An Introductory Guide to Transformations

For an Undergraduate Course in Euclidean Geometry

Cheyenne Ward

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Contents

1 Definitions

Note: Not all of the listed definitions are used in this guide but are still important for the reader to know. It is encouraged that the reader studies them.

1.1 Function

A relation between sets A and B such that, for all $a \in A$, there exists a unique element B that a relates to. We write $f(a) = b$.

1.2 Injective Function

A function f such that all points in a set A maps to a unique point in set B.

1.3 Surjective Function

A function f such that all points in a set B maps to a point in set A.

1.4 Bijective Function

A function f that is both injective and surjecive.

1.5 Transformation of the Plane

A bijective function (in R^2) $\alpha : R^2 \longrightarrow R^2$.

1.6 Isometry

A transformation (in R^2) α such that for all $A, B \in R^2, \overline{AB} = |\overline{\alpha(A)\alpha(B)}|$.

1.7 Direct Isometry

An isometry that preserves the orientation of R^2.

1.8 Opposite Isometry

An isometry that reverses the orientation of R^2.

1.9 Reflection r_m of the plane

A reflection r_m about line m is a transformation of R^2 such that, for all $A \in R^2$, line m is the perpendicular bisector of segment $\overline{Ar_m(A)}$.

1.10 Translation $t_{\overrightarrow{v}}$ of the plane

A translation $t_{\overrightarrow{v}}$ through vector v of R^2 such that for all $\in R^2$, the vector $\overrightarrow{At_{\overrightarrow{v}}(A)}$ is equal to the vector $\overrightarrow{v}$ based at point A.

1.11 Rotation $R_{P,\theta}$ of the plane

A rotation $R_{P,\theta}$ about point P by angle θ is a transformation of R^2 such that, for all $A \in R^2$, either $A = P$ and $r_m(A) = A$, or $|\overline{AP}| = |\overline{R_{P,\theta}(A)P}|$ and $m < APR_{P,\theta}(A) = \theta$. By convention, $\theta > 0°$ if measured in the counterclockwise direction and $\theta < 0°$ otherwise.

1.12 Glide Reflection

A glide reflection is the product of a translation followed by a reflection about a line that is parallel to the vector of translation.

1.13 Congruent

Given two objects in euclidean space, they are said to be congruent if they have the same shape and size but are not equal.

1.14 Similarity S_k of the plane

A transformation S_k is called a similarity if all distances are changed proportionally (multiplied) by the same positive real constant k. S_k is a similarity of the plane with scale factor k. Note: If $k = 1$, then S_k is called an isometry.

1.15 Dilation $H_{P,K}$ of the plane

A transformation of R^2 such that, for each point $A \in R^2$, if $A' = H_{P,k}(A)$, then A, P, and A' are all colinear and $|PA'| = k \cdot |PA|$. Note that $k \neq 0$.

2 Isometries

2.1 Reflection

In this section we provide a brief reminder of what a reflection is: A reflection r_m about line m is a transformation of R^2 such that, for all $A \in R^2$, line m is the perpendicular bisector of segment $\overline{Ar_m(A)}$.

2.2 Composition of two reflections

As we now know, reflections are a special type of isometry: an indirect isometry. An indirect isometry is an isometry that does not preserve the orientation of R^2. Therefore, if we reflect once over some line m and compare the orientation of the points of our pre-image and image, it will be seen their orientation of points are different. In other words, our pre-image with clockwise (or counterclockwise) point orientation ABC could result in an image with clockwise point orientation $C'B'A$ but by definition of a reflection, the image will never have the same clockwise (or counter-clockwise) point orientation as the pre-image when referring to a single reflection. This section is titled "Compositions of two reflections" however, so far, we have only discussed compositions consisting of one reflection. We will now use our knowledge of compositions consisting of one reflection to explore compositions of two reflections. If we look back at the definition of a reflection then we can understand that each reflection reverses orientation. When we reflect across lines n and m such that we have a composition of reflections $r_n r_m$ the second reflection r_n undoes the first reflection r_m. This results in the image stemming from the reflection r_n having the same orientation of points as seen in our original pre-image (that is, the image before reflecting across line m).

2.3 Composition of three reflections

Now that we understand that each reflection in a composition of reflection reverses the orientation of the points in the previous reflection, we can think more about compositions of three reflections. At this point we provide a GeoGebra activity for the reader to explore these compositions themselves. Before beginning the activity however, we encourage you to think about what might a composition of three reflections look like. What is happening to the points in the plane? Do you notice any patterns?

Activity: Composition of Three Reflections

In the Activity above, the reader will reflect a polygon across three parallel lines and use the result to answer the questions: What is happening to the points in the plane? Do you notice any patterns? The following video shows the reader how to reflect the polygon across the three lines:

Activity Aid: Composition of Three Reflections

After you've tried the activity please read on.

Given our exploration of composition of two reflections, and our knowledge that each reflection reverses orientation, it should be no surprise then that a product of three reflections reverses orientation once more. Given a composition of three reflections $r_n r_m r_k$, the orientation of points after r_m will be the same as our pre-image before r_k but will be different after r_n from our pre-image before r_k.

2.4 Compositions of odd and even number of reflections

The pattern that you may notice at this point is that an even number of reflections undoes any reflection! An odd number of reflection reverses orientation. One special case (that you will also see in the other transformations) is when the lines of reflection are equal. This results in the identity reflection resulting in points in the pre-image and image being equal to each other.

2.5 Translation

The next transformation that we will explore is a translation. We remind you of the definition of a translation: A translation $t_{\overrightarrow{v}}$ through vector v of R^2 such that for all $\in R^2$, the vector $\overrightarrow{At_{\overrightarrow{v}}(A)}$ is equal to the vector $\overrightarrow{v}$ based at point A.

2.6 Composition of two translations

A translation is a direct isometry meaning it preserves orientation. Due to this, whether we translate an even or odd number of times, the orientation of points will always be the same. However, one special case we can look at is a composition of two translations $t_{\overrightarrow{z}}t_{\overrightarrow{v}}$ such that $\overrightarrow{v}$ and $\overrightarrow{z}$ are directly on top of each other. This results in an identity translation and points in the pre-image and image will be equal to each other.

2.7 Rotation

The next transformation we will explore is a rotation. We remind you what a rotation is: A rotation $R_{P,\theta}$ about point P by angle θ is a transformation of R^2 such that, for all $A \in R^2$, either $A = P$ and $r_m(A) = A$, or $|\overline{AB}| = |\overline{R_{P,\theta}(A)P}|$ and $m < APR_{P,\theta}(A) = \theta$. By convention, $\theta > 0°$ if measured in the counterclockwise direction and $\theta < 0°$ otherwise.

2.8 Composition of two rotations

Just as we seen with a translation, a rotation is a direct isometry which means it preserves orientation of points in the plane. Due to this, we do not particularly see any interesting behavior in compositions of two or three reflections in terms of orientation of points. However, similar to both translations and reflections, there is in fact an identity for the rotation! According to our definition of a rotation, we need a center point P of rotation and an angle θ. If we have a product of rotations $R_{P,\phi}R_{P,\theta}$ such that $\phi + \theta = 0°$ the rotation reverses and points in the pre-image and image will be equal to each other.

3 Decomposition of Transformations

Before we discuss compositions of isometries, we will first discuss what happens when we decompose them.

If we have a composition of two reflections, then we know that orientation of points is preserved after the second reflection. When the two lines of reflection are not equal to each other and they are instead parallel to each other, the product of reflections result in a translation! This is because a reflection across two parallel lines does indeed reverse orientation but the points in the plane are still being moved in some direction. With this, we can provide another definition of a translation in terms of reflections! That is, a translation $t_{\vec{v}}$ is the product of two reflections $r_n r_m$ where lines n and m are parallel. We can perform a similar process for products of two rotations.

When we have a composition of two rotations $R_{B,\phi}R_{A,\theta}$ such that θ and ϕ are both 180°, we also get a translation! This is because rotating 180° twice results in the points in the plane ending up at the original orientation since we are really rotating by 360°. The direction of translation depends on whether we rotate clockwise or counterclockwise and the magnitude depends on how far apart the center points of rotations are from each other. It should be noted that a composition of rotations, $R_{P,\phi}R_{P,\theta}$ such that θ and ϕ are both $180° = t_{2\overrightarrow{AB}}$.
Proof:
We will use a composition of two rotation matrices to find the general formula for a composition of two rotations about some points A and B by 180°. First we rotate about point $A = (a_1, a_2)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(180°) & -\sin(180°) \\ \sin(180°) & \cos(180°) \end{pmatrix} \begin{pmatrix} x - a_1 \\ y - a_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= \begin{pmatrix} x - a_1 \end{pmatrix} \begin{pmatrix} \cos(180°) \\ \sin(180°) \end{pmatrix} + \begin{pmatrix} y - a_2 \end{pmatrix} \begin{pmatrix} -\sin(180°) \\ \cos(180°) \end{pmatrix}$$

$$= \begin{pmatrix} x\cos(180°) - a_1\cos(180°) - y\sin(180°) + a_2\sin(180°) \\ x\sin(180°) - a_1\sin(180°) + y\cos(180°) - a_2\cos(180°) \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x + a_1 + 0 \\ 0 - y + a_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x + 2a_1 \\ -y + 2a_2 \end{pmatrix}$$

Now we plug in x' and y' into x and y for the rotation matrix for point $B = (b_1, b_2)$:

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} \cos(180°) & -\sin(180°) \\ \sin(180°) & \cos(180°) \end{pmatrix} \begin{pmatrix} (-x + 2a_1) - b_1 \\ (-y + 2a_2) - b_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \begin{pmatrix} -x + 2a_1 - b_1 \end{pmatrix} \begin{pmatrix} \cos(180°) \\ \sin(180°) \end{pmatrix} + \begin{pmatrix} -y + 2a_2 - b_2 \end{pmatrix} \begin{pmatrix} -\sin(180°) \\ \cos(180°) \end{pmatrix}$$

$$= \begin{pmatrix} x - 2a_1 + b_1 + 0 \\ 0 + y - 2a_2 + b_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \begin{pmatrix} x - 2a_1 + 2b_{1)} \\ y - 2a_2 + 2b_2 \end{pmatrix}$$

We can rewrite the right-side of the equation as:

$$\begin{pmatrix} 2a_1 + 2b_1 \\ 2a_2 + 2b_2 \end{pmatrix} + \begin{pmatrix} -x \\ -y \end{pmatrix}$$

Now we will look at the translation $2\overrightarrow{AB}$. We decompose it into a composition of translations $2\overrightarrow{A}$ and $2\overrightarrow{B}$. First we do $2\overrightarrow{A}$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + 2a_1 \\ y + 2a_2 \end{pmatrix}$$

Now we do $2\overrightarrow{B}$:

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} (x + 2a_1) + 2b_1 \\ (y + 2a_2) + 2b_2 \end{pmatrix}$$

The above gives the general equations for translation $2\overrightarrow{AB}$ which we can rewrite the right-side of the equation as:

$$\begin{pmatrix} 2a_1 + 2b_1 \\ 2a_2 + 2b_2 \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

We can see that in both transformations, the coordinates of the point to be reflected is being added to $\begin{pmatrix} 2a_1 + 2b_1 \\ 2a_2 + 2b_2 \end{pmatrix}$.

Thus $R_{P,\phi} R_{P,\theta}$ such that θ and ϕ are both $180° = t_{2\overrightarrow{AB}}$.

We can now provide yet another definition for a translation: A translation is the product of two $R_{P,\phi} R_{P,\theta}$ such that θ and ϕ are both $180°$ such that $R_{P,\phi} R_{P,\theta}$ such that θ and ϕ are both $180° = t_{2\overrightarrow{AB}}$.

4 Compositions Using a Mix of Isometries

In this section we explore various compositions of isometries.

4.1 Glide Reflection

A glide reflection is a composition of a reflection and a translation such that the vector of translation is parallel to the line of reflection. Due to our ability to decompose isometries, glide reflections can be represented in numerous ways. For instance, we can simply see it as a translation through some vector $\overrightarrow{v}$ followed by a reflection across line m. Or we can view it as a product of two rotations $R_{P,\phi}R_{P,\theta}$ such that θ and ϕ are both 180° followed by a reflection. We can even view it as a product of two reflections $r_n r_m$ where lines n and m are parallel followed by another reflection.

4.2 Reflections and Translations

Lets say we had the composition of transformations $t_{\overrightarrow{v}} r_q$ such that the vector $\overrightarrow{v}$ is parallel to the line of reflection q What do you expect to see? Since we are working with reflections and translations in this subsection, you may remember from section 3 that a translation can be decomposed into a composition of reflections $r_n r_k$ such that lines n and k are parallel. You may then conclude that we now have the composition of reflections $r_n r_k r_q$ and that we expect to see the points in the plane reverse orientation from the first reflection r_q and translate from the last two reflections $r_n r_k$. However this is false! At least partially. It is true that a translation can be decomposed into a composition of two reflections $r_n r_k$ such that lines n and k are parallel. However it is important to remember that a vector has both magnitude and direction. So, if we translate the plane about a vector $\overrightarrow{v}$, the plane is being moved by some magnitude and in some direction. When we represent the translation as $r_n r_k$ the magnitude depends on how far apart lines n and k are. The direction depends on the overall orientation of lines n and k. We provide a GeoGebra activity for the reader to further explore this concept:

Activity: Decompostion of Translation into Composition of Two Reflections

In the GeoGebra activity linked above, the user will use the distance slider (d) and the rotation slider (r) to change the distance between the two lines as well as their rotation.

It should also be noted that in any composition of transformations, we read right to left.

So in $r_n r_k r_q$, $r_k r_q$ will form a translation and r_n will form a reflection. Therefore, in $r_n r_k r_q$, we expect to see a translation followed by a reflection IF lines n, k and r are parallel. The magnitude of the translation is dependent on how how far apart lines k and q are from each other and the direction depends on the overall orientation of lines k and q. However, if we were to keep our original composition of transformations $t_{\overrightarrow{v}} r_q$, we expect to see a reflection followed by a transformation.

4.3 Reflections and Rotations

Lets say we had the composition of transformations $R_{P,\theta} r_q$ such that point P is not on our line of reflection q. What do you expect to see? Well it may be no surprise that we will first see the plane reflected about line q then rotated about point P by some angle θ. However you may remember that from section 3 that a rotation can also be viewed as a composition of reflections $r_n r_k$ such that lines n and k are intersecting. With this said, we now explore the composition of reflections $r_n r_k r_q$ such that lines n and k intersect. In the composition of reflections $r_n r_k r_q$ such that lines n and k intersect, what we expect to see is dependent on the orientation of lines n, k and q. If lines q and k are parallel, then we expect to see a translation followed by a reflection also known as a glide reflection. However, if lines q and k are not parallel, then we might see a rotation followed by a reflection. Composition of rotations and reflections can be tricky as what we might expect to see greatly depends on the angle θ chosen and position of the lines. We provide a GeoGebra applet for the reader to further explore this themselves:

Activity: Decomposition of Rotation and Reflection

In the GeoGebra activity linked above, the reader will use the rotation slider (r) to change line q from being parallel to line k to not parallel and observe what happens to our polygon after reflecting across line q then line k. For this part observe the behavior of the blue polygon. The red polygon is a result of our polygon being reflected across all three lines.

4.4 Translations and Rotations

Lets say we had the composition of transformations $R_{P,\theta} t_{\overrightarrow{v}}$ such that point P is not on our vector of translation $t_{\overrightarrow{v}}$. Well, it may be no surprise that we expect to see the plane translate through vector $\overrightarrow{v}$ and then rotate about point P given some angle θ. However, you may remember from section 3 that we can represent a translation as a composition of rotations

$R_{W,\phi}R_{Q,\phi}$ where points W and Q are not equal to each other and $\phi = 180$. Therefore, in the composition of rotations $R_{P,\theta}R_{W,\phi}R_{Q,\phi}$, we expect to see a translation followed by a rotation about an angle θ. Now let us consider the composition of transformations $t_{\vec{v}}R_{P,\theta}$ such that point P is not on our vector of translation $t_{\vec{v}}$. Decomposing into a product of rotations gives $R_{W,\phi}R_{Q,\phi}R_{P,\theta}$. We encourage the reader to think what we can expect to see. Can we say that $R_{P,\theta}t_{\vec{v}} = t_{\vec{v}}R_{P,\theta}$? It is true that $R_{Q,\phi}R_{P,\theta}$ can give us a translation but only if both ϕ and $\theta = 180°$. Otherwise our transformation is changed with the resulting image depending on our angles for ϕ and θ as well as our coordinates for points P, Q and W.

5 Non-Isometries

Up until this point, we have discussed isometries: reflections, glide reflections, rotations and translations. We will now shift our focus to non-isometries: transformations that do not preserve distance. Keep in mind that there are a handful of non-isometries however we will only focus on dilations.

5.1 Dilation

In this subsection we will explore dilations. A dilation is defined as a transformation of R^2 such that, for each point $A \in R^2$, if $A' = H_{P,k}(A)$, then A, P, and A' are all colinear and $|PA'| = k * |PA|$. Note that $k \neq 0$. When $k < 0$ we get a dilation through the point P of dilation such that the point P is between the image and pre-image. In a composition of two dilations $H_{P,K_2}H_{Q,K_1}$, we get a single dilation such that the resulting scale factor is a product of the scale factors k_1 and k_2.

5.1.1 Scale Factor = 1

Similar to isometries, a dilation also has an identity transformation. When the scale factor is equal to one, the coordinates of each point in the plane is essentially being multiplied by 1 which results in the image being equal to the pre-image.

5.1.2 Dilation and Reflection

A dilation on its own, preserves orientation of points. However, in the composition of transformations $r_m H_{P,K}$ for some line m, the resulting image will not have preserved points because the last transformation is a reflection which as we discussed earlier is not orientation preserving. The same can b said for the composition of transformations $H_{P,K}r_m$. We provide a GeoGebra activity for the reader to explore composition of a dilation followed by a reflection:

Activity: Composition of a Dilation Followed by a Reflection

In the GeoGebra activity linked above, the reader will use the scale factor slider (k) to change the scale factor of the dilation through point A. what do you notice? Note: the yellow polygon is after dilation and the red polygon is after dilating then reflecting.

5.1.3 Dilation and Glide-Reflection

A glide reflection is a translation followed by a reflection. Thus, in the composition of transformations $r_m t_{\vec{v}} H_{P,K}$, the image is first dilated then translated then reflected. The dilation occurs after the image is translated then reflected in $H_{P,K} r_m t_{\vec{v}}$.

5.1.4 Dilation and Translation and Rotation

Considering that both a translation and rotation preservers orientation, when combined with a dilation, we are still performing a translation or rotation except the resulting image is dilated through some point P.

6 Conclusion

We've discussed the main isometries: reflections, glide reflections, translations and rotations as well as a non-isometry: dilation. We explored the composition of various isometries and non-isometries such as a translation followed by a reflection or a dilation followed by a rotation. It was found that these compositions can be tricky when extra variables are to be considered such as angle of rotation or orientation of lines. We also found that many isometries can be decomposed (or composed) into other isometries. For example, we can represent a translation as a composition of two reflections $r_n r_m$ where lines n and m are parallel. Below, the reader can find the links to the GeoGebra activities mentioned throughout the reading.

A Appendix 1: Geogebra Activity Links

Activity: Composition of Three Reflections

Activity Aid: Composition of Three Reflections

Activity: Decompostion of Translation into Composition of Two Reflections

Activity: Decomposition of Rotation and Reflection

Activity: Composition of a Dilation Followed by a Reflection

References

[1] Clayton Dodge, *Euclidean Geometry and Transformations*, Version 1, Dover Publications, 2004.