SOUND

33

SIMPLE HARMONIC MOTION FREE, DAMPED AND FORCED VIBRATIONS

33.1 PERIODIC MOTION

If we observe the motion of the pendulum of a clock, piston in the cylinder of an engine, the motion of earth round the sun and the motion of moon round the earth, we find that the same motion along the same path is repeated again and again after equal intervals of time. Such a type of motion is called a periodic motion— *defined as a motion in which the body describes the same path in the same way again and again in equal intervals of time.* If a body moves in a circular path, it is said to describe circular periodic motion and if the motion is repeated along a line it is said to describe linear periodic motion. Periodic motion is also called harmonic motion.

33.2 SIMPLE HARMONIC MOTION

Simple harmonic motion is a particular case of period motion and is most fundamental type of periodic motion having single period. A simple harmonic motion can be (*a*) linear and (*b*) angular depending upon the path described by the body. In general *we can define simple harmonic motion as a motion in which the acceleration of the body is directly proportional to its displacement from a fixed point and is always directed towards the fixed point.* A simple harmonic motion possesses the following characteristics :

(*i*) The motion should be periodic.

(*ii*) When displaced from mean position, a restoring force, tending to bring it to the mean position and directed towards the mean position must act on the body.

(*iii*) The restoring force should be directly proportioanl to the displacement of the body from its mean position.

A simple harmonic motion may also be defined as the projection of a uniform circular motion on a diameter of the circle or any other line in the plane of the circle.

Consider a particle moving along a circle of radius a with a constant speed v as shown in Fig. 33.1. Draw a perpendicular from point P on YY' and

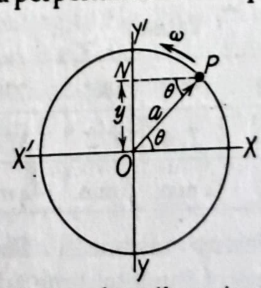

Fig. 33.1. Projection of a uniform circular motion on reference circle.

let N be the projection of P. This circle is known as reference circle and the particle P is known as reference particle.

Now when the particle is at X, the projection is at O. When the particle moves from X to Y' the projection moves from O to Y'. When the particle reaches X' the projection comes back from Y' to O. When the particle reaches Y the projection also reaches Y and when the particle comes to X, the projection comes back to O. Thus as the particle moves along the reference circle, the projection moves along the diameter YY'.

Now if in place of projection, we have a particle to move as the projection moves under the influence of some force, the particle will be said to be performing a linear S.H.M. along YOY'.

33.3 CHARACTERISTICS OF S.H.M.

1. Displacement. The distance of the particle measured along the path of the motion from its mean position, at a given instant is called displacement. Thus as shown in Fig. 33.1, the displacement of particle N is equal to ON or y

$$ON = OP \sin\theta$$

or
$$y = a\sin\theta \quad \text{...(1)}$$

Now if the angular velocity of the reference particle is ω and it describes this displacement in t second, we have

$$\frac{\theta}{t} = \omega \text{ or } \theta = \omega t$$

Hence $$\boxed{y = a\sin\omega t} \quad \text{...(2)}$$

The position of the particle N at any instant can be obtained with the help of displancement curve. A *displacement curve* is a graph between the time that elapses since the particle was at mean position O, and its displacement from its mean position during this time. Table 1 shows displacement of the

Table 1

Angle	0	90°	180°	270°	360°
Time	0	T/4	T/2	3T/4	T
Displacement	0	a max.	0 min.	$-a$ max.	0 min.

particle at different instances. The time is measured in terms of the total time taken for one vibration (T). Fig. 33.2 shows a displacement graph. For plotting the graph the total reference circle is divided into equal eight parts of equal intervals of time $T/8$, T being the time required to go round the circle once.

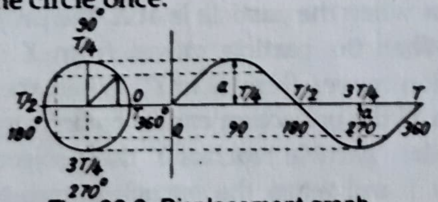

Fig. 33.2. Displacement graph.

2. Amplitude. *The maximum distance covered by the body on either side of the mean position is called its amplitude.* The magnitude of the amplitude is equal to the reference circle *i.e.*, a.

3. Velocity. The velocity of particle at N is equal to the component of the velocity of particle P along the diameter YOY'. If the velocity of the reference particle is v, its component along YOY' will be $v\cos\theta$. Hence the velocity of particle N (Fig. 33.3) at any instant t will be

Velocity of $N = v\cos\omega t$

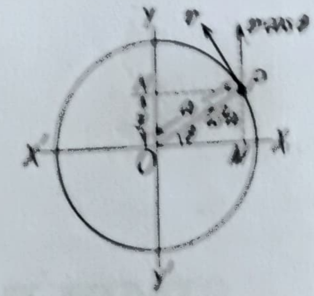

Fig. 33.3

$$= a\omega\cos\omega t \qquad (\because v = a\omega)$$

$$= a\omega\sqrt{(1-\sin^2\omega t)} \qquad (\because \cos^2\omega t + \sin^2\omega t = 1)$$

$$= a\omega\sqrt{\left\{1-\left(\frac{ON}{OP}\right)^2\right\}} \qquad \left(\because \sin\omega t = \frac{ON}{OP} = \frac{y}{a}\right)$$

$$= a\omega\sqrt{\left(1-\frac{y^2}{a^2}\right)} = a\omega\sqrt{\left(\frac{a^2-y^2}{a^2}\right)}$$

Hence $$\boxed{\text{Velocity of } N = \omega\sqrt{a^2-y^2}} \quad \text{...(3)}$$

In eq. (3), a and ω are constant quantities. The velocity of N will depend on the displacement only, being maximum and equal to $a\omega$ at the mean position *i.e.*, when $y = 0$ and minimum and equal to zero at the extreme position when $y = a$.

A graph between instantaneous velocity and time is known as velocity curve and can be plotted in a similar way as the displacement curve. Fig. 33.4 shows a velocity curve. From this curve it is

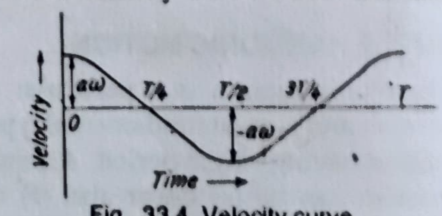

Fig. 33.4. Velocity curve.

clear that the velocity of the particle is maximum when it passes through its mean position and is equal to $a\omega$ and is minimum (zero) when its displacement is equal to the amplitude, *i.e.*, either of the end or extreme positions.

3. Acceleration. *Acceleration of the particle at any instant executing S.H.M. is equal to the component of the acceleration of the reference particle along YOY'.* Since the particle P is moving with a velocity v in a circle of radius a, its acceleration will be v^2/a or $\omega^2 a$ and will be directed towards the centre of the circle (Fig. 33.3). Resolving the acceleration along PN and PM, the component along NO will be $\omega^2 a\sin\theta$. Thus acceleration of N .

$$= -\omega^2 a \sin\theta \qquad \left(\because \sin\theta = \frac{y}{a}\right)$$

or Acceleration of $N = -\omega^2 y$ ...(4)

From the above equation we see that the acceleration of the particle is proportional to the displacement from its mean position and is always directed towards the centre as indicated by the negative sign in the above equation.

A graph between instantaneous acceleration and time is known as acceleration curve and can be drawn in a similar way as that of displancement curve (Fig. 33.5).

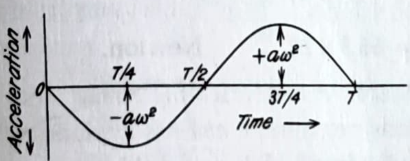

Fig. 33.5. Acceleration curve

From equation (4) and acceleration curve it is clear that the acceleration is maximum at the end points when $y = a$ and is minimum equal to zero at the mean position when $y = 0$.

4. Time period. *The time required to complete one vibration is known as the time period* and is denoted by T. The time taken to complete one revolution by the particle P is same as the particle at N takes to move from O to y, y to y' and y' to O (Fig. 33.3). If ω is the angular velocity and T is time period, then

Angular velocity

$$= \frac{\text{Angle described in one revolution}}{\text{Time taken to complete one revolution}}$$

$$= \frac{2\pi}{T} \text{ or } T = \frac{2\pi}{\omega} \qquad ...(5)$$

Substituting the value of ω from equation (4), we have

$$T = \frac{2\pi}{\sqrt{\text{Acceleration/Displacement}}}$$

$$\text{or } T = \frac{2\pi}{\sqrt{\text{Acceleration per unit displacement}}}$$

$$\text{or } \mathbf{T = 2\pi\sqrt{\left(\frac{Displacement}{Acceleration}\right)}} \qquad ...(6)$$

5. Frequency. *The number of vibrations made by the body in one second is known as the frequency of the vibrating body and is denoted by a letter n.* Frequency is the reciprocal of the time period,

$$\boxed{\mathbf{n = \frac{1}{T}}} \qquad ...(7)$$

6. Phase and phase difference. The phase of a vibrating body gives us an idea of the position of the body at any instant and can be defined as state or conditions as regard to position and direction of motion with reference to its mean or standard position. The phase of particle is measured either in terms of angle described by the reference particle as well as in terms of fraction of the time period which has elapsed since the rating particle last passed through its mean position in the positive direction.

Phase difference between two simple harmonic motions indicates how much the two motions are out of step with each other or by how much angle or by how much time one is ahead of the other.

Referring to Fig. 33.1, let a point P start its motion from the X axis at $t = 0$ and trace an angle θ in time t with angular velocity ω. The displacement y, of N from O at a time t is given by

$$y = ON = a\sin\theta = a\sin\omega t.$$

If, however, the time is measured from the instant when P is at P_0 as shown in Fig. 33.6 where angle $P_0OB = \phi$, then the displacement ON at any time t is given by

$$y = a\sin(\omega t + \phi)$$

where ωt is the angle traced in time t from the position P_0.

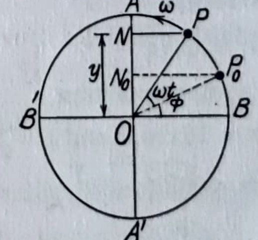

Fig. 33.6

At $t = 0$, the phase is ϕ which is called as *initial phase* or *'epoch'*. If the particle initially makes an angle ϕ below X-axis, then its displacement is given by

$$y = a\sin(\omega t - \phi)$$

Thus, $y = a\sin(\omega t \pm \phi)$ represents the general equation for the displacement in simple harmonic motion. The angle ϕ is called the initial phase.

EXAMPLE 1. *A particle executes a S.H.M. of period* 10 *seconds and amplitude of* 1.5 *metre. Calculate its maximum acceleration and velocity.*

Solution : If ω is the angular velocity, the time to complete one revolution with this speed *i.e.*, to cover 2π radians will be equal to the time period *T*.

Thus $T = \dfrac{2\pi}{\omega}$

or $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{10} = 0.628$ rad./s

Now we know that linear velocity,

$$v = \omega\sqrt{a^2 - y^2}$$

and will be maximum only when $y = 0$ i.e., at the mean position. Thus substituting values of ω (= 0.628) and y (= 0) and a (= 1.5), we have

$$v_{max} = 0.628\sqrt{(1.5)^2 - (0)^2} = 0.942 \text{ m/s.}$$

Similarly we know that the acceleration of a body executing S.H.M. is given by $\omega^2 y$ and will be maximum only when y is maximum, (= a).

Acceleration $= \omega^2 y$

$\therefore$ Maximum acceleration $= (0.628)^2 \times 1.5$

$= \mathbf{0.59\ m/s^2}$

EXAMPLE 2. *A body executing S.H.M. has its velocity* 16 cm/s *when passing through its centre mean position. If it goes* 1 cm *either side of mean position, calculate its time period.*

Solution : We know the velocity of body executing S.H.M. is given by

$$v = \omega\sqrt{a^2 - y^2}$$

Now when at mean position

$y = 0$ and $v = 16$ cm/s and $a = 2$ cm

$\therefore$ $16 = \sqrt{(2)^2 - (0)^2}$

or $\omega = \dfrac{16}{2} = 8$ rad./s.

Since $T = \dfrac{2\pi}{\omega}$

$\therefore$ $T = \dfrac{3 \times 3.14}{8} = $ **0.785 second.**

EXAMPLE 3. *A hydrogen atom has a mass of* 1.68×10^{-27} kg. *When attached to a certain massive molecule it oscillates as a classical oscillator with a frequency of* 10^{14} *cycles per second and with an amplitude of* 10^{-10} m. *Calculate the force acting on the hydrogen atom.*

Solution : The hydrogen atom executes S.H.M. of amplitude, $a = 10^{-10}$ m.

The angular velocity $\omega = 2\pi n$

$= 2\pi \times 10^{14}$ rad./s

The acceleration of the atom $= \omega^2 y$ or $\omega^2 a$

$\therefore$ The force on the hydrogen atom

$= m\omega^2 a$ = mass × acceleration

$= 1.68 \times 10^{-27} \times (2\pi \times 10^{14})^2 \times 10^{-10}$

$= 1.68 \times 4 \times \pi^2 \times 10^{-9}$

$= \mathbf{66.3 \times 10^{-9}}$ **Newton.**

EXAMPLE 4. *A body executing S.H.M. describes* 120 vibrations per minute *and has a velocity of* 5 m/s. *What is the length of its path ? What is the velocity ? When it is half way between its mean position and an extremity of its path ?*

Solution : Time period of the body

$= \dfrac{1}{120}$ min. $= \dfrac{60}{120} = 0.5$ s

Velocity at mean position (when $y = 0$)

$= 5$ m/s

Since $v = \omega\sqrt{a^2 - y^2}$

at the mean position its value will be $a\,\omega$

$\therefore$ $a\,\omega = 5$

or $a.\dfrac{2\pi}{T} = 5$ $\quad (\because \omega = 2\pi/T)$

whence $a = \dfrac{5 \times T}{2\pi} = \dfrac{5 \times 0.5}{2 \times 3.14} = 0.398$ m.

Now total path length will be double the amplitude.

Hence length of path $= 2 \times 0.398 = 0.788$ m.

Velocity of the body executing S.H.M. at any point is given by

$$v = \omega\sqrt{a^2 - y^2}$$

When the body is half way between its mean position and an extremity of its path, $y = a/2$ or 0.197. Substituting the values of y, a and ω, we have

$$v = \frac{2\pi}{T}\sqrt{(0.398)^2 - \left(\frac{0.398}{2}\right)^2}$$

$$= \frac{2\pi}{T} \times 0.398 \sqrt{1-\left(\frac{1}{2}\right)^2},$$

$$\text{or } v = \frac{2\times 3.14}{0.5} \times 0.398 \times \sqrt{\frac{3}{4}}$$

$$= 2\times 3.14 \times 0.398\sqrt{3}$$

or v = **4.329 m/s.**

Thus the length of path of the body of the body is 0.788 m and its velocity when it is half way between its mean position and an extremity of its path is 4.287 m/s.

EXAMPLE 5. *A body executes S.H.M. such that its velocity at the mean position is* 1 m/s *and acceleration at one of the extremity is* 1.57 m/s^2. *Calculate the time period of vibration.*

Solution : Velocity at any point of a body executing S.H.M. is given by

$$v = \omega \sqrt{(a^2 - y^2)}$$

at mean position v = 1 m/s and $y = 0$

$$\therefore \quad 1 = \omega a \quad \text{...(1)}$$

Acceleration at any point of a body executing S.H.M. is given by :

Acceleration $= \omega^2 y$

At extremity $y = a$

and Acceleration = 1.57

$$\therefore \quad 1.57 = \omega^2 a \quad \text{...(2)}$$

Dividing (2) by (1), we have $\omega = 1.57$

Now $T = \frac{2\pi}{\omega} = \frac{2\times 3.14}{1.57}$ = **4 seconds**

33.4 VIBRATION OF SIMPLE SPRINGS MASS SYSTEM

(*a*) **Vertical vibration.** For a light spiral spring within the elastic limit the tension of the spring is proportional to the extension of the spring beyond its length *i.e.* it obeys Hooke's law. Fig. 33.7 (*a*) shows a spring of length L suspended from a support at A. If a mass M is attached to its free and B, it will be stretched downward and its length will increase say by l i.e., $BC = l$ [Fig. 33.7 (*b*)]. Due to the increase in the length of the spring the force exerted by the spring on the mass, according to Hooke's law, will be given by $-kl$, where k is proportionality constant and depends upon the material and size of the spring and balances the tension T in the spring.

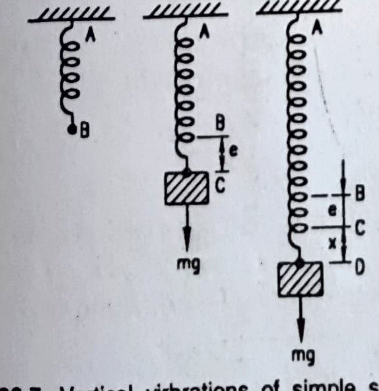

Fig. 33.7. Vertical virbrations of simple spring mass system.

The constant k is known as spring constant or stiffness factor or simply force constant.

$$\text{Hence } T = mg = kl \quad \text{...(1)}$$

Now if the load is displaced downward to a position D through a small distance x ($CD = x$), making the total increase in the length by $(x + l)$, the tension in this position T' will be proportional to $(x + l)$ *i.e.*,

$$T' = k(l + x) \quad \text{...(2)}$$

$\therefore$ The resultant force acting on the mass will be $T - T'$ or $mg - T'$. Hence we have for the resultant force on the mass.

$$F = mg - T' = kl - k(l+x) \text{ or } F = -kx$$

$$\therefore \text{ Acceleration of mass} = \frac{\text{force}}{\text{mass}} = -\frac{k}{m} \quad \text{...(3)}$$

In the above equation (k/m) is constant hecne the acceleration of the mass is proportional to its displacement and is directed towards its mean position C. Thus the mass will execute simple harmonic motion with C as the equilibrium position. From equation (3), we have

$$\frac{k}{m} = \frac{\text{acceleration}}{x}$$

= Acceleration per unit displacement

Now the time period of S.H.M.

$$T = \frac{2\pi}{\sqrt{\text{acceleration per unit displacement}}}$$

$$T = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\left(\frac{m}{k}\right)} \text{ second} \quad \text{...(4)}$$

Since $mg = kl$ we have $k = mg/l$ and substituting this value of k in equation (4), we have

$$T = 2\pi\sqrt{\left(\frac{l}{g}\right)} \quad \text{...(5)}$$

From above equation (4) it is clear that the time period of a spring with large k (strong spring) will be less and is directly proportional to the mass suspended. In the equation (4) we have assumed that the mass of the spring is negligible as compared to the mass hung. In case the mass is not negligible one-third of the total mass of the spring (m_1) will be effective and is to be added to the mass *i.e.*

$$T = 2\pi\sqrt{\left(\frac{m + \frac{m_1}{3}}{k}\right)} \quad ...(6)$$

(*b*) **Horizontal vibrations.** Consider a spring of which one end is fixed to a rigid support and the other end attached to a mass resting on a perfectly smooth horizontal surface. If we pull the mass through a certain distance, we find on releasing the mass that it vibrates about its position. These vibrations are also S.H.M. and this can be shown as follows :

Let the mass of the body be M and on being pulled through a certain distance and let go it starts vibrating between the position A and B. The force applied to pull the body be P, and the tension in the spring be T. The tension T will be the 'restoring force' and its tendency is such that it will try to bring the mass in its original position at O and will be in opposite direction to that of applied force P as shown in Fig. 33.8 *i.e.*,

$$T = -P \quad ...(7)$$

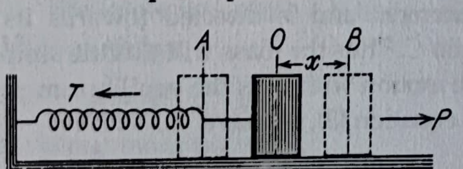

Fig. 33.8. Horizontal vibrations of simple spring mass system.

If the body is pulled through a distance x, the restoring force will be proportional to the displacement x and will be in the direction opposite to the displacement. Hence

$$T = -k\,x \quad ...(8)$$

where k is the proportionality constant and is called stiffness factor or spring constant and depends upon the type of spring used. From equation (8) it is clear that the restoring force which makes the body of mass m to vibrate about its mean position O is $-k\,x$. Hence we have

$$\text{Acceleration of the mass} = -k\,x/m \quad ...(9)$$

Since the acceleration is proportional to the displacement and directed towards the mean position, the body will execute S.H.M. From above equation, we have

$$\frac{\text{Acceleration}}{\text{Displacement}} = \frac{k}{m}$$

Substituting this value of acceleration per unit displacement in the expression for time period of a S.H.M. [Eqn. (6) Sec. 33.3], we have

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}} = 2\pi\sqrt{\left(\frac{k}{m}\right)} \quad ...(10)$$

$$T = 2\pi\sqrt{\left(\frac{\text{mass}}{\text{stiffness factor}}\right)} \quad ...(11)$$

This shows that the vibrations of a spring mass system are S.H.M. in case of vertical as well as horizontal arrangement and the time period is $2\pi\sqrt{\left(\frac{m}{k}\right)}$ in both cases.

33.5 VIBRATION OF SPRING MASS SYSTEM WHEN TWO OR MORE SPRINGS ARE USED

(*a*) **Springs in series.** In case if the mass is connected to a spring which consists of two different springs of different stiffness factor, the time period can be calculated as follows :

Let the stiffness factor of spring S_1 and S_2 be k_1 and k_2 respectively and the increase in the length of spring S_1 be x_1 and that of S_2 be x_2. If x is the total increase in the length of the spring system because of mass m as shown in Fig. 33.9, we have

$$x = x_1 + x_2 \quad ...(1)$$

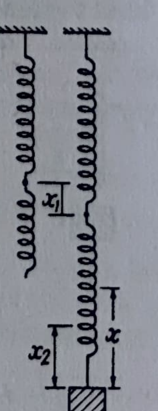

Fig. 33.9. Springs in series.

Now the same weight cause the elongation in each spring

$\therefore\ m g = k_1 x_1$ and also $m g = k_2 x_2$

or $x_1 = \dfrac{m g}{k_1}$ and $x_2 = \dfrac{m g}{k_2}$

If k_{eq} is the equivalent stiffness factor of the combination, we have

$$m g = k_{eq}\, x \text{ or } x = \frac{m g}{k_{eq}}$$

Substituting the value of x, x_1 and x_2 in equation (1), we have

$$\frac{m g}{k_{eq}} = \frac{m g}{k_1} + \frac{m g}{k_2} \text{ or } \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \quad ...(2)$$

From equation (2) we can say in general if a number of springs of different stiffnesses are connected in series the multi-spring system can be regarded as consisting of single spring of equivalent stiffness factor. The equaivalent stiffness factor is given by the following realtion :

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots.. \quad ...(3)$$

The time period of vibrations of such a system can be calcuated by substituting the value of equivalent stiffness factor in equation (11), sec. 33.4.

(*b*) **Springs in parallel.** Consider two springs S_1 and S_2 as shown in Fig. 33.10 connected in parallel. Each spring will share the total load and will have equal elongation say x. If k_1 and k_2 are the

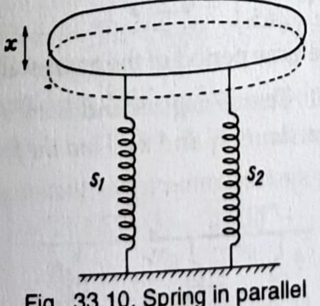

Fig. 33.10. Spring in parallel

stiffness factors for springs S_1 and S_2 and k be the equivalent stiffness factor for the combination.

Total restoring force $= m g = k_{eq}\, x$ ...(4)

Restoring force in spring $S_1 = k_1 x$ ...(5)

and restoring force in spring $S_2 = k_2 x$ ...(6)

Now total restoring force

= Reștoring force in springs S_1 + restoring force in spring S_2

$$k_{eq}\, x = k_1 x + k_2 x$$

or

$$k_{eq} = k_1 + k_2 \quad ...(7)$$

Thus if a number of springs are connected in parallel the equivalent stiffness factor is the sum of individual stiffness factor. The time period can be computed by replacing k by k_{eq} in the expression for time period.

33.6 VIBRATION OF BODIES SUPPORTED ON MORE THAN ONE IDENTICAL SPRINGS

The vibrations of a body supported by a number of springs are S.H.M. and the time period can be calculated in a similar way as calculated for a single spring mass system. Fig. 33.10 shows a mass M supported on two springs S_1 and S_2 of same stiffness factor k. Each spring will share the load equally and will be compressed through the same distance, say x (Fig. 33.10). The total restoring force, which will try to bring springs in their rest position will be equal to the weight of the mass supported by the springs. Since the restoring forces of S_1 and S_2 are same each of them will be equal to half of the weight.

$\therefore$ Force acting on each spring

$$= \frac{m g}{2} = k x \quad ...(1)$$

(If more than two springs are used, force on each spring will be equal to weight divided by the number of springs)

or

$$k = \frac{m g}{2 x}$$

The time period of vibrating spring

$$T = 2\pi \sqrt{\left(\frac{\text{mass}}{\text{stiffness factor}}\right)}$$

$$\text{or } T = 2\pi \sqrt{\left(\frac{m/2}{mg/2g}\right)} = 2\pi \sqrt{\left(\frac{x}{g}\right)}$$

If we have n number of springs and the total mass supported by these springs is m, each spring will support m/n mass and the restoring force on each spring will be $1/n$ of the total restoring force (mg), *i.e.*,

$\therefore$ Restoring force $= \dfrac{m g}{n} = k a$

$\therefore\ k = m g / n k$

$$\text{Time period} = 2\pi\sqrt{\left(\frac{\text{mass}}{\text{stiffness factor}}\right)}$$

$$= 2\pi\sqrt{\left(\frac{m/n}{mg/nx}\right)} = 2\pi\sqrt{\left(\frac{x}{g}\right)}$$

or Time period

$$= 2\pi\sqrt{\left(\frac{\text{compression}}{\text{acceleration due to gravity}}\right)} \quad ...(2)$$

EXAMPLE 6. *A spring of stiffness factor 98 N/m is pulled through 20 cm. Find the restoring force and compute the mass which should be attached so as to stretch in spring by the same amount.*

Solution : $k = 98$ N/m

$$\text{Restoring force} = k\,x = 98 \times \frac{20}{100} = 19.5$$

Let m be the mass attached. We know that the restoring force is equal to the weight of the mass hanged.

Thus restoring force $= k\,x = m\,g$

Substituting $k = 98$, $x = 20/100 = 0.2$ and $g = 9.8$ in the above relation, we have

$$98 \times \frac{20}{100} = m \times 9.8 \text{ or } m = 2\text{ kg}.$$

EXAMPLE 7. *A mass suspended from a light, spiral spring produces an extension of 79 cm. It is observed that the mass on being displaced vertically performs 100 simple harmonic vibrations in 57 s. Compute the value of g from the above experiment.*

Solution : Time period of the spring

$$= \frac{100}{57}\,s \text{ and } l = \frac{79}{100}\text{ m}$$

$$\text{Since } T = 2\pi\sqrt{\left(\frac{l}{g}\right)} \quad \therefore g = \frac{4\pi^2 l}{T^2}$$

Substituting the values of π, l and T, we have

$$g = 4 \times (3.14)^2 \times \frac{79}{100} \times \left(\frac{57}{100}\right)^2 = 10.12\text{ m/s}^2.$$

EXAMPLE 8 *A 4 kg mass is hung on the end of a helical spring and is pulled down and let go so as to vibrate vertically. The mass completes 100 vibrations in 55 seconds. Calcualte the stiffness factor of the spring.*

Solution : Time period of the mass

$$= \frac{100}{55}\text{ second}$$

Let k be the stiffness in N/m and using equation (11), sec. 33.4.

$$T = 2\pi\sqrt{\left(\frac{m}{k}\right)} \text{ or } \frac{100}{55} = 2\pi\sqrt{\left(\frac{4}{k}\right)} \text{ or } \frac{100}{55} = 2\pi \times \frac{2}{\sqrt{k}}$$

$$\text{or } \sqrt{k} = \frac{4 \times \pi \times 55}{100} \text{ or } k = \left(\frac{11\pi}{5}\right)^2 = 47.72\text{ N/m}.$$

EXAMPLE 9. *What mass should be hung on a spiral spring, having a stiffness constant of 89.2 N/m so that it vibrates with a periodic time of one second.*

Solution : Stiffness factor of the spring
$= 89.2$ N/m

Time period (T) = 1 s

If m is the mass hung using the relation

$$T = 2\pi\sqrt{\left(\frac{m}{k}\right)},\ 1 = 2\pi\sqrt{\left(\frac{m}{89.2}\right)}$$

$$\text{or } m = \frac{89.2}{4\pi^2} \text{ or } m = 2.26\text{ kg}.$$

EXAMPLE 10. *A mass of 0.5 kg hangs from a spring. If the mass is pulled downward and let go it executes S.H.M. Calculate the time period if the same spring is stretched 16 cm by 0.4 kg mass.*

Solution : For a simple spring mass system in equilibrium $m\,g = k\,x$

$$\therefore 4 \times 9.8 = k \times \frac{16}{100} \text{ or } k = \frac{4 \times 9.8 \times 100}{16} = 45\text{ N/m}$$

Now 0.5 kg mass is hanged. As the time period of the mass vibrating is given by

$$T = 2\pi\sqrt{\left(\frac{m}{k}\right)}$$

$$\therefore T = 2\pi\sqrt{\left(\frac{0.5}{245}\right)} = 0.28\text{ s}$$

Hence the time period of the mass will be 0.28 s.

EXAMPLE 11. *Two springs A and B each of length l, have a force constants k_1 and k_2. Find the force constant k of the spring system connected as shown in Fig. 33.11.*

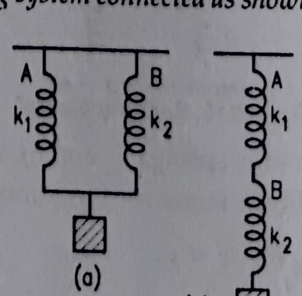

Fig. 33.11

Solution : (*a*) Let x be the extension in each spring. Hence

Tension in $A = k_1 x$

Tension in $B = k_2 x$

$$\therefore \quad m g = (k_1 + k_2) x \quad \text{...(1)}$$

It k be the spring constant of the combination, then

$$m g = k x \quad \text{...(2)}$$

From eqs. (1) and (2)

$$k = (k_1 + k_2) \quad \text{...(3)}$$

(*b*) Let x_1 and x_2 be the extensions in springs A and B respectively.

$$\text{Total extension} = x_1 + x_2 \quad \text{...(4)}$$

$$m g = k_1 x_1 \text{ and } m g = k_2 x_2$$

$$\therefore \; x_1 = \frac{m g}{k_1} \text{ and } x_2 = \frac{m g}{k_2} \quad \text{...(5)}$$

Now total extension

$$= \frac{m g}{k_1} + \frac{m g}{k_2} = m g \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \quad \text{...(6)}$$

$$\text{The force constant } k = \frac{m g}{(x_1 + x_2)} \quad \text{...(7)}$$

From eqs. (6) and (7)

$$k = \frac{m g}{m g \left(\frac{1}{k_1} + \frac{1}{k_2} \right)} = \frac{k_1 k_2}{k_1 + k_2} \quad \text{...(vii)}$$

EXAMPLE 12. *A particle of mass 0.1 kg is held between two rigid supports by two springs of force constants 8 N/m and 2 N/m. If the particle is displaced along the direction of the length of the springs, calculate frequency of vibration.*

Solution : The situation is shown in Fig. 33.12

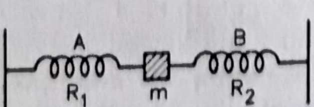

Fig. 33.12

When the mass is displaced along the direction of the length of the spring, one spring is compressed while the other is extended but the force due to both the springs is in the same direction. Hence effective force constant

$$k = k_1 + k_2 = 8 \text{ N/m} + 2 \text{ N/m} = 10 \text{ N/m}$$

The frequency of vibration is given by

$$\nu = \frac{1}{2\pi} \sqrt{\left\{ \frac{k}{m} \right\}} = \frac{1}{2\pi} \sqrt{\left\{ \frac{10}{0.1} \right\}}$$

$$\text{or } \nu = \frac{10}{2\pi} = \frac{5}{\pi} \text{ c/s.}$$

EXAMPLE 13. *Two light springs of force constants k_1 and k_2 and a block of mass m are in one line AB on a smooth horizontal table such that one end of each spring is fixed on rigid supports and the other andis free as shown in Fig. 33.13. The distance CD between the free end of the springs is 60 cm.*

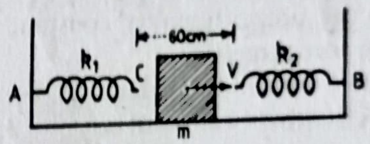

Fig. 33.13

If the block moves along AB with a velocity 120 cm/sec in between the springs, calculate the period of oscillation of the block.

(k_1 = 1 N/m, k_2 = 3.2 N/m and m = 200 gm.)

Solution : The time period E is given by

T = time to travel 30 cm to right + time in contact with spring k_2 + time to travel to 60 cm to left + time in contact with spring k_1 + time to travel 30 cm to right.

$$= \frac{30}{120} + \frac{1}{2}\left[2\pi \sqrt{\left(\frac{m}{k_2} \right)} \right] + \frac{60}{120} + \frac{1}{2}\left[2\pi \sqrt{\left(\frac{m}{k_1} \right)} \right] + \frac{30}{120}$$

$$= 0.25 + \pi \sqrt{\left\{ \frac{0.2}{3.2} \right\}} + 0.5 + \pi \sqrt{\left\{ \frac{0.2}{1.8} \right\}} + 0.25$$

$$= 0.25 + \pi + 0.5 + (\pi/3) + 0.25 = \textbf{2.83 seconds}$$

33.7 VIBRATION OF LOADED BEAMS

(*a*) Beam supproted at one-end—cantilever. A rod of negligible cross-section as compared to its length fixed horizontally at one end and loaded at the other is called *cantilever.* Let *PQ* be such a can-

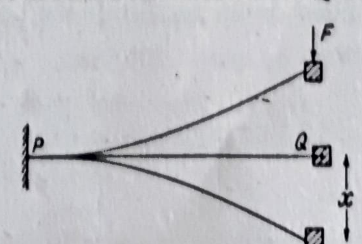

Fig. 33.14. Beam supported at one end—Cantilever.

tilever fixed at P with a rigid support and loaded with a mass m at Q. If the mass is pulled down and let go, the cantilever starts vibrating as shown in Fig. 33.14. As the restoring force which tries to bring the cantilever in its original position is proportional to the displacement and always directed towards the mean position, the beam executes S.H.M. If the beam is displaced through a distance x, the restoring force will be proportional to the displacement x and will be in the direction opposite to the direction of displacement.

$$\therefore \quad F = -kx \qquad ...(1)$$

where k is the proportionality constant (stiffness constant) or force constant.

Since force = mass × acceleration, the acceleration of the vibrating beam will be

$$\frac{F}{M} = \frac{-k}{m}x \qquad ...(2)$$

$$\text{or acceleration of mass} = -\frac{k}{m}x$$

$$\text{or} \quad \frac{\text{Acceleration}}{\text{Displacement}} = \frac{k}{m}$$

$$\therefore \text{Time period of the cantilever} = 2\pi\sqrt{\left(\frac{m}{k}\right)} \qquad ...(3)$$

If the beam is displaced by placing the mass m on other end of the cantilever by l, in equilibrium the restoring force will be equal to the weight to the mass *i.e.* $kl = mg$

$$\text{or} \quad \frac{m}{k} = \frac{l}{g} \qquad ...(4)$$

Substituting the value of m/k in equation (11) Sec. 1.4. we have

$$T = 2\pi\sqrt{\left(\frac{l}{g}\right)} \qquad ...(5)$$

(b) Beam supported at both ends. Let us consider a beam supported on two knife edges as shown in Fig. 33.15. and let it be loaded in the middle by placing a mass m, such that it is depressed through a length l below the mean position. Now if the massis pulled down and let go the beam will

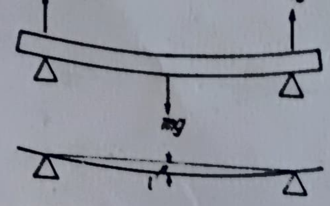

Fig. 33.15. Beam supported on both ends.

start vibrating with a definite time period, which can be calculated as follows :

The restoring force which tends to keep the beam in its mean position will be equal to the weight mg, the applied force. The reaction of supports will be equal and will be half of the restoring force as these reactions at the end points together constitute the total restoring force. This restoring force is proportional to the displacement. If x is the displacement of the beam at the centre, we have

$$F = -kx \qquad ...(6)$$

where k is called stiffness factor or force constant. The acceleration with which the mass m will move up and down will be

$$\text{Acceleration} = \frac{F}{m} = \frac{kx}{m} \qquad ...(7)$$

$$\text{or} \quad \frac{\text{Acceleration}}{\text{Displacment}} = \frac{k}{m}$$

$$\therefore \quad T = 2\pi\sqrt{\left(\frac{\text{displacement}}{\text{acceleration}}\right)} = 2\pi\sqrt{\left(\frac{m}{k}\right)}$$

Since $F = mg$

We have from the above equation (7)

$$\frac{mg}{m} = \frac{kl}{m} \quad \text{or} \quad \frac{m}{k} = \frac{l}{g}$$

Hence the time period of the beam loaded at the centre and supported at both the end points will be

$$\boxed{T = 2\pi\sqrt{\left(\frac{l}{g}\right)}} \qquad ...(8)$$

In the above formula l is the depression caused by the application of load of mass m on the beam.

33.8 AVERAGE KINETIC AND POTENTIAL ENERGIES OF A PARTICLE IN S.H.M.

When a body executes S.H.M. its total energy consists of potential and kinetic energy. The kinetic energy is maximum at the mean position and is minimum (equal to zero) at the extreme position while the potential energy is zero the mean position and maximum at the extreme position, i.e., the energy of a particle executing a simple harmonic motion is on an average half kinetic and half potential in form. Form the above it is clear that the average kinetic energy is equal to the average potential energy and equal to half of the total energy of the particle.

Potential energy. Consider the case of a particle of mass m executing simple harmonic motion

Let at any instant t, the displacement of the particle from the mean position by y. We know that in case of S.H.M. the force F acting on the particle is proportional to its displacement i.e., $F = -ky$, where k is a constant of proportionality. The negative sign indicates that the force is directed towards the equilibrium position.

Let the particle be further displaced through a distance dy, then the work done on the particle against the force F is $-F\,dy = k\,y\,dy$. The total work done on the particle for the displacement y is given by

$$\int_0^y k\,y\,dy = \frac{k\,y^2}{2}$$

The work is stored in the form of potential energy, hence

$$V = \frac{1}{2} k\,y^2\,\omega\,t \qquad ...(1)$$

For S.H.M., $y = a \sin(\omega t + \phi)$

$$\therefore \quad V = \frac{1}{2} k\,[a \sin(\omega t + \phi)]^2$$

$$= \frac{1}{2} k\,a^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2} m\,\omega^2 a^2 \sin^2(\omega t + \phi) \quad (\because \omega^2 = k/m) \qquad ...(2)$$

Equation (2) shows that potential energy V varies with time t. When $\sin^2(\omega t + \phi) = 1$, the potential energy is maximum.

Kinetic energy. The kinetic energy of the particle is given by $\frac{1}{2} m \left(\frac{dy}{dt}\right)^2$

For S.H.M., $y = a \sin(\omega t + \phi)$

$$\therefore \quad \frac{dy}{dt} = a\,\omega \cos(\omega t + \phi)$$

Hence, $T = \frac{1}{2} m\,[a\,\omega \cos(\omega t + \phi)]^2$

or $$T = \frac{1}{2} m\,a^2\,\omega^2 \cos^2(\omega t + \phi) \qquad ...(3)$$

Kinetic energy also depends upon time t. When $\cos^2(\omega t + \phi) = 1$, the kinetic energy is maximum i.e., it is $1/2\, m\,a^2\omega^2$.

Total energy

$$E = V + T = \frac{1}{2} m\,\omega^2 a^2 \sin^2(\omega t + \phi)$$

$$+ \frac{1}{2} m\,a^2\,\omega^2 \cos^2(\omega t + \phi)$$

$$= \frac{1}{2} m\,\omega^2 a^2 = 2\pi^2 m\,n^2 a^2$$

$$\therefore \quad \omega = \frac{2\pi}{T} = 2\pi n$$

Thus the energy of the particle is proportional to the square of frequency and squre of amplitude.

EXAMPLE 14. *A mass of 75 kg is dropped from a height of 10 cm on a pan attached to a light spring. If the spring stretches by 1 cm when loaded with 90 gm, calculate (a) the resulting extension of the spring, and (b) time period.*

Solution : Spring when loaded with 90 gm stretches by 1 cm and is in equilibrium

$$\therefore \quad mg = kx$$

or $$\frac{90}{1000} \times 9.8 = k \cdot \frac{1}{100}$$

or $$k = 9 \times 9.8 \text{ N/m}.$$

(*a*) Let the spring be stretched through x cm. When 75 gm mass falls from the height of 10 cm on the pan.

Work done by the falling weight

$$= m\,g\,h = \frac{75}{1000} \times 9.8 \times \left(\frac{10 + x}{100}\right)$$

The average restoring force offered by the spring

$$= \frac{0 + k\,x}{2} = \frac{1}{2} k\,x$$

$\therefore$ Work done on the spring

$$= \frac{1}{2} k\,x \cdot x = \frac{1}{2} k\,x^2 = \frac{9 \times 9.8}{2} \left(\frac{x}{100}\right)^2 \text{ Joule.}$$

As this work done on the spring is equal to the work done by the falling mass

$$\therefore \quad \frac{75 \times 9.8 \times (10 + x)}{1000 \times 100} = \frac{9 \times 9.8}{2} \times \frac{x^2}{100 \times 100}$$

or $$15\,(10 + x) = 9\,x^2$$

or $$3x^2 - 5x - 50 = 0$$

or $$3\,x^2 - 15\,x + 10\,x - 50 = 0$$

or $$3\,x\,(x - 5) + 10\,(x - 5) = 0$$

or $$(x - 5)\,(3\,x + 10) = 0$$

From above equation, we get $x = 5$ cm, the second value being negative is neglected.

(*b*) The time period

$$T = 2\pi \sqrt{\left(\frac{m}{k}\right)} = 2\pi \sqrt{\left(\frac{75}{1000} \times \frac{1}{9 \times 9.8}\right)}$$

or $T = 0.18$ s.

33.9 FREE VIBRATIONS

If we observe the vibrations of a simple pendulum, we find that the pendulum when let free vibrates with a fixed time period $= 2\pi\sqrt{\left(\frac{l}{g}\right)}$ or with a frequency $n = \frac{1}{2\pi}\sqrt{\left(\frac{g}{l}\right)}$

This frequency is called the natural frequency of the pendulum. The pendulum whenever set to vibrate freely without any resistance, it will always vibrate with the same frequency, given by the above relation. *The frequency with which a body vibrates freely at its own is called its natural frequency.*

Similarly, if we consider the vibration of a simple spring mass system it will vibrate with a natural frequency of

$$n = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)}$$

If no resistance is offered to the motion of the vibrating body by any source such as air friction or internal forces, the body will keep on vibrating indefinitely and such vibrations are called free vibrations. *Free vibrations can be defined as the vibrations in which the body vibrates with its own natural frequency when left free to itself.*

In practice it is not possible to eliminate friction completely. Actually the amplitude of the vibrating body gradually decreases to zero as a result of friction. In the above examples, the friction is less and these examples can be considered as the free vibrations.

Let us consider the motion of a particle of mass m acted upon by a restoring force proportional to its displacement. The restoring force may be expressed by $-\mu y$ where μ is the constant of proportionality and negative sign indicates that the restoring force acts in oppsite direction to the displacement *i.e.*, towards its mean position.

Now according to Newton's law

Force = mass × acceleration $= m\, d^2y/dt^2$

$$\therefore \quad m\frac{d^2y}{dt^2} = -\mu y$$

$$\text{or} \quad \frac{d^2y}{dt^2} = \frac{\mu}{m}y = -\omega^2 y \qquad ...(1)$$

where $\omega^2 = \mu/m$.

From eq (1).

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad ...(2)$$

Let the solution of eq. (2) is given by $y = e^{\alpha t}$

$$\text{Then} \quad \frac{dy}{dt} = \alpha e^{\alpha t}$$

$$\text{and} \quad \frac{d^2y}{dt^2} = \alpha^2 e^{\alpha t}$$

Substituting these values in eq. (2), we get

$$\alpha^2 e^{\alpha t} + \omega^2 e^{\alpha t} = 0$$

$$\text{or} \quad (\alpha^2 + \omega^2)\, e^{\alpha t} = 0$$

Now $e^{\alpha t} \neq 0$, hence $\alpha^2 + \omega^2 = 0$

$$\therefore \quad \alpha^2 = -\omega^2 \text{ or } \alpha = \pm i\omega \qquad ...(3)$$

Thus the general solution is given by

$$y = A\, e^{i\omega t} + B\, e^{-i\omega t} \qquad ...(4)$$

where A and B are some constants.

From eq. (4)

$$y = A(\cos\omega t + i\sin\omega t) + (B(\cos\omega t - i\sin\omega t)$$

$$\text{or} \quad y = \cos\omega t\,(A+B) + i\sin\omega t\,(A-B) \qquad ...(5)$$

Let $(A+B) = R\sin\phi$ and $i(A-B) = R\cos\phi$

$$\therefore \quad y = \cos\omega t.\, R\sin\phi + \sin\omega t.\, R\cos\phi$$

$$\text{or} \quad y = R(\cos\omega t\sin\phi + \sin\omega t\cos\phi)$$

$$\text{or} \quad y = R\sin(\omega t - \phi) \qquad ...(6)$$

Here $R = \sqrt{A^2 + B^2}$ and $\tan\phi = \frac{A}{B}$

From eq. (6) it is obvious that R is the maximum value of y *i.e.* the amplitude.

The value of y repeats when t changes by $2\pi/\omega$ because

$$\sin\{\omega(t + 2\pi/\omega) + \phi\} = \sin\{(\omega t + \phi) + 2\pi\}$$

$$= \sin(\omega t + \phi) = y$$

$\therefore$ Peridic time $T = 2\pi/\omega$

or Frequency $= \omega/2\pi$

Thus in case of the vibrations, the amplitude of vibration is constant with time.

33.10 DAMPED VIBRATIONS

As discussed above for a body executing vibrations, the amplitude keeps on decreasing because of frictional resistance to the motion and hence the vibrations die out after some time. The motion is said to be damped by friction and is called damped vibrations. The resisting frictional force is proportional to the velocity of the body. Fig. 33.16 shows an example of damped vibration. A mass suspended from the spring set to vibrate in air will vibrate for a larger time as compared to the mass which vibrates partially in air and partially in a liquid kept below the mass as shown in Fig. 33.16. The damping force is more when the mass moves in the liquid and hence the vibrations die out quickly in second case as compared to first shown in Fig. 33.16.

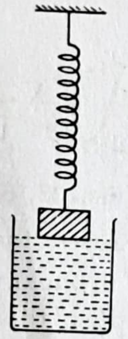

Fig. 33.16

Expressions for the period and amplitude of damped harmonic motion. The damped system is subjected to :

(*i*) a restoring force which is proportional to displacement but oppositely directed. This is written as $-\mu y$, where μ is a constant of proportionality or force constant.

(*ii*) a frictional force proportional to velocity but oppositely directed. This may be written as $-r\, dy/dt$, where r is the frictional force per unit velocity.

Since force = mass × acceleration = $m\, d^2y/dt^2$.

Therefore the equation of motion of the particle is given by

$$m\frac{d^2y}{dt^2} = -\mu y - r\frac{dy}{dt}$$

or $$\frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{\mu}{m} y = 0$$

or $$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0 \qquad ...(1)$$

where $r/m = 2b$ and $\mu/m = \omega^2$

Eq. (1) is a differential equation of second degree. Let its solution is

$$y = A e^{\alpha t}, \qquad ...(2)$$

where A and α are arbitrary constants.

Diffferentiating eq. (2) with respecto to t, we get

$$\frac{dy}{dt} = A\alpha e^{\alpha t} \text{ and } \frac{d^2y}{dt^2} = A\alpha^2 e^{\alpha t}$$

Substituting these values in eq. (1), we have

$$A\alpha^2 e^{\alpha t} + 2b A\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

or $$A e^{\alpha t}(\alpha^2 + 2b\alpha + \omega^2) = 0$$

As $A e^{\alpha t} \neq 0$, $\therefore$ $\alpha^2 + 2b\alpha + \omega^2 = 0$

This gives $\alpha = -b \pm \sqrt{[(b^2 - \omega^2)]}$

The general solution of eq. (1) is given by

$$y = A_1 \exp.[-b + \sqrt{(b^2-\omega^2)}]\,t$$
$$+ A_2 \exp.[-b - \sqrt{(b^2-\omega^2)}]\,t \qquad ...(3)$$

where A_1 and A_2 are arbitrary constants.

Depending upon the relative values of b and ω following three cases are possible.

Case I. When $b^2 > \omega^2$. In this case $\sqrt{(b^2-\omega^2)}$ is real and less than b. Now the powers

$$[-b + \sqrt{(b^2-\omega^2)}] \text{ and } [-b - \sqrt{(b^2-\omega^2)}]$$

in eq. (3) are both negative. Thus the displacement y consists of two terms, both dying off exponentially to zero without performing any osceillations as shown in Fig. 33.17. The rate of decrease of displacement is goveerned by the term

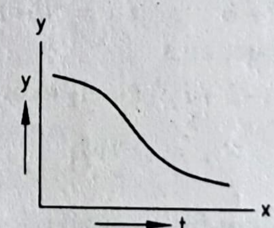

Fig. 33.17

$[-b + \sqrt{(b^2-\omega^2)}]\,t$ as the other term reduced to zero quickly relative to it. This type of motion is called as *over-damped or dead beat.* This type of motion is shown by a pendulum moving in a thick oil or by a dead beat moving coil galvanometer.

Case II. When $b^2 = \omega^2$. If we put $b^2 = \omega^2$ in eq. (3), then this solution does not satisfy the differential eq. (1). Let us consider that $\sqrt{(b^2-\omega^2)}$ is not zero but this is equal to a very small equantity h *i.e.*, $\sqrt{(b^2-\omega^2)} = h \to 0$. Now eq. (3) reduces to

$$y = A_1 \exp.(-b+h)\,t + A_2 \exp.(-b-h)\,t$$

$$= e^{-bt}\,[A_1 e^{ht} + A_2 e^{-ht}]$$

$$= e^{-bt}\,[A_1 (1 + ht + \ldots) + A_2 (1 - ht + \ldots)]$$

$$= e^{-bt}\,[(A_1 + A_2) + ht\,(A_1 - A_2) + \ldots]$$

$$= e^{-bt}\,[p + qt] \qquad \ldots(4)$$

where $p = (A_1 + A_2)$ and $q = h\,(A_1 - A_2)$

Eq. (4) represents a possible form of solution. It is clear from eq. (4) that as t increases, the factor $(p + qt)$ increases but the factor e^{-bt} decreases. In this way the displacement y first increases due to the factor $(p + qt)$ but at the same time reversal occurs due to the exponential term e^{-bt} and the displacement approaches zero as t increases. It is also clear that in this case the exponent is $-bt$ while in the first case it was more than $-bt$, hence in this case the particle tends to acquire its position of equilibrium much more rapidly than in case I. Such a motion is called *critical damped* motion. This type of motion is exhibited by many pointer instruments such as voltmeter, ammeter etc. in which the pointer moves to the correct position and comes to rest without any oscillation.

Case III. When $b^2 < \omega^2$. In this case $\sqrt{(b^2 - \omega^2)}$ is imaginary. Let us write

$$\sqrt{(b^2 - \omega^2)} = i\sqrt{(\omega^2 - b^2)} = i\beta$$

where $\beta = \sqrt{(\omega^2 - b^2)}$ and $i = \sqrt{(-1)}$

Eq. (3) now becomes

$$y = A_1 \exp.(-b + i\beta)\,t + A_2 \exp.(-b - i\beta)\,t$$

$$= e^{-bt}\,[A_1 e^{i\beta t} + A_2 e^{-i\beta t}]$$

$$= e^{-bt}\,[A_1 (\cos\beta t + i\sin\beta t) + A_2(\cos\beta t - i\sin\beta t)]$$

$$= e^{-bt}\,[A_1 + A_2]\cos\beta t + i\,(A_1 - A_2)\sin\beta t]$$

$$= e^{-bt}\,[a\sin\phi\cos\beta t + a\cos\phi\sin\beta t)]$$

where $a\sin\phi = (A_1 + A_2)$ and $a\cos\phi = i\,(A_1 - A_2)$

$$= e^{-bt}\,a\sin(\beta t + \phi)$$

$$= a\,e^{-bt}\sin[\sqrt{(\omega^2 - b^2)}\,t + \phi] \qquad \ldots(5)$$

This equation represents the simple harmonic motion with amplitude $a\,e^{-bt}$ and time period

$$T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{(\omega^2 - b^2)}}$$

The amplitude of the motion is continuously decreasing owing to the factor e^{-bt} which is called the damping factor. Because of the value of sin $[\sqrt{(\omega^2 - b^2)}\,t + \phi]$ varies between $+1$ and -1, therefore, the amplitude also varies between ae^{-bt} and $-ae^{-bt}$. The decay of the amplitude depends upon the damping coefficient b. It is called "*under damped*" motion as shown in Fig. 33.18. In this case the period is slightly increased or frequen-

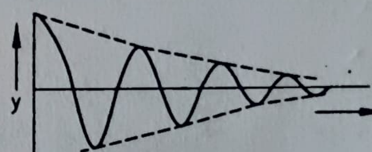

Fig. 33.18

cy decreased because the period is now $2\pi/\sqrt{(\omega^2 - b^2)}$ while in the absence of damping it was $2\pi/\omega$. The example of this type of motion is the motion of a pendulum in air, the motion of the coil of ballistic galvanometer or the electric oscillations of *L-C-R* circuit.

33.11 ATTENUATION COEFFICIENTS OF A VIBRATING SYSTEM

We know that the energy of an oscillator is proportional to the square of its amplitude. In case of a damped oscillator, the amplitude decays exponentially with time as e^{-bt}. So, the energy will decay as $(e^{-bt})^2 = e^{-2bt}$. In this way, the decay rate of energy depends upon $2b$. This is damping force per unit mass at an instant when the vibrating body is moving with a unit velocity. The following three characteristics given the attenuation of a vibrating system completely :

(*a*) Logarithimic decrement

(*b*) Relaxation time, and

(*c*) Quality factor.

These are discussed below

(*a*) **Logarithmic decrement.** Logarithmic decrement measures the rate at which the amplitude dies away. The amplitude of damped harmonic oscillator is given by Amplitude $= ae^{-bt}$ at $t = 0$ amplitude $a = a_0$ Let $a_1, a_2, a_3, \ldots$ be the amplitudes at time $t = T, 2T, 3T, \ldots$ respectively, where T = period of oscillation. Then

$$a_1 = a\,e^{-bt},\ a_2 = a\,e^{-b(2T)},\ a_3 = a\,e^{-b(3T)},\ \ldots$$

Now we get

$$\frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = e^{bT} = e^{\lambda} \text{ (where } bT = \lambda)$$

λ is known as logarithmic decrement.

Taking the natural logarithm, we get

$$\lambda = \log_e \frac{a_0}{a_1} = \log_e \frac{a_1}{a_2} = \log_e \frac{a_2}{a_3}$$

Thus logarithmic decrement is defined as the natural logarithm of the ratio between two successive maximum amplitudes which are separated by one period.

(b) Relaxation time. The relaxation time is defined as the time taken for the total mechanical energy to decay to $(1/e)$ of its original value. The mechanical energy of damped harmonic oscillator is given by

$$E = \frac{1}{2} a^2 \mu e^{-2bt}$$

Let $E = E_0$ when $t = 0$ $\quad \therefore E_0 = 1/2\, a^2\mu$

Now $E = E_0 e^{-2bt}$

Let τ be the relaxation time i.e., at $t = \tau$, $E = E_0/e$. Making this substitution in eq. (1), we get

$$(E_0/e) = E_0 e^{-2b\tau} \qquad \ldots(1$$

$$e^{-1} = e^{-2b\tau} \text{ or } -1 = -2b \therefore \tau = (1/2b) \qquad \ldots(2)$$

From eqs. (1) and (2), we get

$$E = E_0 e^{-t/\tau} \qquad \ldots(3)$$

(c) Quality factor. The quality factor is defined as 2π times the ratio of the energy stored in the system to the energy lost per period

$$Q = 2\pi \frac{\text{energy stored in system}}{\text{energy lost per period}} = 2\pi \frac{E}{PT}$$

where P is the power dissipated and T is peridic time.

$$\therefore\ Q = 2\pi \frac{E}{(E/\tau).T} = \frac{2\pi\tau}{T} = \omega\tau \qquad (\because P = E/\tau)$$

where $\omega = (2\pi/T) =$ angular frequency

So it is clear that higher is the value of Q, higher would be the value of relaxation time τ *i.e.*, lower damping.

In case of low damping, $\omega = \omega_0$, so

$Q = \omega\tau$ (constant of damped motion)

If k be the force constant and m, the mass of vibrating system, then

$$\omega_0 = \sqrt{\left(\frac{k}{m}\right)} \text{ and } \tau = 1/2b$$

$$\therefore\ Q = \frac{1}{2b} \times \sqrt{\left(\frac{k}{m}\right)}$$

This also shows that for low damping, the quality factor is higher.

For the time interval $(1/2b)$, the oscillator executes $(\omega_0\tau/2\pi)$ or $Q/2\pi$ oscillations.

EXAMPLE 15. *An underdamped oscillator has its amplitude reduced to (1/10) th of its initial value after 100 oscillations. If time period is 2 seconds, calculate (1) the damping constant and (2) the decay modulus.*

Solution : (1) The amplitude a of damped oscillator is given by $a = a_0 e^{-bt}$, where a_0 is the amplitude of undamped oscillator. The first amplitude a_1 is observed after $T/4$ sec. where T is the periodic time. Hence

$$a_1 = a_0 e^{-b(T/4)} \qquad \ldots(1)$$

Now the successive amplitudes occur at interval of $T/2$. After 100 oscillations, the number of amplitudes will be 20 and time taken will be $(200\,T/2 + T/4)$. Hence

$$a_{201} = a_0 e^{-b(100\,T + T/4)}$$

Dividing eq. (2) by eq. (1), we get

$$\frac{a_{201}}{a_1} = \frac{e^{-b(100\,T + T/4)}}{e^{-b(T/4)}} = e^{-100\,bT}$$

According to the given problem

$$a_{201}/a_1 = 1/10 \text{ and } T = 2 \text{ sec.}$$

$\therefore\ 10^{-1} = e^{-100 \times b \times 2}$ or $\log_e 10 = 200\,b$

or $2.3 \log_{10} 10 = 200\,b$ or $b = 2.3/200 = \mathbf{0.0115}$

(2) Decay modulus

$= 1/b = 1/0.0115 = \mathbf{86.8\ second}$

EXAMPLE 16. *The amplitude of an oscillator of frequency 200 per second falls to 1/10 of its initial value after 2000 cycles. Calculate (i) its relaxation time (ii) its quality factor, (iii) time in which its energy falls to 1/10 of its initial value, (iv) damping constant.*

Solution : The instantaneous amplitude of damped oscillator $= a\,e^{-bt}$. Let at $t = 0$, initial amplitude $= a_0$

After 10 second, its amplitude $= a_0/10$

$\therefore \quad \frac{a_0}{10} = a_0 e^{-b \times 10}$ or $10 = e^{10b}$

Taking log, we get $\log_e 10 = 10b$

$2.3 \log_{10} 10 = 10b$ or $2.3 = 10b$

$$\therefore \quad b = \frac{2.3}{10} = 0.23$$

(i) Relaxaiton time

$\tau = 1/2b = (1/2 \times 0.23) = 2.174$ sec

(ii) Quality factor $Q = \omega\tau = 2\pi n \times \tau$

$= 2 \times 3.14 \times 200 \times 2.174 = 2730$

(iii) $E = E_0 e^{-t/\tau}$

$\therefore \; E_0/10 = E e^{-t/\tau}$ or $10 = e^{t/\tau}$

Now $\log_e 10 = t/\tau$

or $t = \tau \log_e 10 = 2.174 \times 2.3 = 5$ sec.

EXAMPLE 17. *If the quality factor of an undamped tuning fork of frequency 256 is 10^3, calculate the time in which its energy is reduced to $(1/e)$ of its energy in the absence of damping. How many oscillations the tuning fork will make in this time.*

Solution : When $\tau = t, E = E_0/e$

Now $Q = \omega_0 t$ or $t = Q/\omega_0$

$$\therefore \quad t = \frac{10^3}{2\pi \times 256} = \frac{10^3}{3.14 \times 512} = 0.62 \text{ sec.}$$

So, the energy is reduced to $(1/e)$ in 0.62 sec.

The number of oscillations n made by tuning fork in time 0.62 sec is given by

$$n = \left(\frac{\omega_0}{2\pi}\right)\tau = \frac{Q}{2\pi} = \frac{10^3}{2\pi} \approx 159.$$

EXAMPLE 18 *The Q value of a spring loaded with 0.3 kg is 60. It vibrates with a frequency of 2 Hz. Calculate the force constant and mechanical resistance.*

Solution : The frequency n of an underdampled oscillator is given by

$$n = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)}, k = \text{force constant}$$

$$\therefore \; 2 = \frac{1}{2\pi}\sqrt{\left(\frac{k}{0.3}\right)} \text{ or } 4 = \frac{1}{4\pi^2}\left(\frac{k}{0.3}\right)$$

or $k = 16\pi^2 \times 0.3 = 47.37$ N/m

Further, $Q = \omega\tau = \frac{\omega}{2b} = \frac{\omega m}{r}$

$$\therefore \; r = \frac{\omega m}{Q} = \frac{2\pi n m}{Q} = \frac{2 \times 3.14 \times 2 \times 0.3}{60}$$

$= 0.06282$ kg/m

33.12 FORCED VIBRATIONS

So far we have discussed the vibrations in which the body vibrates at its own frequency without being subjected to any other external force. However, a different situation arised when the body is subejcted to an external force. As an example consider the vibrations of a bridge under the influence of marching soldiers or vibrations of atuning fork when exposed to the periodic force of sound waves. In both the cases the body vibrates because it is subjected to an external periodic force. Such vibrations are called Forced Vibrations. *Forced vibrations can be defined as the vibrations in which the body vibrates with a frequency other than its natural frequency under the action of an external peridic force.*

Theory of forced vibrations. The forces acted upon the particle are :

(i) a restoring force proportional to the displacement but oppositely directed, given by $-\mu y$ where μ is known as the force constant.

(ii) a frictional force proportional to velocity but oppositely directed, given by $-r\, dy/dt$ where r is the frictional force per unit velocity, and

(iii) the external periodic force, represented by $F \sin pt$ where F is the maximum value of this force and $p/2\pi$ is its frequency.

So the total force acting on the particle is given by

$$-\mu y - r\frac{dy}{dt} + F \sin pt$$

By Newton's second law of motion this must be equal to the product of mass m of the particle and its instantaneous acceleration *i.e.*, $m\, d^2y/dt^2$, hence

$$m\frac{d^2y}{dt^2} = -\mu y - r\frac{dy}{dt} + F \sin pt$$

$$\text{or } m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + \mu y = F \sin pt$$

$$\text{or } \frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{\mu}{m}y = \frac{F}{m}\sin pt$$

$$\text{or } \frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + w^2 y = f \sin pt \qquad \text{...(1)}$$

where $\frac{r}{m} = 2b, \frac{\mu}{m} = w^2$ and $\frac{F}{m} = f$

Eq. (1) is the differential equation of the motion of the particle.

In this case, when the steady state is set up, the particle vibrates with the frequency of applied force, and not with its own natural frequency. The solution of differential eq. (1) must be of the type

$$y = A \sin (pt - \theta) \quad ...(2)$$

where A is the steady ampliiude of vibrations and θ is the angle by which the displacement y lags behind the applied force $F \sin pt$. A and θ being arbitrary constants.

Differentiating Eq. (2), we have

$$\frac{dy}{dt} = A p \cos (pt - \theta)$$

and $$\frac{d^2y}{dt^2} = -A p^2 \sin (pt - \theta)$$

Substituting these values in eq. (1), we get

$$-A p^2 \sin (pt - \theta) + 2 bA p \cos (pt - \theta) + \omega^2 A$$
$$\sin (pt - \theta) = f \sin pt = f \sin \{(pt - \theta) + \theta\}$$

or $$A (\omega^2 - p^2) \sin (pt - \theta) + 2 b A p \cos (pt - \theta)$$
$$= f \sin (pt - \theta) \cos \theta + f \cos (pt - \theta) \sin \theta$$

If this relation holds good for all values of t, the coefficients of $\sin (pt - \theta)$ and $\cos (pt - \theta)$ terms on both sides of this equaton must be equal i.e., comparing the coefficients of $\sin (pt - \theta)$ and $\cos (pt - \theta)$ on both sides, we have

$$A (\omega^2 - p^2) = f \cos \theta \quad ...(3)$$

and $$2 b A p = f \sin \theta \quad ...(4)$$

Squaring eqs. (3) and (4)

$$A^2(\omega^2 - p^2)^2 + 4 b^2 A^2 p^2 = f^2$$

or $$A^2[(\omega^2 - p^2)^2 + 4 b^2 p^2] = f^2$$

$$A = \frac{f}{\sqrt{[(\omega^2 - p^2)^2 + 4 b^2 p^2]}} \quad ...(5)$$

While on dividing eq. (4) by eq. (3), we get

$$\tan \theta = \frac{2 b A p}{a (\omega^2 - p^2)}$$

or $$\theta = \tan^{-1} \left(\frac{2 b p}{\omega^2 - p^2} \right) \quad ...(6)$$

Equation (5) gives the amplitude of forced vibration while (6) its phase. Depending upon the relative values of p and ω, the following three cases are possible :

Case I. When driving frequency is low i.e., $p << \omega$. In this case, the amplitude of vibration is given by

$$A = \frac{f}{\sqrt{[(\omega^2 - p^2)^2 + 4 b^2 p^2]}}$$
$$= \frac{f}{\omega^2} = \text{constant}$$

and $$\theta = \tan^{-1} \frac{2 b p}{\omega^2 - p^2} \approx \tan^{-1} (0) \approx 0$$

This shows that the amplitude of vibration is independent of frequency of force. This amplitude depends on the magnitude of the applied force and force constant μ. The force and displacement are always in phase.

Case II. When $p = \omega$ *i.e.*, frequency of the force is equal to the frequency of the body. In this case, the amplitude of vibration is given by

$$A = \frac{f}{2 b p} = \frac{F}{r \omega}$$

$$\left[\because f = \frac{F}{m}, 2b = \frac{r}{m} \text{ and } p = \omega \right]$$

also $$\theta = \tan^{-1} \left(\frac{b p}{0} \right) = \tan^{-1} (\infty) = \pi/2$$

Thus the amplitude of vibration is governed by damping and for small damping forces, the amplitude of vibration will be quite large. The displacement lags behing the force by a phase $\pi/2$.

Case III. When $p >> \omega$ *i.e.*, the frequency of force is greater than the natural frequency ω of the body.

In this case.

$$A = \frac{f}{\sqrt{[p^2 + 4 b^2 p^2]}} \approx \frac{f}{p^2} = \frac{F}{m p^2}$$

and $$\theta = \tan^{-1} \left(\frac{2 b p}{\omega^2 - p^2} \right)$$
$$= \tan^{-1} \left(-\frac{2 b}{p} \right) \approx \tan^{-1} (-0) = \pi$$

Thus in this case, the amplitude A goes on decreasing and phase difference tends towards π. .

33.13 RESONANCE

If we bring a vibrating tuning fork near another stationary tuning fork of the same natural frequency as that of vibrating tuning fork, we find that stationary tuning fork also starts vibrating. This phenomenon is known as resonance.

The phenomenon of making a body vibrate with its natural frequency under the influence of another vibrating body with the same frequency is called resonance.

Consider three springs S_1, S_2 and S_3 suspended from a flexible rod AB such that S_1 and S_2 are identical in all respect and carry equal masses at the ends, while S_3 has different spring constant and carries different mass. Now if S_1 spring is set in vibration by pulling down the attached mass and let go, we find that spring S_2 and S_3 also start vibrating. The vibrations in S_3 die out quickly while the vibrations set in springs S_2 which is identical to S_1, keeps on increasing in its amplitude till it is very nearly equal to the amplitude of the spring S_1. The vibrations of spring S_2 have the same frequency as that of S_1 and are called resonant vibrations and this phenomenon is called resonance.

The other examples of resonance are :

(*a*) Tuning of a radio or transistor, when the natural frequency is so adjusted, by moving the tuning knob of the receiver set that it equals the frequency of the radio waves, the resonance takes place and the incoming sound waves can be listened after being amplified.

(*b*) Musical instruments can be made to vibrate by bringing them in contact with vibrations which have the frequency equal to the natural frequency of the instruments.

(*c*) Soldiers crossing a suspension bridge are prohibited to march in steps and are advised to march on the suspension bridges out of steps so as to avoid the resonance between the natural frequency of the bridge and the frequency of the steps of soldiers which may cause the collapse of the bridge.

Theory of resonant vibrations

(*a*) **Condition of amplitude resonance.** In case of forced vibrations, we have

$$A = \frac{f}{\sqrt{[(\omega^2 - p^2)^2 + 4b^2p^2]}} \qquad ...(1)$$

and
$$\theta = \tan^{-1}\left[\frac{2bp}{(\omega^2 - p^2)}\right] \qquad ...(2)$$

The expression (1) shows that the amplitude varies with the frequency of the force p. For a particular value of p, the amplitude becomes maximum. This phenomenon is known as amplitude resonance. The amplitude is maximum when $\sqrt{[(\omega^2 - p^2)^2 + 4b^2p^2]}$ is minimum

or $$\frac{d}{dp}[(\omega^2 - p^2)^2 + 4b^2p^2] = 0$$

or $$2(\omega^2 - p^2)(-2p) + 4b^2(2p) = 0$$

or $$\omega^2 - p^2 = 2b^2$$

or $$p = \sqrt{(\omega^2 - 2b^2)} \qquad ...(3)$$

Thus the amplitude is maximum when the frequency $p/2\pi$ of the impressed force becomes $\sqrt{(\omega^2 - 2b^2)}/2\pi$. This is the resonmt frequency. This frequency of the system of both in present of damping i.e. $\sqrt{(\omega^2 - b^2)}/2\pi$ and in the absence of damping *i.e.*, $\omega/2\pi$. If the damping is small *i.e.*, b is small then it can be neglected and the condition of maximum amplitude reduced to $p = \omega$

Putting this condition in eq. (1), we get.

$$A_{max} = \frac{f}{\sqrt{(\omega^2 - \omega^2 + 2b^2)^2 + 4b^2(\omega^2 - 2b^2)}}$$

$$= \frac{f}{\sqrt{4b^2\omega^2 - 4b^4}} = \frac{f}{2b\sqrt{(\omega^2 - b^2)}}$$

$$= \frac{f}{2b\sqrt{p^2 + b^2}} \qquad [\because p^2 = \omega^2 - 2b^2]$$

and for low damping it reduces to

$$A_{max} \cong \frac{f}{2bp}$$

Showing that $A_{max} \to \infty$ as $b \to 0$.

(*b*) **Sharpness of the resonance.** We have seen that the amplitude of the forced oscillation is maximum when the frequency of the applied force has a value to satisfy the condition of resonance *i.e.* $p = \sqrt{(\omega^2 - 2b^2)}$. If the frequency changes from this value, the amplitude falls. When the fall in amplitude for a small departure from the rsonance condition is very large, the resonance is said to be *sharp*. On the other hand if the fall in amplitude is small, the resonance is termed as flat. *Thus the term sharpness of resonance means the rate of fall in amplitude, with the change of forcing frequency on each side of resonance frequency.*

Fig. 33.19 shows the variation of amplitude with forcing frequency at different amounts of damping.

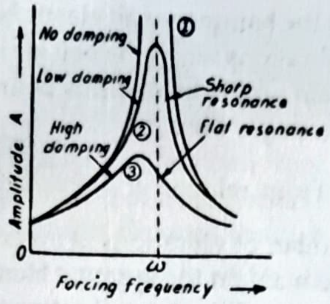

Fig. 33.19

Curve (1) shows the amplitude when there is no damping i.e., $b = 0$. In this case the amplitude becomes infinite at $p = \omega$. This case is never attained in practice due to frictional resistance, as slight damping is always present. Curves (2) and (3) show the effect of damping on the amplitude. It is observed that the peak of the curve moves towards the left. It is also observed that the value of A, which is different for different values of b (damping), diminishes as the value of b increases. For smaller values of b, the fall in the curve about $\omega = p$ is steeper than for large values. This shows that smaller is the value of damping, greater is the departure of amplitude of forced vibration from the maximum value and *vice versa*. Hence *smaller is the damping, sharper is the resonance or large is the damping, flatter is the resonance.*

EXAMPLE 19. *The forced harmonic oscillations have displacement amplitudes at frequencies $\omega_1 = 400\ \text{sec}^{-1}$ and $\omega_2 = 600\ \text{sec}^{-1}$. Find the resonant frequency at which the displacement amplitude is maximum.*

Solution : The displancement in forced oscillation is given by

$$x = \frac{f \sin (pt - \theta)}{\sqrt{[(\omega^2 - p^2)^2 + 4 b^2 p^2]}}$$

The amplitudes at frequencies ω_1 and ω_2 are equal

$$\therefore\ (\omega^2 - \omega_1^2) + 4 b^2 \omega_1^2 = (\omega^2 - \omega_2^2)^2 + 4 b^2 \omega_2^2$$

$$(\omega^2 - \omega_1^2)^2 - (\omega^2 - \omega_2^2)^2 = 4 b^2 (\omega_2^2 - \omega_1^2)$$

$$(2\omega^2 - \omega_1^2 - \omega_2^2)(\omega_2^2 - \omega_1^2) = 4 b^2 (\omega_2^2 - \omega_1^2)$$

$$\therefore\ 4 b^2 = 2\omega^2 - \omega_1^2 - \omega_2^2$$

The resonant frequency is given by

$$= \sqrt{(\omega^2 - 2 b^2)}$$

$$= \sqrt{\left[\omega^2 - \frac{(2\omega^2 - \omega_1^2 - \omega_2^2)}{2}\right]} = \sqrt{\frac{[\omega_1^2 + \omega_2^2]}{2}}$$

$$= \sqrt{\left[\frac{(400)^2 + (600)^2}{2}\right]} = 100\sqrt{(26)} = 510\ \text{sec}^{-1}.$$

OBJECTIVE TYPE QUESTIONS

1. A heavy brass sphere is hung from a weightless inelastic cord and as a simple pendulum its period of oscillation is T. When the sphere is immersed in a non-viscous liquid of density 1/10 th that of brass, it will act as a simple pendulum of period.

(*a*) T (*b*) $\sqrt{\left(\frac{10}{9}\right)}\,T$ (*c*) $\frac{10}{9}T$ (*d*) $\frac{9^2}{10^2}T$

2. Two vibrating systems are said to be in resonance if
(*a*) their amplitudes are equal
(*b*) their temperatures are same
(*c*) their frequencies are equal
(*d*) they are in same phase

3. The energy given out by a vibrating source of sound is proportional to
(*a*) first power of its amplitude
(*b*) square root of its amplitude
(*c*) second power of its amplitude
(*d*) four power of its amplitude.

4. The projection in horizontal plane of a point moving with uniform circular motion in vertical circle is called
(*a*) longitudinal vibrations
(*b*) transverse vibrations
(*c*) simple harmonic motion
(*d*) complex vibrations

5. Which is the case of forced vibrations
(*a*) sound produced in organ pipe
(*b*) sound produced in flute
(*c*) vibrations produced in piano string
(*d*) vibrations produced in telephone transmitter during conversion.

6. In simple harmonic motion, the acceleration is
(*a*) directly proportional to the displacement from central position
(*b*) constant
(*c*) inversely proportional to the displacement from central position
(*d*) inversely proportional to the square of displacement from central portion.

7. A particle moves such that its acceleration a is given by $a = -b\,x$ where x is the displacement from

equilibrium and b is a constant. The period of oscillation is

(a) $2\pi\sqrt{b}$ (b) $2\pi/\sqrt{b}$ (c) $2\pi/b$ (d) $2\sqrt{\pi}/b$

8. A particle is vibrating in simple harmonic motion with an amplitude of 4 cm. At what displancement from the equilibrium position is its energy half potential and half kinetic ?

(a) 1 cm, (b) $\sqrt{2}$ cm, (c) 2 cm, (d) $2\sqrt{2}$ cm.

9. A spring has a force constant k and a mass m is suspended from the spring. The spring is cut in half and the same mass is suspended from one of the halves. If the frequency of oscillation in the first case is α, then the frequency in second case will be

(a) 2α (b) α (c) $\alpha/2$ (d) $\alpha\sqrt{2}$

10. Two masses m_1 and m_2 are suspended together by a massless spring of constant k. When the masses are in equilibrium, m_1 is removed without disturbing the system. Then the angular frequency of oscillation of m_2 is

(a) $\sqrt{(k/m_1)}$ (b) $\sqrt{(k/m_2)}$

(c) $\sqrt{[k/(m_1+m_2)]}$ (d) $\sqrt{[k/(m_1\ m_2)]}$

ANSWERS

1. (b)	2. (c)	3. (c)	4. (c)	5. (d)
6. (a)	7. (b)	8. (d)	9. (d)	10. (b)

PROBLEMS AND EXERCISES

1. Define a simple harmonic motion and derive a relation for velocity and acceleration of a particle executing S.H.M.

2. Explain what do you understand by S.H.M. Define amplitude, time period and phase of a body executing S.H.M.

3. Derive an expression for time period of a simple harmonic motion.

4. With the help of velocity and acceleration curve or mathematically show that in the mean position, velocity is maximum and acceleration is zero while at the extreme positions its velocity is zero but acceleration is maximum.

5. Discuss the theory of a simple spring mass system and derive an expression for its time period and frequency.

6. Distinguish between free and forced vibrations. State the conditions of resonance. Give some important examples of resonance. What are the important engineering applications of resonance ?

7. A spring carrying the mass M is forced horizontally on a smooth table. The mass is slightly pulled and let off. Discuss its motion.

8. Does the hammer of an elastic bell make free or forced vibrations when the bell is ringing.

9. Explain why a loaded bus is more comfortable than an empty bus.

[Hint : From relation $n = 2\pi\sqrt{\left(\frac{k}{m}\right)}$, we find that the number of vibrations is inversely proportional to the mass on the springs. Hence if the bus is loaded there will be less vibrations and will be more comfortable].

10. A particle executes S.H.M. of amplitude 5 cm. When the particle is 3 cm from its mean position its acceleration is found to be 48 cm/s^2. Calculate

(a) its velocity at the same instant,
(b) its time period,
(c) its maximum velocity.

[Ans. (a) 16 cm/s, (b) 1.63, (c) 20 cm/s.]

11. Find the acceleration of a body executing S.H.M. at 20 cm from its mean position. The time period of the body is 10 s and amplitude is 1 m. How long it will take to travel 70 cm from one of its extremities ? [Ans. 7/9 cm/s^2 and 2 s]

12. Calculate the time period of a body executing S.H.M. if it has an acceleration of 2 m/s^2 when its displancment is 10 cm. [Ans. 1.4 s]

13. The velocity of the particle executing S.H.M. is 1 m/s and 0.7 m/s when its distance from its mean position is 30 cm and 60 cm respectively. Find its time period and amplitude.

14. A tuning fork vibrates with a frequency of 130 cycles per second and with an amplitude of 3 mm. Calculate the maximum velocity and acceleration of the prongs of the fork.

15. A body of mass 2.5 kg is hung by a light inextensible string and is found to stretch 0.4 m. The mass is then pulled down a further distance of 5 cm and let go. Find the time period of the mass.

[Ans. 0.7854]

16. A 150 gm mass when hung from a light spring stretches it by 40 cm. Calculate the time period of mass, made to vibrate by pulling the mass down and let go ?

[Ans. 1.27 s]

17. One end of a spiral spring is fixed in a support and the other end carries a pan of 1 kg. Due to the weight of the pan the spring stretches by 5 cm. A mass of 2 kg is placed in the pan and then it is displaced vertically so that it performs S.H.M. Find (a) how far the pan will descend, (b) the tension in

the spring when the pan is at its lowest position, (c) the period of vibration when 2 kg mass is in the pan.

[**Ans.** (*a*) 20 cm, (*b*) 49 N, (*c*) 0.78 s]

18. A 20 kg block is dropped from a height of 0.40 metre on a spring of force constant 1960 N/m. Find the maximum distance through which the spring will be compressed ($g = 9.8$ m/s^2).

[**Ans.** 10 cm.]

19. A mass of 100 kg is supported on a spring of stiffness constant 980 N/m. Find its compression and time period of vibration. [**Ans.** 1 cm and 0.2 s]

20. In a car seat there are 20 springs. If a man weighing 60 kg sits into drive, a compression of 0.3 mm is noted. Find the frequency of the vibration of the driver if there is no other passenger in the car.

21. A car can be considered to be mounted on a spring as far as vertical vibrations are considered. If the spring's vibrations frequency is 3 vibrations per second, calculate the spring's stiffness constant if the car weighs 1500 kg. What will the vibration frequency be if five passengers averaging 75 kg each, ride in the car ?

22. A spring of spring constant 980 N/m is attached to a mass of 10 kg and is placed on a smooth table. It is pulled through 10 cm and let go. Find restoring force of the system and time-period of vibration. [**Ans.** 98 N and 0.63 s]

23. Fill up the blank :

When the impressed frequency coincides with natural frequency ....takes place.

[**Ans.** resonance]

24. If the differential equation representing the free oscillations of a body is given by $d^2y/dt^2 + \omega^2 y = 0$, then the natural frequency of the body is given by (*i*) ω (*ii*) π/ω (*iii*) $2\pi/\omega$ (*iv*) $\omega/2\pi$ [**Ans.** (*iv*)]