Appendix G

# Kalman filter using the orthogonality principle

*This appendix presents derivation steps for obtaining the discrete Kalman filter equations using a method based on the orthogonality principle. The content is mainly based on Jazwinski [14, chap. 7]. For simplification, the system is considered time-invariant and the noises* $\mathbf{w}$ *and* $\mathbf{v}$ *are taken independent.*

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Following the presentation in section 4.3.1, preliminaries are set in place.

Given the state-space linear system:

$$\mathbf{x}_{k+1} = \Phi\mathbf{x}_k + \Gamma\mathbf{w}_k \tag{G.1}$$

$$\mathbf{z}_k = H\mathbf{x}_k + \mathbf{v}_k \tag{G.2}$$

and a sequence of measurements $\mathbf{Z}_k = \{\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_k\}$, the objective is to find the best estimate of the real state at time $k$, $\hat{\mathbf{x}}_k$, and also to find a practical method of updating this estimate to $\hat{\mathbf{x}}_{k+1}$ once a new measurement $\mathbf{z}_{k+1}$ is available.

The best estimate is defined in the context of this derivation as the minimum variance estimate. Taking into account that both the state and estimates are vectors, the minimum variance condition for an estimator of $\mathbf{x}$ can be written as:

$$J(\mathbf{a}) = \mathrm{E}\langle(\mathbf{x}_k - \mathbf{a})^T(\mathbf{x}_k - \mathbf{a})|\mathbf{Z}_k\rangle \tag{G.3}$$

$$\hat{\mathbf{x}}_k = \mathrm{argmin}_{\mathbf{a}} J(\mathbf{a}) \tag{G.4}$$

$$J(\hat{\mathbf{x}}_k) = \mathrm{min}_{\mathbf{a}} J(\mathbf{a}) \tag{G.5}$$

$J(\mathbf{a})$ is a scalar cost value. In the report, all vectors are vertical elements, and their transposes are horizontal.

## G.1 Conditional mean

It was mentioned in section 4.3 that the best estimate for the filtering problem is the conditional mean. A property of the conditional mean is that it is the solution of the minimum variance criterion.

This can be argued in the following way. Given $\mathbf{x}$ a vector random variable, $\mathbf{Z}_p = \{\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_p\}$ a sequence of measurements that are realizations of $p$ other random vector variables, and vector $\mathbf{a}$ an estimator of $\mathbf{x}$, if the conditional mean $\mathrm{E}\langle \mathbf{x}|\mathbf{Z}_p\rangle$ denoted by $\bar{\mathbf{x}}$ is subtracted and added in (G.3):

$$\begin{aligned} J(\mathbf{a}) = \mathrm{E}\langle (\mathbf{x}-\mathbf{a})^T(\mathbf{x}-\mathbf{a})|\mathbf{Z}_p\rangle &= \mathrm{E}\langle \Big((\mathbf{x}-\bar{\mathbf{x}})^T + (\bar{\mathbf{x}}-\mathbf{a})^T\Big)\Big((\mathbf{x}-\bar{\mathbf{x}}) + (\bar{\mathbf{x}}-\mathbf{a})\Big)|\mathbf{Z}_p\rangle \\ &= \mathrm{E}\langle (\mathbf{x}-\bar{\mathbf{x}})^T(\mathbf{x}-\bar{\mathbf{x}})|\mathbf{Z}_p\rangle + \mathrm{E}\langle (\mathbf{x}-\bar{\mathbf{x}})^T(\bar{\mathbf{x}}-\mathbf{a})|\mathbf{Z}_p\rangle + \\ &\quad + \mathrm{E}\langle (\bar{\mathbf{x}}-\mathbf{a})^T(\mathbf{x}-\bar{\mathbf{x}})|\mathbf{Z}_p\rangle + \mathrm{E}\langle (\bar{\mathbf{x}}-\mathbf{a})^T(\bar{\mathbf{x}}-\mathbf{a})|\mathbf{Z}_p\rangle = \\ &= \mathrm{E}\langle (\mathbf{x}-\bar{\mathbf{x}})^T(\mathbf{x}-\bar{\mathbf{x}})|\mathbf{Z}_p\rangle + \mathrm{E}\langle (\mathbf{x}-\bar{\mathbf{x}})^T|\mathbf{Z}_p\rangle_{=0}(\bar{\mathbf{x}}-\mathbf{a}) + \\ &\quad + (\bar{\mathbf{x}}-\mathbf{a})^T\mathrm{E}\langle (\mathbf{x}-\bar{\mathbf{x}})|\mathbf{Z}_p\rangle_{=0} + (\bar{\mathbf{x}}-\mathbf{a})^T(\bar{\mathbf{x}}-\mathbf{a}) = \\ &= \mathrm{E}\langle (\mathbf{x}-\bar{\mathbf{x}})^T(\mathbf{x}-\bar{\mathbf{x}})|\mathbf{Z}_p\rangle + (\bar{\mathbf{x}}-\mathbf{a})^T(\bar{\mathbf{x}}-\mathbf{a}) \end{aligned} \tag{G.6}$$

We want to minimize the final expression (G.6) in variable $\mathbf{a}$. The first term does not depend on $\mathbf{a}$, thus the minimization will only be related to the second term. Since the second term is quadratic, it is $\geq 0$ and its minimum value is 0.

$$\bar{\mathbf{x}}_k - \mathbf{a} = 0 \Rightarrow \mathbf{a} = \mathrm{E}\langle \mathbf{x}|\mathbf{Z}_p\rangle \tag{G.7}$$

The minimum variance estimator is the conditional mean. The conditional mean is also an unbiased estimator.

$$\mathrm{E}\langle \mathbf{a}-\mathbf{x}\rangle = \mathrm{E}\langle \mathbf{a}\rangle - \mathrm{E}\langle \mathbf{x}\rangle = \mathrm{E}\langle \mathrm{E}\langle \mathbf{x}|\mathbf{Z}_p\rangle\rangle - \mathrm{E}\langle \mathbf{x}\rangle = \mathrm{E}\langle \mathbf{x}\rangle - \mathrm{E}\langle \mathbf{x}\rangle = 0 \tag{G.8}$$

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Thus, the solution for the optimum, minimum variance estimator for the linear state-space system is the conditional mean:

$$\hat{\mathbf{x}}_k = \mathrm{E}\langle \mathbf{x}_k|\mathbf{Z}_k\rangle \tag{G.9}$$

This expression is however not transparent, and needs to be explained further before it can be used. How does the conditional mean look like?

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If the random variables $\mathbf{x}$ and $\{\mathbf{z}_i | i \geq 1\}$ have Gaussian probability distributions, then it is known that the conditional probability function is also Gaussian and that its mean is a linear combinations in the observations $\mathbf{z}_i$:

$$p(\mathbf{x}|\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_p) \in \mathcal{N}(0, \cdot) \tag{G.10}$$

$$\mathrm{E}\langle \mathbf{x}|\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_p\rangle = \sum_{i=1}^{p} L_i \mathbf{z}_i \tag{G.11}$$

Thus, $\hat{\mathbf{x}}_k$ can be developed one step further:

$$\hat{\mathbf{x}}_k = \sum_{i=1}^{k} L_i \mathbf{z}_i \tag{G.12}$$

## G.2 Orthogonal Projections

In continuing with the derivation, Kalman introduced Orthogonal Projections, a mechanism that uses vector spaces formalism to show properties of how the linear combinations making up the estimator $\hat{\mathbf{x}}_k$ evolve with each new measurement data.

Without using a detailed and rigorous mathematical context, the Orthogonal Projections can be described in the following way.

## A Vector Space of Random Variables

First, a collection of $n$-random (m-vector) variables (it is not necessary that they are uncorrelated) together with the second moment operator as inner product can define a vector space $\mathcal{Z}_n$.

$$\{\mathrm{z}_1, \mathrm{z}_2, ..., \mathrm{z}_n\} \tag{G.13}$$

$$\langle \mathrm{z}_i, \mathrm{z}_i \rangle = \mathrm{E}\langle \mathrm{z}_i \mathrm{z}_j^T \rangle \tag{G.14}$$

The vector space $\mathcal{Z}_n$ contains exactly all the linear combinations of the $n$-random variables.

$$\mathcal{Z}_n = \{\sum_{i=1}^{n} A_i \mathbf{z}_i \mid \forall A_i \in \mathbf{R}^{m \times m}\} \tag{G.15}$$

On this space an orthonormal basis can be defined. A set of $n$ random variables $\{\mathrm{u}_1, \mathrm{u}_2, ..., \mathrm{u}_n\}$ can be determined to fulfill properties of orthogonality and unit-norm.

$$\{\mathrm{u}_1, \mathrm{u}_2, ..., \mathrm{u}_n\} \tag{G.16}$$

$$\langle \mathrm{u}_i, \mathrm{u}_j \rangle = \mathrm{E}\langle \mathrm{u}_i \mathrm{u}_j^T \rangle = 0 \tag{G.17}$$

$$\langle \mathrm{u}_i, \mathrm{u}_i \rangle = \|\mathrm{u}_i\| = \mathbf{I} \tag{G.18}$$

The basis random variables $\mathrm{u}_i$ are thus uncorrelated. They can be obtained using techniques such as the Gram-Schmidt process. The vector space $\mathcal{Z}_n$ can be thought to be similar in many ways with a common Euclidean space $\mathbf{R}^n$. A parallel can be drawn in the following way:

| Space | $\mathcal{Z}_n$ | $\mathbf{R}^n$ |
|---|---|---|
| Elements | a random variable | a deterministic point |
| Dimensions or Degrees of Freedom | $n$ probabilistic directions | $n$ spatial dimensions |
| Orthogonality | probabilistic un-correlation | vector perpendicularity |
| Norm | variance of the r.v. | distance from origin |

Two elements of the $\mathcal{Z}_n$ space are orthogonal if they are uncorrelated, and minimum variance is similar to a minimum distance condition in $\mathbf{R}^n$.

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In the filtering problem, the vector space defined above is used the following way.

The sequence of measurements $\mathbf{Z}_k = \{\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_k\}$ is a realization of the set of random variables denoted for simplicity with the same notation $\{\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_k\}$. The associated vector space is $\mathcal{Z}_k$. An orthonormal basis can be determined for this space and contain vectors $\{\mathrm{u}_1, \mathrm{u}_2, ..., \mathrm{u}_k\}$.

Next, an infinite dimensional vector space $\mathcal{X}$ is considered to contain the random variable of interest $\mathbf{x}_k$.

The filtering problem is to find a random variable element in the $\mathbf{Z}_k$ space such that it best approximates in the minimum variance sense $\mathbf{x}_k$. Using terms loosely borrowed from the Euclidean parallel made above, it is needed to project the $\mathbf{x}_k$ element in the $\mathcal{Z}_k$ space such that the distance between them is minimum.

$\hat{\mathbf{x}}_k$ is the approximation element belonging to the $\mathcal{Z}_k$ space. Notation $\epsilon_k$ is introduced for the approximation error $\hat{\mathbf{x}}_k - \mathbf{x}_k$, which to be optimum needs be orthogonal to the $\mathbf{Z}_k$ space.

$$\langle \hat{\mathbf{x}}_k - \mathbf{x}_k, \mathbf{a} \rangle = \langle \epsilon_k, \mathbf{a} \rangle = 0, \forall \mathbf{a} \in \mathbf{Z}_k \Leftrightarrow \tag{G.19}$$

$$\mathrm{E}\langle \epsilon_k \mathbf{a}^T \rangle = 0, \forall \mathbf{a} \in \mathbf{Z}_k \tag{G.20}$$

In other words, the optimum estimation error $\epsilon_k$ is uncorrelated with any of the measurements $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ and also to any linear combination of these measurements.

Figure G.1 below shows a geometric interpretation, where the infinite dimension $\mathcal{X}$ space is a 3-D space, and the approximation space is a two-dimensional plane. The best approximation for element $\mathbf{x}$ in the plane $\mathcal{Z}$ and in the minimum distance sense is the projected element $\hat{\mathbf{x}}$.

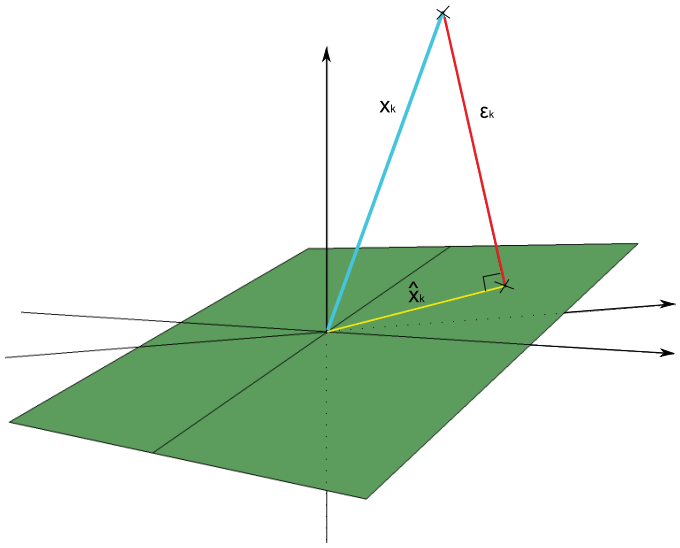


**Figure G.1:** Orthogonal Projections

## G.3 Recursive relation for the estimate

Using Orthogonal Projections, the minimum variance estimator $\hat{\mathbf{x}}_k$ is first expressed in relation to $\hat{\mathbf{x}}_{k-1}$. Then coefficients of this recursive relation are determined.

$\hat{\mathbf{x}}_k$, the projection of the unknown r.v. $\mathbf{x}_k$ belonging to the infinite space $\mathcal{X}$ into the $\mathcal{Z}_k$ finite observation space is the sum of the component wise projections onto the basis vectors $\mathbf{u}_i$.

$$\hat{\mathbf{x}}_k = \sum_{i=1}^{k} \mathrm{E}\langle \mathbf{x}_k \mathbf{u}_i^T \rangle \mathbf{u}_i \tag{G.21}$$

This has geometric meaning, but is also shown below.

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From the Orthogonal Projections it is known that the estimator $\hat{\mathbf{x}}_k$ is uncorrelated with any of the elements of the $\mathcal{Z}_k$ space, and in particular also with the basis vectors $\mathbf{u}_i$. Thus:

$$\mathrm{E}\langle (\hat{\mathbf{x}}_k - \mathbf{x}_k)\mathbf{u}_i^T \rangle = 0; \ \ \forall i \Leftrightarrow \tag{G.22}$$

$$\mathrm{E}\langle \mathbf{x}_k \mathbf{u}_i^T \rangle = \mathrm{E}\langle \hat{\mathbf{x}}_k \mathbf{u}_i^T \rangle; \ \ \forall i \tag{G.23}$$

By multiplying with $\mathbf{u}_i$ and summing across $i$:

$$\sum_{i=1}^{k} \mathrm{E}\langle \mathbf{x}_k \mathbf{u}_i^T \rangle \mathbf{u}_i = \sum_{i=1}^{k} \mathrm{E}\langle \hat{\mathbf{x}}_k \rangle \mathbf{u}_i^T \mathbf{u}_i \tag{G.24}$$

Further, $\hat{\mathbf{x}}_k$ belongs to the $\mathcal{Z}_k$ space and therefore can be written as a linear combination of the basis vectors:

$$\hat{\mathbf{x}}_k = \sum_{i=1}^{k} A_i \mathbf{u}_i \tag{G.25}$$

By replacing this expression of $\hat{\mathbf{x}}_k$ in (G.24) above, the entire right hand term will reduce to $\hat{\mathbf{x}}_k$, resulting in:

$$\sum_{i=1}^{k} \mathrm{E}\langle \mathbf{x}_k \mathbf{u}_i^T \rangle \mathbf{u}_i = \hat{\mathbf{x}}_k \tag{G.26}$$

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Next, to obtain a recursive dependence, two steps are performed. First, the sum is split by separating the term containing the last basis vector.

$$\hat{\mathbf{x}}_k = \sum_{i=1}^{k-1} \mathrm{E}\langle \mathbf{x}_k \mathbf{u}_i^T \rangle \mathbf{u}_i + \mathrm{E}\langle \mathbf{x}_k \mathbf{u}_k^T \rangle \mathbf{u}_k \tag{G.27}$$

Secondly, in the partial sum term, $\mathbf{x}_k$ is expanded according to the dynamic equation of the system (G.1).

$$\hat{\mathbf{x}}_k = \sum_{i=1}^{k-1} \mathrm{E}\langle (\Phi \mathbf{x}_{k-1} + \Gamma \mathbf{w}_{k-1}) \mathbf{u}_i^T \rangle \mathbf{u}_i + \mathrm{E}\langle \mathbf{x}_k \mathbf{u}_k^T \rangle \mathbf{u}_k \tag{G.28}$$

$$= \Phi \sum_{i=1}^{k-1} \mathrm{E}\langle \mathbf{x}_{k-1} \mathbf{u}_i^T \rangle \mathbf{u}_i + \mathrm{E}\langle \mathbf{x}_k \mathbf{u}_k^T \rangle \mathbf{u}_k \tag{G.29}$$

$$= \Phi \hat{\mathbf{x}}_{k-1} + \mathrm{E}\langle \mathbf{x}_k \mathbf{u}_k^T \rangle \mathbf{u}_k \tag{G.30}$$

The term $\mathrm{E}\langle \mathbf{w}_{k-1} \mathbf{u}_i^T \rangle$ disappeared because the measurements $\mathcal{Z}_{k-1}$ are independent of the random variable $\mathbf{w}_{k-1}$ and the mean of this r.v. is 0 by the problem context.

Also, the following underlying assumption about the basis vectors has been made. A set of $k-1$ measurement is considered. The space defined by the $k-1$ random variables $\mathbf{z}_i$ can be described by an orthonormal basis $\{\mathbf{u}_i, \ldots, \mathbf{u}_{k-1}\}$. When a new measurement is considered, and a new random variable $\mathbf{z}_k$, the space is extended with one dimension. Consequently the orthonormal basis is extended but in such a way as to keep the previous vectors and only add a new one $\mathbf{u}_k$. Thus $\{\mathbf{u}_i, \ldots, \mathbf{u}_{k-1}\}$ still describes the space before the last measurement. Thus the identification of $\hat{\mathbf{x}}_{k-1}$ can be made using (G.21).

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The next objective is to better express the term $\mathrm{E}\langle \mathbf{x}_k \mathbf{u}_k^T \rangle \mathbf{u}_k$ so as not to use the unknown the basis term $\mathbf{u}_k$ which is impractical to calculate.

It is noted first than by splitting the sum in the first step, $\hat{\mathbf{x}}_k$ has become expressed as a sum of two terms, the first belonging to space $\mathcal{Z}_{k-1}$ and the second along the uncorrelated probabilistic-direction $\mathbf{u}_k$.

Thus we are looking for another vector that is placed along $\mathbf{u}_k$, i.e. belonging to $\mathcal{Z}_k$ and orthogonal (probabilistic uncorrelated) with $\mathcal{Z}_{k-1}$. Then $\mathrm{E}\langle \mathbf{x}_k \mathbf{u}_k^T \rangle \mathbf{u}_k$ can be replaced by this new vector-term scaled by a matrix coefficient.

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It will be shown next that

$$\mathbf{z}_k - H\Phi\hat{\mathbf{x}}_{k-1} \tag{G.31}$$

is probabilistic uncorrelated to $\mathcal{Z}_{k-1}$ and that it belongs to $\mathcal{Z}_k$.

The Orthogonality Projections are expressed for the estimation at time step $k-1$:

$$\mathrm{E}\langle(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1})\mathbf{z}_i^T\rangle = 0;\ \ \forall i = \overline{1, k-1} \tag{G.32}$$

$$\tag{G.33}$$

It is then multiplied by $\Phi$:

$$\mathrm{E}\langle(\Phi\mathbf{x}_{k-1} - \Phi\hat{\mathbf{x}}_{k-1})\mathbf{z}_i^T\rangle = 0;\ \ \forall i = \overline{1, k-1} \tag{G.34}$$

Using the system dynamic equation (G.1) and because, as mentioned before, $\mathrm{w}_{k-1}$ is independent of all $\mathbf{z}_i$ with $i \leq k-1$ and with mean 0, it can written as:

$$\mathrm{E}\langle(\mathbf{x}_k - \Phi\hat{\mathbf{x}}_{k-1})\mathbf{z}_i^T\rangle = 0;\ \ \forall i = \overline{1, k-1} \tag{G.35}$$

This is now multiplied with matrix $H$. Using the system observation equation (G.2) and because $\mathrm{v}_k$ is independent of all $\mathbf{z}_i$ with $i \leq k-1$ and with mean 0:

$$\mathrm{E}\langle(\mathbf{z}_k - H\Phi\hat{\mathbf{x}}_{k-1})\mathbf{z}_i^T\rangle = 0;\ \ \forall i = \overline{1, k-1} \tag{G.36}$$

Thus $\mathbf{z}_k - H\Phi\hat{\mathbf{x}}_{k-1}$ is probabilistic uncorrelated with the space $\mathcal{Z}_{k-1}$. It also belongs to $\mathcal{Z}_k$ since it is a linear combination of terms belonging to $\mathcal{Z}_k$.

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Therefore:

$$\mathrm{E}\langle\mathbf{x}_k\mathbf{u}_k^T\rangle\mathbf{u}_k = K_k(\mathbf{z}_k - H\Phi\hat{\mathbf{x}}_{k-1}) \tag{G.37}$$

where $K_k$ is an unknown scaling matrix, and the recursive relation for $\hat{\mathbf{x}}_k$ is:

$$\hat{\mathbf{x}}_k = \Phi\hat{\mathbf{x}}_{k-1} + K_k(\mathbf{z}_k - H\Phi\hat{\mathbf{x}}_{k-1}) \tag{G.38}$$

## G.4 Determination of the matrix $K$

The Orthogonal Projections are used in relation with the most actual measurement $\mathbf{z_k}$:

$$\mathrm{E}\langle(\hat{\mathbf{x}}_k - \mathbf{x}_k)\mathbf{z}_k^T\rangle = 0 \tag{G.39}$$

The expression of the estimation error using the recursive formulation has already been expanded in 4.3 and given in (4.41). It is repeated below for visibility:

$$\hat{\mathbf{x}}_k - \mathbf{x}_k = \epsilon_k = (I - K_kH)\Phi\epsilon_{k-1} - (I - K_kH)\Gamma\mathbf{w}_{k-1} + K_k\mathbf{v}_k$$

The measurement $\mathbf{z}_k$ is also expressed in an appropriate form for calculations:

$$\mathbf{z}_k = H\Phi(\hat{\mathbf{x}}_k - \epsilon_{k-1}) + H\Gamma\mathbf{w}_{k-1} + \mathbf{v}_k \tag{G.40}$$

This are now replaced in (G.39):

$$\mathrm{E}\langle\Big((I - K_kH)\Phi\epsilon_{k-1} - (I - K_kH)\Gamma\mathbf{w}_{k-1} + K_k\mathbf{v}_k\Big)\Big(H\Phi(\hat{\mathbf{x}}_k - \epsilon_{k-1}) + H\Gamma\mathbf{w}_{k-1} + \mathbf{v}_k\Big)^T\rangle = 0$$

The following terms become 0:

| | |
|---|---|
| $\mathrm{E}\langle\epsilon_{k-1}\hat{\mathbf{x}}_{k-1}^T\rangle$ | - from the O.P., the error is uncorrelated with any vector in $\mathcal{Z}$ space |
| $\mathrm{E}\langle\epsilon_{k-1}\mathbf{w}_{k-1}^T\rangle$ | - $\mathbf{w}_{k-1}$ is not correlated with the error term before time index $k$ |
| $\mathrm{E}\langle\epsilon_{k-1}\mathbf{v}_k^T\rangle$ | - $\mathbf{v}_k$ is not correlated with the error term before time index $k$ |
| $\mathrm{E}\langle\mathbf{v}_k\mathbf{w}_{k-1}^T\rangle$ | - the noise processes $\mathbf{v}$ and $\mathbf{w}$ are assumed uncorrelated |

For the second moment of the estimation error, notation $\mathrm{E}\langle\epsilon_{k-1}\epsilon_{k-1}^T\rangle = P_{k-1}^+$ is introduced. Also, $\mathrm{E}\langle\mathbf{w}_{k-1}\mathbf{w}_{k-1}^T\rangle = Q$ and $\mathrm{E}\langle\mathbf{v}_k\mathbf{v}_k^T\rangle = R$. And expression (G.39) becomes:

$$(\Phi P_{k-1}^+\Phi^T + \Gamma Q\Gamma^T)H^T = K_k\Big(R + H(\Gamma Q\Gamma^T + \Phi P_{k-1}^+\Phi^T)H^T\Big) \tag{G.41}$$

The term $\Phi P_{k-1}\Phi^T + \Gamma Q\Gamma^T$ reflects the covariance propagation through the system dynamics, as shown in section 4.3.3. It is noted with $P_k^-$.

Thus finally, the expression of the gain matrix $K_k$ is found:

$$K_k = P_k^- H^T(R + HP_k^- H^T) \tag{G.42}$$

## G.5 Covariance relations

The only missing element now is a propagation expression for $P_k^+$. This can be obtained from (4.41).

$$\begin{aligned} P_k^+ = \mathrm{E}\langle\epsilon_k\epsilon_k^T\rangle = \mathrm{E}\langle\Big((I - K_kH)\Phi\epsilon_{k-1} - (I - K_kH)\Gamma\mathbf{w}_{k-1} + K_k\mathbf{v}_k\Big)\cdot \\ \cdot\Big(\epsilon_{k-1}^T\Phi^T(I - K_kH)^T - \mathbf{w}_{k-1}^T\Gamma^T(I - K_kH)^T + \mathbf{v}_k^TK_k^T\Big)\rangle \end{aligned} \tag{G.43}$$

The following terms become 0:

| | |
|---|---|
| $\mathrm{E}\langle\epsilon_{k-1}\mathbf{w}_{k-1}^T\rangle$ | - $\mathbf{w}_{k-1}$ is not correlated with the error term before time index $k$ |
| $\mathrm{E}\langle\epsilon_{k-1}\mathbf{v}_k^T\rangle$ | - $\mathbf{v}_k$ is not correlated with the error term before time index $k$ |
| $\mathrm{E}\langle\mathbf{v}_k\mathbf{w}_{k-1}^T\rangle$ | - the noise processes $\mathbf{v}$ and $\mathbf{w}$ are assumed uncorrelated |

By noting that $\mathrm{E}\langle\epsilon_{k-1}\epsilon_{k-1}^T\rangle = P_{k-1}^+$, $\mathrm{E}\langle\mathbf{w}_{k-1}\mathbf{w}_{k-1}^T\rangle = Q$ and $\mathrm{E}\langle\mathbf{v}_k\mathbf{v}_k^T\rangle = R$, it is obtained:

$$\begin{aligned} P_k^+ &= (I - K_kH)\Phi P_{k-1}^+\Phi^T(I - K_kH)^T + (I - K_kH)\Gamma Q\Gamma^T(I - K_kH)^T + K_kRK_k^T \\ &= (I - K_kH)(\Phi P_{k-1}^+\Phi^T + \Gamma Q\Gamma^T)(I - K_kH)^T + K_kRK_k^T && \text{(G.44)} \\ &= (I - K_kH)P_k^-(I - K_kH)^T + K_kRK_k^T && \text{(G.45)} \end{aligned}$$

(G.45) is the Joseph form presented and used in the report.

If in the Joseph form the optimal gain matrix expression (G.42) in used, the following simplified and familiar expression is found:

$$P_k^+ = (I - K_kH)P_k^- \tag{G.46}$$

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Appendix H

# Kalman filter using least squares

*This appendix presents derivation steps for obtaining the discrete Kalman filter equations using a least squares method. In this method, noises* $\mathbf{w}$ *and* $\mathbf{v}$ *are not considered to be random variables, but deterministic errors quantities that are unknown. The content of this chapter is mainly based on Jazwinski [14, chap. 7]. For simplification, the system is time-invariant and the modeling errors and disturbances* $\mathbf{w}$ *are considered 0.*

## H.1 Simplified LS approach

The filtering problem can be viewed as composed of a system of linear observation equations with $\mathbf{x}_k$ the unknowns:

$$\mathbf{z}_k = H\mathbf{x}_k \tag{H.1}$$

and a set of constraints coming from the dynamics:

$$\mathbf{x}_{k+1} = \Phi\mathbf{x}_k \tag{H.2}$$

It can be seen that by using the constraints there is only one unknown, for example $\mathbf{x}_0$ since all the $\mathbf{x}_k$ can then be expressed in terms of $\mathbf{x}_0$ using the propagation relation. Thus there are N measurement constraints and one unknown and the system is overdetermined.

For all practical cases, no matter how the unknown is chosen only one of the equations will be guaranteed to be fulfilled and the others will not. To solve this problem, a least squares approach can be used, which to chose solution $\hat{\mathbf{x}}_0$ that has the property that residual errors from the observation equations $\epsilon_k = \mathbf{z}_k - H\hat{\mathbf{x}}_k = \mathbf{z}_k - H\Phi^k\hat{\mathbf{x}}_0$ are smaller than those of any other choice of $\mathbf{x}_0$.

In the estimation problem it is not $\hat{\mathbf{x}}_0$ that is of interest, but rather $\hat{\mathbf{x}}_N$. $\hat{\mathbf{x}}_N$ can be easily found by $(\Phi^N\mathbf{x}_0)$, however in keeping with the derivations line from [14] we can use $\hat{\mathbf{x}}_N$ as the unknown from the beginning, by noting that $\mathbf{x}_k = \Phi^{k-N}\mathbf{x}_N$.

The least squares method is to minimize a cost function that is the sum of the residual errors. Because the error elements can be both negative and positive, a sum of quadratic terms is

appropriate for the minimization. It is also noted that $\epsilon_k$ are vectors.

$$J_N = \frac{1}{2}\sum_{k=1}^{N}\epsilon_k^T\epsilon_k = \frac{1}{2}\sum_{k=1}^{N}(\mathbf{z}_k - H\Phi^{k-N}\mathbf{x}_N)^T(\mathbf{z}_k - H\Phi^{k-N}\mathbf{x}_N) \tag{H.3}$$

$$\hat{\mathbf{x}}_N = \text{argmin}_{\mathbf{x}_N} J_N \tag{H.4}$$

To minimize $J_N$, its derivative with respect to $\hat{\mathbf{x}}_N$ is set to 0.

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For convenience, vector derivation rules are summarized below:

$$\frac{d}{dx}\Big(\mathbf{x}^T A\mathbf{x}\Big) = \mathbf{x}^T(A^T + A) = \mathbf{x}^T A^T + \mathbf{x}^T A \tag{H.5}$$

$$\frac{d}{dx}\Big(\mathbf{a}^T\mathbf{x}\mathbf{x}^T\mathbf{b}\Big) = \mathbf{x}^T(\mathbf{a}\mathbf{b}^T + \mathbf{b}\mathbf{a}^T) \tag{H.6}$$

$$\frac{d}{dx}\Big((A\mathbf{x}+\mathbf{b})^T C(D\mathbf{x}+\mathbf{e})\Big) = (D\mathbf{x}+\mathbf{e})^T C^T A + (A\mathbf{x}+\mathbf{b})^T CD \tag{H.7}$$

$$\frac{d}{dx}(\mathbf{a}^T\mathbf{x}) = \frac{d}{dx}(\mathbf{x}^T\mathbf{a}) = \mathbf{a} \tag{H.8}$$

$$\frac{d}{dx}(A\mathbf{x}) = \frac{d}{dx}(\mathbf{x}^T A^T) = A \tag{H.9}$$

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Using the rule in (H.7), the minimization of $J_N$ is equivalent to:

$$\sum_{k=1}^{N}(\mathbf{z}_k - H\Phi^{k-N}\hat{\mathbf{x}}_N)^T H\Phi^{k-N} = 0 \tag{H.10}$$

$$\sum_{k=1}^{N}\mathbf{z}_k^T H\Phi^{k-N} = \mathbf{x}_N^T\sum_{k=1}^{N}(H\Phi^{k-N})^T H\Phi^{k-N} \tag{H.11}$$

Matrix terms $(H\Phi^{k-N})^T H\Phi^{k-N}$ are square of size $n \times n$ where $n$ is the size of vector $\mathbf{x}_k$. If their sum, the *information matrix* $\mathcal{I}_N$,

$$\mathcal{I}_N = \sum_{k=1}^{N}(H\Phi^{k-N})^T H\Phi^{k-N} \tag{H.12}$$

is an invertible matrix, then the solution $\mathbf{x}_N$ can now be found from:

$$\left(\sum_{k=1}^{N}\mathbf{z}_k^T H\Phi^{k-N}\right)\left(\sum_{k=1}^{N}(H\Phi^{k-N})^T H\Phi^{k-N}\right)^{-1} = \mathbf{x}_N^T \tag{H.13}$$

by using the property of the inverse operation and of the transpose operation of having interchangeable order:

$$\hat{\mathbf{x}}_N = \left(\sum_{k=1}^{N}(H\Phi^{k-N})^T H\Phi^{k-N}\right)^{-1}\left(\sum_{k=1}^{N}(\Phi^{k-N})^T H^T\mathbf{z}_k\right) \tag{H.14}$$

This method of estimation is called batch processing because all measurements are processed at the same time. When a new observation is available the process must be repeated from the beginning with a now extended by one data set. Thus the computation load is increases from one step to the next. It can quickly become very slow and unpractical.

Also a large matrix, the information matrix, must be inverted at every step.

## H.2 Recursive Least Squares

In this section, the steps of deriving the recursive formulation of the least square estimator are presented.

By using the notation $\mathcal{I}_N$ and introducing a new notation $M_N$, equation (H.14) is expressed as:

$$\hat{\mathbf{x}}_N = \mathcal{I}_N^{-1} M_N; \text{where} \tag{H.15}$$

$$M_N = \sum_{k=1}^{N} (\Phi^{k-N})^T H^T \mathbf{z}_k \tag{H.16}$$

Consequently,

$$\hat{\mathbf{x}}_{N+1} = \mathcal{I}_{N+1}^{-1} M_{N+1} \tag{H.17}$$

Recursive or propagation expressions can be derived for $\mathcal{I}_N$ and $M_N$ in the following way.

$$\mathcal{I}_{N+1} = \sum_{k=1}^{N+1} (H\Phi^{k-N-1})^T H\Phi^{k-N-1} \tag{H.18}$$

$$= (\Phi^{-1})^T \left( \sum_{k=1}^{N} (H\Phi^{k-N})^T H\Phi^{k-N} \right) (\Phi^{-1}) + H^T H \tag{H.19}$$

$$= (\Phi^{-1})^T \mathcal{I}_N \Phi^{-1} + H^T H \tag{H.20}$$

$$M_{N+1} = \sum_{k=1}^{N+1} (\Phi^{k-N-1})^T H^T \mathbf{z}_k \tag{H.21}$$

$$= (\Phi^{-1})^T \sum_{k=1}^{N} (\Phi^{k-N})^T H^T \mathbf{z}_k + H^T \mathbf{z}_k \tag{H.22}$$

$$= (\Phi^{-1})^T M_N + H^T \mathbf{z}_k \tag{H.23}$$

The expression of $\hat{\mathbf{x}}_{N+1}$ from (H.17) becomes:

$$\hat{\mathbf{x}}_{N+1} = \left( (\Phi^{-1})^T \mathcal{I}_N \Phi^{-1} + H^T H \right)^{-1} \left( (\Phi^{-1})^T M_N + H^T \mathbf{z}_k \right) \tag{H.24}$$

Two notation changes are made at this point that remind of the conventional *a priori* Kalman notation:

$$(P_N)^{-1} = \mathcal{I}_N \tag{H.25}$$

$$(P_{N+1}^-)^{-1} = (\Phi^{-1})^T (P_N^+)^{-1} \Phi^{-1} \tag{H.26}$$

$$\hat{\mathbf{x}}_{N+1}^- = \Phi \hat{\mathbf{x}}_N \tag{H.27}$$

Also, it can be checked that:

$$(\Phi^{-1})^T M_N = (P_{N+1}^-)^{-1} \hat{\mathbf{x}}_{N+1}^- \tag{H.28}$$

Thus in the next step the expression of $\hat{\mathbf{x}}_{N+1}$ from (H.24) becomes:

$$\hat{\mathbf{x}}_{N+1} = \left( (P_{N+1}^-)^{-1} + H^T H \right)^{-1} \left( (P_{N+1}^-)^{-1} \hat{\mathbf{x}}_{N+1}^- + H^T \mathbf{z}_k \right) \tag{H.29}$$

It is noted here that the term $\left((P_{N+1}^{-})^{-1}+H^TH\right)^{-1}$ is still $\mathcal{I}_{N+1}$ and needs to be invertible.

A matrix equality is used next to reformulate the above term. If $P > 0$, then the following two equalities holds[14]:

$$(P^{-1} + H^TR^{-1}H)^{-1} = P - PH^T(HPH^T + R)^{-1}HP \tag{H.30}$$

$$(P^{-1} + H^TR^{-1}H)^{-1}H^TR^{-1} = PH^T(HPH^T + R)^{-1} \tag{H.31}$$

Equation (H.29) will first be expanded by breaking the second parenthesis, and then the two matrix equalities are used:

$$\hat{\mathbf{x}}_{N+1} = \left((P_{N+1}^{-})^{-1} + H^TH\right)^{-1}(P_{N+1}^{-})^{-1}\hat{\mathbf{x}}_{N+1}^{-} + \left((P_{N+1}^{-})^{-1} + H^TH\right)^{-1}H^T\mathbf{z}_k \tag{H.32}$$

$$= \left(P_{N+1}^{-} - P_{N+1}^{-}H^T(HP_{N+1}^{-}H^T + I)^{-1}HP_{N+1}^{-}\right)(P_{N+1}^{-})^{-1}\hat{\mathbf{x}}_{N+1}^{-} + \\ + P_{N+1}^{-}H^T(HP_{N+1}^{-}H^T + I)^{-1}\mathbf{z}_k \tag{H.33}$$

$$= \hat{\mathbf{x}}_{N+1}^{-} + P_{N+1}^{-}H^T(HP_{N+1}^{-}H^T + I)^{-1}(\mathbf{z}_k - H\hat{\mathbf{x}}_{N+1}^{-}) \tag{H.34}$$

The term $P_{N+1}^{-}H^T(HP_{N+1}^{-}H^T + I)^{-1}$ is defined as $K_k$, the Kalman gain, and the recursive expression of the estimator is found in the known form:

$$\hat{\mathbf{x}}_{N+1} = \hat{\mathbf{x}}_{N+1}^{-} + K_k(\mathbf{z}_k - H\hat{\mathbf{x}}_{N+1}^{-}) \tag{H.35}$$

## H.3 Formulation of the complete problem

The simplified problem omitted to take into considerations the system noise $\mathbf{w}$ and consider weights for the initial state and the process noise. The complete problem is formulated in this section, but the calculations are not performed.

The system of linear observation equations with $\mathbf{x}_k$:

$$\mathbf{z}_k = H\mathbf{x}_k \tag{H.36}$$

and a set of constraints coming from the dynamics in which the errors are addressed explicitly.

$$\mathbf{x}_{k+1} = \Phi\mathbf{x}_k + \Gamma\mathbf{w}_k \tag{H.37}$$

The unknowns of the system $\mathbf{x}_k$ but also the constraint error $\mathbf{w}_k$.

Also, the initial state $\bar{\mathbf{x}}_0$ is known and given together with confidence measure in the form of a uncertainty weight (equivalent to a covariance) matrix $P_0$.

The cost function of the least squares method is constructed in such a way as to express three criteria: the unknowns must pass as close as possible to the observations i.e. the observation errors should be small, the constraint errors $\mathbf{w}_k$ should be small, and the error in the estimated initial state should also be small. Because the notion "small" can have different proportions between these three types of errors, weights are also introduced. Finally the 0.5 factor, which has been also seen in the simplified formulation, is sometimes introduced for aesthetic reasons as it eliminates a scaling factor of 2 that appears after taking the derivative.

The complete cost function is:

$$J_N = \frac{1}{2}\sum_{k=1}^{N}(\mathbf{z}_k - H\Phi^{k-N}\mathbf{x}_N)^TR^{-1}(\mathbf{z}_k - H\Phi^{k-N}\mathbf{x}_N) + \\ + \frac{1}{2}\sum_{k=1}^{N}\mathbf{w}_k^TQ^{-1}\mathbf{w}_k + \frac{1}{2}(\bar{\mathbf{x}}_0 - \mathbf{x}_0)^TP_0^{-1}(\bar{\mathbf{x}}_0 - \mathbf{x}_0) \tag{H.38}$$

## H.4 Considerations for the multi-MEMS system

The information matrix introduced above is linked with the system property of observability. It is a scaled version of the observability gramian matrix.

For the multi-MEMS state space system, the information matrix is not invertible. This can be easily checked, for example with MATLAB®.

$$\mathcal{I}_N = \sum_{k=1}^{N} H^T H; \quad \text{where} \quad H = \begin{bmatrix} \mathbf{1}_{N_s} & \mathbf{I}_{N_s} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{H.39}$$

If the number of sensors $N_s = 8$, then a numerical example shows:

$$\mathcal{I}_N = \begin{pmatrix} N & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N \\ 0 & N & 0 & 0 & 0 & 0 & 0 & 0 & N \\ 0 & 0 & N & 0 & 0 & 0 & 0 & 0 & N \\ 0 & 0 & 0 & N & 0 & 0 & 0 & 0 & N \\ 0 & 0 & 0 & 0 & N & 0 & 0 & 0 & N \\ 0 & 0 & 0 & 0 & 0 & N & 0 & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 & N & 0 & N \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & N & N \\ N & N & N & N & N & N & N & N & 8N \end{pmatrix} \tag{H.40}$$

This matrix can easily be seen to be singular as the last line is a linear combination of the the above other 7. For all $N \geq 1$, $\mathcal{I}_N$ has one eigen value that is zero.

Thus, batch processing cannot be applied as a method of estimation in the form presented here. It is however interesting that the recursive expression does not have this problem.

This comes from the change introduced by using the two matrix equalities. The matrix equalities (H.30) and (H.31) have meaning only when the term $P > 0$, and thus in fact do not apply when $P^{-1} = \mathcal{I}_N$ is not invertible. The general recursive formulation remains valid because knowledge of the initial state is introduced and it increases the information of the system. By adding the contribution of the initial state in the cost function $J$, the term to be inverted becomes $\left( (\Phi^{-N})^T P_0^{-1} \Phi^- N + I_N \right)$.