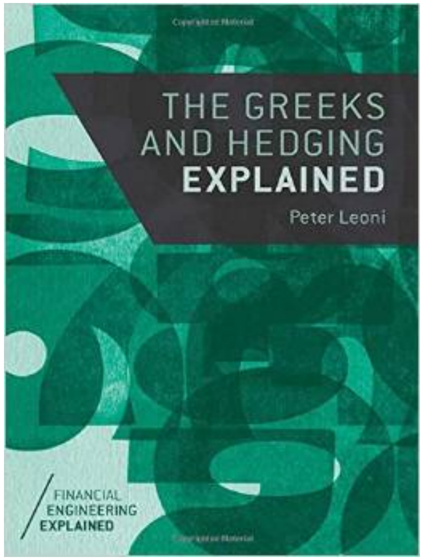
Option Hedging Simulation

The Greeks & Hedging Explained

Course Leader: Peter Leoni

The Greeks and Hedging Explained

1. Hedging Contingent Claims
2. Delta-hedging in the perfect world
3. The balance between Gamma and Theta
4. Trading is the answer to the unknown
5. Competition: Trading in practise
6. Vega as a crucial Greek
7. The Greek Expansion
8. Volatility Termstructure
9. Skew and Smile
10. Vanna-Volga
11. Volatility Trading
12. Competition: Option hedging



1. Hedging Contingent Claims
    - Hedging
    - Hedging Forwards
    - Put-Call parity
    - Binomial Tree model
    - Black-Scholes-Merton Model
    - Terminology

Hedging

- Hedging strategy
  - *A trading strategy aimed at minimizing exposures*
  - *Partial or full hedge*
  - *Static or dynamic hedge*

- Replicating Strategy:
  - *A trading strategy that matches all cashflows of another strategy or financial instrument*

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hedge

Line breaks: hedge

Pronunciation: /hɛdʒ /

Definition of *hedge* in English:

NOUN

1 A fence or boundary formed by closely growing bushes or shrubs:

'a privet hedge'

MORE EXAMPLE SENTENCES

SYNONYMS

2 A way of protecting oneself against financial loss or other adverse circumstances:

*'index-linked gilts are a useful **hedge against** inflation'*

MORE EXAMPLE SENTENCES

SYNONYMS

3 A word or phrase used to avoid overprecise commitment, for example *etc.*, *often*, or *sometimes*.

Hedging: Example

- FORWARD:
  - Investopedia Definition

DEFINITION OF 'FORWARD CONTRACT'

A customized contract between two parties to buy or sell an asset at a specified price on a future date. A forward contract can be used for hedging or speculation, although its non-standardized nature makes it particularly apt for hedging. Unlike standard futures contracts, a forward contract can be customized to any commodity, amount and delivery date. A forward contract settlement can occur on a cash or delivery basis. Forward contracts do not trade on a centralized exchange and are therefore regarded as over-the-counter (OTC) instruments. While their OTC nature makes it easier to customize terms, the lack of a centralized clearinghouse also gives rise to a higher degree of default risk. As a result, forward contracts are not as easily available to the retail investor as futures contracts.

  - Current Price of the contract: $S(t_0)$
  - Future Date: T
  - Specified Price: K

Hedging: Example

- FORWARD:
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  - Current Price of the contract: $S(t_0)$
  - Future Date: T
  - Specified Price: K
- How to best hedge?
  - Transact in the underlying immediately
- What is the fair level for K?
  - Cost of hedging

Hedging: Example (2)

- Trader *X* agrees to sell forward a stock at ***T***. What price ***K*** should he charge for this?

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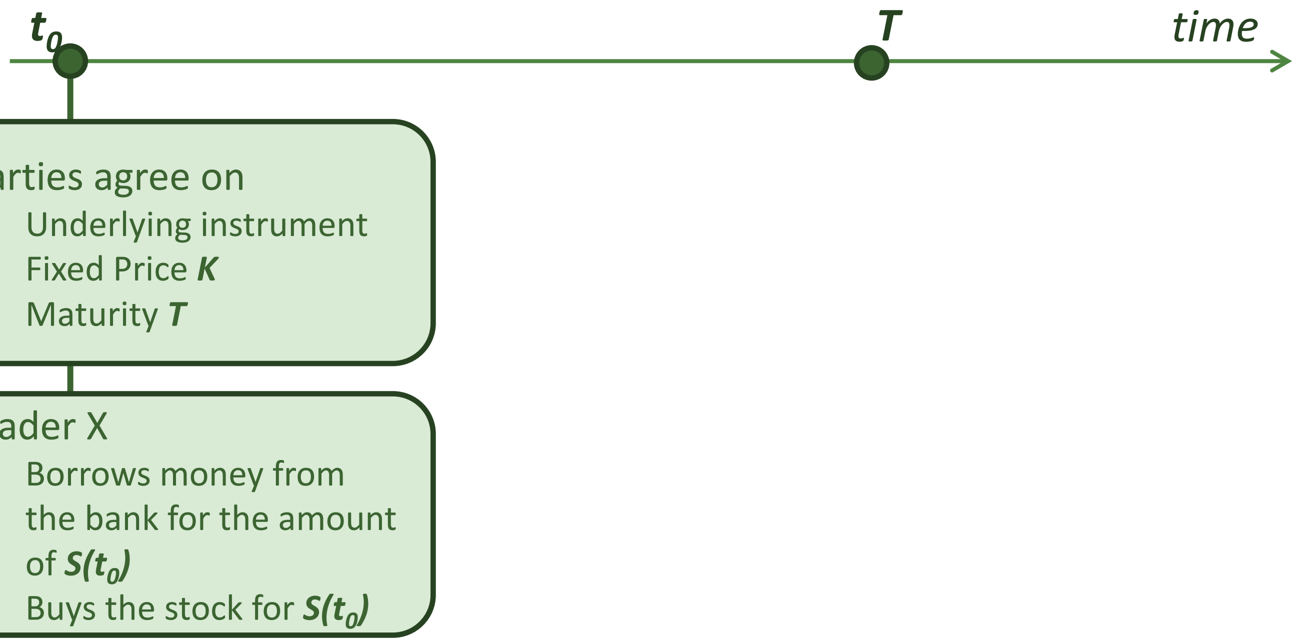

Hedging: Example (2)

- Trader *X* agrees to sell forward a stock at ***T***. What price ***K*** should he charge for this?

t_0 ***T*** *time*

Parties agree on
- Underlying instrument
- Fixed Price ***K***
- Maturity ***T***

Trader X
- Borrows money from the bank for the amount of ***$S(t_0)$***
- Buys the stock for ***$S(t_0)$***

Trader X
- Sells the stock for Fixed Price ***K***
- Pays back his loan to bank for an amount of of ***$S(t_0)$***
- Pays back the interest for the amount of ***$S(t_0)(\exp(r(T-t_0)) - 1)$***

Hedging: Example (2)

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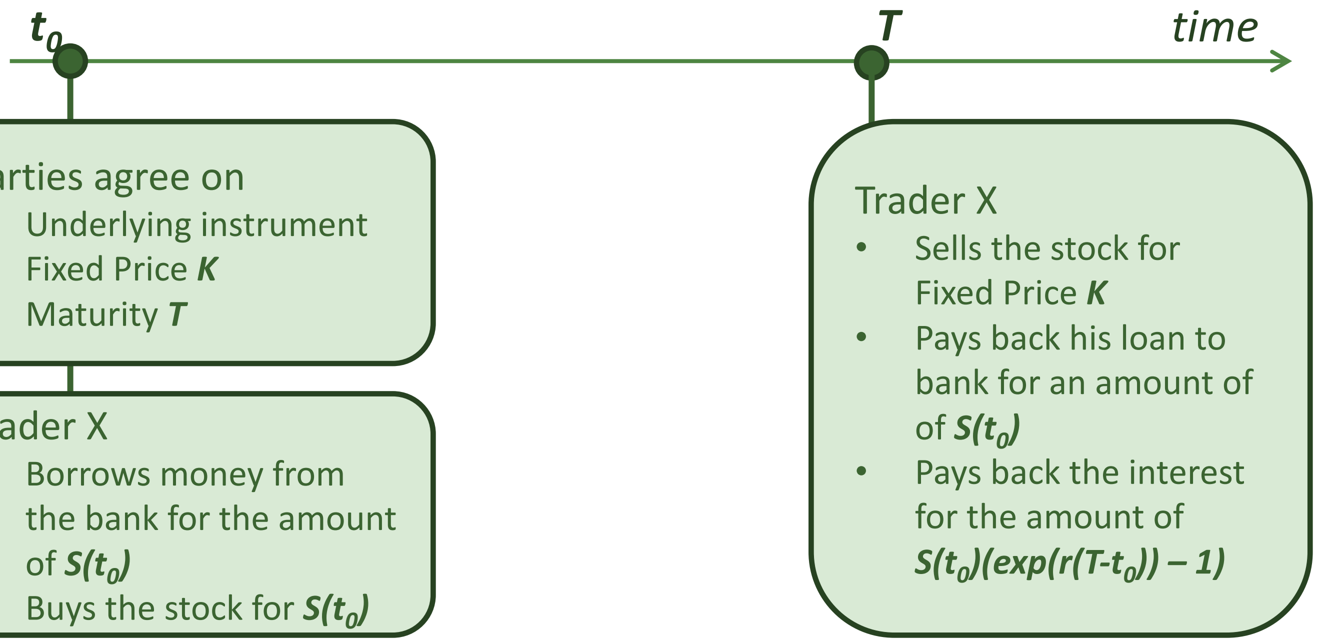

Cost of hedging/Fair Price:
$K = S(t_0) \exp(r(T-t_0))$

- **Perfect replicating Strategy**

Hedging: Example (3)

- Complications
  - Trader receives Dividends during ***[t_0, T]***: *i.e. forward price can be lower*
  - Good estimate for the dividends? Tax regime?
  - Which Interest rate?
  - Payment dates
  - Liquidity: can the trader really buy at present?
  - Replicating Strategy turns into Hedging strategy (imperfect)

Options and derivatives

A: A derivative is a contract between two or more parties whose value is based on an agreed-upon underlying financial asset, index or security. Common underlying instruments include: bonds, commodities, currencies, interest rates, market indexes and stocks.

Futures contracts, forward contracts, options, swaps and warrants are common derivatives. A futures contract, for example, is a derivative because its value is affected by the performance of the underlying contract. Similarly, a stock option is a derivative because its value is "derived" from that of the underlying stock.

Call option

Put option

An option is a contract that gives the buyer the right, but not the obligation, to buy or sell an underlying asset at a specific price on or before a certain date. An option, just like a stock or bond, is a security. It is also a binding contract with strictly defined terms and properties.

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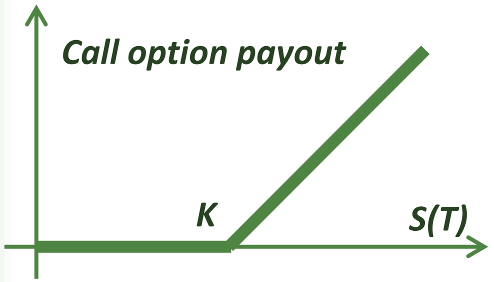
- Payout Profiles
  - Call Option: $\max(0, S(T) - K)$

Non-linear

Linear



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- Payout Profiles
  - Put Option: ***max(0, K – S(T))***

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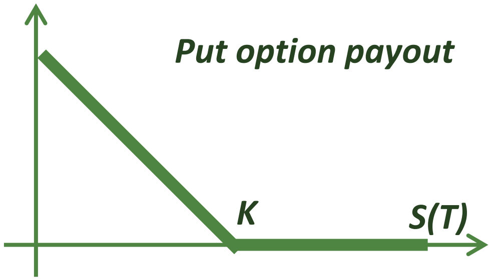
Linear



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- Payout Profiles

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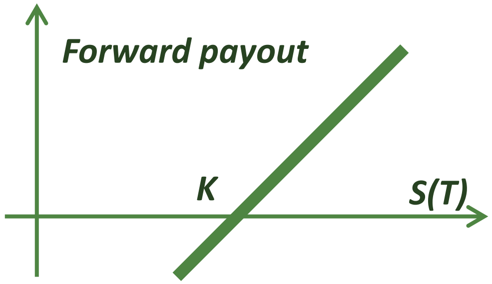
  – Forward Contract: ***S(T) – K*** *Linear*



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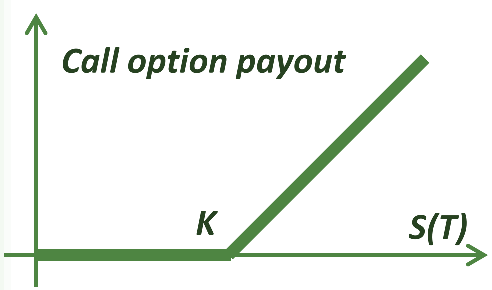
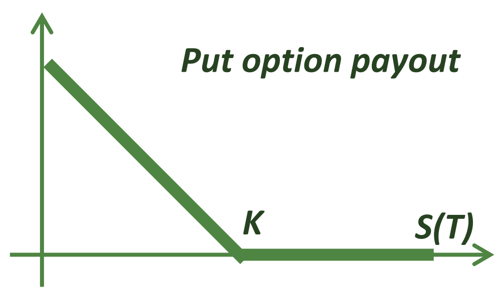
  - Put Option: ***max(0, K – S(T))***

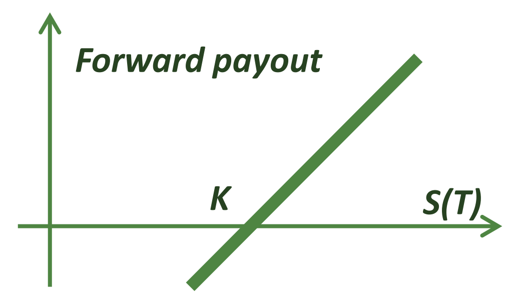
  Non-linear

  - Forward Contract: ***S(T) – K***

  Linear



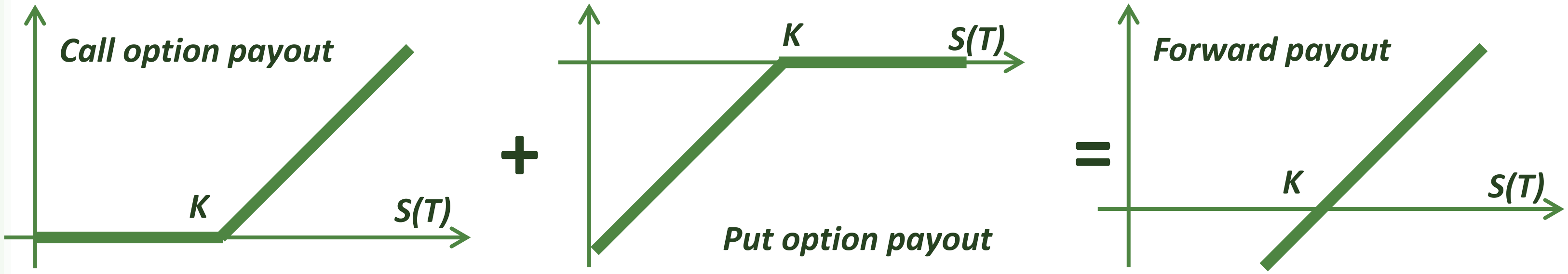




Options and derivatives

- Put – Call Parity

$$S(T) - K = \max(0, S(T) - K) - \max(0, K - S(T))$$



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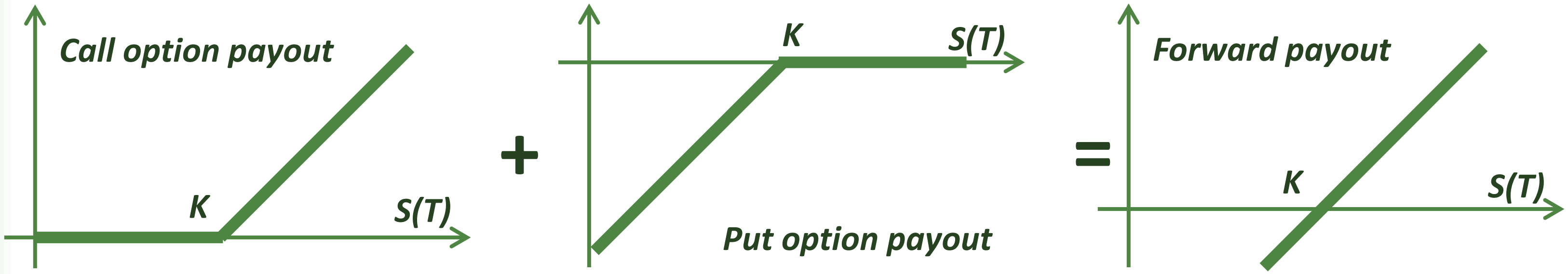

- Replicating strategy

$$S(t_0) - K\exp(-r(T-t_0)) = Call(K,T) - Put(K,T)$$

Binomial Tree

- Very simple universe for stock price:
  - Current Price: $S(t_0)=S_0$
  - Single timestep: t_0 to T
  - Only 2 price-outcomes:
    - Price up: $S(T)=S \times u = S_{up}$ with **u > 1**
    - Price down: $S(T)=S \times d = S_{down}$ with **d < 1**

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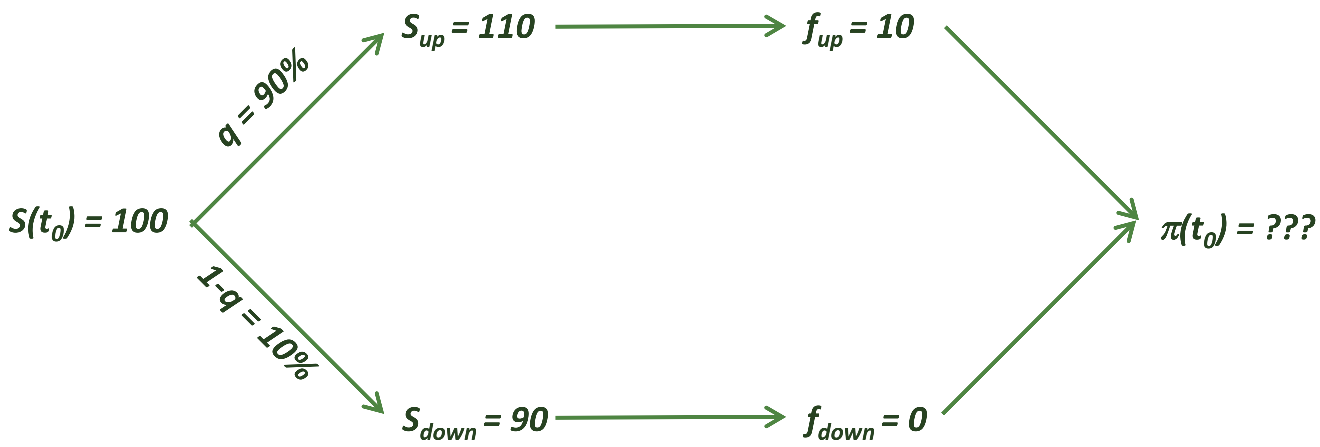
- Probabilistic description
  - Probability of "up" is q
  - Probability of "down" is $1-q$
- Options can be replicated and cost of hedging will set the price of the option
- Option/Derivatives payout in 2 states:
    - Price up $f(T)$
    - Price down $f(T)$

Binomial Tree (2)

- Example: large probability for up-move



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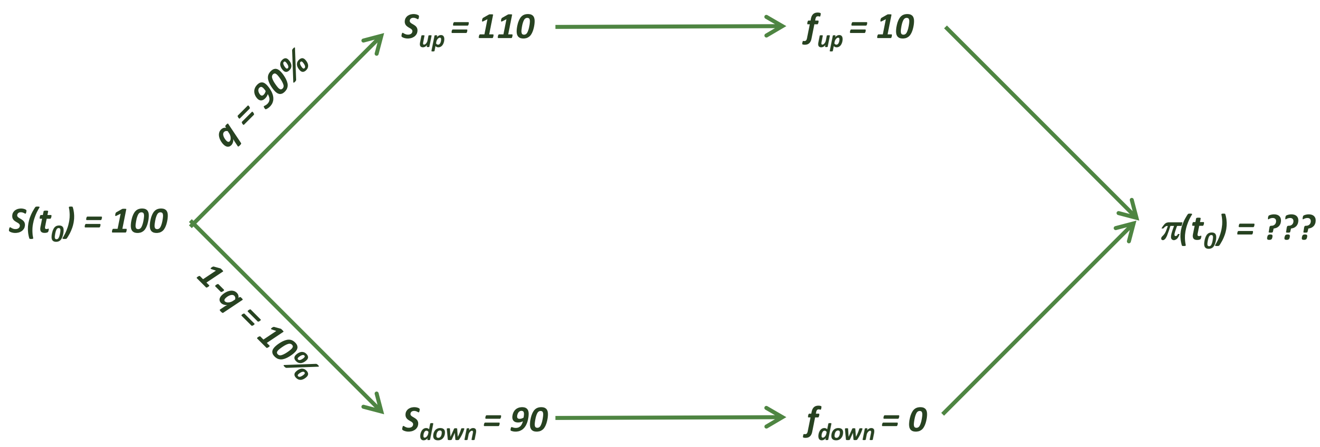
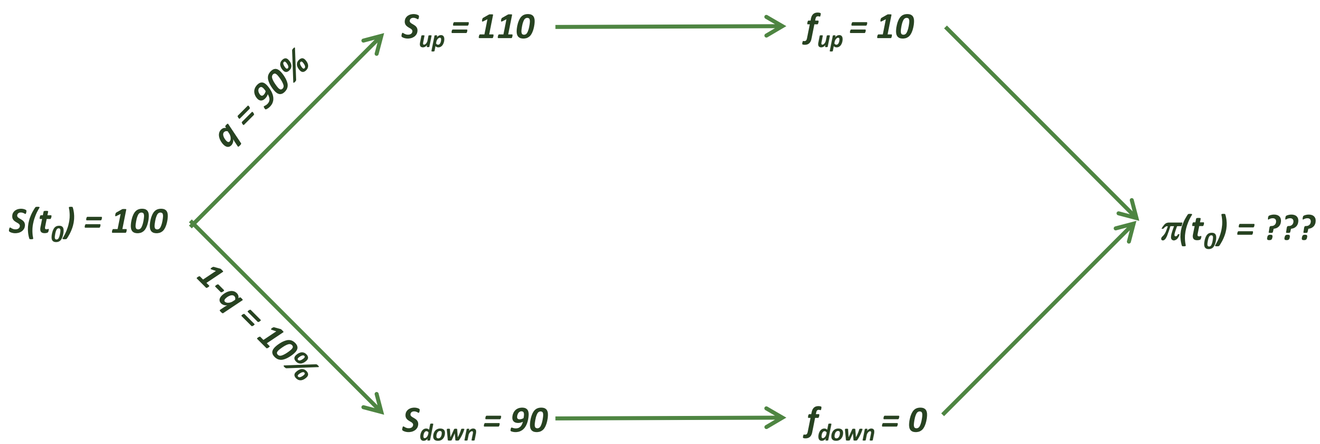


Actuarial Calculation

$$\pi = q\, fu + (1-q) f_d$$
$$= 0.9 \times 10 + 0.1 \times 0$$
$$= 9$$

Binomial Tree (4)

- Example: large probability for up-move

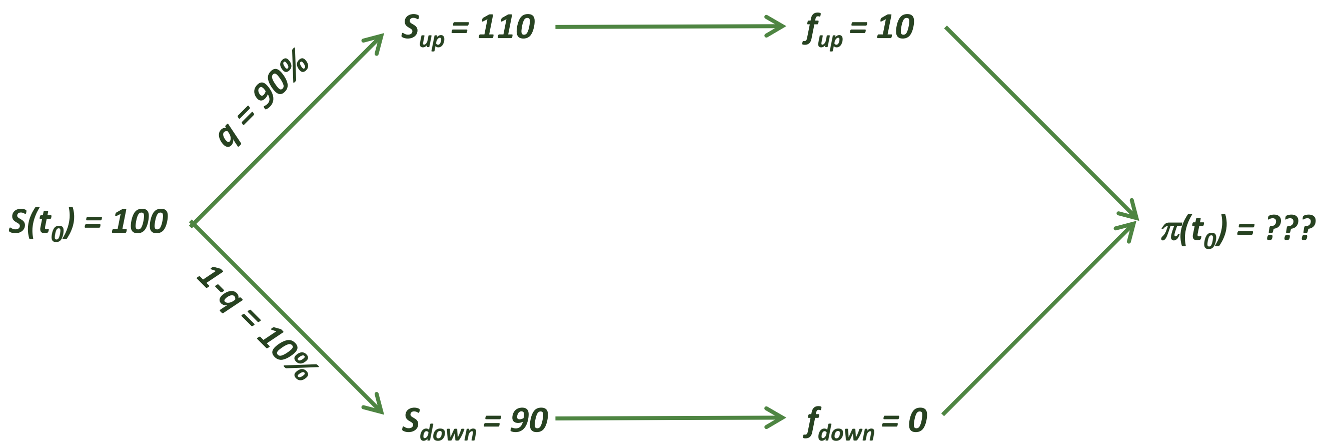


Hedging Strategy:

- Loan Δ ***S(t₀) +*** π
- Buy option π
- Buy Δ stocks

Binomial Tree (4)

- Example: large probability for up-move



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Match Δ to eliminate risk

$$\Delta\, S(T) + f(T)$$

or,

$$110\,\Delta + 10 = 90\,\Delta + 0$$

or,

$$\Delta = -0.50$$

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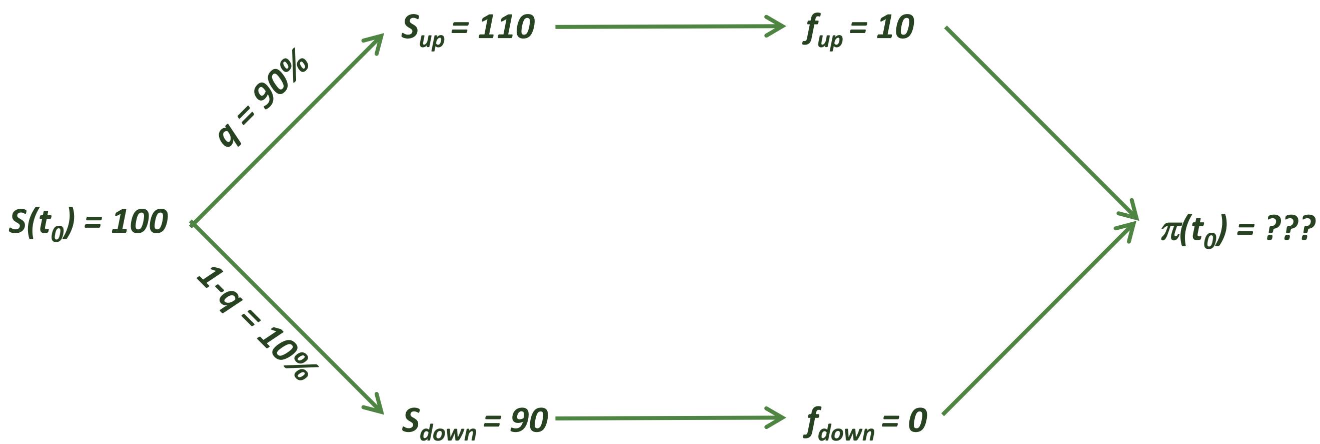

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Value of hedge at T

$$\Delta\, S(T) + f(T) = -45$$

Cost of hedging at t_0:

$$\Delta\, S(t_0) + \pi = -50 + \pi$$

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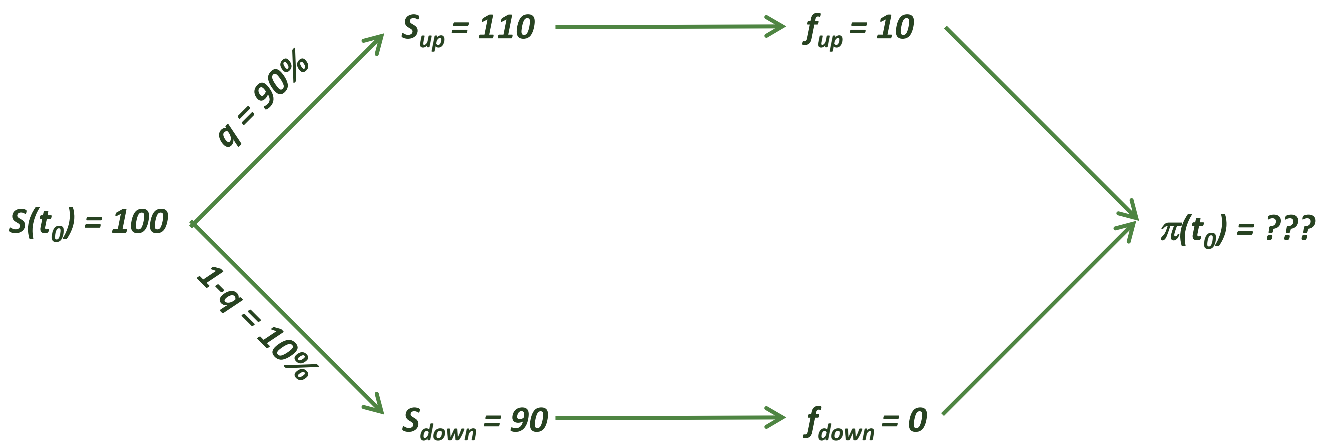

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Value of hedge at T

$$\Delta\, S(T) + f\,(T) = -45$$

Cost of hedging at t_0:

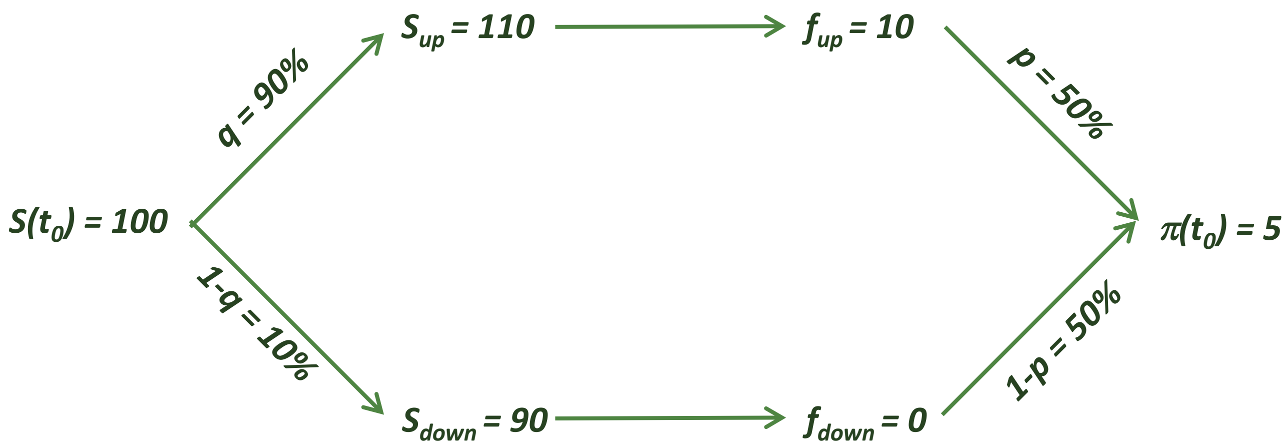
$$\Delta\, S(t_0) + \pi = -50 + \pi$$

Cost of the option

$$\pi = 5$$

Binomial Tree (5)

- Example: large probability for up-move



Actuarial Calculation

$$\begin{aligned}\pi &= q\, f_u + (1-q) f_d \\ &= 0.9 \times 10 + 0.1 \times 0 \\ &= 9\end{aligned}$$

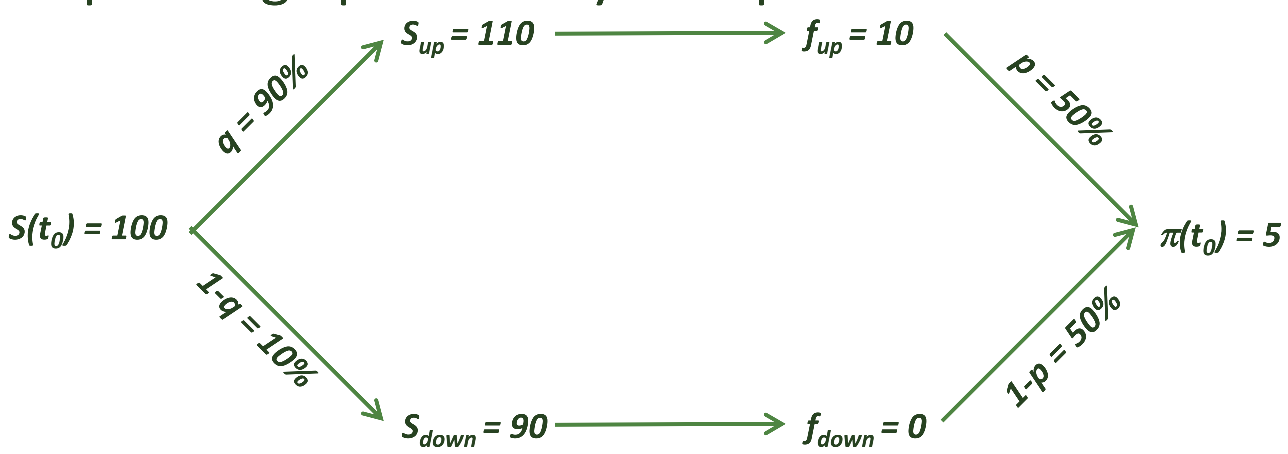
Cost of hedging

$$\begin{aligned}\pi &= p f_u + (1-p) f_d \\ &= 0.5 \times 10 + 0.5 \times 0 \\ &= 5\end{aligned}$$

- Risk neutral valuation

Binomial Tree (5)

- Example: large probability for up-move



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$$= 0.9 \times 10 + 0.1 \times 0$$
$$= 9$$

Cost of hedging

$$\pi = p\, fu + (1-p) f_d$$
$$= 0.5 \times 10 + 0.5 \times 0$$
$$= 5$$

- Risk neutral valuation
- Discount effect should be added to borrow money

Cost of hedging

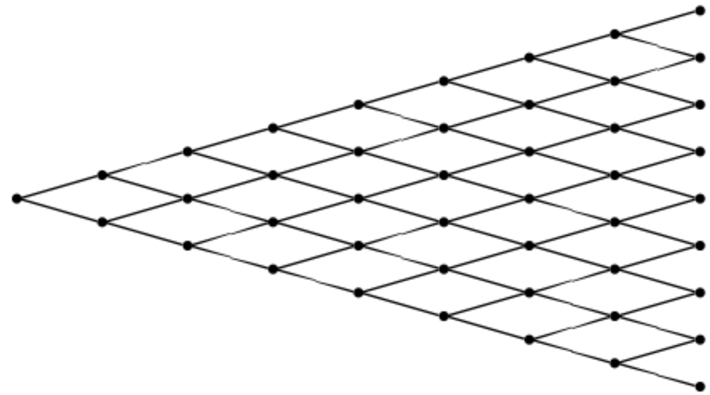
$$\pi = \exp(-rdt))(pfu + (1-p) f_d)$$

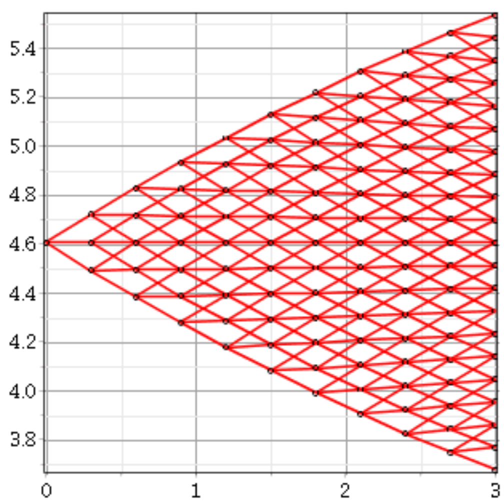
and

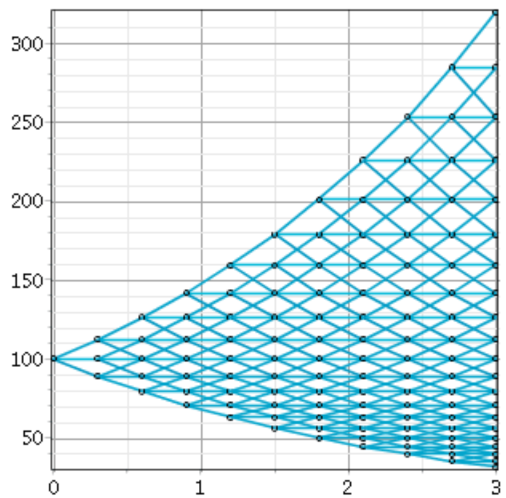
$$\Delta = \frac{\exp(rdt) - d}{u - d}$$

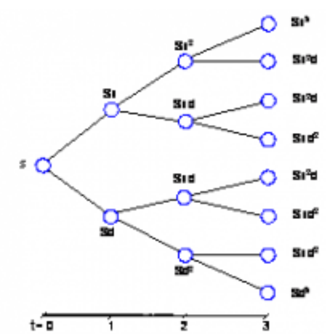
Binomial Tree (6)

- Extension: multi-step binomial tree
- Recombining trees
- Non-recombining trees
- Trinomial Trees
- Non-linear vs linear trees





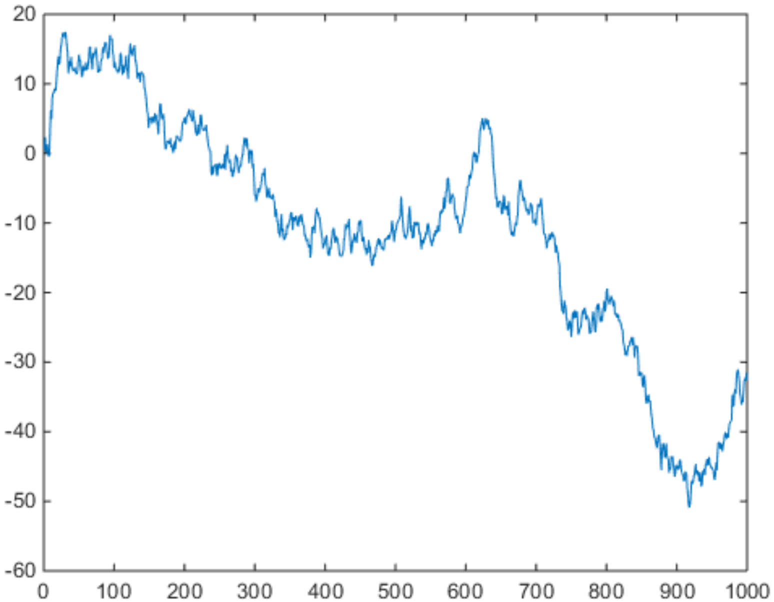




Black-Scholes-Merton Model

- Same hedging principles as the Binomial Tree model
- Continuous model based on Brownian motion ***W(t)***
  - ***W(0) = 0***
  - *Increments independent*
  - *Increments stationary*
  - *Increments normally distributed*

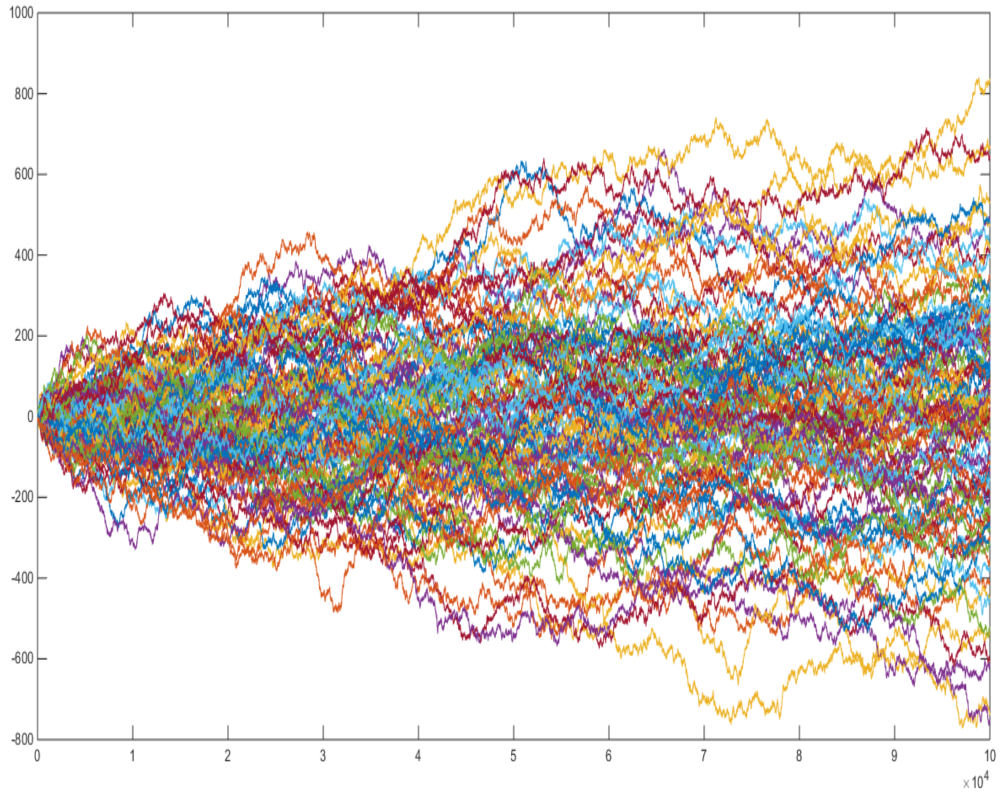
$$W(t+\Delta t) - W(t) \sim N(0, \Delta t)$$

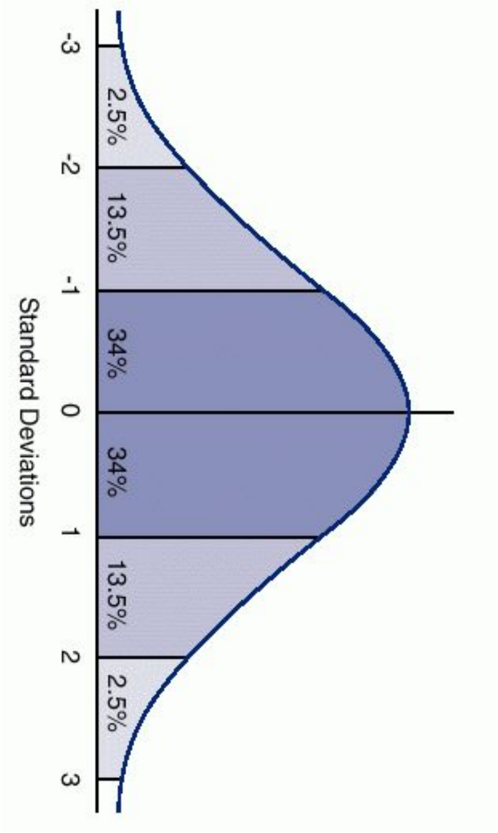


Black-Scholes-Merton Model (2)

- Continuous model based on Brownian motion ***W(t)***
- Statistical properties

$$E[W(t)] = 0$$
$$Var(W(t+\Delta t) - W(t)) = \Delta t$$

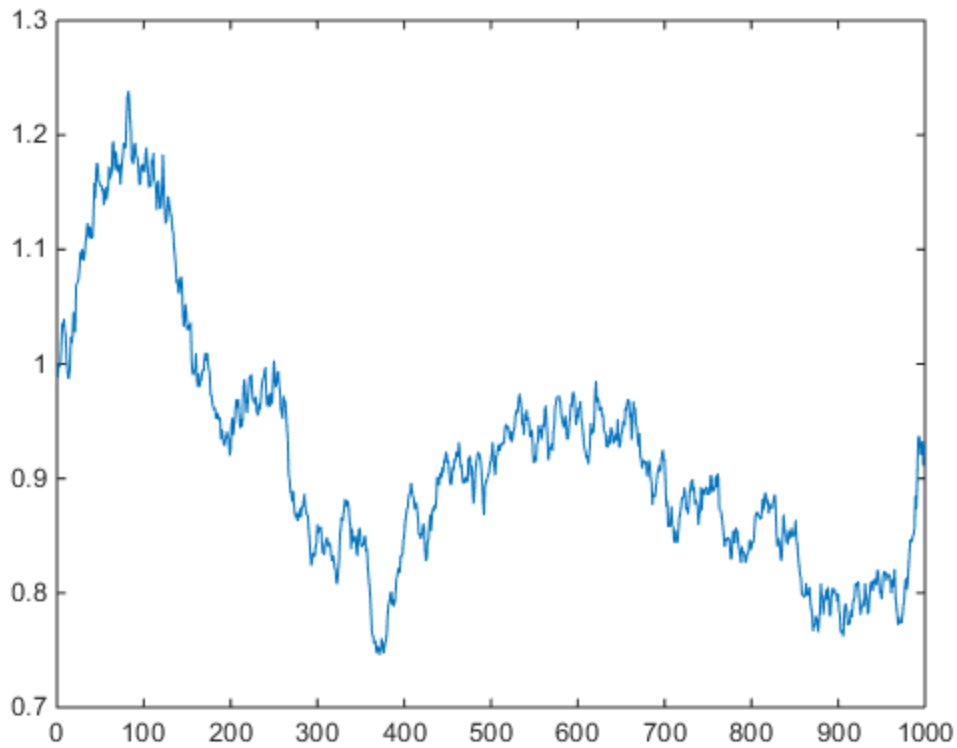




Black-Scholes-Merton Model (3)

- SDE description + Ito calculus
- Price of a stock follows

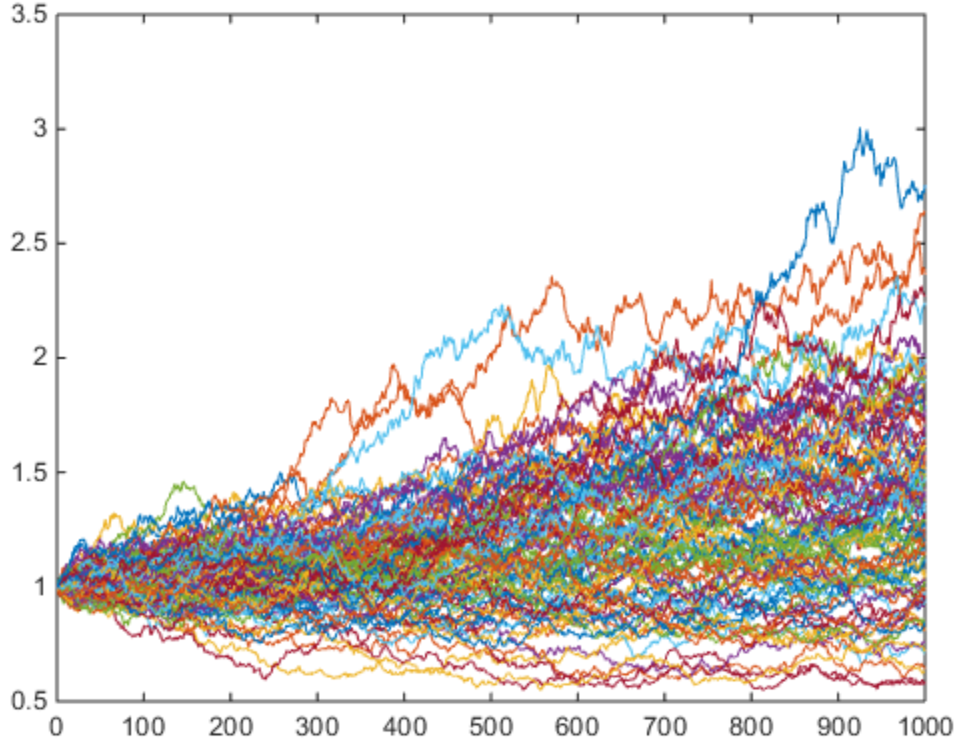
$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$



Black-Scholes-Merton Model (4)

- SDE description + Ito calculus
- Price of a stock follows a log-normal distribution

$$S(t) = S(0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right]$$



Black-Scholes-Merton Model (5)

- Hedging argument:
  - Construct Portfolio ***V*** with derivative π and Δ stocks
  - Eliminate risk (in each time step)
  - Use the risk-neutral value of the portfolio
  - Find the price of the derivative

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- Variation of the portfolio ***V***: $dV = \Delta \cdot dS - d\pi\,.$
- Apply Ito's lemma to this

$$dV = \Delta dS - \left(\frac{\partial \pi}{\partial S} dS + \frac{\partial \pi}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 dt \right)$$

$$= \left(\Delta - \frac{\partial \pi}{\partial S} \right) dS + \left(\frac{\partial \pi}{\partial t} + \frac{1}{2} \sigma^2 S^2 \right) dt.$$

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$$= \left(\Delta - \frac{\partial \pi}{\partial S} \right) dS + \left(\frac{\partial \pi}{\partial t} + \frac{1}{2} \sigma^2 S^2 \right) dt.$$

- Eliminate Risk by choosing Δ

$$\Delta = \frac{\partial \pi}{\partial S}.$$

Black-Scholes-Merton Model (6)

- For European Call/Put options, one can derive a closed form expression for the premium π and the delta Δ

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$$F = S(t_0)\exp\left((r-q)(T-t_0)\right)$$

$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t_0} = \frac{\log(F/K) - \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}.$$

	Premium π
π_C	$\exp(-r(T-t_0))(FN(d_1) - KN(d_2))$
π_P	$\exp(-r(T-t_0))(KN(-d_2) - FN(-d_1))$

	Delta Δ
Δ_C	$\exp\left(-q(T-t_0)\right)N(d_1)$
Δ_P	$-\exp\left(-q(T-t_0)\right)N(-d_1)$

Black-Scholes-Merton Model (6)

- For European Call/Put options, one can derive a closed form expression for the premium π and the delta Δ

$$F = S(t_0)\exp\left((r-q)(T-t_0)\right)$$

$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t_0} = \frac{\log(F/K) - \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}.$$

	Premium π
π_C	$\exp(-r(T-t_0))(FN(d_1) - KN(d_2))$
π_P	$\exp(-r(T-t_0))(KN(-d_2) - FN(-d_1))$

	Delta Δ
Δ_C	$\exp\left(-q(T-t_0)\right)N(d_1)$
Δ_P	$-\exp\left(-q(T-t_0)\right)N(-d_1)$

- Neither depends on the drift-term μ in the SDE, equivalent to the risk-neutral probability for the binomial tree model

EXERCISE : Black-Scholes calculator

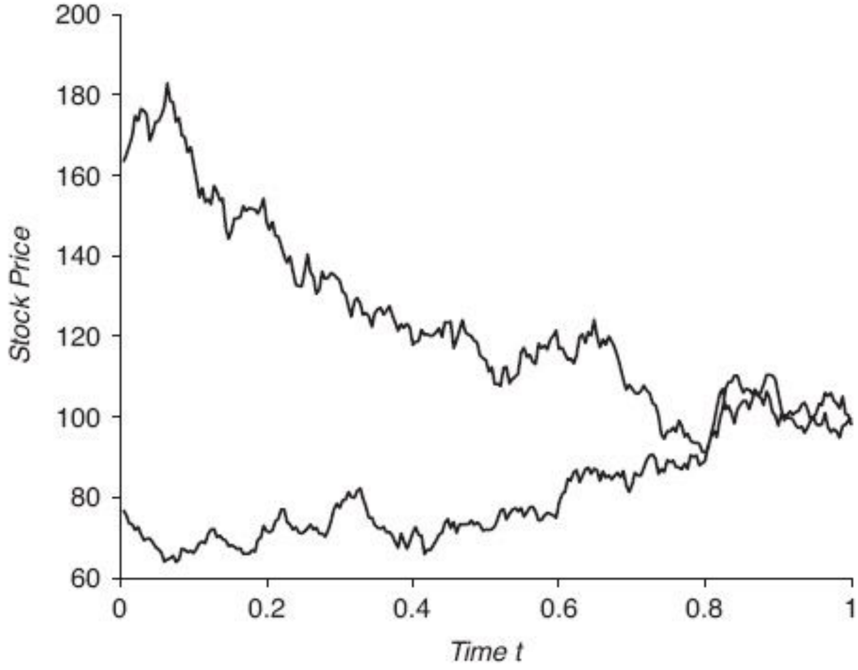
- Set up a Black-Scholes calculator in excel

Price	100
Strike	100
interest rate	2%
dividend yield	0%
duration	1.5
Volatility	20%
Call/Put	Call
FWD	
d1	
d2	
N(d1)	
N(d2)	
Price	
Delta	
Gamma	
Vega	
Theta	
Vanna	
Volga	

Black-Scholes-Merton Model (7)

- Risk neutral implies Call options for both stocks have the same price
- Investors might be willing to pay more for Call optior on Stock 1 than for call options on Stock 2
- Cost of hedging for both is identical

Stock 1	Stock 2
$\mu_1 = +10\%$	$\mu_2 = -10\%$
$\sigma_1 = 30\%$	$\sigma_2 = 30\%$
$q_1 = 0\%$	$q_2 = 0\%.$



Black-Scholes-Merton Model (8)

- The risk-neutral equivalent of the binomial tree model specifies μ should be replaced with r in the SDE

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

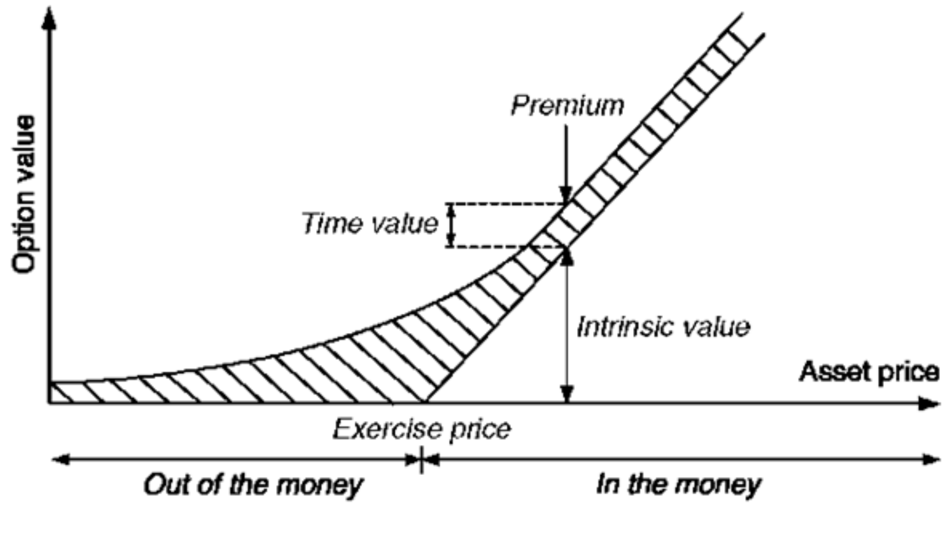
- The solution of the SDE depends on the Brownian motion and gives a simple sampling tool to sample points at any time *t*

$$S_i(t) = S(0)\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\cdot Z_i\right], i = 1, \ldots, N$$

where Z_i are independent standard normal numbers.

Terminology

- Intrinsic Value of an option: $\pi(S(t_0), T)$
- Options can be
  - In the money (ITM)
  - Out of the money (OTM)
  - At the money (ATM)
- Options have additional premium above the intrinsic value, called time-value



Terminology (2)

- Positions can be
  - Long: positive position in the underlying quantity
  - Short: negative position in the underlying quantity
  - Neutral/Flat: no position in the underlying quantity

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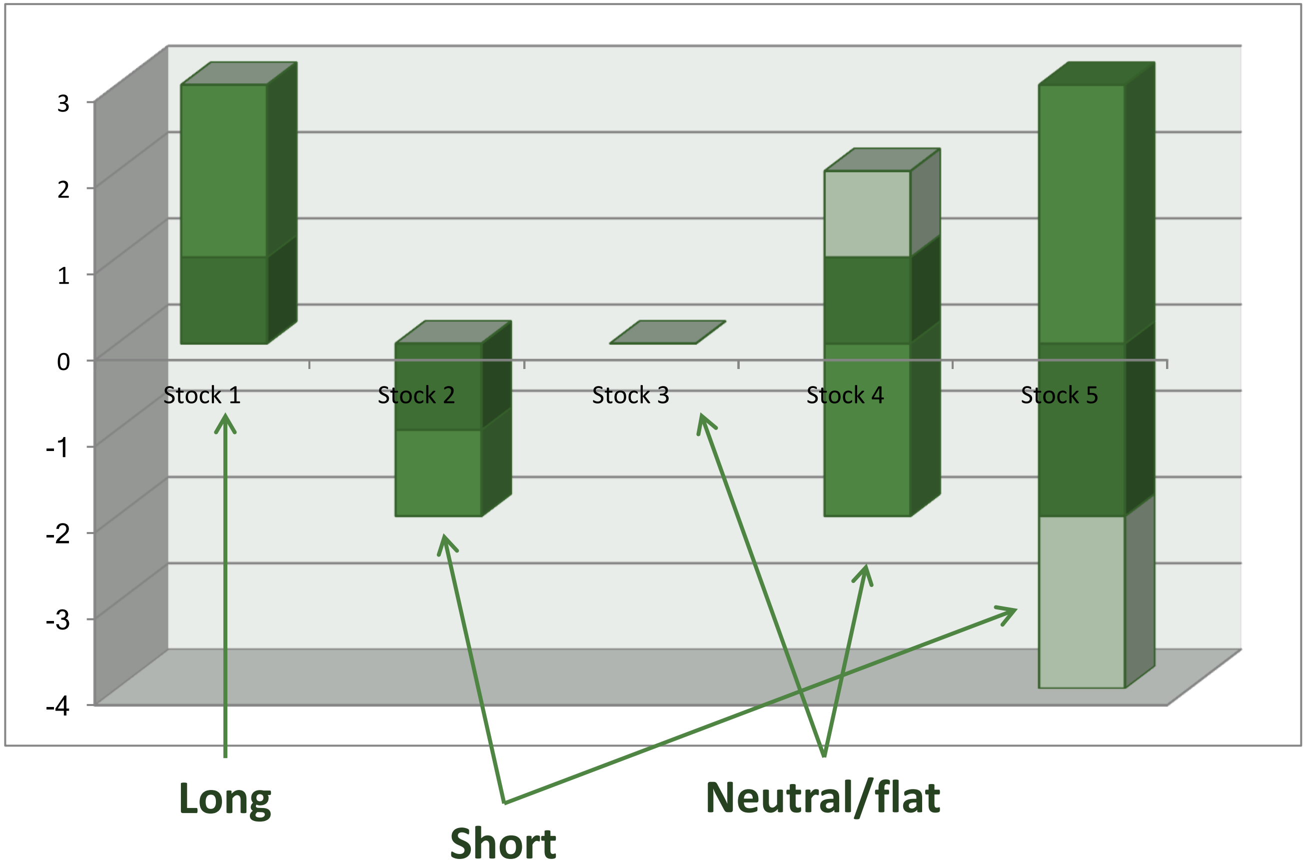
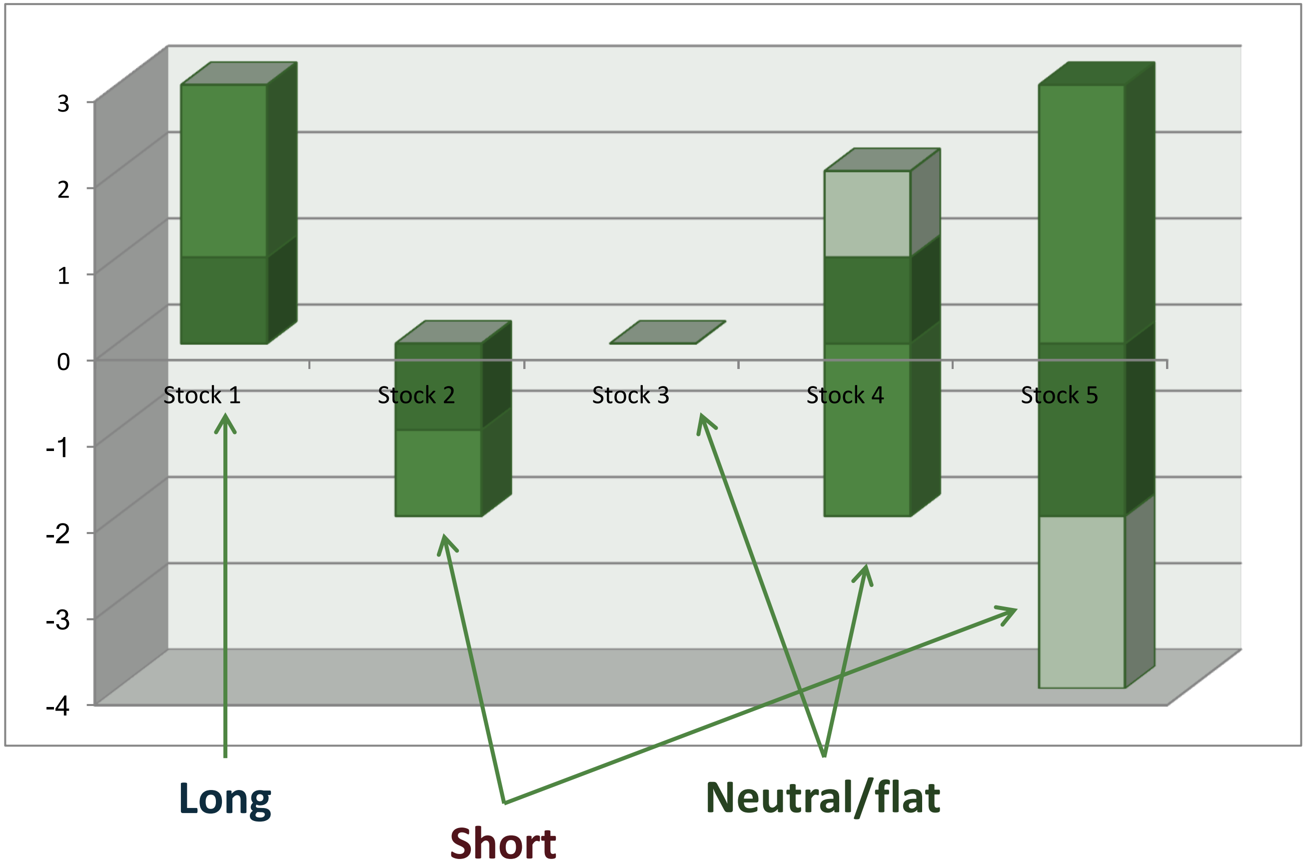


Terminology (3)

- Transactions*

	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5
Day 1	1	-1	-	1	-2
Day 2	2	-1	-	-2	3
Day 3	-	-	-	1	-2

** Positive numbers mean buy, negative numbers mean sell*



Terminology (4)

- Mark-to-market:

DEFINITION OF 'MARK TO MARKET - MTM'

1. A measure of the fair value of accounts that can change over time, such as assets and liabilities. Mark to market aims to provide a realistic appraisal of an institution's or company's current financial situation.

2. The accounting act of recording the price or value of a security, portfolio or account to reflect its current market value rather than its book value.

3. When the net asset value (NAV) of a mutual fund is valued based on the most current market valuation.

- P&L: Profit and Loss

DEFINITION OF 'PROFIT AND LOSS STATEMENT - P&L'

A financial statement that summarizes the revenues, costs and expenses incurred during a specific period of time - usually a fiscal quarter or year. These records provide information that shows the ability of a company to generate profit by increasing revenue and reducing costs. The P&L statement is also known as a "statement of profit and loss", an "income statement" or an "income and expense statement".

INVESTOPEDIA EXPLAINS 'PROFIT AND LOSS STATEMENT - P&L'

The statement of profit and loss follows a general form as seen in this example. It begins with an entry for revenue and subtracts from revenue the costs of running the business, including cost of goods sold, operating expenses, tax expense and interest expense. The bottom line (literally and figuratively) is net income (profit). Many templates can be found online for free, that can be used in creating your profit and loss, or income statement.

The balance sheet, income statement and statement of cash flows are the most important financial statements produced by a company. While each is important in its own right, they are meant to be analyzed together.

2. Delta-Hedging in the perfect world

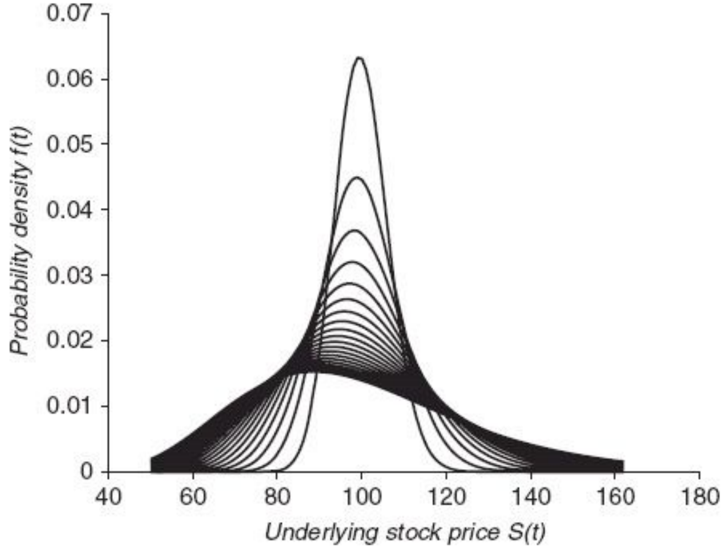
- Black-Scholes-Merton model revisited
- Flaws of the BSM model
- Popularity of the BSM model
- Flavours of volatility
- Volatility Estimation
- The Delta-hedging Experiment

Black-Scholes-Merton model revisited

- In the BSM model, the stock price ***S(t)*** follows a lognormal distribution at each time.

$$f(s,t;t_0,\mu,\sigma)=\frac{1}{\sqrt{2\pi(t-t_0)}\sigma s}\exp\left[-\frac{\left(\log s/S(t_0)-\mu(t-t_0)\right)^2}{2\sigma^2(t-t_0)}\right]$$

- The dynamics display a dispersion around the initial/current level
- In the option calculation, one can calculate the premium by integrating the payout function over the density (discounted)



$$\pi_C(S(t_0),t_0;K,T)=\exp(-r(T-t_0))\cdot\int_0^\infty \max(0,s-K)\cdot f\left(s,T;t_0,\mu,\sigma\right)ds$$

Risk-neutral valuation: replace μ with $\boldsymbol{r}$

Black-Scholes-Merton model revisited (2)

- Black-Scholes-Merton formula for European Call option with strike ***K*** and maturity ***T***:

	Premium π
π_C	$\exp(-r(T-t_0))(FN(d_1)-KN(d_2))$
π_P	$\exp(-r(T-t_0))(KN(-d_2)-FN(-d_1))$

with

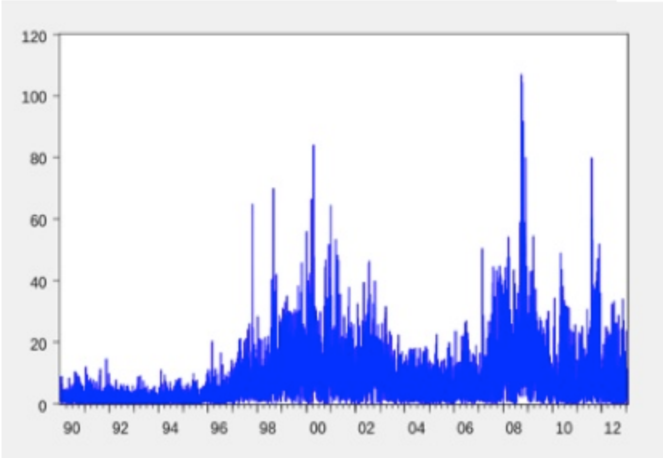
$$F = S(t_0)\exp\left((r-q)(T-t_0)\right)$$
$$d_1 = \frac{\log(F/K)+\frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t_0} = \frac{\log(F/K)-\frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}.$$

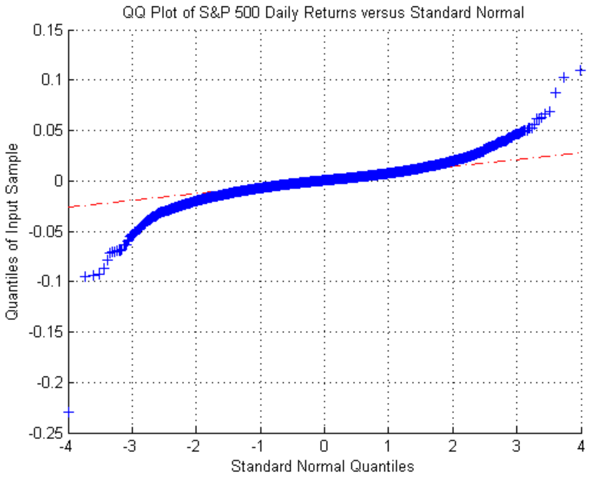
- Inputs:

Input Parameter	Description	Unit	Example
r	Discount rate	Numeric	0.02
t_0	Valuation date/time	Years	0
T	Expiry date/time	Years	1/12 (month), 1 (year)
$S(t_0)$	Level of stock at the t_0	In a given currency	100 EUR, USD, AUD,…
σ	Annualized volatility	Numeric	0.20 (20%)
K	Strike level	In the currency of the stock price	100 EUR, USD, AUD,…

Flaws of the BSM model

- Independent increments implies no memory. In reality there is a positive (and significant) slowly-varying autocorrelation which suggests volatility clustering
- Parameter **σ** (called the volatility) is constant
- Perfect Scaling between daily, weekly, monthly returns
- (Log)Returns are normally distributed: frequency of tail events can be calculated:





σ	probability	exceedance frequency	Empirical probability S&P 500 (2008–13)
1σ	68.269%	once in three days	81.79%
2σ	95.450%	once per month	94.64%
3σ	99.730%	once per year	98.08%
4σ	99.994%	once per century	99.21%

Popularity of the BSM Model

- Easy to understand
- Single parameter: *volatility*: relatively universal
- Closed formulas for European options + range of exotics
- Can be extended to include *(transparency)*
  - Bid/offer
  - Slippage
  - Volatility structures
  - Stochastic volatility (Heston)
- Explicit hedging strategy
- Flaws of the model are expressed in volatility adjustments
  (option premium π is monotonic in volatility σ for European options)

Flavours of volatility

- **Realised volatility**:
  - the volatility as observed in the past over a certain time interval at a certain granularity (eg daily observations)
  - *Calculation* method: use Ito's lemma, the statistical properties of the Brownian motion to find:

$$d\log S(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW(t).$$

  - Practically: calculate daily logreturns (or returns by approximation) and find the standard deviation as the daily volatility. Remember:

$$\log\left(S(t+\Delta t)/S(t)\right) \approx \frac{S(t+\Delta t) - S(t)}{S(t)} \text{ for } \Delta t \text{ given as 1 day}$$

  - How to annualize?
    - Relationship: $\sigma\sqrt{\Delta t} = \sigma_{\Delta t}$
    - What is 1 day?
      - 1/365? Or 1/366? Or 1/256?
    - Consistency
  - Past is no representation for the future!

Flavours of volatility (2)

- **Historical volatility**:
  - the future realized volatility as it will be observed over a certain time interval at a certain granularity (eg daily observations)
  - *Estimation* method:
    - Past is no representation for the future!
    - Garch methods are *models* and depend on the *past* as well

Flavours of volatility (3)

- **Instantaneous volatility**:
  - The volatility function ***σ(t)*** that expresses the amount of volatility at any given point ***t*** (past, present and future)
  - *Estimation* method:
    - Past is no representation for the future!
    - Functional fits are difficult, but in some cases it can be relevant (seasonality, fixing (eg zero-coupon bonds, commodity futures))
  - Link with the historical volatility (in BSM model)

$$\sigma_{hist}(t_0;T) = \sqrt{\frac{1}{T-t_0}\int_{t_0}^{T} \sigma_{inst}^2(u)\,du}$$

Flavours of volatility (4)

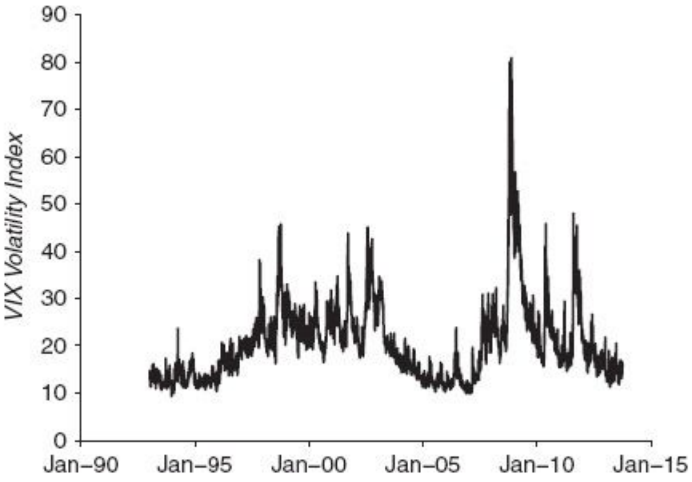
- **Imposed volatility**:
  - The volatility function σ used in experiments or simulations
  - In the trading games/exercises we will start from a particular model volatility.
  - Finite size effects can make the historical volatility of the sample different from the imposed volatility.

Flavours of volatility (5)

- **Implied volatility**:
  - Option prices are set by the market participants by supply and demand. They are a reflection of the following parameters of each participant
    - Estimate of the historical volatility
    - Current portfolio fit
    - Risk appetite
    - Open interest of buyers versus sellers
    - Market sentiment
    - Known and unknown events
    - …
  - One can find the parameter σ in the BSM option formula that matches this market price. Each option (strike, maturity) will have one unique matching volatility parameter to match the price. This is called the implied volatility.
  - *The wrong number to put into the wrong formula to get to the right price!!*

Flavours of volatility (6)

- **Hedging volatility**:
  - If the implied volatility gives the right option price, will the implied volatility also be the parameter to use in the hedging ratio Δ?
  - BSM model is very robust, but most market participants follows this common practice.
- **VIX, the volatility index:**
  - Now a traded instrument
  - Closed formula, depends on the market implied volatility of a prescribed set of options.



Volatility estimation

- Volatility is indivertibly linked to the BS model. In the BS model, there is a perfect scaling in the time-dimension coming from the independent and stationary increments and the linear variance.
- How do we measure/estimate this?
- We use historical data: there is more data granularity/detail than just close-to-close data

Notation	Explanation
C_i	Closing price of the underlying
H_i	High price of the underlying
L_i	Low price of the underlying
O_i	Opening price of the underlying

Volatility estimation

- Traditional Close-to-Close data (standard deviation of the log returns:

$$\sigma_{cc}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\log \left(\frac{C_i}{C_{i-1}} \right) \right)^2$$

- Or explicitly taking into account the drift:

$$\sigma_{acc}^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(\left(\log \left(\frac{C_i}{C_{i-1}} \right) \right)^2 - \frac{\left(\log \left(\frac{C_n}{C_0} \right) \right)^2}{n(n-1)} \right)$$

- Parkinson estimator: 5 times more efficient

$$\sigma_p^2 = \frac{1}{4n \log(2)} \sum_{i=1}^{n} \left(\log \left(\frac{H_i}{L_i} \right) \right)^2$$

Volatility estimation

- Garman & Klass estimator: 7.4 times more efficient

$$\sigma_{gk}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(0.511\left(\log\left(\frac{H_i}{L_i}\right)\right)^2 - 0.019\log\left(\frac{C_i}{O_i}\right)\log\left(\frac{H_i L_i}{O_i^2}\right) - 2\log\left(\frac{H_i}{O_i}\right)\log\left(\frac{L_i}{O_i}\right)\right).$$

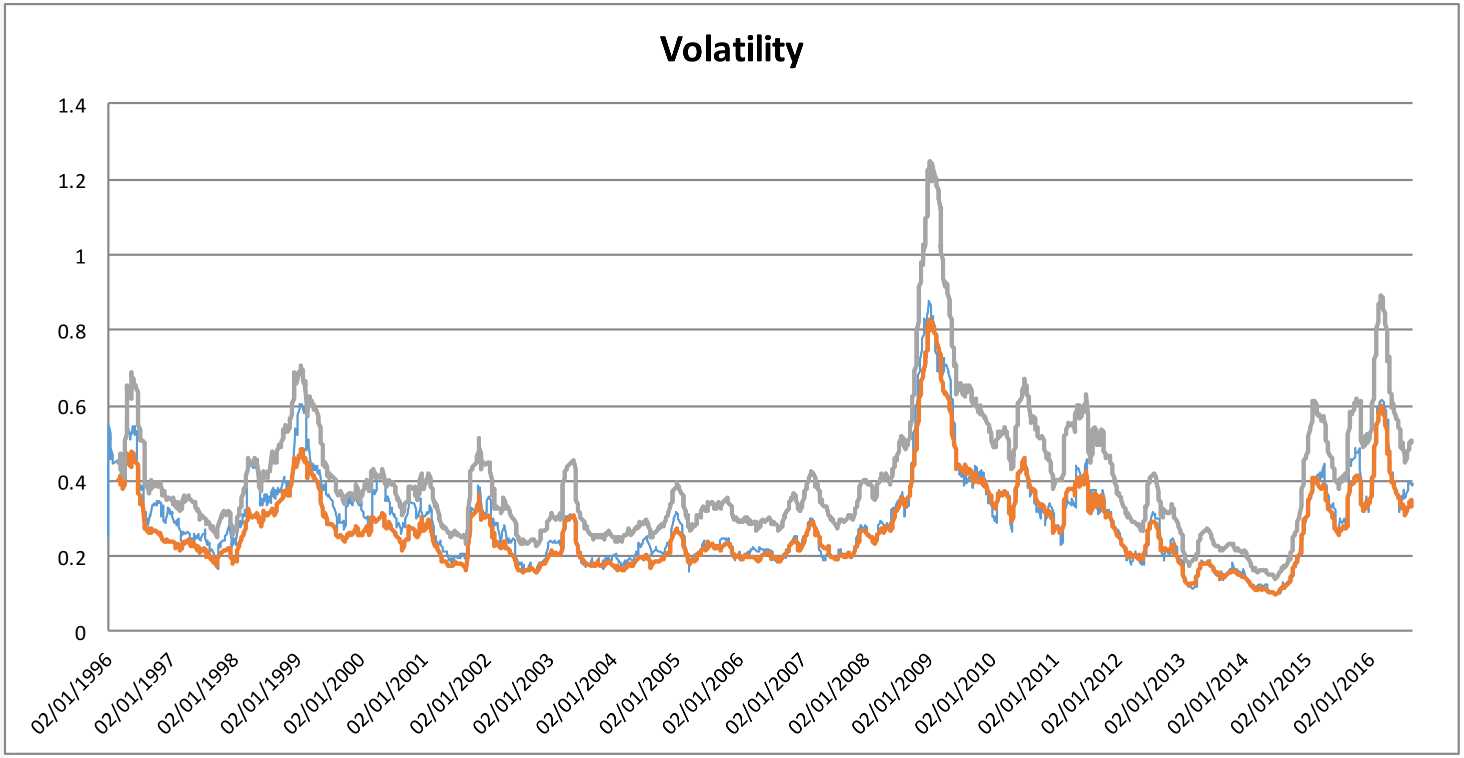
- Rogers & Satchell estimator: neutralise the drift dependency

$$\sigma_{rs}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\log\left(\frac{H_i}{C_i}\right)\log\left(\frac{H_i}{O_i}\right) + \log\left(\frac{L_i}{C_i}\right)\log\left(\frac{L_i}{O_i}\right)\right)$$

EXERCISE : Volatility estimation

Use the provided Brent Prompt Month (Roll-adjusted) data.

1. Use a moving window of 50 days, calculate the Close-close volatility over the time series
2. Use the Parkinson estimator to do the same
3. Use the Garman & Klass estimator to do the same



EXERCISE : Conclusions

- You need a large enough sample set to calculate averages/standard deviations that are statistically meaningful. Any dataset should be at least contain 30 sample points.
- Close-to-Close: simplest and most common estimator. Usually closing prices are more reliable (liquidity).
- Parkinson: this was the first more advanced estimator using more price information that only close-close. Because it uses H/L on each day, it assumes you continuously trade and have no gaps. Therefore in markets that gap (large jumps from close on one day to the open on the next day), this estimator tends to underestimate the volatility.
- Garman Klass: a further extension of Parkinson that takes into account the full information set on each day. The calculations are still done by distilling data per day (no cross days data is being used), the gap volatility is still ignored.

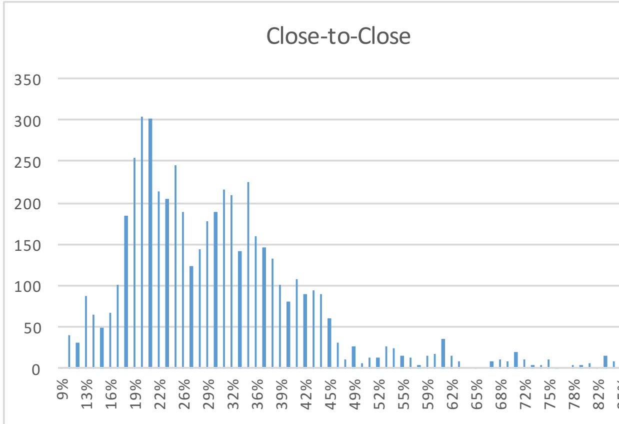
EXERCISE : Conclusions

- There are even more measures but the formulas tend to be more complicated
- The formulas often ignore drift in the returns. Of course usually drift is $O(\Delta t)$ and volatility is $O\left(\sqrt{\Delta t}\right)$ and the latter is more important.
- Similar formulas exist for correlation estimation as well or can easily be derived by transformation theory.
- The above gives us at each point a view of the historically realised volatility over the window and its dynamic behaviour

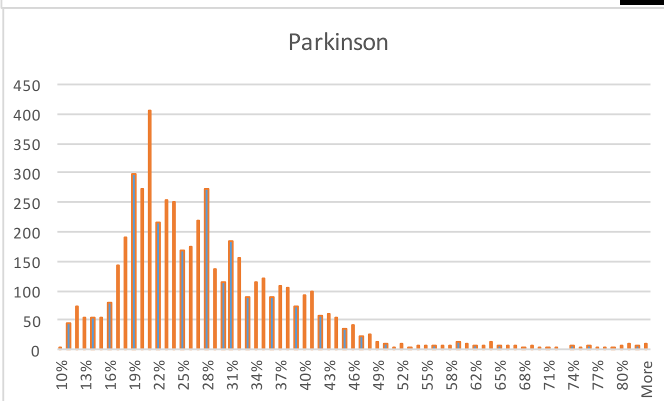
EXERCISE : continued

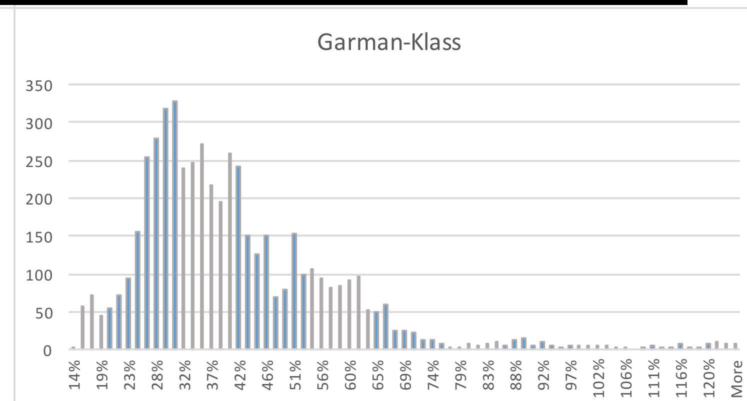
4. For each of the time series of the volatility estimator classes, derive statistical properties
   - Mean/Average
   - Standard deviation
   - High/low
   - Percentiles: 25%, 40%, 50%, 60%, 75%

EXERCISE : Conclusions



	Close-to-Close	Parkinson	Garman-Klass
Mean	29.78%	27.64%	40.38%
Standard deviation	12.55%	11.49%	17.03%
Median	27.99%	25.27%	36.36%
Low	9.28%	9.55%	14.01%
High	88.07%	82.95%	124.77%
25% percentile	20.65%	19.85%	29.03%
40% percentile	24.35%	22.60%	33.25%
50% percentile	27.99%	25.27%	36.36%
60% percentile	30.89%	27.51%	40.10%
75% percentile	35.55%	33.19%	48.45%





- Latest volatility is on the high compared to historical data (sell signal?)
- Some care needs to be taken to decide on which historical data is relevant for which purpose
- Some assessment on uncertainty of volatility can be derived

Exponentially weighted moving average

- Daily return is the building stone for the variance (and hence volatility)

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

- And as before the variance on day *n* is estimated by

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2$$

- Are all historical days equally important? Memory?
- How about something like this:

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} \alpha_i \cdot u_{n-i}^2$$

where the α_i add up to 1 but more weight is on the recent return contributions

Exponentially weighted moving average

- The EWMA model uses

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

- And as before the variance on day *n* is estimated by

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_n^2$$

- One can easily see by extending the recursion:

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_n^2$$

$$\sigma_n^2 = \lambda(\lambda\sigma_{n-2}^2 + (1-\lambda)u_{n-1}^2) + (1-\lambda)u_n^2$$

$$\sigma_n^2 = \lambda^2\sigma_{n-2}^2 + (1-\lambda)(u_n^2+\lambda u_{n-1}^2)$$

So that finally:

$$\sigma_n^2 = \lambda^m\sigma_{n-m}^2 + (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}\cdot u_{n-i}^2$$

Exponentially weighted moving average

$$\sigma_n^2 = \lambda^m \sigma_{n-m}^2 + (1-\lambda)\sum_{i=1}^{m} \lambda^{i-1} \cdot u_{n-i}^2$$

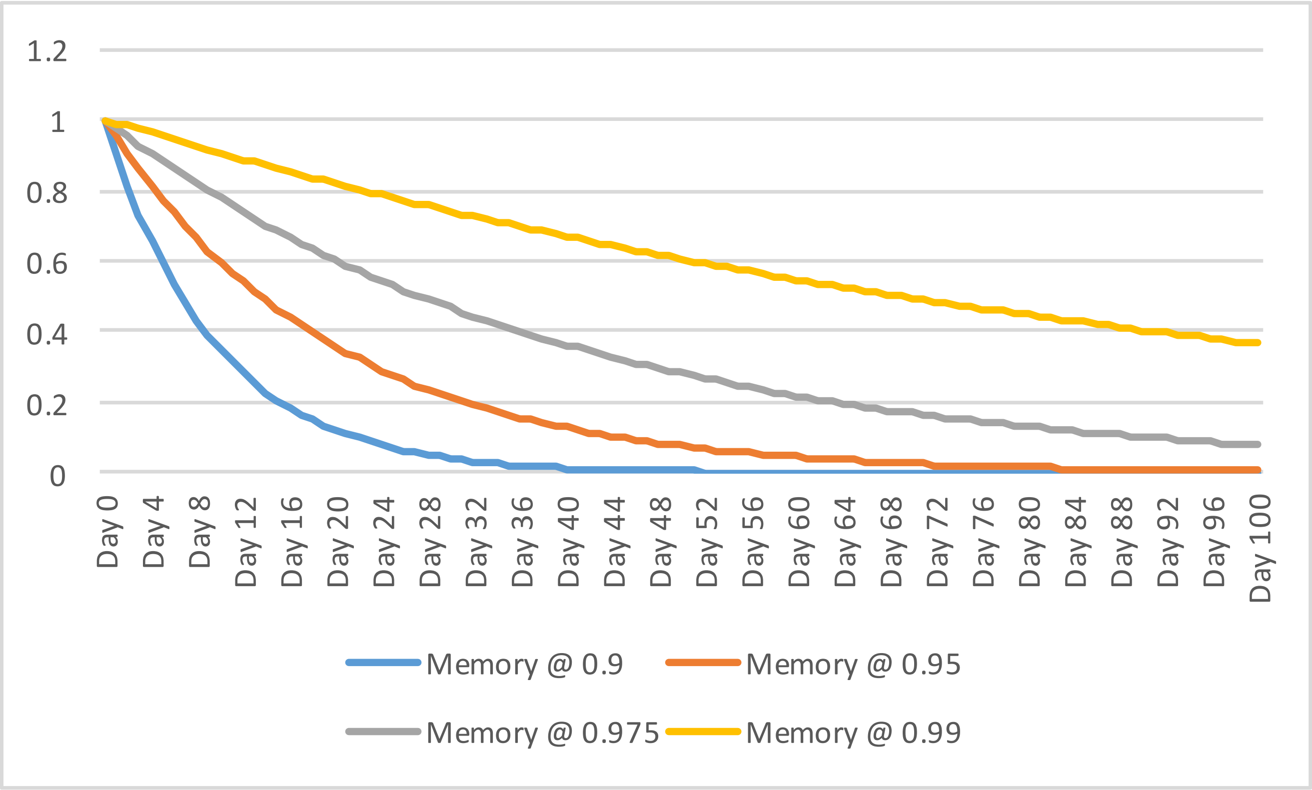
- The Long-term memory term faded exponentially
- The weights add up to 1
- Extreme cases :
  - $\lambda = 0$ only current return contributes
  - $\lambda = 1$ only the initial variance contribues

Exercise:

- Determine for the following parameters what the memory profile is: $\lambda = 0.90, 0.95, 0.975, 0.99$

Exponentially weighted moving average

$$\sigma_n^2 = \lambda^m \sigma_{n-m}^2 + (1-\lambda)\sum_{i=1}^{m} \lambda^{i-1} \cdot u_{n-i}^2$$



Exponentially weighted moving average

Recursive formula:

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_n^2$$

Actual full formula:

$$\sigma_n^2 = \lambda^m\sigma_{n-m}^2 + (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}\cdot u_{n-i}^2$$

- Very little data needs to be stored: very easy updating calculation
- RiskMetrics database (JPMorgan, 1994) uses $\lambda = 0.94$ as a good overall, across asset classes good forecast rate for the realised variance rate (for the subsequent 25 days)

Exponentially weighted moving average

Recursive formula:

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_n^2$$

Actual full formula:

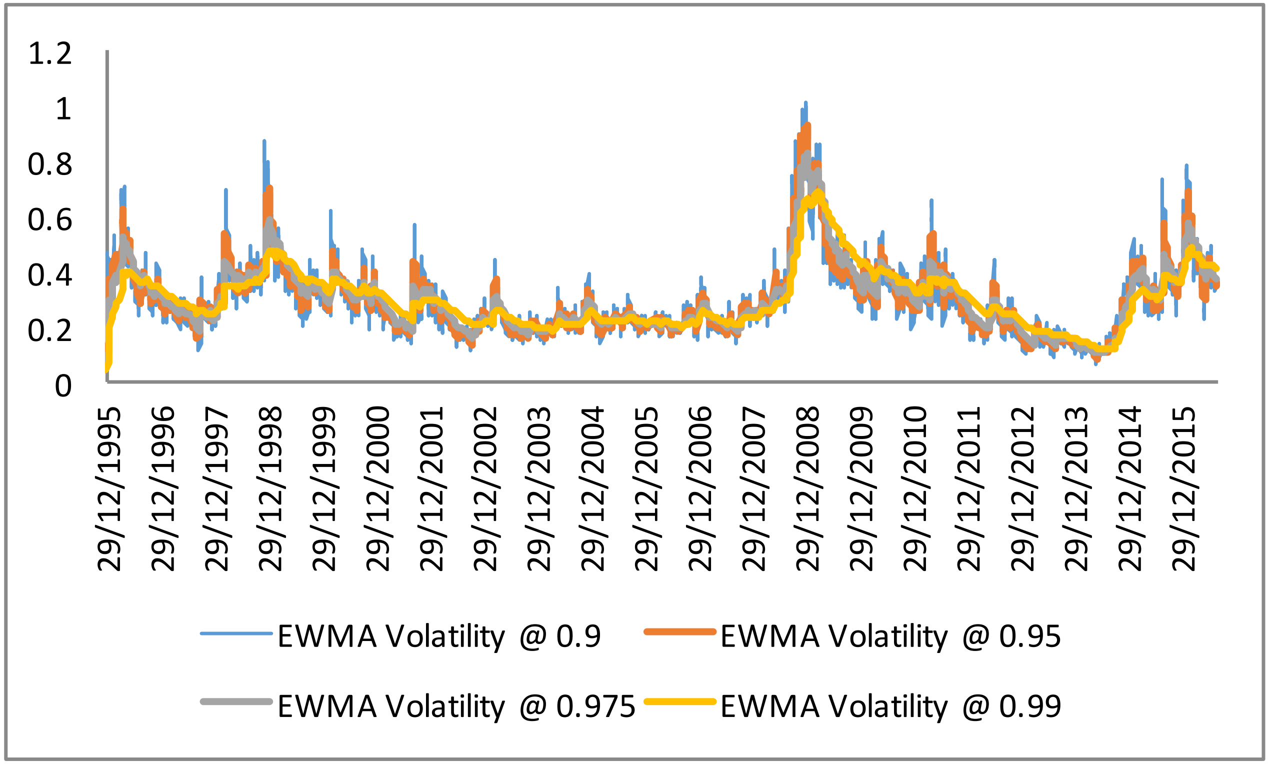
$$\sigma_n^2 = \lambda^m \sigma_{n-m}^2 + (1-\lambda)\sum_{i=1}^{m} \lambda^{i-1} \cdot u_{n-i}^2$$

Exercise:

- Implement the EWMA estimator in excel (applied to the close-close data of the Brent Crude Prices)

Exponentially weighted moving average

Conclusions:



- Smaller lambda, more volatile estimate, faster reaction time to spikes
- Bigger lambda, more weight on the memory term
- You can also derive statistics from the data

Further extensions

- GARCH models (Generalised AutoRegressive Conditional Heteroskedasticity)
- In its simplest representation, known as GARCH(1,1) there are 3 contributions to the variance estimate:
  - Previous estimate (memory)
  - Latest return
  - Long-term variance

$$\sigma_n^2 = \gamma V_L + \beta \sigma_{n-1}^2 + \alpha u_n^2$$

- Reasoning: volatility (and hence variance) are mean reverting, so they need to be reverting back to the long term variance driven term V_L.
- There is a whole stream of Garch models with roughly 2 branches of applications

Further extensions

- Somewhere in the theory, there is a strong assumption on the normal distribution.
- There is a whole stream of Garch models with roughly 2 branches of applications
  - Forecasting variance
  - Using the underlying stochastic volatility model to price derivatives
- The first applications has merit although nowadays better forecasters (not all based on the normal distribution)
- Stochastic volatility models such as Heston outperform the Garch-like models, especially when it comes to exotics.

Delta-hedging Experiment

- Parameters

Stock drift	μ = 10%
Stock Volatility (imposed)	σ = 20%
Div yield	q = 0%
Int rate	r = 2%
Timestep	$\Delta t = 0.005$
Duration	$T\text{-}t_0 = 0.10$
Initial stock level	$S(t_0) = 100$
Instrument	ATM Call option
Currency	EUR

- Initial Actions
  - Sell the ATM Call option for $\pi(t_0 ; K, T)$
  - Buy Δ stocks for $\Delta\, S(t_0)$ EUR
  - Borrow money from the bank: $\Delta\, S(t_0) - \pi(t_0 ; K, T)$

Delta-hedging Experiment (2)

INITIAL

- Sell the ATM Call option for $\pi(t_0 ; K, T)$
- Buy $\Delta(t_0)$ stocks for $\Delta(t_0)\ S(t_0)$ EUR
- Borrow money from the bank:

$$\Delta(t_0)\ .\ S(t_0) - \pi(t_0 ; K, T) \text{ EUR}$$

Delta-hedging Experiment (2)

INITIAL

- Sell the ATM Call option for $\pi(t_0 ; K, T)$
- Buy $\Delta(t_0)$ stocks for $\Delta(t_0)\ S(t_0)$ EUR
- Borrow money from the bank:

  $\Delta(t_0)\ .\ S(t_0) - \pi(t_0 ; K, T)$ EUR

Cash settle

- Pay interest on the cash account for an amount of

  $r\Delta t\ (\Delta(t_0)\ .\ S(t_0) - \pi(t_0 ; K, T))$ EUR

Delta-hedging Experiment (2)

INITIAL

- Sell the ATM Call option for $\pi(t_0\,;K,T)$
- Buy $\Delta(t_0)$ stocks for $\Delta(t_0)\,S(t_0)$ EUR
- Borrow money from the bank:

  $\Delta(t_0)\,.\,S(t_0) - \pi(t_0\,;K,T)$ EUR

Cash settle

- Pay interest on the cash account for an amount of

  $r\Delta t\,(\Delta(t_0)\,.\,S(t_0) - \pi(t_0\,;K,T))$ EUR

MtM

- Evaluate the ~~ATM~~ Call option : $\pi(t_1\,;K,T)$
- Evaluate the $\Delta(t_0)$ stocks position : $\Delta(t_0)\,S(t_1)$ EUR

Delta-hedging Experiment (2)

INITIAL

- Sell the ATM Call option for $\pi(t_0\,; K, T)$
- Buy $\Delta(t_0)$ stocks for $\Delta(t_0)\ S(t_0)$ EUR
- Borrow money from the bank:

  $\Delta(t_0)\ .\ S(t_0) - \pi(t_0\,; K, T)$ EUR

Cash settle

- Pay interest on the cash account for an amount of

  $r\Delta t\ (\Delta(t_0)\ .\ S(t_0) - \pi(t_0\,; K, T))$ EUR

MtM

- Evaluate the ~~ATM~~ Call option : $\pi(t_1\,; K, T)$
- Evaluate the $\Delta(t_0)$ stocks position : $\Delta(t_0)\ S(t_1)$ EUR

Rebalance

- Evaluate the new $\Delta(t_1)$
- Rebalance the delta position: buy

  $\Delta(t_1) - \Delta(t_0)$ stocks
- Borrow additional money from the bank:

  $(\Delta(t_1) - \Delta(t_0))\ S(t_1)$ EUR

Delta-hedging Experiment (2)

INITIAL

- Sell the ATM Call option for $\pi(t_0\,; K, T)$
- Buy $\Delta(t_0)$ stocks for $\Delta(t_0)\ S(t_0)$ EUR
- Borrow money from the bank:

$$\Delta(t_0)\ .\ S(t_0) - \pi(t_0\,; K, T) \text{ EUR}$$

Cash settle

- Pay interest on the cash account for an amount of

$$r\Delta t\ (\Delta(t_0)\ .\ S(t_0) - \pi(t_0\,; K, T)) \text{ EUR}$$

MtM

- Evaluate the ~~ATM~~ Call option : $\pi(t_1\,; K, T)$
- Evaluate the $\Delta(t_0)$ stocks position : $\Delta(t_0)\ S(t_1)$ EUR

Rebalance

- Evaluate the new $\Delta(t_1)$
- Rebalance the delta position: buy

$$\Delta(t_1) - \Delta(t_0) \text{ stocks}$$

- Borrow additional money from the bank:

$$(\Delta(t_1) - \Delta(t_0))\ S(t_1) \text{ EUR}$$

FINAL

- Stock position long $\Delta(t_1)$ stocks
- Cash position short

$$\Delta(t_0)\ .\ S(t_0) - \pi(t_0\,; K, T) + (\Delta(t_1) - \Delta(t_0))\ S(t_1) \text{ EUR}$$

$$\Delta(t_0)\ .\ (S(t_0) - S(t_1)) - \pi(t_0\,; K, T) + \Delta(t_1)\ S(t_1) \text{ EUR}$$

$$\Delta(t_0)\ .\ \Delta S(t_1) - \pi(t_0\,; K, T) + \Delta(t_1)\ S(t_1) \text{ EUR}$$

Delta-hedging Experiment (3)

- MtM, Settling and rebalancing is equivalent to
  - Buying back the option (cash)
  - Sell out all stocks (cash)
  - Paying back loan (cash)

  - Selling the option
  - Buy all stocks
  - Take new loan

Delta-hedging Experiment (4)

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
	Imposed Parameters			Discretisation			Option Settings			Market Parameters			Hedging Parameter		
	mu	10%		dt	0.002739726		Maturity	5		interest	2%		volatility	20%	
	sigma	20%		StartLevel	100		Strike	100							
							Type	Call							
	Realised	19.01%					Pricing Vo	20%							
														TOTAL	0.03
Step	Z	time	S(t)		Value	Delta		Stock	Cash		Stock	Cash		P&L	Value Book
1		0.000	100.00		22.02	67.26%	0.32				67.26	-45.24			0.00
2	-0.09691	0.003	99.92	-0.0008	21.96	67.20%		67.21	-45.24	0.05	67.14	-45.18		0.00	0.00
3	-0.23372	0.005	99.70	-0.00223	21.81	67.01%		66.99	-45.18		66.81	-45.00		0.00	0.01
4	0.588582	0.008	100.34	0.006381	22.23	67.52%		67.24	-45.01		67.75	-45.52		0.00	0.01
5	-0.57431	0.011	99.76	-0.00579	21.83	67.05%		67.36	-45.52		66.89	-45.05		0.00	0.01
6	-0.75611	0.014	98.99	-0.0077	21.31	66.42%		66.37	-45.06		65.75	-44.44		0.00	0.02
7	0.375607	0.016	99.40	0.004151	21.58	66.75%		66.02	-44.44		66.35	-44.77		0.00	0.02
8	1.166225	0.019	100.65	0.012428	22.41	67.75%		67.18	-44.78		68.19	-45.78		0.00	0.02
9	-0.04939	0.022	100.62	-0.0003	22.38	67.72%		68.17	-45.78		68.14	-45.76		0.00	0.02
10	0.782224	0.025	101.47	0.008408	22.95	68.39%		68.72	-45.76		69.40	-46.44		0.00	0.02
11	-0.65639	0.027	100.79	-0.00665	22.49	67.86%		68.94	-46.45		68.40	-45.91		0.00	0.03
12	-0.33365	0.030	100.46	-0.00327	22.26	67.59%		68.17	-45.91		67.90	-45.64		0.00	0.03
13	0.037384	0.033	100.53	0.000611	22.29	67.63%		67.94	-45.65		67.99	-45.70		0.00	0.04
14	0.677461	0.036	101.26	0.007311	22.79	68.22%		68.49	-45.70		69.08	-46.29		0.00	0.04
15	1.436294	0.038	102.82	0.015255	23.85	69.42%		70.14	-46.29		71.38	-47.53		0.00	0.03
16	1.329221	0.041	104.28	0.014134	24.87	70.52%		72.40	-47.53		73.55	-48.68		0.00	0.03
17	-0.74284	0.044	103.50	-0.00756	24.31	69.93%		72.99	-48.68		72.38	-48.07		0.00	0.03
18	-1.67086	0.047	101.73	-0.01727	23.08	68.57%		71.14	-48.07		69.75	-46.67		-0.01	0.02
19	1.270401	0.049	103.11	0.013518	24.03	69.63%		70.70	-46.68		71.80	-47.77		0.00	0.02
20	0.755205	0.052	103.95	0.008125	24.61	70.26%		72.38	-47.77		73.04	-48.43		0.00	0.02
21	-0.219	0.055	103.74	-0.00207	24.45	70.10%		72.89	-48.44		72.72	-48.27		0.00	0.03
22	1.383083	0.058	105.27	0.014698	25.53	71.23%		73.79	-48.27		74.99	-49.46		0.00	0.02
23	-0.23932	0.060	105.03	-0.00229	25.35	71.05%		74.82	-49.46		74.63	-49.28		0.00	0.03
24	1.249919	0.063	106.44	0.013304	26.35	72.07%		75.63	-49.28		76.71	-50.36		0.00	0.02
25	-1.07761	0.066	105.27	-0.01106	25.50	71.22%		75.86	-50.36		74.97	-49.47		0.00	0.02

Sheet1 Sheet2 Sheet3

Delta hedging

	Imposed Parameters			Discretisation				Option Settings			Market Parameters			Hedging Parameter	
	mu	3%		dt	########			Maturity	####		interest	2%		volatility	20%
	sigma	20%		StartLeve	100			Strike	105						
								Type	Call						
	Realised	#####						Pricing Vo	20%						
														TOTAL	0.00
Step	Z	time	S(t)	Return	Value	Delta		Stock	Cash		Stock	Cash		P&L	Value Boo
1		0.000	100.00												0.00
2	#####	0.003		#####										0.00	0.00
3	#####	0.005		#####										0.00	0.00
4	#####	0.008		#####										0.00	0.00
5	#####	0.011		#####										0.00	0.00

1. Fill out the yellow formulas
   - Stock price follows a geometric brownian motion
   - Option Value as per Black—Scholes—Merton formula
   - Option Delta as per Black—Scholes—Merton formula
   - Portfolio value of the stocks/cash
   - Revalued Portfolio value of the stocks/cash
2. Set the parameters to
   - Imposed volatility 20%, drift 3%, interest rate 2%
   - Implied volatility 20%, hedging volatility 20%
   - Lifetime 1 year
   - Daily rebalancing
   - Call option @ 105, sold to customer

Delta hedging

Imposed Parameters		Discretisation		Option Settings		Market Parameters		Hedging Parameter	
mu	3%	dt	#########	Maturity	####	interest	2%	volatility	20%
sigma	20%	StartLeve	100	Strike	105				
				Type	Call				
Realised	#####			Pricing Vc	20%				
								TOTAL	0.00

Step	Z	time	S(t)	Return	Value	Delta		Stock	Cash		Stock	Cash		P&L	Value Boc
1		0.000	100.00												0.00
2	#####	0.003		#####										0.00	0.00
3	#####	0.005		#####										0.00	0.00
4	#####	0.008		#####										0.00	0.00
5	#####	0.011		#####										0.00	0.00

3. Suppose the trader doesn't hedge: what is the final P&L?
4. Change the pricing volatility to 25%, what happens to the P&L?
5. Change the pricing volatility to 15%, what happens to the P&L?
6. Change the hedging volatility to 0%, what happens to the delta?
7. Find a day where the daily P&L is significant (hint near expiry, near strike) and alter the value of ***Z*** to find the balancing point (numerically) such that the P&L on that day becomes zero instead.
8. Change the rebalancing frequency to twice a day (dt)
9. Change the rebalancing frequency to 5 times a day (dt)
10. Change the drift term from 3% to 15%. What is the impact on the P&L?

Delta-hedging Experiment (6)

- While LONG the option
  - If realized volatility < implied volatility, Delta hedge will lose money
  - If realized volatility > implied volatility, Delta hedge will gain money
  - Break-even point is 1 standard deviation move

- While delta hedging an option over many days
  - There will be days with more gamma profit than theta loss
  - There will be days with less gamma profit than theta loss
  - Overall, on average the gains and losses cancel each other out

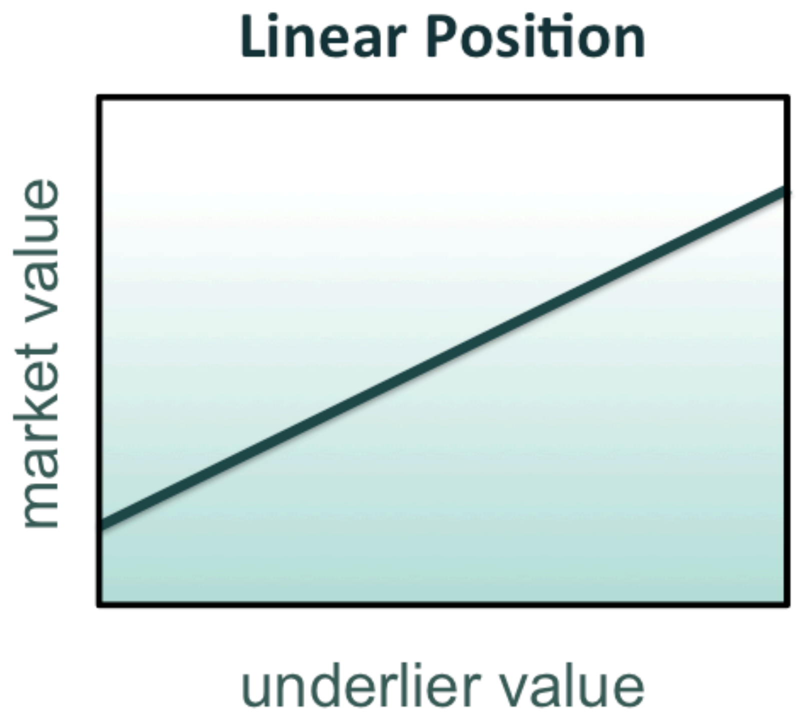
- Hedging Frequency
  - By hedging more frequently, the distribution narrows

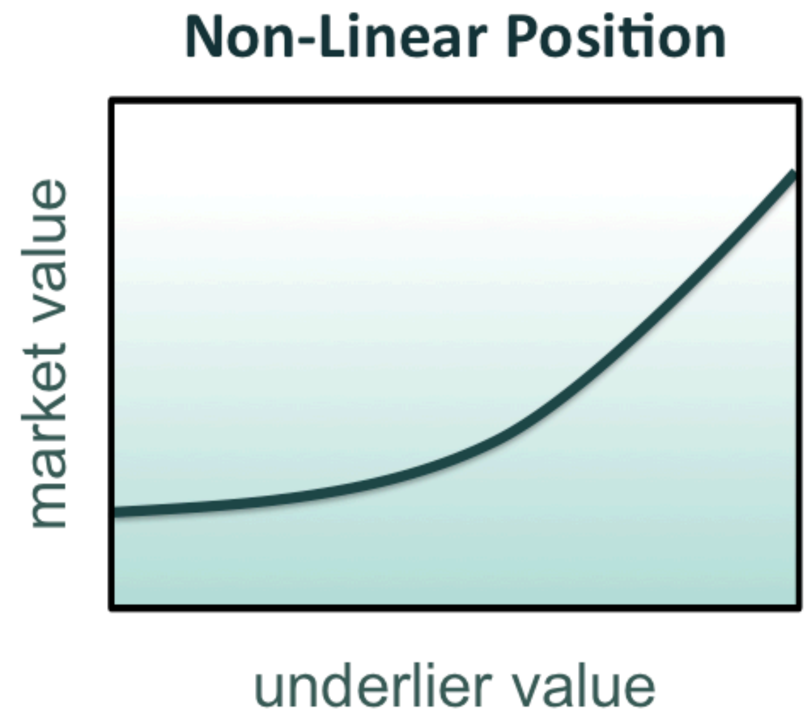
3. The balance between Gamma and Theta

- Taylor Expansion
- Gamma
- Rebalancing Delta hedge
- Theta
- Greek appearances
- Gamma-Theta balance

Taylor Expansion

- Options are non-linear in the underlying price





- Delta-hedge: local hedge *(time and space)*

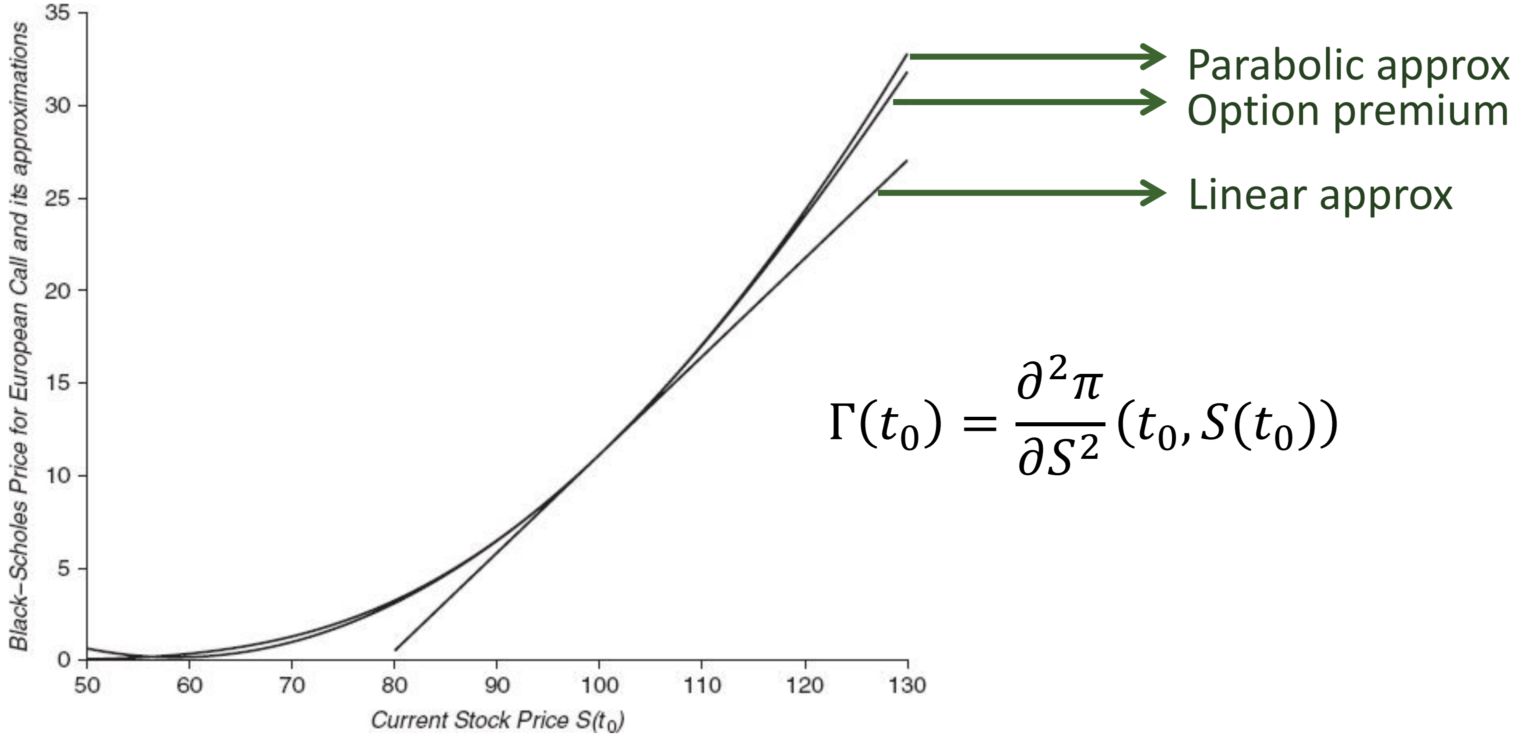
$$\pi(t_0, S(t_0) + \Delta S) - \pi\big(t_0, S(t_0)\big) = \Delta(t_0) \cdot \Delta S$$

- Recognise the partial derivative:

$$\frac{\pi(t_0, S(t_0) + \Delta S) - \pi\big(t_0, S(t_0)\big)}{\Delta S} = \Delta(t_0) = \frac{\partial \pi}{\partial S}\big(t_0, S(t_0)\big)$$

Taylor Expansion (2)

- Graphical representation of the delta-hedge



$$\Gamma(t_0) = \frac{\partial^2 \pi}{\partial S^2}(t_0, S(t_0))$$

$$\pi(t_0, S(t_0) + \Delta S) - \pi\big(t_0, S(t_0)\big) = \Delta(t_0) \cdot \Delta S + \frac{1}{2}\Gamma(t_0) \cdot \Delta S^2$$

Gamma

- Difference between option and Delta-hedge

$$\pi(t_0, S(t_0) + \Delta S) - \pi\big(t_0, S(t_0)\big) - \Delta(t_0) \cdot \big(S(t_0) + \Delta S - S(t_0)\big)$$

- Parameters:

Stock Volatility (implied)	σ = 20%
Div yield	q = 0%
Int rate	r = 0%
Duration	$T\text{-}t_0 = 1$
Initial stock level	$S(t_0) = 50$
Instrument	ATM Call option
Currency	EUR

- Hedging ratio:

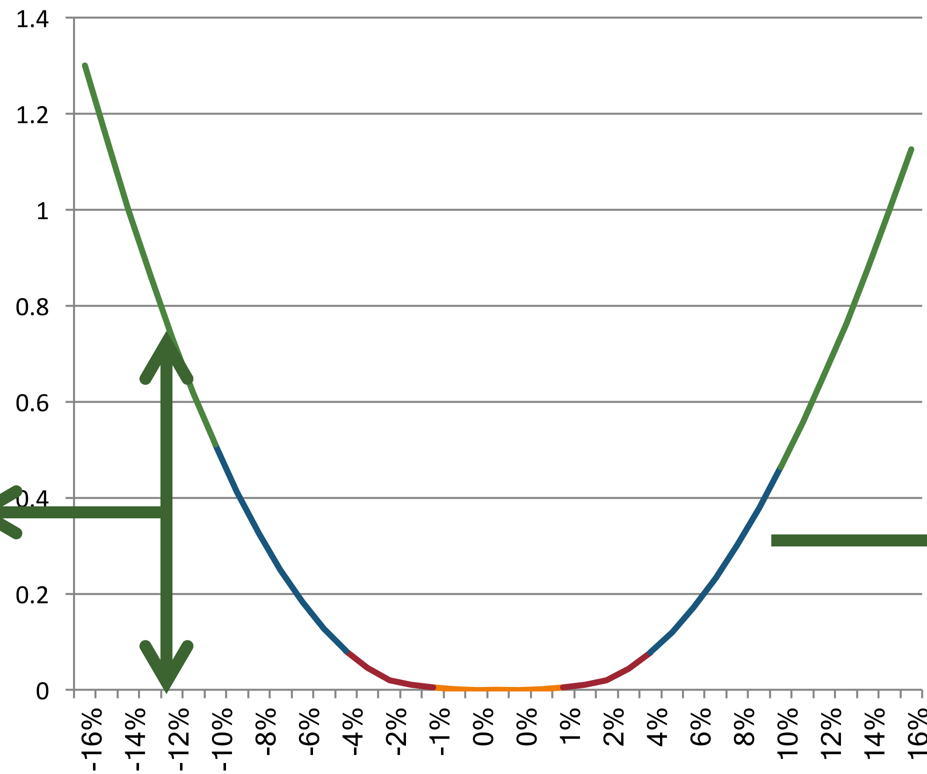
$$\Delta(t_0, S(t_0)) = 53.99\%$$

Gamma (2)

- Variations of **ΔS**

Stock price move	−8.00	−4.00	−2.00	−1.00	−0.50	0.00
Call option price move	−3.32	−1.99	−1.08	−0.56	−0.29	0.00
Put option price move	4.68	2.01	0.92	0.44	0.21	0.00
Stock price move	0.00	0.50	1.00	2.00	4.00	8.00
Call option price move	0.00	0.29	0.60	1.23	2.61	5.72
Put option price move	0.00	−0.21	−0.40	−0.77	−1.39	−2.28

$$\Gamma_{P\&L} = +\frac{1}{2} \cdot \Gamma \cdot \Delta S^2$$



Parabolic shape

Gamma (3)

- Gamma for Put and Call are identical. From the put-call parity we can easily derive:

$$\Delta_C - \Delta_C = 1$$

and hence

$$\Gamma_C = \Gamma_P$$

- Black-Scholes-Merton Gamma:

	Gamma Γ
Γ_C	$\exp\left(-q\left(T-t_0\right)\right)\dfrac{\phi\left(d_1\right)}{S\left(t_0\right)\sigma\sqrt{T-t_0}}$
Γ_P	$\exp\left(-q\left(T-t_0\right)\right)\dfrac{\phi\left(d_1\right)}{S\left(t_0\right)\sigma\sqrt{T-t_0}}$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right).$$

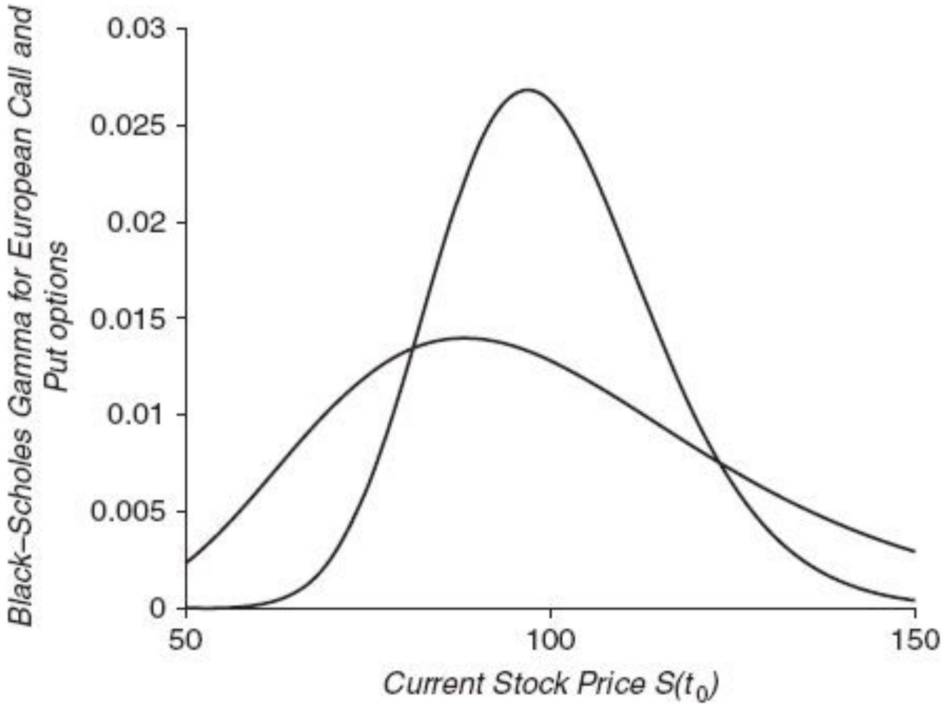
EXERCISE : Black-Scholes calculator

- Extend the Black-Scholes calculator in excel

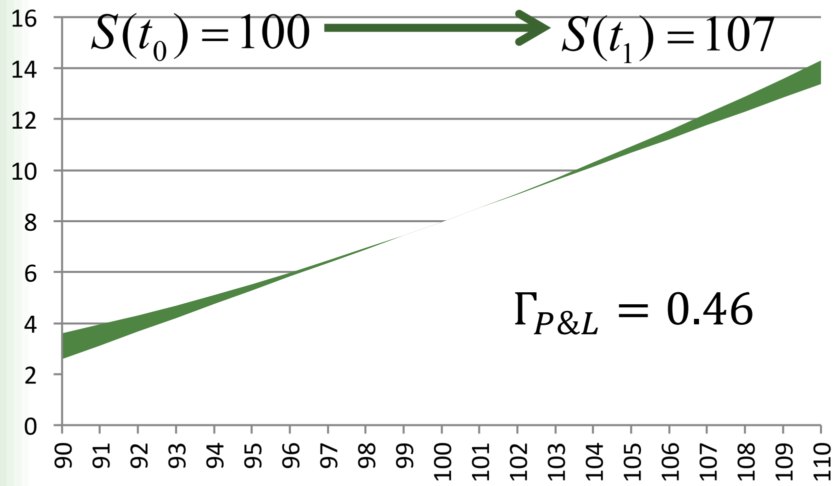
Price	100
Strike	100
interest rate	2%
dividend yield	0%
duration	1.5
Volatility	20%
Call/Put	Call
FWD	
d1	
d2	
N(d1)	
N(d2)	
Price	
Delta	
Gamma	
Vega	
Theta	
Vanna	
Volga	

Gamma (4)

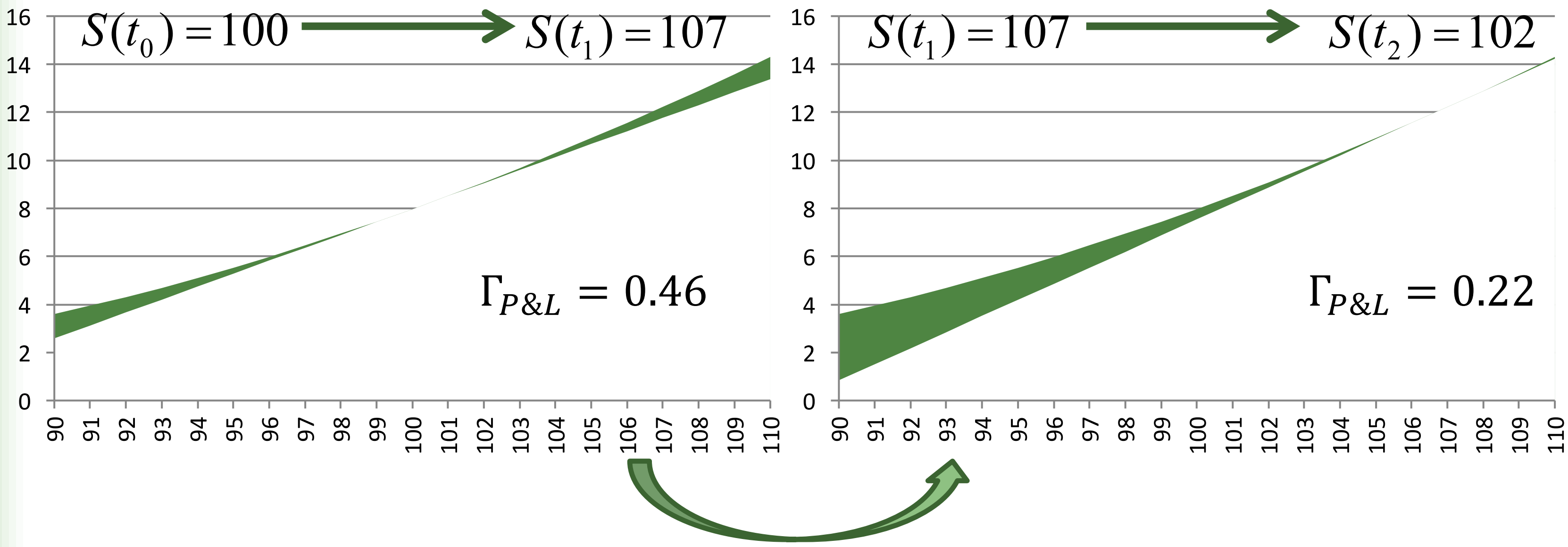
- Properties
  - Flatter for longer-dated options
  - Peaks around the ATM point
  - Identical for Puts and Calls



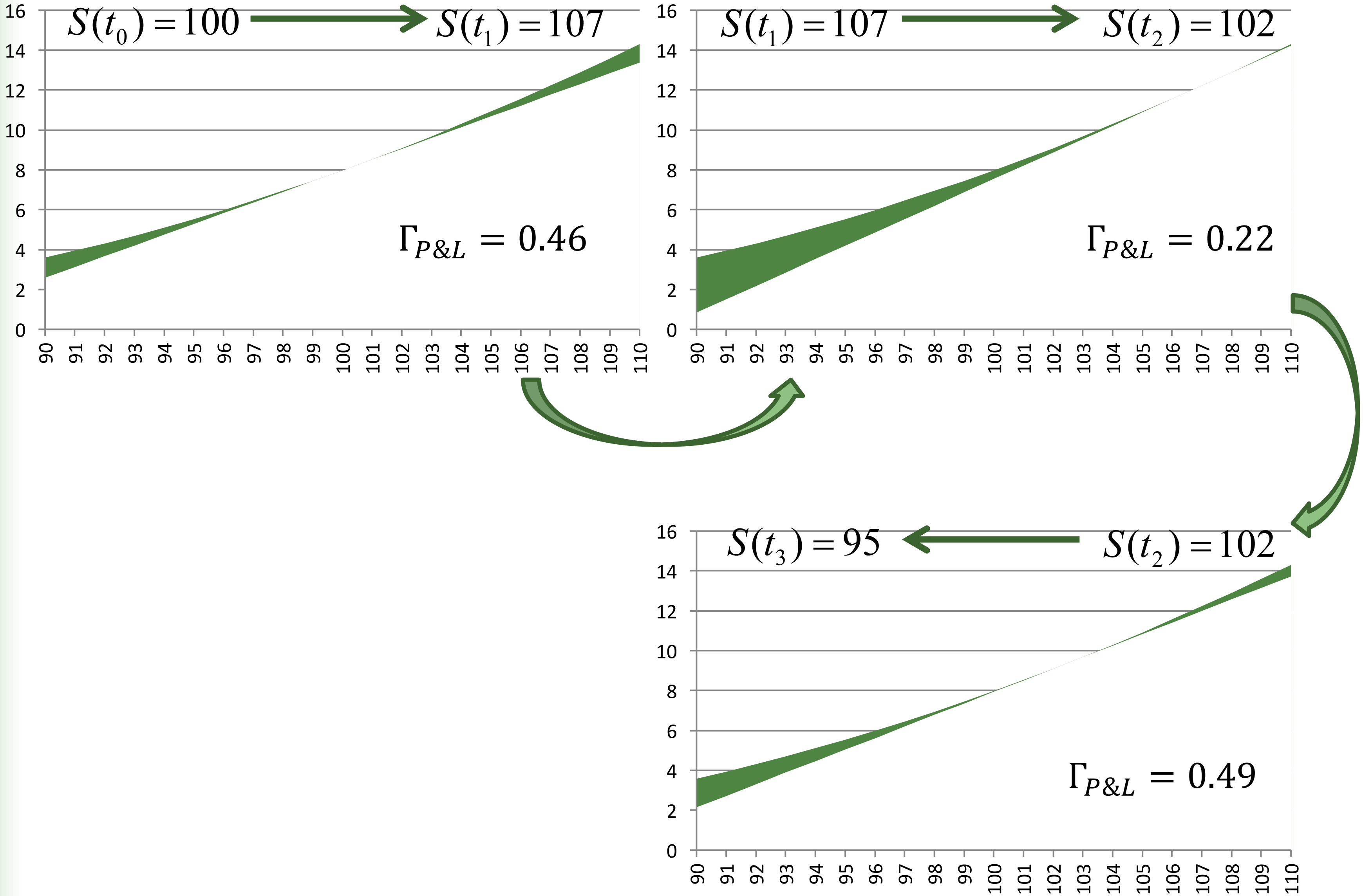
Rebalancing Delta-hedge



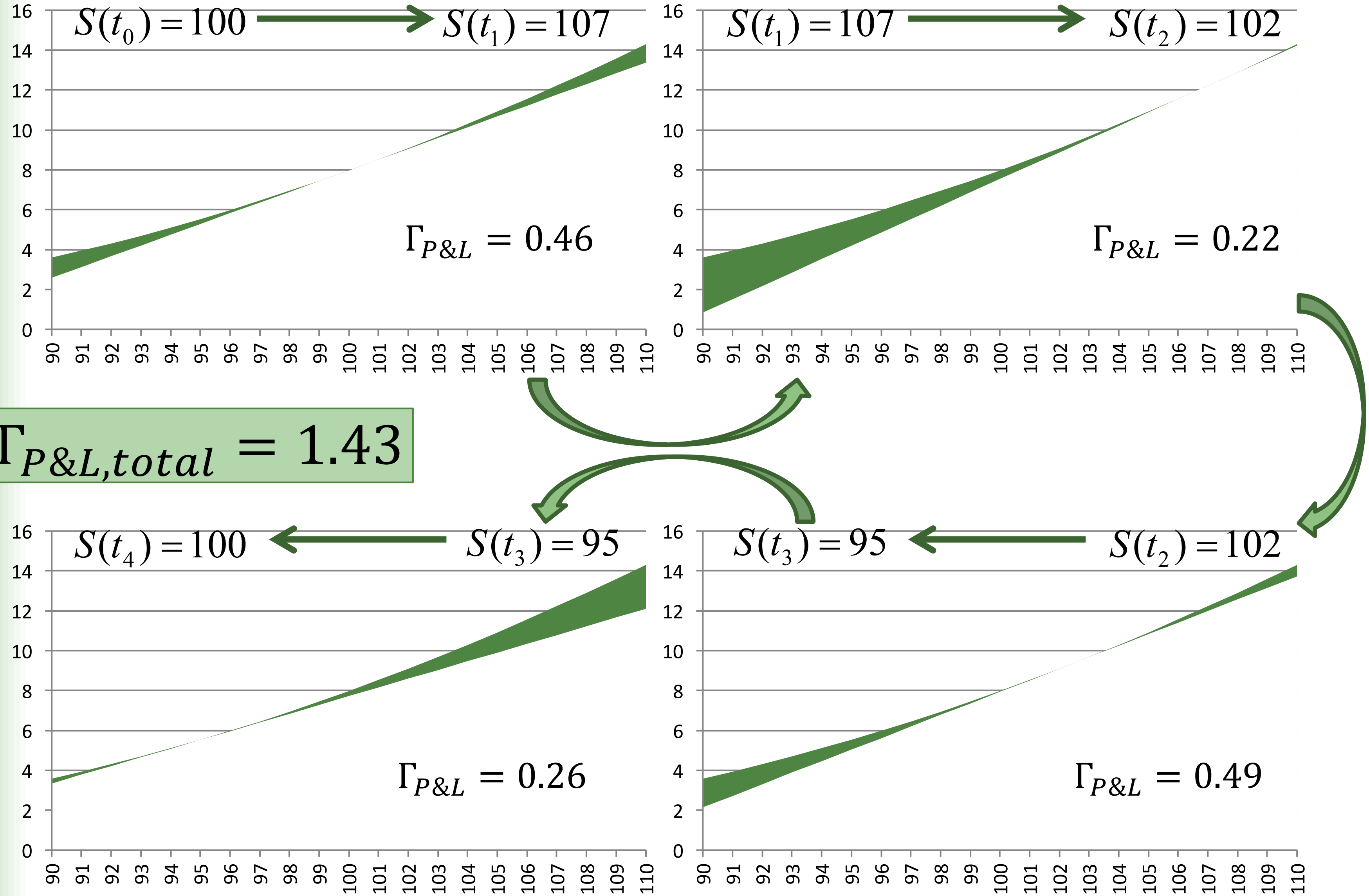
Rebalancing Delta-hedge



Rebalancing Delta-hedge



Rebalancing Delta-hedge



$$\Gamma_{P\&L,total} = 1.43$$

Rebalancing Delta-hedge (2)

- For european call/put options, rebalancing after a move ΔS results in positive P&L for the trader holding the option (long the option).
- Is this an arbitrage? Option lose value over time:

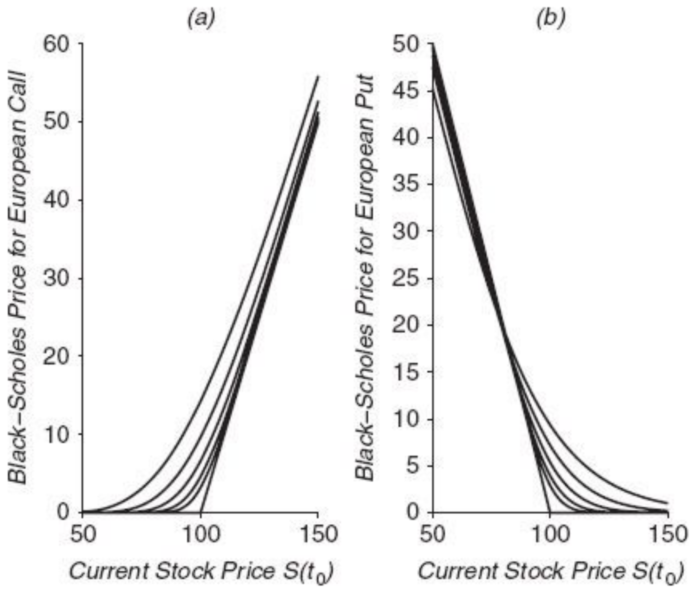
Stock Volatility	σ = 35%
Div yield	q = 0%
Int rate	r = 2.5%
Duration	$T-t_0$ = 250 days
Initial stock level	$S(t_0)$ = 20
Instrument	ATM Put option
Currency	EUR

Rebalancing Delta-hedge (3)

- Fixed Stock level

Value	Days left	Value	Days left	Value	Days left
2.50	250	1.99	150	1.66	100
2.48	245	1.96	145	1.62	95
2.46	240	1.93	140	1.58	90
2.44	235	1.90	135	1.54	85
1.20	50	0.94	30	0.55	10
1.14	45	0.86	25	0.39	5
1.07	40	0.77	20	0.18	1
1.01	35	0.67	15	0.00	0

- On a graph:



- Various stock levels

Stock	Value	Discounted intrinsic value	Time value	Days left
16	4.00	3.92	0.08	50
16	3.99	3.92	0.07	45
18	2.37	1.96	0.41	50
18	2.33	1.96	0.37	45
20	1.20	0.00	1.20	50
20	1.14	0.00	1.14	45
22	0.52	0.00	0.52	50
22	0.46	0.00	0.46	45
24	0.19	0.00	0.19	50
24	0.16	0.00	0.16	45

Theta

- Theta Definition: how does the option lose value over time:
  - Change valuation time

$$\theta = \frac{\partial \pi}{\partial t}.$$

  - Change maturity time

$$\theta = -\frac{\partial \pi}{\partial T}.$$

- Daily theta (time decay)

$$\theta_d = \theta \cdot \Delta t$$

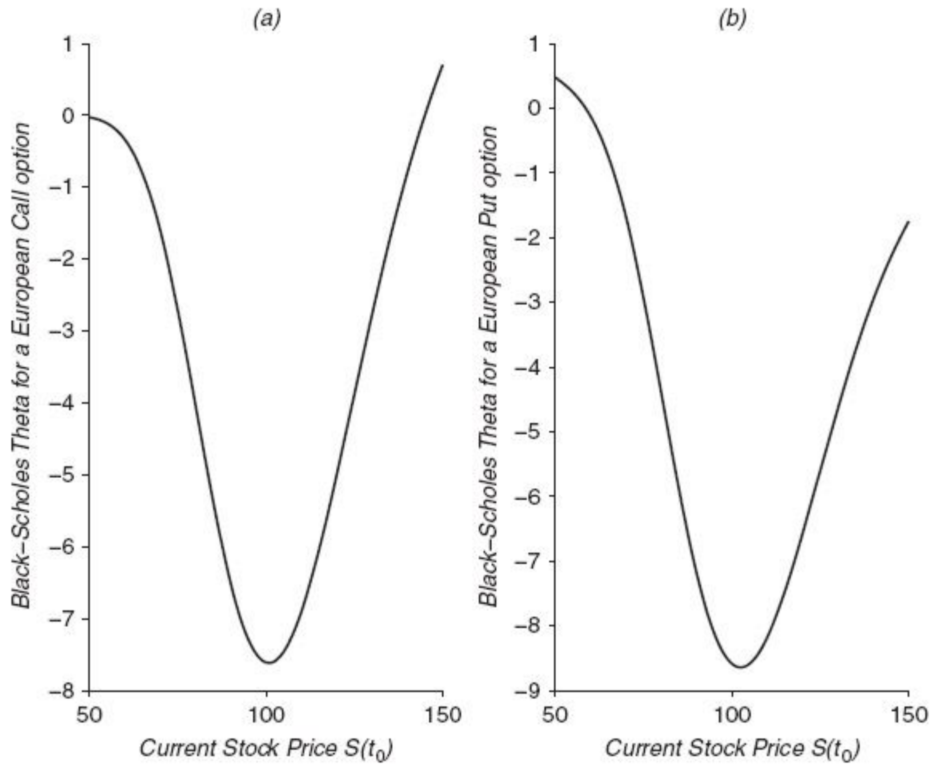
- BSM model is stationary: *both concepts are identical*

$$\begin{aligned}\theta_d &= \pi\left(S_0, t_0 + \Delta t; T, K\right) - \pi\left(S_0, t_0; T, K\right) \\ &= \pi\left(S_0, t_0; T - \Delta t, K\right) - \pi\left(S_0, t_0; T, K\right).\end{aligned}$$

Theta (2)

- In the BSM model, the theta is a closed formula:

	Theta θ
θ_C	$\exp(-r(T-t_0))\left(-\frac{1}{2}F\sigma\phi(d_1) - rKN(d_2) + qFN(d_1)\right)$
θ_P	$\exp(-r(T-t_0))\left(-\frac{1}{2}F\sigma\phi(d_1) + rKN(-d_2) - qFN(-d_1)\right).$



EXERCISE : Black-Scholes calculator

- Extend the Black-Scholes calculator in excel

Price	100
Strike	100
interest rate	2%
dividend yield	0%
duration	1.5
Volatility	20%
Call/Put	Call
FWD	
d1	
d2	
N(d1)	
N(d2)	
Price	
Delta	
Gamma	
Vega	
Theta	
Vanna	
Volga	

Greeks Relationships

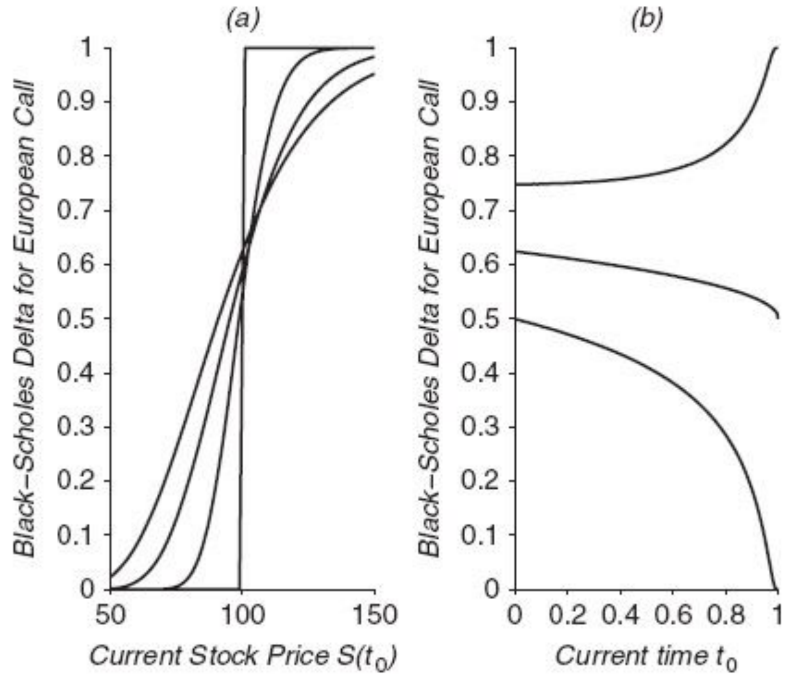


Figure 3.7 (*a*) The call option delta for three different maturities, as compared to the intrinsic delta. (*b*) The delta for an ITM (top), an ATM (middle) and an OTM (bottom) call option as a function of time t_0.

Greeks Relationships (2)

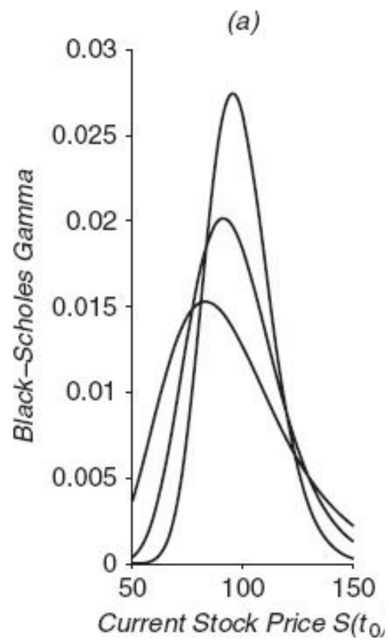


Figure 3.8 (*a*) The gamma for three different maturities. (

Greeks Relationships (3)

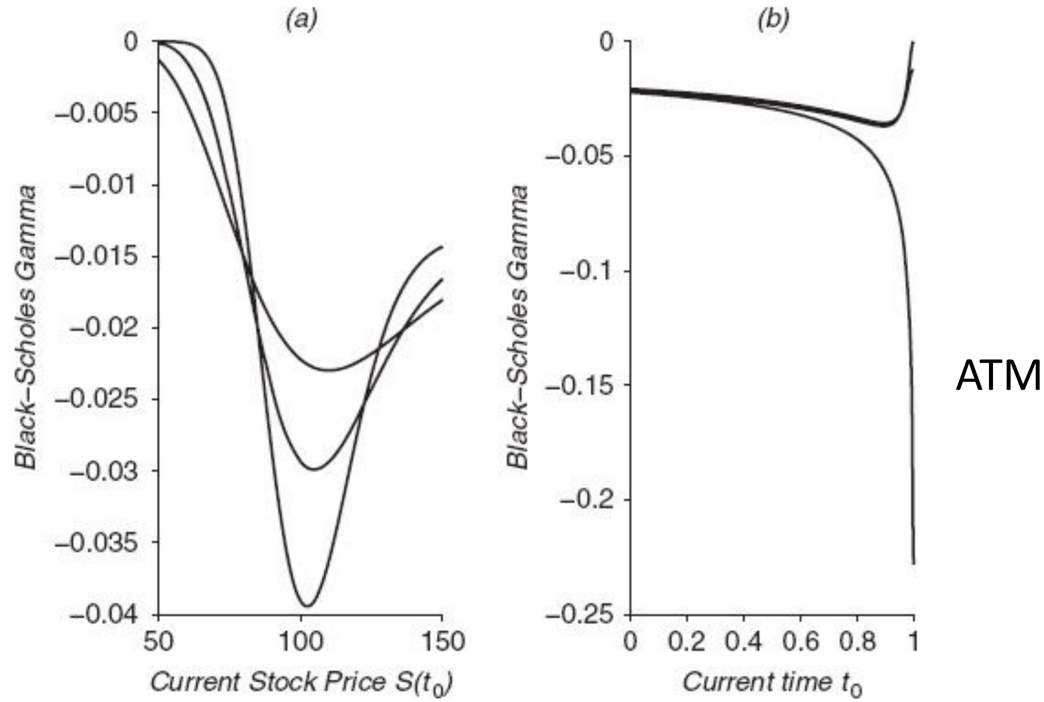


Figure 3.9 (a) The call option theta for three different maturities. (b) The theta for an ITM (top), an ATM ( ) and an OTM ( ) call option as a function of time t_0.

Greeks Relationships (4)

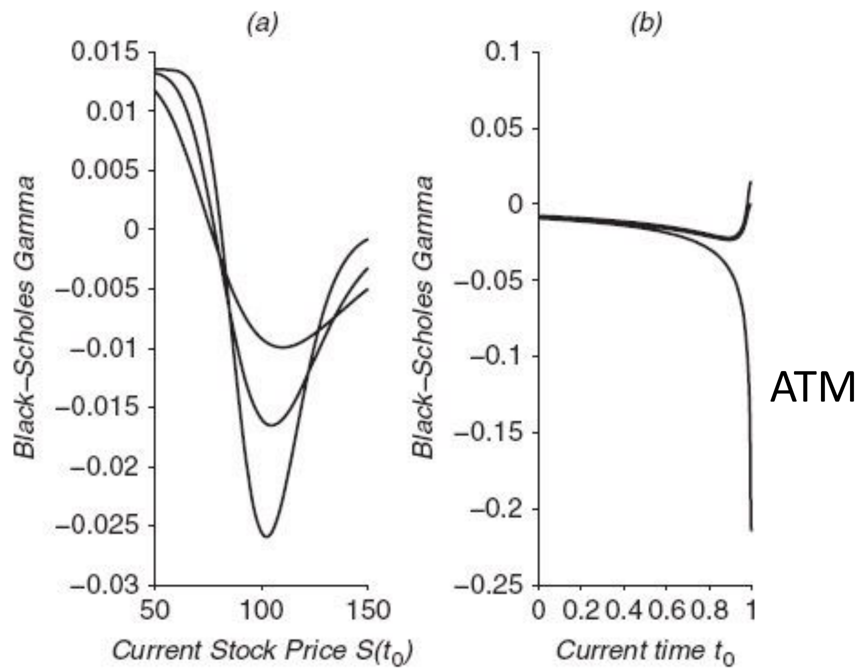


Figure 3.10 (a) The put option theta for three different maturities. (b) The theta for an ITM (top), an ATM :) and an OTM n) put option as a function of time t_0.

Gamma-theta balance

- When stock price moves: *gamma profit*

$$\Gamma_{P\&L} = \frac{1}{2} \cdot \Gamma(t_0) \cdot (\Delta S)^2$$

- When time moves: *theta loss*

$$\theta_{\Delta t} = \theta\ (t_0) \cdot (\Delta t)$$

- In the BSM model, space and time are linked:

$$(\Delta S)^2 \sim \left(\sqrt{\Delta t} S \sigma\right)^2$$

- Cash account:
  - Option premium π
  - Delta-hedge Δ x $S(t_0)$
  - Interest payment: r Δt ($\pi - \Delta$ x $S(t_0)$)

Gamma-theta balance

- All balance out in the theta-gamma balance:

$$\theta + \frac{1}{2}(S(t_0)\sigma)^2\Gamma = r \cdot (\pi - S(t_0) \cdot \Delta)$$

- Balance
  - individual instruments: stock, forwards, options
  - Portfolio containing any instrument
  - Only holds in the BSM model, but approximately in other models
- Rewritten with partial derivatives: **BS PDE**

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 \pi}{\partial S^2} - r\pi + rS \cdot \frac{\partial \pi}{\partial S} = 0$$

Gamma theta balance

- Another way to ask the question, while hedging:
- How much should my underlying move such that Gamma and Theta perfectly balance?
- What does this balancing point mean statistically?

$$\Gamma_{P\&L} = \frac{1}{2} \cdot \Gamma(t_0) \cdot (\Delta S)^2$$

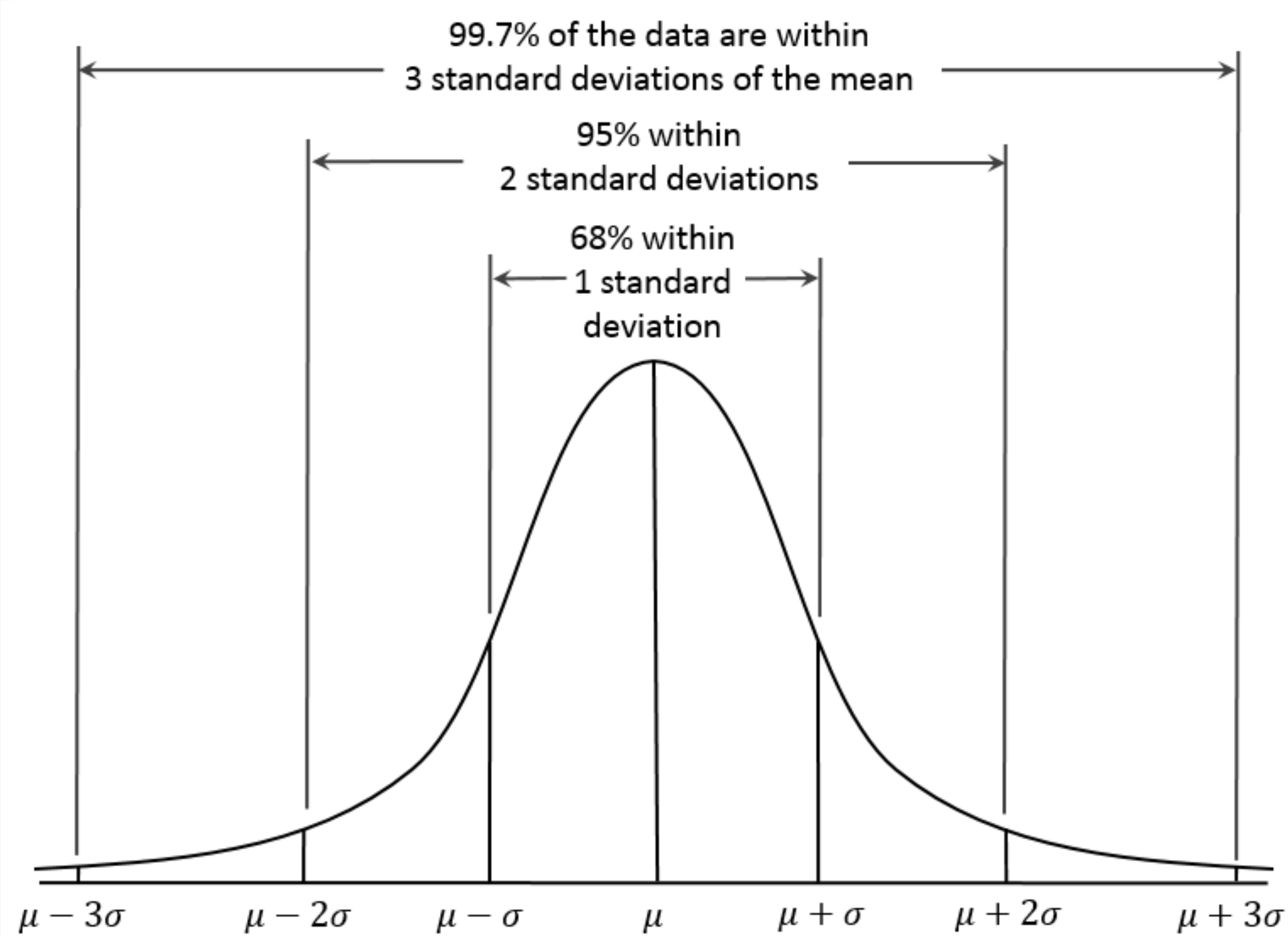
$$\theta_{\Delta t} = \theta\,(t_0) \cdot (\Delta t)$$

$$(\Delta S)^2 = -2 \cdot \frac{\theta\,(t_0)}{\Gamma(t_0)} \cdot \Delta t \qquad \text{and} \qquad (\Delta S)^2 \sim \left(\sigma S \sqrt{\Delta t} Z\right)^2$$

$$\Longrightarrow Z \sim \pm \sqrt{2 \cdot \frac{\theta\,(t_0)}{\sigma^2 S^2(t_0)\Gamma(t_0)}} = 1$$

- 1-standard deviation points are the balancing points

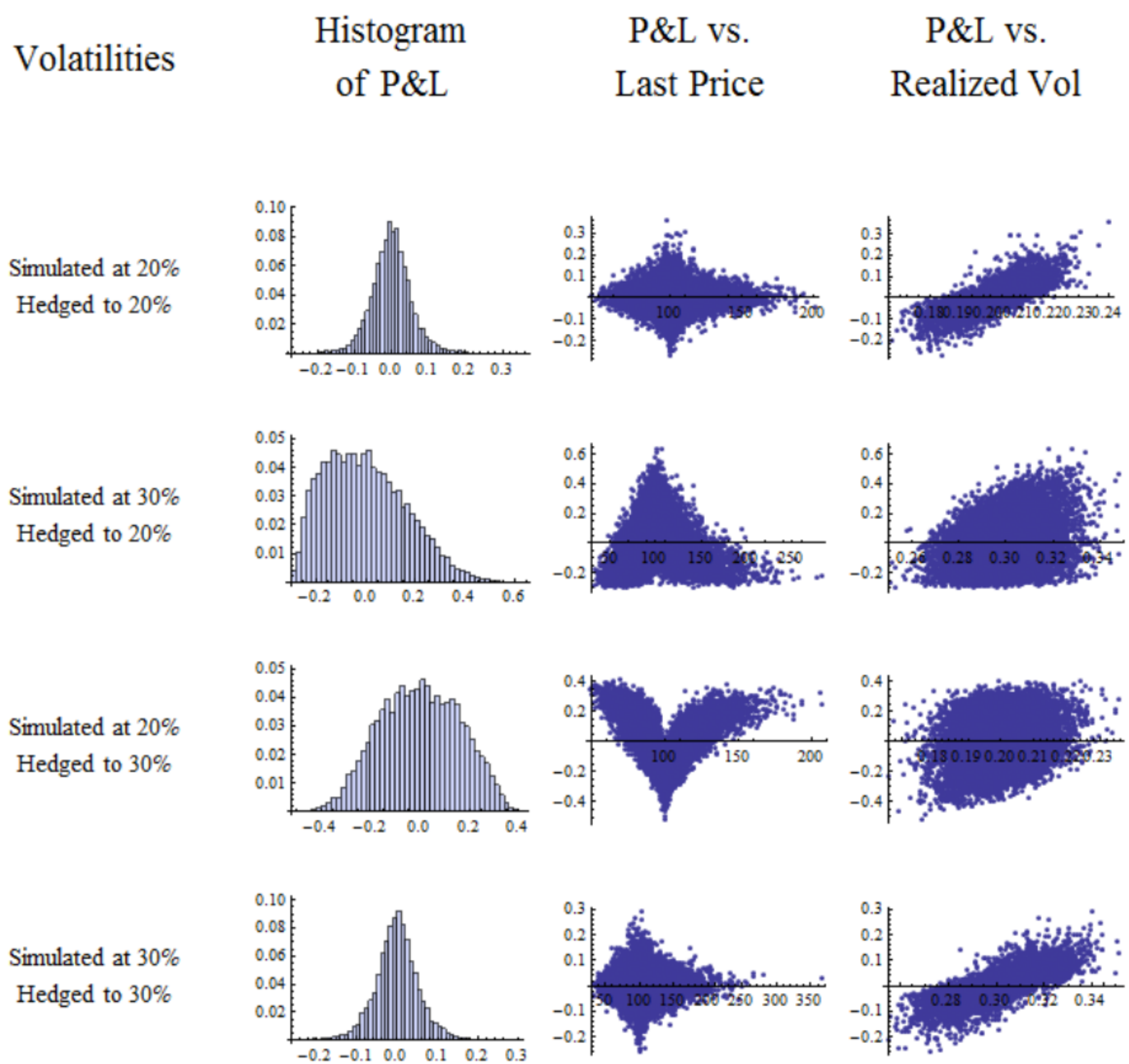
Gamma theta balance



Gamma theta balance

Range	Expected Fraction of Population Inside Range	Approximate Expected Frequency Outside Range	Approximate Frequency for Daily Event
μ ± 0.5σ	0.382 924 922 548 026	2 in 3	Four times a week
μ ± σ	0.682 689 492 137 086	1 in 3	Twice a week
μ ± 1.5σ	0.866 385 597 462 284	1 in 7	Weekly
μ ± 2σ	0.954 499 736 103 642	1 in 22	Every three weeks
μ ± 2.5σ	0.987 580 669 348 448	1 in 81	Quarterly
μ ± 3σ	0.997 300 203 936 740	1 in 370	Yearly
μ ± 3.5σ	0.999 534 741 841 929	1 in 2149	Every six years
μ ± 4σ	0.999 936 657 516 334	1 in 15 787	Every 43 years (twice in a lifetime)
μ ± 4.5σ	0.999 993 204 653 751	1 in 147 160	Every 403 years (once in the modern era)
μ ± 5σ	0.999 999 426 696 856	1 in 1 744 278	Every 4776 years (once in recorded history)
μ ± 5.5σ	0.999 999 962 020 875	1 in 26 330 254	Every 72 090 years (thrice in history of modern humankind)
μ ± 6σ	0.999 999 998 026 825	1 in 506 797 346	Every 1.38 million years (twice in history of humankind)
μ ± 6.5σ	0.999 999 999 919 680	1 in 12 450 197 393	Every 34 million years (twice since the extinction of dinosaurs)
μ ± 7σ	0.999 999 999 997 440	1 in 390 682 215 445	Every 1.07 billion years (a quarter of Earth's history)
μ ± xσ	$\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$	1 in $\frac{1}{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}$	Every $\frac{1}{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}$ days

Delta hedging Performance



Delta hedging Performance

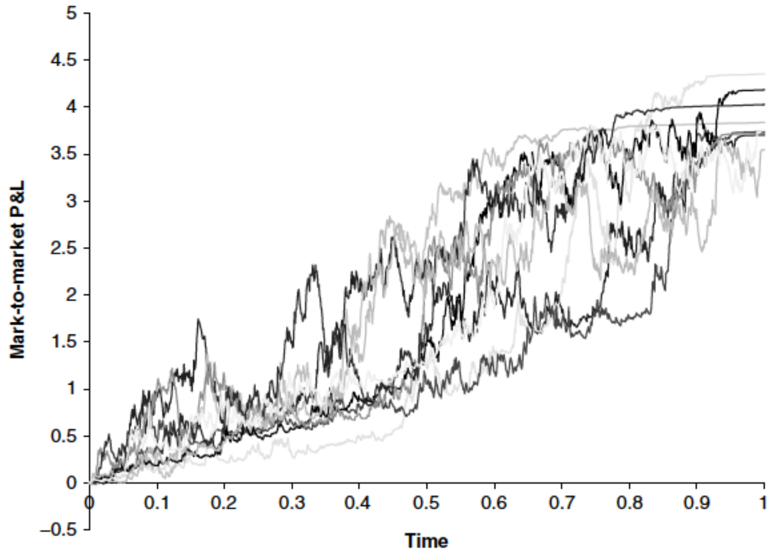


Figure 12.2 P&L for a delta-hedged option on a mark-to-market basis, hedged using actual volatility.

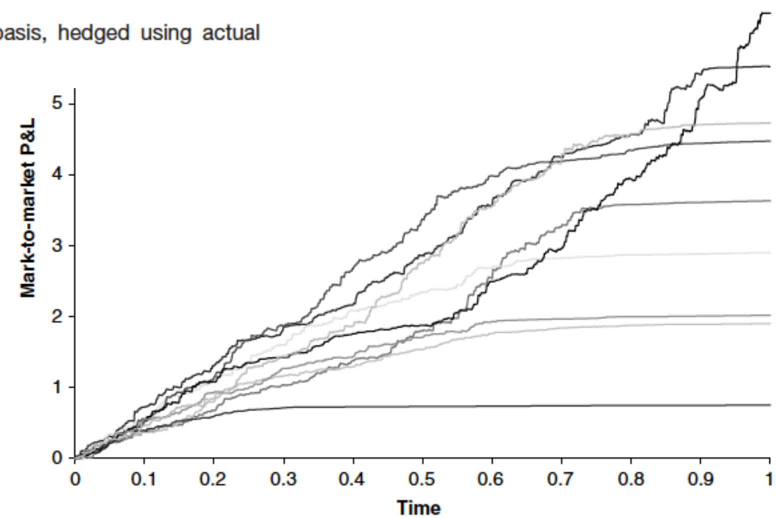


Figure 12.3 P&L for a delta-hedged option on a mark-to-market basis, hedged using implied volatility.

Delta hedging Performance

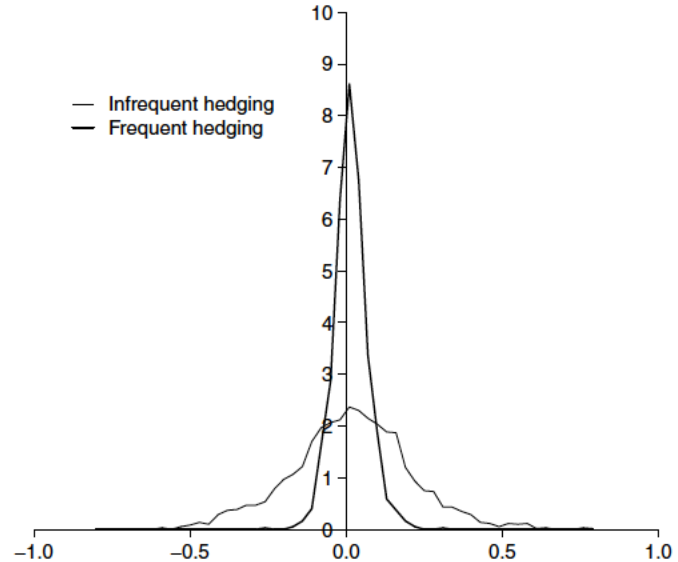
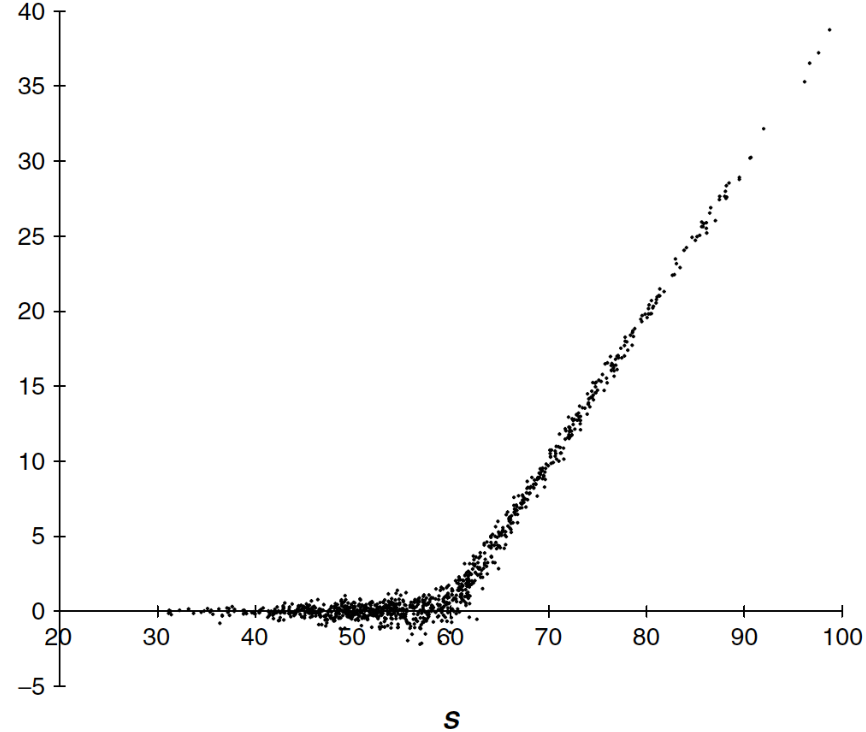


Figure 47.6 The distribution of the total hedging error for a call hedged frequently and infrequently.

Discrete Hedging

- So far we learned about hedging:
  - If we know the volatility
  - if continuous rebalancing takes place,

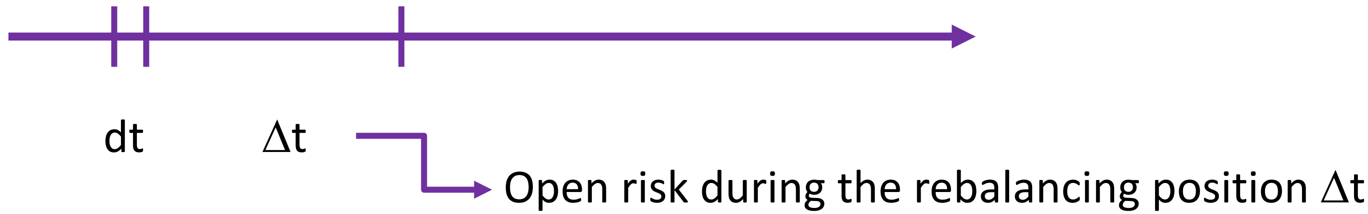
  the hedge works
- Discrete hedging introduces noise
- ATM options have more noise
- If there is a trend, your discrete hedge can become biased



The results of discrete hedging.

Discrete Hedging

- Hedging with a view: while hedging discretely, we can even find a "better" hedge, which is no longer risk-neutral.
- Minimize the variance of the portfolio over the next time step



- The minimisation of the variance over the time-step leads to setting delta:

$$\Delta = \frac{\partial V}{\partial S} + \delta t \left(\mu - r + \tfrac{1}{2}\sigma^2\right) S \frac{\partial^2 V}{\partial S^2}.$$

- And hence to another option value, which can be shown to again be the BS formula, but with adjusted volatility

$$\sigma^* = \sigma \left(1 + \frac{\delta t}{2\sigma^2}(\mu - r)\left(r - \mu - \sigma^2\right)\right)$$

4. Trading is the answer to the unknown

- Input Parameters
- Current Price Level
- Interest rate
- Time
- Volatility

Input Parameters

- The Black-Scholes-Merton formula:

$$F = S(t_0)\exp\left((r-q)(T-t_0)\right)$$

$$d_1 = \frac{\log(F/K) + \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t_0} = \frac{\log(F/K) - \frac{1}{2}\sigma^2(T-t_0)}{\sigma\sqrt{T-t_0}}.$$

	Premium π
π_C	$\exp(-r(T-t_0))(FN(d_1) - KN(d_2))$
π_P	$\exp(-r(T-t_0))(KN(-d_2) - FN(-d_1))$

	Delta Δ
Δ_C	$\exp\left(-q(T-t_0)\right)N(d_1)$
Δ_P	$-\exp\left(-q(T-t_0)\right)N(-d_1)$

Current Price Level

- Do we really know $S(t_0)$?
- Bid/Offer spread. Price per volume / volume per price
- Mid Price does *not* exist
- Examples: Purchase price
  - Volume = 10000: Price = 50.10
  - Volume = 20000: Price = 50.125

Bid		Offer	
Value	Volume	Value	Volume
49.90	10000	50.10	15000
49.85	20000	50.20	5000
49.82	10000	50.25	10000
49.80	5000	50.30	20000

- Is there always enough volume to delta hedge?
  - **Small** volumes don't trade
  - **Large** Volumes can be problematic too

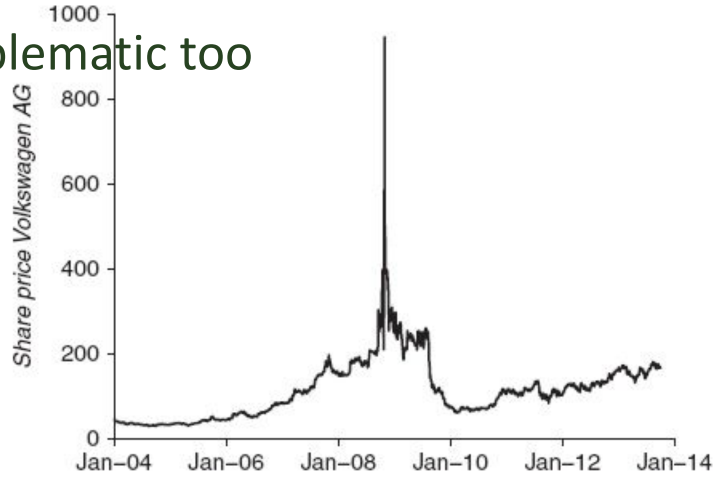


Figure 4.1 The historical share prices for Volkswagen AG from January 2004 till October 2013. One can clearly see the spike of 28 October 2008.

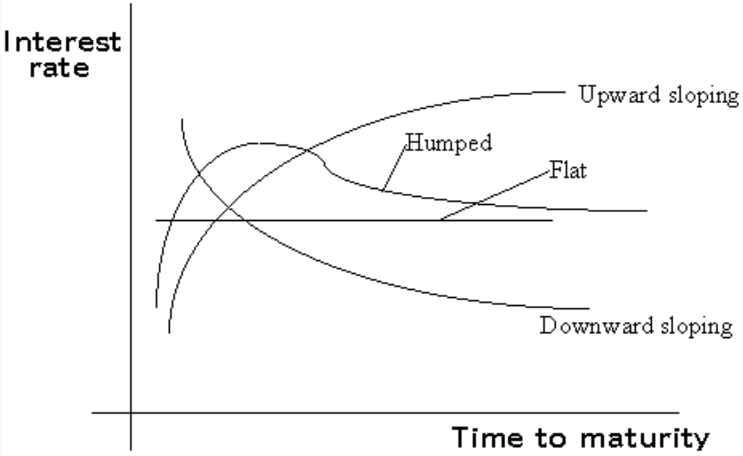
Current Price Level (2)

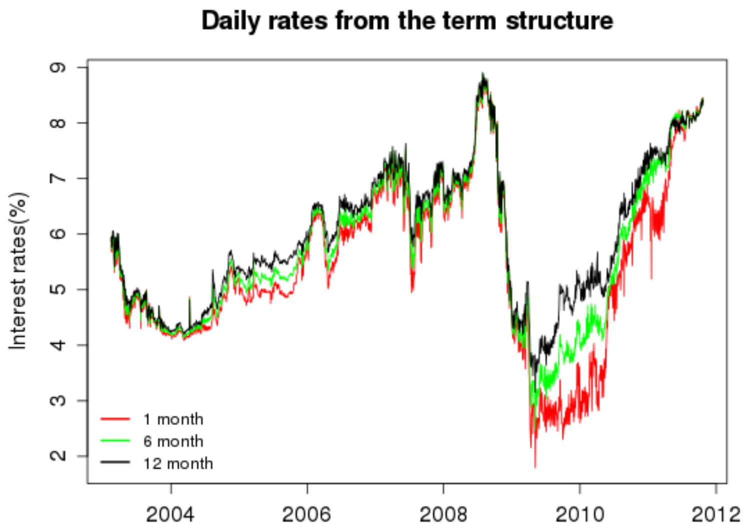
- Standard practice: use mid level for valuation + hedging
- Volatility/fluctuations are larger than bid/offer
- Volume constraints are ignored (or priced in)
  - Leland model
  - Δt: timestep of hedging
  - κ: percentage transaction cost (slippage)
  - Bid volatility and offer volatility:
    - gamma profits will be smaller when long the option due to slippage.
    - gamma losses will be larger when short the option due to slippage.

$$\check{\sigma} = \sigma\sqrt{1-\sqrt{\frac{8}{\pi\,\Delta t}}\frac{\kappa}{\sigma}} \qquad \hat{\sigma} = \sigma\sqrt{1+\sqrt{\frac{8}{\pi\,\Delta t}}\frac{\kappa}{\sigma}}$$

Interest rate

- Is the interest rate really fixed?





- Is lending and borrowing done at the same rate?
- Is interest rate really risk-free?
- There is another greek: rho

$$\rho = \frac{\partial \pi}{\partial r}.$$

- ***Funding Cost:*** Typically artificially set or risk-sharing with other desks (rates, treasury,…)

Time

- Exercise (individual and confidential)

1. Build the Black-Scholes calculator in excel

Price	100
Strike	100
interest rate	2%
dividend yield	0%
duration	1.5
Volatility	20%
Call/Put	Call
FWD	
d1	
d2	
N(d1)	
N(d2)	
Price	

Time

- Exercise (individual and confidential)

1. Build the Black-Scholes calculator in excel

Price	100
Strike	100
interest rate	2%
dividend yield	0%
duration	1.5
Volatility	20%
Call/Put	Call
FWD	
d1	
d2	
N(d1)	
N(d2)	
Price	

	Price
Particpant 1	
Particpant 2	
Particpant 3	
Particpant 4	
Particpant 5	
Particpant 6	
Particpant 7	
Particpant 8	
Particpant 9	
Particpant 10	
Particpant 11	
Particpant 12	

2. Valuation Date = 28/Feb/2016 – Expiry Date = 1/Oct/2017, what is the sales price of the call option?

Time

- How to measure time in years?
  - Actual/365
  - Actual/366 (leap years)
  - Trading days / 365
  - Trading days / Total trading days
  - …
- Example: ATM put option
  - $S(t_0) = 75$; $r = 2\%$; $\sigma = 20\%$
  - Valuation: 5/february/2012
  - Expiry: 10/april/2013
  - Actual / 365 = 432 / 365
  - Actual / 366 = 432 / 366 (leap year 2012)
  - Actual Weekdays / Total weekdays year = 308 / 261
  - Option price ranges from 5.5766 to 5.5752
- Consistency between volatility estimation and calculation $\sigma\sqrt{T-t_0}$.

Volatility

- Black-Scholes statistical properties are flawed (See 1.)
- Volatility is a concept within the BS model, visible in the numerical presence of higher order moments (SP500)

	Empirical	Theoretical
Mean	−0.00042	$\mu \cdot \Delta t$
Stdev	0.0291	$\sigma \cdot \sqrt{\Delta t}$
Skew	−0.3771	0
Kurtosis	14.0283	3

- Non-constant, random volatility

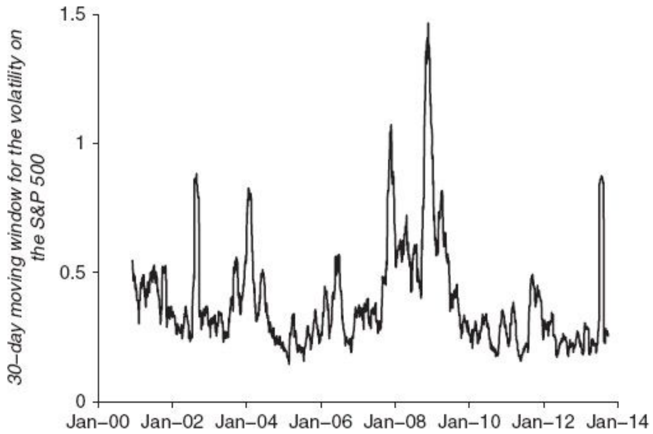


Figure 4.2 The 30-day moving window realised volatility for the S&P 500 over the period 19 October 2000 till 1 October 2013.

	min	max	mean	median	std
Volatility	14.47%	146.47%	38.75%	32.73%	19.98%

Volatility (2)

- Randomized risk factor
- Monetizeable through
  - Delta hedging
  - Buying/Selling options
  - VIX index
  - Variance swaps
- Skew and termstructure (see 7. and 8.)
- Hedging under uncertainty?

BS is robust

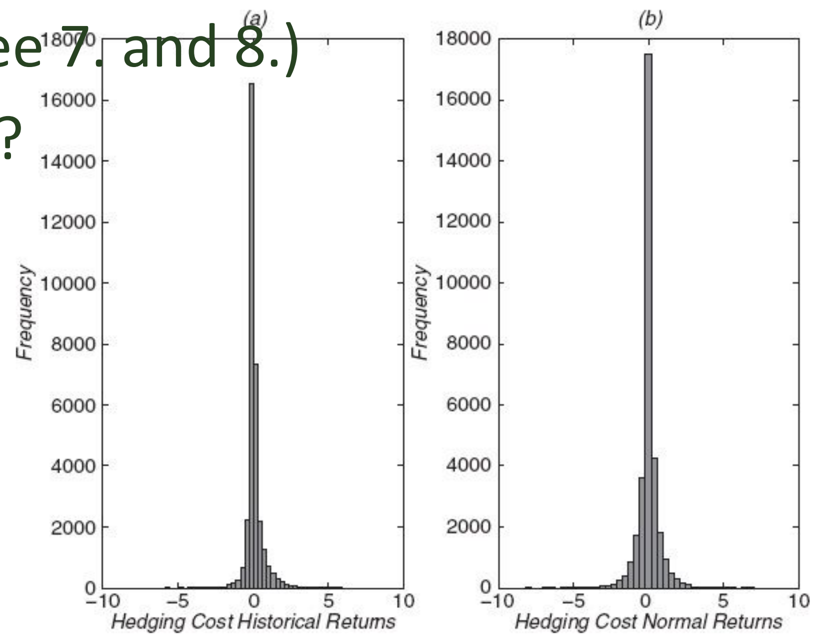


Figure 4.3 The hedging cost for put and call options with various strikes $K = 75\%, 90\%, 100\%, 110\%, 125\%$, each with a one month maturity. (a) shows the backtest using historical returns for the S&P 500 and (b) shows the result using normal returns.

6. Vega as the crucial Greek

- Definition
- Vega spreads
- Vega effect
- Vega profile
- ATM confusion
- Vega over time
- Option Price Approximation
- Old Greeks revisited

Definition

- Volatility is random
- Volatility has become a market parameter
- Vega is born:

$$\nu_{inst} = \frac{\partial \pi}{\partial \sigma} \quad \text{or} \quad \nu = \nu_{inst} \cdot \Delta\sigma$$

- For European Call/Put options

	Vega ν
$\nu_{inst,C}$	$S(t_0)\exp\left(-q(T-t_0)\right)\phi(d_1)\sqrt{T-t_0}$
$\nu_{inst,P}$	$S(t_0)\exp\left(-q(T-t_0)\right)\phi(d_1)\sqrt{T-t_0}$

- In practice, $\Delta\sigma$ = 1% *(typical vol move)*

EXERCISE : Black-Scholes calculator

- Extend the Black-Scholes calculator in excel

Price	100
Strike	100
interest rate	2%
dividend yield	0%
duration	1.5
Volatility	20%
Call/Put	Call
FWD	
d1	
d2	
N(d1)	
N(d2)	
Price	
Delta	
Gamma	
Vega	
Theta	
Vanna	
Volga	

Vega spreads

- Volatility is often used to define option bid offer spreads
- Example
  - $S(t_0) = 100$
  - $T - t_0 = 1$
  - $r = 3\%$
  - $\sigma = 20\%$

	$K=75$	$K=100$	$K=125$
ν_C	0.959	0.3867	0.2743
ν_P	0.959	0.3867	0.2743.

- Use 1 or 2 vega spread between bid and offer of the option, depending on the liquidity.
- Exercise: For the above 6 options (3 strikes, put/call), apply a 2 vega bid/offer spread. Calculate the bid/offer prices and derive the bid-offer spread:

Vega spreads

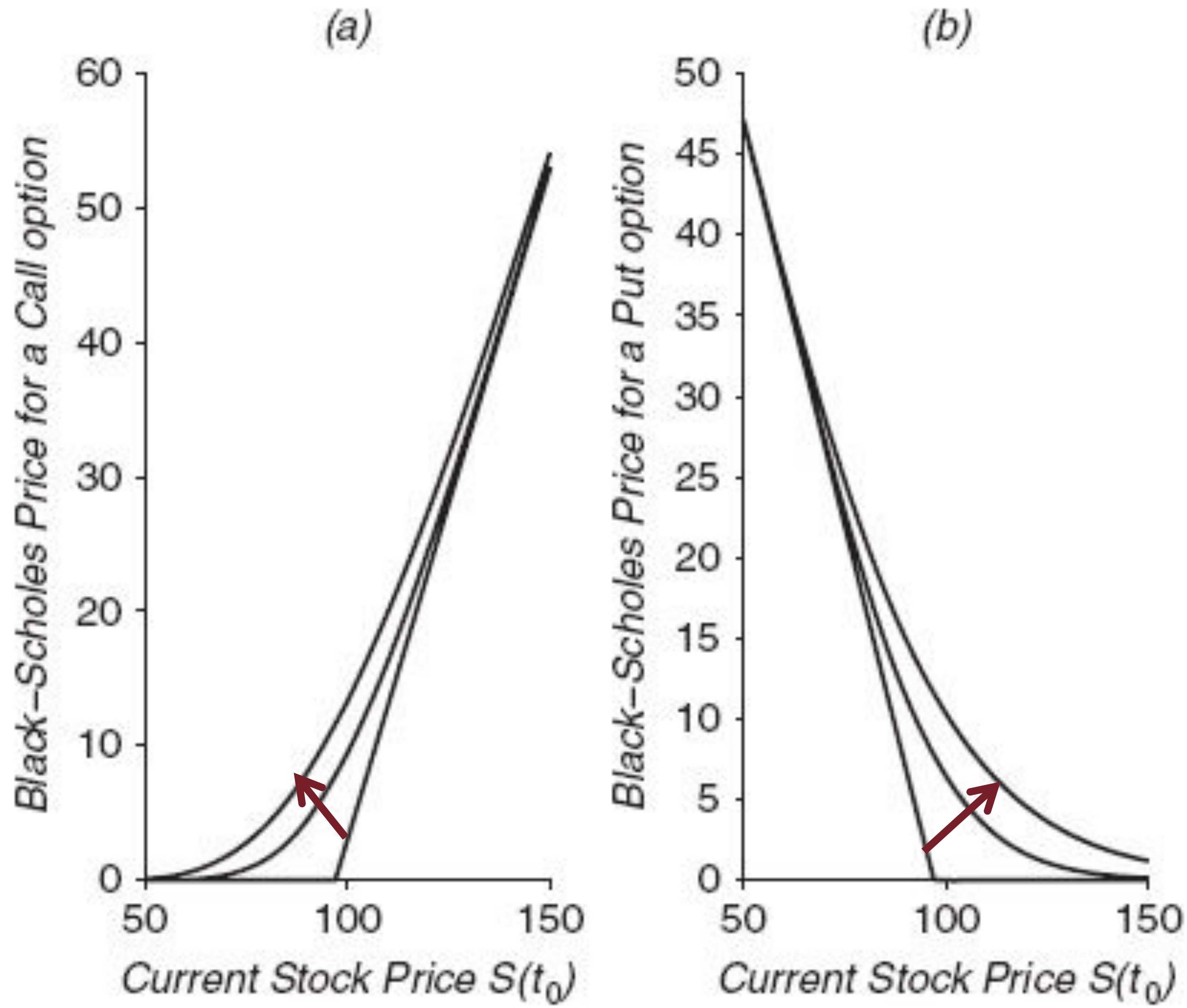
- Volatility is often used to define option bid offer spreads
- Example
  - $S(t_0) = 100$
  - $T\text{-}t_0 = 1$
  - $r = 3\%$
  - $\sigma = 20\%$

	$K = 75$	$K = 100$	$K = 125$
ν_C	0.959	0.3867	0.2743
ν_P	0.959	0.3867	0.2743.

- Exercise: For the above 6 options (3 strikes, put/call), apply a 2 vega bid/offer spread. Calculate the bid/offer prices and derive the bid-offer spread:

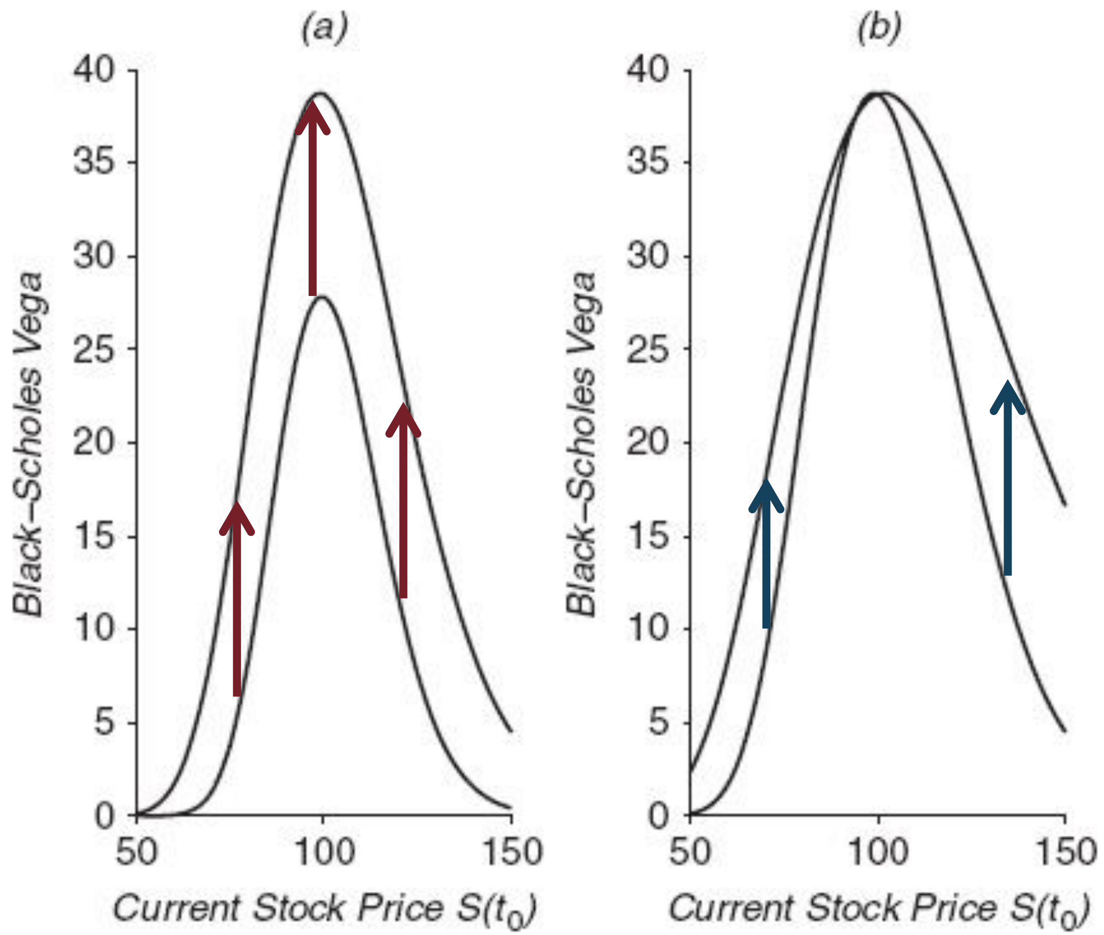
	K = 75			K = 100			K = 125		
Call	B:	O:		B:	O:		B:	O:	
Put	B:	O:		B:	O:		B:	O:	

Vega effect



Increased volatility = increased option price

Vega profile



Increased maturity

Increased volatility

ATM confusion

- Different ATM definitions
  - $K = S(t_0)$
  - $K = F(t_0)$
  - $|\Delta| = 50\%$
  - Gamma peak
  - Vega peak
- All definitions lead to another value of the ATM point
- Large volatility increases the ATM "region"

Vega over time

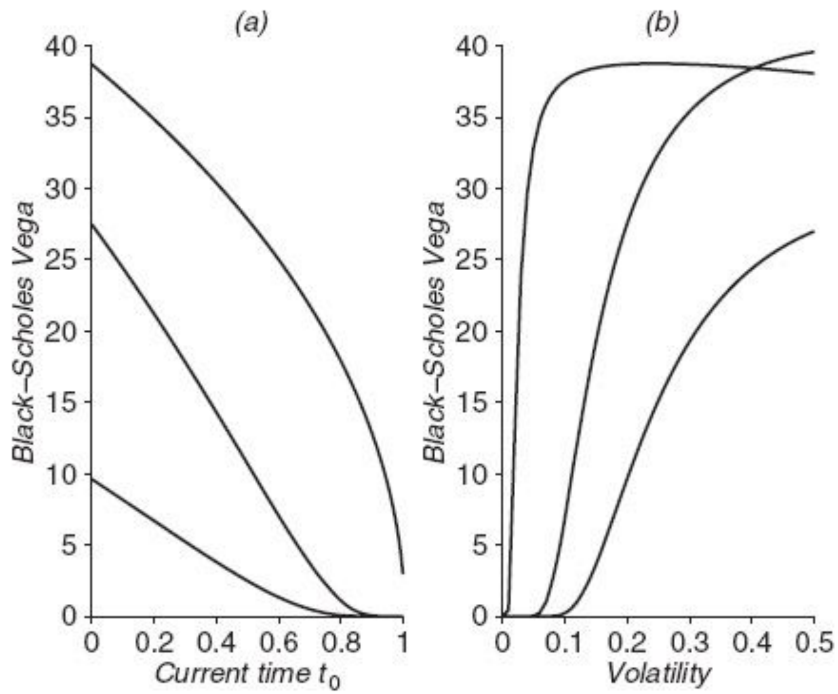


Figure 5.3 (a) The vega as we move through time to the expiration date (right-end point where $T = 1$) for various levels of moneyness. From the top to bottom: ATM, high strike, low strike. (b) The vega for ATM (top), high strike and low strike as a function of the volatility σ.

Option Price approximation

- Option Traders often need to
  - make quick decisions
  - Anticipate moves
  - Manage options on many different underlying instruments
- Option Price approximations can be a great tool

$$\pi_C \approx 0.4 \cdot S(t_0) \exp\left(q(T - t_0)\right) \cdot \sigma \cdot \sqrt{T - t_0}.$$

  - Linear in the underlying level
  - Linear in volatility
  - Proportional to the square root of time (can be neglected during a trading day)

Option Price approximation (2)

- Derivation of the approximation:

For (FWD) ATM call options, we have that $S(t_\circ) = K \exp((r-q)(T-t_\circ))$,which we can plug into the BS formula for a call option (1.10):

$$\begin{aligned}\pi_C &= S(t_0)\exp\left(q(T-t_0)\right)\left(N\left(\frac{1}{2}\sigma\sqrt{T-t_0}\right)-N\left(-\frac{1}{2}\sigma\sqrt{T-t_0}\right)\right)\\ &= S(t_0)\exp\left(q(T-t_0)\right)\left(2\cdot N\left(\frac{1}{2}\sigma\sqrt{T-t_0}\right)-1\right)\end{aligned}$$

so that for small $\sigma\sqrt{T-t_0}$, we can use the following approximation for the cumulative normal distribution:

$$N(\varepsilon) = N(0) + \varepsilon N'(0) + \frac{1}{2}\varepsilon^2 N''(0) + \mathcal{O}\left(\varepsilon^3\right)$$

and of course

$$N'(0) = \phi(0) = \frac{1}{\sqrt{2\pi}} \approx 0.4,$$

so we can finally derive

$$\pi_C \approx 0.4\cdot S(t_0)\exp\left(q(T-t_0)\right)\cdot\sigma\cdot\sqrt{T-t_0}. \qquad (5.3)$$

We can derive the same result for the put option or just observe from the put–call parity that for (FWD) ATM options, the premium for a put and call have to coincide:

$$\begin{aligned}\pi_C - \pi_P &= S\exp\left(q(T-t_0)\right) - K\exp\left(r(T-t_0)\right)\\ &= 0.\end{aligned}$$

Option Price approximation

- Exercise

$$\pi_C \approx 0.4 \cdot S(t_0) \exp\left(q(T-t_0)\right) \cdot \sigma \cdot \sqrt{T-t_0}.$$

- $S_0 = 100$; $r=q = 0$; $\sigma = 0.20$; $T\text{-}t_0 = 1$

1. Calculate (with the BS calculator) the price of the ATM call option
2. Calculate the approximation and compare
3. Use the approximation to estimate the price if the strike was $K = 110$
4. Calculate (with the BS calculator the price of the option with $K = 110$ and compare

Old Greeks revisited

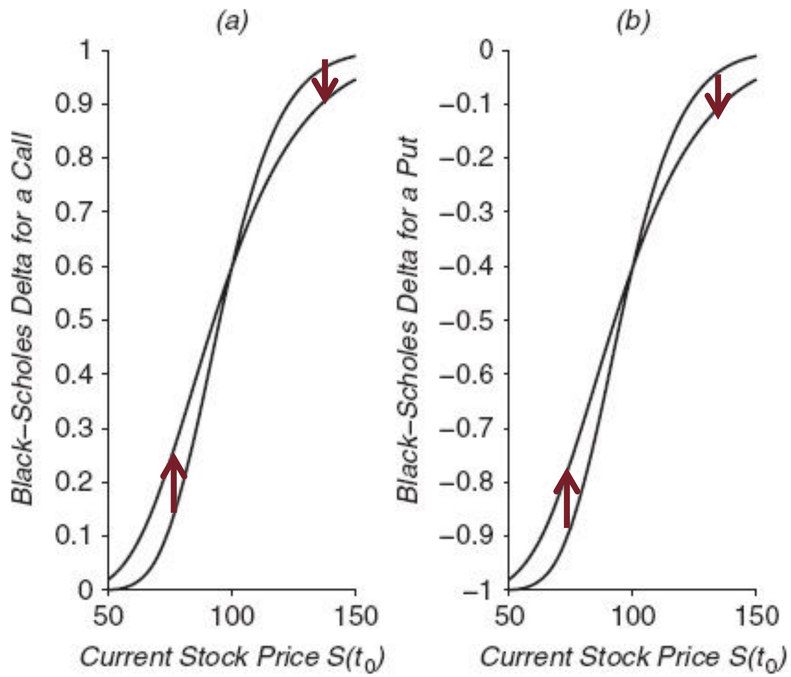


Figure 5.4 The delta Δ for a low and high value of the volatility parameter σ for call options (a) and put options (b).

Increased volatility = Flattening delta

Old Greeks revisited (2)

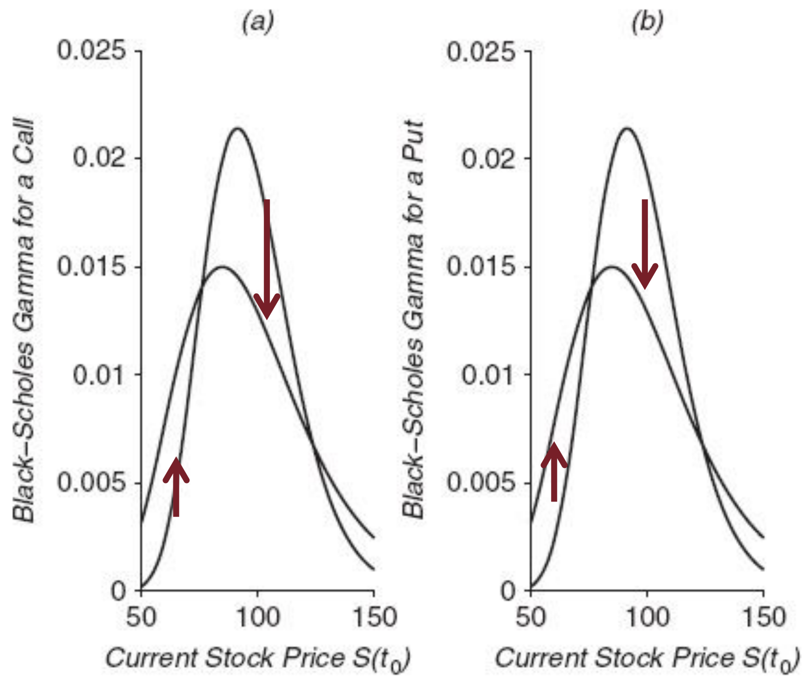


Figure 5.5 The gamma Γ for a low and high value of the volatility parameter σ for call options (a) and put options (b).

Increased volatility = Flattening gamma

Old Greeks revisited (3)

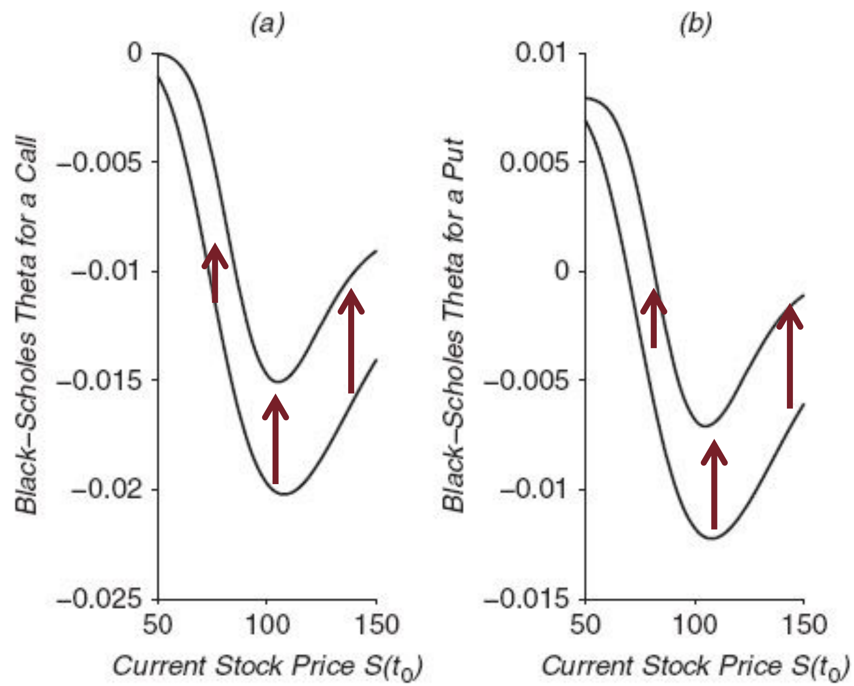


Figure 5.6 The theta θ for a low and high value of the volatility parameter σ for call options (a) and put options (b).

Increased volatility = increased theta / time value / extrinsic

7. The Greek Expansion

- Delta hedge revisited
- Taylor Expansion up to order 1
- Taylor Expansion up to higher order
- P&L in Greek
- Vega matrix
- Portfolio hedging
- Long and short the Greeks
- Trading the Greeks

Delta hedge revisited

- Hedging ration Delta revisited

$$\Delta(S(t_0), t_0) = \frac{\partial \pi}{\partial S}(S(t_0), t_0)$$

- Delta hedge revisited:

$$\pi(S) = \pi(S_0) + \frac{\partial \pi}{\partial S}(S_0) \cdot (S - S_0).$$

or

$$\frac{\pi(S_0 + \Delta S) - \pi(S_0)}{\Delta S} \simeq \frac{\partial \pi}{\partial S}(S_0).$$

or

$$\pi(S_0 + \Delta S) \simeq \pi(S_0) + \Delta(S_0) \cdot \Delta S.$$

Taylor Expansion up to order 1

- Taylor Expansion up to **order 1**:

$$\pi\left(S_0+\Delta S\right) \simeq \pi\left(S_0\right)+\frac{\partial \pi}{\partial S}\left(S_0\right)\left(S_0+\Delta S-S_0\right)$$
$$=\pi\left(S_0\right)+\frac{\partial \pi}{\partial S}\left(S_0\right) \cdot \Delta S.$$

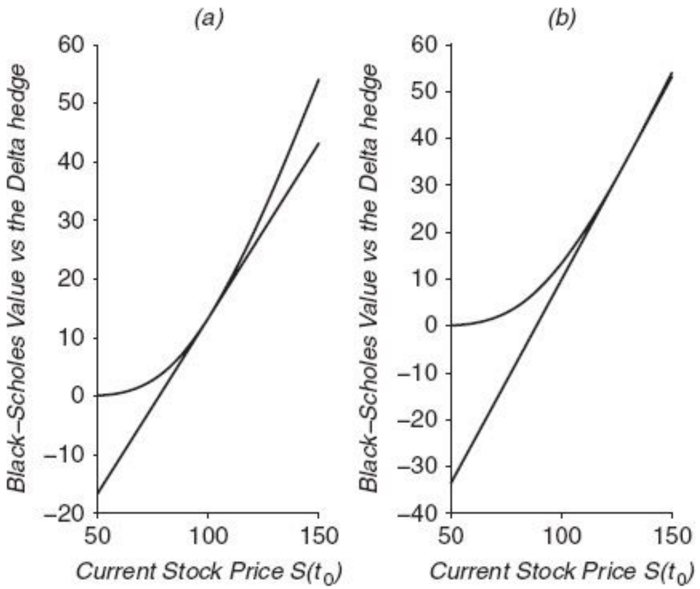


Figure 6.1 The delta hedge procedure allows you to lock in $0.5 \cdot \Gamma \cdot \Delta S^2$ when the stock price moves by ΔS. In the figure (a) the initial delta hedge is shown. After the stock moves up, the trader will rebalance his hedge to (b). The option will always outperform the hedge if gamma long or $\Gamma > 0$.

Taylor Expansion up to order 1 (2)

- Next term in space:

$$\pi\left(S_0+\Delta S\right) \simeq \pi\left(S_0\right)+\Delta\left(S_0\right) \cdot \Delta S+\frac{1}{2} \Gamma\left(S_0\right) \cdot(\Delta S)^2$$

- Space and time are linked in the BSM model:

$$(\Delta S)^2 \sim \left(S(t_0) \cdot \sigma \cdot \sqrt{\Delta t}\right)^2 = O(\Delta t)$$

- Taylor including the time dimension:

$$\begin{aligned}
\pi\left(S_0+\Delta S, t_0+\Delta t\right) \simeq\ & \pi\left(S_0, t_0\right)+\Delta\left(S_0, t_0\right) \cdot \Delta S+\frac{\partial \pi}{\partial t}\left(S_0, t_0\right) \cdot \Delta t \\
& +\frac{1}{2} \Gamma\left(S_0, t_0\right) \cdot(\Delta S)^2 \\
= & \pi\left(S_0, t_0\right)+\Delta\left(S_0, t_0\right) \cdot \Delta S+\theta\left(S_0, t_0\right) \cdot \Delta t \\
& +\frac{1}{2} \Gamma\left(S_0, t_0\right) \cdot(\Delta S)^2 .
\end{aligned}$$

- Taylor including the new market parameter σ:

$$\begin{aligned}
\pi\left(S_0+\Delta S, t_0+\Delta t, \sigma_0+\Delta \sigma\right) \simeq\ & \pi\left(S_0, t_0, \sigma_0\right)+\Delta\left(S_0, t_0, \sigma_0\right) \cdot \Delta S \\
& +\theta\left(S_0, t_0, \sigma_0\right) \cdot \Delta t+\frac{1}{2} \Gamma\left(S_0, t_0, \sigma_0\right) \cdot(\Delta S)^2 \\
& +\nu\left(S_0, t_0, \sigma_0\right) \cdot \Delta \sigma .
\end{aligned}$$

Taylor Expansion up to higher order

- How to order terms:

Delta Gamma Theta

$$\boxed{\mathcal{O}(\Delta S)} < \boxed{\mathcal{O}(\Delta S^2)} \sim \boxed{\mathcal{O}(\Delta t)}$$
$$< \mathcal{O}(\Delta S^3) \sim \mathcal{O}(\Delta S \Delta t)$$
$$< \mathcal{O}(\Delta S^4) \sim \mathcal{O}(\Delta t^2) \sim \mathcal{O}(\Delta S^2 \Delta t);$$

- Corresponding Greeks

First order Greeks		Second order Greeks		Third order Greeks	
Delta	$\Delta = \frac{\partial \pi}{\partial S}$	Gamma	$\Gamma = \frac{\partial^2 \pi}{\partial S^2}$	Colour	$\frac{\partial^3 \pi}{\partial S^2 \partial t}$
Theta	$\theta = \frac{\partial \pi}{\partial t} = -\frac{\partial \pi}{\partial T}$	Inertia	$\frac{\partial^2 \pi}{\partial t^2}$	Ultima	$\frac{\partial^3 \pi}{\partial \sigma^3}$
Vega	$\nu = \frac{\partial \pi}{\partial \sigma}$	Charm	$\frac{\partial^2 \pi}{\partial t \partial S} = \frac{\partial \Delta}{\partial t} = \frac{\partial \theta}{\partial S}$	Zomma	$\frac{\partial^3 \pi}{\partial S^2 \partial \sigma}$
Rho	$\rho = \frac{\partial \pi}{\partial r}$	Veta	$\frac{\partial^2 \pi}{\partial t \partial \sigma} = \frac{\partial \nu}{\partial t} = \frac{\partial \theta}{\partial \sigma}$		
Phi	$\phi = \frac{\partial \pi}{\partial q}$	Volga	$\frac{\partial^2 \pi}{\partial \sigma^2}$		
		Vanna	$\frac{\partial^2 \pi}{\partial S \partial \sigma}$		
		Vera	$\frac{\partial^2 \pi}{\partial \sigma \partial r}$		

Taylor Expansion up to higher order

- Taylor expansion including more Greeks:

$$
\begin{aligned}
\pi(\Delta S, \Delta t, \Delta\sigma) \simeq\ & \pi_0 + \Delta \cdot \Delta S + \frac{1}{2}\Gamma \cdot (\Delta S)^2 \\
& + \theta \cdot \Delta t + \nu \cdot \Delta\sigma \\
& + \frac{1}{2}\frac{\partial^2 \pi}{\partial t^2} \cdot (\Delta t)^2 + \frac{1}{2}\frac{\partial^2 \pi}{\partial \sigma^2} \cdot (\Delta\sigma)^2 + \frac{\partial^2 \pi}{\partial t \partial S} \cdot \Delta t \Delta S \\
& + \frac{\partial^2 \pi}{\partial t \partial \sigma} \cdot \Delta t \Delta\sigma + \frac{\partial^2 \pi}{\partial S \partial \sigma} \cdot \Delta S \Delta\sigma \\
=\ & \pi_0 + \Delta \cdot \Delta S + \frac{1}{2}\Gamma \cdot (\Delta S)^2 \\
& + \theta \cdot \Delta t + \nu \cdot \Delta\sigma \\
& + \rho \cdot dr + \phi \cdot dq \\
& + \frac{1}{2}\text{Inertia} \cdot (\Delta t)^2 + \frac{1}{2}\text{Volga} \cdot (\Delta\sigma)^2 + \text{Charm} \cdot \Delta t \Delta S \\
& + \text{Veta} \cdot \Delta t \Delta\sigma + \text{Vanna} \cdot \Delta S \Delta\sigma + \ldots
\end{aligned}
$$

P&L in Greek

- Taylor explains how option premium changes (approximately) when market parameters move
- Greek expansion does not only apply to European options, but also to portfolios of instruments
- Examples:
  - Greeks for a **stock** are simple

Δ	1
Γ	0
θ	0
ν	0

  - Greeks for a **forward** are linear

Δ	$\exp((r-q)(T-t_0))$
Γ	0
θ	$-(r-q)F(t0)$
ν	0

  - Greeks of the cash account *B(t)*

θ	$rB(t0)$
Δ,Γ,ν	0

P&L in Greek (2)

- Example: Portfolio: 100,000 call options (*K=105* and 1 year maturity)
  - Market Levels: $S_0 = 100, \sigma_0 = 20\%, q = 0, r = 2\%.$ and $\pi_0=6.70$
  - Portfolio value: 670,477.48 EUR

1 week later

  - $S_1=107$ and $\sigma_1=22\%$ and hence $\pi_0=11.27$
  - Portfolio value: 1,126,797.45 EUR *(increase of 456,319.97 EUR)*
- Greek explanation

biggest driver

	Greek value		Change	Effect on the option value	
Delta	$\Delta =$	48,247.18	$\Delta S = 7.00$	$\boxed{\Delta \cdot \Delta S =}$	$\boxed{337{,}730.26}$
Gamma	$\Gamma =$	1,992.79		$\frac{1}{2}\Gamma(\Delta S)^2 =$	48,823.25
Theta	$\theta_d =$	−1,319.56	$\Delta t = 7$	$\theta \cdot \Delta t =$	−9,236.97
Vega	$\nu =$	39,855.72	$\Delta\sigma = 2\%$	$\nu \cdot \Delta\sigma =$	79,711.43
Cash		−670,477.48		Interest	−257.22
				Total	456,770.75

P&L in Greek (3)

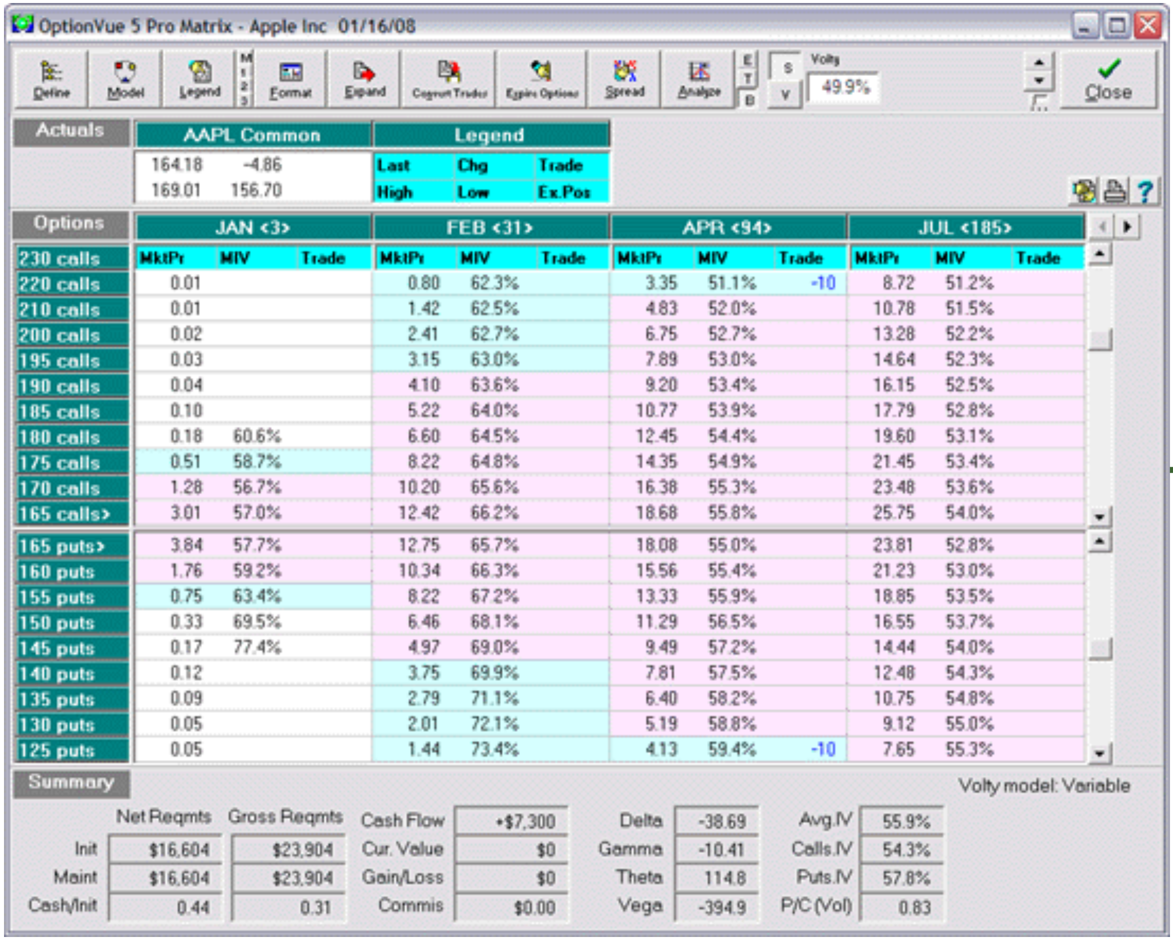
- Example (continued): delta hedged portfolio
  - Sell stocks
  - Collect cash
- Greek explanation

	Greek value		Change	Effect on the option value	
Delta	$\Delta =$	0.18	$\Delta S = 7.00$	$\Delta \cdot \Delta S =$	1.26
Gamma	$\Gamma =$	$1{,}992.79$		$\frac{1}{2}\Gamma\,(\Delta S)^2 =$	$48{,}823.25$
Theta	$\theta_d =$	$-1{,}319.56$	$\Delta t = 7$	$\theta \cdot \Delta t =$	$-9{,}236.97$
Vega	$\nu =$	$39{,}855.72$	$\Delta\sigma = 2\%$	$\nu \cdot \Delta\sigma =$	$79{,}711.43$
Cash		$+4{,}154{,}222.52$		Interest	$+1{,}593.71$
				Total	$120{,}892.68$

biggest driver

Vega matrix

- Each strike/maturity has its own implied volatility ***(volatility surface)***



Options	JAN <3>			FEB <31>			APR <94>			JUL <185>		
230 calls	MktPr	MIV	Trade	MktPr	MIV	Trade	MktPr	MIV	Trade	MktPr	MIV	Trade
220 calls	0.01			0.80	62.3%		3.35	51.1%	-10	8.72	51.2%	
210 calls	0.01			1.42	62.5%		4.83	52.0%		10.78	51.5%	
200 calls	0.02			2.41	62.7%		6.75	52.7%		13.28	52.2%	
195 calls	0.03			3.15	63.0%		7.89	53.0%		14.64	52.3%	
190 calls	0.04			4.10	63.6%		9.20	53.4%		16.15	52.5%	
185 calls	0.10			5.22	64.0%		10.77	53.9%		17.79	52.8%	
180 calls	0.18	60.6%		6.60	64.5%		12.45	54.4%		19.60	53.1%	
175 calls	0.51	58.7%		8.22	64.8%		14.35	54.9%		21.45	53.4%	
170 calls	1.28	56.7%		10.20	65.6%		16.38	55.3%		23.48	53.6%	
165 calls>	3.01	57.0%		12.42	66.2%		18.68	55.8%		25.75	54.0%	
165 puts>	3.84	57.7%		12.75	65.7%		18.08	55.0%		23.81	52.8%	
160 puts	1.76	59.2%		10.34	66.3%		15.56	55.4%		21.23	53.0%	
155 puts	0.75	63.4%		8.22	67.2%		13.33	55.9%		18.85	53.5%	
150 puts	0.33	69.5%		6.46	68.1%		11.29	56.5%		16.55	53.7%	
145 puts	0.17	77.4%		4.97	69.0%		9.49	57.2%		14.44	54.0%	
140 puts	0.12			3.75	69.9%		7.81	57.5%		12.48	54.3%	
135 puts	0.09			2.79	71.1%		6.40	58.2%		10.75	54.8%	
130 puts	0.05			2.01	72.1%		5.19	58.8%		9.12	55.0%	
125 puts	0.05			1.44	73.4%		4.13	59.4%	-10	7.65	55.3%	

Summary — Volty model: Variable

	Net Reqmts	Gross Reqmts						
Init	$16,604	$23,904	Cash Flow	+$7,300	Delta	-38.69	Avg.IV	55.9%
Maint	$16,604	$23,904	Cur. Value	$0	Gamma	-10.41	Calls.IV	54.3%
Cash/Init	0.44	0.31	Gain/Loss	$0	Theta	114.8	Puts.IV	57.8%
			Commis	$0.00	Vega	-394.9	P/C (Vol)	0.83

$\boxed{\sigma_{imp}(K, T)}$

- What is vega? Which vega?

Vega matrix (2)

- Implied volatility surface

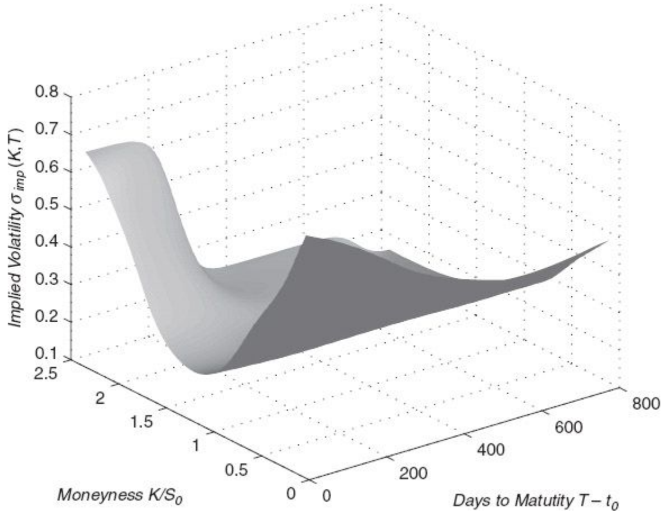
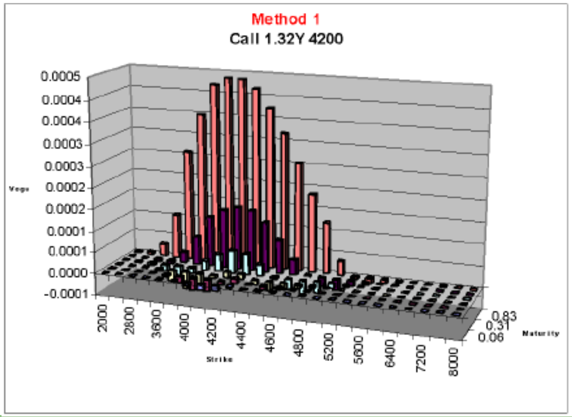


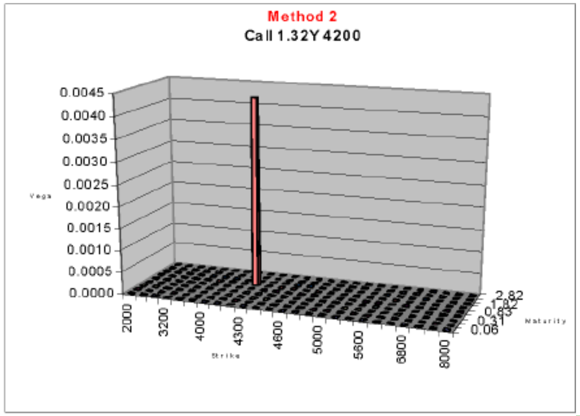
Figure 8.2 The volatility surface $\sigma_{imp}(K, T)$ for the S&P 500 on 19 October 2009.

- Vega matrix

$$\begin{array}{c|ccc} & \tau_1 & \cdots & \tau_N \\ \hline K_1 & \nu_{11} & \cdots & \nu_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ K_M & \nu_{M1} & \cdots & \nu_{MN}. \end{array}$$

- Interdependent sensitivities





Portfolio hedging

- Exotic options: less straight forward
  - Is the option correctly priced?
  - Closed formula available?
  - Numerical Greek Calculation?
  - Exotic models?
- Examples with closed formula under the BSM model
  - Option constructions *(call spreads, butterflies,…)*
  - Lookback options
  - Barrier options
  - Digital options
- Greek representation is still the trader’s approach, even though the Greeks are rooted in BSM model

Portfolio hedging (2)

- Example
  - Market Parameters:

S_0	$=1500.00$
r	$=2.5\%$
q	$=0\%$
σ	$=25\%.$

  - Greek representation

Δ_V	$=300{,}000$
Γ_V	$=2{,}500$
θ_V	$=-500{,}000$
ν_V	$=750{,}000.$

Portfolio hedging (3)

- Can we hedge this portfolio
  - Delta hedge ⟶ *Transact in the underlying stock/index/instrument*
  - Other Greeks? ⟶ *Options are needed to hedge portfolio Greeks*

	OTM put	ATM call	OTM call
K	1200	1500	1800
τ	1	1.5	0.5
π	27.85	207.98	25.95
Δ	−0.13	0.61	0.19
Γ	0.0006	0.0008	0.0010
θ	−0.09	−0.21	−0.22
ν	3.20	7.05	2.90.

- Take this portfolio:

	OTM puts	ATM calls	OTM calls	Stocks
Volume	−542,270	−1,157,791	3,673,006	227,891
Position	Short	Short	Long	Long

Portfolio hedging (4)

- General framework
  - Given portfolio Greeks

$$\begin{array}{c}\hline\hline \Delta_V \\ \Gamma_V \\ \theta_V \\ \nu_V \\ \hline\hline\end{array}$$

  - How do we find the hedging ratios: $(\omega_1, \omega_2, \omega_3, \omega_4)$
  - Mathematically: linear problem

$$\begin{cases} \omega_1 \Delta_1 + \omega_2 \Delta_2 + \omega_3 \Delta_3 + \omega_4 = \Delta_V \\ \omega_1 \Gamma_1 + \omega_2 \Gamma_2 + \omega_3 \Gamma_3 = \Gamma_V \\ \omega_1 \theta_1 + \omega_2 \theta_2 + \omega_3 \theta_3 = \theta_V \\ \omega_1 \nu_1 + \omega_2 \nu_2 + \omega_3 \nu_3 = \nu_V. \end{cases}$$

  - Only has a solution if

$$\begin{vmatrix} \Delta_1 & \Delta_2 & \Delta_3 & 1 \\ \Gamma_1 & \Gamma_2 & \Gamma_3 & 0 \\ \theta_1 & \theta_2 & \theta_3 & 0 \\ \nu_1 & \nu_2 & \nu_3 & 0 \end{vmatrix} = \begin{vmatrix} \Gamma_1 & \Gamma_2 & \Gamma_3 \\ \theta_1 & \theta_2 & \theta_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} \neq 0.$$

Portfolio hedging (5)

- Comments
  - In a flat-vol model, not every Greek portfolio can be hedged as the option surface is equivalent to a single option (market is complete). This results in a badly conditioned matrix and unstable solutions (typically with very large weights in the option positions)

Portfolio hedging (5)

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  - Short-dated options have more gamma than vega

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  - ATM options have more vega, gamma and theta value

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  - The resulting delta hedge is not the same as the original delta hedge as each option needs to be delta hedged as well
  - Long-dated options have more vega than gamma
  - Short-dated options have more gamma than vega
  - ATM options have more vega, gamma and theta value
  - Traders don’t necessarily hedge, they can trade as one can construct any portfolio from a Greek perspective if multiple options are available

Long and Short the Greeks

- Terminology revisited: Positions can be
  - Long: positive position in the underlying quantity
  - Short: negative position in the underlying quantity
  - Neutral/Flat: no position in the underlying quantity
- Delta Long
  - Equivalent to a long position in the underlying
  - Portfolio $\Delta > 0$
  - Examples
    - Long the underlying
    - Long a call option; delta equivalent
    - Short a put option; delta equivalent
  - Increase in the underlying are beneficial ; decreases are harmful
- Delta short
  - The opposite of Delta long
  - Portfolio $\Delta < 0$
- Delta neutral/flat
  - Portfolio $\Delta = 0$ or $\Delta \sim 0$

Long and Short the Greeks (2)

- Gamma Long
  - Position with $\Gamma > 0$
  - Examples
    - Long a call option
    - Long a put option
  - Every move introduces Gamma P&L: big moves are quadratically better
- Gamma short
  - Opposite of gamma long: $\Gamma < 0$
  - Small moves hurt less than big moves
- Gamma flat/neutral
  - Portfolio Γ = 0 (or $\Gamma \sim 0$)

Long and Short the Greeks (3)

- Vega Long
  - Position with $\nu > 0$
  - Examples
    - Long a call option
    - Long a put option
  - Increases in volatility are beneficial for the portfolio
- Vega short
  - Portfolio $\nu < 0$
- Vega flat/neutral
  - Portfolio vega $\nu \sim 0$
  - Insensitive to implied Volatility changes

Long and Short the Greeks (4)

- Theta short
  - Position with $\theta < 0$
  - Examples
    - Long a call option
    - Long a put option
    - Short cash
  - Over time, the portfolio will automatically lose value
- Theta Long
  - Position with $\theta > 0$
  - Examples
    - Short a call option
    - Short a put option
    - Long cash
  - Over time, the portfolio will automatically gain value
- Theta flat/neutral
  - Position with $\theta \sim 0$

Trading the Greeks

- Traders can increase the Greek value in the portfolio

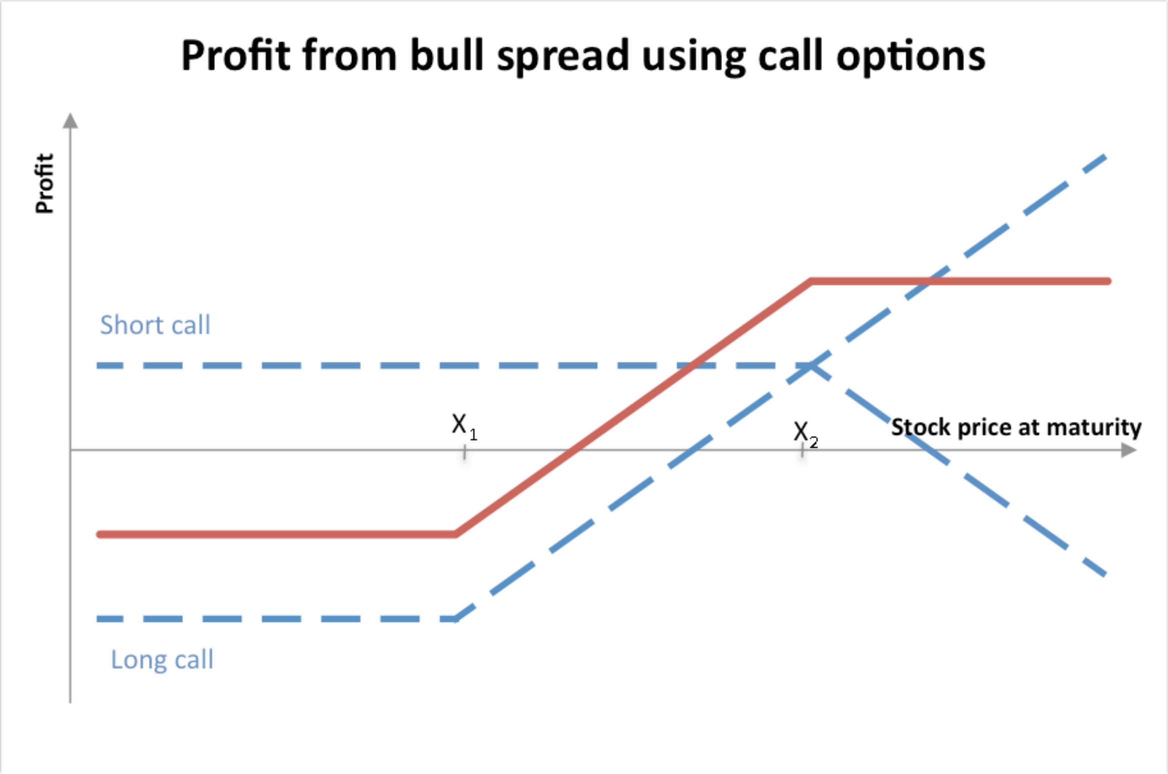
Greek	Examples
Buy Delta	Buy stock, sell a put, buy a call
Buy Gamma	Buy a call, buy a put
Buy Theta	Sell a call, sell a put
Buy Vega	Buy a call, buy a put

- Traders can decrease the Greek value in the portfolio

Greek	Examples
Sell Delta	Sell stock, buy a put, sell a call
Sell Gamma	Sell a call, sell a put
Sell Theta	Buy a call, buy a put
Sell Vega	Sell a call, sell a put

Portfolio hedging revisited

- Example:
  - Long ATM call option
  - Short OTM call option



Portfolio hedging revisited (2)

- Example:
  - Long ATM call option
  - Short OTM call option

Portfolio analysis/scenario corresponds to higher order Greeks

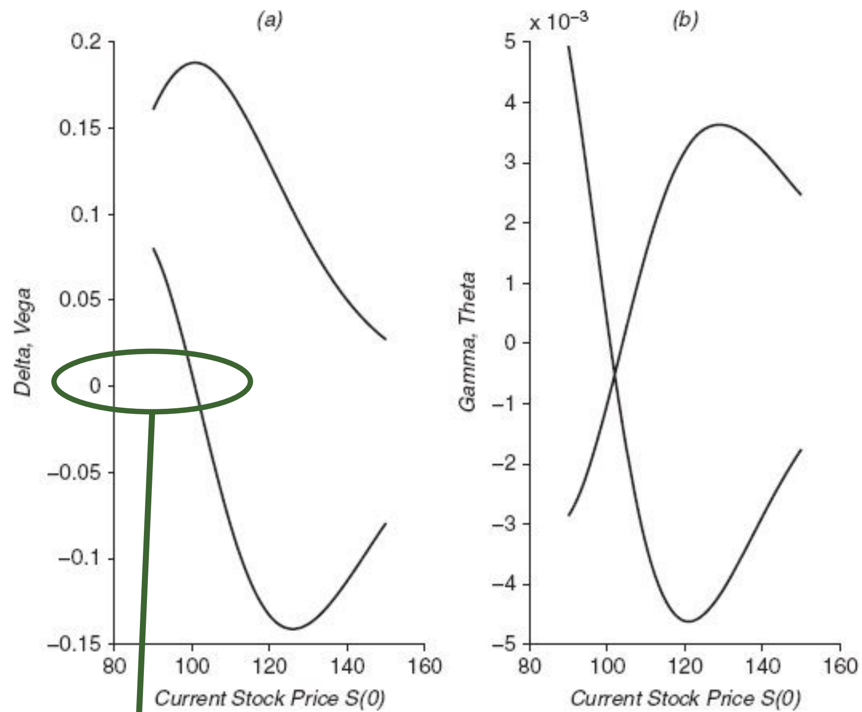


Figure 6.2 (*a*) The top graph is the delta Δ_V and the bottom one depicts the vega ν_V. (*b*) The convex shaped curve is the gamma Γ_V and the concave shape is the theta θ_V.

$$\frac{\partial \nu_V}{\partial S} = \frac{\partial^2 \pi_V}{\partial \sigma \partial S}$$

ν~0 but vega won't be neutral/flat if the stock moves $\longrightarrow$ Vanna < 0 measures this

8. Volatility Term structure

- Deterministic instantaneous volatility
- Hedging under term structure
- Market Term Structure

Deterministic instantaneous volatility

- BSM model has constant (instantaneous) variance. The total volatility over a certain time interval is

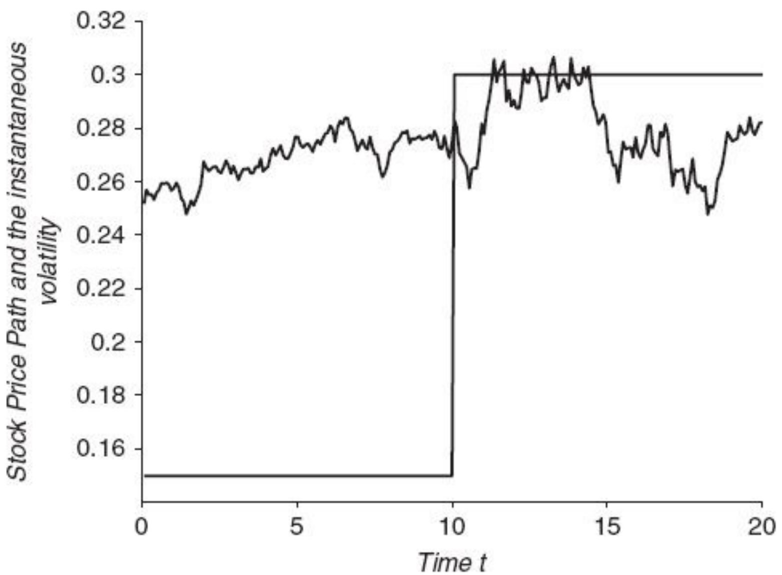
$$\bar{\sigma} = \sqrt{\frac{1}{T - t_0} \int_{t_0}^{T} \sigma^2(u)\, du},$$

- Example:
  - Two-volatility regimes: low and high
  - Deterministic cross-over

$$\sigma(t) = \begin{cases} \sigma_1 & \text{if} \quad t \leq T_1 \\ \sigma_2 & \text{if} \quad T_1 < t \leq T_2. \end{cases}$$

  - Average volatility:

$$\bar{\sigma} = \sqrt{\frac{1}{\tau(t_0, T_2)} \left(\sigma_1^2 \cdot \tau(t_0, T_1) + \sigma_2^2 \cdot \tau(T_1 + 1, T_2)\right)}.$$



Deterministic instantaneous volatility (2)

- The “implied” volatility will change over time as the low-volatility regime within the remaining time period gets smaller.

Day	0	1	2	3	4	5
$\sigma(t)$	15%	15%	15%	15%	15%	15%
$\bar{\sigma}$	23.38%	23.72%	24.09%	24.49%	24.94%	25.43%
Day		6	7	8	9	10
$\sigma(t)$		15%	15%	15%	15%	15%
$\bar{\sigma}$		25.98%	26.59%	27.28%	28.06%	28.96%
Day		11	12	13	14	15
$\sigma(t)$		30%	30%	30%	30%	30%
$\bar{\sigma}$		30%	30%	30%	30%	30%
Day		16	17	18	19	20
$\sigma(t)$		30%	30%	30%	30%	30%
$\bar{\sigma}$		30%	30%	30%	30%	30%

- Exercise: Recalculate this table and plot both volatility functions over the 20 days.

Deterministic instantaneous volatility (2)

- Market Parameters for the *put* option
  - S(t0) = 100 EUR
  - K = 99 EUR
  - r =q=0%
  - Maturity in 20 days

Deterministic instantaneous volatility (3)

- Hedging Paradox: which σ do we use in the delta?

Argument 1: For hedging, one has to use the same volatility as is used to calculate the option premium. After all, if it's good enough to find the price, it surely must capture the essentials of the hedging as well, since the Black–Scholes price formula is always the cost of hedging.

Argument 2: The above might be true, but I already know that the realised volatility is going to be a lot lower than the 23.38% I am using to price the option. So if I know this is true, then why would I put on a hedge using the wrong volatility or in other words, how can I trust my delta hedge? I would prefer hedging against movements of the lower volatility as I know these are real (expected) movements. I will adjust my volatility to 30 per cent after the initial 10 days are over.

- Realized volatility = 15%
- Implied volatility = 23.38%
- Gamma profit less than pricing volatility
- Theta loss will be bigger than gamma profit
- Deterministic loss?

Deterministic instantaneous volatility (4)

- Hedging Paradox: which σ do we use in the delta?
  - The 1-standard deviation move can be used for analysis:

$$S(t_0+1) = S(t_0) \cdot \exp\left(\sigma_1\sqrt{1/365}\right)$$

  - P&L explained by the Greeks:

$\pi(t_\circ) = 1.76.$

When moving to the next day $t_\circ + 1$, there are two effects that take place on the valuation of the same option, now with an expiry in 19 days. First, there is the usual loss of theta. We know how much this is by plugging all the parameters into (3.6) : $\theta = -0.05$. The value for gamma is $\Gamma = 0.07$, so this means that the gamma profit (remember it is quadratic?) is $1/2 \cdot \Gamma \cdot \Delta S^2$ where $\Delta S = S(t_\circ + 1) - S(t_\circ) = 0.79$ in our example. This gives a cash gamma (3.5) profit of $\Gamma_{P\&L} = 0.02$, which is clearly not enough to compensate the theta loss of 5 cents.

Deterministic instantaneous volatility (5)

- Vega solves the paradox when we move to the next day:
  - Theta loss
  - (Too) small gamma gain
  - Implied volatility will increase: vega kicks in

When we have moved to the second day, the pricing volatility is no longer $\bar{\sigma}(t_0) = 23.38\%$, as we can cross off one day from the low-volatility calendar. From the previous table, we see that the new pricing volatility is in fact higher and given by $\bar{\sigma}(t_0+1) = 23.72\%$. So the same option is priced with a higher volatility on the next day. Wait, that's great, so the option becomes relatively more expensive by itself. And we know how much more expensive as we know the vega concept. For this particular option, on day t_0, the vega is given by $\nu = 0.09$. So the vega effect of $\nu \cdot (23.72 - 23.38) = 0.03$.

- Distorted Gamma-Theta balance gets restored by (deterministic) vega correction

Deterministic instantaneous volatility (6)

We can actually formalise the above by going back to the Taylor [7] expansion (5.1) :

$$\pi\left(S_0+\Delta S, t_0+\Delta t, \sigma_0+\Delta\sigma\right)=\pi_0+\Delta\cdot\Delta S+\frac{1}{2}\Gamma\left(\Delta S\right)^2+\theta\cdot\Delta t+\nu\cdot\Delta\nu.$$

In the case of our time-dependent volatility model, we can say that the average volatility function becomes time-dependent as well (as its value depends on the day we are using it), $\bar{\sigma}=\bar{\sigma}(t)$, and we know $\Delta\sigma$ over one day is related to the change in time. In fact, by using (7.1) and some simple algebraic approximations such as

$$\frac{b}{a+\varepsilon}=\frac{b}{a}-\frac{b}{a^2}\varepsilon \text{ for } \varepsilon \text{ small} \tag{7.4}$$

$$\sqrt{a+b\delta}=\sqrt{a}+\frac{1}{2}\frac{b}{\sqrt{a}}\delta \text{ for } \delta \text{ small} \tag{7.5}$$

one can find an approximation for the change in volatility overnight:

$$\begin{aligned}\Delta\sigma &= \sqrt{\frac{1}{T-(t_0+\Delta t)}\int_{t_0+\Delta t}^{T}\sigma^2(t)\,dt}-\sqrt{\frac{1}{T-t_0}\int_{t_0}^{T}\sigma^2(t)\,dt}\\ &=\frac{1}{2}\frac{\bar{\sigma}^2(t_0)-\sigma^2(t_0)}{\bar{\sigma}(t_0)}\frac{\Delta t}{T-t_0}\\ &=\hat{\theta}\cdot\Delta t\end{aligned} \tag{7.6}$$

with

$$\hat{\theta}=\frac{1}{2}\frac{\bar{\sigma}^2(t_0)-\sigma^2(t_0)}{\bar{\sigma}(t_0)(T-t_0)}.$$

So this means that we can in fact rearrange the terms in the Taylor expansion above and include the vega term in the theta term, and get an effective theta which is a combination of the regular Black–Scholes theta and the vega term addition:

$$\theta_{\text{eff}}=\theta+\nu\cdot\hat{\theta}.$$

Hedging under term structure

- What are we really hedging?
  - Stock moves, or
  - Option price?
- Make the example more simple to eliminate some complexities: $\boxed{\sigma_1 = 0\%}$
  - Zero volatility of the stock: no need to hedge
  - Option price will change because
    - the theta loss, and
    -

Day	$\bar{\sigma}$	Put option	$\Delta(\bar{\sigma})$	θ	ν	θ_{eff}
0	20.70%	1.51	−0.41	−0.05	0.09	0.00
1	21.21%	1.51	−0.41	−0.05	0.09	0.00
2	21.76%	1.51	−0.41	−0.05	0.09	0.00
3	22.36%	1.51	−0.41	−0.05	0.09	0.00
4	23.01%	1.51	−0.41	−0.06	0.08	0.00
5	23.72%	1.51	−0.41	−0.06	0.08	0.00
6	24.49%	1.51	−0.41	−0.07	0.08	0.00
7	25.35%	1.51	−0.41	−0.07	0.08	0.00
8	26.31%	1.51	−0.41	−0.08	0.07	0.00
9	27.39%	1.51	−0.41	−0.08	0.07	0.00
10	28.60%	1.51	−0.41	−0.09	0.07	0.00

Perfect balance,
no option premium change

Hedging under term structure (2)

- What if there is an interest rate *r > 0%*
- Same principles hold, but the bank account theta needs to be included and the option price will change over time *(no constant price while volatility is zero because of the time-value of money)*
- What if there is a non-zero volatility?
- The effective theta still does the trick. Going back to the example:

Day	$\sigma(t_0)$	$\bar{\sigma}(t_0)$	Put option	$\Delta(\sigma)$	$\Delta(\bar{\sigma})$	Γ	ν	θ	θ_{eff}
0	15%	23.38%	1.76	−0.38	−0.42	0.07	0.09	−0.05	−0.02
1	15%	23.72%	1.43	−0.30	−0.36	0.07	0.09	−0.05	−0.02

Hedging under term structure (3)

- What would have happened if we hedge with instantaneous volatility?
  - Our delta position would have been adjusted: more stocks hedged
  - Our actual gamma profit would be larger
  - Our overall P&L would be positive, not zero
- Is this a way to make sure money?
  - In this example: yes
  - We *know* the implied volatility will increase, so if we only hedge against stock moves, we collect free vega rent. This is similar to accrual of the bank account.
  - This is not a hedge however.
  - When $\sigma(t)$ is stochastic/random we are at risk as we don't know what direction the vega correction will take us

Market Term Structure

- Can we derive the instantaneous volatility function $\sigma(t)$ from the market information?
- Yes, special case of the Dupire Local Volatility formula

$$\sigma(t) = \sqrt{\sigma_{imp}^2(t_0, t) + 2(t - t_0)\,\sigma_{imp}(t_0, t)\,\frac{\partial \sigma_{imp}(t_0, t)}{\partial t}}.$$

- Typical structure $T \longmapsto \sigma(K, T)$
  - Increasing term structure
  - Disconnect on prompt

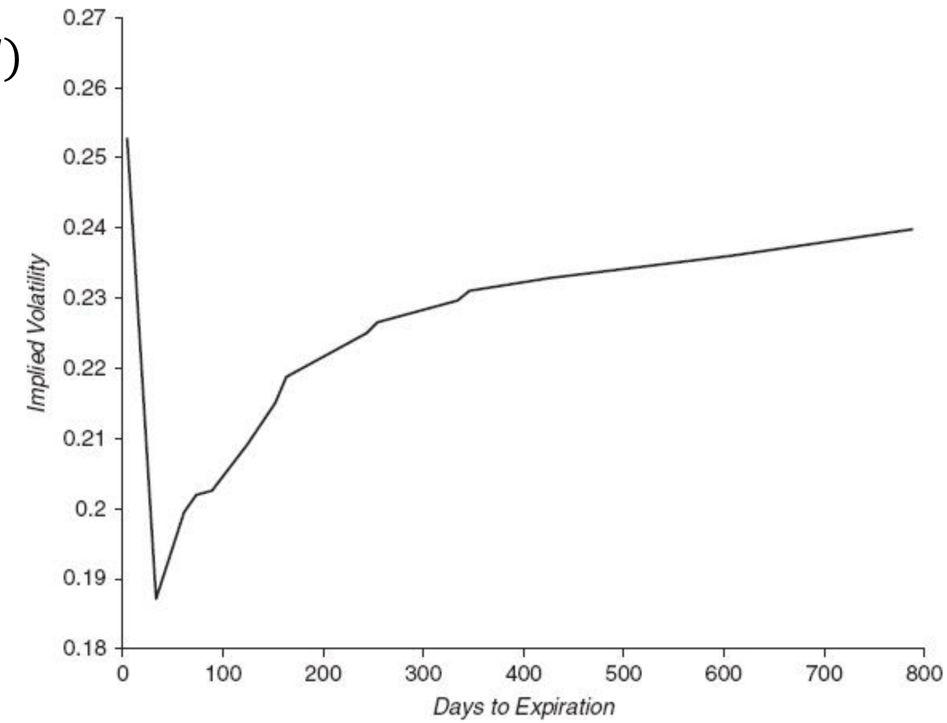


Figure 7.2 The ATM volatility term structure of the S&P 500 on 19 October 2009.

Market Term Structure (2)

- Can we have any structure for the implied volatility $T \longmapsto \sigma(K,T)$?
- <u>No</u>, the term under the square root cannot be negative
- Example

Example 25 *Assume we have two maturities given $T_{1,2} = 1{,}2$ and the corresponding volatilities are $\sigma_1 = 20\%$ and $\sigma_2 = 10\%$. As in Section 7.2.1, we turn to the variance. The variance over the first year is given by $\sigma_1^2(T_1 - t_0) = 0.04$. The variance over the two years combined is given by $\sigma_2^2(T_2 - t_0) = 0.02$. This means that the variance in the second year has to be negative, which gives a contradiction.*

- What does a negative variance mean from a trading perspective? **Arbitrage**

Example 26 *Take an initial stock price of $S_0 = 250$ and for simplicity assume $r = q = 0\%$. Let's consider the following strategy. Sell a one-year ATM call option with volatility $\sigma_1 = 10\%$, priced 19.91 euro and buy a two year ATM call option with volatility $\sigma_2 = 20\%$, priced 14.08 euro. By using the prices, one can immediately see we are collecting money to put this strategy in place. However, at the expiry of the first option, we know that the second option will have a value above the intrinsic level, hence no matter what the payout is for the short position we can then sell the second option (which still has a lifetime of one year) and collect extra premium above the intrinsic value. This leads to an arbitrage.*

Market Term Structure (3)

- Necessary condition (later part of a sufficient condition) for avoidance of arbitrage

Now that we clarified the arbitrage opportunity, it has become clear that the relationship that we are using to rule out the arbitrage is

$$T_1 \leq T_2 \Longrightarrow \pi_C(t_0, K, T_1) \leq \pi_C(t_0, K, T_2) \tag{7.8}$$

or that calendar spreads are positive. This is also known as the convex order.

One can show that this is one of the sufficient conditions to be arbitrage free as well [28, 41, 110]. We will state the other condition in the next chapter.

- One can also state this in terms of put options.
- Different maturities have different volatilities. This makes sense as they will be delta hedged over different time intervals (overlapping).

9. Skew and smile

- Market skew
- How do we start smiling?
- How does a smile turn into a smirk?
- Implied distribution
- DIY Skew
- Measuring Skew
- Can we smile any way we want?
- Skew across maturities
- Hedging under skew

Market skew

- Implied volatility skew:

$$T \longmapsto \sigma(K,T)$$

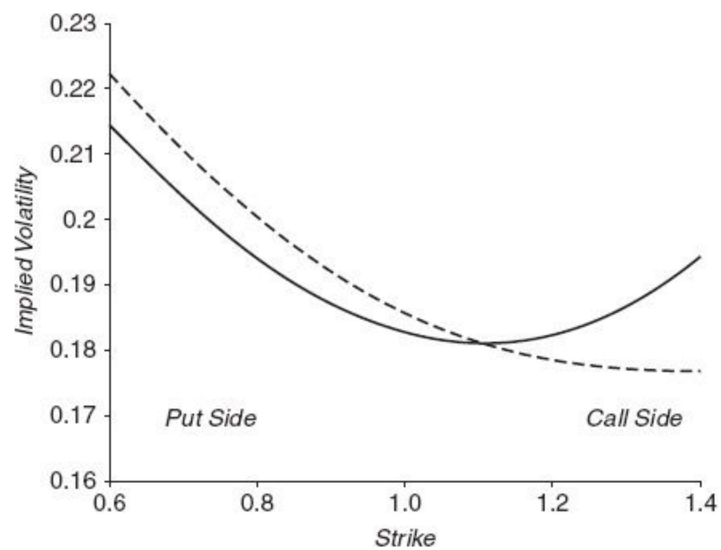


Figure 8.1 A typical smile (skewed) shape in the full line and a skew shape in the dashed line for the implied volatility. The put-side corresponds to the low strikes and the call-side to the high strikes.

- Other shapes

Smile	Frown	+Skew	Flat Vol	- Skew	Arb
					

How do we start smiling?

- Why would the cost of hedging depend on the Strike?
  - In BSM *all* options can be replicated while following the dynamic delta hedging strategy.
  - Hedging period is the same for all options
- Example:
  - $S(t_0) = 1500$; $r = 2.5\%$; $q = 1.2\%$; $\sigma = 18\%$
  - Options to be priced:
    - ATM call
    - 10%-OTM call
    - 10%-OTM put
    - 20%-OTM put

	20 put	10 put	ATM call	10 call
Strike	1200	1350	1500	1650
Model price	10.49	38.49	115.45	58.62
Margin-adjusted price	12.00	40.00	116.50	60.00
Margin implied volatility	18.66%	18.33%	18.18%	18.25%

- Vega spread naturally leads to smile

How does a smile turn into a smirk?

- Risk premium is typically charged for the following scenario
  - Market crashes
  - OTM put options become ATM
  - Volatility increased
  - Hedging cost for the ATM puts is more expensive

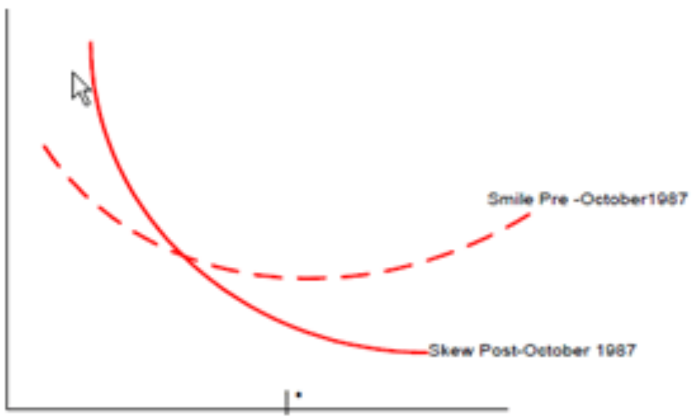
DEFINITION OF 'BLACK MONDAY'

October 19, 1987, when the Dow Jones Industrial Average (DJIA) lost almost 22% in a single day. That event marked the beginning of a global stock market decline, making Black Monday one of the most notorious days in recent financial history. By the end of the month, most of the major exchanges had dropped more than 20%.

INVESTOPEDIA EXPLAINS 'BLACK MONDAY'

Interestingly enough, the cause of the massive drop cannot be attributed to any single news event because no major news event was released on the weekend preceding the crash. While there are many theories that attempt to explain why the crash happened, most agree that mass panic caused the crash to escalate.

Since Black Monday, a number of protective mechanisms have been built into the market to prevent panic selling, such as trading curbs and circuit breakers.



Implied distribution

RISK - *Volatility Has Become a Shadow Currency*

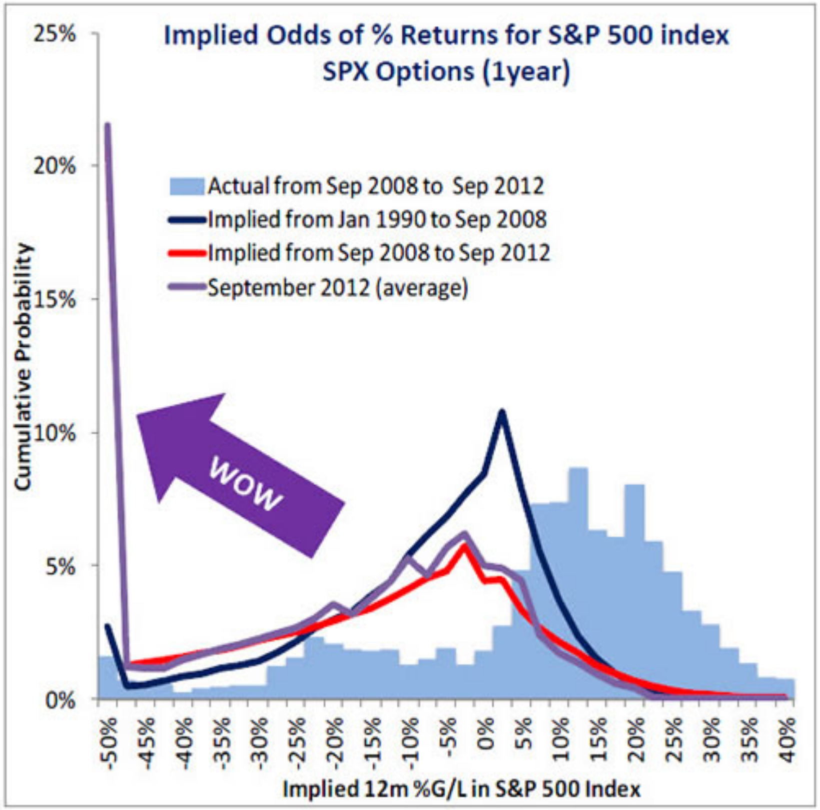
A 21% Chance Of A 50% Plunge In The S&P 500? 11-05-12 *Artemis Capital Management* vai ZH

Investors' perceptions of risks, both normal (volatility) and tail (event), have intriguingly run to both extremes at the same time. 'Normal' volatility has been so suppressed by Central-Bank action as to become an almost useless indicator (or at best contemporaneous) -

or as Artemis Capital notes

> *"volatility has become a shadow currency" with the USD (safe-haven) becoming considerably more correlated with volatility. Extreme volatility concerns are where the 'unintended' consequence has appeared.*

In a somewhat stunning market realization, **options markets currently suggest a 1 in 4.7 chance of a greater-than-50% drop in the S&P over the next year**. That is more likely than the lifetime risk of a heart attack. The question then is, are tail-risks over-priced? Or **are investors willing to overpay for that kind of 'deflation' insurance since we now know that the impossible is possible**!



Source: Gordon T Long, Market Research & Analytics

SIMULATION

SET YOUR OWN SKEW

Market Maker - Skew

Market settings: $S(t_0) = 1500$; $r = 2.5\%$; $q = 1.2\%$; $\sigma = 18\%$

1. Price these options ”fair”

	20 put	10 put	ATM call	10 call
Strike	1200	1350	1500	1650

Market Maker - Skew

Market settings: $S(t_0) = 1500$; $r = 2.5\%$; $q = 1.2\%$; $\sigma = 18\%$

1. Price these options "fair"

	20 put	10 put	ATM call	10 call
Strike	1200	1350	1500	1650
Model price	10.49	38.49	115.45	58.62

2. Apply some rounding

Margin-adjusted price	12.00	40.00	116.50	60.00

Market Maker - Skew

Market settings: $S(t_0) = 1500$; $r = 2.5\%$; $q = 1.2\%$; $\sigma = 18\%$

1. Price these options ”fair”

	20 put	10 put	ATM call	10 call
Strike	1200	1350	1500	1650
Model price	10.49	38.49	115.45	58.62

2. Apply some rounding

Margin-adjusted price	12.00	40.00	116.50	60.00

3. Calculate the BS implied volatility

Margin-adjusted price	12.00	40.00	116.50	60.00
Margin implied volatility	18.66%	18.33%	18.18%	18.25%

Market Maker - Skew

4. Exercise: Take the 10% OTM **put**: ***find the risk premium***

- Fair volatility σ = 18%
- Fair premium π_P = XXXX
- Γ = YYYY
- θ = ZZZZ

Market Maker - Skew

4. Exercise: Take the 10% OTM **put** : ***find the risk premium***

- Fair volatility σ = 18%
- Fair premium π_P = 38.49
- Γ = 0.0011
- θ = -0.0956

- Set a crash level: –10%
- Find the cost of this crash

$$\Gamma_{P\&L} = \frac{1}{2}\Gamma(\Delta S)^2$$

- Theta-balanced move would be ΔS = XXXX, as the solution from:

$$0.0956 = \frac{1}{2}\Gamma(\Delta S)^2$$

- Crash move of 10% (or 150) would cost

$$\Gamma_{P\&L} = \frac{1}{2}\Gamma(\Delta S)^2 = YYYY$$

Market Maker - Skew

4. Exercise: Take the 10% OTM **put** : ***find the risk premium***

- Fair volatility σ = 18%
- Fair premium π_P = 38.49
- Γ = 0.0011
- θ = -0.0956

- Set a crash level: –10%
- Find the cost of this crash

$$\Gamma_{P\&L} = \frac{1}{2}\Gamma(\Delta S)^2$$

- Theta-balanced move would be ΔS = 13.16, as the solution from:

$$0.0956 = \frac{1}{2}\Gamma(\Delta S)^2$$

- Crash move of 10% (or 150) would cost

$$\Gamma_{P\&L} = \frac{1}{2}\Gamma(\Delta S)^2 = 12.375$$

- Change premium to

π_P = 38.49 + 12.38 = 50.77

Market Maker - Skew

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- Fair volatility σ = 18%
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- Change premium to

π_P = 38.49 + 12.38 = 50.77

- Implied volatility has become σ = XXXX%

Market Maker - Skew

4. Exercise: Take the 10% OTM **put** : ***find the risk premium***

- Fair volatility σ = 18%
- Fair premium π_P = 38.49
- Γ = 0.0011
- θ = -0.0956

- Set a crash level: –10%
- Find the cost of this crash

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- Theta-balanced move would be ΔS = 13.16, as the solution from:

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- Crash move of 10% (or 150) would cost

$$\Gamma_{P\&L} = \frac{1}{2}\Gamma(\Delta S)^2 = 12.375$$

- Change premium to

π_P = 38.49 + 12.38 = 50.77

- Implied volatility has become σ = 20.67%

DIY Skew (2)

- Example: 10% OTM **put** : ***More risk premia***
  - After the crash, volatility might remain higher, eg. 20%
  - BS Price with 20% volatility: 47.63
  - Charge 20% volatility + risk premium: 47.63 + 12.375 = 59.91 EUR
  - Final implied volatility: σ = 22.59%
- Further refinements
  - Timing of the crash
  - Multiple crash events

DIY Skew (3)

- Example: 10% OTM **call**: ***find the implied volatility***
  - Similar hedging cost argument
  - No crash event: calls respond to different factors than puts
  - OTM calls become ITM after market increase
  - Usually volatility decreases gradually (after crashes)
  - Use a time-dependent volatility model $\sigma(t)$

DIY Skew (4)

- Under σ = 18%, a normal (daily) move equals 13.16 EUR
- The strike of the OTM call is 150 EUR above current market, or 11 “normal” moves
- Assume
  - 11 “normal” consecutive up moves (1 standard deviation)
  - After each step, the actual move is reduced to 99% (to model the reduction of the realized volatility)
  - This will give P&L each day (as the realized volatility is lower than implied)

Day	$S(t)$	Γ	θ_{1d}	ΔS	$\log(S(t+\Delta t)/S(t)$	99% applied	P&L
1	1500.00	0.001104	−0.0956	13.16	0.87%	1500.00	
2	1513.16	0.001055	−0.0935	13.31	0.88%	1513.03	0.002
3	1526.47	0.001006	−0.0911	13.46	0.88%	1526.17	0.004
4	1539.93	0.000956	−0.0885	13.61	0.88%	1539.46	0.006
5	1553.54	0.000906	−0.0858	13.76	0.88%	1552.91	0.008
6	1567.30	0.000857	−0.0829	13.91	0.88%	1566.50	0.010
7	1581.21	0.000808	−0.0798	14.06	0.89%	1580.24	0.011
8	1595.27	0.000760	−0.0767	14.21	0.89%	1594.13	0.012
9	1609.48	0.000712	−0.0734	14.36	0.89%	1608.18	0.013
10	1623.84	0.000666	−0.0701	14.52	0.89%	1622.37	0.014
11	1638.36	0.000620	−0.0667	14.67	0.89%	1636.72	0.015
12	1653.02						
						Total	0.10

DIY Skew (5)

- Calculations to link this back to the implied volatility adjustment:
  - Hedging gain: +0.10 over 11 days
  - 1^{st} day theta: -0.0956
  - 11 day theta, approximated as 11 x (-0.0956) = -1.05
  - Corrected 1 day theta = (11 x (-0.0956) + 0.10) / 11 = -0.0869
  - BS implied volatility corresponding to this theta σ = 16.90%

- Both approaches are very intuitive. Option traders use BS to measure sensitivities and as a tool in their toolkit to then adjust the price, leading to the implied volatility.

Measuring Skew

- Skew measures the difference in implied volatility between ATM options and OTM puts/calls

$$\lambda = \frac{\partial \sigma_{imp}(K,T)}{\partial K}.$$

- Numerical data

S/K	90%	100%	110%
σ_{imp}	22.59%	18.18%	16.90%
π_C	225.36	116.50	52.58
π_P	59.91	97.35	179.73

$$\lambda = \frac{\sigma_{imp}(K_{high},T) - \sigma_{imp}(K,T)}{K_{high} - K} = \frac{16.90\% - 18.18\%}{10\%} = -0.1280$$

$$\lambda = \frac{\sigma_{imp}(K,T) - \sigma_{imp}(K_{low},T)}{K - K_{low}} = \frac{18.18\% - 22.59\%}{10\%} = -0.4410$$

$$\lambda = \frac{\sigma_{imp}(K_{high},T) - \sigma_{imp}(K_{low},T)}{K_{high} - K_{low}} = \frac{16.90\% - 22.59\%}{20\%} = -0.2845.$$

Can we smile any way we want?

- Non-arbitrage conditions
  - Call/Put spread condition $K < K' \Longrightarrow \begin{cases} \pi_C(K,T) \geq \pi_C(K',T) \\ \pi_P(K,T) \leq \pi_P(K',T). \end{cases}$
  - Butterfly condition $K < K' < K'' \Longrightarrow \pi_C(K,T) - 2\pi_C(K',T) + \pi_C(K'',T) \geq 0,$

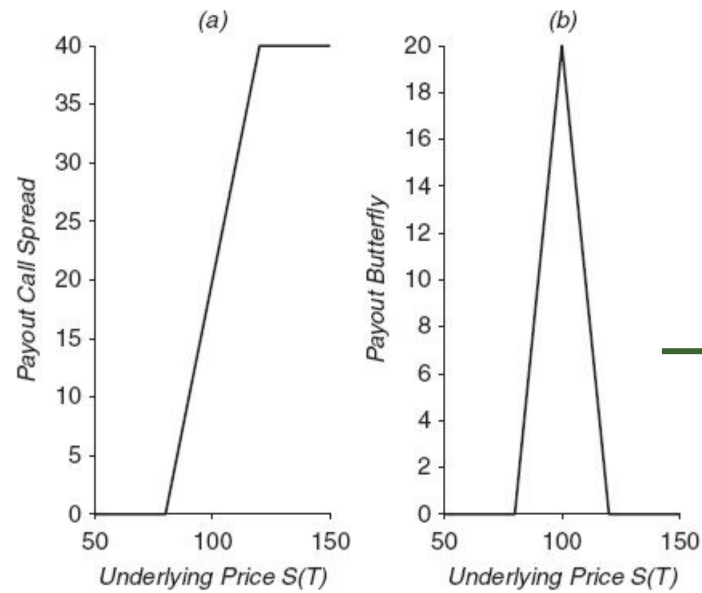


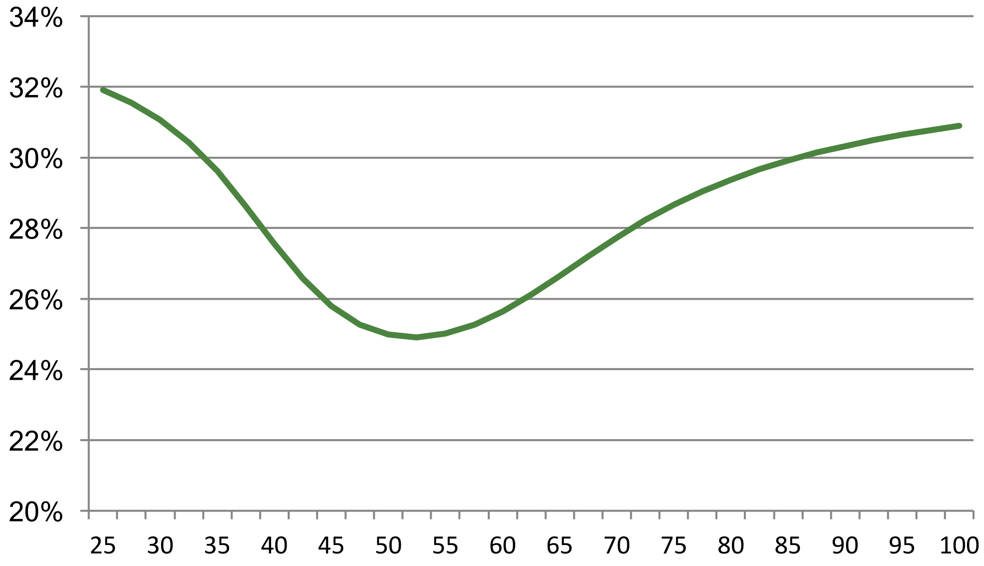
Figure 8.3 (*a*) A call spread strategy: long a low strike call and short a higher strike call. (*b*) A butterfly strategy: long a low strike call, long a high strike call and short two calls with a strike in the middle of the low and high. Both strategies have a payout function (intrinsic value) that is always positive.

Corresponds to numerical approximation for the density

Can we smile any way we want?

- Tail behaviour cannot be linear [**Roger** W. **Lee**.]

$$\left(\sigma_{imp}\left(K,T\right)\right)^2 \sim \log K, \text{ as } K \rightarrow 0, \infty$$



Skew across maturities

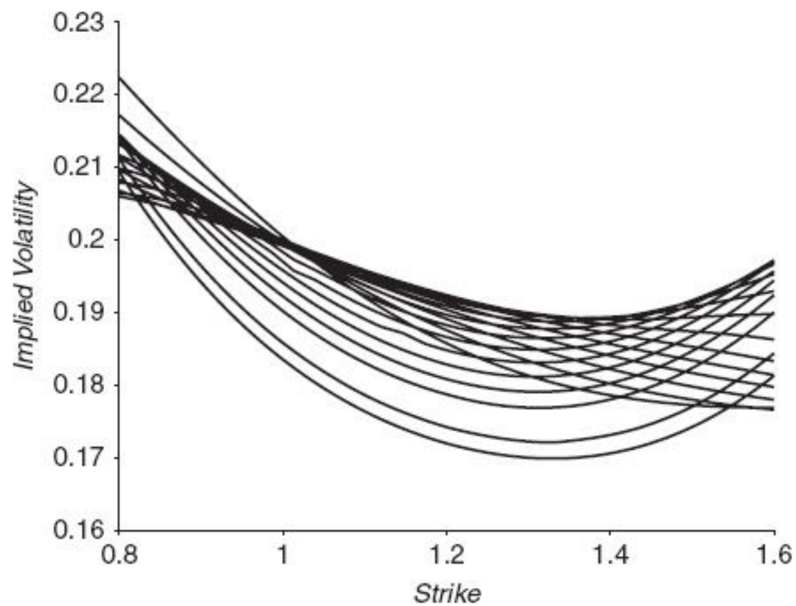
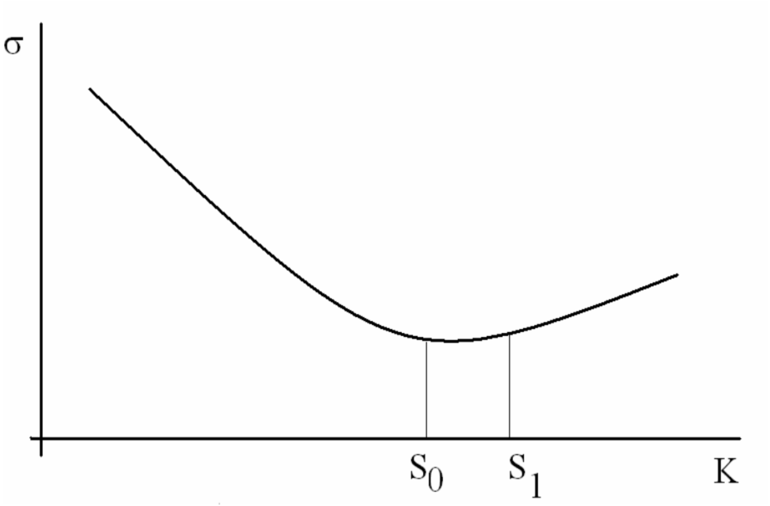


Figure 8.4 A wide variety of skew functions $K \to \sigma_{imp}(K, T)$ across a variety of maturities ranging from six months to five years.

Skew steepens close to maturity

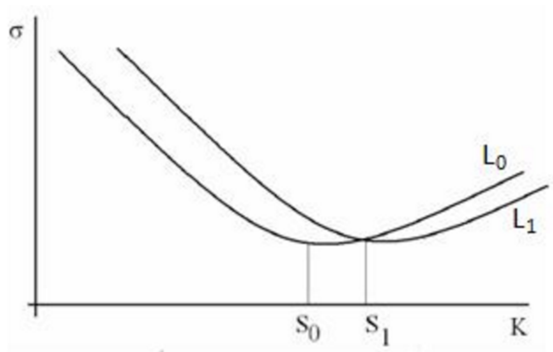
Hedging under skew

- The Sticky strike rule:
  - "Some market players believe that when the stock/index moves, the volatility skew for an option remains unchanged with strike. This behaviour is referred to as the the sticky strike rule. The rule is appliacable when the markets are expected to range bound in near future without significant change in realized volatility."
  - The actual strike sets the options volatility, i.e. if the market moves the vol surface changes if you plot it as a function of moneyness but remains the same if you plot it as a function of the strike itself
  - Equity markets behave a little bit more sticky strike



Hedging under skew

- The Sticky delta rule:
  - "There are some market players that tend to believe that the volatility skew remains unchanged with moneyness. For example lets say that the implied volatility for an ATM option is 30% with the index level being at 100. Now if the index declines to 90, this rule would predict that the implied volatility for 90 stike option would now be 30%. Hence the behaviour is known as sticky moneyness or sticky delta.
  - The sticky delta rule is more applicable when the markets are trending without a significant change in realized volatility.
  - The moneyness strike sets the options volatility, i.e. if the market moves the vol surface changes if you plot it as a function of strike but remains the same if you plot it as a function of the delta
  - FX markets



$$\Delta_{\text{eff}} = \Delta - \lambda \cdot \nu.$$

11. Volatility Trading

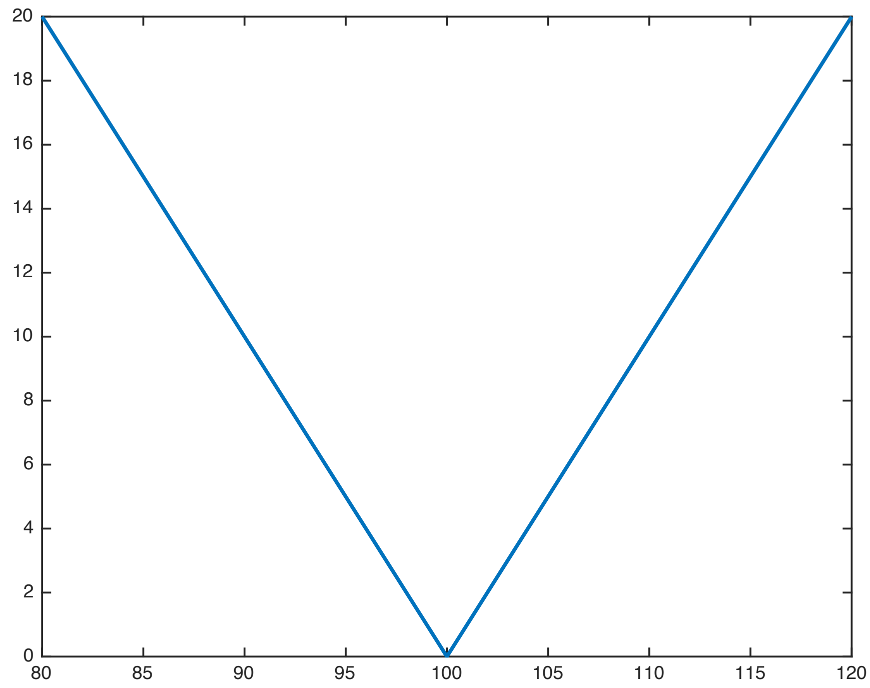
- From hedging to trading
- Straddles
- Risk reversals
- Butterflies

Option Trading strategies

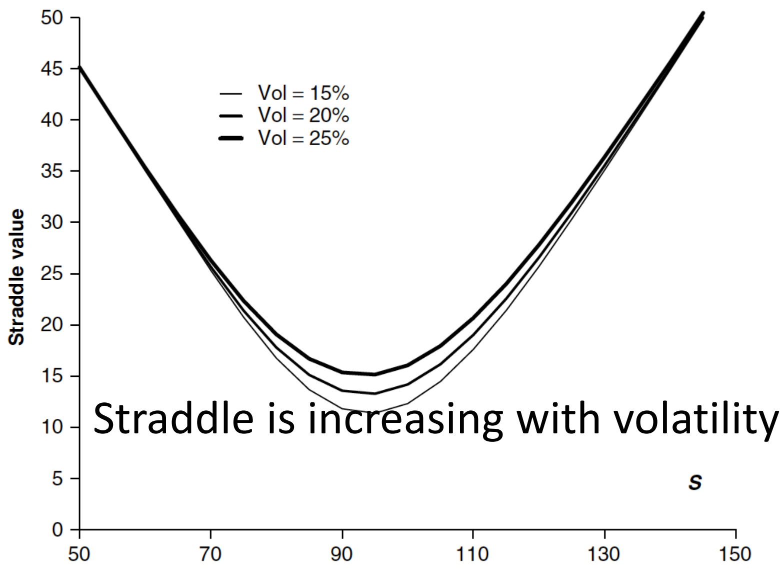
- Everything so far has been about “hedging” and “risk management”
- What about trading? What if I believe (through my estimations) that the market is mispricing volatility features, how do I trade this?
  - Trading Volatility
  - Trading Skew
  - Trading Termstructure

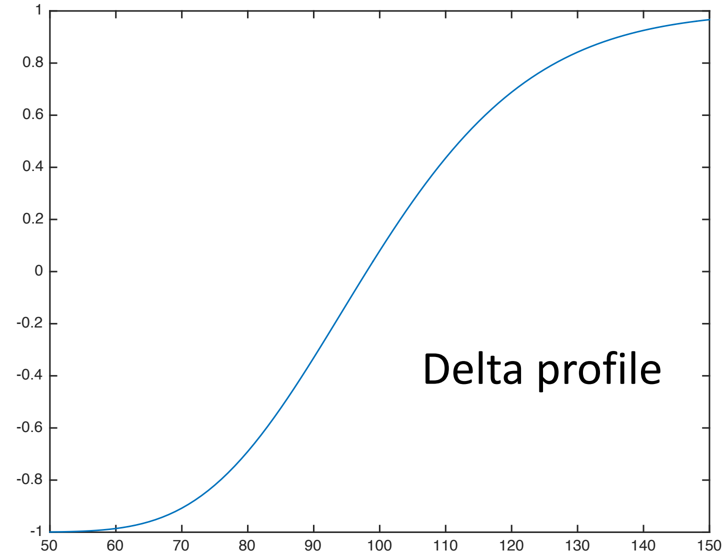
Straddle

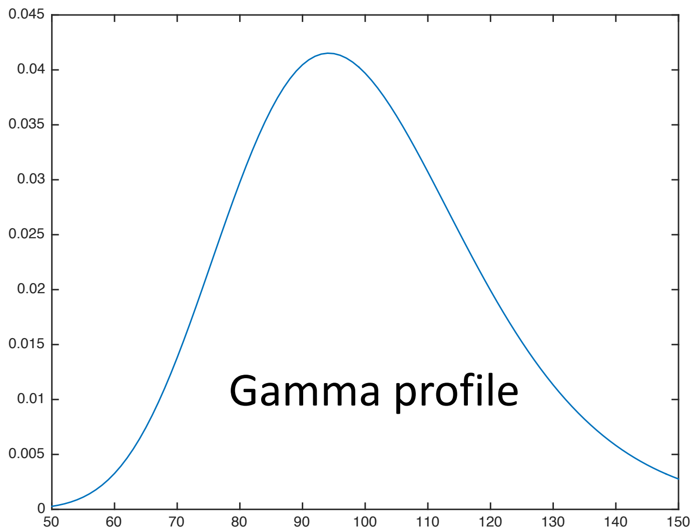
- Portfolio made up a <u>long call</u> and a <u>long put</u>, both having the same strike and maturity
- Put-call parity states that both put and call have to have the same implied volatility



Straddle



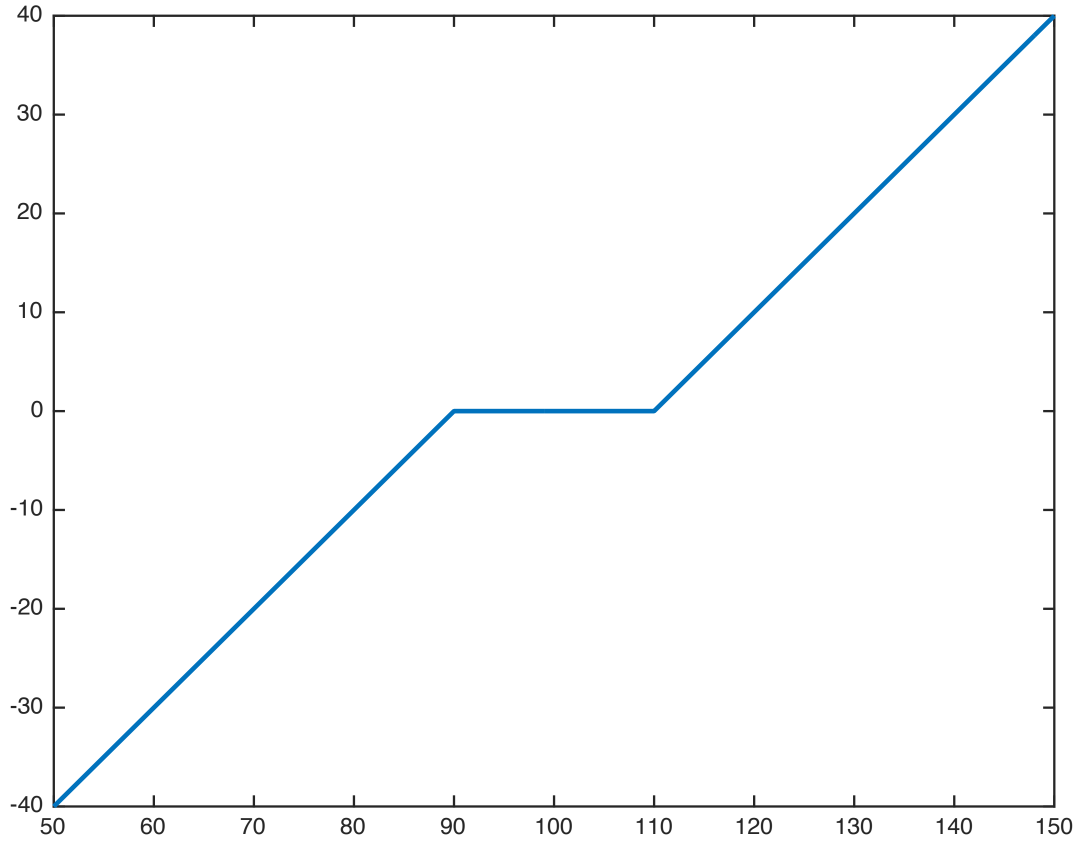




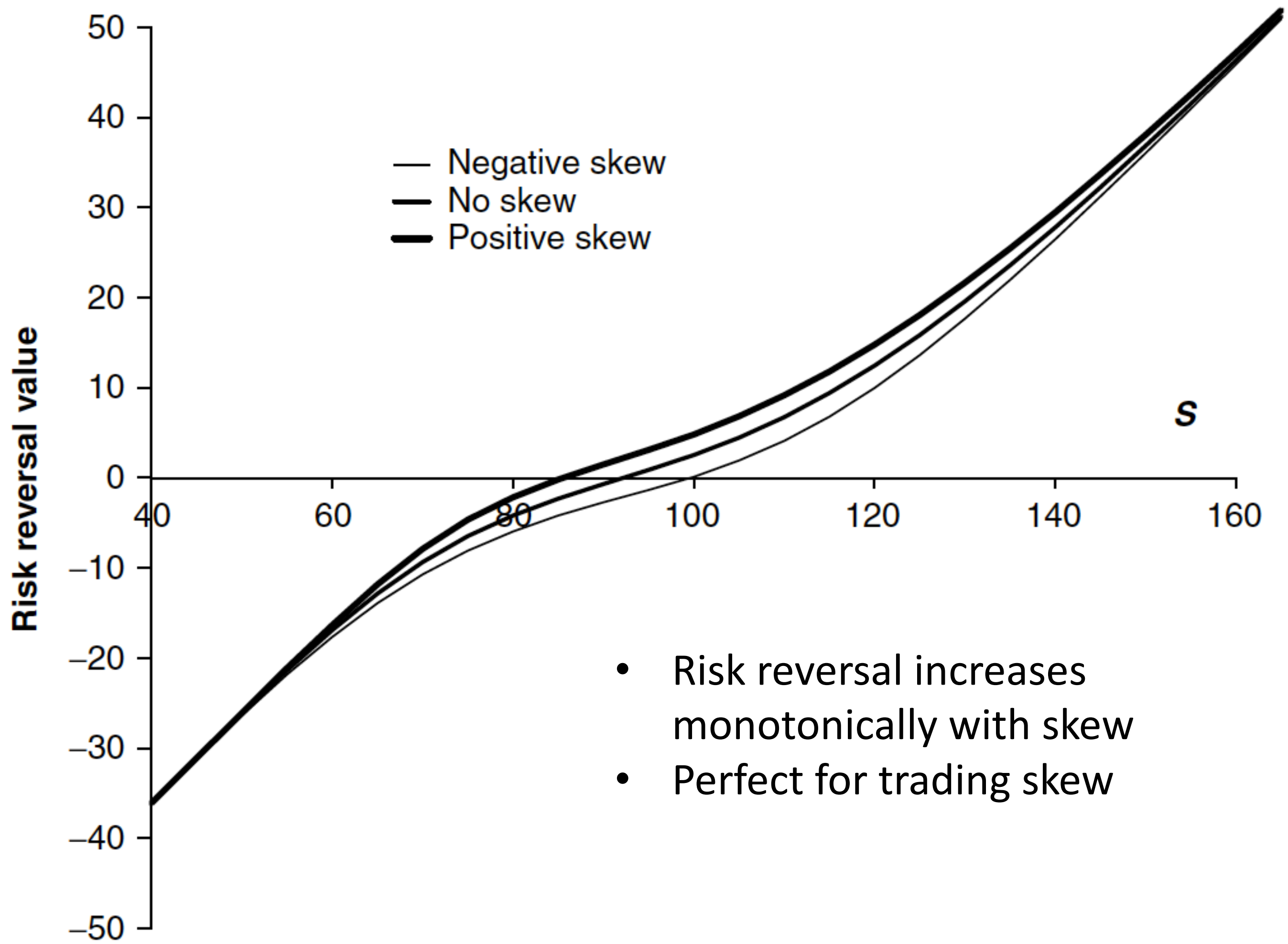
- Straddle increases with volatility
- Gamma profile extended
- Vega profile extended

Risk reversal

- Portfolio made up a long call and a short put, both having the same maturity but the call having a higher strike
- Sensitive to skew (different volatilities)



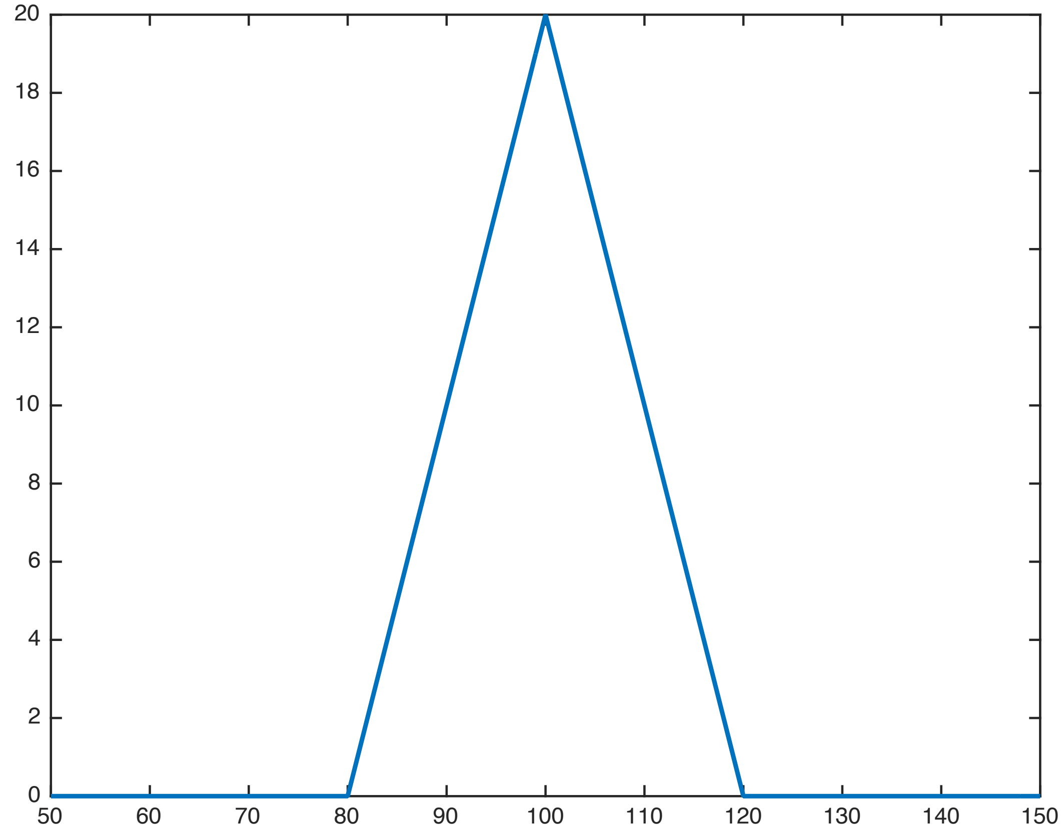
Risk reversal



- Risk reversal increases monotonically with skew
- Perfect for trading skew

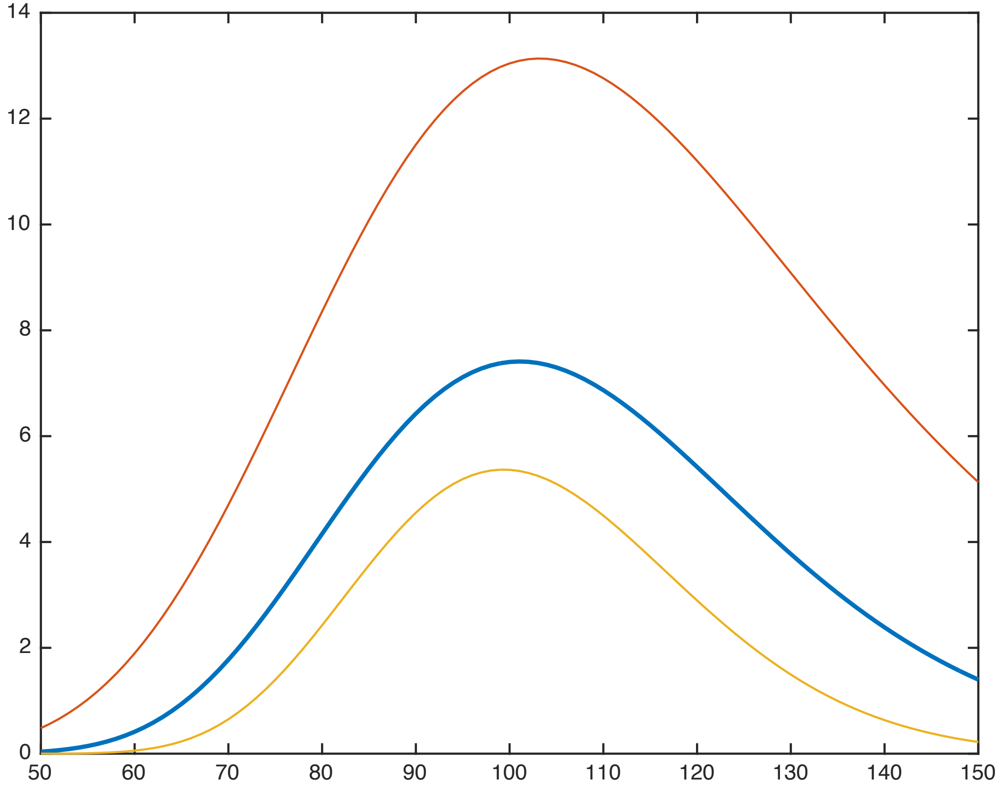
Butterfly

- Portfolio made up a long low-strike call, short 2 middle strike calls and long a high strike call, all having the same maturity.
- Sensitive to smile (different volatilities)



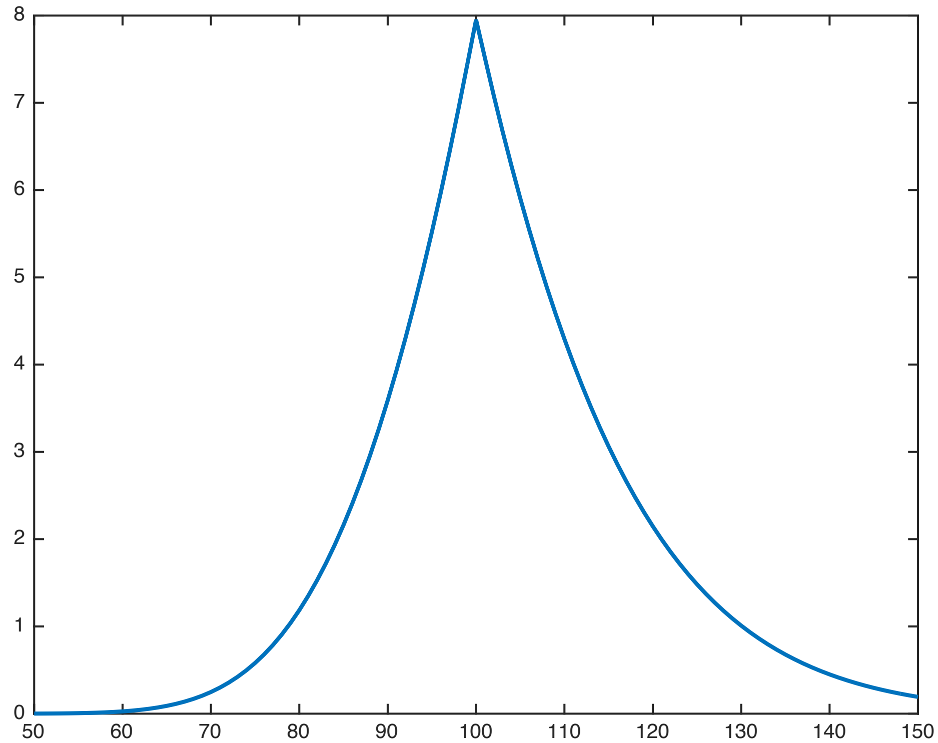
Butterfly

- Portfolio made up a long low-strike call, short 2 middle strike calls and long a high strike call, all having the same maturity.
- Sensitive to smile (different volatilities) – (red = more smile ; blue = no smile, yellow = frown)



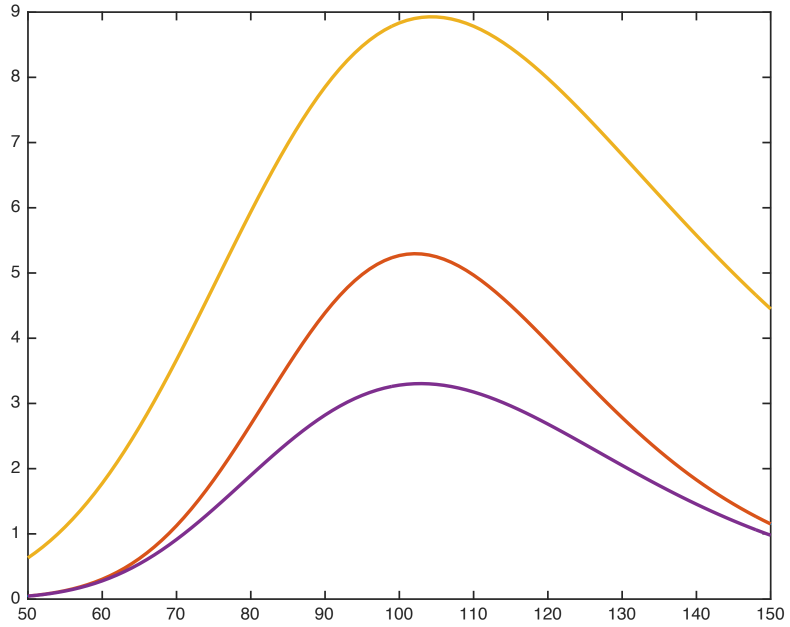
Calendar Spread

- Portfolio made up a long long-dated call and short a short-dated call, both having the same strike.
- Sensitive to term structure (different volatilities)
- At expiry of the first call option: payout profile:



Calendar Spread

- Portfolio made up a long long-dated call and short a short-dated call, both having the same strike.
- Sensitive to term structure (different volatilities) – (purple = no term structure, red = some term structure ; yellow = strong term structure)



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Q & A