x, 2z = w - 1, w = 5$$

$$\text{Hence } x = 3, y = 4, z = 2, w = 5.$$

**Example 2.10.** Express  $\begin{vmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{vmatrix}$  as the sum of a lower triangular matrix with zero leading diagonal and an upper triangular matrix.

**Solution:** Let  $L = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix}$  be the lower triangular matrix with zero leading diagonal.

and

$U = \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$  be the upper triangular matrix.

Then  $\begin{bmatrix} 3 & 5 & -7 \\ -8 & 11 & 4 \\ 13 & -14 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix} + \begin{bmatrix} l & m & n \\ 0 & p & q \\ 0 & 0 & r \end{bmatrix}$

Equating corresponding elements from both sides, we obtain  $3 = l$ ,  $5 = m$ ,  $-7 = n$ ,  $-8 = a$ ,  $11 = p$ ,  $4 = q$ ,  $13 = b$ ,  $-14 = c$ ,  $6 = r$ .

Hence  $L = \begin{bmatrix} 0 & 0 & 0 \\ -8 & 0 & 0 \\ 13 & -14 & 0 \end{bmatrix}$  and  $U = \begin{bmatrix} 3 & 5 & -7 \\ 0 & 11 & 4 \\ 0 & 0 & 6 \end{bmatrix}$

**(4) Multiplication of matrices.** Two matrices can be multiplied only when the number of columns in the first is equal to the number of rows in the second. Such matrices are said to be **conformable**.

For instance, the product  $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \times \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix}$

is defined as the matrix  $\begin{bmatrix} a_1l_1 + b_1m_1 + c_1n_1 & a_1l_2 + b_1m_2 + c_1n_2 \\ a_2l_1 + b_2m_1 + c_2n_1 & a_2l_2 + b_2m_2 + c_2n_2 \\ a_3l_1 + b_3m_1 + c_3n_1 & a_3l_2 + b_3m_2 + c_3n_2 \\ a_4l_1 + b_4m_1 + c_4n_1 & a_4l_2 + b_4m_2 + c_4n_2 \end{bmatrix}$

In general, if  $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$  and  $B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix}$

be two  $m \times n$  and  $n \times p$  conformable matrices, then their product is defined as the  $m \times p$  matrix

$$AB = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1p} \\ c_{21} & c_{22} & \dots & c_{2p} \\ \dots & \dots & \dots & \dots \\ c_{m1} & c_{m2} & \dots & c_{mp} \end{bmatrix}$$

where  $c_{ij} = a_{1j}b_{1j} + a_{2j}b_{2j} + a_{3j}b_{3j} + \dots + a_{nj}b_{nj}$ , i.e., the element in the  $i$ th row and the  $j$ th column of the matrix  $AB$  is obtained by weaving the  $i$ th row of  $A$  with  $j$ th column of  $B$ . The expression for  $c_{ij}$  is known as the **inner product** of the  $i$ th row with the  $j$ th column.

**Post-multiplication and Pre-multiplication.** In the product  $AB$ , the matrix  $A$  is said to be **post-multiplied** by the matrix  $B$ . Whereas in  $BA$ , the matrix  $A$  is said to be **pre-multiplied** by  $B$ . In one case the product may exist and in the other case it may not. Also the product in both cases may exist yet may or may not be equal.

**Obs. 1.** Multiplication of matrices is associative, i.e.,  $(AB)C = A(BC)$  provided  $A, B$  are conformable for the product  $AB$  and  $B, C$  are conformable for the product  $BC$ . (Ex. 2.15).

**Obs. 2.** Multiplication of matrices is distributive, i.e.,  $A(B + C) = AB + AC$ ,

provided  $A, B$  are conformable for the product  $AB$  and  $A, C$  are conformable for the product  $AC$ .

**Obs. 3.** Power of a matrix. If  $A$  be a square matrix, then the product  $AA$  is defined as  $A^2$ . Similarly, we define higher powers of  $A$ , i.e.,  $A \cdot A^2 = A^3$ ,  $A^2 \cdot A^3 = A^4$  etc.

If  $A^2 = A$ , then the matrix  $A$  is called **Idempotent**.

**Example 2.14.** If  $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$ , form the product of  $AB$ . Is  $BA$  defined?

**Solution:** Since the number of columns of  $A$  = the number of rows of  $B$  (each being = 3).

∴ The product  $AB$  is defined and

$$= \begin{bmatrix} 0 \cdot 1 + 1 \cdot -1 + 2 \cdot 2, & 0 \cdot -2 + 1 \cdot 0 + 2 \cdot -1 \\ 1 \cdot 1 + 2 \cdot -1 + 3 \cdot 2, & 1 \cdot -2 + 2 \cdot 0 + 3 \cdot -1 \\ 2 \cdot 1 + 3 \cdot -1 + 4 \cdot 2, & 2 \cdot -2 + 3 \cdot 0 + 4 \cdot -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 3 & -3 \\ 7 & -8 \end{bmatrix}$$

Again since the number of columns of  $B$  ≠ the number of rows of  $A$ .

∴ The product  $BA$  is not possible.

**Example 2.15.** If  $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$ , find the matrix  $B$  such that  $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$ .

(Mumbai, 2008)

**Solution:** Let  $AB = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} l & m & n \\ p & q & r \\ u & v & w \end{bmatrix}$

$$= \begin{bmatrix} 3l + 2p + 2u & 3m + 2q + 2v & 3n + 2r + 2w \\ l + 3p + u & m + 3q + v & n + 3r + w \\ 5l + 3p + 4u & 5m + 3q + 4v & 5n + 3r + 4w \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix} \quad (\text{given})$$

Equating corresponding elements, we get

$$3l + 2p + 2u = 3, \quad l + 3p + u = 1, \quad 5l + 3p + 4u = 5 \quad \dots(i)$$

$$3m + 2q + 2v = 4, \quad m + 3q + v = 6, \quad 5m + 3q + 4v = 6 \quad \dots(ii)$$

$$3n + 2r + 2w = 2, \quad n + 3r + w = 1, \quad 5n + 3r + 4w = 4 \quad \dots(iii)$$

Solving Eqn. (i), we get  $l = 1, p = 0, u = 0$

Similarly Eqn. (ii) give  $m = 0, q = 2, v = 0$

and Eqn. (iii) give  $n = 0, r = 0, w = 1$

$$\text{Thus, } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**Example 2.16.** Prove that  $A^3 - 4A^2 - 3A + 11I = 0$ , where  $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ .

$$\text{Solution: } A^2 = A \times A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1+6+2 & 3+0+4 & 2-3+6 \\ 2+0-1 & 6+0-2 & 4+0-3 \\ 1+4+3 & 3+0+6 & 2-2+9 \end{bmatrix} = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 9+14+5 & 27+0+10 & 18-7+15 \\ 1+8+1 & 3+0+2 & 2-4+3 \\ 8+18+9 & 24+0+18 & 16-9+27 \end{bmatrix} = \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix}$$

$$A^3 - 4A^2 - 3A + 11I$$

$$= \begin{bmatrix} 28 & 37 & 26 \\ 10 & 5 & 1 \\ 35 & 42 & 34 \end{bmatrix} - 4 \begin{bmatrix} 9 & 7 & 5 \\ 1 & 4 & 1 \\ 8 & 9 & 9 \end{bmatrix} - 3 \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 28-36-3+11 & 37-28-0+0 & 26-20-6+0 \\ 10-4-6+0 & 5-16+0+11 & 1-4+3+0 \\ 35-32-3+0 & 42-36-6+0 & 34-36-9+11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

**Example 2.17.** By mathematical induction, prove that if

$$A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}$$

**Solution.** When  $n = 1$ ,  $A^n$  gives  $A^1 = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$  ... (i)

Let us assume that the result is true for any positive integer  $k$ , so that

$$A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$$

$$\therefore A^{k+1} = A^k \cdot A^1 = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix} \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \\ = \begin{bmatrix} 11(1+10k) - 100k & -25(1+10k) + 225k \\ 44k + 4(1-10k) & -100k - 9(1-10k) \end{bmatrix} = \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix} \quad \dots (ii)$$

This is true for  $n = k + 1$

We have seen in Eqn. (i) that the result is true for  $n = 1$ .

$\therefore$  It is true for  $n = 1 + 1 = 2$

Similarly, it is true for  $n = 2 + 1 = 3$  and so on.

Hence by mathematical induction, the result is true for all positive integers  $n$ .

**Example 2.18.** Prove that  $(AB)C = A(BC)$ , where  $A, B, C$  are matrices conformable for the products.

**Solution.** Let  $A = [a_{ij}]$  be of order  $m \times n$ ,  $B = [b_{ij}]$  be of order  $n \times p$  and  $C = [c_{ij}]$  be of order of  $p \times q$ .

$$\text{Then } AB = [a_{ik}] \cdot [b_{kj}] = \sum_{k=1}^n a_{ik} b_{kj}$$

$$\therefore (AB)C = \left[ \sum_{k=1}^n a_{ik} b_{kj} \right] \cdot [c_{lj}] = \left[ \sum_{l=1}^p \left( \sum_{k=1}^n a_{ik} b_{kj} \right) c_{lj} \right] = \left[ \sum_{k=1}^n \sum_{l=1}^p a_{ik} b_{kl} c_{lj} \right]$$

$$\text{Similarly, } BC = [b_{kl}] \cdot [c_{lj}] = \sum_{l=1}^p b_{kl} c_{lj}$$

$$\therefore A(BC) = [a_{ik}] \left[ \sum_{l=1}^p b_{kl} c_{lj} \right] = \left[ \sum_{k=1}^n a_{ik} \left( \sum_{l=1}^p b_{kl} c_{lj} \right) \right] = \left[ \sum_{k=1}^n \left( \sum_{l=1}^p a_{ik} b_{kl} c_{lj} \right) \right]$$

Hence  $(AB)C = A(BC)$ .

### PROBLEMS 2.2

- For what values of  $x$ , the matrix  $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$  is singular?
- Find the values of  $x, y, z$  and  $a$  which satisfy the matrix equation  $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$ .
- Matrix  $A$  has  $x$  rows and  $x+5$  columns. Matrix  $B$  has  $y$  rows and  $11-y$  columns. Both  $AB$  and  $BA$  exist. Find  $x$  and  $y$ .
- If  $A+B = \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}$  and  $A-B = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}$ , calculate the product  $AB$ .
- If  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ , find  $AB$  or  $BA$ , whichever exists.
- If  $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$  and  $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ , verify that  $(AB)C = A(BC)$  and  $A(B+C) = AB + AC$ .
- Evaluate (i)  $[x, y, z] \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ; (ii)  $\begin{bmatrix} 2 & 1 & -1 \\ 4 & -5 & 6 \\ -3 & 7 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ -6 & 4 \\ -2 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$ ; (iii)  $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 & 5 & 2 \\ -3 & 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$

8. Prove that the product of two matrices

$$\begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$$

is a null matrix when  $\theta$  and  $\phi$  differ by an odd multiple of  $\pi/2$ .

9. If  $A = \begin{bmatrix} 0 & -\tan \alpha/2 \\ \tan \alpha/2 & 0 \end{bmatrix}$ , show that  $I + A = (I + A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ .

10. If  $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ , find the value of  $A^2 - 6A + 8I$ , where  $I$  is a unit matrix of second order.

11. If  $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ , and  $I$  is the unit matrix of order 3, evaluate  $A^2 - 3A + 9I$ .

12. If  $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 4 & -3 & 2 \end{bmatrix}$ , verify the result  $(A + B)^2 = A^2 + BA + AB + B^2$ .

13. If  $E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and  $F = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ,

calculate the products  $EF$  and  $FE$  and show that  $E^2F + FE^2 = E$ .

14. If  $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ , show that  $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ , when  $n$  is a positive integer.

15. Factorize the matrix  $A = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$  into the form  $LU$ , where  $L$  is lower triangular and  $U$  is upper triangular matrix.

## 2.6 RELATED MATRICES

(1) **Transpose of a matrix.** The matrix obtained from any given matrix  $A$ , by interchanging rows and columns is called the transpose of  $A$  and is denoted by  $A'$ .

Thus the transposed matrix of  $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$  is  $A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$

Clearly, the transpose of an  $m \times n$  matrix is an  $n \times m$  matrix. Also the transpose of the transpose of a matrix coincides with itself, i.e.,  $(A')' = A$ .

For a symmetric matrix,  $A' = A$  and for a skew-symmetric matrix,  $A' = -A$ .

**Obs. 1.** The transpose of the product of the two matrices is the product of their transposes taken in the reverse order i.e.,  $(AB)' = B'A'$ .

For, the element in the  $i$ th row and  $j$ th col. of  $(AB)'$

= element in the  $j$ th row and  $i$ th col. of  $AB$  = inner product of  $j$ th row of  $A$  with  $i$ th col. of  $B$

= inner product of  $j$ th col. of  $A'$  with  $i$ th row of  $B'$  = element in the  $i$ th row and  $j$ th col. of  $B'A'$

Hence  $(AB)' = B'A'$ .

**Obs. 2.** Every square matrix can be uniquely expressed as a sum of a symmetric and a skew-symmetric matrix.

Let  $A$  be the given square matrix, then  $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ .

Let  $B = \frac{1}{2}(A + A')$  and  $C = \frac{1}{2}(A - A')$

$\therefore B = \left[ \frac{1}{2}(A + A') \right] = \frac{1}{2}[A' + (A')'] = \frac{1}{2}(A' + A) = B$ , i.e.,  $B = \frac{1}{2}(A + A')$  is a symmetric matrix.

Again,  $C' = \left[ \frac{1}{2}(A - A') \right]' = \frac{1}{2}[A' - (A')'] = \frac{1}{2}(A' - A) = -C$ , i.e.,  $C = \frac{1}{2}(A - A')$  is a skew-symmetric matrix.

Hence  $A$  can be expressed as the sum of a symmetric and a skew-symmetric matrix.

To prove the uniqueness, assume that  $P$  is a symmetric matrix and  $Q$  is a skew-symmetric matrix such that  $A = P + Q$ . Then  $A' = (P + Q)' = P' + Q' = P - Q$ .

Thus,  $P = \frac{1}{2}(A + A')$  and  $Q = \frac{1}{2}(A - A')$

which shows that there is one and only one way of expressing  $A$  as the sum of a symmetric and skew-symmetric matrix.

**Example 2.19.** Express the matrix  $A$  as the sum of a symmetric and a skew-symmetric matrix where

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$$

**Solution:** We have  $A' = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

Then  $A + A' = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$  and  $A - A' = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

$$\therefore A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \begin{bmatrix} 4 & 1.5 & -4 \\ 1.5 & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 & 1 \\ -0.5 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}.$$

**(2) Adjoint of a square matrix.** The determinant of the square matrix

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ is } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The matrix formed by the cofactors of the elements in  $\Delta$  is

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}. \text{ Then the transpose of this matrix, i.e., } \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

is called the *adjoint of the matrix  $A$*  and is written as *Adj.  $A$* .

Thus the **adjoint of  $A$**  is the transposed matrix of cofactors of  $A$ .

**(3) Inverse of a matrix.** If  $A$  be any matrix, then a matrix  $B$  if it exists, such that  $AB = BA = I$ , is called the **Inverse of  $A$**  which is denoted by  $A^{-1}$  so that  $AA^{-1} = I$ .

Also  $A^{-1} = \frac{\text{Adj. } A}{|\mathbf{A}|}$

For  $A(\text{Adj. } A) = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} |\mathbf{A}| & 0 & 0 \\ 0 & |\mathbf{A}| & 0 \\ 0 & 0 & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

or  $A \cdot \frac{\text{Adj. } A}{|\mathbf{A}|} = I \quad [\because |\mathbf{A}| \neq 0] \quad \text{or} \quad \frac{\text{Adj. } A}{|\mathbf{A}|} \text{ is the inverse of } A.$

**Obs. 1.** Inverse of a matrix, is unique.

If possible, let the two inverses of the matrix  $A$  be  $B$  and  $C$ ,

then

$$AB = BA = I \quad \text{and} \quad AC = CA = I$$

$$CAB = (CA)B = IB = B \quad \text{and} \quad CAB = C(AB) = CI = C$$

∴

$$\text{Thus, } B = C.$$

**Obs. 2:** The reciprocal of the product of two matrices is the product of their reciprocals taken in the reverse order i.e.,

$$(AB)^{-1} = B^{-1}A^{-1}$$

If  $A, B$  be two matrices, then the reciprocal of their product is  $(AB)^{-1}$ .

$$\begin{aligned}\text{Clearly, } (AB) \cdot (B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} = AA^{-1} = I.\end{aligned}$$

[by Associative law]

$$\text{Similarly, } (B^{-1}A^{-1}) \cdot (AB) = I$$

Hence  $B^{-1}A^{-1}$  is the reciprocal of  $AB$ .

**Obs. 3:** Multiplication by an inverse matrix plays the same role in matrix algebra that division plays in ordinary algebra.

i.e., if

$$[A][B] = [C][D], \text{ then } [A]^{-1}[A][B] = [A^{-1}][C][D]$$

or

$$B = A^{-1}[C][D], \text{ i.e., } \frac{[C][D]}{[A]} = A^{-1}[C][D].$$

**Note:** For other methods see Examples 2.25 ; 2.28 and 2.46.

**Example 2.20:** Find the matrix  $A$  if  $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

**Solution:** If  $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = B$ ,  $\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = C$  and  $\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} = D$ , then

$$BAC = D \quad \text{or} \quad AC = B^{-1}D$$

$$\therefore A = B^{-1}DC^{-1}$$

Now,

$$B^{-1} = \frac{\text{adj } B}{|B|} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Similarly,

$$C^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$$

$$\text{Hence, } A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 14 & 8 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 13 \\ -34 & -18 \end{bmatrix}.$$

**Example 2.21:** Show that  $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$  (GTU, 2019)

**Solution:** Applying  $C_1 - C_3$  and  $C_2 - C_3$  and denoting the given determinant by  $\Delta$ . we have

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}.$$

on taking  $(a+b+c)$  common from  $C_1$  and  $C_2$

Next applying  $R_3 - R_2 - R_1$ , we have

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

finally, applying  $\left(c + \frac{1}{a}c_3 \text{ and } c_2 + \frac{1}{b}c_3\right)$ . We have

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c & a^2/b & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$

$$\begin{aligned}\Delta &= (a+b+c)^2 \cdot 2ab \begin{vmatrix} b+c & a^2/b \\ b^2/a & c+a \end{vmatrix} \\ &= 2ab(a+b+c)^2 [(b+c)(c+a) - ab] \\ &= 2abc(a+b+c)^3\end{aligned}$$

**Example 2.22.** Show that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abc d \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right). \quad (\text{B.P.T.U. 2017})$$

**Solution:** Denoting the given determinant by  $\Delta$  and taking  $a, b, c, d$  common from  $R_1, R_2, R_3, R_4$  respectively. We have,

$$\Delta = abc d \begin{vmatrix} a^{-1}+1 & a^{-1} & a^{-1} & a^{-1} \\ b^{-1} & b^{-1}+1 & +b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

Now applying  $R_1 + (R_2 + R_3 + R_4)$  and taking out the common factor from  $R_1$ , we have

$$\Delta = abc d (1 + a^{-1} + b^{-1} + c^{-1} + d^{-1}) \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

Next, applying  $C_2 - C_1, C_3 - C_1$  and  $C_4 - C_1$ , we have

$$\begin{aligned}\Delta &= abc d (1 + a^{-1} + b^{-1} + c^{-1} + d^{-1}) \begin{vmatrix} 1 & 0 & 0 & 0 \\ b^{-1} & 1 & 0 & 0 \\ c^{-1} & 0 & 1 & 0 \\ d^{-1} & 0 & 0 & 1 \end{vmatrix} \\ &= abc d \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)\end{aligned}$$

$$\text{Example 2.23. } \Delta = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} \text{ as product of two determinants, and find its value.}$$

**Solution:** Since  $(x-y), (y-z), (z-x)$  and also  $(a-b), (b-c), (c-a)$  are factors of  $\Delta$  by factor theorem, we have by trial and adjustment.

$$\begin{aligned}\Delta &= \begin{vmatrix} x^2 & -2x & 1 \\ y^2 & -2y & 1 \\ z^2 & -2z & 1 \end{vmatrix} \times \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \\ &= 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \\ &= 2(x-y)(y-z)(z-x) \times (a-b)(b-c)(c-a)\end{aligned}$$

## PROBLEMS 2.3

If  $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$ , verify that  $AA' = I = A'A$ , where  $I$  is the unit matrix.

Express each of the following matrices as the sum of a symmetric and a skew-symmetric matrix:

(i)  $\begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 3 & 4 & 0 \end{bmatrix}$

(ii)  $\begin{bmatrix} a & a & b \\ c & b & b \\ c & a & c \end{bmatrix}$

Show that

$$\begin{vmatrix} a & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix} = ((a-b)(b-c)(c-a))^2 \quad (\text{U.P.T.U. 2018})$$

Show that

$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix} = \lambda^3(a^2+b^2+c^2+d^2+\lambda). \quad (\text{Mumbai, 2018})$$

If  $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$ , compute  $\text{adj } A$  and  $A^{-1}$ . Also find  $B$  such that  $AB = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 6 & 1 \\ 5 & 6 & 4 \end{bmatrix}$ .

If  $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ ,

(i) find  $A^{-1}$ ;

(ii) show that  $A^8 = A^{-1}$ .

Find the inverse of the matrix

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and if } A = \frac{1}{2} \begin{bmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \\ 2 & 1 & 5 \end{bmatrix},$$

show that  $SAS^{-1}$  is a diagonal matrix dig (2, 3, 1).

If  $A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ , prove that  $A^{-1} = A'$ .

Show that  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan \theta/2 \\ \tan \theta/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan \theta/2 \\ -\tan \theta/2 & 1 \end{bmatrix}^{-1}$ .

If  $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ , verify that  $(AB)' = B'A'$ , where  $A'$  is the transpose of  $A$ .

Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

where  $A, B$  etc are the co-factors of  $a, b$  etc. in the determinant. ( $a_1, b_2, c_3$ ).

(RGTU, 2018)

If  $A$  is a square matrix, show that (i)  $A + A'$  is symmetric, and (ii)  $A - A'$  is skew-symmetric.

If  $A$  and  $B$  are square matrices of the same order and  $A$  is symmetrical, show that  $B'AB$  is also symmetrical.  
[Hint: Show that  $(B'AB)' = B'AB$ ]

14. If a non-singular matrix  $A$  is symmetric, show that  $A^{-1}$  is also symmetric.

15. Let  $\Delta = \begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$

Express  $\Delta$  as the product of two determinants. Hence show that  $\Delta = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$ .

16. Show that

$$(i) \begin{vmatrix} 1 & \cos(\beta - \alpha) & \cos(\gamma - \alpha) \\ \cos(\alpha - \beta) & 1 & \cos(\gamma - \beta) \\ \cos(\alpha - \gamma) & \cos(\beta - \gamma) & 1 \end{vmatrix} = 0$$

$$(ii) \begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix} = \{(a-b)(b-c)(c-a)\}^2$$

## 2.7 (1) RANK OF A MATRIX

If we select any  $r$  rows and  $r$  columns from any matrix  $A$ , deleting all the other rows and columns, then the determinant formed by these  $r \times r$  elements is called the *minor of  $A$  of order  $r$* . Clearly, there will be a number of different minors of the same order, got by deleting different rows and columns from the same matrix.

**Def.** A matrix is said to be of rank  $r$  when

- (i) it has at least one non-zero minor of order  $r$ , and
- (ii) every minor of order higher than  $r$  vanishes.

Briefly, the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

The rank of a matrix  $A$  shall be denoted by  $\rho(A)$ .

**Obs. 1.** Rank of  $A$  and its transpose is the same i.e.  $\rho(A) = \rho(A')$ .

**Obs. 2.** Rank of a null matrix is zero.

**Obs. 3.** Rank of a non-singular square matrix of order  $r$  is  $r$ .

**Obs. 4.** If a matrix has a non-zero minor of order  $r$ , its rank is  $\geq r$ .

If all minors of a matrix of order  $r+1$  are zero, its rank is  $\leq r$ .

**(2) Elementary transformation of a matrix.** The following operations, three of which refer to rows and three to columns are known as *elementary transformations*:

I. The interchange of any two rows (columns).

II. The multiplication of any row (column) by a non-zero number.

III. The addition of a constant multiple of the elements of any row (column) to the corresponding elements of any other row (column).

**Notation.** The elementary row transformations will be denoted by the following symbols :

(i)  $R_{ij}$  for the interchange of the  $i$ th and  $j$ th rows.

(ii)  $kR_i$  for multiplication of the  $i$ th row by  $k$ .

(iii)  $R_i + pR_j$  for addition to the  $i$ th row,  $p$  times the  $j$ th row.

The corresponding column transformation will be denoted by writing  $C$  in place of  $R$ .

Elementary transformations do not change either the order or rank of a matrix. While the value of the minors may get changed by the transformation I and II, their zero or non-zero character remains unaffected.

**(3) Equivalent matrix.** Two matrices  $A$  and  $B$  are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order and the same rank. The symbol  $\sim$  is used for equivalence.

**Example 2.24.** Determine the rank of the following matrices :

$$(i) \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

**Solution:** (i) Operate  $R_2 - R_1$  and  $R_3 - 2R_1$  so that the given matrix

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} = A \text{ (say)}$$

Obviously, the 3rd order minor of  $A$  vanishes. Also its 2nd order minors formed by its 2nd and 3rd rows are all zero. But another 2nd order minor is  $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$ .

$\therefore \rho(A) = 2$ . Hence the rank of the given matrix is 2.

(ii) Given matrix

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & -3 & -1 \\ 1 & 1 & -3 & -1 \end{bmatrix}$$

[Operating  $C_3 - C_1, C_4 - C_1$ ]

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

[Operating  $R_3 - R_1, R_4 - R_1$ ]

$$\sim \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A \text{ (say)}$$

[Operating  $R_3 - 3R_2, R_4 - R_2$ ]

[Operating  $C_3 + 3C_2, C_4 + C_2$ ]

Obviously, the 4th order minor of  $A$  is zero. Also every 3rd order minor of  $A$  is zero. But, of all the 2nd order minors, only  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \neq 0$ .  $\therefore \rho(A) = 2$ .

Hence the rank of the given matrix is 2.

**(4) Elementary matrices.** An elementary matrix is that, which is obtained from a unit matrix, by subjecting it to any of the elementary transformations.

Examples of elementary matrices obtained from

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ are } R_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = C_{23}; kR_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}; R_1 + pR_2 = \begin{bmatrix} 1 & p & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**(5) Theorem.** Elementary row (column) transformations of a matrix  $A$  can be obtained by pre-multiplying (post-multiplying)  $A$  by the corresponding elementary matrices.

Consider the matrix  $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\text{Then } R_{23} \times A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

So a pre-multiplication by  $R_{23}$  has interchanged the 2nd and 3rd rows of  $A$ . Similarly, pre-multiplication by  $kR_2$  will multiply the 2nd row of  $A$  by  $k$  and pre-multiplication by  $R_1 + pR_2$  will result in the addition of  $p$  times the 2nd row of  $A$  to its 1st row.

Thus the pre-multiplication of  $A$  by elementary matrices results in the corresponding elementary row transformation of  $A$ . It can easily be seen that post multiplication will perform the elementary column transformations.

(6) **Gauss-Jordan method of finding the inverse\***. Those elementary row transformations which reduce a given square matrix  $A$  to the unit matrix, when applied to unit matrix  $I$  give the inverse of  $A$ .

Let the successive row transformations which reduce  $A$  to  $I$  result from pre-multiplication by the elementary matrices  $R_1, R_2, \dots, R_i$  so that

$$R_i R_{i-1} \dots R_2 R_1 A = I$$

$$\therefore R_i R_{i-1} \dots R_2 R_1 A A^{-1} = I A^{-1}$$

or

$$R_i R_{i-1} \dots R_2 R_1 I = A^{-1}$$

$$[\because AA^{-1} = I]$$

Hence the result.

**Working rule to evaluate  $A^{-1}$ .** Write the two matrices  $A$  and  $I$  side by side. Then perform the same row transformations on both. As soon as  $A$  is reduced to  $I$ , the other matrix represents  $A^{-1}$ .

**Example 2.25.** Using the Gauss-Jordan method, find the inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

(C.S.V.T.U., 2015 ; Anna, 2012)

**Solution:** Writing the same matrix side by side with the unit matrix of order 3, we have

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array} \right]$$

(Operate  $R_2 - R_1$  and  $R_3 + 2R_1$ )

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array} \right]$$

(Operate  $\frac{1}{2}R_2$  and  $\frac{1}{2}R_3$ )

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array} \right]$$

(Operate  $R_1 - R_2$  and  $R_3 + R_2$ )

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array} \right]$$

[Operate  $R_1 + 3R_3$ ,  $R_2 - \frac{3}{2}R_3$  and  $\left(-\frac{1}{2}\right)R_2$ ]

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array} \right]$$

\*Named after the great German mathematician Carl Friedrich Gauss (1777–1855) who made his first great discovery as a student at Göttingen. His important contributions are to algebra, number theory, mechanics, complex analysis, differential equations, differential geometry, non-Euclidean geometry, numerical analysis, astronomy and electromagnetism. He became director of the observatory at Göttingen in 1807.

Name after another German mathematician and geodesist Wilhelm Jordan (1842–1899).

Hence the inverse of the given matrix is  $\begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

[cf. Example 2.21]

**(7) Normal form of a matrix.** Every non-zero matrix  $A$  of rank  $r$ , can be reduced by a sequence of elementary transformations, to the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ called the normal form of } A.$$

**Cor. 1.** The rank of a matrix  $A$  is  $r$  if and only if it can be reduced to the normal form (i).

**Cor. 2.** Since each elementary transformation can be affected by pre-multiplication or post-multiplication with suitable elementary matrix and each elementary matrix is non-singular, therefore, we have the following result:

Corresponding to every matrix  $A$  of rank  $r$ , there exist non-singular matrices  $P$  and  $Q$  such that  $P_A Q$  equals (i).

If  $A$  be a  $m \times n$  matrix, then  $P$  and  $Q$  are square matrices of orders  $m$  and  $n$  respectively.

**Example 2.26.** Reduce the following matrix into its normal form and hence find its rank.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}. \quad (\text{J.N.T.U., 2015 ; V.T.U., 2012 ; U.P.T.U., 2011})$$

**Solution:**

$$\begin{aligned}
 A &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} && [\text{By } R_1] \\
 &\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} && [\text{By } R_2 - 2R_1, R_3 - 3R_1, R_4 - 6R_1] \\
 &\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} && [\text{By } C_2 + C_1, C_3 + 2C_1, C_4 + 4C_1] \\
 &\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} && [\text{By } R_4 - R_2 - R_3] \\
 &\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} && [\text{By } R_2 - R_1] \\
 &\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} && [\text{By } R_3 - 4R_2] \\
 &\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} && [\text{By } C_3 + 6C_2, C_4 + 3C_2]
 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[By  $\frac{1}{33} C_3$ ]

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[By  $C_4 - 22C_3$ ]

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence  $r(A) = 3$ .

**Example 2.27.** Find non-singular matrices  $P$  and  $Q$  such that  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$  is reduced to normal form. Also find the rank of  $A$ .

**Solution:** Total number of rows = 3  $\therefore$  Consider unit matrix  $I_3$   
 Total number of columns = 4  $\therefore$  Consider unit matrix  $I_4$

$$A_3 \times 4 = I_3 A I_4$$

$$A = I_3 A I_4$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Our aim is to reduce  $A$  to normal form by using elementary transformations.

$$R_2 - 2R_1, R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2 - 2C_1, C_3 - 3C_1, C_4 - 4C_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } -R_2 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 5 \\ 0 & 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} A \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } C_2 - C_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } C_3 - 2C_2, C_4 - 5C_2 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } C_3 - 2C_2, C_4 - 5C_2 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -5 & -5 \\ 0 & 1 & -2 & -5 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } \frac{1}{12} R_3 \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \end{bmatrix} A \begin{bmatrix} 1 & 1 & -5 & -9 \\ 0 & 1 & -2 & -5 \\ 0 & -1 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } C_{34} \quad \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 1 & : & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \end{bmatrix} A \begin{bmatrix} 1 & 1 & -9 & -5 \\ 0 & 1 & -5 & -2 \\ 0 & -1 & 5 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[I_3 : O] = PAQ$$

Here  $PAQ$  is in normal form  $P(A) = 3$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -\frac{1}{12} & \frac{1}{6} & -\frac{1}{12} \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & -9 & -5 \\ 0 & 1 & -5 & -2 \\ 0 & -1 & 5 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Here  $P$  and  $Q$  are non-singular matrices

**Example 2.28.** For the matrix  $A$ , find non-singular matrices  $P$  and  $Q$  in the normal form.

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ -1 & 1 & 1 & 2 \end{bmatrix}$$

**Solution:** Order of  $A = 3 \times 4$

Total number of rows in  $A = 3 \quad \therefore$  Consider Unit matrix  $I_3$

Total number of columns in  $A = 4 \quad \therefore$  Consider unit matrix  $I_4$

$$A_{3 \times 4} = I_3 A I_4$$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ -1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } R_{13} \quad \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By  $R_2 - 3R_1, R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By  $C_2 - C_1, C_3 - C_1, C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 1 \end{bmatrix}$$

By  $-R_2, -R_3$  and then followed by  $R_{23}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 6 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & -1 & 3 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By  $R_3 - 6R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & -28 & -56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 6 & -1 & -9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By  $C_3 - 5C_2, C_4 - 10C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -28 & -56 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 6 & -1 & -9 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & -8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By

$$-\frac{1}{28} R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -\frac{6}{8} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By

$$C_4 - 2C_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ -\frac{6}{28} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_3 : 0] = PAQ$$

$PAQ$  is in normal form  $P, Q$  are non-singular matrices.

$$P(A) = 3.$$

## PROBLEMS 2.4

Determine the rank of the following matrix :

$$1. \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

2. Reduce the following matrix to its normal form and hence find its rank.

$$(i) \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$$

(P.T.U., 2014)

(Andhra, 2022)

$$3. \begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$$

$$4. \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

(JNTU, 2022, C.S.V.T.U., 2012)

5. Determine the values of  $p$  such that the rank of  $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ p & 2 & 2 & 2 \\ 9 & 9 & p & 3 \end{bmatrix}$  is 3.

6. Use Gauss-Jordan method to find the inverse of the following matrices :

$$(i) \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad (\text{U.P.T.U., 2012})$$

$$(ii) \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad (\text{P.T.U., 2015})$$

7. Find non singular matrices  $P$  and  $Q$  such that  $PAQ$  in normal form. Hence find the rank of  $A$  where,

$$(i) A = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 2 & 3 \\ 0 & -1 & -3 & -2 \end{bmatrix} \quad (\text{Rohtak, 2019})$$

$$(ii) A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix} \quad (\text{Bhopal, 2019})$$

$$(iii) A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix} \quad (\text{Kurukhetra, 2018})$$

$$(iv) A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

(UTU, 2018)

8. If  $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ , find  $A^{-1}$ . Also find two non-singular matrices  $P$  and  $Q$  such that  $PAQ = I$ , where  $I$  is the unit matrix and verify that  $A^{-1} = QP$ .

9. Reduce each of the following matrices to normal form and hence find their ranks :

$$(i) \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix} \quad (\text{Delhi, 2015 ; Kurukshetra, 2005}) \quad (ii) \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} \quad (\text{V.T.U., 2014 ; Delhi, 2012})$$

$$(iii) \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix} \quad (\text{U.K.T.U., 2010})$$

## 2.8 PARTITION METHOD OF FINDING THE INVERSE

According to this method of finding the inverse, if the inverse of a matrix  $A_n$  of order  $n$  is known, then the inverse of the matrix  $A_{n+1}$  can easily be obtained by adding  $(n+1)$ th row and  $(n+1)$ th column to  $A_n$ .

Let

$$A = \begin{bmatrix} A_1 & : & A_2 \\ \dots & : & \dots \\ A_3' & : & \alpha \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \dots & : & \dots \\ X_3' & : & x \end{bmatrix}$$

where  $A_1, X_2$  are column vectors and  $A_3', X_3'$  are row vectors (being transposes of column vectors  $A_3, X_3$ ) and  $\alpha, x$  are ordinary numbers. We also assume that  $A_1^{-1}$  is known.

$$\text{Then, } AA^{-1} = I_{n+1}, \text{ i.e., } \begin{bmatrix} A_1 & : & A_2 \\ \dots & : & \dots \\ A_3' & : & \alpha \end{bmatrix} \begin{bmatrix} X_1 & : & X_2 \\ \dots & : & \dots \\ X_3' & : & x \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}$$

gives

$$A_1 X_1 + A_2 X_3' = I_n \quad \dots(i)$$

$$A_1 X_2 + A_2 x = 0 \quad \dots(ii)$$

$$A_3' X_1 + \alpha X_3' = 0 \quad \dots(iii)$$

$$A_3' X_2 + \alpha x = 1 \quad \dots(iv)$$

From Eqn. (ii),  $X_2 = -A_1^{-1} A_2 x$  and using this, Eqn. (iv) gives  $x = (\alpha - A_3' A_1^{-1} A_2)^{-1}$

Hence  $x$  and then  $X_2$  are given.

Also from Eqn. (i),  $X_1 = A_1^{-1} (I_n - A_2 X_3')$

and using this, Eqn. (iii) gives  $X_3' = -A_3' A_1^{-1} (\alpha - A_3' A_1^{-1} A_2)^{-1} = -A_3' A_1^{-1} x$

Then  $X_1$  is determined and hence  $A^{-1}$  is computed.

**Obs.** This is also known as the 'Escalator method'. For evaluation of  $A^{-1}$  we only need to determine two inverse matrices  $A_1^{-1}$  and  $(\alpha - A_3' A_1^{-1} A_2)^{-1}$ .

**Example 2.29.** Using the partition method, find the inverse of  $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$ .

**Solution:** Let  $A = \begin{bmatrix} 1 & 1 & : & 1 \\ 4 & 3 & : & -1 \\ \dots & \dots & : & \dots \\ 3 & 5 & : & 3 \end{bmatrix} = \begin{bmatrix} A_1 & : & A_2 \\ \dots & : & \dots \\ A_3' & : & \alpha \end{bmatrix}$

so that

$$A_1^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$$

Let  $A^{-1} = \begin{bmatrix} X_1 & : & X_2 \\ \dots & : & \dots \\ X_3' & : & x \end{bmatrix}$  so that  $AA^{-1} = I$ .

$$\alpha - A_3' A_1^{-1} A_2 = 3 + [3 \ 5] = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -10$$

$$\therefore x = (\alpha - A_3' A_1^{-1} A_2)^{-1} = -\frac{1}{10}$$

Also,  $X_2 = -A_1^{-1} A_2 x = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10} \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

Then  $X_3' = -A_3' A_1^{-1} x = [3 \ 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10} [-11 \ 2]$

Finally,  $X_1 = A_1^{-1}(I - A_2 X_3') = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} [-11 \ 2]$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} -44 & 8 \\ 55 & -10 \end{bmatrix} = \begin{bmatrix} 1.4 & 0.2 \\ -1.5 & 0 \end{bmatrix}$$

Hence  $A^{-1} = \begin{bmatrix} 1.4 & 0.2 & -0.4 \\ -1.5 & 0 & 0.5 \\ 1.1 & -0.2 & -0.1 \end{bmatrix}$ .

**Example 2.30.** If  $A$  and  $C$  are non-singular matrices, then show that  $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$

Hence find inverse of  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix}$ .

**Solution:** Let the given matrix be  $M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$  and its inverse be  $M^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$  both in the partitioned form where  $A, B, C, P, Q, R, S$  are all matrices.

$$\therefore MM^{-1} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = I$$

$$\text{or } \begin{bmatrix} AP + OR & AQ + OS \\ BP + CR & BQ + CS \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$\therefore$  Equating corresponding elements, we have

$$AP + OR = I, AQ + OS = 0, BP + CR = 0, BQ + CS = I.$$

$$AQ = 0, \text{ i.e., } Q = 0 \text{ as } A \text{ is non-singular.}$$

$$AP = I, \text{ i.e., } P = A^{-1}.$$

$$BP + CR = 0, \text{ i.e., } CR = -BP = -BA^{-1}$$

$$C^{-1}CR = -C^{-1}BA^{-1} \text{ or } IR = -C^{-1}BA^{-1} \text{ or } R = -C^{-1}BA^{-1}$$

$$BQ + CS = I, \text{ or } CS = I \text{ or } S = C^{-1}$$

Hence

$$M^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}.$$

Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

Whence

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, C^{-1} = \frac{1}{12} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\therefore -C^{-1}(BA^{-1}) = -\frac{1}{12} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \left\{ \frac{1}{2} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$= -\frac{1}{24} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} = -\frac{1}{24} \begin{bmatrix} 18 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence,

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ -3/4 & 0 & 1/4 & 0 \\ 0 & -1/6 & 0 & 1/3 \end{bmatrix}.$$

### PROBLEMS 2.5

Find the inverse of each of the following matrices using the partition method :

$$1. \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

$$2. \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

## 2.9 SOLUTION OF LINEAR SYSTEM OF EQUATIONS

### (1) Method of determinants—Cramer's\* rule

Consider the equations  $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$  ... (i)

If the determinant of coefficient be  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\text{then } x\Delta = \begin{vmatrix} xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \\ xa_3 & b_3 & c_3 \end{vmatrix} \quad [\text{Operate } C_1 + yC_2 + zC_3]$$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad [\text{By (i)}]$$

$$\text{Thus } x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} + \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad \text{provided } \Delta \neq 0. \quad \dots (\text{ii})$$

$$\text{Similarly, } y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad \dots (\text{iii})$$

$$\text{and } z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad \dots (\text{iv})$$

Equations (ii), (iii) and (iv) giving the values of  $x, y, z$  constitute the **Cramer's rule**, which reduces the solution of the linear Eqn. (i) to a problem in evaluation of determinants.

### (2) Matrix inversion method

$$\text{If } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

then the Eqn. (i) are equivalent to the matrix equation  $AX = D$  ... (v)  
where  $A$  is the *coefficient matrix*.

Multiplying both sides of Eqn. (v) by the reciprocal matrix  $A^{-1}$ , we get

$$A^{-1}AX = A^{-1}D \quad \text{or} \quad IX = A^{-1}D \quad [\because A^{-1}A = I]$$

$$\text{or } X = A^{-1}D \quad \text{i.e., } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \dots (\text{vi})$$

where,  $A_1, B_1$  etc. are the cofactors of  $a_1, b_1$  etc. in the determinant  $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$  ( $\Delta \neq 0$ )

Hence equating the values of  $x, y, z$  to the corresponding elements in the product on the right side of (vi), we get the desired solutions.

**Obs.** When  $A$  is a singular matrix, i.e.,  $\Delta = 0$ , the above methods fail. These also fail when the number of equations and the number of unknowns are unequal. Matrices can, however, be usefully applied to deal with such equations as will be seen in § 2.10(2).

\*Gabriel Cramer (1704–1752), a Swiss mathematician.

**Example 2.31.** Solve the equations  $3x + y + 2z = 3$ ,  $2x - 3y - z = -3$ ,  $x + 2y + z = 4$  by (i) determinants (ii) matrices.

**Solution:** (i) By determinants :

$$\text{Here } \Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3(-3+2) - 2(1-4) + (-1+6) = 8 \quad [\text{Expanding by } C_3]$$

$$\therefore x = \frac{1}{\Delta} \begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} \quad [\text{Expand by } C_1]$$

$$= \frac{1}{8} [3(-3+2) + 3(1-4) + 4(-1+6)] = 1$$

$$\text{Similarly, } y = \frac{1}{\Delta} \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 2 \quad \text{and } z = \frac{1}{\Delta} \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$$

$$\text{Hence } x = 1, y = 2, z = -1.$$

**Note.** The use of Cramer's rule involves a lot of labour when the number of equations exceeds four. In such and other cases, the numerical methods given in § 28.4 to 28.6 are preferable.

(ii) By matrices :

$$\text{Here } \Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad (\text{say}).$$

$$\text{Then } A_1 = -1, A_2 = 3, A_3 = 5; B_1 = -3, B_2 = 1, B_3 = 7; C_1 = 7, C_2 = -5, C_3 = -11.$$

$$\text{Also } \Delta = a_1 A_1 + a_2 A_2 + a_3 A_3 = 8.$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -3 - 9 + 20 \\ -9 - 3 + 28 \\ 21 + 15 - 44 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\text{Hence } x = 1, y = 2, z = -1.$$

**Example 2.32.** Solve the equations  $x_1 - x_2 + x_3 + x_4 = 2$ ;  $x_1 + x_2 - x_3 + x_4 = -4$ ;  $x_1 + x_2 + x_3 - x_4 = 4$ ;  $x_1 + x_2 + x_3 + x_4 = 0$ , by finding the inverse by elementary row operations.

**Solution:** Given system can be written as  $AX = B$ , where

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, B = \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix}$$

To find  $A^{-1}$ , we write

$$\begin{aligned} [A : I] &= \left[ \begin{array}{cccc|cccc} 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{c} R_2 - R_1 \\ R_3 + R_1 \\ R_4 + R_1 \\ \hline \end{array} \right] \\ &= \left[ \begin{array}{cccc|cccc} 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & -1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 2 & 1 & 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{c} \frac{1}{2}R_2 \\ \frac{1}{2}R_3 \\ \frac{1}{2}R_4 \\ \hline \end{array} \right] \\ &= \left[ \begin{array}{cccc|cccc} 1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 1 & 0 & 1 & 1 & 1/2 & 0 & 0 & 1/2 \end{array} \right] \left[ \begin{array}{c} R_3 - R_2 \\ R_4 - R_3 \\ \hline \end{array} \right] \end{aligned}$$

$$\begin{aligned}
 &= \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array} \right] \begin{bmatrix} R_1 - R_4 \\ R_2 + R_3 \end{bmatrix} \\
 &= \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 & +1/2 & -1/2 \\ 1 & 1 & 0 & 0 & 0 & 1/2 & 1/2 & 0 \\ 1 & 0 & 1 & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array} \right] \begin{bmatrix} R_2 - R_1 \\ R_3 - R_1 \end{bmatrix} \\
 &= \left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & 0 & -1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1/2 & 1/2 \end{array} \right]
 \end{aligned}$$

Thus,  $A^{-1} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}$

Hence,  $X = A^{-1}B = \begin{bmatrix} 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$

i.e.,  $x_1 = 1, x_2 = -1, x_3 = 2, x_4 = -2.$

### PROBLEMS 2.6

Solve the following equations with the help of determinants (1 to 5) :

1.  $x + y + z = 1, 3x + y - 3z = 5, x - 2y - 5z = 0$  (Andhra, 2016)
2.  $x + 3y + 6z = 2 ; 3x - y + 4z = 9 ; x - 4y + 2z = 7.$
3.  $x + y + z = 6.6 ; x - y + z = 2.2 ; x + 2y + 3z = 15.2.$
4.  $x^2 z^3/y = e^8 ; y^2 z/x = e^4 ; x^3 y/z^4 = 1.$
5.  $2vw - wu + uv = 3uvw ; 3vw + 2wu + 4uv = 19uvw ; 6vw + 7wu - uv = 17uvw.$

Solve the following system of equations by matrix method (6 to 8) :

6.  $x_1 + x_2 + x_3 = 1, x_1 + 2x_2 + 3x_3 = 6, x_1 + 3x_2 + 4x_3 = 6.$
7.  $x + y + z = 3 ; x + 2y + 3z = 4 ; x + 4y + 9z = 6.$
8.  $x + y + z = 6$   
 $2x + y + 3z = 13$   
 $5x + 2y + z = 12$   
 $2x - 3y - 2z = -0$

9.  $2x - y + 3z = 8 ; x - 2y - z = -4 ; 3x + y - 4z = 0.$  (UPTU, 2019)

10.  $2x_1 + x_2 + 2x_3 + x_4 = 6, 4x_1 + 3x_2 + 3x_3 - 3x_4 = -1, 6x_1 - 6x_2 + 6x_3 + 12x_4 = 36, 2x_1 + 2x_2 - x_3 + x_4 = 10.$

11. By finding  $A^{-1}$ , solve the linear equation  $AX = B$ , where  $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 0 \\ 5 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $B = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}.$

12. In a given electrical network, the equations for the currents  $i_1, i_2, i_3$  are  
 $3i_1 + i_2 + i_3 = 8 ; 2i_1 - 3i_2 - 2i_3 = -5 ; 7i_1 + 2i_2 - 5i_3 = 0.$   
Calculate  $i_1$  and  $i_3$  by Cramer's rule.

13. Using the loop current method on a circuit, the following equations are obtained :  
 $7i_1 - 4i_2 = 12, -4i_1 + 12i_2 - 6i_3 = 0, -6i_2 + 14i_3 = 0.$   
By matrix method, solve for  $i_1, i_2$  and  $i_3$ .

14. Solve the following equations by calculating the inverse by elementary row operations :  
 $2x_1 + 2x_2 + 2x_3 - 3x_4 = 2 ; 3x_1 + 6x_2 - 2x_3 + x_4 = 8 ; x_1 + x_2 - 3x_3 - 4x_4 = -1 ; 2x_1 + x_2 + 5x_3 + x_4 = 5.$

## 2.10

## (1) CONSISTENCY OF LINEAR SYSTEM OF EQUATIONS

Consider the system of  $m$  linear equations

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = k_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = k_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = k_m \end{array} \right\} \quad \text{...(i)}$$

containing the  $n$  unknowns  $x_1, x_2, \dots, x_n$ . To determine whether the Eqn. (i) are consistent (i.e., possess a solution) or not, we consider the ranks of the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & k_1 \\ a_{21} & a_{22} & \dots & a_{2n} & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & k_m \end{bmatrix}$$

$A$  is the co-efficient matrix and  $K$  is called the augmented matrix of the Eqn. (i).

(2) **Rouche's theorem.** The system of Eqn. (i) is consistent if and only if the coefficient matrix  $A$  and the augmented matrix  $K$  are of the same rank otherwise the system is inconsistent.

*Proof.* We consider the following two possible cases :

I. Rank of  $A$  = rank of  $K = r$  ( $r \leq$  the smaller of the numbers  $m$  and  $n$ ). The Eqn. (i) can, be suitable row operations, be reduced to

$$\left. \begin{array}{l} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n = l_1 \\ 0.x_1 + b_{22}x_2 + \dots + b_{2n}x_n = l_2 \\ \dots \\ 0.x_1 + 0.x_2 + \dots + b_{rn}x_n = l_r \end{array} \right\} \quad \text{...(ii)}$$

and the remaining  $m - r$  equations being all of the form  $0.x_1 + 0.x_2 + \dots + 0.x_n = 0$ .

The Eqn. (ii) will have a solution, though  $n - r$  of the unknowns may be chosen arbitrarily. The solution, will be unique only when  $r = n$ . Hence the Eqn. (i) are consistent.

II. Rank of  $A$  (i.e.,  $r$ )  $<$  rank of  $K$ . In particular, let the rank of  $K$  be  $r + 1$ . In this case, the Eqn. (i) will reduce, by suitable row operations, to

$$\left. \begin{array}{l} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n = l_1, \\ 0.x_1 + b_{22}x_2 + \dots + b_{2n}x_n = l_2, \\ \dots \\ 0.x_1 + 0.x_2 + \dots + b_{rn}x_n = l_r, \\ 0.x_1 + 0.x_2 + \dots + 0.x_n = l_{r+1}, \end{array} \right.$$

and the remaining  $m - (r + 1)$  equations are of the form  $0.x_1 + 0.x_2 + \dots + 0.x_n = 0$ .

Clearly, the  $(r + 1)$ th equation cannot be satisfied by any set of values for the unknowns. Hence the Eqn. (i) are inconsistent.

[Procedure to test the consistency of a system of equations in  $n$  unknowns :

Find the ranks of the coefficient matrix  $A$  and the augmented matrix  $K$ , by reducing  $A$  to the triangular form by elementary row operations. Let the rank of  $A$  be  $r$  and that of  $K$  be  $r'$ .

- (i) If  $r \neq r'$ , the equations are inconsistent, i.e., there is no solution.
- (ii) If  $r = r' = n$ , the equations are consistent and there is a unique solution.
- (iii) If  $r = r' < n$ , the equations are consistent and there are infinite number of solutions. Giving arbitrary values to  $n - r$  of the unknowns, we may express the other  $r$  unknowns in terms of these.]

**Example 2.33.** Test for consistency and solve

$$5x + 3y + 7z = 4, \quad 3x + 26y + 2z = 9, \quad 7x + 2y + 10z = 5.$$

**Solution:** We have

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

Operate  $3R_1, 5R_2,$

$$\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$$

Operate  $R_2 - R_1,$

$$\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$$

Operate  $\frac{7}{3}R_1, 5R_3, \frac{1}{11}R_2,$

$$\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$$

Operate  $R_3 - R_1 + R_2, \frac{1}{7}R_1,$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

The ranks of coefficient matrix and augmented matrix for the last set of equations, are both 2. Hence the equations are consistent. Also the given system is equivalent to

$$5x + 3y + 7z = 4, 11y - z = 3, \quad \therefore y = \frac{3}{11} + \frac{z}{11} \quad \text{and} \quad x = \frac{7}{11} - \frac{16}{11}z$$

where  $z$  is a parameter.

Hence  $x = \frac{7}{11}, y = \frac{3}{11}$  and  $z = 0$ , is a particular solution.

**Obs.** In the above solution, the coefficient matrix is reduced to an upper triangular matrix by row-transformations.

**Example 2.34.** Investigate the values of  $\lambda$  and  $\mu$  so that the equations

$$2x + 3y + 5z = 9, 7x + 3y - 2z = 8, 2x + 3y + \lambda z = \mu$$

have (i) no solution, (ii) a unique solution and (iii) an infinite number of solutions.

(Andhra, 2022; Hazaribag, 2015 ; C.S.V.T.U., 2014 ; V.T.U., 2013 ; D.T.U., 2011)

**Solution:** We have  $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$

The system admits of unique solution if, and only if, the coefficient matrix is of rank 3. This requires that

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5 - \lambda) \neq 0$$

Thus for a unique solution  $\lambda \neq 5$  and  $\mu$  may have any value. If  $\lambda = 5$ , the system will have no solution for those values of  $\mu$  for which the matrices

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix}$$

and

$$K = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

are not of the same rank. But  $A$  is of rank 2 and  $K$  is not of rank 2 unless  $\mu = 9$ . Thus if  $\lambda = 5$  and  $\mu \neq 9$ , the system will have no solution.

If  $\lambda = 5$  and  $\mu = 9$ , the system will have an infinite number of solutions.

**Example 2.35.** Test for consistency the following equations and solve them if consistent :  $x - 2y + 3t = 2$ ;  $2x + y + z + t = -4$ ;  $4x - 3y + z + 7t = 8$ .

**Solution:** Given equation can be written as

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$$

Operate  $R_2 - 2R_1$ ,  $R_3 - 4R_1$ ,

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 5 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Operate  $R_3 - R_2$ ,

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Clearly, rank of the coefficient matrix is 2 and the rank of augmented matrix is also 2. Hence the given equations are consistent. But the rank  $2 < 4$ , the number of unknowns.

∴ The number of parameters is  $4 - 2 = 2$

Thus the equations have doubly infinite solutions. Now putting  $t = k_1$  and  $z = k_2$  in

$$x - 2y + 3t = 2 \quad \text{and} \quad 5y + z - 5t = 0,$$

we get  $x - 2y + 3k_1 = 2$  and  $5y + k_2 - 5k_1 = 0$

Hence  $y = k_1 - k_2/5$

and

$$\begin{aligned} x &= 2 + 2y - 3k_1 \\ &= 2 + 2(k_1 - k_2/5) - 3k_1 \\ &= 2 - k_1 - \frac{2}{5}k_2 \end{aligned}$$

**(3) System of linear homogeneous equations.** Consider the homogeneous linear equations

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \dots & \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned} \right\} \quad \dots(iii)$$

Find the rank  $r$  of the coefficient matrix  $A$  by reducing it to the triangular form by elementary row operations.

I. If  $r = n$ , the equations (iii) have only a trivial zero solution

$$x_1 = x_2 = \dots = x_n = 0$$

If  $r < n$ , the equation (iii) have  $(n - r)$  linearly independent solutions.

The number of linearly independent solutions is  $(n - r)$  means, if arbitrary values are assigned to  $(n - r)$  of the variables, the values of the remaining variables can be uniquely found.

Thus the equations (iii) will have an infinite number of solutions.

II. When  $m < n$  (i.e., the number of equations is less than the number of variables), the solution is always other than  $x_1 = x_2 = \dots = x_n = 0$ . The number of solutions is infinite.

III. When  $m = n$  (i.e., the number of equations = the number of variables), the necessary and sufficient condition for solutions other than  $x_1 = x_2 = \dots = x_n = 0$ , is that the determinant of the coefficient matrix is zero. In this case the equations are said to be consistent and such a solution is called non-trivial solution. The determinant is called the eliminant of the equations.

**Example 2.36.** Solve the equations

- (i)  $x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0$   
(ii)  $4x + 2y + z + 3w = 0, 6x + 3y + 4z + 7w = 0, 2x + y + w = 0$ .

**Solution:** (i) Rank of the coefficient matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 7 & 10 & 12 \end{bmatrix} \quad [\text{Operating } R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix} \quad [\text{Operating } R_3 - 7R_1 - 2R_2]$$

is 3 which = the number of variables (i.e.,  $r = n$ )

$\therefore$  The equations have only a trivial solution :  $x = y = z = 0$ .

(ii) Rank of the coefficient matrix

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix} \quad [\text{Operating } R_2 - \frac{3}{2}R_1, R_3 - \frac{1}{2}R_1]$$

$$\sim \begin{bmatrix} 4 & 2 & 1 & 3 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\text{Operating } R_3 + \frac{1}{5}R_2]$$

is 2 which  $<$  the number of variable (i.e.,  $r < n$ )

$\therefore$  Number of independent solutions =  $4 - 2 = 2$ . Given system is equivalent to

$$4x + 2y + z + 3w = 0, z + w = 0.$$

$\therefore$  We have  $z = -w$  and  $y = -2x - w$

which give an infinite number of non-trivial solutions,  $x$  and  $w$  being the parameters.

**Example 2.37.** Show that if  $\mu \neq 0$ , the system of equations.

$$2x + y = a$$

$$x + \mu y - z = b$$

$$y + 2z = c$$

has unique solution for every choice of  $a, b, c$ . If  $\mu = 0$ , determine the relation satisfy by  $a, b, c$  such that the system is inconsistent. Find the general solution by taking  $\mu = 0, a = 1, b = 1, c = -1$ . (JBTU, 2019)

**Solution:** Part 1. The above system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & \mu & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Here 3 equations in 3 unknowns are there, hence we will apply another method.

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & \mu & -1 \\ 0 & 1 & 2 \end{vmatrix} = 2(2\mu + 1) - 1(2)$$

$$|A| = 4\mu$$

If  $\mu = 0$ ,  $\therefore |A| \neq 0$  the system  $AX = B$  always possesses a unique solution for every choice of  $a, b, c$ .

Part 2: If  $\mu = 0$  then.

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\text{Perform } R_{12} \sim \left[ \begin{array}{ccc|c} 1 & 0 & -1 & b \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \ell \end{array} \right]$$

$$\text{Perform } R_2 \sim 2R_1 \sim \left[ \begin{array}{ccc|c} 1 & 0 & -1 & b \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & \ell \end{array} \right]$$

$$\text{Perform } R_3 \sim R_2 \sim \left[ \begin{array}{ccc|c} 1 & 0 & -1 & b \\ 0 & 1 & 2 & a-2b \\ 0 & 0 & 0 & c-a+2b \end{array} \right]$$

If  $\mu = 0$  and  $c = a + 2b \neq 0$  then

$$p(A) = 0 = 1 = 2, p(A, B) = 0 = 0 = 3$$

$p(A) \neq p(A, B)$  system is inconsistent.

Hence if  $\mu = 0$  and  $c = a + 2b \neq 0$  then system possesses no solution.

**Part 3:** If  $\mu = 0, c = a + 2b = 0$  then  $p(A) = p(A, B) = 2 < 3$ . System possesses an infinite no. of solutions.  
Hence, system is consistent if  $\mu = 0$  and

$$c = a + 2b = 0.$$

**Part 4:** Take  $\mu = 0, a = 1, b = 1, c = -1$ , substituting in equation (i), we get

$$(A, B) \sim \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$p(A) = p(A, B) = 2 < 3$ , system possesses an infinite no. of solutions.

$$\text{By } R_3 \quad y + 2z = -1 \quad \text{let } z = t, y = -1 - 2t$$

$$\text{By } R_1 \quad x - z = 1 \quad x = 1 + t$$

Hence, if  $\mu = 0, a = 1, b = 1, c = -1$ ,

then solution, set is  $x = 1 + t$

$$y = -1 - 2t,$$

$$z = t.$$

**Example 2.38.** By considering the rank of relevant matrices, examine for consistency the system of equations

$$2x - y - z = 2$$

$$x + 2y + z = 2$$

$$4x - 7y - 5z = 2 \text{ and solve if found consistent.}$$

**Solution:** Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

i.e.,

$$AX = B$$

Consider an augmented matrix

$$(A, B) = \left[ \begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array} \right]$$

$$\text{Perform } R_{12} \sim \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{array} \right]$$

(MDU, 2013)

$$\text{Perform } R_2 - 2R_1, R_3 - 4R_1 \sim \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 15 & -5 & -6 \end{array} \right]$$

$$\text{Perform } R_3 - 3R_2 \sim \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rho(A) = 3 - 1 = 2, \quad \rho(A, B) = 3 - 1 = 2$$

Here,  
 $\rho(A) = \rho(A, B) \therefore \text{System is Consistent}$

$$\rho(A) = \rho(A, B) = 2 < 3, \text{ number of Variables}$$

System possesses an infinite number of solutions given as follows.

$$\text{By } R_2, \quad -5y - 3z = -2 \quad \text{Let } z = t, \quad y = \frac{2 - 3t}{5}$$

$$\text{By } R_1, \quad x + 2y + z = 2x = 2 - \frac{4 - 6t}{t} - t = \frac{10 - 4 + 6t - 5t}{5} = \frac{6 + t}{5}$$

$$\text{Solution set is } x = \frac{6+t}{5}, y = \frac{2-3t}{5}, z = t.$$

**Example 2.39.** Examine for consistency the following set of equations and obtains the solution if consistent.

$$2x_1 + x_2 - x_3 + 3x_4 = 11$$

$$x_1 - 2x_2 + x_3 + x_4 = 8$$

$$4x_1 + 7x_2 + 2x_3 - x_4 = 0$$

$$3x_1 + 5x_2 + 4x_3 + 4x_4 = 17$$

**Solution:** Given system of equation in matrix form can be written as

$$\left[ \begin{array}{cccc|c} 2 & 1 & -1 & 3 & x_1 \\ 1 & -2 & 1 & 1 & x_2 \\ 4 & 7 & 2 & -1 & x_3 \\ 3 & 5 & 4 & 4 & x_4 \end{array} \right] = \left[ \begin{array}{c} 11 \\ 8 \\ 0 \\ 17 \end{array} \right]$$

$$AX = B$$

Consider an augmented matrix

$$(A, B) = \left[ \begin{array}{cccc|c} 2 & 1 & -1 & 3 & 11 \\ 1 & -2 & 1 & 1 & 18 \\ 4 & 7 & 2 & -1 & 0 \\ 3 & 5 & 4 & 4 & 17 \end{array} \right]$$

By  $R_{12}$

$$\sim \left[ \begin{array}{cccc|c} 1 & -2 & 1 & 1 & 11 \\ 2 & 1 & -1 & 3 & 0 \\ 4 & 7 & 2 & -1 & 17 \\ 3 & 5 & 4 & 4 & 8 \end{array} \right]$$

By  $R_2 - 2R_1, R_3 - 4R_1, R_4 - 3R_1$

$$\sim \left[ \begin{array}{cccc|c} 1 & -2 & 1 & 1 & 8 \\ 0 & 5 & -3 & 1 & -5 \\ 0 & 15 & -2 & -5 & -32 \\ 0 & 11 & 1 & 1 & -7 \end{array} \right]$$

By  $R_4 - \frac{6}{5}R_2$

$$\sim \left[ \begin{array}{cccc|c} 1 & -2 & 1 & 1 & 8 \\ 0 & 5 & -3 & 1 & -5 \\ 0 & 0 & 7 & -8 & -17 \\ 0 & 0 & \frac{38}{5} & \frac{-6}{5} & 4 \end{array} \right]$$

By  $R_3 - 3R_2, R_4 - R_2$

$$\sim \left[ \begin{array}{cccc|c} 1 & -2 & 1 & 1 & 8 \\ 0 & 5 & -3 & 1 & -5 \\ 0 & 0 & 7 & -8 & -17 \\ 0 & 6 & 4 & 0 & -2 \end{array} \right]$$

By  $R_4 - \frac{38}{35} R_3$

$$\sim \left[ \begin{array}{cccc|c} 1 & -2 & 1 & 1 & 8 \\ 0 & 5 & -3 & 1 & -5 \\ 0 & 0 & 7 & -8 & -17 \\ 0 & 0 & 0 & \frac{262}{35} & \frac{786}{35} \end{array} \right]$$

Here

$$\rho(A) = 4 - 0 = 4, \quad \rho(A, B) = 4 - 0 = 4$$

$$\rho(A) = \rho(A, B) = 4 = 4,$$

Number of variable system is consistant and posses a unique solution.

By  $R_4$

$$\frac{262}{35} x_4 = \frac{786}{35}, \quad x_4 = 3$$

By  $R_3$

$$7x_3 - 8x_4 = -17, \quad 7x_3 = -17 + 24 = 7, \quad x_3 = 1$$

By  $R_2$

$$5x_2 - 3x_3 + x_4 = -5, \quad 5x_2 = -5 + 3 - 3 = -5, \quad x_2 = -1,$$

$$x_1 = -2 - 1 - 3 + 8 = 2, \quad x_1 = 2$$

Required solution set is  $x_1 = 2, x_2 = -1, x_3 = 1, x_4 = 3$ .

**Example 2.40.** Find the values of  $\lambda$  for which the equations

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$

$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$

$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

are consistent, and find the ratios of  $x : y : z$  when  $\lambda$  has the smallest of these values. What happens when  $\lambda$  has the greatest of these values.

(J.N.T.U., 2015)

**Solution:** The given equations will be consistent, if

$$\left| \begin{array}{ccc|c} \lambda - 1 & 3\lambda + 1 & 2\lambda & 0 \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 & 0 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) & 0 \end{array} \right| = 0$$

[Operate  $R_2 - R_1$ ]

$$\text{or if, } \left| \begin{array}{ccc|c} \lambda - 1 & 3\lambda + 1 & 2\lambda & 0 \\ 0 & \lambda - 3 & 3 - \lambda & 0 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) & 0 \end{array} \right| = 0$$

[Operate  $C_3 + C_1$ ]

$$\text{or if, } \left| \begin{array}{ccc|c} \lambda - 1 & 3\lambda + 1 & 5\lambda + 1 & 0 \\ 0 & \lambda - 3 & 0 & 0 \\ 2 & 3\lambda + 1 & 6\lambda - 2 & 0 \end{array} \right| = 0$$

[Expand by  $R_3$ ]

$$\text{or if, } (\lambda - 3) \left| \begin{array}{cc|c} \lambda - 1 & 5\lambda + 1 & 0 \\ 2 & 2(3\lambda - 1) & 0 \end{array} \right| = 0 \quad \text{or if, } 2(\lambda - 3) [(\lambda - 1)(3\lambda - 1) - (5\lambda + 1)] = 0$$

$$\text{or if, } 6\lambda(\lambda - 3)^2 = 0 \quad \text{or if, } \lambda = 0 \text{ or } 3.$$

(a) When  $\lambda = 0$ , the equations become  $-x + y = 0$

$$-x - 2y + 3z = 0$$

$$2x + y - 3z = 0$$

Solving (ii) and (iii), we get  $\frac{x}{6-3} = \frac{y}{6-3} = \frac{z}{-1+4}$ . Hence  $x = y = z$ .

(b) When  $\lambda = 3$ , equations becomes identical.

## PROBLEMS 2.7

1. Investigate for consistency of the following equations and if possible find the solutions :  
 $4x - 2y + 6z = 8, x + y - 3z = -1, 15x - 3y + 9z = 21.$  (D.T.U., 2012)
2. Find the values of  $a, b$  for which the equations  $x + ay + z = 3, x + 2y + 2z = b, x + 5y + 3z = 9$  are consistent. When will these equations have a unique solution. (Andhra, 2021)
3. Investigate for what values of  $\lambda$  and  $\mu$  the simultaneous equations  
 $x + y + z = 6, x + 2y + 3z = 10, x + 2y + \lambda z = \mu,$   
have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions. (Andhra, 2022, 2018)

4. Test for consistency and solve,  
(i)  $x + 2y + z = 3, 2x + 3y + 2z = 5, 3x - 5y + 5z = 2, 3x + 9y - z = 4.$  (C.S.V.T.U., 2015; V.T.U., 2014; P.T.U., 2013)  
(ii)  $2x + 6y + 11 = 0, 6x + 20y - 6z + 3 = 0, 6y - 18z + 1 = 0.$  (C.S.V.T.U., 2011S)  
(iii)  $3x + 3y + 2z = 1, x + 2y = 4, 10y + 3z = -2, 2x - 3y - z = 5.$  (Rohtak, 2012)
5. Investigate for what values of  $a$  and  $b$ , the system of simultaneous equations. (U.K.T.U., 2010)  

$$\begin{aligned} 2x - y + 3z &= 2 \\ x + y + 2z &= 2 \\ 5x - y + az &= b \end{aligned}$$
 (DTU, 2018)

- have (i) No solution (ii). A unique solution (iii) An infinite number of solutions.  
6. Show that if  $\lambda \neq -5$ , the system of equations

$$3x - y + 4z = 3, x + 2y - 3z = -2, 6x + 5y + \lambda z = -3,$$

- have a unique solution. If  $\lambda = -5$ , show that the equations are consistent. Determine the solutions in each case.  
7. Show that the equations

$$3x + 4y + 5z = a, 4x + 5y + 6z = b, 5x + 6y + 7z = c$$

- do not have a solution unless  $a + c = 2b$  i.e.  $a, b, c$  are in A.P. (Andhra., 2015; J.N.T.U., 2015; D.T.U., 2013)
8. Prove that the equations  $5x + 3y + 2z = 12, 2x + 4y + 5z = 2, 39x + 43y + 45z = c$  are incompatible unless  $c = 74$ ; and in that case the equations are satisfied by  $x = 2 + t, y = 2 - 3t, z = -2 + 2t$ , where  $t$  is any arbitrary quantity.
9. Find the values of  $\lambda$  for which the equations  $(2 - \lambda)x + 2y + 3 = 0, 2x + (4 - \lambda)y + 7 = 0, 2x + 5y + (6 - \lambda) = 0$  are consistent and find the values of  $x$  and  $y$  corresponding to each of these values of  $\lambda$ . (Andhra, 2014)
10. Show that there are three real values of  $\lambda$  for which the equations  $(a - \lambda)x + by + cz = 0, bx + (c - \lambda)y + az = 0, cx + ay + (b - \lambda)z = 0$  are simultaneously true and that the product of these values of  $\lambda$  is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

11. Determine the values of  $k$  for which the following system of equations has non-trivial solutions and find them :  
 $(k - 1)x + (4k - 2)y + (k + 3)z = 0, (k - 1)x + (3k + 1)y + 2kz = 0, 2x + (3k + 1)y + 3(k - 1)z = 0.$  (Mumbai, 2005)
12. Show that the following system of equations.

$$\begin{aligned} 3x + 4y + 5z &= a \\ 4x + 5y + 6z &= b \\ 5x + 6y + 7z &= c \end{aligned}$$

will be consistent only when  $a + c = 2b$ .

13. Investigate for what values of  $k$  the equations.

$$\begin{aligned} x + y + z &= 1 \\ 2x + y + 4z &= k \\ 4x + y + 10z &= k^2 \end{aligned}$$

have infinite number of solution? Hence find solutions

14. Solve completely the system of equations

$$\begin{aligned} (i) \quad x + y - 2z + 3w &= 0; x - 2y + z - w = 0; 4x + y - 5z + 8w = 0; 5x - 7y + 2z - w = 0. \\ (ii) \quad 3x + 4y - z - 6w &= 0; 2x + 3y + 2z - 3w = 0; 2x + y - 14z - 9w = 0; x + 3y + 13z + 3w = 0. \end{aligned}$$

15. Show that the system

$$\begin{aligned} 3x + 4y + 5z &= \alpha \\ 4x + 5y + 6z &= \beta \\ 5x + 6y + 7z &= \gamma \end{aligned}$$

is consistent only when  $\alpha, \beta, \gamma$  are in arithmetic progression.

16. Examine for consistency and if consistent then solve it.

$$\begin{aligned} (i) \quad x + 2y + z &= -1 \\ 6x + y + z &= -4 \\ 2x - 3y - z &= 0 \\ -x - 7y - 2z &= 7 \\ x - y &= 1 \end{aligned}$$

(UPTU, 2018)

$$(ii) \quad \begin{aligned} x + y + z &= 6 \\ 2x + y + 3z &= 13 \\ 5x + 2y + z &= 12 \\ 2x - 3y - 2z &= -10 \end{aligned} \quad (iii) \quad \begin{aligned} 2x_1 + x_2 + 2x_3 + x_4 &= 6 \\ 6x_1 - 3x_2 + 6x_3 + 12x_4 &= 36 \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 &= -1 \\ 2x_1 + 2x_2 - x_3 + x_4 &= 10 \end{aligned}$$

## 2.11 (1) LINEAR AND ORTHOGONAL TRANSFORMATIONS

Let  $(x, y)$  be the co-ordinates of a point  $P$  referred to set of rectangular axes  $OX, OY$ . Then its co-ordinates  $(x', y')$  referred to  $OX', OY'$ , obtained by rotating the former axes through an angle  $\theta$  given by

$$\left. \begin{aligned} x' &= x \cos \theta + y \sin \theta, \\ y' &= -x \sin \theta + y \cos \theta \end{aligned} \right\}$$

A more general transformation than (i) is

$$\left. \begin{aligned} x' &= a_1 x + b_1 y \\ y' &= a_2 x + b_2 y \end{aligned} \right\}$$

which in matrix notation is  $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Such transformations as Eqns. (i) and (ii), are called *linear transformations* in two dimensions.

$$\text{Similarly, the relations of the type } \left. \begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned} \right\}$$

give a *linear transformation* from  $(x, y, z)$  to  $(x', y', z')$  in three dimensional problems.

$$\text{In general, the relation } Y = AX \text{ where } Y = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}, A = \begin{bmatrix} a_1 & b_1 & c_1 & \dots & k_1 \\ a_2 & b_2 & c_2 & \dots & k_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & k_n \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

give *linear transformation* from  $n$  variables  $x_1, x_2, \dots, x_n$  to the variables  $y_1, y_2, \dots, y_n$  i.e., the transformation of vector  $X$  to the vector  $Y$ .

This transformation is called linear because the linear relations  $A(X_1 + X_2) = AX_1 + AX_2$  and  $A(bX) = bAX$  hold for this transformation.

If the transformation matrix  $A$  is singular, the transformation also is said to be singular otherwise non-singular. For a non-singular transformation  $Y = AX$ , we can also write the inverse transformation  $X = A^{-1}Y$ . A non-singular transformation is also called a *regular* transformation.

**Cor.** If a transformation from  $(x_1, x_2, x_3)$  to  $(y_1, y_2, y_3)$  is given by  $Y = AX$  and another transformation of  $(y_1, y_2, y_3)$  to  $(z_1, z_2, z_3)$  is given by  $Z = BY$ , then the transformation from  $(x_1, x_2, x_3)$  to  $(z_1, z_2, z_3)$  is given by  $Z = BY = B(AX) = (BA)X$ .

**(2) Orthogonal transformation.** The linear transformation Eqn. (iv), i.e.,  $Y = AX$ , is said to be *orthogonal* if, it transforms

$$y_1^2 + y_2^2 + \dots + y_n^2 \text{ into } x_1^2 + x_2^2 + \dots + x_n^2$$

The matrix of an orthogonal transformation is called an *orthogonal matrix*.

We have,

$$XX = [x_1 \ x_2 \ \dots \ x_n] \times \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \dots + x_n^2$$

and similarly,  $YY = y_1^2 + y_2^2 + \dots + y_n^2$ .

∴ If  $Y = AX$  is an orthogonal transformation, then

$X'X = Y'Y = (AX)'(AX) = X'A'AX$  which is possible only if  $A'A = I$ .  
But  $A^{-1}A = I$ , therefore,  $A' = A^{-1}$  for an orthogonal transformation.

Hence a square matrix  $A$  is said to be *orthogonal* if  $AA' = A'A = I$ .

**Obs. 1.** If  $A$  is orthogonal,  $A'$  and  $A^{-1}$  are also orthogonal.  
Since  $A$  is orthogonal,  $A' = A^{-1}$ .

∴  $(A')' = (A^{-1})' = (A')^{-1}$ , i.e.,  $B' = B^{-1}$  where  $B = A'$

Hence  $B$  (i.e.,  $A'$ ) is orthogonal. As  $A' = A^{-1}$ ,  $A^{-1}$  is also orthogonal.

**Obs. 2.** If  $A$  is orthogonal, then  $|A| = \pm 1$ .

$$\text{Since } AA' = A'A = I \quad \therefore |A| |A'| = |I|$$

$$\text{But } |A'| = |A|, \quad \therefore |A| |A| = |1|$$

$$\text{or} \quad |A|^2 = 1 \quad \text{i.e.,} \quad |A| = \pm 1.$$

**Example 2.41.** Show that the transformation

$$y_1 = 2x_1 + x_2 + x_3, y_2 = x_1 + x_2 + 2x_3, y_3 = x_1 - 2x_3$$

is regular. Write down the inverse transformation.

(V.T.U., 2016)

**Solution:** The given transformation may be written as

$$Y = AX$$

$$\text{where, } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

$$\text{Now } |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1$$

Thus the matrix  $A$  is non-singular and hence the transformation is regular.

$\therefore$  The inverse transformation is given by

$$X = A^{-1} Y$$

$$\text{where, } A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Thus  $x_1 = 2y_1 - 2y_2 - y_3; x_2 = -4y_1 + 5y_2 + 3y_3; x_3 = y_1 - y_2 - y_3$   
is the inverse transformation.

**Example 2.42.** Prove that the following matrix is orthogonal :

$$\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

(Delhi, 2012)

$$\begin{aligned} \text{Solution: We have } AA' &= \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \times \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} \\ &= \begin{bmatrix} 4/9 + 1/9 + 4/9 & -4/9 + 2/9 + 2/9 & -2/9 - 2/9 + 4/9 \\ -4/9 + 2/9 + 2/9 & 4/9 + 4/9 + 1/9 & 2/9 - 4/9 + 2/9 \\ -2/9 - 2/9 + 4/9 & 2/9 - 4/9 + 2/9 & 1/9 + 4/9 + 4/9 \end{bmatrix} = I. \end{aligned}$$

Hence the matrix is orthogonal.

$$\text{Example 2.43. If } A = \frac{1}{3} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \text{ is orthogonal, find } a, b, c \text{ and } A^{-1}.$$

**Solution:** As  $A$  is orthogonal,  $AA' = I$

$$\therefore \frac{1}{3} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 1+4+a^2 & 2+2+ab & 2-4+ac \\ 2+2+ab & 4+1+b^2 & 4-2+bc \\ 2-4+ac & 4-2+bc & 4+4+c^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\therefore 5 + a^2 = 9, 5 + b^2 = 9, 8 + c^2 = 9, \text{ i.e., } a^2 = 4, b^2 = 4, c^2 = 1. \quad \text{Thus } a = 2, b = 2, c = 1.$$

$$\text{Since } A \text{ is orthogonal, } A^{-1} = A' = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix}.$$

## 2.12 (1) VECTORS

Any quantity having  $n$ -components is called a *vector of order  $n$* . Therefore, the coefficients in a linear equation or the elements in a row or column matrix will form a vector. Thus any  $n$  numbers  $x_1, x_2, \dots, x_n$  written in a particular order, constitute a vector  $\mathbf{x}$ .

(2) **Linear dependence.** The vectors  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  are said to be **linearly dependent**, if there exist numbers  $\lambda_1, \lambda_2, \dots, \lambda_r$ , not all zero, such that

$$\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \dots + \lambda_r\mathbf{x}_r = \mathbf{0}.$$

If no such numbers, other than zero, exist, the vectors are said to be **linearly independent**. If we transpose  $\lambda_1\mathbf{x}_1$  to the other side and dividing by  $-\lambda_1$ , we write Eqn. (i) in the form

$$\mathbf{x}_1 = \mu_2\mathbf{x}_2 + \mu_3\mathbf{x}_3 + \dots + \mu_r\mathbf{x}_r.$$

Then the vector  $\mathbf{x}_1$  is said to be a linear combination of the vectors  $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_r$ .

**Example 2.44.** Are the vectors  $\mathbf{x}_1 = (1, 3, 4, 2)$ ,  $\mathbf{x}_2 = (3, -5, 2, 2)$  and  $\mathbf{x}_3 = (2, -1, 3, 2)$  linearly dependent? If so express one of these as a linear combination of the others.

**Solution:** The relation  $\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \lambda_3\mathbf{x}_3 = \mathbf{0}$ .

$$\text{i.e., } \lambda_1(1, 3, 4, 2) + \lambda_2(3, -5, 2, 2) + \lambda_3(2, -1, 3, 2) = \mathbf{0}$$

$$\text{is equivalent to } \lambda_1 + 3\lambda_2 + 2\lambda_3 = 0, 3\lambda_1 - 5\lambda_2 - \lambda_3 = 0,$$

$$4\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0, 2\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$$

As these are satisfied by the values  $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -2$  which are not zero, the given vectors are linearly dependent. Also we have the relation,

$$\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 = \mathbf{0}$$

by means of which any of the given vectors can be expressed as a linear combination of the others.

**Obs.** Applying elementary row operations to the vectors  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ , we see that the matrices

$$A = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3 \end{bmatrix}$$

have the same rank. The rank of  $B$  being 2, the rank of  $A$  is also 2. Moreover  $\mathbf{x}_1, \mathbf{x}_2$  are linearly independent and  $\mathbf{x}_3$  can be expressed as a linear combination of  $\mathbf{x}_1$  and  $\mathbf{x}_2$  [ $\because \mathbf{x}_3 = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$ ]. Similar results will hold for column operations and for any matrix. In general, we have the following results :

If a given matrix has  $r$  linearly independent vectors (rows or columns) and the remaining vectors are linear combinations of these  $r$  vectors, then rank of the matrix is  $r$ . Conversely, if a matrix is of rank  $r$ , it contains  $r$  linearly independent vectors and remaining vectors (if any) can be expressed as a linear combination of these vectors.

## PROBLEMS 2.8

1. Represent each of the transformations

$$x_1 = 3y_1 + 2y_2, y_1 = z_1 + 2z_2 \text{ and } x_2 = -y_1 + 4y_2, y_2 = 3z_1$$

by the use of matrices and find the composite transformation which express  $x_1, x_2$  in terms of  $z_1, z_2$ .

2. If  $\xi = x \cos \alpha - y \sin \alpha$ ,  $\eta = x \sin \alpha + y \cos \alpha$ , write the matrix  $A$  of transformation and prove that  $A^{-1} = A'$ . Hence write the inverse transformation.
3. A transformation from the variables  $x_1, x_2, x_3$  to  $y_1, y_2, y_3$  is given by  $Y = AX$ , and another transformation from  $y_1, y_2, y_3$  to  $z_1, z_2, z_3$  is given by  $Z = BY$ , where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -2 \\ -1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}. \text{ Obtain the transformation from } x_1, x_2, x_3 \text{ to } z_1, z_2, z_3.$$

4. Find the inverse transformation of  $y_1 = x_1 + 2x_2 + 5x_3; y_2 = 2x_1 + 4x_2 + 11x_3; y_3 = -x_2 + 2x_3$ . (V.T.U., 2015)
5. Show that the transformation

$$y_1 = 2x_1 - 2x_2 - x_3, y_2 = -4x_1 + 5x_2 + 3x_3, y_3 = x_1 - x_2 - x_3$$

is regular and find the inverse transformation.

(V.T.U., 2015)

6. Verify that the following matrix is orthogonal :

$$(i) \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

$$(ii) \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}$$

7. Find the values of  $a, b, c$  if  $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$  is orthogonal ?

8. Prove that  $\begin{bmatrix} l & m & n & 0 \\ 0 & 0 & 0 & -1 \\ n & l & -m & 0 \\ -m & n & -l & 0 \end{bmatrix}$  is orthogonal, when  $l = 2/7, m = 3/7, n = 6/7$ .

9. If  $A$  and  $B$  are orthogonal matrices, prove that  $AB$  is also orthogonal.

10. Are the following vectors linearly dependent. If so, find the relation between them :

$$(i) (2, 1, 1), (2, 0, -1), (4, 2, 1).$$

$$(ii) (1, -1, 0), (0, 1, -1), (0, 0, 1).$$

$$(iii) \mathbf{x}_1 = (1, 1, -1, 1), \mathbf{x}_2 = (1, -1, 2, -1), \mathbf{x}_3 = (3, 1, 0, 1)$$

$$(iv) \mathbf{x}_1 = (1, 2, 4), \mathbf{x}_2 = (2, -1, 3), \mathbf{x}_3 = (0, 1, 2), \mathbf{x}_4 = (-3, 7, 2).$$

(P.T.U., 2015 S)

(U.P.T.U., 2013)

## 2.13 (1) EIGEN VALUES

If  $A$  is any square matrix of order  $n$ , we can form the matrix  $A - \lambda I$ , where  $I$  is the  $n$ th order unit matrix. The determinant of this matrix equated to zero,

$$\text{i.e., } |A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

is called the *characteristic equation* of  $A$ . On expanding the determinant, the characteristic equation takes the form

$$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_n = 0,$$

where  $k$ 's are expressible in terms of the elements  $a_{ij}$ . The roots of this equation are called the *eigenvalues* or *latent roots* or *characteristic roots* of the matrix  $A$ .

### (2) Eigen vectors

$$\text{If } X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \text{ then the linear transformation } Y = AX \quad \dots(i)$$

carries the column vector  $X$  into the column vector  $Y$  by means of the square-matrix  $A$ . In practice, it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let  $X$  be such a vector which transforms into  $\lambda X$  by means of the transformation (i).

$$\text{Then } \lambda X = AX \text{ or } AX - \lambda X = 0 \text{ or } [A - \lambda I]X = 0 \quad \dots(ii)$$

This matrix equation represents  $n$  homogeneous linear equations

$$\left. \begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 + \dots + a_{2n}x_n &= 0 \\ \dots & \\ a_{n1}x_1 + a_{n2}x_2 + \dots + (a_{nn} - \lambda)x_n &= 0 \end{aligned} \right\} \quad \dots(iii)$$

which will have a non-trivial solution only if the coefficient matrix is singular, i.e., if  $|A - \lambda I| = 0$ .

This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix  $A$ . It has  $n$  roots and corresponding to each root, the equation (ii) [or (iii)] will have a non-zero solution.

$X = [x_1, x_2, \dots, x_n]',$  which is known as the *eigen vector* or *latent vector*.

**Obs. 1.** Corresponding to  $n$  distinct eigen values, we get  $n$  independent eigen vectors. But when two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

**Obs. 2.** If  $X_i$  is a solution for a eigen value  $\lambda_i$ , then it follows from (ii) that  $cX_i$  is also a solution, where  $c$  is arbitrary constant. Thus the eigen vector corresponding to a eigen value is not unique but may be any one of the vectors  $cX_i$ .

$$\text{Example 2.45. } A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

Determine the Eigen values and Eigen vectors of the following matrix.

**Solution:** Step 1: The characteristic equation of  $A$  is  $|A - \lambda I| = 0$

$$\text{i.e., } \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0, \text{ where } S_1 = 4 + 3 - 3 = 4$$

$$S_2 = \begin{vmatrix} 3 & 2 \\ -4 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} 4 & 6 \\ 1 & 3 \end{vmatrix}$$

$$S_2 = -1 - 6 + 6 = -1$$

$$|A| = 4(-1) - 6(-1) + 6(-1) = -4$$

$$\text{The characteristic equation of } A \text{ is } \lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

**Step 2:** To find the roots of (1) we note that coefficients = 0

$\therefore (\lambda - 1) = 0$  must be the factor of (1)

Now use synthetic division

$$\begin{array}{c|ccccc} 1 & 1 & -4 & -1 & 4 \\ & & +1 & -3 & -4 \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

$$\text{The factors are } (\lambda - 1)(\lambda^2 - 3\lambda - 4) = 0$$

$$(\lambda - 1)(\lambda + 1)(\lambda - 4) = 0$$

Hence  $\lambda = -1, \lambda = 1, \lambda = 4$  are the required Eigen values of  $A$ .

Here  $\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$ , i.e.,  $-1 + 1 + 4 = 4 + 3 - 3$ .

**Step 3:** Let  $\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 4$ , be the Eigen values of  $A$ .

To find the Eigen vectors for the corresponding Eigen values we will consider the matrix equation  $(A - \lambda_i I)X = 0$ , where,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ i.e., } \begin{bmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \dots(2)$$

**Step 4:** For  $\lambda = \lambda_1 = -1$ , Let the corresponding eigen vector be  $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

put  $\lambda = \lambda_1 = -1$  in (2). The matrix equation (2) will become  $\begin{bmatrix} 5 & 6 & 6 \\ 1 & 4 & 2 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  which is homogeneous of

equations. Consider first two equations.

$$5x + 6y + 6z = 0$$

$$x + 4y + 2z = 0$$

Solving by Cramer's rule

$$\frac{x}{-12} = \frac{-y}{4} = \frac{z}{-2}$$

$$\frac{x}{-12} = \frac{-y}{4} = \frac{z}{-2} \text{ or } \frac{x}{-6} = \frac{y}{-2} = \frac{z}{-1} = t$$

$$x = -6t, y = -2t, z = t$$

For

$$X_1 = \begin{bmatrix} -6t \\ -2t \\ t \end{bmatrix} \text{ or } t \begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix}$$

Hence for  $\lambda_1 = -1$ , corresponding Eigen vector  $X_1 = \begin{bmatrix} -6 \\ -2 \\ 1 \end{bmatrix}$ .

Step 5: For  $\lambda = \lambda_2 = 1$  let the corresponding Eigen vector be  $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

put  $\lambda = \lambda_2 = 1$  in (2)

$\therefore$  The matrix equation (2) will become

$$\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = Z$$

which is homogeneous system of equations. Here first two equations are identical.

$\therefore$  Consider last two equations

$$x + 2y + 2z = 0$$

$$-x - 4y - 4z = 0$$

Solving by Cramer's rule,

$$\frac{x}{-2} = \frac{y}{-2} = \frac{z}{-2}$$

$$\frac{x}{0} = \frac{-y}{-2} = \frac{z}{-2} \text{ or, } \frac{x}{0} = \frac{y}{1} = \frac{z}{-1} = t$$

$$x = 0, y = t, z = -t$$

$$\text{For } \lambda_2 = 1, X_2 = \begin{bmatrix} 0 \\ t \\ -t \end{bmatrix} \text{ or } t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Hence for  $\lambda_1 = -1$ , corresponding eigen vectors is  $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ .

Step 6: For  $\lambda = \lambda_3 = 4$ , let the corresponding Eigen vector be  $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

$$\lambda = \lambda_3 = 4 \text{ in (2)}$$

$\therefore$  The matrix equation (2) will become

$$\begin{bmatrix} 0 & 6 & 6 \\ 1 & -1 & 2 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e., } AX = -Z$$

which is Homogeneous system of equations.

$$6y + 6z = 0, \quad y + z = 0 \quad \text{Let } z = t, y = -t$$

By  $R_1$

$$x - y + 2z = 0, \quad x = -3t$$

$$\text{For } \lambda_3 = 4, X_3 = \begin{bmatrix} -3t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Hence for } \lambda_3 = 4, \text{ the Eigen vector is } X_3 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

$$\text{Step 7. } \lambda_1 = -1 \rightarrow X_1 = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}; \lambda_2 = 1 \rightarrow X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_3 = 4 \rightarrow X_3 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

**Example 2.46.** Find the eigen values and eigen vectors of the matrix  $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ .

(Andhra, 2018; Anna, 2016; V.T.U., 2013; Calicut, 2012; C.S.V.T.U., 2012)

**Solution.** The characteristic equation is  $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix}$ , i.e.,  $\lambda^3 - 7\lambda^2 + 36 = 0$

Since,  $\lambda = -2$  satisfies it, we can write this equation as

$$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0 \quad \text{or} \quad (\lambda + 2)(\lambda - 3)(\lambda - 6) = 0.$$

Thus the eigen values of  $A$  are  $\lambda = -2, 3, 6$ .

If  $x, y, z$  be the components of an eigen vector corresponding to the eigen value  $\lambda$ , we have

$$[A - \lambda I] X = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \dots(i)$$

Putting  $\lambda = -2$ , we have  $3x + y + 3z = 0, x + 7y + z = 0, 3x + y + 3z = 0$ .

The first and third equations being the same, we have from the first two

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20} \quad \text{or} \quad \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

Hence the eigen vector is  $(-1, 0, 1)$ . Also every non-zero multiple of this vector is an eigen vector corresponding to  $\lambda = -2$ .

Similarly, the eigen vectors corresponding to  $\lambda = 3$  and  $\lambda = 6$  are the arbitrary non-zero multiples of the vectors  $(1, -1, 1)$  and  $(1, 2, 1)$  which are obtained from (i).

Hence the three eigen vectors may be taken as  $(-1, 0, 1), (1, -1, 1), (1, 2, 1)$ .

**Example 2.47.** Find the eigen values and eigen vectors of the matrix  $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$  (AU, 2023)

**Solution.** The characteristic equation is

$$|A - \lambda I| = 0, \text{ i.e., } \begin{vmatrix} 3 - \lambda & 1 & 4 \\ 0 & 2 - \lambda & 6 \\ 0 & 0 & 5 - \lambda \end{vmatrix} = 0$$

or

$$(3 - \lambda)(2 - \lambda)(5 - \lambda) = 0$$

Thus the eigen values of  $A$  are 2, 3, 5.

If  $x, y, z$  be the components of an eigen vector corresponding to the eigen value  $\lambda$ , we have

$$[A - \lambda I] X = \begin{bmatrix} 3 - \lambda & 1 & 4 \\ 0 & 2 - \lambda & 6 \\ 0 & 0 & 5 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Putting  $\lambda = 2$ , we have  $x + y + 4z = 0, 6z = 0, 3z = 0$ , i.e.,  $x + y = 0$  and  $z = 0$ .

$$\therefore \frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = k_1 \text{ (say)}$$

Hence the eigen vector corresponding to  $\lambda = 2$  is  $k_1(1, -1, 0)$ .

Putting  $\lambda = 3$ , we have  $y + 4z = 0, -y + 6z = 0, 2z = 0$ , i.e.,  $y = 0, z = 0$ .

$$\therefore \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k_2$$

Hence the eigen vector corresponding to  $\lambda = 3$  is  $k_2(1, 0, 0)$ .

Similarly, the eigen vector corresponding to  $\lambda = 5$  is  $k_3(3, 2, 1)$ .

## 2.14 PROPERTIES OF EIGEN VALUES

I. Any square matrix  $A$  and its transpose  $A'$  have the same eigen values.

We have

$$(A - \lambda I)' = A' - \lambda I' = A' - \lambda I$$

$$|(A - \lambda I)'| = |A' - \lambda I|$$

$$|A - \lambda I| = |A' - \lambda I|$$

$$\therefore |A - \lambda I| = 0 \text{ if and only if } |A' - \lambda I| = 0$$

i.e.,  $\lambda$  is an eigen value of  $A$  if and only if it is an eigen value of  $A'$ .

II. The eigen values of a triangular matrix are just the diagonal elements of the matrix.

Let  $A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix}$  be a triangular matrix of order  $n$ .

Then  $|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$ .

$\therefore$  Roots of  $|A - \lambda I| = 0$  are  $\lambda = a_{11}, a_{22}, \dots, a_{nn}$ .

Hence the eigen values of  $A$  are the diagonal elements of  $A$ , i.e.,  $a_{11}, a_{22}, \dots, a_{nn}$ .

Cor. The eigen values of a diagonal matrix are just the diagonal elements of the matrix.

III. The eigen values of an idempotent matrix are either zero or unity.

Let  $A$  be an idempotent matrix so that  $A^2 = A$ . If  $\lambda$  be an eigen value of  $A$ , then there exists a non-zero vector  $X$  such that

$$AX = \lambda X \quad \dots(1)$$

$$\therefore A(AX) = A(\lambda X), \text{ i.e., } A^2X = \lambda(AX)$$

$$AX = \lambda(\lambda X)$$

$$[\because A^2 = A \text{ and } AX = \lambda X]$$

$$\therefore AX = \lambda^2 X$$

$$\dots(2)$$

From (1) and (2), we get  $\lambda^2 X = \lambda X$  or  $(\lambda^2 - \lambda) X = 0$

or

$$\lambda^2 - \lambda = 0 \text{ whence } \lambda = 0 \text{ or } 1.$$

Hence the result.

*IV. The sum of the eigen values of a matrix is its trace i.e. the sum of the elements of the principal diagonal.*  
 [This property will be proved for a matrix of order 3, but the method will be capable of easy extension to matrices of any order.]

Consider the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

so that

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} \\ &= -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \dots \end{aligned}$$

$$\text{If } \lambda_1, \lambda_2, \lambda_3 \text{ be the eigen values of } A, \text{ then } |A - \lambda I| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \\ = -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \dots$$

Equating the right hand sides of Eqns. (ii) and (iii) and comparing coefficients of  $\lambda^2$ , we get

$$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}. \text{ Hence the result.}$$

*V. The product of the eigen values of a matrix A is equal to its determinant.*

Putting  $\lambda = 0$  in Eqn. (iii), we get the result.

*VI. If  $\lambda$  is an eigen value of a matrix A, then  $1/\lambda$  is the eigen value of  $A^{-1}$ .*

If  $X$  be the eigen vector corresponding to  $\lambda$ , then  $AX = \lambda X$

Premultiplying both sides by  $A^{-1}$ , we get  $A^{-1}AX = A^{-1}\lambda X$

i.e.,

$$IX = \lambda A^{-1}X \quad \text{or} \quad X = \lambda(A^{-1}X), \quad \text{i.e.,} \quad A^{-1}X = (1/\lambda)X$$

This being of the same form as Eqn. (i), shows that  $1/\lambda$  is an eigen value of the inverse matrix  $A^{-1}$ .

*VII. If  $\lambda$  is an eigen value of an orthogonal matrix, then  $1/\lambda$  is also its eigen value.*

We know that if  $\lambda$  is an eigen value of a matrix  $A$ , then  $1/\lambda$  is an eigen value of  $A^{-1}$ . [Property V]. Since  $A$  is an orthogonal matrix,  $A^{-1}$  is same as its transpose  $A'$ .

$\therefore 1/\lambda$  is an eigen value of  $A'$ .

But the matrices  $A$  and  $A'$  have the same eigen values, since the determinants  $|A - \lambda I|$  and  $|A' - \lambda I|$  are the same.

Hence  $1/\lambda$  is also an eigen value of  $A$

*VIII. If  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the eigen values of a matrix A, then  $A^m$  has the eigen values  $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$  (m being a positive integer).*

(J.N.T.U., 2015 ; Mumbai, 2016)

Let  $A_i$  be the eigen value of  $A$  and  $X_i$  the corresponding eigen vector. Then

$$AX_i = \lambda_i X_i$$

We have

$$A^2 X_i = A(AX_i) = A(\lambda_i X_i) = \lambda_i(AX_i) = \lambda_i(\lambda_i X_i) = \lambda_i^2 X_i$$

Similarly,

$$A^3 X_i = \lambda_i^3 X_i. \text{ In general, } A^m X_i = \lambda_i^m X_i \text{ which is of the same form as Eqn. (i).}$$

Hence  $\lambda_i^m$  is an eigen value of  $A^m$ .

The corresponding eigen vector is the same  $X_i$ .

## 2.15 CAYLEY-HAMILTON THEOREM\*

*Every square matrix satisfies its own characteristic equation ; i.e., if the characteristic equation for the n<sup>th</sup> order square matrix A is*

$$\begin{aligned} |A - \lambda I| &= (-1)^n \lambda^n + k_1 \lambda^{n-1} + \dots + k_n = 0 \\ &(-1)^n A^n + k_1 A^{n-1} + \dots + k_n = 0. \end{aligned}$$

Let the adjoint of the matrix  $A - \lambda I$  be P. Clearly, the elements of P will be polynomials of the  $(n-1)^{th}$  degree in  $\lambda$ , for the cofactors of the elements in  $|A - \lambda I|$  will be such polynomials.

$\therefore P$  can be split up into a number of matrices, containing terms with the same powers of  $\lambda$ , such that

$$P = P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \dots + P_{n-1} \lambda + P_n$$

where  $P_1, P_2, \dots, P_n$  are all the square matrices of order  $n$  whose elements are functions of the elements of A.

\*William Rowan Hamilton (1805–1865) an Irish mathematician who is known for his work in dynamics.

Since the product of a matrix by its adjoint = determinant of the matrix  $\times$  unit matrix.

$$\therefore [A - \lambda I]P = |A - \lambda I| \times I$$

$\therefore$  by Eqns. (i) and (ii),  $[A - \lambda I] [P_1 \lambda^{n-1} + P_2 \lambda^{n-2} + \dots + P_{n-1} \lambda + P_n]$   
 $= [(-1)^n \lambda^n + k_1 \lambda^{n-1} + \dots + k_{n-1} \lambda + k_n] I.$

Equating the coefficients of various powers of  $\lambda$ , we get

$$\begin{aligned} -P_1 &= (-1)^n I & [\because IP_1 = P_1] \\ AP_1 - P_2 &= k_1 I, \\ AP_2 - P_3 &= k_2 I, \\ \dots & \\ AP_{n-1} - P_n &= k_{n-1} I, \\ AP_n &= k_n I. \end{aligned}$$

Now pre-multiplying the equations by  $A^n, A^{n-1}, \dots, A, I$  respectively and adding, we get

$$(-1)^n A^n + k_1 A^{n-1} + \dots + k_{n-1} A + k_n I = 0, \quad \dots(iii)$$

for the terms on the left cancel in pairs. This proves the theorem.

**Cor. Another method of finding the inverse.**

Multiplying (iii) by  $A^{-1}$ , we get

$$(-1)^n A^{n-1} + k_1 A^{n-2} + \dots + k_{n-1} I + k_n A^{-1} = 0$$

whence

$$A^{-1} = -\frac{1}{k_n} [(-1)^n A^{n-1} + k_1 A^{n-2} + \dots + k_{n-1} I].$$

This result gives the inverse of  $A$  in terms of  $n-1$  powers of  $A$  and is considered as a practical method for the computation of the inverse of the large matrices. As a by-product of the computation, the characteristic equation and the determinant of the matrix are also obtained.

**Example 2.48.** Verify Cayley-Hamilton theorem for the matrix  $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$  and find its inverse.

Also express  $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$  as a linear polynomial in  $A$

(Andhra, 2022; P.T.U., 2015; Delhi, 2012)

**Solution.** The characteristic equation of  $A$  is

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^2 - 4\lambda - 5 = 0 \quad \dots(i)$$

By Cayley-Hamilton theorem,  $A$  must satisfy its characteristic equation (i), so that

$$A^2 - 4A - 5I = 0 \quad \dots(ii)$$

Now

$$\begin{aligned} A^2 - 4A - 5I &= \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \end{aligned}$$

This verifies the theorem.

Multiplying (ii) by  $A^{-1}$ , we get  $A - 4I - 5A^{-1} = 0$

$$\text{or } A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5} \left\{ \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Now dividing the polynomial  $\lambda^5 - 4\lambda^4 - 7\lambda^3 + 11\lambda^2 - \lambda - 10I$  by the polynomial  $\lambda^2 - 4\lambda - 5$ , we obtain

$$\begin{aligned} \lambda^5 - 4\lambda^4 - 7\lambda^3 - \lambda - 10I &= (\lambda^2 - 4\lambda - 5)(\lambda^3 - 2\lambda + 3) + \lambda + 5 \\ &= \lambda + 5 \end{aligned} \quad [\text{By (i)}]$$

Hence  $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A + 5I$ , which is a linear polynomial in  $A$ .

**Example 2.49.** Find the characteristic equation of the matrix  $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$  and hence find its inverse. (C.S.V.T.U., 2015; P.T.U., 2014)

**Solution.** The characteristic equation is  $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0$ , i.e.,  $\lambda^3 - 20\lambda + 8 = 0$ .

By Cayley-Hamilton theorem,  $A^3 - 20A + 8I = 0$ , whence  $A^{-1} = \frac{5}{2}I - \frac{1}{8}A^2$ ,

$$= \frac{5}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8} \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

**Example 2.50.** Find the characteristic equation of the matrix,  $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$  and hence compute  $A^{-1}$ . (U.T.U., 2018)

Also find the matrix represented by

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

(Andhra, 2021, 2018; C.S.V.T.U., 2018)

**Solution.** The characteristic equation of the matrix  $A$  is

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0 \text{ or } [\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0]$$

According to Cayley-Hamilton theorem, we have  $A^3 - 5A^2 + 7A - 3I = 0$

Multiplying Eqn. (i) by  $A^{-1}$ , we get

$$A^2 - 5A + 7I - 3A^{-1} = 0 \text{ or } A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

$$\text{But } A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\text{Hence from Eqn. (ii), } A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ = A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I \\ = A^2 + A + I \end{aligned} \quad [\because A^3 - 5A^2 + 7A - 3I = 0]$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}.$$

**Example 2.51.** Find the Eigen values and Eigen vectors of the following matrix  $A = \begin{bmatrix} 14 & -10 \\ 5 & -1 \end{bmatrix}$ .

**Solution.** Step 1: The characteristic equation of  $A$  is  $|A - \lambda I| = 0$ .

$$\begin{vmatrix} 14-\lambda & -10 \\ 5 & -1-\lambda \end{vmatrix} = 0$$

$$(14-\lambda)(-1-\lambda) + 50 = 0 \text{ or } \lambda^2 - 13\lambda + 36 = 0 \text{ or } (\lambda - 9)(\lambda - 4) = 0$$

or,  $\lambda^2 - S_1\lambda + |A| = 0$ , where  $S_1 = 14 - 1 = 13$  and  $|A| = -14 + 50 = 36$

$$\lambda^2 - 13\lambda + 36 = 0$$

Hence  $\lambda = 9, \lambda = 4$  are the required Eigen values of  $A$ .

$$\lambda_1 + \lambda_2 = a_{11} + a_{22} \text{ i.e., } 9 + 4 = 14 - 1.$$

**Step 2:** Let  $\lambda_1 = 9, \lambda_2 = 4$  be the Eigen values of  $A$ . To find the Eigen vectors for the corresponding Eigen values, we will consider the matrix equation.

$$(A - \lambda I) X = 0$$

Where

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ i.e., } \begin{bmatrix} 14 - \lambda & -10 \\ 5 & -1 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots(1)$$

**Step 3:** For  $\lambda = \lambda_1 = 9$ , Let the corresponding Eigen vector be  $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

Put  $\lambda = \lambda_1 = 9$  in Eqns. (1).

$$\therefore \text{The matrix Eqns. (1) will be become } \begin{bmatrix} 5 & -10 \\ 5 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By  $R_1; 5x - 10y = 0, x - 2y = 0$ , we set  $y = t, x = 2t$

$$\text{For } \lambda_1 = 9, X_1 = \begin{bmatrix} 2t \\ t \end{bmatrix} \text{ or } t \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \text{ Hence for } \lambda_1 = 9 \text{ the eigen vector is } X_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

**Step 4:** For  $\lambda = \lambda_2 = 4$ , Let the corresponding Eigen vector be  $X_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

put  $\lambda = \lambda_2 = 4$  in Eqns. (1)

$$\therefore \text{The matrix Eqns. (1) will become } \begin{bmatrix} 10 & -10 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By  $R_1 \quad 10x - 10y = 0, x - y = 0, x = y$ .

We set  $y = t, x = t$ .

$$\text{For } \lambda_2 = 4, X_2 = \begin{bmatrix} t \\ t \end{bmatrix} \text{ or } t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hence for  $\lambda_2 = 4$ , the Eigen vector is  $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

**Step 5.**  $\lambda_1 = 9 \rightarrow X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ,

$$\lambda_2 = 4 \rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

**Example 2.52.** Find the Eigen values and Eigen vectors of the following matrix  $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ .

**Solution. Step-1 :** The characteristic equation of  $A$  is  $|A - \lambda I| = 0$ .

$$\text{i.e., } \begin{vmatrix} 1 - \lambda & 0 & -1 \\ 1 & 2 - \lambda & 1 \\ 2 & 2 & 3 - \lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

$$\text{Where, } S_1 = 1 + 2 + 3, S_1 = 6$$

$$S_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$$\begin{aligned} &= 4 + 3 + 2 \\ S_2 &= 11 \\ |A| &= |(3-2) - 2(2-5)| = 8 \end{aligned}$$

The characteristic equation of  $A$  is  $\lambda^3 - 8\lambda^2 + 11\lambda - 8 = 0$ .  
Step 2: To find the roots of (1), we note that sum of coefficients = 1.  
 $\lambda - 1 = 0$  must be the factor of (1).

$$\begin{array}{r} 1 \ 1 \ 1 \ -8 \ 11 \ -8 \\ \quad \quad \quad 1 \ -5 \ 6 \\ \hline 1 \ -5 \ 6 \ 0 \end{array}$$

The factors are  $(\lambda - 1)(\lambda^2 - 5\lambda + 8) = 0$   
 $(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$

$\lambda = 1, \lambda = 2, \lambda = 3$  are the required Eigen values of  $A$ ,

$$\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= a_{11} + a_{22} + a_{33} \\ 1 + 2 + 3 &= 1 + 2 + 3 \end{aligned}$$

Step 3: Let  $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$  be the Eigen values of  $A$ .

To find the Eigen vectors for the corresponding Eigen values. We will consider the matrix equation.

$$(A - \lambda I)X = 0, \quad \text{where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Step 4: For Eigen values  $\lambda = \lambda_1 = 1$ , Let the corresponding Eigen vector be  $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

put  $\lambda = \lambda_1 = 1 \therefore$  The matrix Eqn. (2) will become

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e., } AX = Z$$

which is Homogeneous system of equations.

$$\begin{aligned} \text{By } R_1 & \quad 0(x) + 0(y) - 1(z) = 0 \quad \therefore Z = 0 \\ & \quad x + y + z = 0 \quad x + y = 0 \end{aligned}$$

Let  $y = t, x = -t$

Hence for  $\lambda_1 = 1$ , the corresponding Eigen Vector is

$$X_1 = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

By considering the particular solution (vector), we have of  $\lambda_1 = 1$ ,

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

**Step 5:** For Eigen value  $\lambda = \lambda_2 = 2$ . Let the corresponding Eigen vector be  $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ .

put  $\lambda = \lambda_2 = 2$ ,  $\therefore$  The matrix equation (2) will become  $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .

i.e.,  $AX = Z$ , which is homogeneous system of equation by  $R_1 - x - y = 0$

Let  $Z = t$ ,  $\therefore x = -t$

By  $R_3$ ,  $2x + 2y + z = 0$ ,  $y = t/2$

$$\therefore X_2 = \begin{bmatrix} -t \\ t/2 \\ t \end{bmatrix} \text{ or } \begin{bmatrix} -2t \\ t \\ 2t \end{bmatrix} \text{ or } t \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Hence for  $\lambda_2 = 2$ , the corresponding Eigen vector is  $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ .

**Step 6:** For Eigen value  $\lambda = \lambda_3 = 3$ . Let the corresponding.

Eigen vector be  $X_3 = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ . Put  $\lambda = \lambda_3 = 3$ .

$\therefore$  The matrix equation (2) will become

$$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX = Z$ , which is homogeneous system of equations.

By  $R_1$ ,  $-2x - z = 0$  Let  $x = t, z = -2t$

By  $R_3$ ,  $2x + 2y = 0, x + y = 0, y = -t$

$$X_3 = \begin{bmatrix} t \\ -t \\ 2t \end{bmatrix} \text{ or } t = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Hence for  $\lambda_3 = 3$  the corresponding Eigen vector is  $X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

**Step 7:**  $\lambda_1 = 1 \rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \lambda_2 = 2 \rightarrow X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \lambda_3 = 3 \rightarrow X_3 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

## PROBLEMS 2.9

1. Find the sum and product of the eigen values of  $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ .

(Andhra, 2015; C.S.V.T.U., 2014)

2. Find the eigen values and eigen vectors of the matrices :

(a)  $\begin{bmatrix} 4 & 8 \\ 2 & 9 \end{bmatrix}$

- (b) Find the Eigen values and the corresponding Eigen vectors for the following matrices.

$$\begin{bmatrix} 15 & 0 & -15 \\ -3 & 6 & 9 \\ 5 & 0 & -5 \end{bmatrix}$$

(U.P.T.U., 2019)

3. Find the latent roots and the latent vectors of the matrices :

(a)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(P.T.U., 2015)

(b)  $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & 4 \\ 2 & -4 & 3 \end{bmatrix}$

(Delhi, 2015; V.T.U., 2014; Calicut, 2013; U.P.T.U., 2011)

(c)  $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(Andhra, 2022; C.S.V.T.U., 2014; D.T.U., 2011)

(d)  $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

(Andhra, 2021; D.T.U., 2015; P.T.U., 2013 S; Rohtak, 2012)

(e)  $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(Andhra, 2022; JNTUK, 2022, D.T.U., 2016; C.S.V.T.U., 2015;

V.T.U., 2013)

4. If  $\lambda$  be an eigen value of a non-singular matrix  $A$ , show that  $|A|/\lambda$  is an eigen value of the matrix  $\text{adj } A$ .

5. Find the eigen values of  $\text{adj } A$  and of  $A^2 - 2A + I$ , where  $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

6. Two eigen values of the matrix  $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$  are = 1 each. Find the eigen values of  $A^{-1}$ .

7. Show that if  $\lambda_1, \lambda_2, \dots, \lambda_n$  are the latent roots of a matrix  $A$ , then (i)  $A^2$  has the latent roots  $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ . (Anna, 2016)  
 (ii)  $\lambda_i \lambda_j$  are the eigen values of  $A^2$ .  
 (iii)  $1/\lambda_i$  are the eigen values of  $A^{-1}$ .

(Anna, 2012)

8. The eigen vectors of a  $3 \times 3$  matrix  $A$  corresponding to the eigen values 1, 1, 3 are  $[1, 0, -1]^T$ ,  $[0, 1, -1]^T$  and  $[1, 1, 0]^T$  respectively, find the matrix. (U.K.T.U., 2013)

9. Using Cayley-Hamilton theorem, find the inverse of

(i)  $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$

(ii)  $\begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(iii)  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$  (Andhra, 2022)

(iv)  $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  (Andhra, 2023)

10. (i) Matrix A is Non-symmetric. Eigen values are different. Find the Eigen values and the corresponding Eigen vectors for the following matrices.

$$\begin{bmatrix} 9 & -1 & 9 \\ 3 & -1 & 3 \\ -7 & 1 & -7 \end{bmatrix}$$

(Bhopal, 2018)

- (ii) Find the Eigen values and the corresponding Eigen vectors of the following matrices.

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

11. Find the characteristic equation of the matrix  $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ . Show that the equation is satisfied by A and hence obtain the inverse of the given matrix. (Calicut, 2013 ; Bhopal, 2008 ; Anna, 2005)

12. Verify Cayley-Hamilton theorem for the matrix A and find its inverse.

(a)  $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

(Andhra, 2021; Anna, 2013 ; C.S.V.T.U., 2012 ; U.P.T.U., 2012)

(b)  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$

(Anna, 2016 ; Andhra, 2014)

(c)  $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$  (J.N.T.U., 2015)

(d)  $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

(D.T.U., 2016)

13. Using Cayley-Hamilton theorem, find  $A^3$ , if  $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ .

14. If  $A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ , find  $A^4$ .

15. Using Cayley-Hamilton theorem, find  $A^{-2}$ , where  $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ . (Andhra, 2022)

16. If  $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ , evaluate  $A^{-1}$ ,  $A^{-3}$  and  $A^{-5}$ .

17. If  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ , show that  $A^n = A^{n-2} + A^2 - 1$ . Hence find  $A^{50}$ .

18. Matrix A is Non-Symmetric, Eigen values are different. Find the Eigen values and the correspond Eigen vectors for the following matrices.

(CSVTU, 2017)

$$\begin{bmatrix} 1 & 20 & 0 \\ -1 & 7 & 1 \\ 3 & 6 & -2 \end{bmatrix}$$

## 2.16 (1) REDUCTION TO DIAGONAL FORM

If a square matrix  $A$  of order  $n$  has  $n$  linearly independent eigen vectors, then a matrix  $P$  can be found such that  $P^{-1}AP$  is a diagonal matrix.

[This result will be proved for a square matrix of order 3 but the method will be capable of easy extension to matrices of any order.]

Let  $A$  be a square matrix of order 3. Let  $\lambda_1, \lambda_2, \lambda_3$  be its eigen values and

$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$  and  $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$  be the corresponding eigen vectors.

Denoting the square matrix  $[X_1 X_2 X_3] = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$  by  $P$ , we have

$$AP = A[X_1 X_2 X_3] = [AX_1, AX_2, AX_3] = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD, \text{ where } D \text{ is the diagonal matrix}$$

$$\therefore P^{-1}AP = P^{-1}PD = D, \text{ which proves the theorem.}$$

**Obs. 1.** The matrix  $P$  which diagonalises  $A$  is called the **modal matrix** of  $A$  and the resulting diagonal matrix  $D$  is known as the **spectral matrix** of  $A$ .

**2.** The diagonal matrix has the eigen values of  $A$  as its diagonal elements.

**3.** The matrix  $P$ , which diagonalise  $A$ , constitutes the eigen vectors of  $A$ .

(2) **Similarity of matrices.** A square matrix  $\hat{A}$  of order  $n$  is called **similar** to a square matrix  $A$  of order  $n$ ,

$$\hat{A} = P^{-1}AP \text{ for some non-singular } n \times n \text{ matrix } P.$$

This transformation of a matrix  $A$  by a non-singular matrix  $P$  to  $\hat{A}$  is called a **similarity transformation**.

**Obs.** If the matrix  $\hat{A}$  is similar to the matrix  $A$ , then  $\hat{A}$  has the same eigen values as  $A$ .

If  $\mathbf{x}$  is an eigen vector of  $A$ , then  $y = P^{-1}\mathbf{x}$  is an eigen vector of  $\hat{A}$  corresponding to the same eigen value.

(3) **Powers of a matrix.** Diagonalisation of a matrix is quite useful for obtaining powers of a matrix. Let  $A$  be the square matrix. Then a non-singular matrix  $P$  can be found such that

$$D = P^{-1}AP$$

$$D^2 = (P^{-1}AP)(P^{-1}AP) = P^{-1}A^2P$$

Similarly,

$$D^3 = P^{-1}A^3P \text{ and in general, } D^n = P^{-1}A^nP$$

To obtain  $A^n$ , premultiply Eqn. (i) by  $P$  and post-multiply by  $P^{-1}$ .

Then  $PD^nP^{-1} = PP^{-1}A^nPP^{-1} = A^n$  which gives  $A^n$ .

Thus,

$$A^n = PD^nP^{-1} \text{ where, } D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

**Working procedure :**

1. Find the eigen values of the square matrix  $A$ .
2. Find the corresponding eigen vectors and write the modal matrix  $P$ .
3. Find the diagonal matrix  $D$  from  $D = P^{-1}AP$
4. Obtain  $A^n$  from  $A^n = PD^nP^{-1}$ .

**Example 2.53.** Reduce the matrix  $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$  to the diagonal form.

(Andhra, 2023, 2021; V.T.U., 2011; U.T.U., 2010)

**Solution:** The characteristic equation of  $A$  is

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0 \quad \text{or} \quad \lambda^3 - \lambda^2 - 5\lambda + 5 = 0.$$

Solving, we get  $\lambda_1 = 1$ ,  $\lambda_2 = \sqrt{5}$ ,  $\lambda_3 = -\sqrt{5}$  as the eigen values of  $A$ .

When  $\lambda = 1$ , the corresponding eigen vector is given by

$$-2x + 2y - 2z = 0, x + y + z = 0, -x - y - z = 0$$

Solving the first two equations, we get  $\frac{x}{2} = \frac{y}{0} = \frac{z}{-2}$  giving the eigen vector  $(1, 0, -1)$

When  $\lambda = \sqrt{5}$ , the corresponding eigen vector is given by

$$(-1 - \sqrt{5})x + 2y - 2z = 0, x + (2 - \sqrt{5})y + z = 0, -x - y - \sqrt{5}z = 0.$$

Solving 2nd and 3rd equations, we get

$$\frac{x}{6 - 2\sqrt{5}} = \frac{y}{-1 + \sqrt{5}} = \frac{z}{1 - \sqrt{5}} \quad \text{or} \quad \frac{x}{\sqrt{5} - 1} = \frac{y}{1} = \frac{z}{-1}$$

giving the eigen vector  $(\sqrt{5} - 1, 1, -1)$ .

Similarly the eigen vector corresponding to  $\lambda = -\sqrt{5}$ , is  $(\sqrt{5} + 1, -1, 1)$ .

Writing the three eigen vectors as the three columns, we get the transformation (*modal*) matrix as

$$P = \begin{bmatrix} 1 & \sqrt{5} - 1 & \sqrt{5} + 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Hence the diagonal matrix is

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}.$$

**Example 2.54.** Find the matrix  $P$  which transforms the matrix  $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$  to the diagonal form.

Hence calculate  $A^4$ .

**Solution:** The eigen values of  $A$  (found in Ex. 2.43) are  $-2, 3, 6$  and the eigen vectors are  $(-1, 0, 1)$ ,  $(1, -1, 1)$ ,  $(1, 2, 1)$ . Writing these eigen vectors as the three columns, the required transformation matrix (*modal* matrix) is

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{To find } P^{-1}, \quad |P| = \begin{vmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad (\text{say})$$

$$A_1 = -3, B_1 = 2, C_1 = 1, A_2 = 0, B_2 = -2, C_2 = 2, A_3 = 3, B_3 = 2, C_3 = 1$$

$$\text{Also } |P| = a_1 A_1 + b_1 B_1 + c_1 C_1 = 6$$

$$\therefore P^{-1} = \frac{1}{|P|} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Thus

$$D = P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$D^4 = \begin{bmatrix} (-2)^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix} = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix}$$

Hence

$$A^4 = PD^4P^{-1} = \frac{1}{6} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -8 & 0 & 8 \\ 27 & -27 & 27 \\ 216 & 512 & 216 \end{bmatrix} = \begin{bmatrix} 251 & 485 & 235 \\ 485 & 1051 & 485 \\ 235 & 485 & 251 \end{bmatrix}$$

**Example 2.55.** Find  $e^A$  and  $A^4$  if  $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$ .

**Solution:** The characteristic equation of  $A$  is

$$\begin{vmatrix} 3/2 - \lambda & 1/2 \\ 1/2 & 3/2 - \lambda \end{vmatrix} = 0, \quad i.e., (3/2 - \lambda)^2 - 1/4 = 0.$$

$$\lambda^2 - 3\lambda + 2 = 0 \quad \text{whence } \lambda = 1, 2.$$

When  $\lambda = 1$ ,  $[A - \lambda I] X = 0$ , gives

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore x_1 + x_2 = 0$ . If  $x_2 = -1$ ,  $x_1 = 1$ , i.e., the eigen vector is  $[1, -1]'$ .

When  $\lambda = 2$ ,  $[A - \lambda I] X = 0$ , gives  $\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -x_1 + x_2 = 0$ , i.e.,  $x_1 = x_2$

If  $x_2 = 1$ ,  $x_1 = 1$ , i.e., the eigen vector is  $[1, 1]'$

Now  $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  and  $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$$P^{-1} = \frac{\text{adj } P}{|P|} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

If

$$f(A) = e^A, f(D) = e^D = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$$

$\therefore$

$$e^A = P f(D) P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e & e^2 \\ -e & e^2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e + e^2 & -e + e^2 \\ -e + e^2 & e + e^2 \end{bmatrix}$$

Replacing  $e$  by 4, we get

$$4^A = \frac{1}{2} \begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

## 2.17 REDUCTION OF QUADRATIC FORM TO CANONICAL FORM

A homogeneous expression of the second degree in any number of variables is called a *quadratic form*.

For instance, if  $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $X' = [x \ y \ z]$ , then

$$X'AX = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \quad \dots(i)$$

which is a *quadratic form*.

Let  $\lambda_1, \lambda_2, \lambda_3$  be the eigen values of the matrix  $A$  and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

be its corresponding eigen vectors in the normalized form (*i.e.*, each element is divided by square root of sum of the squares of all the three elements in the eigen vector).

$$\text{Then by } \S 2.16(1), P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \text{ where, } P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Hence the *quadratic form (i)* is reduced to a **canonical form** (or **sum of squares form** or **Principal axes form**).

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$$

and  $P$  is the **matrix of transformation** which is an orthogonal matrix.

**Note. Congruent (or orthogonal) transformation.** The diagonal matrix  $D$  and the matrix  $A$  are called *congruent matrices* and the above method of reduction is called **congruent (or orthogonal) transformation**.

Remember that the matrix  $A$  corresponding to the quadratic form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$$\text{is } \begin{bmatrix} \text{coeff. of } x^2 & \frac{1}{2} \text{ coeff. of } xy & \frac{1}{2} \text{ coeff. of } zx \\ \frac{1}{2} \text{ coeff. of } xy & \text{coeff. of } y^2 & \frac{1}{2} \text{ coeff. of } yz \\ \frac{1}{2} \text{ coeff. of } zx & \frac{1}{2} \text{ coeff. of } yz & \text{coeff. of } z^2 \end{bmatrix}, \text{ i.e., } \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.$$

**Example 2.56.** Reduce the quadratic form  $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$  to the canonical form and specify the matrix of transformation.

(JNTUK, 2022; V.T.U., 2016; Andhra, 2014; Anna, 2013; D.T.U., 2013)

**Solution:** The matrix of the given quadratic form is  $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Its characteristic equation is  $|A - \lambda I| = 0$ , *i.e.*,  $\begin{vmatrix} 3 - \lambda & -1 & 1 \\ -1 & 5 - \lambda & -1 \\ 1 & -1 & 3 - \lambda \end{vmatrix} = 0$

which gives  $\lambda = 2, 3, 6$  as its eigen values. Hence the given quadratic form reduces to the canonical form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2, \quad \text{i.e.,} \quad 2x^2 + 3y^2 + 6z^2.$$

To find the matrix of transformation

From  $[A - \lambda I] X = 0$ , we obtain the equations  
 $(3 - \lambda)x - y + z = 0 ; -x + (5 - \lambda)y - z = 0 ; x - y + (3 - \lambda)z = 0$ .

Now corresponding to  $\lambda = 2$ , we get  $x - y + z = 0, -x + 3y - z = 0$ , and  $x - y + z = 0$ ,

whence

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$

$\therefore$  The eigen vector is  $X_1(1, 0, -1)$  and its normalised form is  $(1/\sqrt{2}, 0, -1/\sqrt{2})$ .

Similarly, corresponding to  $\lambda = 3$ , the eigen vector is  $X_2(1, 1, 1)$  and its normalised form is  $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ .

Finally, corresponding to  $\lambda = 6$ , the eigen vector is  $X_3(1, -2, 1)$  and its normalised form is  $(1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6})$ .

Hence the matrix of transformation is  $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$ .

## 2.18 NATURE OF A QUADRATIC FORM

Let  $Q = X'AX$  be a quadratic form in  $n$  variables  $x_1, x_2, \dots, x_n$ .

**Index.** The number of positive terms in its canonical form is called the index of the quadratic form.

**Signature (s) of the quadratic form** is the difference of positive and negative terms in the canonical form.

If the rank of the matrix  $A$  is  $r$  and the index of the quadratic form  $Q$  is  $s$ , then the quadratic form is said to be

- (i) positive definite if  $r = n$  and  $s = n$
- (ii) negative definite if  $r = n$  and  $s = 0$
- (iii) positive semidefinite if  $r < n$  and  $s = r$
- (iv) negative semidefinite if  $r < n$  and  $s = 0$
- (v) indefinite in all other cases.

In other words a real quadratic form  $X'AX$  in a variable is said to be

- (i) positive definite if all the eigen values of  $A > 0$ .
- (ii) negative definite if all the eigen values of  $A < 0$ .
- (iii) positive semidefinite if all the eigen values of  $A \geq 0$  and at least one eigen value = 0.
- (iv) negative semidefinite if all the eigen values of  $A \leq 0$  and at least one eigen value = 0.
- (v) indefinite if some of the eigen values of  $A$  are positive and others negative.

**Example 2.57.** Reduce the quadratic form  $2x_1x_2 + 2x_1x_3 - 2x_2x_3$  to a canonical form by an orthogonal reduction and discuss its nature.

Also find the modal matrix.

(Andhra, 2022)

**Solution:** (i) The matrix of the given quadratic form is  $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Its characteristic equation is  $[A - \lambda I] = 0$ , i.e.,  $\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{bmatrix} = 0$

which gives  $\lambda^3 - 3\lambda + 2 = 0$

Solving, we get  $\lambda = 1, 1, -2$  as the eigen values. Hence the given quadratic form reduces to the canonical form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0, \text{ i.e., } x^2 + y^2 - 2z^2 = 0$$

(ii) Since some of the eigen values of  $A$  are positive and others are negative, the given quadratic form is Indefinite.

(iii) To find the matrix of transformation

From  $[A - \lambda I] X = 0$ , we get the equations

$$-\lambda x + y + z = 0, x - \lambda y + z = 0, x - y - \lambda z = 0$$

When  $\lambda = -2$ , we get  $2x + y + z = 0, x + 2y - z = 0, x - y + 2z = 0$ .

Solving first and second equations, we get

$$\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$$

$\therefore$  The corresponding eigen vector  $X_1 = (-1, 1, 1)$  and its normalised form is  $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

When  $\lambda = 1$ , we get  $-x + y + z = 0, x - y - z = 0, x - y - z = 0$ .

These equations are same. Take  $y = 0$  so that  $x = z$ .

$\therefore$  The corresponding eigen vector  $X_2 = (1, 0, 1)$  and its normalised form is  $(1/\sqrt{2}, 0, 1/\sqrt{2})$

To find the eigen vector  $X_3 = (l, m, n)$  (say)

Since  $X_3$  is orthogonal to  $X_1$ ,  $\therefore -l + m + n = 0$

Since  $X_3$  is orthogonal to  $X_2$ ,  $\therefore l + n = 0$

These equations give  $\frac{l}{1} = \frac{m}{2} = \frac{n}{-1}$ .

$\therefore$  The eigen vector  $X_3 = (1, 2, -1)$  and normalised form is  $(1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6})$ .

Hence the modal matrix is

$$P = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}.$$

**Example 2.58.** Find the characteristic equation for  $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$  and use it to find the simplified expression for  $A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I$ . (V.T.U. 2019)

**Solution.** The characteristic equation of  $A$  is  $|A - \lambda I| = 0$

**Step 1:**

$$\begin{vmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - S_1\lambda + |A| = 0$$

$$|A| = 3 - 8 = -5$$

$\lambda^2 - 4\lambda - 5 = 0$  which is the required characteristic equation of  $A$ . ... (i)

**Step 2:** By Cayley - Hamilton theorem  $A^2 - 4A - 5I = 0$

**Step 3:** To verify it, we require  $A^2$

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**Step 4:** From Eqn. (i)  $A^2 = 4A + 5I$

$$A^3 = 4A^2 + 5A = 4(4A + 5I) + 5A = 21A + 20I$$

$$A^4 = 21A^2 + 20A = 21(4A + 5I) + 20A = 104A + 105I$$

$$A^5 = 104A^2 + 105A = 104(4A + 5I) + 105A = 521A + 520I$$

$$\therefore A^5 + 5A^4 - 6A^3 + 2A^2 - 4A + 7I = 521A + 520I + 5(104A + 105I) - 6(21A + 20I) + 2(4A + 5I) - 4A + 7I$$

$$= 919A + 942I = 919 \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 942 & 0 \\ 0 & 942 \end{bmatrix}$$

$$= \begin{bmatrix} 1861 & 3676 \\ 1838 & 3699 \end{bmatrix}.$$

1. If  $A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  and  $P = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , show that  $P^{-1}AP$  is a diagonal matrix.

2. Show that the linear transformation

$H = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ , where  $\theta = \frac{1}{2} \tan^{-1} \frac{2h}{a-b}$ , changes the matrix

$C = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$  to the diagonal form  $D = HCH'$ .

3. Reduce the following matrices to the diagonal form :

(a)  $A = \begin{bmatrix} 19 & 7 \\ -42 & 16 \end{bmatrix}$  (Andhra, 2022)

(b)  $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

(Andhra, 2023; V.T.U., 2013)

4. If  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ , then obtain the matrix  $P$  such that  $P^{-1}AP$  is a diagonal matrix.

(Andhra, 2022, Delhi, 2013)

5. If  $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$ , find  $A^n$  and  $A^4$ .

6. If  $A = \begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ , then prove that  $3 \tan A = A \tan 3$ .

7. Find the eigen vectors of the matrix  $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$  and hence reduce  $6x^2 + 3y^2 + 3z^2 - 2yz + 4zx - 4xy$  to a 'sum of squares'. Also write the nature of the matrix.

(JNTUK, 2022; Andhra, 2018; Anna, 2013)

(Anna, 2013)

8. Reduce the quadratic form  $2xy + 2yz + 2zx$  into canonical form.

9. (a) Find the eigen values, eigen vectors and the modal of matrix  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ .

(J.N.T.U., 2015)

- (b) Reduce the quadratic form  $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$  to a canonical form.

10. Reduce the following quadratic forms into a 'sum of squares' by an orthogonal transformation and give the matrix of transformation.

(i)  $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ .

(Andhra, 2021)

(ii)  $x^2 + 5y^2 + z^2 + 2yz + 6zx + 2xy$ .

(V.T.U., 2013)

11. Find the index and signature of the quadratic form  $x_1^2 + 2x_2^2 - 3x_3^2$ .

12. Find the nature of the quadratic form  $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6zx$ .

13. Show that the form  $5x_1^2 + 26x_2^2 + 10x_3^2 + 4x_2x_3 + 14x_3x_1 + 6x_1x_2$  is a positive semi-definite and find a non-zero set of values of  $x_1, x_2, x_3$  which make the form zero.

## 2.19 COMPLEX MATRICES

So far, we have considered matrices whose elements were real numbers. The elements of a matrix can however, be complex numbers also.

(1) **Conjugate of a matrix.** If the elements of a matrix  $A = [a_{rs}]$  are complex numbers  $a_{rs} + i\beta_{rs}$ ,  $a_{rs}$  and  $\beta_{rs}$  being real, then the matrix

$$\bar{A} = [\bar{a}_{rs}] = [a_{rs} - i\beta_{rs}] \text{ is called the conjugate matrix of } A.$$

The transpose of a conjugate of a matrix  $A$  is denoted by  $A^*$  or  $A^\theta$ , i.e.,  $(\bar{A})^* = A^*$ .

**(2) Hermitian matrix.** A square matrix  $A$  such that  $A' = \bar{A}$  is said to be a **Hermitian matrix\***. The elements of the leading diagonal of a Hermitian matrix are evidently real, while every other element is the complex conjugate of the element in the transposed position. For instance  $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & -5 \end{bmatrix}$  is a Hermitian matrix, since  $A' = \begin{bmatrix} 2 & 3-4i \\ 3+4i & -5 \end{bmatrix} = \bar{A}$

**(3) Skew-Hermitian matrix.** A square matrix  $A$  such that  $A' = -\bar{A}$  is said to be a **skew-Hermitian matrix**. This implies that the leading diagonal elements of a skew-Hermitian matrix are either all zeros or all purely imaginary.

**Obs.** A Hermitian matrix is a generalisation of a real symmetric matrix as every real symmetric matrix is Hermitian. Similarly, a skew-Hermitian matrix is a generalisation of a real skew-symmetric matrix.

### Properties

I. Any square matrix  $A$  can be written as the sum of a Hermitian and skew-Hermitian matrices.

$$\text{Take } B = \frac{1}{2}(A + \bar{A}') \text{ and } C = \frac{1}{2}(A - \bar{A}')$$

$$\text{Then } B' = \frac{1}{2}(A + \bar{A}') = \frac{1}{2}(A' + \bar{A})$$

$$\text{and } \bar{B} = \frac{1}{2}\overline{(A + \bar{A}')} = \frac{1}{2}(\bar{A} + A') = B'$$

i.e.,  $B$  is a Hermitian matrix.

$$\text{Again, } C' = \frac{1}{2}(A - \bar{A}')' = \frac{1}{2}(A' - \bar{A})$$

$$\text{and } \bar{C} = \frac{1}{2}\overline{(A - \bar{A}')} = \frac{1}{2}(\bar{A} - A') = -C'$$

$\therefore C' = -C$ , i.e.,  $C$  is a skew-Hermitian matrix.

$$\text{Thus, } A = \frac{1}{2}(A + \bar{A}') + \frac{1}{2}(A - \bar{A}') = B + C$$

Hence the result.

II. If  $A$  is a Hermitian matrix, then  $(iA)$  is a skew-Hermitian matrix.

$$\begin{aligned} \text{We have } (i\bar{A})' &= (i\bar{A})' = (-i\bar{A})' = -i\bar{A}' \\ &= -iA \end{aligned}$$

[ $\because \bar{A}' = A$ ]

Thus  $(iA)$  is a skew-Hermitian matrix.

Similarly if  $A$  is a skew-Hermitian matrix then  $(iA)$  is a Hermitian matrix.

III. The eigen values of a Hermitian matrix are real. (see Fig. 2.1)

Let  $\lambda$  be the eigen value and  $X$  the corresponding eigen vector of a Hermitian matrix  $A$ , so that

$$AX = \lambda X$$

$$\bar{X}'AX = \bar{X}'\lambda X = \lambda \bar{X}'X \quad \text{or} \quad \lambda = \bar{X}'AX / \bar{X}'X$$

Since  $\bar{X}'X = \bar{x}_1x_1 + \bar{x}_2x_2 + \dots + \bar{x}_nx_n = |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$  is real and non-zero. Also  $\bar{X}'AX$  is a Hermitian form which is always real.

$\therefore \lambda$ , the eigen value of a Hermitian matrix is real.

IV. The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

Let  $\lambda$  be the eigen value and  $X$  the corresponding eigen vector of a skew-Hermitian matrix  $B$  so that  $BX = \lambda X$ .

$$\therefore \bar{X}'BX = \bar{X}'\lambda X = \lambda \bar{X}'X \quad \text{or} \quad \lambda = \bar{X}'BX / \bar{X}'X$$

Since  $\bar{X}'X$  is real and non-zero. Also  $\bar{X}'BX$  is a skew-Hermitian form which is purely imaginary or zero.

$\therefore \lambda$ , the eigen value of a skew-Hermitian matrix is purely imaginary or zero.

\* Named after the French mathematician Charles Hermite (1822–1901), known for his contributions to algebra and number theory.

**4. Unitary matrix.** A square matrix  $U$  such that  $\bar{U} = U^{-1}$  is called a **unitary matrix**. For a unitary matrix,  $U \cdot U^* = U^* \cdot U = I$ .

This is a generalisation of the orthogonal matrix in the complex field.

### Properties

I. Inverse of a unitary matrix is unitary

If  $U$  is a unitary matrix, then

$$\bar{U} = U^{-1}$$

$$U = \bar{U}^{-1}$$

$$[(U^{-1})^{-1}]' = \bar{U}^{-1}$$

Writing,  $U^{-1} = V$ , we have

$$[V^{-1}]' = \bar{V} \quad \text{or} \quad V^{-1} = \bar{V}$$

Thus  $V (= U^{-1})$  is also unitary.

Cor. Inverse of an orthogonal matrix is orthogonal.

II. Transpose of a unitary matrix is unitary

If  $U$  is a unitary matrix,

$$\bar{U}' = U^{-1}$$

$$(\bar{U}') = U^{-1}$$

$$[(\bar{U}')]' = [U^{-1}]' = [U']^{-1}$$

Writing  $U' = V$ , we have  $\bar{V}' = V^{-1}$

Thus  $V$  (i.e.,  $U'$ ) is also unitary.

Cor. Transpose of an orthogonal matrix is orthogonal.

III. Product of two unitary matrices is a unitary matrix.

If  $U$  and  $V$  are unitary matrices then

$$U' = \bar{U}^{-1}, V' = \bar{V}^{-1}$$

Now,

$$\begin{aligned} (\bar{U}\bar{V})^{-1} &= (\bar{U}\bar{V})^{-1} = \bar{V}^{-1}\bar{U}^{-1} \\ &= V'U' \\ &= (UV)' \end{aligned}$$

(Delhi, 2013)

[∴  $U, V$  are unitary]

Thus,  $UV$  is a unitary matrix.

Cor. Product of two orthogonal matrices is an orthogonal matrix.

IV. The eigen value of a unitary matrix has absolute value 1.

If  $U$  is a unitary matrix then  $UX = \lambda X$

Taking conjugate transpose of Eqn. (1),

$$(\bar{U}\bar{X})' = (\bar{U}\bar{X})' = \bar{X}'\bar{U}' = \bar{X}'\bar{U}^{-1}$$

$$(\bar{U}\bar{X})' = (\bar{\lambda}\bar{X})' = \bar{\lambda}\bar{X}'$$

$$\bar{X}'\bar{U}^{-1} = \bar{\lambda}\bar{X}'$$

Also

i.e.,

Post-multiplying Eqn. (2) by Eqn. (1), we get

$$(\bar{X}'\bar{U}^{-1})(UX) = (\bar{\lambda}\bar{X}') = (\lambda X)$$

$$\bar{X}'(U^{-1}U)X = (\bar{\lambda}\lambda)(\bar{X}'X)$$

$$\bar{X}'X = (\lambda\bar{\lambda})(\bar{X}'X)$$

Thus

$$\lambda\bar{\lambda} = |\lambda|^2 = 1.$$

Hence the result.

Cor. The eigen value of an orthogonal matrix has absolute value 1.

[∴  $U^{-1}U = I$ ]

[∴  $\bar{X}X = I$ ]

**Example 2.59.** If  $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$ , show that  $AA^*$  is a Hermitian matrix, where  $A^*$  is the conjugate transpose of  $A$ .

**Solution.** We have  $A' = \begin{bmatrix} 2+i & -5 \\ 3 & i \\ -1+3i & 4-2i \end{bmatrix}$

and  $A^* = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$

$$\therefore AA^* = \begin{bmatrix} 2-i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix} \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$= \begin{bmatrix} 4-i^2+9+1-9i^2 & -10-5i-3i-10+10i \\ -10+5i+3i-10-10i & 25-i^2+16-4i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & -20+2i \\ -20-2i & 46 \end{bmatrix}, \text{ which is a Hermitian matrix.}$$

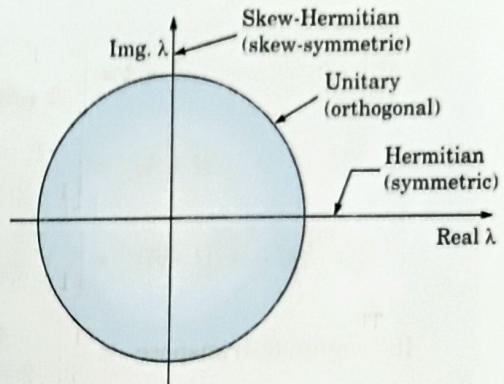


Fig. 2.1. Eigen values of various matrices.

**Example 2.60.** Prove that the matrix  $A = \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \end{bmatrix}$  is unitary and find  $A^{-1}$ .

(Andhra, 2021; D.T.U., 2013)

**Solution.** Conjugate of  $A$ , i.e.,  $\bar{A} = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(-1-i) \\ \frac{1}{2}(1-i) & \frac{1}{2}(1+i) \end{bmatrix}$

$\therefore$  Transpose of  $\bar{A}$ , i.e.,  $A^0 = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix}$

Now  $A^0 \cdot A = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ \frac{1}{2}(-1-i) & \frac{1}{2}(1+i) \end{bmatrix} \begin{bmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) \\ \frac{1}{2}(1+i) & \frac{1}{2}(1-i) \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{4}(1+1) + \frac{1}{4}(1+1) & -\frac{1}{4}(1-i)^2 + \frac{1}{4}(1-i)^2 \\ -\frac{1}{4}(1+i)^2 + \frac{1}{4}(1+i)^2 & \frac{1}{4}(1+1) + \frac{1}{4}(1+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly,  $AA^0 = I$ .

Hence  $A$ , is a unitary matrix.

Also  $A^{-1} = A^0 = \begin{bmatrix} \frac{1}{2}(1-i) & \frac{1}{2}(1-i) \\ -\frac{1}{2}(1+i) & \frac{1}{2}(1+i) \end{bmatrix}$

**Example 2.61.** Given that  $A = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ , show that  $(I-A)(I+A)^{-1}$  is a unitary matrix.  
 (Andhra, 2021; U.P.T.U., 2011; Mumbai, 2011)

**Solution.**  $I + A = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$ ,  $|I + A| = 1 - (-1 - 4) = 6$

$$(I + A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6. \text{ Also } I - A = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$\therefore (I - A)(I + A)^{-1} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \div 6 = \frac{1}{6} \begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Its conjugate-transpose  $= \frac{1}{6} \begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$

$$\therefore \text{Product of Eqns. (i) and (ii)} = \frac{1}{36} \begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix} \begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = I.$$

Hence the result.

### PROBLEMS 2.11

1. Prove that every Hermitian matrix can be written as  $A + iB$ , where  $A$  is real and symmetric and  $B$  is real and skew-symmetric.

2. Show that every square matrix can be uniquely expressed as  $P + iQ$ , where  $P$  and  $Q$  are Hermitian matrices.

3. Show that a Hermitian matrix remains Hermitian when transformed by an orthogonal matrix.

4. Show that the matrix  $\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$  is a unitary matrix, if  $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$ . (U.P.T.U., 2011)

5. Show that  $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$  is a Hermitian matrix.

6. Prove that  $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$  is a skew Hermitian matrix and also unitary. (Andhra, 2016)

7. If  $A = \begin{bmatrix} -1 & 2+i & 5-3i \\ 2-i & 7 & 5i \\ 5+3i & -5i & 2 \end{bmatrix}$ , show that  $A$  is a Hermitian matrix and  $iA$  is a skew-Hermitian matrix.

8. Show that the following matrix is unitary

(a)  $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$  (U.P.T.U., 2013)

(b)  $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$

9. Express  $A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$  as  $P + iQ$  where  $P$  is real and skew-symmetric and  $Q$  is real and symmetric.

10. If  $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$ , where  $a = e^{2i\pi/3}$ , prove that  $S^{-1} = \frac{1}{3}\bar{S}$ .

## 2.20 OBJECTIVE TYPE OF QUESTIONS

### PROBLEMS 2.12

Choose the correct answer or fill up the blanks in the following problems :

1. To multiply a matrix by scalar  $k$ , multiply  
 (a) any row by  $k$       (b) every element by  $k$       (c) any column by  $k$ .
2. If  $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ , then  $A^2$  is  
 (a)  $\begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$       (b)  $\begin{bmatrix} 3^n & (-4)^n \\ 1 & (-1)^n \end{bmatrix}$       (c)  $\begin{bmatrix} 1+3n & 1-4n \\ 1+n & 1-n \end{bmatrix}$       (d)  $\begin{bmatrix} 1+2n & -4n \\ 1+n & 1-2n \end{bmatrix}$
3. The inverse of the matrix  $\begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  is  
 (a)  $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$       (b)  $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$       (c)  $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$       (d)  $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
4. If  $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ , then the determinant  $AB$  has the value  
 (a) 4      (b) 8      (c) 16      (d) 32
5. The system of equations  $x + 2y + z = 9$ ,  $2x + y + 3z = 7$  can be expressed as  
 (a)  $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$   
 (b)  $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$   
 (c)  $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$   
 (d) none of the above.
6. If  $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ , then  $X$  equals  
 (a)  $\begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$       (b)  $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$       (c)  $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$       (d)  $\begin{bmatrix} 3 & -14 \\ 4 & -17 \end{bmatrix}$
7. If  $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ , then  $A(\text{adj } A)$  equals  
 (a)  $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$       (b)  $\begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$       (c)  $\begin{bmatrix} 10 & 1 \\ 1 & 10 \end{bmatrix}$   
 (d) none of the above.
8. If  $3x + 2y + z = 0$ ,  $x + 4y + z = 0$ ,  $2x + y + 4z = 0$ , be a system of equations, then  
 (a) it is inconsistent  
 (b) it has only the trivial solution  $x = 0, y = 0, z = 0$ .  
 (c) it can be reduced to a single equation and so a solution does not exist.  
 (d) determinant of the matrix of coefficients is zero.
9. If  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and  $C = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ , then  
 (a)  $C = A \cos \theta - B \sin \theta$       (b)  $C = A \sin \theta + B \cos \theta$   
 (c)  $C = A \sin \theta - B \cos \theta$       (d)  $C = A \cos \theta + B \sin \theta$ .

10. Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$ , then
- $A$  is row equivalent to  $B$  only when  $\alpha = 2$ ,  $\beta = 3$ , and  $\gamma = 4$
  - $A$  is row equivalent to  $B$  only when  $\alpha \neq 0$ ,  $\beta \neq 0$ , and  $\gamma = 0$
  - $A$  is not row equivalent to  $B$
  - $A$  is row equivalent to  $B$  for all value of  $\alpha, \beta, \gamma$ .
11. If  $A \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$  where  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then  $A$  is
- $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$
  - $\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$
  - $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$
  - $\begin{bmatrix} 2 & 1 \\ -1/2 & -1/2 \end{bmatrix}$
12. Matrix has a value. This statement
- is always true
  - depends upon the matrices
  - is false
13. If  $A$  is a square matrix such that  $AA' = I$ , then value of  $A'A$  is
- $A^2$
  - $I$
  - $A^{-1}$
14. If every minor of order  $r$  of a matrix  $A$  is zero, then rank of  $A$  is
- greater than  $r$
  - equal to  $r$
  - less than or equal to  $r$
  - less than  $r$ .
15. A square matrix  $A$  is called orthogonal if
- $A = A^2$
  - $A' = A^{-1}$
  - $AA^{-1} = I$
16. The rank of matrix  $\begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$  is .....
17. The sum of the eigen values of a matrix is the ..... of the elements of the principal diagonal.
18. The sum and product of the eigen values of the matrix  $\begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix}$  are ..... and ..... respectively.
19. Inverse of  $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$  is  $\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & k \\ 2 & 2 & 5 \end{bmatrix}$  then  $k$  is .....
20. Using Cayley-Hamilton theorem, the value of  $A^4 - 4A^3 - 5A^2 - A + 2I$  when  $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$  is .....
21. If two eigen values of  $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$  are 3 and 15, then the third eigen value is .....
22. A quadratic form is positive semi-definite when .....
23.  $A_{m \times n}$  and  $B_{p \times q}$  are two matrices. When will
- $A \cdot B$  exist
  - $A + B$  exist ?
24. The product of the eigen values of (a)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$  is ..... (b)  $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \dots$  (C.S.V.T.U., 2011)
25. The quadratic form corresponding to the diagonal matrix  $\text{diag } (\lambda_1, \lambda_2, \dots, \lambda_n)$  is
- $x_1^2 + x_2^2 + \dots + x_n^2$
  - $\lambda_1^2 x_1^2 + \lambda_2^2 x_2^2 + \dots + \lambda_n^2 x_n^2$
26. An example of a  $3 \times 3$  matrix of rank one is .....
27. The quadratic form corresponding to the symmetric matrix  $\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix}$  is .....
28. Solving the equations  $x + 2y + 3z = 0$ ,  $3x + 4y + 4z = 0$ ,  $7x + 10y + 12z = 0$ ,  $x = \dots$ ,  $y = \dots$ ,  $z = \dots$



50. Pick the correct answer :

(i) The eigen values of  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$  are

(a) 0, 0, 0

(b) 0, 0, 1

(c) 0, 0, 3

(d) 1, 1, 1. (U.P.T.U., 2012)

(ii) Two square matrices  $A$  and  $B$  are similar if

(a)  $A = B$

(b)  $B = P^{-1}AP$

(c)  $A' = B'$

(d)  $A^{-1} = B^{-1}$ . (V.T.U., 2012)

51. The eigen values of  $A = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$  are the roots of the equation .....

52. A system of linear non-homogeneous equations is consistent, if and only if the rank of coefficient matrix is equal to ..... rank of .....

53. The matrix of the quadratic form  $q = 4x^2 - 2y^2 + z^2 - 2xy + 6zx$  is .....

54. If  $\lambda_1, \lambda_2, \lambda_3$  are the eigen values of a matrix  $A$ , then  $A^3$  has the eigen values .....

55. If  $\lambda$  is an eigen value of a non-singular matrix  $A$ , then the eigen value of  $A^{-1}$  is .....

(U.P.T.U., 2012)

56. Value of  $P$  for which  $A = \begin{bmatrix} 3 & P & P \\ P & 3 & P \\ P & P & 3 \end{bmatrix}$  is of rank 1 is .....

57. The sum of the squares of the eigen values of  $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$  is .....

58. If the rank of a matrix  $A$  is 2, then the rank of  $A'$  is .....

(True or False)

59. The index and signature of the quadratic form  $x_1^2 + 2x_2^2 - 3x_3^2$  are respectively .....

(True or False)

60. The equations  $x + 2y = 1, 7x + 14y = 12$  are consistent.

(True or False)

61. If  $\text{rank}(A) = 2, \text{rank}(B) = 3$ , then  $\text{rank}(AB) = 6$ .

(True or False)

62. Any set of vectors which includes the zero vector is linearly independent.

(True or False)

63. If  $\lambda$  is an eigen value of a symmetric matrix, then  $\lambda$  is real.

(True or False)

64. Every square matrix does not satisfy its own characteristic equation.

(True or False)

65. If  $\lambda$  is an eigen value of an orthogonal matrix, then  $1/\lambda$  is also its eigen value. (J.N.T.U., 2015)

(True or False)

66. If the rank of a matrix  $A$  is 3, then the rank of  $3A^T$  is 1.

(True or False)

67. The vectors  $[1, 1, -1, 1], [1, -1, 2, -1], [3, 1, 0, 1]$  are linearly dependent.

(True or False)

68. The eigen values of a triangular matrix are just the diagonal elements of the matrix. (D.T.U., 2016)

(True or False)

69. Inverse of a unitary matrix is a unitary matrix.

(True or False)

70.  $A$  is a non-zero column matrix and  $B$  is a non-zero row matrix, then rank of  $AB$  is one.

(True or False)

71. The sum of the eigen values of  $A$  equals to the trace of  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ .

(True or False)

72. The matrix,  $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$  is Skew-Hermitian and unitary.

(True or False) (U.P.T.U., 2012)

73. The quadratic form corresponding to the matrix  $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{bmatrix}$  is .....

(Anna, 2013)