RV College of Engineering®

DEPARTMENT OF MATHEMATICS

UNIT-I

ELEMENTARY LINEAR ALGEBRA

Topic Learning Objectives:

With Completion of this unit, students will be able to:

- Understand the fundamentals of the rank of a matrix, echelon form of the matrix.
- Check the consistency of system of linear equations.
- Apply elementary operations to solve homogeneous and non-homogeneous linear equations.
- Solve system of linear equations by using Gauss elimination, Gauss Jordan and Gauss Seidel methods.
- Find eigenvalues and eigenvectors of a given square matrix
- Properties of eigenvalues and eigenvectors.
- Apply power method to obtain largest eigenvalue.

Introduction:

Linear algebra is a vital area of mathematics that explores the theory and application of vectors, vector spaces, matrices, and linear transformations. It is a fundamental tool used in many fields of science and engineering to model and solve problems involving multiple variables and relationships. Linear algebra provides a powerful framework for understanding and manipulating complex systems by representing them in a simpler, more structured way. It is particularly essential for engineers as it forms the mathematical backbone for concepts like stress and strain analysis, fluid dynamics, control systems and data analysis.

In aerospace engineering, linear algebra is crucial for aerodynamics, flight dynamics, and spacecraft trajectory analysis. It is used to solve the governing equations of fluid flow around an aircraft's wings (aerodynamics) and to analyze the stability and control of an aircraft during flight. Linear algebra is also essential for satellite orbital mechanics, where it helps in calculating and predicting satellite positions and trajectories. In structural analysis, it is used to study the stress distribution on an aircraft's fuselage and wings.

Chemical engineers rely on linear algebra to model and solve complex reaction kinetics, mass transfer, and process control problems. It is used to analyze systems of differential equations that describe chemical reactions and to perform material and energy balances for chemical processes. In process optimization, linear programming, a key application of linear algebra, is used to find the most efficient way to operate a chemical plant, such as maximizing product yield while minimizing costs.

Industrial engineers use linear algebra for optimization, supply chain management, and resource allocation. Linear programming is a cornerstone of industrial engineering, used to solve problems like minimizing transportation costs, scheduling production runs, and optimizing factory layouts. It is also used in data analytics and statistical modeling to analyze large datasets, improve efficiency, and make informed decisions about resource management and quality control.

Mechanical engineers use linear algebra to solve problems in statics, dynamics and solid mechanics. For instance, in finite element analysis (FEA), a method used to predict how a product reacts to forces, heat, or vibrations, linear algebra is used to solve large systems of equations that describe the behavior of a structure. In robotics, it is used to calculate the position and orientation of robot arms (kinematics). Furthermore, in vibrational analysis, linear algebra helps in determining the natural frequencies and mode shapes of a system, which is crucial for designing stable and safe mechanical components.

Basic concepts and definitions:

Definition: A matrix is a rectangular arrangement of numbers in rows and columns represented by

$$\begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix}.$$

If a matrix has m rows and n columns, then it is said to be of order $m \times n$ (read as "m by n" matrix). The elements a_{ij} of a matrix are identified by double subscript notation ij, where i denotes the row and j denotes the column.

Elementary row transformations:

These are operations that are carried out on the rows of a given matrix. The following operations constitute the three row transformations.

1. Interchange of i^{th} and j^{th} rows: R_{ij} or $R_i \leftrightarrow R_j$
2. Multiplying each element of the i^{th} row by a non-zero constant k : $R_i' \to kR_i$.
3. Adding a constant k multiple of j^{th} row to i^{th} row:

   $R_i{}' \to R_i + kR_j$.

Example:

$$\text{Let } A = \begin{bmatrix} 2 & 8 & 6 & 7 \\ 1 & 5 & 6 & 7 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

Performing row operation R_{13} (i.e., $R_1 \leftrightarrow R_3$) of A, results

$$A \sim \begin{bmatrix} 3 & 1 & 4 & 2 \\ 1 & 5 & 6 & 7 \\ 2 & 8 & 6 & 7 \end{bmatrix}$$

Next, by performing row operation on R_2 of A by multiplying $\frac{1}{3}$ to it

(i.e., $R_2' \to \frac{1}{3}R_2$), we get.

$$A \sim \begin{bmatrix} 3 & 1 & 4 & 2 \\ \frac{1}{3} & \frac{5}{3} & 2 & \frac{7}{3} \\ 2 & 8 & 6 & 7 \end{bmatrix}$$

The row operation $R_2' \to R_2 + (-2)\, R_3$ on A gives the matrix

$$A \sim \begin{bmatrix} 3 & 1 & 4 & 2 \\ -3 & -16 & -12 & -14 \\ 2 & 8 & 6 & 7 \end{bmatrix}$$

Equivalent matrices:

Two matrices are said to be equivalent if one of these can be obtained by applying a finite number of successive elementary row/column transformations to the other.

Rank of a matrix:

A matrix is said to be of rank $\boldsymbol{r}$ if

1. It has at least one non-zero minor of order $\boldsymbol{r}$ and
2. Every minor of order higher than $\boldsymbol{r}$ vanishes.

The rank of a matrix A is denoted by $\rho(A)$.

Echelon form or Row Echelon Form:

A non-zero matrix A is said be in echelon form, if

i) leading entry in each row is non zero
ii) all the entries below the leading entry are zero.
iii) Number of zeros before the leading entry in any row exceeds number of zeros before the leading entry in the previous row.
iv) all the zero rows are below non zero rows.

If the leading entry is 1 and the other entries in this column are zero then the echelon form is called row reduced echelon form.

$$\textbf{Example: } B = \begin{bmatrix} 1 & 3 & 1 & 5 & 0 \\ 0 & 1 & 5 & 1 & 5 \\ 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ is in row-echelon form.}$$

Note: The rank of an echelon matrix is the number of non-zero rows in it. i.e., $\rho(B) = 3$.

<u>Problems:</u>

1. Determine the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

Solution: Since the given matrix A is of order 3×4, $\rho(A) \leq 3$.

Consider all the minors of order 3.

They are

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -2 & 1 \\ 3 & 0 & 4 \end{vmatrix} = 0, \begin{vmatrix} 2 & 3 & -2 \\ -2 & 1 & 3 \\ 0 & 4 & 1 \end{vmatrix} = 0, \begin{vmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 3 & 4 & 1 \end{vmatrix} = 0, \begin{vmatrix} 1 & 2 & -2 \\ 2 & -2 & 3 \\ 3 & 0 & 1 \end{vmatrix} = 0.$$

Therefore, the rank is less than3.

Now consider minors of order 2.

$$\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} = -6 \neq 0$$

(considered only one minor of order 2 since its non-zero).

Therefore, $\rho(A) = 2$.

Note: The method of finding the rank of a matrix by using the definition of the rank of a matrix is very tedious. However, it would be better to apply the definition to find the rank, after bringing the given matrix to echelon form.

2. Determine the rank of the matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.

Solution: The rank of the matrix can be obtained by reducing it to row echelon form.

Given matrix $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

Perform R_{12} i.e., interchanging row 1 and row 2 we get

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_2' \to R_2 - 2\mathrm{R}_1,\ \mathrm{R}_3' \to R_3 - 3\mathrm{R}_1,\ \mathrm{R}_4' \to R_4 - 6\mathrm{R}_1$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} R_2' \to -4\mathrm{R}_2,\ R_3' \to 5\mathrm{R}_3$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 20 & 12 & 28 \\ 0 & 20 & 45 & 50 \\ 0 & 9 & 12 & 17 \end{bmatrix} R_2' \to \mathrm{R}_3 - \mathrm{R}_2$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 20 & 12 & 28 \\ 0 & 0 & 33 & 22 \\ 0 & 9 & 12 & 17 \end{bmatrix} R_2' \to \frac{1}{4}\mathrm{R}_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 9 & 12 & 17 \end{bmatrix} R_4' \to R_4 - \frac{9}{5}R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix} R_4' \to R_4 - 5R_3$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33/5 & 22/5 \end{bmatrix} R_4' \to 5R_4$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{bmatrix} R_4' \to R_4 - R_3$$

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As there are no elements below the fourth diagonal element the process is complete.

$\rho(A) =$ Rank of A =number of non-zero rows= 3.

3. In operations research and resource allocation problems, the rank of a constraint matrix shows the number of independent constraints in an optimization model. Consider the following constraint matrix:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Reduce this matrix to echelon form and hence find the rank.

Solution: Given matrix is

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Perform R_{12} i.e., interchanging row 1 and row 2 we get

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Perform $R_3' \to R_3 - 2R_1, R_4' \to R_4 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & -1 & 3 \end{bmatrix}$$

Perform $R_3' \to R_3 + R_2, R_4' \to R_4 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

Perform $R_4' \to R_4 - R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Above matrix is in the echelon from, therefore rank of matrix $A = 3$ (no. of non-zero rows).

4. In mechanical systems, the number of independent equilibrium equations often determines the stability of structures or mechanisms. Consider the system represented by the following coefficient matrix:

$$B = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}.$$

Using the elementary transformations find the rank of the matrix.

Solution: Given matrix is

$$B = \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$$

$$R_2' \to R_2 + 2\mathrm{R}_1,\ \mathrm{R}_3' \to R_3 + 3\mathrm{R}_1,\ \mathrm{R}_4' \to R_4 + 5\mathrm{R}_1$$

$$B \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & -2 & 14 & -4 \\ 0 & -2 & 14 & -4 \end{bmatrix}$$

$R_3' \to R_3 - 2\mathrm{R}_2$, $\mathrm{R}_4' \to R_4 - 2\mathrm{R}_2$, we get

$$B \sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank of matrix $B = 2$.

5. In a chemical reaction network, the stoichiometric coefficients of reactants and products are often represented in the form of a matrix. The rank of this matrix indicates the number of independent chemical reactions in the system (i.e., reactions that cannot be obtained by linear combinations of others). The stoichiometric coefficient matrix for a set of reactions is given as:

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Reduce this matrix to echelon form and hence find the rank of the matrix, which corresponds to the number of independent reactions in the network.

Solution: $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}, R_4' \to R_4 - (R_3 + R_2 + R_1)$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_2' \to R_2 - 2R_1,\ R_3' \to R_3 - 3R_1$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_2 \leftrightarrow R_3$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) =$ Rank of A =number of non-zero rows= 3.

The rank = 3 means there are 3 linearly independent reactions in this chemical network. One reaction is redundant (can be expressed as a combination of the others.

Exercise:

Objective type of questions:

1. If A is a 3×4 matrix then rank of A cannot exceed _____________.
2. Rank of the matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is ___________.
3. Rank of identity matrix of order 5 is ____________.
4. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is____________________.
5. If every minor of order r of a matrix A is zero, then rank of A is___________.
6. If the rank of the transpose matrix A is 3 then the rank of matrix A is __________.
7. If $A = \begin{bmatrix} 2 & -1 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix}$ then $\rho(A) =$ ___________________.
8. Rank of singular matrix of order 5 is___________________.

9. If $= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, $\rho(A) =$ __________.

Let $A = [a_{ij}]$ be the matrix, $a_{ij} = k \neq 0$, for every i, j then rank of A is________.

Descriptive type of questions:

1. Reduce the following matrix into echelon form $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \\ 1 & 3 & 4 & 2 \end{bmatrix}$.
2. The stiffness coefficients of a 3-DOF (degree of freedom) mechanical system are given by the following matrix K=$\begin{bmatrix} 2 & -1 & 13 \\ 1 & 4 & -2 \\ 5 & 2 & 4 \end{bmatrix}$. Determine the rank of the stiffness matrix K.
3. Find the rank of the matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 4 \\ -4 & 3 & 1 \\ -6 & -4 & -8 \end{bmatrix}$.
4. Determine the rank of the following matrices by transforming them into row echelon form.

$$(i) \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} (ii) \begin{bmatrix} 8 & 2 & 1 & 6 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & 1 & 1 & 4 \end{bmatrix} (iii) \begin{bmatrix} 1 & -2 & -1 & 3 \\ 2 & 5 & -4 & 7 \\ -1 & -2 & -1 & 2 \\ 3 & 3 & -5 & 10 \end{bmatrix}$$

$$(iv) \begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 3 & 7 & -1 \\ 1 & 9 & 16 & -13 \end{bmatrix}$$

5. Find the value of b in the matrix $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$ given that its rank is 2.
6. In the study of aerospace structures, stiffness and flexibility matrices often depend on design parameters. Consider the following matrix, which represents the coefficients of a structural system.

   $K = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & K \\ 1 & 4 & 10 & K^2 \end{bmatrix}$. Determine the values of K for which the rank of the matrix K is (a) 3 (system fully constrained) (b) 2 (system has one degree of redundancy)

   Answers:

   Objective type of questions:

   1) 3 2) 3 3) 5 4) 2 5) <r 6) 3 7) 2 8) ≤ 4 9) 2. 10) 1

Descriptive type of questions:

1) $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ 2) 2 3) 3 4) (i) 2 (ii) 3 (iii) 3 (iv) 2 5) b = 2

6) Rank=3 if $K \neq 1, K \neq= 2$, Rank=2 if $K = 1$ or $K = 2$.

Applications:

1. One useful application of calculating the rank of a matrix is in the computation of solutions of a system of linear equations.
2. In the area of source enumeration.
3. In the classification of an image.
4. If we view a square matrix as specifying a transformation, the rank tells you about the dimension of the image.

Solution of simultaneous linear equations:

A linear system of simultaneous equations of m equations in n unknowns can be expressed as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \quad \text{-----------------}(\mathbf{1})$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n$$

The above system in the matrix equivalent form can be expressed as $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & . & . & . & a_{mn} \end{bmatrix}_{m\times n}$$ is called the coefficient matrix,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$ is called the matrix of unknowns and $$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$ is column matrix of constants.

If all b_i's for $i = 1 \cdots m$ are zero i.e.,$b_1 = \mathrm{b}_2 = \cdots = \mathrm{b}_m = 0$, then the system is said to be homogenous and is said to be non-homogeneous if at least one b_i is non-zero.

Augmented matrix:

Suppose we form a matrix of the form[A:B] by appending to A an extra column whose elements are columns of B i.e,

$$[\mathrm{A:B}] = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n}: b_1 \\ a_{21} & a_{22} & . & . & . & a_{2n}: b_2 \\ . & . & . & . & . & . \\ a_{m1} & a_{m2} & . & . & . & a_{mn}: b_m \end{bmatrix}$$

is called the augmented matrix associated with the system and is denoted by $[A|B]$ or [A:B].

Solution of simultaneous linear equations:

A system of linear equations such as $(\mathbf{1})$ may or may not have a solution. However, existence of solution is guaranteed only if the system is homogeneous.

Solution of non-homogeneous system of linear equations:

A non-homogeneous system of equations $\boldsymbol{AX = B}$ is **consistent** if $\boldsymbol{r}$, the rank of coefficient matrix $\boldsymbol{A}$ is equal to $\boldsymbol{r'}$, the rank of the augmented matrix $[\mathbf{A:B}]$ and has **unique** solution if $\boldsymbol{r = r' = n}$, the number of unknowns. If $\boldsymbol{r = r' < n}$then the system possesses **infinite** number of solutions. The system is **inconsistent** if $\boldsymbol{r \neq r'}$.

Solution of homogeneous system of linear equations:

A homogeneous system of linear equations $\boldsymbol{AX} = \mathbf{0}$ is always consistent as for such a system, $\boldsymbol{A} =$ [A:0] and hence rank of coefficient matrix is equal to the rank of the augmented matrix. If rank of $\boldsymbol{A}$ is equal to the number of unknowns $\boldsymbol{n}$, the system has trivial solution i.e., all unknowns $\boldsymbol{x_1, x_2, \cdots, x_n}$ are zero. A non-trivial solution exists to a system if and only if $|\boldsymbol{A}| = \mathbf{0}$ and hence the system has infinite number of solution. The following block diagram illustrates connection between rank of a matrix and consistence of that system.

NON-HOMOGENOUS SYSTEM

AX=B, where $B \neq 0$

Consistent if $\rho(\mathrm{A}) = \rho[\mathrm{A{:}B}]$

Inconsistent if $\rho(\mathrm{A}) \neq \rho[\mathrm{A{:}B}]$

Unique solution: if $\rho(\mathrm{A}) = \mathrm{n}$

Infinite number of solutions: if $\rho(\mathrm{A}) = \mathrm{r} < \mathrm{n}$

$(n-r)$ free variables in the solution.

HOMOGENOUS SYSTEM

AX = B, where B = 0

Always consistent

Unique solution (trivial): X = 0 if $\rho(A) = n$

Infinite number of solutions: if $\rho(\mathrm{A}) = \mathrm{r} < \mathrm{n}$, then $(n-r)$ free variables in the solution. If $\mathbf{A}$ is a square matrix, then $|\mathbf{A}| = \mathbf{0}$.

The following examples illustrate the above concepts.

1. Test the consistency of the following system and solve

$$2x_1 - x_2 + 3x_3 = 1$$

$$-3x_1 + 4x_2 - 5x_3 = 0$$

$$x_1 + 3x_2 - 6x_3 = 0$$

Solution: Consider the augmented matrix

$$[A:B]=\begin{bmatrix}2 & -1 & 3 & :1\\ -3 & 4 & -5 & :0\\ 1 & 3 & -6 & :0\end{bmatrix}$$

$$R_2' \to R_2+(3/2)R_1,\ R_3' \to R_3-(1/2)R_1$$

$$[A:B]\sim\begin{bmatrix}2 & -1 & 3 & :1\\ 0 & 5 & -1 & :3\\ 0 & 7 & -15 & :-1\end{bmatrix}$$

$$R_3' \to R_3-(7/5)R_2$$

$$[A:B]\sim\begin{bmatrix}2 & -1 & 3 & :1\\ 0 & 5 & -1 & :3\\ 0 & 0 & -68 & :-26\end{bmatrix}$$

$\rho(A)=\rho([A:B])=3$= number of unknowns.

Thus, the system of linear equations is consistent and possesses a unique solution.

To find the unknowns, consider the rows of [A:B] in the last step in terms of its equivalent equations,

$$2x_1-x_2+3x_3=1$$

$$5x_2-x_3=3$$

$$-68x_3=-26$$

Here, we make use of **back substitution** in order to find the unknowns by considering, last equation to findx_3, next second to find x_2 and finally first equation to findx_1.

Therefore, from last equation we obtainx_3

i.e.,$-68x_3=-26 \Rightarrow x_3=\frac{13}{34}$.

Next, from second equation we findx_2,

i.e.,$5x_2-x_3=3 \Rightarrow x_2=\frac{3+x_3}{5} \Rightarrow x_2=\frac{23}{34}$.

Finally, to find the$x_1$we make use first equation

$$2x_1-x_2+3x_3=1 \Rightarrow x_1=\frac{1}{2}(1+x_2-3x_3)=\frac{1}{2}\left(1+\frac{23}{34}-3\frac{13}{34}\right) \Rightarrow x_1=\frac{9}{34}.$$

There the solution is given by

$$x_1=\frac{9}{34}, x_2=\frac{23}{34}, x_3=\frac{13}{34}.$$

2. Check the following system of equations for consistency and solve, if consistent.

$$x+2y+2z=1,\ 2x+y+z=2,\ 3x+2y+2z=3,\ y+z=0$$

Solution : The augmented matrix is given by

$$[\text{A:B}] = \begin{bmatrix} 1 & 2 & 2 & :1 \\ 2 & 1 & 1 & :2 \\ 3 & 2 & 2 & :3 \\ 0 & 1 & 1 & :0 \end{bmatrix}$$

$$R_2' \to R_2 - 2\text{R}_1, \text{R}_3' \to R_3 - 3\text{R}_1$$

$$[\text{A:B}] \sim \begin{bmatrix} 1 & 2 & 2 & :1 \\ 0 & -3 & -3 & :0 \\ 0 & -4 & -4 & :0 \\ 0 & 1 & 1 & :0 \end{bmatrix}$$

$$R_2' \to (-1/3)\ \text{R}_2, \text{R}_3' \to (-1/4)\ \text{R}_3$$

$$[\text{A:B}] \sim \begin{bmatrix} 1 & 2 & 2 & :1 \\ 0 & 1 & 1 & :0 \\ 0 & 1 & 1 & :0 \\ 0 & 1 & 1 & :0 \end{bmatrix}$$

$$R_3' \to R_3 - R_2, R_4' \to R_4 - R_2$$

$$[\text{A:B}] \sim \begin{bmatrix} 1 & 2 & 2 & :1 \\ 0 & 1 & 1 & :0 \\ 0 & 0 & 0 & :0 \\ 0 & 0 & 0 & :0 \end{bmatrix}$$

$\rho(A) = \rho([\text{A:B}]) = 2 < 3$, number of unknowns.

Thus, the given system is consistent and possesses infinite number of solutions by assigning arbitrary values to $(n - r) = 3 - 2 = 1$free variable.

$$\Rightarrow \quad x + 2y + 2z = 0$$

$$y + z = 0.$$

Three unknowns are here, we should take $z$as the free variable and let $z = k$ (arbitrarily value).

From second equation, $y + z = 0 \Rightarrow y = -z = -k.$

Finally, from first equation,

$$x + 2y + 2z = 1 \Rightarrow x = 1 - 2y - 2z = 1 - 2(-k) - 2k$$

$$\Rightarrow x = 1.$$

Therefore, the solution is given by:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -k \\ k \end{bmatrix}.$$

3. Show that the following system of equations is not consistent.

$$x + 2y + 3z = 6, 3x - y + z = 4, 2x + 2y - z = -3, -x + y + 2z = 5$$

Solution: Consider the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 2 & 3 & :6 \\ 3 & -1 & 1 & :4 \\ 2 & 2 & -1 & :-3 \\ -1 & 1 & 2 & :5 \end{bmatrix}$$

$$R_2' \to R_2 - 3R_1, R_3' \to R_3 - 2R_1, R_4' \to R_4 + R_1$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -8 & :-14 \\ 0 & -2 & -7 & :-15 \\ 0 & 3 & 5 & :11 \end{bmatrix}$$

$$R_3' \to R_3 - (2/7)R_2, R_4' \to R_4 + (3/7)R_2$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -8 & :-14 \\ 0 & 0 & -33 & :-77 \\ 0 & 0 & 11 & :35 \end{bmatrix}$$

$$R_4' \to R_4 + R_3$$

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -8 & :-14 \\ 0 & 0 & -33 & :-77 \\ 0 & 0 & 0 & :28 \end{bmatrix}$$

$$\rho(A) = 3 \text{ and } \rho([A:B]) = 4$$

$$\rho(A) \neq \rho([A:B]).$$

Therefore, the given system is inconsistent and it has no solution.

4. Check the following system of equations for consistency and solve, if consistent.

$$x + y - 2z = 3, 2x - 3y + z = -4,$$

$$3x - 2y - z = -1, y - z = 2.$$

Solution: Consider the augmented matrix,

$$[A:B] = \begin{bmatrix} 1 & 1 & -2 & :3 \\ 2 & -3 & 1 & :-4 \\ 3 & -2 & -1 & :-1 \\ 0 & 1 & -1 & :2 \end{bmatrix}$$

$$R_2' \to R_2 - 2R_1, R_3' \to R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & :3 \\ 0 & -5 & 5 & :-10 \\ 0 & -5 & 5 & :-10 \\ 0 & 1 & -1 & :2 \end{bmatrix}$$

$$R_3' \to R_3 - R_2, \quad R_4' \to R_4 + (1/5)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -2 & : & 3 \\ 0 & -5 & 5 & : & -10 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

We see that $\rho(A) = \rho([A:B]) = 2 < 3$ number of unknowns.

Thus, the equations are consistent and possesses infinite number of solutions with $(n - r) = 3 - 2 = 1$ free variable.

The corresponding equations are:

$$x + y - 2z = 3$$

$$-5y + 5z = -10.$$

Let us choose $z = k$ (arbitrary constant).

Then from second equation:

i.e., $-5y + 5z = -10 \Rightarrow y = -\frac{1}{5}(-10 - 5z) = -\frac{1}{5}(-10 - 5k)$

$$= 2 + k$$

From first equation: $x + y - 2z = 3 \Rightarrow x = 3 - y + 2z$

$$= 3 - (2 + k) + 2k \Rightarrow x = 1 + k.$$

Therefore, the solution is given by:

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 + k \\ 2 + k \\ k \end{bmatrix}.$$

5. Determine whether the following system of equations possesses a non-trivial solution

$$x_1 + 2x_2 - x_3 = 0$$

$$4x_1 - x_2 + x_3 = 0$$

$$5x_1 + x_2 - 2x_3 = 0.$$

Solution:

Since the given system of linear equations is homogeneous for which the rank of the coefficient matrix is same as rank of the augmented matrix, therefore we consider only the coefficient matrix and reduce it to row echelon form and solve the system as we did in the case of non-homogeneous system.

Method 1: Consider the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 4 & -1 & 1 \\ 5 & 1 & -2 \end{bmatrix}$$

$$R_2' \to R_2 - 4R_1, R_3' \to R_3 - 5R_1$$

$$\sim\begin{bmatrix}1 & 2 & -1\\0 & -9 & 5\\0 & -9 & 3\end{bmatrix}$$

$$R_3' \to R_3 - R_2$$

$$\sim\begin{bmatrix}1 & 2 & -1\\0 & -9 & 5\\0 & 0 & -2\end{bmatrix}$$

Thus, the given system is equivalent to

$$x_1 + 2x_2 - x_3 = 0,$$
$$-9x_2 + 5x_3 = 0,$$
$$-2x_3 = 0.$$

From last equation,

$$\Rightarrow x_3 = 0.$$

From second equation, we have $x_2 = x_3 = 0$.

Finally, from first equation $x_1 = -2x_2 + x_3 = 0$.

i.e., $X = \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$, which is a trivial solution.

Hence the system does not possess a non-trivial solution.

<u>Method 2</u>: We have

$|A| = \begin{vmatrix}1 & 2 & -1\\4 & -1 & 1\\5 & 1 & -2\end{vmatrix}$ = 1(2-1)-2(8-5)-1(4+5)=14 $\neq 0$.

Hence, the system does not possess non-trivial solutions.

6. Find the values of λ for which the following system has a solution and solve it in each case.

$$x + y + z = 1, x + 2y + 4z = \lambda, x + 4y + 10z = \lambda^2$$

Solution: The augmented matrix is given by

$$[A:B] = \begin{bmatrix}1 & 1 & 1 & :1\\1 & 2 & 4 & :\lambda\\1 & 4 & 10 & :\lambda^2\end{bmatrix}$$

$$R_2' \to R_2 - R_1, R_3' \to R_3 - R_1$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 1\\0 & 1 & 3 & : & \lambda - 1\\0 & 3 & 9 & : & \lambda^2 - 1\end{bmatrix}$$

$$R_3' \to R_3 - 3R_2$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 1\\0 & 1 & 3 & : & \lambda-1\\0 & 0 & 0 & : & \lambda^2-3\lambda+2\end{bmatrix}$$

We observe that $\rho(A)=2$ and $\rho([A:B])$ will be equal to 2 if and only if $\lambda^2-3\lambda+2=0$,

i.e., for $\lambda=1$or $\lambda=2$.

⇒System will possess a solution if $\lambda=1$or 2 and in both the cases the system will have infinite number of solutions as $\rho(A)=\rho([A:B])=2<3$ number of unknowns and hence 1 free variable.

Let us consider these cases one by one.

Case (i): When $\lambda=1$, the reduced system gives

$$x+y+z=1$$

$$y+3z=1-1=0.$$

Let $z=k$ be arbitrary and from second equation we have

$$y=-3z=-3k.$$

From first equation, we have

$x=1-y-z=1-(-3k)-k=1+2k.$

Therefore, the solution in this case is $X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1+2k\\-3k\\k\end{bmatrix}$

Case (ii): When $\lambda=2$, the reduced system gives,

$$x+y+z=1$$

$$y+3z=2-1=1$$

Let z = a, then y = 1-3a and x = 1-y-z = 1-1+3a-a = 2a

Where a is an arbitrary constant.

Therefore, the solution in this case is $X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2a\\1-3a\\a\end{bmatrix}$

7. Find the values of λ and μ for which the system $x+y+z=6$,

$x+2y+3z=10, x+2y+z=\mu$ has

(i) a unique solution (ii) infinitely many solutions (iii) no solution.

Solution: Consider the augmented matrix

$$[A:B]=\begin{bmatrix}1 & 1 & 1 & :6\\1 & 2 & 3 & :10\\1 & 2 & \lambda & :\mu\end{bmatrix}$$

$$R_2'\to R_2-R_1, R_3'\to R_3-R_1$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 1 & \lambda-1 & : & \mu-6\end{bmatrix}$$

$$R_3' \to R_3 - R_2$$

$$\sim\begin{bmatrix}1 & 1 & 1 & : & 6\\ 0 & 1 & 2 & : & 4\\ 0 & 0 & \lambda-3 & : & \mu-10\end{bmatrix}.$$

Here we observe that

a) If $\lambda - 3 = 0$ and $\mu - 10 \neq 0$i.e., $\lambda = 3$ and $\mu \neq 10$, then the system will be inconsistent and possesses no solution.

b) If $\lambda - 3 = 0$ and $\mu - 10 = 0$i.e., $\lambda = 3$ and $\mu = 10$ the system will reduce to

$$x + y + z = 6$$

$$y + 2z = 4.$$

Hence in this case the system possesses infinite solutions.

c) If $\lambda - 3 \neq 0$ i.e., $\lambda \neq 3$, the system will possess a unique solution, irrespective of the value of μ.

8. Show that the equations

$$-2x + y + z = a, x - 2y + z = b, x + y - 2z = c$$

have a solution only if $a + b + c = 0$. Find all possible solutions when $a = 1, b = 1, c = -2$.

Solution: Consider the augmented matrix

$$[A:B] = \begin{bmatrix}-2 & 1 & 1 & :a\\ 1 & -2 & 1 & :b\\ 1 & 1 & -2 & :c\end{bmatrix}$$

$$R_2' \to 2R_2 + R_1, R_3' \to 2R_3 + R_1$$

$$\sim\begin{bmatrix}-2 & 1 & 1 & : & a\\ 0 & -3 & 3 & : & 2b+a\\ 0 & 3 & -3 & : & 2c+a\end{bmatrix}$$

$$R_3' \to R_3 + R_2$$

$$\sim\begin{bmatrix}-2 & 1 & 1 & : & a\\ 0 & -3 & 3 & : & 2b+a\\ 0 & 0 & 0 & : & 2a+2b+2c\end{bmatrix}$$

i.e.,$-2x + y + z = a; -3y + 3z = 2b + a; 0 = 2a + 2b + 2c$ (1)

The above system of equations will be consistent if

$2a + 2b + 2c - 0$ i.e., if $a + b + c = 0$.

To find the solution when $a = 1, b = 1, c = -2$.

The reduced equations in (1) give,

$$-2x + y + z = 1 \text{and} - 3y + 3z = 3.$$

Let $z = k$. then $y = k - 1$ and $x = k - 1$.

9. Find the value of λ such that the system of equations

$3x - y + 4z = 3, x + 2y - 3z = -2, 6x + 5y + \lambda z = -3$ have

i) unique solution, ii) infinite solutions.

Solution:

$$[A:B] = \begin{bmatrix} 3 & -1 & 4 & : 3 \\ 1 & 2 & -3 & :-2 \\ 6 & 5 & \lambda & :-3 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & :-2 \\ 3 & -1 & 4 & : 3 \\ 6 & 5 & \lambda & :-3 \end{bmatrix}, R'_2 \to R_2 - 3R_1, R'_3 \to R_3 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & :-2 \\ 0 & -7 & 13 & : 9 \\ 0 & -7 & \lambda + 18 & : 9 \end{bmatrix} R'_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -3 & :-2 \\ 0 & -7 & 13 & : 9 \\ 0 & 0 & \lambda + 5 & : 0 \end{bmatrix}$$

Case(i) if $\lambda \neq -5$

$$\rho(A) = \rho([A:B]) = 3 = n(\text{number of unknowns})$$

The system is consistent and have unique solution.

Case(ii) if $\lambda = -5$

$$\rho(A) = \rho([A:B]) = 2$$

The system is consistent and have infinite number of solutions.

Exercise:

1. Solve the system of equations

   $x + 2y + 3z = 0;\ 3x + 4y + 4z = 0;\ 7x + 10y + 12z = 0.$

2. Show that the system of equations

   $x + y + z = 4; 2x + y - z = 1; x - y + 2z = 2$ is consistent and hence find the solution.

3. Test for consistency and hence solve $x + y + z = 9; 2x - 3y + 4z = 13; 3x + 4y + 5z = 40.$

4. Find the value of λ for which the system $x + y + z = 1,\ x + 2y + 4z = \lambda,$

   $x + 4y + 10z = \lambda^2$ has a solution. Solve it in each case.

5. Find the values of μ and λ for which the system

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$2x + 3y + \mu z = \lambda$$

has (i) no solution (ii) unique solution (iii) Infinitely many solutions.

6. If the following system,

$ax + by + cz = 0; bx + cy + az = 0; cx + ay + bz = 0$ has non-trivial solution then prove that $a + b + c = 0 \ \ or \ \ a = b = c$.

Answers:

1) Trivial solution

2) $x = \frac{3}{7}, y = \frac{13}{7}, z = \frac{12}{7}$

3) $x = 1, y = 3, z = 5$

4) $(i)\ \lambda = 1, x = 2k + 1, y = -3k, z = k$

4) $(ii)\ \lambda = 2, x = 2k, y = 1 - 3k, z = k$

5) $(i)\ \lambda = 5,\ (ii)\ \lambda \neq 5\ (iii)\ \lambda = 5, \mu = 9$

6) i) $\lambda \neq -5, \lambda = -5$.

Gauss elimination method:

In this method the unknowns are eliminated successively and the system is reduced to upper triangular system from which the unknowns are found by back substitution.

For the linear system $AX = B$ with 'n' unknown and 'm' equations.

1. Solve the following system by Gauss elimination method

$$x + y - z = 0; 2x - 3y + z = -1; x + y + 3z = 12; y + z = 5.$$

Solution : The augmented matrix is given by

$$[A:B] = \begin{bmatrix} 1 & 1 & -1 & :0 \\ 2 & -3 & 1 & :-1 \\ 1 & 1 & 3 & :12 \\ 0 & 1 & 1 & :5 \end{bmatrix}$$

$$R_2' \to R_2 - 2R_1, R_3' \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & :0 \\ 0 & -5 & 3 & :-1 \\ 0 & 0 & 4 & :12 \\ 0 & 1 & 1 & :1 \end{bmatrix}$$

$$R_4' \to R_4 + (1/5)R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & :0 \\ 0 & -5 & 3 & :-1 \\ 0 & 0 & 4 & :12 \\ 0 & 0 & 8 & :24 \end{bmatrix}$$

$R_4' \to R_4$-2R$_3$

$$\sim \begin{bmatrix} 1 & 1 & -1 & :0 \\ 0 & -5 & 3 & :-1 \\ 0 & 0 & 4 & :12 \\ 0 & 0 & 0 & :0 \end{bmatrix}.$$

By back substitution

$$4z = 12 \Rightarrow z = 3,$$

$$5y + 3z = -1 \Rightarrow y = 2,$$

$$x + y - z = 0 \Rightarrow x = 1.$$

2. The temperature u_1, u_2, u_3 of a metal plate follow the linear equations

$$2u_1 - u_2 + 3u_3 = 1$$

$$-3u_1 + 4u_2 - 5u_3 = 0$$

$$u_1 + 3u_2 - 6u_3 = 0.$$

Compute u_1, u_2, u_3 using Gauss elimination method.

Solution: Consider the augmented matrix

$$[A:B] = \begin{bmatrix} 2 & -1 & 3 & :1 \\ -3 & 4 & -5 & :0 \\ 1 & 3 & -6 & :0 \end{bmatrix}$$

$R_2' \to R_2 + (3/2)R_1$, R$_3' \to R_3 - (1/2)R_1$

$$\sim \begin{bmatrix} 2 & -1 & 3 & :1 \\ 0 & 5 & -1 & :3 \\ 0 & 7 & -15 & :-1 \end{bmatrix}$$

$R_3' \to R_3 - (7/5)R_2$

$$\sim \begin{bmatrix} 2 & -1 & 3 & :1 \\ 0 & 5 & -1 & :3 \\ 0 & 0 & -68 & :-26 \end{bmatrix}$$

$$2u_1 - u_2 + 3u_3 = 1$$

$$\Rightarrow 5u_2 - u_3 = 3$$

$$-68u_3 = -26.$$

By back substitution the solution is given by

$$u_3 = \frac{13}{34}, u_2 = \frac{23}{34}, \ u_1 = \frac{9}{34}.$$

Gauss-Jordan elimination method:

The procedure for Gauss-Jordan elimination is as follows:

1. Find the left most column that is not all zeros.
2. Interchange the top row (if necessary) with another row to bring a non-zero entry to the top of the column.
3. If the top entry is a, then multiply the top row by $\frac{1}{a}$ to form a leading 1 in that row.
4. Add multiples of this row to the other rows so that all other rows have a zero in this column.
5. Cover up the top row and go back to step 1, considering only the rows below this one (until step 4). Continue until the matrix is in row reduced echelon form.

1. Solve for x, y and z in the following system of equations:

$$2y - 3z = 2, x + z = 3, x - y + 3z = 1$$

Solution: Let the augmented matrix of the given system is

$$[A:B] = \begin{bmatrix} 0 & 2 & -3: & 2 \\ 1 & 0 & 1: & 3 \\ 1 & -1 & 3: & 1 \end{bmatrix}$$

Interchange first and second row (to make top left entry non-zero)

$$\sim \begin{bmatrix} 1 & 0 & 1: & 3 \\ 0 & 2 & -3: & 2 \\ 1 & -1 & 3: & 1 \end{bmatrix}$$

$$R_3' \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1: & 3 \\ 0 & 2 & -3: & 2 \\ 0 & -1 & 2: & -2 \end{bmatrix}$$

$$R_3' \to R_3 + (1/2)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1: & 3 \\ 0 & 2 & -3: & 2 \\ 0 & 0 & 1/2: & -1 \end{bmatrix}$$

$$R_3' \to 2\mathrm{R}_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1: & 3 \\ 0 & 2 & -3: & 2 \\ 0 & 0 & 1: & -2 \end{bmatrix}$$

$$R_1' \to R_1 - R_3, \mathrm{R}_2' \to R_2 + 3\mathrm{R}_3$$

$$\sim\begin{bmatrix}1 & 0 & 0: & 5\\ 0 & 2 & 0: & -4\\ 0 & 0 & 1: & -2\end{bmatrix}$$

$$R_2' \to (1/2)R_2$$

$$\sim\begin{bmatrix}1 & 0 & 0: & 5\\ 0 & 1 & 0: & -2\\ 0 & 0 & 1: & -2\end{bmatrix}$$

This is now in reduced row reduced echelon form.

$$\therefore x = 5, y = -2 \text{ and } z = -2.$$

2. Use Gauss-Jordon method to find the inverse of the following matrix:

$$A = \begin{bmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{bmatrix}$$

Solution: Consider the matrix $[A\,|\,I]$and apply elementary row operations on both A and I until A gets transformed to I.

i.e, $[A\,|\,I] = \left[\begin{array}{ccc|ccc}1 & 1 & 3 & 1 & 0 & 0\\ 1 & 3 & -3 & 0 & 1 & 0\\ -2 & -4 & -4 & 0 & 0 & 1\end{array}\right]$

$$R_2' \to R_2 - R_1 \text{ and } R_3' \to R_3 + 2R_1$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 1 & 3 & 1 & 0 & 0\\ 0 & 2 & -6 & -1 & 1 & 0\\ 0 & -2 & 2 & 2 & 0 & 1\end{array}\right]$$

$$R_3' \to R_3 + R_2$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 1 & 3 & 1 & 0 & 0\\ 0 & 2 & -6 & -1 & 1 & 0\\ 0 & 0 & -4 & 1 & 1 & 1\end{array}\right]$$

$$R_2' \to \frac{1}{2}R_2 \text{ and } R_3' \to \frac{-1}{4}R_3$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 1 & 3 & 1 & 0 & 0\\ 0 & 1 & -3 & \frac{-1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4}\end{array}\right]$$

$$R_1' \to R_1 - 3R_3 \text{ and } R_2' \to R_2 + 3R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{7}{4} & \frac{3}{4} & \frac{3}{4} \\ 0 & 1 & 0 & \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right]$$

$R_1' \to R_1 - R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

Exercise:

1. Gauss-Jordon method aims in reducing the coefficient matrix of a system of equations to ______________ matrix.
2. Gauss elimination method aims in reducing the coefficient matrix of a system of equations to______________ matrix.
3. Solve using Gauss-elimination and Gauss-Jordan method:

i. $2x + y - z = 8;\ -3x - y + 2z = -11;\ -2x + y + 2z = -3.$

ii. $x + y + z = 3;\ x + 2y + 2z = 5;\ 3x + 4y + 5z = 12.$

iii. $-5x + 9y - z = 3, -2x - 5y + 2z = -1, -2x + 6y + 4z = -8.$

4. A chemical engineer is designing a network of three interconnected continuous stirred-tank reactors (CSTRs). A steady-state mass balance for a specific component (e.g., a reactant or product) is performed for each reactor. The mass flow rate of this component into each reactor must equal the mass flow rate out. This gives a system of linear equations relating the unknown concentrations in each reactor (c_1, c_2, c_3) to the known inlet concentrations and flow rates as follows.

Reactor 1: $3c_1 - c_2$=100, Reactor 2: $-c_1 + 4c_2 - 2c_3 = 50$, Reactor 3: $-2c_2 + 5c_3 = 200$. Solve for c_1, c_2 and c_3 using Gauss elimination method.

iv. Using the loop current method on a circuit, the following equations are obtained $7i_1 - 4i_2 = 12; -4i_1 + 12i_2 - 6i_3 = 0; -6i_2 + 14i_3 = 0$. Solve for i_1, i_2, i_3 using Gauss elimination method.

v. Find the inverse of the given matrix A by Gauss-Jordon method.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}.$$

Answers:

1) $x = 2, y = 3, z = -1$ 2) $x = 1, y = 1, z = 1$

3) $x = -0.7247, y = -0.2844, z = -1.9357$

4) $i_1 = 2.2628, i_2 = 0.96, i_3 = 0.4114$

5) $A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$

Diagonally dominant form:

A system of n linear equations in n unknowns given by

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \qquad (1)$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \qquad (2)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

is said to be in diagonally dominant form if in equation (1), $|a_{11}|$is greater the sum of the absolute values of the remaining coefficients; in (2), $|a_{22}|$ is greater than the sum of the absolute of the remaining coefficients and so on.

$$|a_{11}| > |a_{12}| + |a_{13}| + \ldots\ldots\ldots |a_{1n}|$$

$$|a_{22}| > |a_{21}| + |a_{23}| + \ldots\ldots\ldots |a_{2n}|$$

$$|a_{nn}| > |a_{n1}| + |a_{n2}| + \ldots\ldots\ldots |a_{n(n-1)}|.$$

Gauss-Seidel Method:

The Gauss-Seidel method is an iterative method that can be used to solve a system of n linear equations in 'n' unknowns. A starting or an initial solution is first assumed, which is then improved through successive iteration. A convergence to the actual solution is ensured if the given system of equations is arranged in the diagonally dominant form. The following example illustrates the working procedure of this method.

1. Solve the following system of equations using Gauss-Seidel method.

$$6x + 15y + 2z = 72, x + y + 54z = 110, 27x + 6y - z = 85.$$

Solution: In the above equations, we have

$$|15| > |6| + |2|, |54| > |1| + |1| \ \& \ |27| > |6| + |-1|.$$

Hence the equations are arranged in the diagonally dominant form as:

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$x + y + 54z = 110.$

The first equation is used to determine *x* and is therefore rewritten as

$x = \frac{85-6y+z}{27}.$ (1)

The second equation is used to determine *y* and is rewritten as

$y = \frac{72-6x-2z}{15}.$ (2)

The third equation used to determine *z* is rearranged as

$z = \frac{110-x-y}{54}.$ (3)

Equations (1), (2), (3) are used to find sequentially *x*, *y* and *z* in each of the iterations.

Let the initial approximation as $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0.$

First iteration:

$$x^{(1)} = \frac{1}{27}[85 - 0 + 0] = 3.148$$

$$y^{(1)} = \frac{1}{15}[72 - 6(3.1481) - 0] = 3.5407$$

$$z^{(1)} = \frac{1}{54}[110 - 3.1481 - 3.5407] = 1.9132.$$

Note that in finding $y^{(1)}$ the latest value $x^{(1)} = 3.1481$ is used and not $x^{(0)} = 0$. Similarly, in finding $z^{(1)}$, $x^{(1)} = 3.148, y^{(1)} = 3.5407$ are used.

Second iteration:

$$x^{(2)} = \frac{1}{27}[85 - 6(3.5407) + 1.9132] = 2.4322$$

$$y^{(2)} = \frac{1}{15}[72 - 6(2.4322) - 2(1.9132] = 3.5720$$

$$z^{(2)} = \frac{1}{54}[110 - 2.4322 - 3.5720] = 1.9258.$$

Third iteration:

$$x^{(3)} = \frac{1}{27}[85 - 6(3.5720) + 1.9258] = 2.4257$$

$$y^{(3)} = \frac{1}{15}[72 - 6(2.4257) - 2(1.9258)] = 3.5729$$

$$z^{(3)} = \frac{1}{54}[110 - 2.4257 - 3.5729] = 1.9259.$$

Therefore, $[x, y, z] = [2.4257, 3.5729, 1.9259]$.

Fourth iteration:

$$x^{(4)} = \frac{1}{27}[85 - 6(3.5729) + 1.9259] = 2.4255$$

$$y^{(4)} = \frac{1}{15}[72 - 6(2.4255) - 2(1.9259)] = 3.5730$$

$$z^{(4)} = \frac{1}{54}[110 - 2.4255 - 3.5730] = 1.9259.$$

Since the solutions in 3^{rd} and 4^{th} iterations agree upto 3 places of decimals, the solution can be taken as

$$[x, y, z] = [2.4255, 3.5730, 1.9259].$$

6. Solve the following system of equations using Gauss-Seidel method.

$$8x - 3y + 2z = 20, 4x + 11y + z = 33, 6x + 3y - 12z = 35.$$

Perform four iterations.

Sol: The first equation is used to determine *x* and is therefore rewritten as

$x = \frac{20+3y-2z}{8}$. (1)

The second equation is used to determine *y* and is rewritten as

$y = \frac{33-4x+z}{11}$. (2)

The third equation used to determine *z* is rearranged as

$z = \frac{35-6x-3y}{12}$. (3)

Equations (1), (2), (3) are used to find sequentially *x*, *y* and *z* in each of the iterations.

Starting solution: Let us choose $[x, y, z] = [0,0,0]$ as the starting solution.

First iteration:

$$x^{(1)} = \frac{1}{8}[20 + 0 - 0] = 2.5$$

$$y^{(1)} = \frac{1}{11}[33 - 4(2.5) - 0] = 2.0909$$

$$z^{(1)} = \frac{1}{12}[35 - 6(2.5 - 3(2.0909)] = 1.1439.$$

Second iteration:

$$x^{(2)} = \frac{1}{8}\left[20 + 3y^{(1)} - 2z^{(1)}\right] = 2.9981$$

$$y^{(2)} = \frac{1}{11}\left[33 - 4x^{(2)} + z^{(1)}\right] = 2.0137$$

$$z^{(2)} = \frac{1}{12}\left[35 - 6x^{(2)} - 3y^{(2)}\right] = 0.9141.$$

Third iteration:

$$x^{(3)} = \frac{1}{8}\left[20 + 3y^{(2)} - 2z^{(2)}\right] = 3.0266$$

$$y^{(3)} = \frac{1}{11}\left[33 - 4x^{(3)} + z^{(2)}\right] = 1.9825$$

$$z^{(3)} = \frac{1}{12}\left[35 - 6x^{(3)} - 3y^{(3)}\right] = 0.9077.$$

Fourth iteration:

$$x^{(4)} = \frac{1}{8}\left[20 + 3y^{(3)} - 2z^{(3)}\right] = 3.0165$$

$$y^{(4)} = \frac{1}{11}\left[33 - 4x^{(3)} + z^{(2)}\right] = 1.9859$$

$$z^{(4)} = \frac{1}{12}\left[35 - 6x^{(3)} - 3y^{(3)}\right] = 0.9118.$$

After four iterations the solutions can be taken as

$$[x, y, z] = [3.0165, 1.9859, 0.9118].$$

Exercise:

1. Use Gauss -Seidel method to solve the system

   $10x + y + z = 12;\ 2x + 10y + z = 13; 2x + 2y + 10z = 14.$

2. Solve the following system of equations using Gauss -Seidel method starting with $(2, 2, -1)$ as initial approximation:

   $5x_1 - x_2 + x_3 = 10; x_1 + x_2 + 5x_3 = -1; 2x_1 + 4x_2 = 12.$

3. Solve the following system of equations by Gauss -Seidel method

   $10x - 2y - z - w = 3\,, -2x + 10y - z - u = 15, -x - y + 10z - 2u = 27$

   $-x - y - 2z + 10u = -9.$

4. A manufacturing plant produces three different products (P1, P2, P3) that each require a certain amount of time on three different machines (M_1, M_2, M_3). The plant manager needs to determine the optimal number of units of each product to manufacture (x_1, x_2, x_3) to fully utilize the available machine time each day without overbooking. The total available time on each machine is fixed.

   The resource constraints can be modelled as a system of linear equations, where each equation represents the total time used on a specific machine as follows:

$$5x_1 + 2x_2 + 3x_3 = 1000$$
$$2x_1 + 6x_2 + x_3 = 1200$$
$$x_1 + 3x_2 + 8x_3 = 1500.$$

   Solve for the unknown production quantities x_1, x_2 and x_3.

Answers:

1) $x = 0.9995, y = 1.0001, z = 1.0000$
2) $x_1 = 2.5555, x_2 = 1.7222, z = -1.0555$
3)) $x = 0.9971, y = 1.9985, z = 2.9988, u = -0.0006$

EIGENVALUES & EIGENVECTORS:

Eigenvalues are a fundamental concept in linear algebra with extensive applications across various engineering disciplines. They represent the natural frequencies or modes of a system. Finding the largest and smallest eigenvalues helps engineers understand a system's most dominant and least significant behaviours, respectively.

In aerospace engineering, eigenvalues are critical for **stability and control systems analysis**. The eigenvalues of an aircraft's or spacecraft's flight dynamics model determine its stability. A system is stable if all its eigenvalues have negative real parts. The **dominant eigenvalues** (often the largest in magnitude) dictate the most critical modes of motion. For example, an aircraft has a short-period mode and a phugoid mode. The short-period mode, characterized by a large negative real part of its eigenvalue, is a rapidly damped oscillation, while the phugoid mode has a small negative real part, resulting in a slow, long-term oscillation. Engineers analyze these eigenvalues to ensure the aircraft is stable and controllable. The smallest and largest eigenvalues are used to design autopilots and stability augmentation systems that adjust flight controls to keep the system stable and responsive.

Chemical engineers use eigenvalues to analyse the **stability and dynamics** of chemical processes and reaction systems. The eigenvalues of a system's Jacobian matrix (which describes how small changes in one part of the system affect other parts) determine its stability. If all eigenvalues have negative real parts, the system is **stable** and will return to equilibrium after a disturbance. If even a single eigenvalue has a positive real part, the system is **unstable** and can lead to runaway reactions or oscillations. The **largest eigenvalue** (or the one with the largest positive real part) indicates the system's rate of instability, helping engineers design control systems to stabilize the process. For instance, in a continuous stirred-tank reactor (CSTR), analyzing the eigenvalues of the system can help determine if the temperature or concentration will remain stable or spiral out of control.

In mechanical engineering, the largest and smallest eigenvalues are crucial for **vibration analysis** and **structural dynamics**. The eigenvalues represent the natural frequencies at which a structure or machine will vibrate.

Definition:

Let A be an $n \times n$ matrix. A number λ is said to be an eigenvalue of A if there exists a non-zero solution vector X of the system of equations.

$AX = \lambda X$or $AX - \lambda IX = 0$, Ibeing an identity matrix of order n.

The non-zero solution vector X is said to be an eigenvector corresponding to the eigenvalueλ. The word "eigenvalue" is a combination of German and English terms Eigenwerte (Proper value). Eigenvalues and eigenvectors are also called characteristic values and characteristic vectors, respectively.

The characteristic equation of the matrix A is defined to be

$|A - \lambda I| = 0$.

Properties of Eigen values and Eigen vectors:

The following are a few important properties of eigenvalues.

(1) Square matrix of order n will always possess n eigenvalues which may be distinct or not.

(2) If all the $n$eigenvalues of A are distinct, then there exists exactly one eigenvector corresponding to each one of them.

(3) For eigenvalues that are repeated, there may be exactly one or more than one eigenvector.

(4) A^{-1}exists if and only if 0is not an eigenvalue of A.

(5) If λ is the eigenvalue of A, then $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.

(6) The same characteristic vector cannot correspond to two distinct eigenvalues.

Example:

Let $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$ and $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Consider the product

$$AX = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5+4 \\ 2-4 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 2 \end{bmatrix} = (-1)\,X$$

The above is of the form $AX = \lambda X$, where $\lambda = -1$and $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Hence $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue -1.

<u>Determination of eigenvalue and eigenvector:</u>

The eigenvalues and eigenvectors can be determined by

(a) Solving the characteristic equation $det(A - \lambda I) = 0$ (direct method) to find λ and using $AX = \lambda X$ to find X.

(b) Power method (an iterative technique). The following examples illustrates each of the above two techniques.

1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Solution: The characteristic equation of A is $det(A - \lambda I) = 0$,

i.e.,$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 6 = 0$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0.$$

Thus, the eigenvalues are $\lambda = 4, -1$.

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ denote the characteristic vector. Then

(i) Characteristic vector corresponding to λ =4 is the solution of the system

$$(1-4)x + 2y = 0$$

$$3 + (2-4)y = 0$$

$$\text{i.e.,} \ -3x + 2y = 0$$

$$3x - 2y = 0.$$

Note that the above two equations are identical.

Let $= k(k \neq 0)$, then $y = \frac{3k}{2}$.

Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ \frac{3k}{2} \end{bmatrix}$ is the characteristic vector corresponding to eigenvalue 4.

(ii) Characteristic vector corresponding to $\lambda = -1$ is the solution of the system

$$(1+1)x + 2y = 0, 3 + (2+1)y = 0$$

$$\text{i.e., } 2x + 2y = 0, 3x + 3y = 0.$$

Note that the above two equations are identical as in earlier case.

Set $x = k$ (a non-zero parameter), then $y = -k$.

Thus $\begin{bmatrix} k \\ -k \end{bmatrix}$ is the eigenvector corresponding to the eigenvalue -1.

2. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Solution: The characteristic equation is $det(A - \lambda I) = 0$,

$$\text{i.e., } \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0,$$

$(2-\lambda)[(2-\lambda)^2 - 0] - 0 + 1[0 - (2-\lambda)] = 0,$

i.e., $(2-\lambda)[\lambda^2 - 4\lambda + 4] - 2 + \lambda = 0,$

$$2\lambda^2 - 8\lambda + 8 - \lambda^3 + 4\lambda^2 - 4\lambda + \lambda - 2 = 0,$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

$\lambda = 1$ is a root of the above equation (by inspection).

Let us use synthetic division to find the other two roots.

$$\begin{array}{c|cccc} \lambda = 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

Therefore, the other two are determined by the equation

$$1.\lambda^2 - 5.\lambda + 6 = 0.$$

$$\Rightarrow \quad \lambda = 2, 3.$$

Thus, the Eigen values are $\lambda = 1,2,3$.

(i) Characteristic vector corresponding to $\lambda = 1$ is the solution of the system of equations;

$$x_1 + 0.x_2 + x_3 = 0$$

$$0.x_1 + x_2 + 0.x_3 = 0$$

$$x_1 + 0.x_2 + x_3 = 0.$$

Selecting two equations that are linearly independent, (for example the last two equations, the non-zero solution is:

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1} = k_1 \text{ (non-zero parameter).}$$

Therefore $[x_1 x_2 x_3]^T = [k_1 0 - k_1]^T$ is the characteristic vector

(ii) Characteristic vector corresponding to λ =2 is the solution of the system of equations

$$0x_1 + 0x_2 + 1x_3 = 0$$

$$x_3 = 0$$

$$x_1 + 0x_2 + 0x_3 = 0.$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{0} = k_2 \text{(non-zeroparameter)}$$

Therefore $[x_1 x_2 x_3]^T = [0, k_2, 0]^T$ is the characteristic vector.

(iii) Characteristic vector corresponding to λ =3 is the solution of the system of equations

$$-x_1 + 0x_2 + 1x_3 = 0$$

$$0x_1 - x_2 + 0x_3 = 0$$

$$x_1 + 0x_2 - x_3 = 0.$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1} = k_3 \text{(non-zeroparameter)}$$

Therefore $[x_1 x_2 x_3]^T = [k_3, 0, k_3]^T$ is the characteristic vector.

Rayleigh's Power Method:

In many problems we need to calculate only the largest eigenvalue, this method is very useful to find the largest eigenvalue and its corresponding eigenvectors. Note inverse power method is used to find the smallest eigenvalue and its corresponding eigenvector.

Rayleigh's Power Method is an **iterative method** to approximate the **dominant eigenvalue** (largest in magnitude) and the corresponding eigenvector of a matrix.

The following steps are followed while finding the largest eigenvalue:

1. Let X_1 be an arbitrary vector which is normalized by choosing its first or last coefficient as 1. We form the product AX_1 and express it in the form $AX_1 = \lambda_1 X_2$ where X_2 is also normalized in the same manner. Here λ_1 is the numerically largest value in the X_1.
2. Form the product AX_2 and express it as $AX_2 = \lambda_2 X_3$, where X_3 is normalized in the same manner.
3. Continuing the iterations till two successive iterations having the same values up to the desired degree of accuracy.

Examples:

1. In aerospace structural dynamics, the vibration of an aircraft component can be represented by the system matrix

$$A = \begin{bmatrix} 4 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

Using the Power Method, estimate the largest eigenvalue and the corresponding eigenvector of this system. (Perform 7 iterations)

Solution:

Let $X_0 = [1 \quad 1 \quad 1]^T$ be the initial approximation.

$$AX_0 = \begin{bmatrix} 7 \\ 3 \\ 4 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 3/7 \\ 4/7 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = 5.28571\begin{bmatrix} 1 \\ 0.24324 \\ 0.48649 \end{bmatrix}$$

$$A\,X_2 = 4.72972\begin{bmatrix} 1 \\ 0.15428 \\ 0.46857 \end{bmatrix}$$

$$A\,X_3 = 4.46284\begin{bmatrix} 1 \\ 0.10371 \\ 0.46863 \end{bmatrix}$$

$$A\,X_4 = 4.31113\begin{bmatrix} 1 \\ 0.07217 \\ 0.47342 \end{bmatrix}$$

$$A\,X_5 = 4.02433\begin{bmatrix} 1 \\ 0.00605 \\ 0.4970 \end{bmatrix}$$

$$A\,X_6 = 4.01815\begin{bmatrix} 1 \\ 0.0045 \\ 0.49775 \end{bmatrix}$$

Therefore, the largest eigenvalue is 4.02.

The corresponding eigenvector is $\begin{bmatrix} 1 \\ 0.0045 \\ 0.49775 \end{bmatrix}$.

2. In chemical reactor stability analysis, the Jacobian matrix of reaction rates are given by

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

The dominant eigenvalue of this Jacobian determines whether the reactor exhibits instability (thermal runaway) or remains stable, while the corresponding eigenvector represents the most sensitive concentration mode. Using the Rayleigh Power Method, and starting with the initial concentration vector as $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, estimate the dominant eigenvalue and its corresponding eigenvector. Perform 5 iterations.

Solution: $AX_0 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 0.3333 \\ 0.3333 \end{bmatrix} = \lambda_1 X_1$

$$AX_1 = \begin{bmatrix} 4 \\ -1.3333 \\ -1.3333 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -0.3333 \\ -0.3333 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 4 \\ -1.3333 \\ -1.3333 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -0.3333 \\ -0.3333 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} -3.9999 \\ 1.3333 \\ 1.3333 \end{bmatrix} = 4 \begin{bmatrix} -1 \\ 0.3333 \\ 0.3333 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 4 \\ -1.3333 \\ -.3333 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -0.3333 \\ -0.3333 \end{bmatrix} = \lambda_4 X_4$$

After 4 iterations, numerically largest eigen value is $\lambda = 4$, corresponding eigen vector is $X = \begin{bmatrix} 1 \\ -0.3333 \\ -0.3333 \end{bmatrix}$.

Smallest Eigenvalue:

The following steps are followed while finding the smallest eigenvalue (Inverse power method):

1. Find inverse of the matrix for which smallest eigenvalue is to be found.
2. Find largest eigenvalue λ for that inverse matrix using usual Rayleigh power method.
3. $\frac{1}{\lambda}$ is the smallest eigen value of given matrix and the corresponding eigenvector is same as that of the λ.

Examples:

2. Using inverse power method, find an approximate value of smallest eigenvalue and the corresponding eigenvector of the matrix by taking initial vector as $[1 \quad 1 \quad 1]^T$.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Solution: $A^{-1} = \frac{1}{12}\begin{bmatrix} 5 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 5 \end{bmatrix}$

Let $X_0 = [1 \quad 1 \quad 1]^T$ be the initial approximation.

$$A^{-1}X_0 = \frac{1}{12}\begin{bmatrix} 9 \\ 15 \\ 9 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 1.25 \\ 0.75 \end{bmatrix} = 1.25\begin{bmatrix} 0.6 \\ 1 \\ 0.6 \end{bmatrix} \lambda_1 X_1$$

$$A^{-1}X_1 = \begin{bmatrix} 0.55 \\ 1.05 \\ 0.55 \end{bmatrix} = 1.05\begin{bmatrix} 0.5238 \\ 1 \\ 0.5238 \end{bmatrix} = \lambda_2 X_2$$

$$A^{-1}\,\mathrm{X}_2 = \begin{bmatrix} 0.5199 \\ 1.0199 \\ 0.5199 \end{bmatrix} = 1.0199\begin{bmatrix} 0.5059 \\ 1 \\ 0.5059 \end{bmatrix} = \lambda_3 X_3$$

$$A^{-1}\,\mathrm{X}_3 = \begin{bmatrix} 0.5030 \\ 1.0030 \\ 0.5030 \end{bmatrix} = 1.0030\begin{bmatrix} 0.5015 \\ 1 \\ 0.5015 \end{bmatrix} = \lambda_4 X_4$$

$$A^{-1}\mathrm{X}_4 = \begin{bmatrix} 0.5008 \\ 1.0008 \\ 0.5008 \end{bmatrix} = 1.0008\begin{bmatrix} 0.5004 \\ 1 \\ 0.5004 \end{bmatrix} = \lambda_5 X_5$$

Therefore, the smallest eigenvalue of A is $\frac{1}{1.0008} = 0.9992$

The corresponding eigenvector is $\begin{bmatrix} 0.5004 \\ 1 \\ 0.5004 \end{bmatrix}$.

3. A simplified model of a three-mass-spring system can be described by a stiffness matrix $\mathbf{K} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.Find the smallest eigenvalue and the corresponding eigenvector by performing 5 iterations of the Inverse Power Method on the given stiffness matrix, $\mathbf{K}$ and initial vector as $[1 \quad 0 \quad 0]^T$.

Solution: $A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Let $X_0 = [1 \quad 0 \quad 0]^T$ be the initial approximation.

$$A^{-1}X_0 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -0.3333 \\ -0.3333 \end{bmatrix} = \lambda_1 X_1$$

$$A^{-1}X_1 = \begin{bmatrix} 3.6666 \\ -1.3333 \\ -1.3333 \end{bmatrix} = 3.6666\begin{bmatrix} 1 \\ -0.3636 \\ -0.3636 \end{bmatrix} = \lambda_2 X_2$$

$$A^{-1}\,\mathrm{X}_2 = \begin{bmatrix} 3.7272 \\ -1.3636 \\ -1.3636 \end{bmatrix} = 3.7272\begin{bmatrix} 1 \\ -0.3659 \\ -0.3659 \end{bmatrix} = \lambda_3 X_3$$

$$A^{-1}X_3 = \begin{bmatrix} 3.7318 \\ -1.3659 \\ -1.3659 \end{bmatrix} = 3.7318 \begin{bmatrix} 1 \\ -0.3660 \\ -0.3660 \end{bmatrix} = \lambda_4 X_4$$

$$A^{-1}X_4 = \begin{bmatrix} 3.7320 \\ -1.3660 \\ -1.3660 \end{bmatrix} = 3.7320 \begin{bmatrix} 1 \\ -0.3630 \\ -0.3630 \end{bmatrix} = \lambda_5 X_5$$

Therefore, the smallest eigenvalue of A after 5 iteration is $\frac{1}{3.7320} = 0.2679$

The corresponding eigenvector is $\begin{bmatrix} 1 \\ -0.3630 \\ -0.3630 \end{bmatrix}$.

Further Applications:

1. It allows people to find important subsystems or patterns inside noisy data sets.
2. Eigenvalues and eigenvectors have widespread practical application in multivariate statistics.
3. Powers of a Diagonal Matrix, Matrix Factorization.
4. Eigenvalue analysis is also used in the design of the car stereo systems, where it helps to reproduce the vibration of the car due to the music.
5. Electrical Engineering: The application of eigenvalues and eigenvectors is useful for decoupling three-phase systems through symmetrical component transformation.
6. Model population growth using an age transition matrix and an age distribution vector, and find a stable age distribution vector.

<u>Exercise:</u>

Objective Type Questions:

1. The sum of the eigenvalues of a matrix is the ____ of the matrix.
2. If λ is an eigenvalue of a matrix A, then ________ is the eigenvalue of A^{-1} matrix.
3. If 3, 4 are the eigenvalues of a matrix A, then the eigenvalues of A^4 has __________.
4. An iterative method to find the largest eigenvalue and its corresponding eigenvector is_______.
5. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, then the eigenvalues of A^2 are ________________.
6. The Largest Eigen value of the Matrix $\begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}$ is _________________.
7. If λ is an eigenvalue of a square matrix A with X as a corresponding eigenvector, then the eigenvalue of A^n is equal to ___________.

8. Given that two of the eigenvalues of 3 by 3 matrix are equal to 1& determinant of the matrix is equal to 5 then the eigenvalues of its inverse are __________.

Descriptive type questions:

1. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$.
2. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$.
3. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.
4. Obtain the dominant eigenvalue and corresponding eigenvector after two iterations for the matrix $\begin{bmatrix} 3 & -1 \\ 4 & -6 \end{bmatrix}$ with initial eigenvector $[1 \quad 0]^T$.
5. Find the numerically largest eigenvalue and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ Starting with $[1,0,0]^T$as the initial approximation to the corresponding eigenvector carry out 7 iterations.
6. Use the power method to find the dominant eigenvalue and eigenvector for the matrix$A = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$
7. Find the numerically largest eigenvalue and the corresponding eigen vector of the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ Starting with $[1,0,0]^T$as the initial approximation to the corresponding eigenvector carry out 4 iterations.
8. Using inverse power method, find an approximate value of smallest eigenvalue and the corresponding eigenvector of the matrix $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 5 \end{bmatrix}$ by taking initial vector as $[1 \quad 1 \quad 1]^T$. Perform 5 iterations.

Answers:

Objective Type Questions:

1) Trace 2) $\frac{1}{\lambda}$ 3) 81, 256 4) Power method 5) 1, 9, 4 6) 7 7) λ^n

8) 1, 1, $\frac{1}{5}$

Descriptive type questions:

1) $\lambda = 4{,}6,\ \ X_1 = [k_1, k_1]^T, X_2 = [k_2, 2k_2]^T$

2) $\lambda = 1{,}6,\ \ X_1 = [k_1, -k_1]^T, X_2 = [4k_2, k_2]^T$

3) $\lambda = -2{,}3{,}6,\ \ X_1 = [-k_1, 0, k_1]^T, X_2 = [k_2, -k_2, k_2]^T, X_2 = [k_3, 2k_3, k_3]^T$

4) $\lambda = 3, X = [0.4166, -1]^T$ 5) $\lambda = 3, X = [1, 0, 1]^T$

6) $\lambda = 4$ and X $= \begin{bmatrix} 0.4 \\ 0.6 \\ 1 \end{bmatrix}$ 7) $\lambda = 4$ and X $= \begin{bmatrix} 2.2 \\ 1.1 \\ 0 \end{bmatrix}$

8) $\lambda = 1.7140$ and X $= \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$.

Video links:

1. Visualize Different Matrices part1 | SEE Matrix, Chapter 1
2. Geometry of Linear Algebra
3. Eigenvalues and Eigenvectors
4. Eigenvalues and Eigen vectors:https://youtu.be/PFDu9oVAE-g+-