Financial Engineering

KU Leuven 2024

PART 1: EQUITY DERIVATIVES MODELLING

This part introduces and applies advanced models for the pricing of equity derivatives.

Students should develop a solid understanding of the these frameworks for pricing these instruments and the mathematical and practical background necessary to apply the various pricing methodologies to the market.

OUTLINE PART 1

- **Exotic Options**

- **Financial Models**

- **Fast Pricing of Vanilla Options under Advanced Equity Models**
  - Advanced pricing formulas
  - Application, pros and cons
  - Fast Fourier Transform (FFT) techniques versus direct integration
  - Modelling stochastic volatility with the Heston and Bates model
  - Other examples of advanced models for pricing (jump models)

OUTLINE PART 1

- **Calibration on Option Surfaces**
  - Basics
  - Objective functions
  - Calibration algorithm
- **Monte Carlo Simulation and Pricing of Exotics**
  - Basics of Monte-Carlo Simulation
  - Sampling of Heston paths : Euler and Milstein scheme
  - Pricing of exotics under Heston using Monte Carlo Simulation
- **Model Risk and Calibration Risk**
  - Model risk
  - Calibration Risk
- **Bid and Ask Pricing**
  - Introduction to bid-ask pricing
  - Distorted expectations and the concept of acceptability
- **Conclusion**

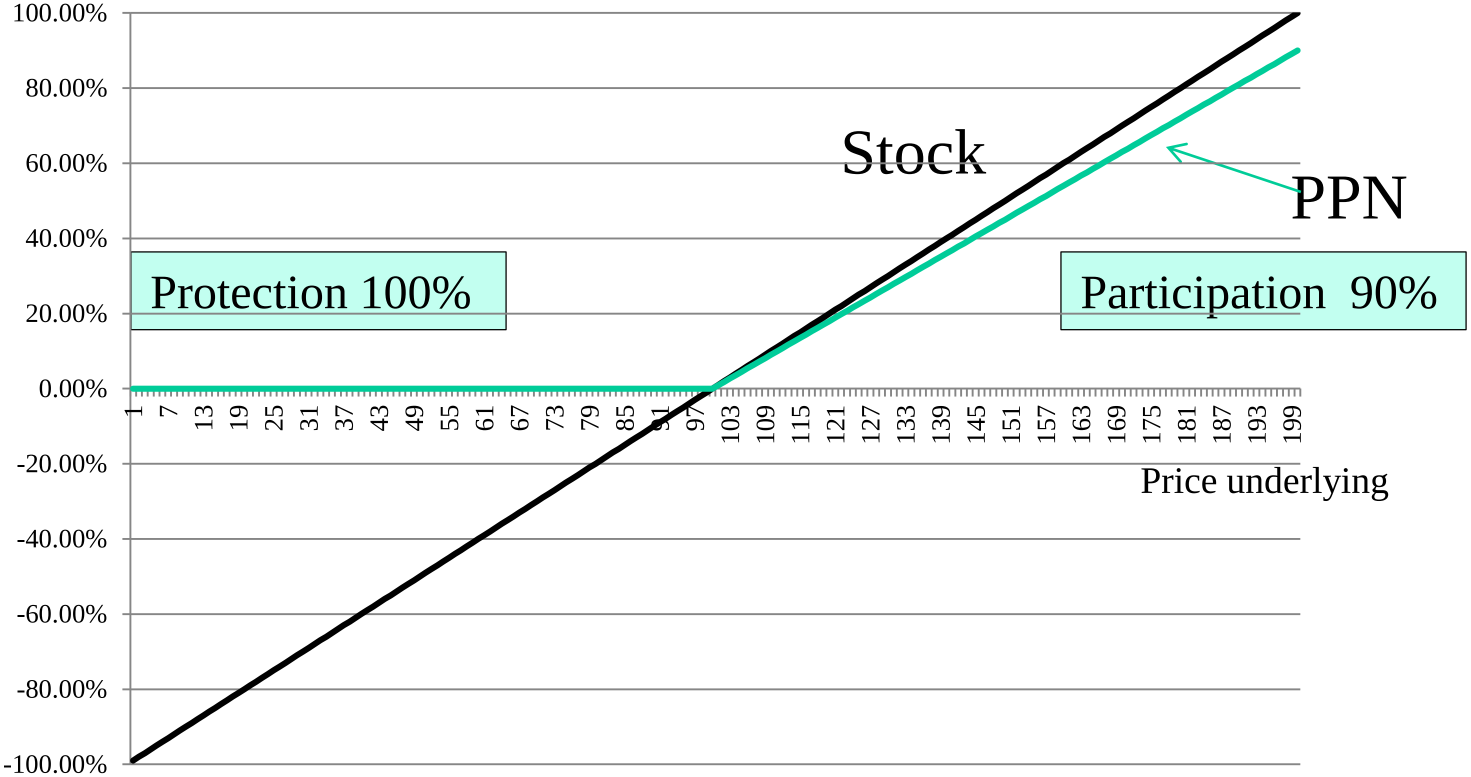
- We start with overviewing a few traditional exotic derivatives

  - **Modest path-dependent structures**
    - Asian options
    - Cliquets
    - …

  - **Heavily path-dependent structures**
    - Lookback
    - Barriers
    - …

- We start with some basic Structured Products.
- A Principal Proctected Note (PPN) for example may be suitable for those seeking full protection of their original investment and for investors who have long-term financial obligations.
- PPNs generally offer a return at maturity linked to an underlying asset.
- Investors typically give up a portion of the potential gains in exchange for principal protection.
- In its most basic form, a principal protected note typically consists of a zero-coupon bond and an exotic derivative. At maturity, the zero-coupon bond is redeemed at par, while the derivative offers participation in an underlying reference asset.

- ZCB + “Fraction” of ATM Call
- Typical Principal Protection is 100 %
- Most of the time fraction = fixed percentage of participation < 1.



- EC is ATM

$$100 + A * \max(S_T - S_0, 0) =$$
$$100 + A * S_0 * \max((S_T - S_0)/ S_0, 0)$$

Depending on implied volatility

Participation rate

- Example :
  - EC = 9; ZCB = 82; S_0=40
  - Participation rate = 2 * 40 /100 = 80 %
  - Buy A = 2 ATM EC

- EC is ATM

$$100 + A * \max(S_T - S_0, 0) =$$
$$100 + A^* S_0 * \max((S_T - S_0)/ S_0, 0)$$

Participation rate

- If EC is more expensive (high vol):
  - EC = 18; ZCB = 82; S_0=40
  - Participation rate = 1 * 40 /100 = 40 % is NOT ATTRACTIVE
  - Buy A = 1 ATM EC

- How to make note more attractive (higher particpation rate)
- **SOLUTION 1**: Principal is only protected to 90:

$$90 + A^*\max((S_T - S_0), 0)$$

- Example :
  - EC = 18; ZCB = 74; S_0=40
  - A = 26/18 = 1.4444
  - Participation rate = 1.4444 * 40 /100 = 57.78 %

Allocate less for ZCB

Increase participation rate

- **SOLUTION 2**: Make payoff barrier dependent:

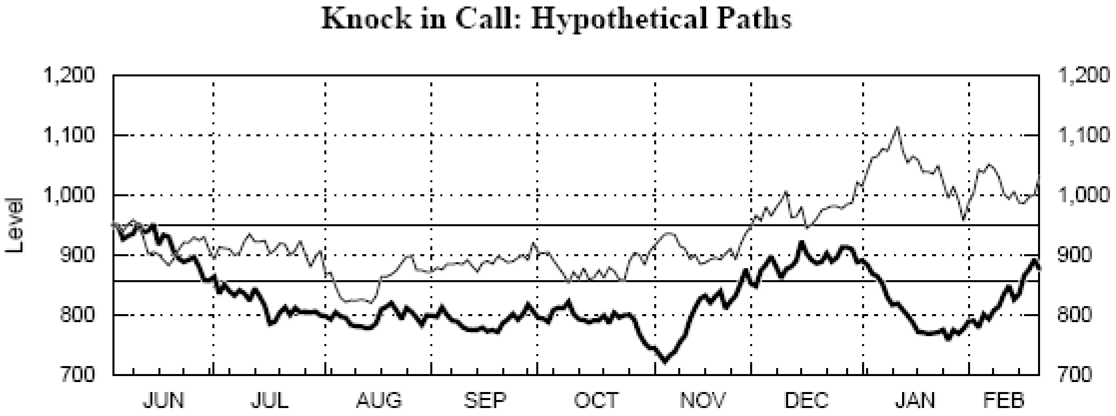
100 + ($A^*\max((S_T - S_0), 0)$ if barrier H has never been hit)

- Example :
  - DOBC = 12; ZCB = 82; S_0=40
  - A = 18/12 = 1.5
  - Participation rate = 1.5 * 40 /100 = 60 %

DOBC < EC

- Some traditional Exotic option structures
  - **Barriers**: Payoff is conditional on whether a low or high barrier is hit.
    - Down-out-Barrier Call : DOBC = $(S_T\text{-}K)^+ \, 1(\min\{S_{t,} ,\ 0 \le t \le T\} \ge H)$
    - Down-in-Barrier Call : DIBC = $(S_T\text{-}K)^+ \, 1(\min\{S_{t,} ,\ 0 \le t \le T\} \le H)$
    - Up-out-Barrier Call : UOBC = $(S_T\text{-}K)^+ \, 1(\max\{S_{t,} ,\ 0 \le t \le T\} \le H)$
    - Up-in-Barrier Call : UIBC = $(S_T\text{-}K)^+ \, 1(\max\{S_{t,} ,\ 0 \le t \le T\} \ge H)$
    - Down-out-Barrier Put : DOBP = $(K\text{-}S_T)^+ \, 1(\min\{S_{t,} ,\ 0 \le t \le T\} \ge H)$
    - Down-in-Barrier Put : DIBP = $(K\text{-}S_T)^+ \, 1(\min\{S_{t,} ,\ 0 \le t \le T\} \le H)$
    - Up-out-Barrier Put : UOBP = $(K\text{-}S_T)^+ \, 1(\max\{S_{t,} ,\ 0 \le t \le T\} \le H)$
    - Up-in-Barrier Put : UIBP = $(K\text{-}S_T)^+ \, 1(\max\{S_{t,} ,\ 0 \le t \le T\} \ge H)$
  - **Basic Relations:**
    - DOBC+DIBC = EC
    - UOBC+UIBC = EC
    - DOBP+DIBP = EP
    - UOBP+UIBP = EP

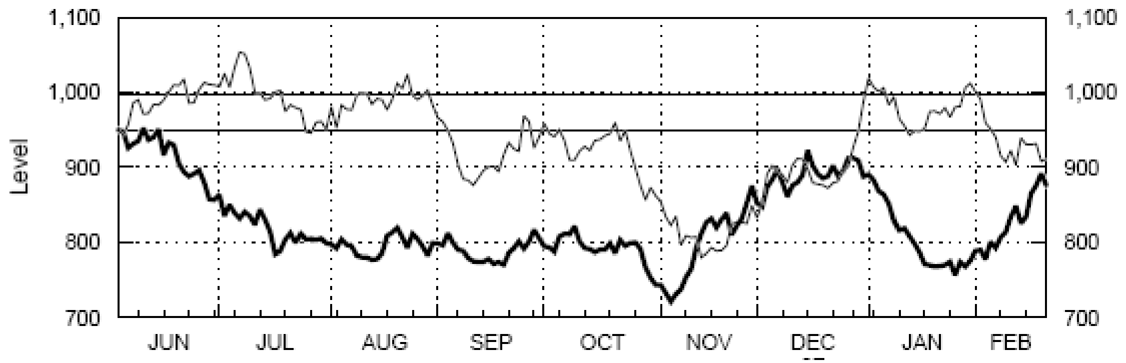
- An investor buys a DIBC on the S&P 500. The call has six months to expiration, is struck at the money and knocks in when the index falls 10% from its initial level.
- Assume that this is offered for 0.94%. With the index at 950, this puts the strike at 950, the barrier at 855 and the premium is 8.93 index points.



- The first path crosses the barrier at 855 in June. At expiration, the index is at 876.05. Since the strike is at 950, the option expires worthless.
- The second path knocks in in August. At expiration, the index is at 1030.78, so the option pays out 80.78 = 1030.78 - 950.00 index points.

- An investor buys a UIBP on the S&P 500. The put has six months to expiration, is struck at the money and knocks in when the index rises 5% from its initial level.
- The DIBP is offered for 2.60%. With the index at 950, this puts the strike at 950, the barrier at 997.50 and the premium is 24.70 index points.

Knock in Put: Hypothetical Paths



- The first path never reaches a level greater than 952.53, which is much less than the knock in barrier at 997.50. Hence, the option expires worthless even though the index at expiration is below the strike.
- The second path does knock in, crossing above the 977.50 level in mid June. At expiration, the level of the index is 911.40, so the option pays out 38.60 = 950.00 - 911.40 index points.

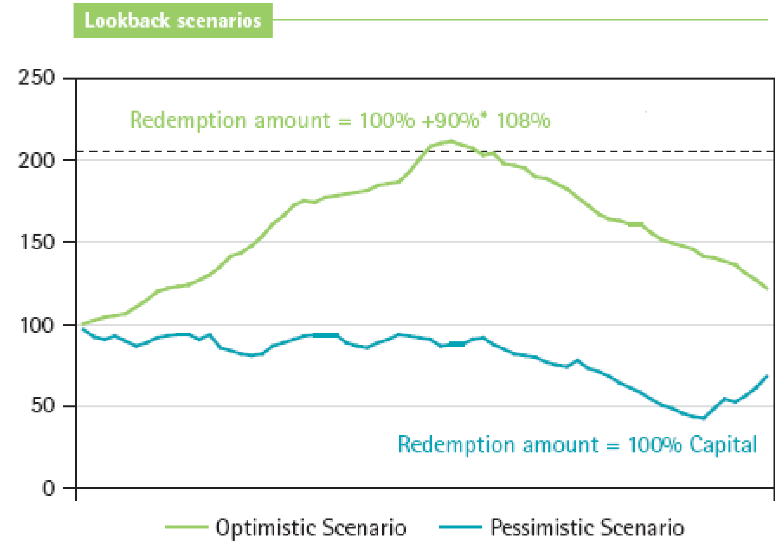
Lookbacks: Payoff is depending on the minimum or maximum of the underlying over the lifetime:

- Lookback Call : LBC = $(\max\{S_{t,}, \ 0 <= t <= T\} – K)^+$
- Lookback Put : LBC = $(K - \min\{S_{t,}, \ 0 <= t <= T\})^+$
- Conditional Lookback Call : $(\max\{S_{t,}, \ 0 <= t <= T\} – K)^+ 1(S_T>H)$
- Conditional Lookback Put : $(K- \min\{S_{t,}, \ 0 <= t <= T\} – K)^+ 1(S_T>H)$

Example:

An investor purchases a six-year lookback on the FTSE 100, with monthly observation.

At maturity, the investor gets 90% of the highest performance of the FTSE over the investment period, with full capital protection



- A Bonus Certificate (BC) pay out a **bonus** at maturity if certain conditions are satisfied, e.g. the underlying stock has never hit a low barrier during the life time of the certificate.
- It gives typically a reduction of the downside risk, a potential higher bonus in case markets move sideways and an unlimited upside potential.

Standard example: BC on FTSE-100, T=3y

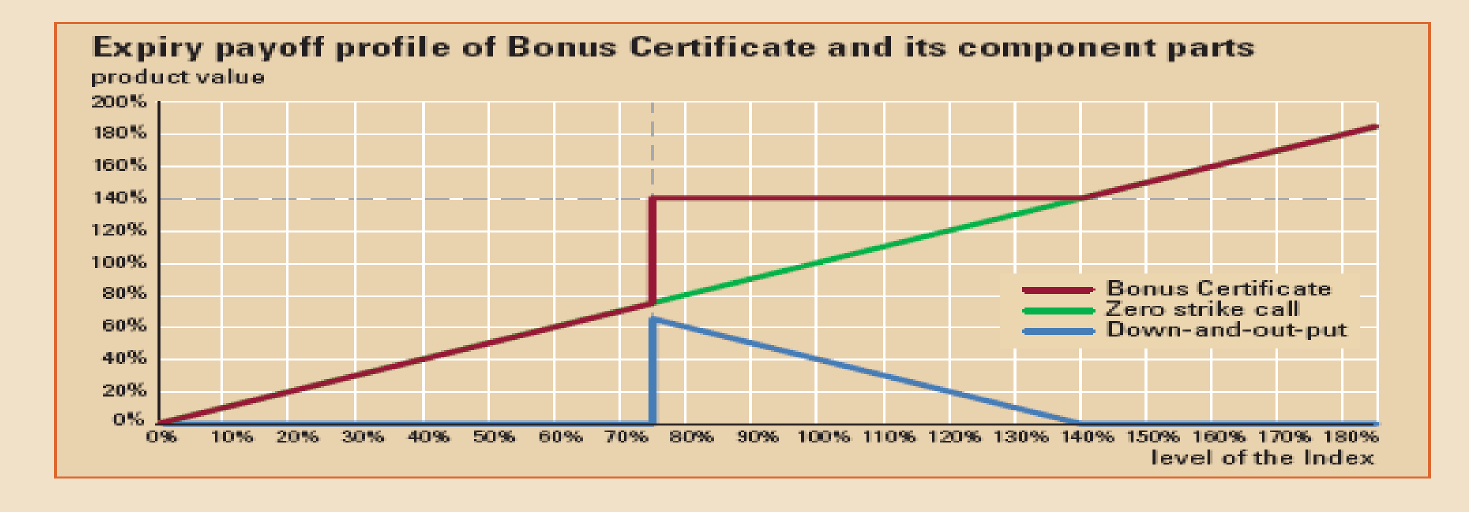
At expiry a holder recives

A. the final index level if it is above 140% of the initial level, or

B. 140% of the initial level, if the final level is between 75% and 140% of the initial level, unless the index level has fallen below 75% of the initial level during the lifetime of the certificate in which case…

C. one receives just the final level

- The Bonus Certifcate (BC) is a combination of a
  - Zero strike European call,
  - Barrier option: down-and-out barrier put (DOBP).

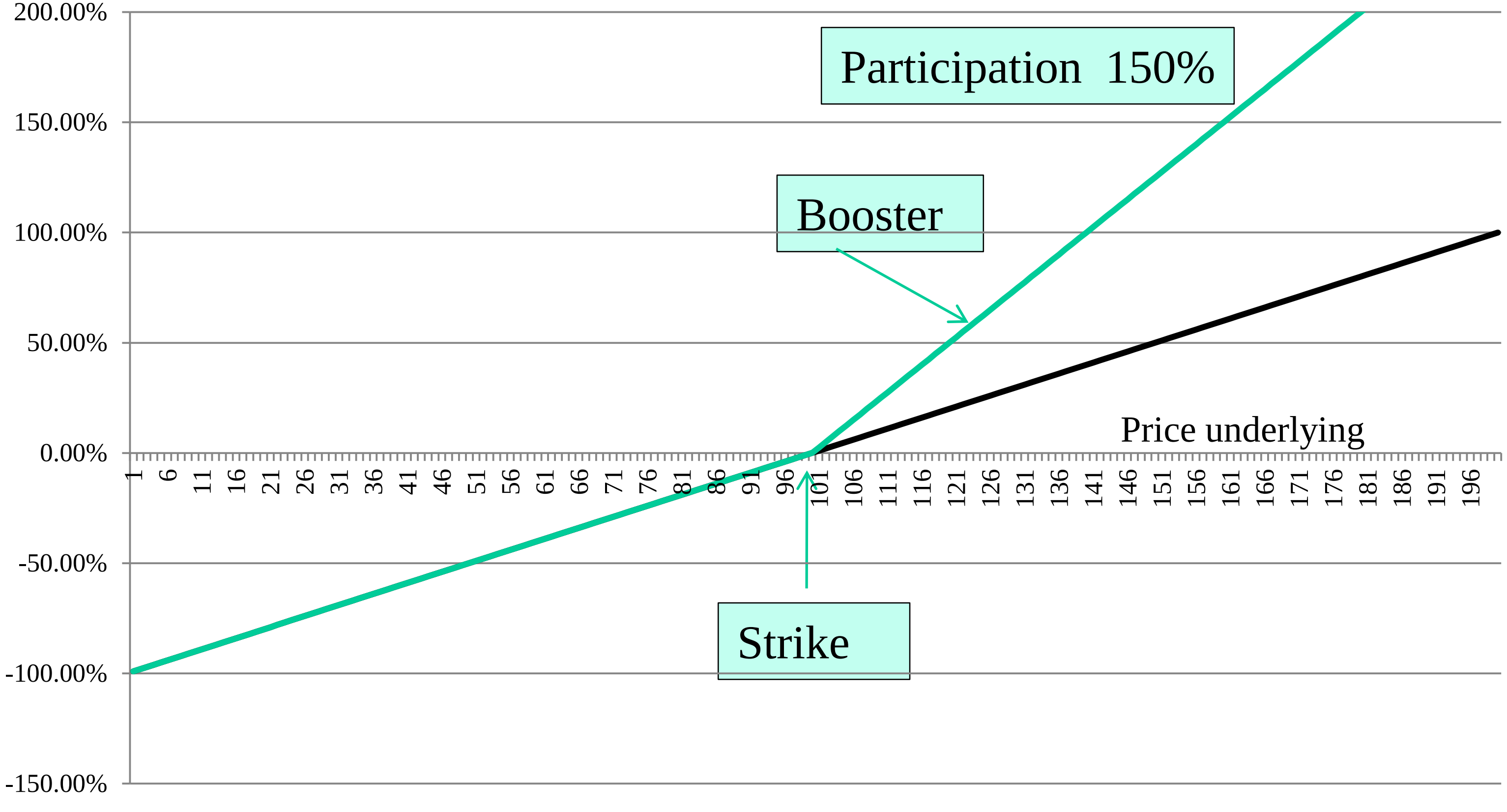


QUESTION *: Is the payoff always better than a direct investment ?*

ANSWER: No, a BC doesn't pay dividends. These dividends are used to finance the DOBP.

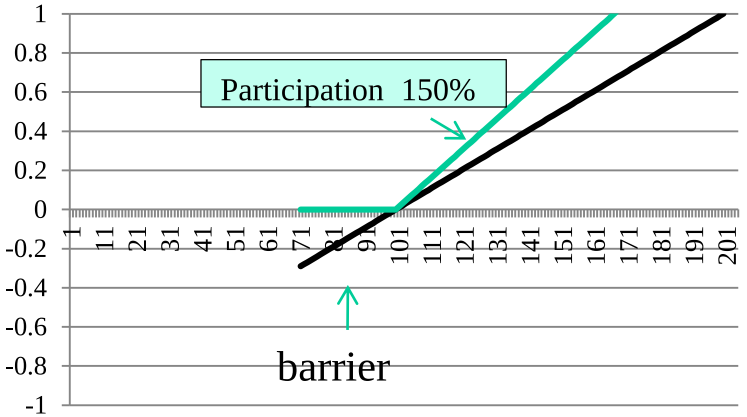
This makes high yield stocks/indices appealing in to be used in BCs.

- Booster :
  - Zero strike European call + Fraction of ATM call

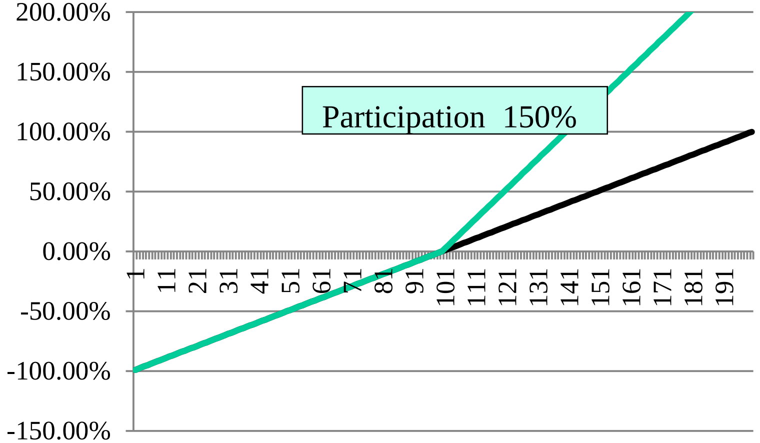


- Variation of a Booster with Barrier clausule
- Zero strike Call + fraction of ATM Call + DOBP

  – **<u>If underlying always above barrier level</u>**:
    - geared exposure if final level is above initial level
    - nomial if final level is below initial level
    - Call works if in the money
    - Put works if out of the money

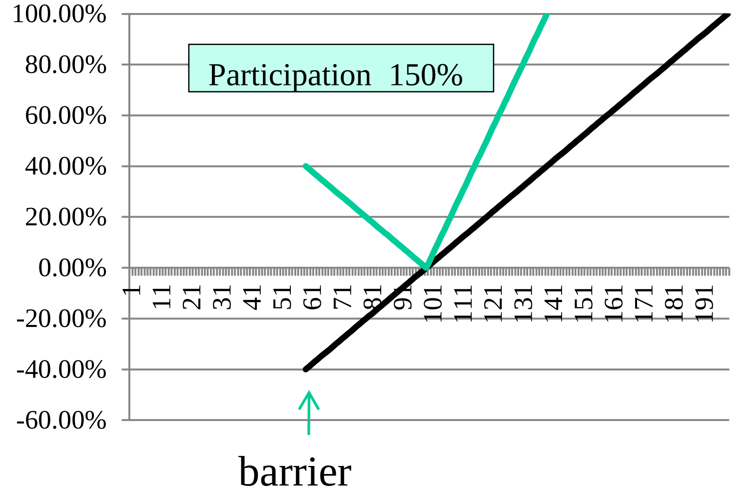


  – **<u>If barrier level has been breached</u>**:
    - geared exposure if final level is above initial level
    - underlying if below initial level
    - Call works if in the money

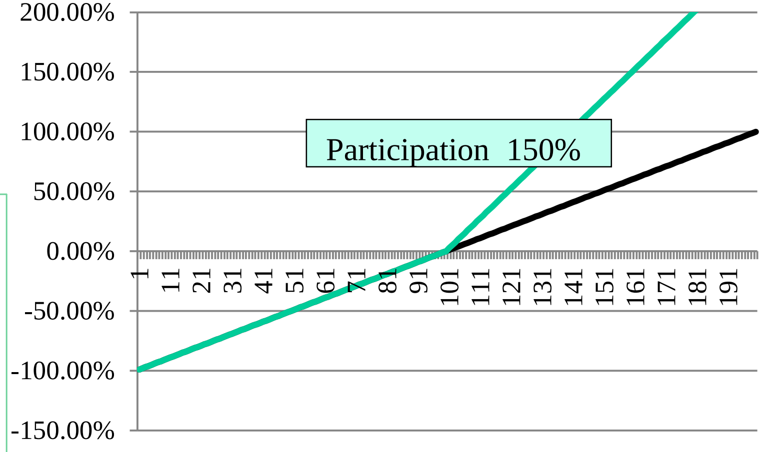


Important : Note is non-principal protected; entire capital is at risk

- Zero strike Call + fraction of ATM Call + 2 DOBP
  - **<u>If underlying always above barrier level</u>**:
    - geared exposure if final level is above initial level
    - positive exposuref to a fall if final below initial level
    - Call works if in the money
    - 2 Put works if out of the money



  - **<u>If barrier level has been breached</u>**:
    - geared exposure if final level is above initial level
    - underlying if below initial level
    - Call works if in the money

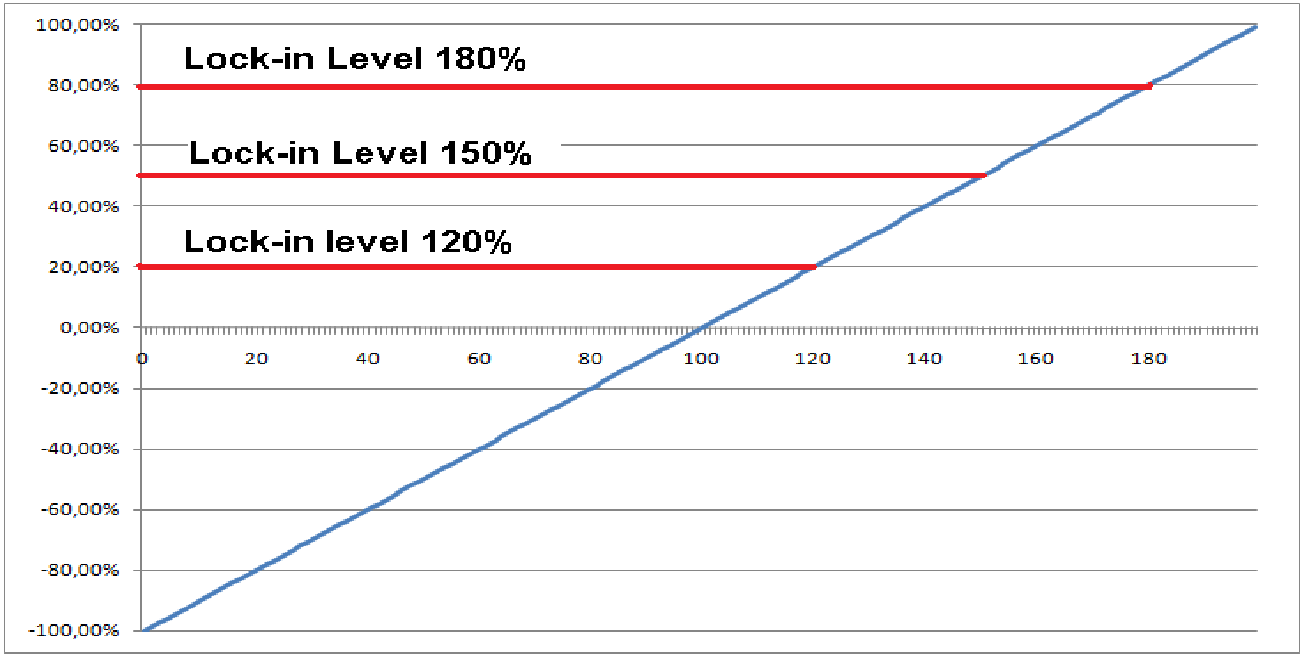


<u>Question:</u>

Compare with Power (same setting).

Is Power-Barrier higher/lower than Twin-Twin-Barrier ?

- Zero strike Call + series of UIBP (with strikes=barrier).
- Investor receives at maturity the greater of the lock-in level if achieved or the performance of underlying.



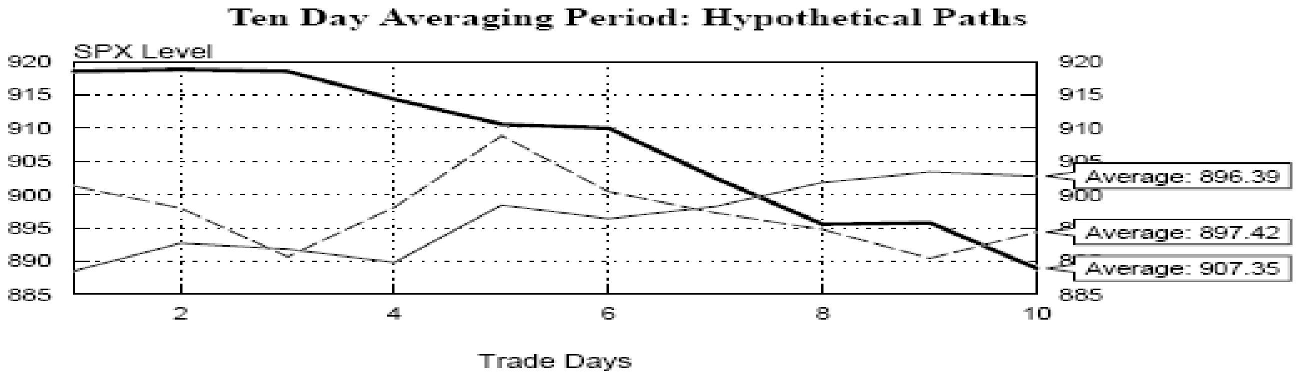
Question: Is this note capital protected ?

- **Asian Call** : Gives the buyer exposure to the upside in the underlying stock above the strike, with settlement based on an average stock price over some period.

  AsianCall = $(A-K)^+$, where A is the average of the stock levels in the averaging period.
- **Asian Put** : Gives the buyer exposure to the downside in the underlying stock below the strike, with settlement based on an average stock price over some period.

  AsianPut = $(K-A)^+$, where A is the average of the stock levels in the averaging period.
- The effect of averaging is to make the observed settlement level less volatile. Many investors use Asian options in case the market makes a large move on the day of expiration.

- Asian put on the S&P 500 with 6 months to expiration and struck 5% out of the money. The strike is fixed on the trade date (not based on averaging) and the terminal level is taken as the average closing price over the last ten trading days up to expiration.
- Assume this is offered for 4.25% of spot. With the index at 950, this puts the strike at 902.50 and the cost is 40.375 index points.



- The first path begins well below the strike at 888.48, and ends about 15 points higher at 902.79. Although the index closes above the strike, the average over the ten trading days is 896.39, so the investor receives 6.11 (902.50 - 896.39) index points.
- The second path starts around 901.The average is 897.42, so the investor receives 5.08 index points.
- In the third case, the index begins the averaging period at 918.54.
  The average is 907.35, so the investor receives nothing. If the option were ordinary (not Asian), then the payoff would have been positive: 13.58 (902.50 - 888.92) index points.

- A VARIANCE SWAPS is a forward contract on annualized variance with payoff at expiration equals to:

$$N \times \left(\sigma_R^2 - K_{var}\right)$$

where σ^2_R is the realized variance (in annual terms) over a period specified by the contract

$$\sigma_R^2 = \frac{252}{n} \sum_{i=1}^{n} \left(\log\left(\frac{S_i}{S_{i-1}}\right)\right)^2$$

and K_{var} is the agreed fixed variance and N is the notional amount.

The K_{var} for a fair price of the variance swap is given by

$$K_{var} = \frac{2\exp(rT)}{T}\left[\int_0^{F_T} \frac{1}{K^2} P(K,T)\mathrm{d}K + \int_{F_T}^{\infty} \frac{1}{K^2} C(K,T)\mathrm{d}K\right]$$

where $F_T = S_0 \exp(rT)$ is the current forward price.

All this is related to the VIX calculation:

$$\sigma^2 = \frac{2\exp(rT)}{T}\sum_i \frac{\Delta K_i}{K_i^2} Q(K_i,T) - \frac{1}{T}\left[\frac{F_T}{K_0} - 1\right]^2$$

OTM C(K,T) or OTM P(K,T)

Strike near F_T

This expression comes from the fact that:

$$(\log x)^2 \approx 2(x - 1 - \log(x))$$

in combination with due to telescoping that

$$\begin{aligned}
\sum_{i=1}^{n} \left(\log\left(\frac{S_i}{S_{i-1}}\right)\right)^2 &\approx 2\sum_{i=1}^{n}\left(\frac{S_i}{S_{i-1}} - 1 - \log\left(\frac{S_i}{S_{i-1}}\right)\right) \\
&= 2\sum_{i=1}^{n}\left(\frac{S_i}{S_{i-1}} - 1\right) - 2(\log(S_T) - \log(S_0)) \\
&\approx 2\sum_{i=1}^{n}(r-q)\Delta t - 2(\log(S_T) - \log(S_0)) \\
&= 2(r-q)T - 2(\log(S_T) - \log(S_0))
\end{aligned}$$

Hence the VS is essentially minus twice a log-contract.

Further the log-contract can be priced by the Breeden-Litzenberger formula:

$$E[f(S_T)] = f(F_T) + \exp(rT)\left(\int_0^{F_T} f''(K)P(K,T)\mathrm{d}K + \int_{F_T}^{\infty} f''(K)C(K,T)\mathrm{d}K\right)$$

where $\text{forward} = F_T = \exp((r-q)T)S_0$

Note that for $f(K) = \log(K),\ f(F_T) = (r-q)T + \log(S_0),\ f''(K) = -\dfrac{1}{K^2}$

Hence the appearance of the strike-square in the fair strike formula.

Note that the B-L-formula provides a static hedge.

Financial Models

About Mean and Variance

- Mean is the arithmetic average: $S^* = \frac{\sum_{i=1}^{n} S_i}{n}$
- Variance is the deviation from that average
  - S = Stock return for a given period t (in years)
  - S* = mean of stock returns
  - n = number of observations

Variance of stock returns: $$\theta_s^2 = \frac{\sum_{i=1}^{n} \left(S_i - S^*\right)^2}{n}$$

Sum of squared deviations from the mean

Annualised volatility: $\sigma_s = \frac{\theta_s}{\sqrt{t}}$

How stock prices move

Proportional return over Δt

Expected return

Stochastic Part. Mean = 0, standard deviation = $\sigma\sqrt{\Delta t}$

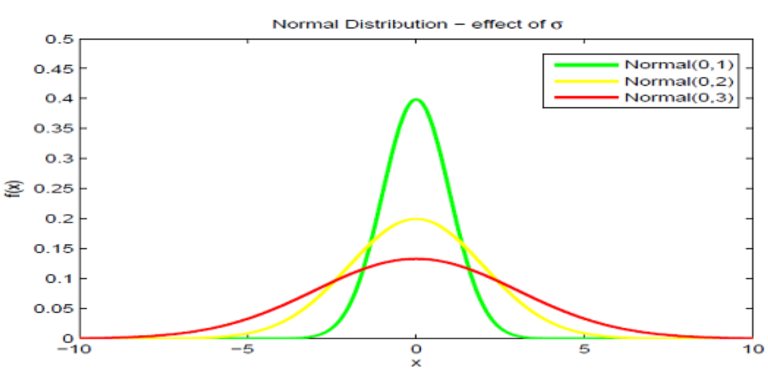
$$\frac{\Delta S}{S} = \mu \Delta t + \overbrace{\sigma \varepsilon \sqrt{\Delta t}}$$

Volatility

ε = Random number from a normal distribution with mean = 0 and standard deviation = 1

Normal Distribution $Normal(\mu, \sigma^2)$ on $(-\infty, +\infty)$:

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



Overall the proportional return of the stock over a small interval of time is normally distributed with mean $\mu \Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$

About Volatility

Volatility is the annualized standard deviation of asset returns (or s.d. of log of asset price).

– **Implied volatility:**

  The volatility that makes the output of an option pricing model equal to the market value

– **Historical volatility:**

  Annualized standard deviation of observed asset returns

  Calculation period and observation frequency affect result; not simple to measure properly.

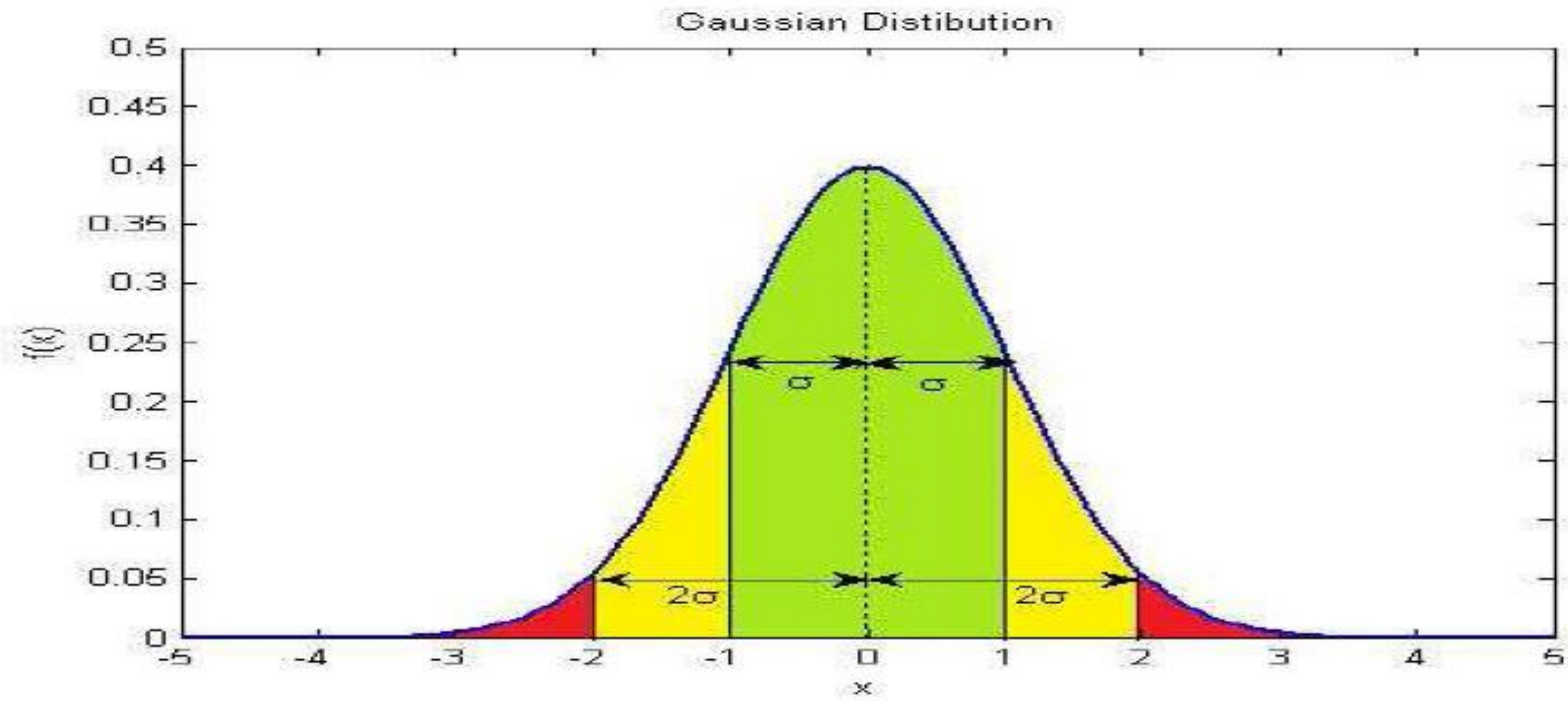
QUESTION : If a stock has a volatility of 16% how much is it likely to move in one day?

If the one year return is normally distributed the stock has 68% probability of moving by 16% or less up or down.

Over one day the standard deviation will be:

Use 256 trading days so daily vol. is 1% ; Really 252 trading days in a year, but 256 is a good and easy approximation

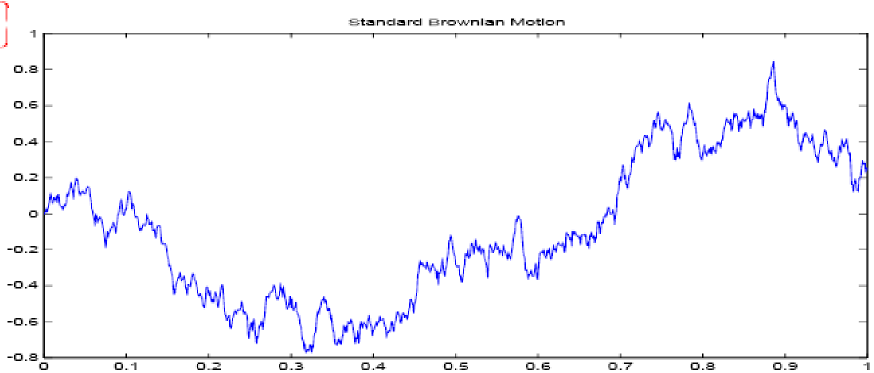
About Volatility



σ	probability	freq.
1σ	0.68269	exceed once in 3 days
2σ	0.95450	exceed once per month
3σ	0.99730	exceed once per year
4σ	0.99994	exceed once per century

Black and Scholes

Brownian Motion : $W = \{W_t, t \geq 0\}$



1. starts at zero
2. independent distributed increments
3. stationary distributed increments
4. normally distributed increments :

$$W_{t+s} - W_t \sim \mathrm{Normal}(0, s)$$

5. continuous sample paths

SDE : $\mathrm{d}S_t = S_t(\mu \mathrm{d}t + \sigma \mathrm{d}W_t), \quad S_0 > 0.$

Geometric BM : $S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$

Normal log-returns : $\log S_t - \log S_0 = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$

Vanilla Black-Scholes Pricing

OPTION PRICES EQUAL DISCOUNTED EXPECTED PAYOFFS

Delta

Probability that option will finish in the money

$$C(K,T) = \exp(-qt)S_0\mathrm{N}(d_1) - K\exp(-rT)\mathrm{N}(d_2)$$

$$d_1 = \frac{\log(S_0/K) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

c = Call option price
r = Risk free interest rate
q = dividend yield
S = spot
K = strike
T = time to maturity
σ = volatility
N( ) = cumulative normal probability distribution

$$\boxed{\text{Call Delta} = N(d_1) \quad \text{Put Delta} = -N(-d_1)}$$

Fast Pricing of Vanillas

- For the Black-Scholes model we have a nice close-form pricing formula for calls and puts:

$$C(K,T) = \exp(-qT)S_0\mathrm{N}(d_1) - \exp(-rT)K\mathrm{N}(d_2)$$

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$

- For more advanced models such a closed-form expression is not readily available.

Fast Pricing of Vanillas

- For many advanced models expressions based on Fourier transforms are available.

- The Fourier-Stieltjes Transform (or characteristic function) of a distribution function $F(x) = P(X \leq x)$:

$$\phi_X(u) = E[\exp(\mathrm{i}uX)] = \int_{-\infty}^{+\infty} \exp(\mathrm{i}ux)\mathrm{d}F(x).$$

- If the density function f of X is available, one can write :

$$\phi_X(u) = E[\exp(\mathrm{i}uX)] = \int_{-\infty}^{+\infty} \exp(\mathrm{i}ux) f_X(x)\mathrm{d}x.$$

Fast Pricing of Vanillas

- The characteristic function always exists, is continuous and it DETERMINES THE DISTRIBUTION FUNCTION UNIQUELY.

- Out of the characteristic function, one can easily derive the moments of X. If $E[|X|^k] < \infty$, then

$$E[X^k] = \mathrm{i}^{-k} \frac{\mathrm{d}}{\mathrm{d}u^k}\phi(u)\Big|_{u=0} .$$

Fast Pricing of Vanillas

- Examples :

Distributions on the Nonnegative Integers

Distribution	$\phi(u) = E[\exp(\mathrm{i}uX_1)]$
Poisson(λ)	$\exp(\lambda(\exp(\mathrm{i}u) - 1))$

Distributions on the Positive Half-Line

Distribution	$\phi(u) = E[\exp(\mathrm{i}uX_1)]$
Gamma(a, b)	$(1 - \mathrm{i}u/b)^{-a}$
Exp(λ)	$(1 - \mathrm{i}u/\lambda)^{-1}$
IG(a, b)	$\exp(-a(\sqrt{-2\mathrm{i}u + b^2} - b))$
GIG(λ, a, b)	$\mathrm{K}_\lambda^{-1}(ab)(1 - 2\mathrm{i}u/b^2)^{\lambda/2}\mathrm{K}_\lambda(ab\sqrt{1 - 2\mathrm{i}u/b^2})$
TS(κ, a, b)	$\exp(ab - a(b^{1/\kappa} - 2\mathrm{i}u)^\kappa)$

Fast Pricing of Vanillas

- Examples :

Distributions on the Real Line

Distribution	$\phi(u) = E[\exp(\mathrm{i}uX_1)]$
$\mathrm{Normal}(\mu, \sigma^2)$	$\exp(\mathrm{i}u\mu)\exp(-\sigma^2 u^2/2)$
$\mathrm{VG}(\sigma, \nu, \theta)$	$(1 - \mathrm{i}u\theta\nu + \sigma^2\nu u^2/2)^{-1/\nu}$
$\mathrm{VG}(C, G, M)$	$(GM/(GM + (M-G)\mathrm{i}u + u^2))^C$
$\mathrm{NIG}(\alpha, \beta, \delta)$	$\exp(-\delta(\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2} - \sqrt{\alpha^2 - \beta^2}))$
$\mathrm{CGMY}(C, G, M, Y)$	$\exp(C\Gamma(-Y)((M - \mathrm{i}u)^Y - M^Y + (G + \mathrm{i}u)^Y - G^Y))$
$\mathrm{Meixner}(\alpha, \beta, \delta)$	$(\cos(\beta/2)/\cosh((\alpha u - \mathrm{i}\beta)/2))^{2\delta}$
$\mathrm{GZ}(\alpha, \beta_1, \beta_2, \delta)$	$\left(\dfrac{B(\beta_1 + \mathrm{i}\alpha u/2\pi, \beta_2 - \mathrm{i}\alpha u/2\pi)}{B(\beta_1, \beta_2)}\right)^{2\delta}$
$\mathrm{HYP}(\alpha, \beta, \delta)$	$\left(\dfrac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + \mathrm{i}u)^2}\right)^{1/2} \dfrac{\mathrm{K}_1(\delta\sqrt{\alpha^2 - (\beta + \mathrm{i}u)^2})}{\mathrm{K}_1(\delta\sqrt{\alpha^2 - \beta^2})}$
$\mathrm{GH}(\lambda, \alpha, \beta, \delta)$	$\left(\dfrac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + \mathrm{i}u)^2}\right)^{\lambda/2} \dfrac{\mathrm{K}_\lambda(\delta\sqrt{\alpha^2 - (\beta - \mathrm{i}u)^2})}{\mathrm{K}_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}$

Heston Stochastic Volatility Model

- Detailing the HESTON Stochastic Volatility Model.
  - Stock price process

$$\frac{dS_t}{S_t} = (r-q)dt + \sqrt{v_t}dW_t, \quad S_0 \geq 0$$

(negatively) correlated Brownian motions

  - (squared) volatility process

$$dv_t = \kappa\,(\eta - v_t)\,dt + \theta\sqrt{v_t}\,d\tilde{W}_t, \quad v_0 = \sigma_0^2 \geq 0,$$

Mean reverting property of volatility

”vol-of-vol”

Heston Stochastic Volatility Model

- Heston parameters**:**
  - Spead of mean reversion: $\kappa > 0$
  - Level of mean reversion: $\eta > 0$
  - Vol-of vol : $\theta > 0$ *(BETTER: vol of var)*
  - Correlation vol-stock : $-1 < \rho < 1$
  - Initial vol : $v_0 > 0$
- Model dates back to 1993
- Heston is popular among practitioners: it incorporates stochastic volatility in an tractable and intuitive way.
- Exotic option pricing can be done using MC; also PDE methods are doable.
- Pure Heston does not include jumps.

Heston Stochastic Volatility Model

- The correlation ρ governs joint movements in the stock and its variance.
- ρ is typically negative for equity: if stock drops, vol rises.
- Typically, the more negative ρ the steeper the skew.
- The price of an option C under Heston satisfies the PDE:

$$\frac{\partial C}{\partial t} + (r-q)S\frac{\partial C}{\partial S} + \frac{S^2 v}{2}\frac{\partial^2 C}{\partial S^2} + \underbrace{\overbrace{\kappa(\eta - v)\frac{\partial C}{\partial v}}^{\text{Vega}} + \overbrace{\frac{\theta^2 v}{2}\frac{\partial^2 C}{\partial v^2}}^{\text{Vega convexity}} + \overbrace{\rho\theta v S\frac{\partial^2 C}{\partial v \partial S}}^{\text{Vanna}}}_{\text{Additional Terms due to variance being stochastic}} = (r-q)C$$

Heston Stochastic Volatility Model

– One says that the Feller condition is satisfied if :

$$2\kappa\eta \geq \theta^2$$

– In that case, theoretically the process never hits 0.

– Doing MC simulation however, one must take care that, due to discretization errors, negative variance is avoided (even under the Feller condition)

Equity Exotic Options

Heston Stochastic Volatility Model

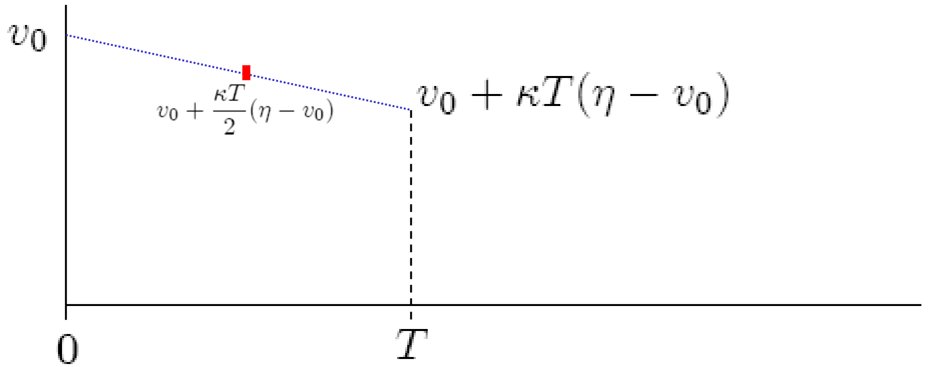
– The fair strike K_{var} for a variance swap under Heston equals.

$$K_{var} \approx E\left[\frac{1}{T}\int_0^T v_t \mathrm{d}t\right] = \frac{1-\exp(-\kappa T)}{\kappa T}(v_0-\eta)+\eta$$

– Note the fair variance swap strike is independent of vol-of-var and the correlation between stock and variance.

– Further :

$$K_{var} \approx \frac{1-\exp(-\kappa T)}{\kappa T}(v_0-\eta)+\eta \approx \left(1-\frac{\kappa T}{2}\right)(v_0-\eta)+\eta = v_0+\frac{\kappa T}{2}(\eta-v_0)$$



Characteristic Function HESTON:

- The characteristic function of Heston is available closed-form.
- Be aware of the Heston trap and the axis of evil !

$$
\begin{aligned}
\phi(u,t) &= E[\exp(\mathrm{i}u\log(S_t))|S_0,\sigma_0^2] \\
&= \exp(\mathrm{i}u(\log S_0 + (r-q)t)) \\
&\quad \times \exp(\eta\kappa\theta^{-2}((\kappa-\rho\theta u\mathrm{i}-d)t - 2\log((1-g\mathrm{e}^{-dt})/(1-g)))) \\
&\quad \times \exp(\sigma_0^2\theta^{-2}(\kappa-\rho\theta\mathrm{i}u-d)(1-\mathrm{e}^{-dt})/(1-g\mathrm{e}^{-dt})),
\end{aligned}
$$

where

$$
\begin{aligned}
d &= ((\rho\theta u\mathrm{i}-\kappa)^2 - \theta^2(-\mathrm{i}u-u^2))^{1/2}, \\
g &= (\kappa-\rho\theta u\mathrm{i}-d)/(\kappa-\rho\theta u\mathrm{i}+d).
\end{aligned}
$$

BATES Model (Heston with Jumps)

$$\frac{\mathrm{d}S_t}{S_t} = (r - q - \lambda\mu_J)\mathrm{d}t + \sigma_t \mathrm{d}W_t + J_t \mathrm{d}N_t, \quad S_0 \geq 0,$$

$$\mathrm{d}\sigma_t^2 = \kappa(\eta - \sigma_t^2)\mathrm{d}t + \theta\sigma_t \mathrm{d}\tilde{W}_t, \quad \sigma_0 \geq 0,$$

- $N = \{N_t, t \geq 0\}$ is an independent Poisson process with intensity parameter $\lambda > 0$
- the percentage jump size J_t follows

$$\log(1 + J_t) \sim \mathrm{Normal}\left(\log(1 + \mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2\right),$$

- $W = \{W_t, t \geq 0\}$ and $\tilde{W} = \{\tilde{W}_t, t \geq 0\}$ are two correlated standard Brownian motions such that $\mathrm{Cov}[\mathrm{d}W_t \mathrm{d}\tilde{W}_t] = \rho \mathrm{d}t$,
- J_t and N are independent, as well as of W and of $\tilde{W}$.

BATES Model (Heston with Jumps)

$$\frac{\mathrm{d}S_t}{S_t} = (r - q - \lambda\mu_J)\mathrm{d}t + \sigma_t \mathrm{d}W_t + J_t \mathrm{d}N_t, \quad S_0 \geq 0,$$

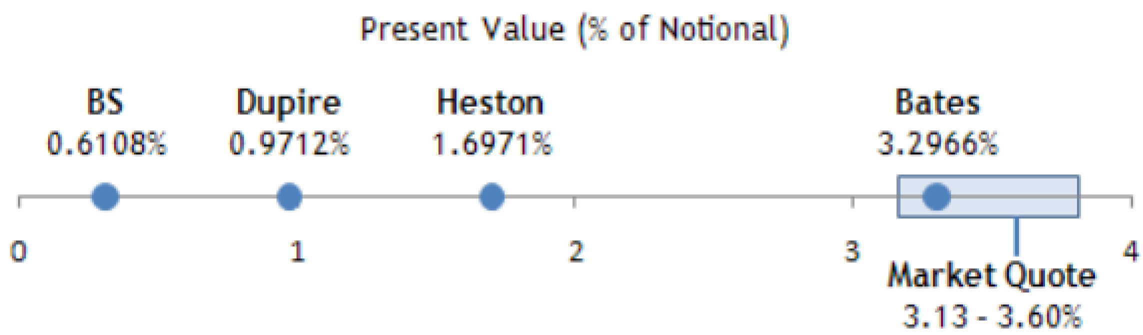
$$\mathrm{d}\sigma_t^2 = \kappa(\eta - \sigma_t^2)\mathrm{d}t + \theta\sigma_t \mathrm{d}\tilde{W}_t, \quad \sigma_0 \geq 0,$$

Characteristic function:

$$\begin{aligned}
\phi(u,t) &= E[\exp(\mathrm{i}u\log(S_t))|S_0, \sigma_0^2] \\
&= \exp(\mathrm{i}u(\log S_0 + (r-q)t)) \\
&\quad \times \exp(\eta\kappa\theta^{-2}((\kappa - \rho\theta u\mathrm{i} - d)t - 2\log((1 - g\mathrm{e}^{-dt})/(1-g)))) \\
&\quad \times \exp(\sigma_0^2\theta^{-2}(\kappa - \rho\theta\mathrm{i}u - d)(1 - \mathrm{e}^{-dt})/(1 - g\mathrm{e}^{-dt})), \\
&\quad \times \exp(-\lambda\mu_J\mathrm{i}ut + \lambda t((1+\mu_J)^{\mathrm{i}u}\exp(\sigma_J^2(\mathrm{i}u/2)(\mathrm{i}u-1)) - 1)),
\end{aligned}$$

BATES Model (Heston with Jumps)

Trade: 1-year cliquet on SPX Index with 1-month resets and a local cap of 3%, trading on 24-Apr-2009 and expiring on 26-Apr-2010.



Prices quoted are for demonstration purposes only and should not to be used for trading.

Source: Numerix

VG model

BASIC IDEA: replace in BS the Normal distribution by a more flexible distribution

$$S_t = S_0 \exp((r - q + \omega)t + X_t), t \geq 0,$$

$$\omega = \nu^{-1} \log\left(1 - \frac{1}{2}\sigma^2\nu - \theta\nu\right)$$

Characteristic function of VG law:

$$E[\exp(\mathrm{i}uX)] = \phi_{VG}(u; \sigma, \nu, \theta) = (1 - \mathrm{i}u\theta\nu + \sigma^2\nu u^2/2)^{-1/\nu}.$$

Fast Pricing of Vanillas

- Vanilla pricing under advanced models (Heston, Lévy, Jump diffusions, Sato, Savy, …) can be done using Carr-Madan formula

$$C(K,T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-\mathrm{i}v \log(K)) \varrho(v) \mathrm{d}v$$

$$\varrho(v) = \frac{\exp(-rT) E[\exp(\mathrm{i}(v - (\alpha+1)\mathrm{i}) \log(S_T))]}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v}$$

Characteristic function of the logarithm of the stock price in the point *(v-(α+1)i)*

Fast Pricing of Vanillas

- The proof of the Carr-Madan formula is not so-difficult.
- We write $k = \log(K)$ for log-strike.
- Let $q(x,t)$ be the density of the log stock price at maturity.
- We assume we have the characteristic function of the log stock price at T available:

$$\phi(u,T) = \int_{-\infty}^{+\infty} \exp(\mathrm{i}ux) q(x;T) \mathrm{d}x$$

- We have :

$$\begin{aligned} C(k,T) &= \exp(-rT) E_Q[(S_T - K)^+] \\ &= \exp(-rT) \int_k^{\infty} (\mathrm{e}^x - \mathrm{e}^k) q(x;T) \mathrm{d}x \end{aligned}$$

Fast Pricing of Vanillas

- However, the call function C(k, T) converges to a non-zero constant if $k \to -\infty$ (the zero strike call price). Hence this function is not square integrable and Fourier theory would not apply.

- To obtain a square integrable function, we consider the modified call price for some $\alpha > 0$.

$$c(k;T) = \exp(\alpha k) C(k;T)$$

- Next we take the Fourier-Stieltjes transform of $c(k;T)$.

Fast Pricing of Vanillas

$$
\begin{aligned}
\varrho(v) = \psi(v;T) &= \int_{-\infty}^{+\infty} \exp(\mathrm{i}vk)c(k;T)\mathrm{d}k \\
&= \int_{-\infty}^{+\infty} \exp(\mathrm{i}vk)\exp(\alpha k)C(k;T)\mathrm{d}k \\
&= \int_{-\infty}^{+\infty} \exp(\mathrm{i}vk)\exp(-rT)\exp(\alpha k)\int_{k}^{\infty}(\mathrm{e}^{x}-\mathrm{e}^{k})q(x;T)\mathrm{d}x\mathrm{d}k \\
&= \exp(-rT)\int_{-\infty}^{+\infty} q(x;T)\int_{-\infty}^{x} \exp(\mathrm{i}vk)\exp(\alpha k)(\mathrm{e}^{x}-\mathrm{e}^{k})\mathrm{d}k\mathrm{d}x \\
&= \exp(-rT)\int_{-\infty}^{+\infty} q(x;T)\left(\frac{\exp\left((\alpha+1+\mathrm{i}v)x\right)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}\right)\mathrm{d}x \\
&= \frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i},T)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v} \qquad (*)
\end{aligned}
$$

Fast Pricing of Vanillas

- The inverse Fourier transform recovers the original function out of the Fourier transform:

Inverse Fourier Transform

$$\begin{aligned} C(k,T) &= \exp(-\alpha k)c(k;T) \\ &= \exp(-\alpha k)\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-\mathrm{i}vk)\psi(v;T)\mathrm{d}v \\ &= \exp(-\alpha k)\frac{1}{\pi}\int_{0}^{+\infty}\exp(-\mathrm{i}vk)\psi(v;T)\mathrm{d}v \end{aligned}$$

$$\varrho(v) = \psi(v;T) = \frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i},T)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}$$

Fast Pricing of Vanillas

- Characteristic function of the logarithm of the stock price is available in closed form for many models.

- The integral is typically calculated using a Fast Fourier Transform.

- Using the FFT one can actually calculate the prices for a whole range of strikes in one run.

- The algorithm chooses these strikes in a cleaver way.

- The price in your strike is obtained via interpolation (the grid of strikes is quite dense).

Fast Pricing of Vanillas

Fast Pricing of Vanillas

- FFT is an efficient algorithm for computing the following transform of a vector $(\alpha_n, n = 1, \ldots, N)$ into a vector $(\beta_n, n = 1, \ldots, N)$:

$$\beta_n = \sum_{j=1}^{N} \exp\left(-\frac{i2\pi(j-1)(n-1)}{N}\right)\alpha_j$$

- Typically N is a power of 2.

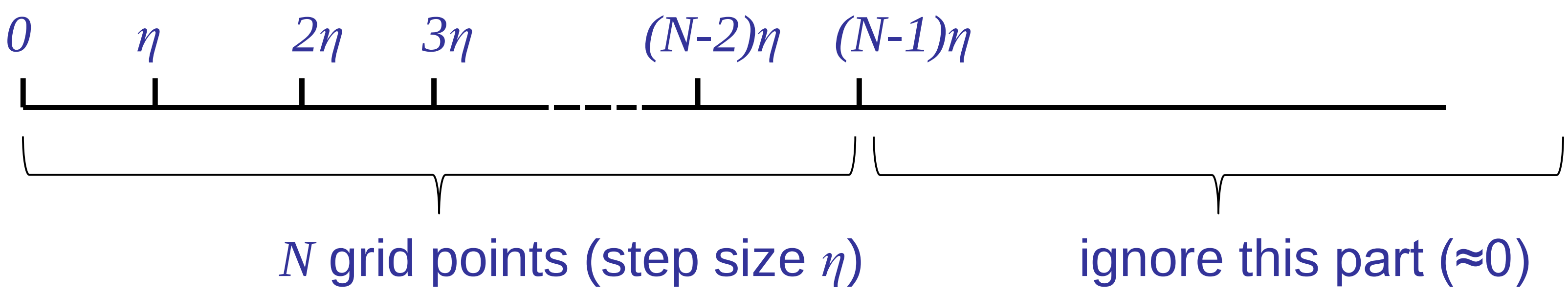
- The number of calculations of the FFT is of order $N \log(N)$ and this in contrast to the straightforward evaluation of the above sums, which give rise to order N^2 of calculations.

Fast Pricing of Vanillas

- Let us apply the FFT for the calculations of the Call prices via the Carr-Madan formula:

$$C(k,T) \;=\; \exp(-\alpha k)\frac{1}{\pi}\int_0^{+\infty} \exp(-\mathrm{i}vk)\varrho(v)\mathrm{d}v$$

- Make a grid for the range of the integral :



$$C(k,T) \;\approx\; \exp(-\alpha k)\frac{1}{\pi}\sum_{j=1}^{N} \exp(-\mathrm{i}v_j k)\varrho(v_j)\eta, \qquad v_j = \eta(j-1)$$

Fast Pricing of Vanillas

- We are going to calculate the values for a whole range of strikes:

$$k_n = -b + \lambda(n-1), \qquad n = 1, \ldots, N, \quad \text{where } \lambda = 2b/N$$

- This gives

$$\begin{aligned} C(k_n, T) &\approx \exp(-\alpha k_n)\frac{1}{\pi}\sum_{j=1}^{N} \exp(-\mathrm{i} v_j(-b+\lambda(n-1)))\varrho(v_j)\eta, \\ &= \exp(-\alpha k_n)\frac{1}{\pi}\sum_{j=1}^{N} \exp(-\mathrm{i}\eta\lambda(j-1)(n-1))\exp(\mathrm{i} v_j b)\varrho(v_j)\eta \end{aligned}$$

- This is almost in the form of the FFT.

FFT :

$$\beta_n = \sum_{j=1}^{N} \exp\left(-\frac{\mathrm{i}2\pi(j-1)(n-1)}{N}\right)\alpha_j$$

Fast Pricing of Vanillas

$$
\begin{aligned}
C(k_n, T) &\approx \exp(-\alpha k_n)\frac{1}{\pi}\sum_{j=1}^{N}\exp(-\mathrm{i}v_j(-b+\lambda(n-1)))\varrho(v_j)\eta, \\
&= \exp(-\alpha k_n)\frac{1}{\pi}\sum_{j=1}^{N}\exp(-\mathrm{i}\eta\lambda(j-1)(n-1))\exp(\mathrm{i}v_j b)\varrho(v_j)\eta
\end{aligned}
$$

- We just need to take $\lambda\eta = 2\pi/N$ to obtain :

$$
C(k_n, T) \approx \exp(-\alpha k_n)\frac{1}{\pi}\sum_{j=1}^{N}\exp\left(-\frac{\mathrm{i}2\pi(j-1)(n-1)}{N}\right)\exp(\mathrm{i}v_j b)\varrho(v_j)\eta
$$

- Hence the sum is just the FFT of the vector

$$
(\exp(\mathrm{i}v_j b)\varrho(v_j)\eta, j = 1, \ldots, N)
$$

FFT : $\beta_n = \sum_{j=1}^{N}\exp\left(-\frac{\mathrm{i}2\pi(j-1)(n-1)}{N}\right)\alpha_j$

Fast Pricing of Vanillas

- There is a whole theory to choose optimally $\alpha>0$
- Carr-Madan (1995) report that the following values give satisfactory results :

$$\begin{aligned} \eta &= 0.25 \\ N &= 4096 \\ \alpha &= 1.5 \end{aligned}$$

- These values lead to :

$$\begin{aligned} \lambda &= 0.0061 \text{ or an interstrike range a little over a half a percentage} \\ b &= 12.57 \end{aligned}$$

Fast Pricing of Vanillas

- An more refined weighting (Simpson's rule) for the integral in the Carr-Madan formula on the N points-grid $(0, \eta, 2\eta, 3\eta, \ldots, (N-1)\eta)$ leads to the following approximation:

$$C(k,T) \approx \exp(-\alpha k)\frac{1}{\pi}\sum_{j=1}^{N} \exp(-\mathrm{i}v_j k)\varrho(v_j)\eta\left(\frac{3+(-1)^j-\delta_{j-1}}{3}\right), \qquad v_j = \eta(j-1)$$

  where $\delta_{j-1} = 1$ if $j = 1$ and zero otherwise.

- This approximation gives a much more accurate integration.

Fast Pricing of Vanillas

- Examples : Black-Scholes setting

$$
\begin{aligned}
S_T &= S_0 \exp((r-q-\sigma^2/2)T + \sigma W_T), \text{ with } W \text{ standard Brownian motion} \\
s_T &= \log(S_0) + (r-q-\sigma^2/2)T + \sigma W_T \\
s_T &\sim \text{Normal}(\log(S_0) + (r-q-\sigma^2/2)T, \sigma^2 T) \\
\phi_{BS}(u;T) &= \exp\left(\mathrm{i}u\left(\log(S_0) + (r-q-\sigma^2/2)T\right)\right) \exp\left(-\frac{1}{2}\sigma^2 T u^2\right)
\end{aligned}
$$

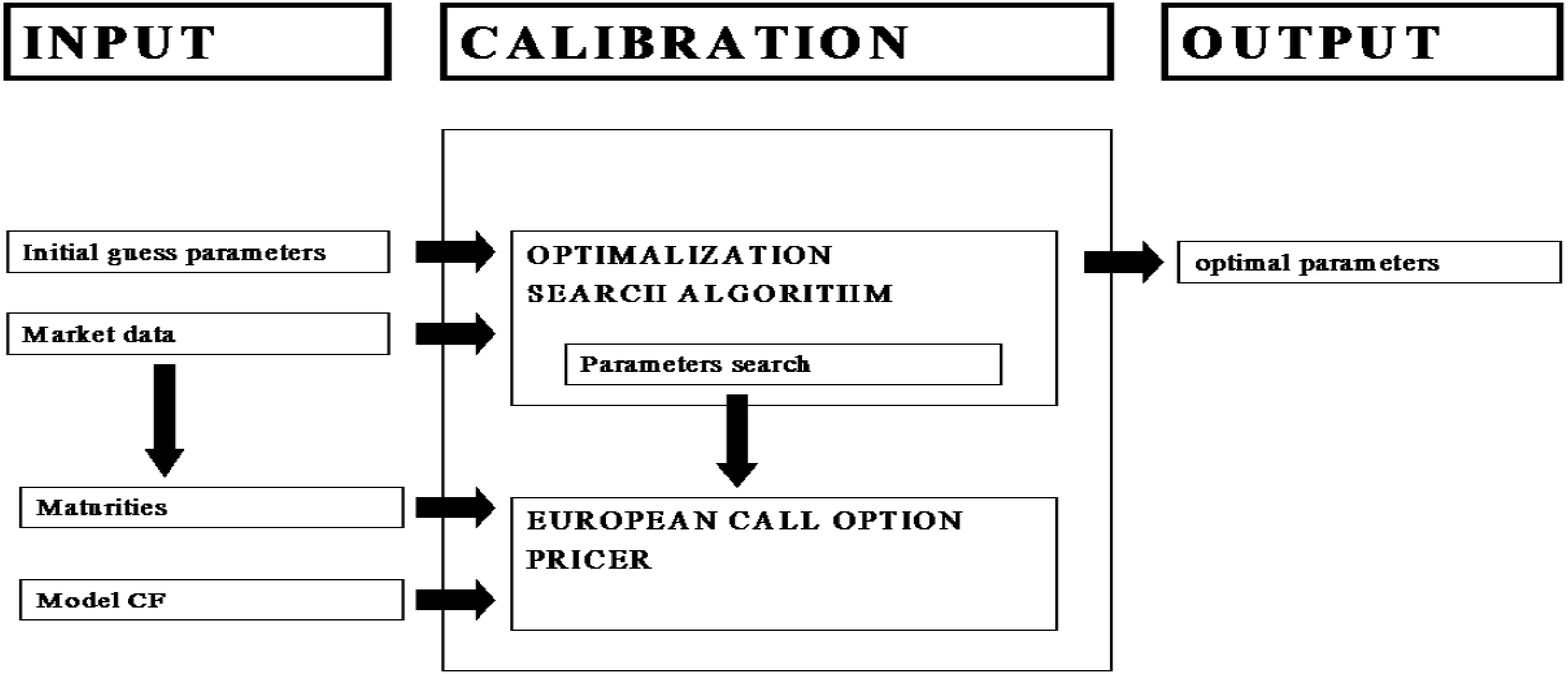
Fast Pricing of Vanillas

```
function y =bsfftcf2(u,p,r,t,x)
sig=x;
y=exp(i.*u*(log(p)+r.*t-(1/2).*sig.^2.*t)).*exp(-(1/2).*sig.^2.*u.^2*t);
return
```

```
N=4096; alpha=1.5; eta=0.25;
p=100; strike=90; sig=0.2; r=0.03; q=0; t=1;
lambda=2*pi/N/eta;
b=lambda*N/2;
k=[-b:lambda:b-lambda];
KK=exp(k);
v=[0:eta:(N-1)*eta];
sw=(3+(-1).^(1:1:N));
sw(1)=1;
sw=sw/3;
rho=exp(-r*t)*bsfftcf2(v-(alpha+1)*i,p,r,t,sig)./(alpha^2+alpha-v.^2+i*(2*alpha+1)*v);
A=rho.*exp(i*v*b)*eta.*sw;
Z=real(fft(A));
CallPricesBS=exp(-alpha*k)/pi.*Z;
CallPriceBSFFT=spline(KK,CallPricesBS,strike);
```

Calibration

Calibration



Basics of Calibration

- Calibration in a nutshell

- Finding minima and maxima of a function

- The problem with local minima and maxima

- Some basic search algorithms

- Examples

Basics of Calibration

- Calibration in a nutshell :

Pricing model

Model parameters

Derivatives

market prices

Find:
model parameters
that match
model prices
as best as possible
with **market prices**

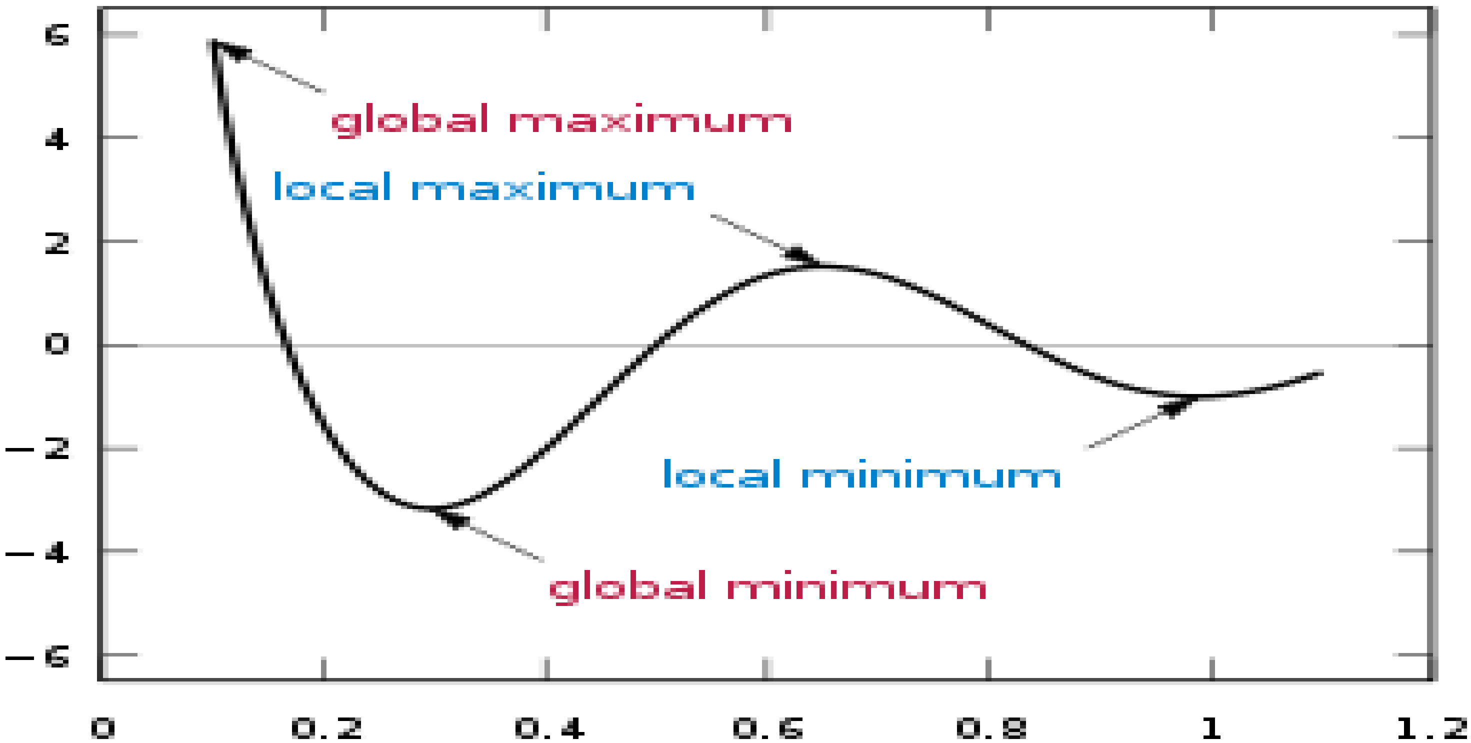
Derivatives

model prices

Optimization problem:
Finding minimum "distance"

Basics of Calibration

- A function can have global and local maxima/minima



Basics of Calibration

in the neighbourhood of x^*

- A function f defined is said to have a **local maximum point** at the point x^*, if there exists some $\varepsilon > 0$ such that

$$f(x^*) \geq f(x) \text{ when } |x - x^*| < \varepsilon.$$

- Similarly, f has a **local minimum point** at x^*, if there exists some $\varepsilon > 0$ such that

$$f(x^*) \leq f(x) \text{ when } |x - x^*| < \varepsilon.$$

- A function has a **global maximum point** at x^* if

$$f(x^*) \geq f(x) \text{ for all } x.$$

- Similarly, a function has a **global minimum point** at x^* if

$$f(x^*) \leq f(x) \text{ for all } x.$$

- The global maximum and global minimum points are also known as the **arg max** and **arg min**: the argument (input) at which the maximum (respectively, minimum) occurs.

Basics of Calibration

- Finding global maxima and minima is the goal of optimization.
- One does this by systematically choosing the values from within an allowed set.
- The first optimization technique, which is known as steepest descent and goes back to Gauss.
- **Gradient descent** is a 1st order optimization algorithm.

Basics of Calibration

- To find a local minimum, one takes steps proportional to the *negative* of the gradient of the function at the current point.

- If instead one takes steps proportional to the gradient, one approaches a local maximum (**gradient ascent)**.

- The gradient of *f* is defined to be the vector whose components are the partial derivatives of *f*. That is:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

- So for a univariate function $\nabla f = f'$ and can be approximated by

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$ for *h* small

Basics of Calibration

Step size $\gamma > 0$ and is allowed to change in each iteration

- We let the algorithm look for a local minumum.
- We start with an x_1 and calculate x_2 (which is closer to the minimum) by the formula:

$$x_2 = x_1 - \gamma_1 \nabla f(x_1) \text{ , with } \gamma_1 \text{ small.}$$

- Next, we reiterate this procedure :

$$x_3 = x_2 - \gamma_2 \nabla f(x_2) \text{ , with } \gamma_2 \text{ small.}$$

- In general :

$$x_n = x_{n-1} - \gamma_{n-1} \nabla f(x_{n-1}) \text{ , with } \gamma_{n-1} \text{ small.}$$

- We stop if we don't see much improvement anymore (say less than a preset tolerance).

Basics of Calibration

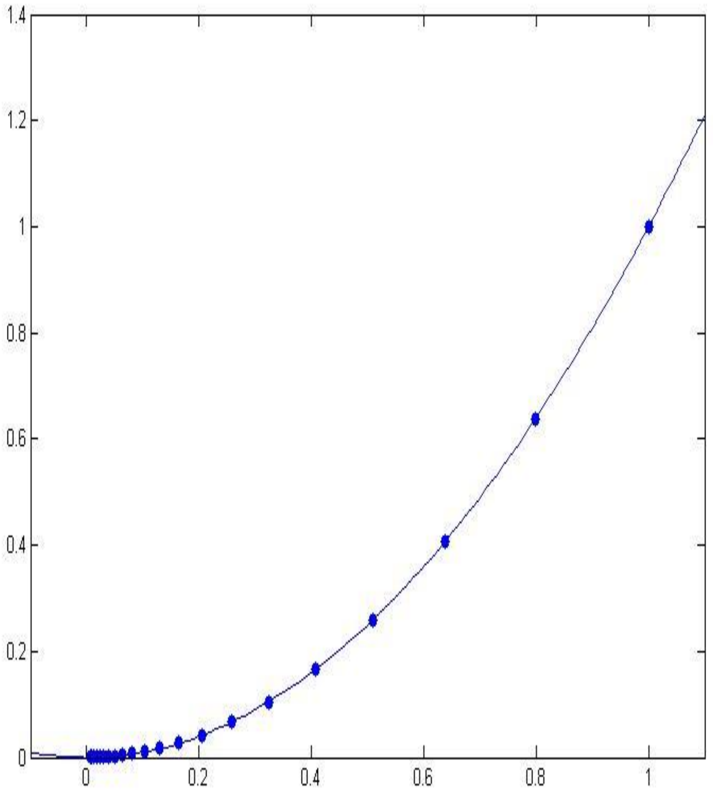
- Let $f(x)= x^2$. We know $f'(x)=2x$ and f has a minimum in zero.
- We start with an x_1 and calculate x_2 (which is closer to the minimum).
- We start with x_1 =1; take $h=0.01$ and $\gamma_1 =0.1.$
- $f(1)=1^2=1$.
- $f'(1) \approx (1+0.01)^2 - 1^2) / 0.01 = 2.01$.
- Note that $f'(1)=2.$
- $x_2 = 1 – 0.1 * 2.01 = 0.7990$

Basics of Calibration

- After 20 iterations (with a constant γ)

h	0,01
gamma	0,1

n	*x*	*f(x)*	*f'(x)* true	*f'(x)* approx
1	1,0000	1,0000	2,0000	2,0100
2	0,7990	0,6384	1,5980	1,6080
3	0,6382	0,4073	1,2764	1,2864
4	0,5096	0,2597	1,0191	1,0291
5	0,4066	0,1654	0,8133	0,8233
6	0,3243	0,1052	0,6486	0,6586
7	0,2585	0,0668	0,5169	0,5269
8	0,2058	0,0423	0,4115	0,4215
9	0,1636	0,0268	0,3272	0,3372
10	0,1299	0,0169	0,2598	0,2698
11	0,1029	0,0106	0,2058	0,2158
12	0,0813	0,0066	0,1627	0,1727
13	0,0641	0,0041	0,1281	0,1381
14	0,0503	0,0025	0,1005	0,1105
15	0,0392	0,0015	0,0784	0,0884
16	0,0304	0,0009	0,0607	0,0707
17	0,0233	0,0005	0,0466	0,0566
18	0,0176	0,0003	0,0353	0,0453
19	0,0131	0,0002	0,0262	0,0362
20	0,0095	0,0001	0,0190	0,0290



Basics of Calibration

First order approximation

Why does this work, i.e. $f(x_1) > f(x_2)$?

We have

$$f(x_2) \approx f(x_1) + f'(x_1)(x_2 - x_1) \quad \text{and} \quad x_2 = x_1 - \gamma\, f'(x_1)$$

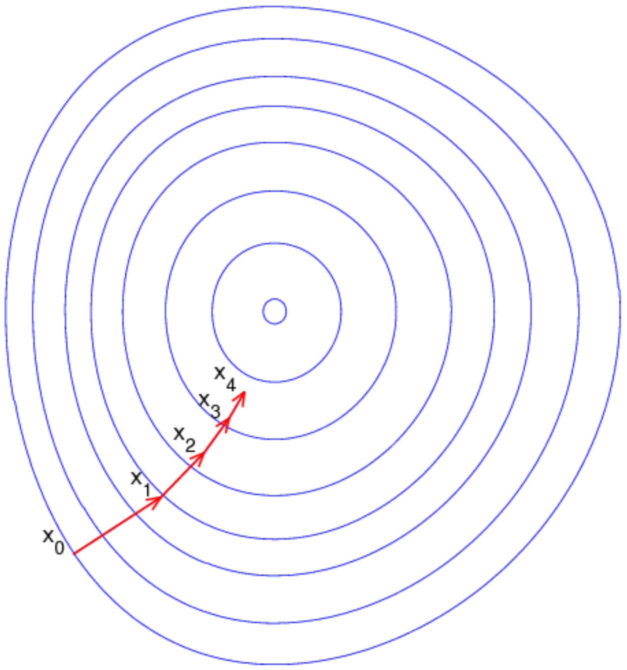
Hence

$$f(x_2) - f(x_1) \approx f'(x_1)(x_2 - x_1) = f'(x_1)(-\gamma\, f'(x_1)) = -\gamma\, (f'(x_1))^2 < 0$$

Basics of Calibration

In practice, we do this in higher dimensions.

- Assume f has a bowl shape.
- The blue curves are the contour lines, that is, the regions on which the value of f is constant.
- A red arrow originating at a point shows the direction of the negative gradient at that point.
- Note that the (negative) gradient at a point is orthogonal to the contour line going through that point.
- We see that gradient *descent* leads us to the bottom of the bowl, the minimum.



Basics of Calibration

- Recall:

$$x_2 = x_1 - \gamma f'(x_1), \text{ for some } \gamma$$

- Rewrite:

$$x_2 - x_1 = -\gamma f'(x_1).$$

- Let's find γ such that $f(x_2)$, is as small as possible (we are looking for a minimum).
- We use : $f(x_2) \approx f(x_1) + f'(x_1)(x_2 - x_1) + \frac{1}{2} f''(x_1)(x_2 - x_1)^2$.
- This is a quadratic function in $z=(x_2 - x_1)$: $f(x_1)+f'(x_1)z + \frac{1}{2} f''(x_1)z^2$, which has a min or max if its derivative is zero: $f'(x_1) + f''(x_1)z = 0$
- Hence it reaches its extrema if: $(x_2 - x_1) = z = -f'(x_1) / f''(x_1)$.
- Combining gives $\gamma = 1/ f''(x_1)$.
- This is Newton's method.

Make sure you go downhill !

Basics of Calibration

- We try Newton out on our quadratic function $f(x)= x^2$.
- $f'(x)=2x$ and $f''(x)=2$. Recall the minimum is in zero.
- Now $\gamma = 0.5$.
- $x_2 = 1 - 0.5 * 2 = 0$
- We find the minimum just after one iteration !
- One can prove that if f is quadratic the minimum is found after one step.
- How does one calculate second derivative numerically:

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Equity Exotic Options

Basics of Calibration

- For $f(x)=x^4$; $f'(x)=4x^3$; $f''(x)=12x^2$

Gradient Descent	Newton's Method

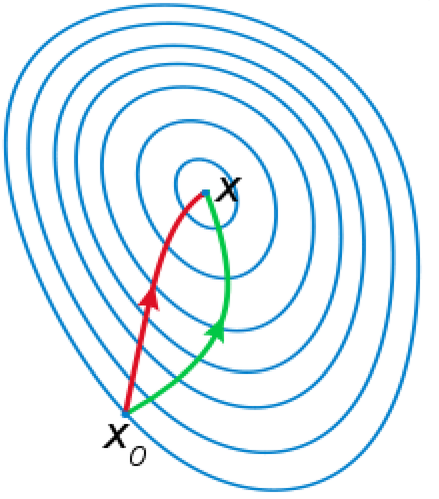
h	0,01
gamma	0,1

n	x	f(x)	f' true	f' approx
1	1,0000	1,0000	4,0000	4,0604
2	0,59396	0,1245	0,8382	0,8596
3	0,50800	0,0666	0,5244	0,5401
4	0,45399	0,0425	0,3743	0,3868
5	0,41531	0,0298	0,2865	0,2971
6	0,38561	0,0221	0,2293	0,2384
7	0,36176	0,0171	0,1894	0,1974
8	0,34203	0,0137	0,1600	0,1672
9	0,32531	0,0112	0,1377	0,1442
10	0,31089	0,0093	0,1202	0,1261
11	0,29828	0,0079	0,1061	0,1116
12	0,28712	0,0068	0,0947	0,0997
13	0,27714	0,0059	0,0851	0,0899
14	0,26815	0,0052	0,0771	0,0816
15	0,26000	0,0046	0,0703	0,0745
16	0,25255	0,0041	0,0644	0,0684
17	0,24572	0,0036	0,0593	0,0631
18	0,23941	0,0033	0,0549	0,0584
19	0,23357	0,0030	0,0510	0,0543
20	0,22813	0,0027	0,0475	0,0507

n	x	f(x)	f' true	f' approx	f'' true	f'' approx
1	1,0000	1,0000	4,0000	4,0604	12	12,0002
2	0,66164	0,1916	1,1586	1,1851	5,253192	5,253392
3	0,43605	0,0362	0,3316	0,3432	2,281682	2,281882
4	0,28564	0,0067	0,0932	0,0982	0,97906	0,97926
5	0,18533	0,0012	0,0255	0,0276	0,412154	0,412354
6	0,11840	0,0002	0,0066	0,0075	0,168228	0,168428
7	0,07370	0,0000	0,0016	0,0020	0,065181	0,065381
8	0,04376	0,0000	0,0003	0,0005	0,022977	0,023177
9	0,02354	0,0000	0,0001	0,0001	0,006651	0,006851
10	0,00955	0,0000	0,0000	0,0000	0,001094	0,001294

Basics of Calibration

- Newton’s method takes curvature into account to take a more direct route.
- Is therefore faster.
- Can lead to zig-zagging.
- Both methods cannot be started at points where derivative is zero.
- If the initial value is too far from the true zero, Newton's method may fail to converge.



Basics of Calibration

- The **Nelder–Mead method** or **downhill simplex method** or **amoeba method** is a commonly used nonlinear optimization technique.
- The Nelder–Mead technique was proposed by John Nelder & Roger Mead (1965) and is a technique for minimizing an objective function in a many-dimensional space.
- Nelder–Mead technique is only a heuristic, since it can converge to non-stationary points.
- The method uses the concept of a simplex, which is a special geometric object of $N + 1$ vertices in N dimensions. Examples of simplices include a line segment on a line, a triangle on a plane, a tetrahedron in 3D etc.

Basics of Calibration

- Many variations exist depending on the actual nature of the problem being solved.
- Nelder–Mead generates a in each iteration a new test position by extrapolating the behavior of the objective function measured at each test point arranged as a simplex.
- The algorithm then chooses to replace one of these test points with the new test point and so the technique progresses.
- It is based on reflection, expansion, contraction and shrink operations of the simplex.

Basics of Calibration

STOPPING CRITIRIA : When does the algorithm needs to stop ?

- **Criterion 1**: reach the number of iteration specified by the user;

$$i > N$$

- **Criterion 2**: when the change of x value is smaller than a user specified threshold;

$$| x_i - x_{i-1} | < \varepsilon_1$$

- **Criterion 3**: when the change of function value is smaller than a user specified threshold;

$$| f(x_i) - f(x_{i-1}) | < \varepsilon_2$$

Basics of Calibration

STARTING VALUES:

- Many of the optimization techniques depend (heavily) on the starting values.

- Good intuition for the involved parameters is utmost important:

  e.g.: volatility is a number around 20% for equity, not a number of around 10 Bn.

- Many adhoc method exist to choose good starting values :
  - Pick a whole grid of them and do a very quick optimization (just a few steps) and then take best result.
  - Take previous day results or parameters of similar instruments.
  - Etc.

Basics of Calibration

CALIBRATION INSTRUMENTS:

- In financial application the instruments (derivatives) on which we calibrate the model are the available (vanilla) instruments in the market, for which we have fast model pricers available
- In equity world these are the European Calls and Puts available in the market.
- There are here two possibilities :
  - Calibrate on the prices themselves
  - Calibrate on implied volatilities
- Sometimes other derivative (exotic) instruments are also traded, however the pricing of them under the model may take too long… (recall a calibration will try-out a huge amount of parameter combinations).

Basics of Calibration

OBJECTIVE FUNCTIONS:

- Root Mean Square Error (RMSE) : $$\text{RMSE} = \sqrt{\sum_{j=1}^{N} \frac{(P_j - \hat{P}_j)^2}{N}}$$
- Average Percentage Error (APE) : $$\text{APE} = \frac{1}{\text{mean}_j \hat{P}_j} \sum_{j=1}^{N} \frac{\left| P_j - \hat{P}_j \right|}{N}$$
- Average Relative Price Error (ARPE) $$\text{ARPE} = \frac{1}{N} \sum_{j=1}^{N} \frac{\left| P_j - \hat{P}_j \right|}{\hat{P}_j}$$
- Etc.

Market Price

Model Price

Basics of Calibration

WEIGHTINGS: $$\textbf{Weighted RMSE} = \sqrt{\sum_{j=1}^{N} w_j (P_j - \hat{P}_j)^2}$$

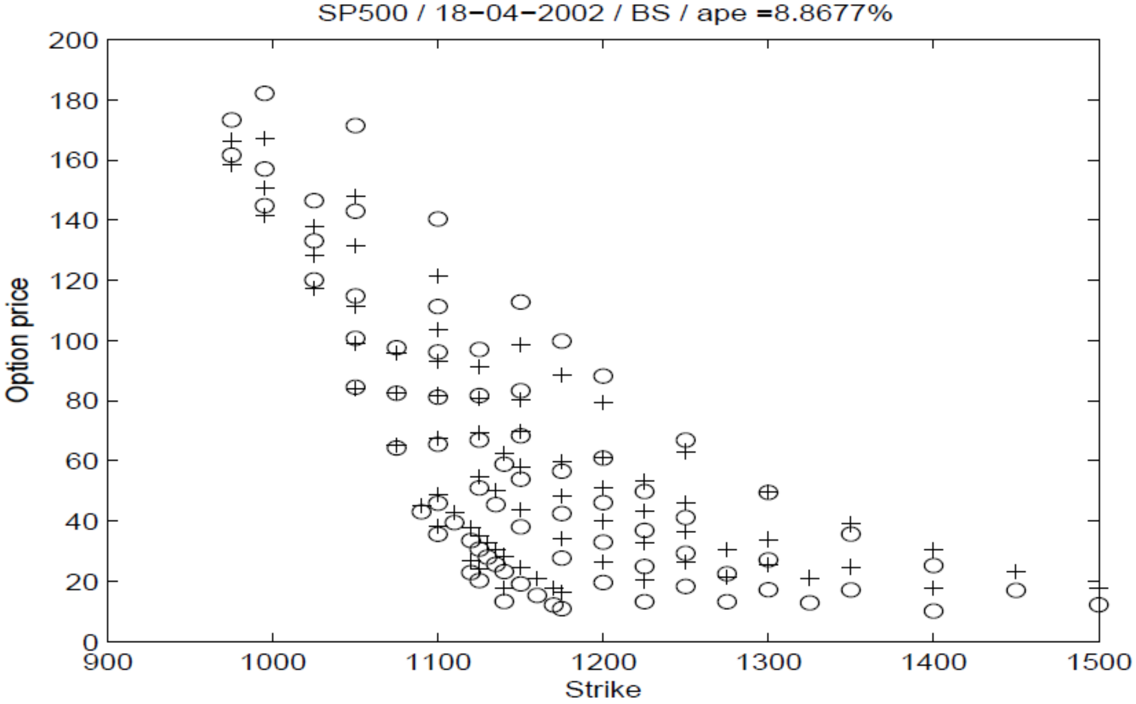
- Equal weighting.

- Weighting according to open interest.

- Weighting inverse to bid-ask spread: If the spread is great, we have a wider range of prices that the model can imply. In other words the model is allowed to imply a wider range of prices around the mid-price. This means that less weight should be placed on such a price, and vice-versa.

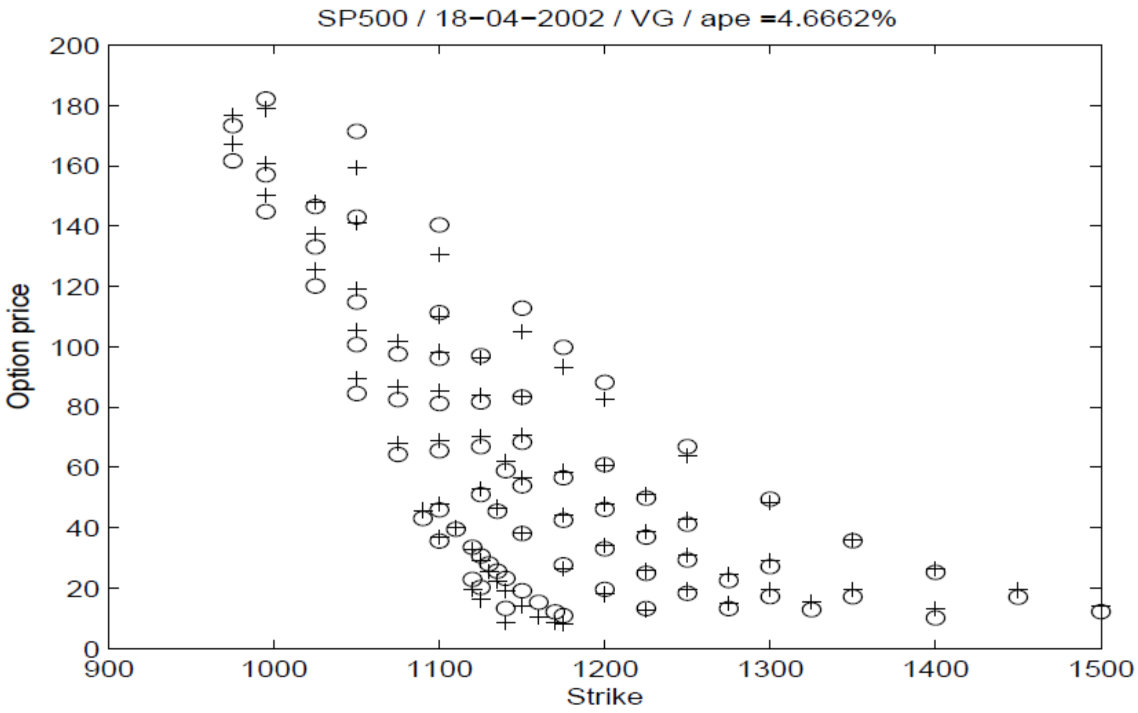
- implied volatilities as weights.

Calibration



- initial parameters: $(\sigma = 0.1)$
- optimal parameters: $(\sigma = 0.181)$

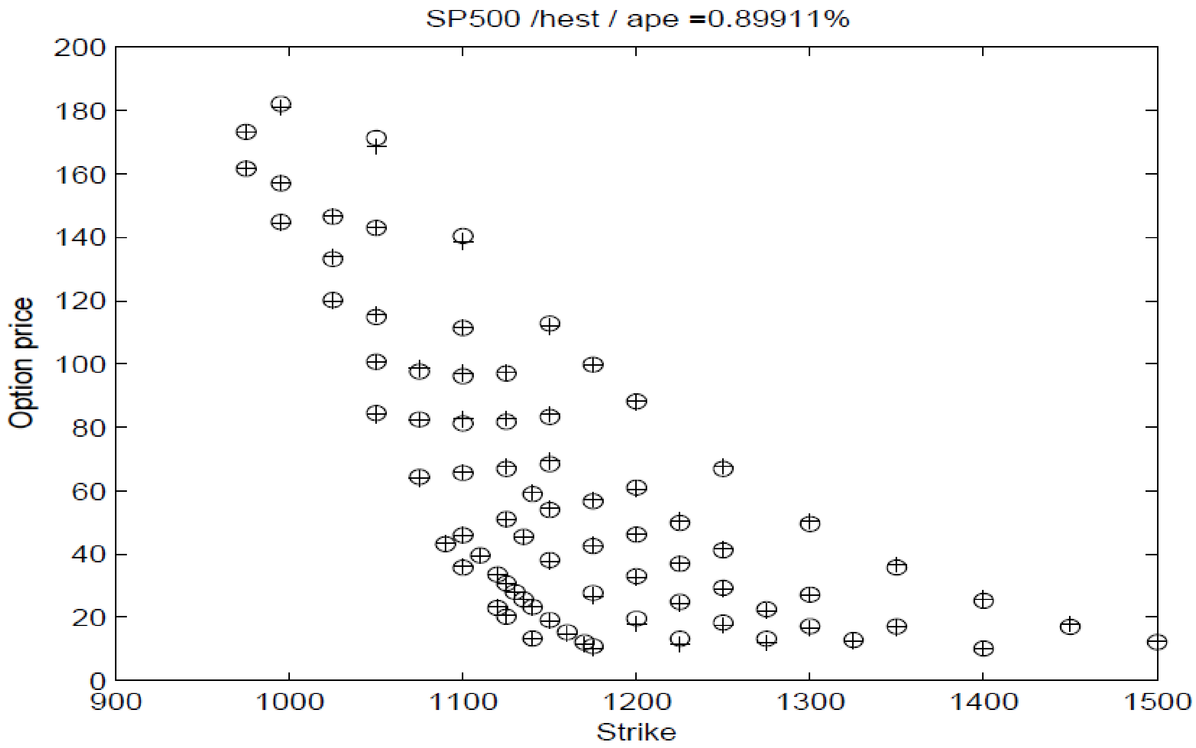
Calibration



- initial parameters: $(C = 1, \quad G = 5, \quad M = 5)$
- optimal parameters: $(C = 1.3574, \quad G = 5.8703, \quad M = 14.2699)$

Note : Better to calibrate on OTM Call and Puts than just on Calls

Calibration



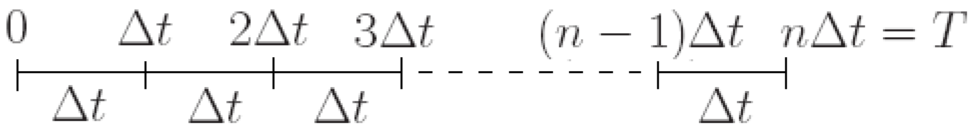
- initial parameters: $(v0 = 0.05, \quad \kappa = 0.5, \quad \eta = 0.05, \quad \lambda = 0.2, \quad \rho = -0.75)$
- optimal parameters: $(v0 = 0.0227, \kappa = 0.2478, \eta = 0.1827, \lambda = 0.3012, \rho = -0.7520)$

Monte Carlo Simulation

BASIC IDEA:

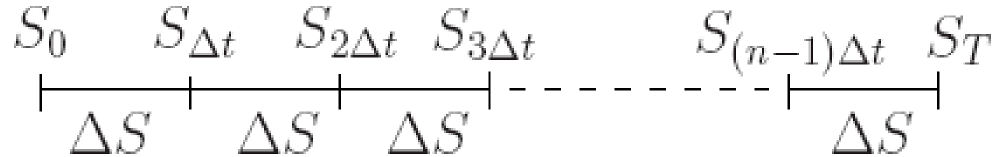
- – Each simulation consists of a series of steps and is one possible sequence of asset prices.
- – No decisions can be made between start and end of time period (will not work for American style options)
- – Calculation time depends upon the product, the number of steps and the number of simulations.
- – Typical number of simulations: 10.000-100.000
- – Therefore fairly slow to calculate.
- – Calculating greeks or American styles by MC very slow.

- Simple Example
  - – Divide whole time interval (T) into a number of periods (n) so Δt = T/n :



  - –Calculate asset values at the end of each period by picking at random from a N(0,1) distribution and using:

$$\frac{\Delta S}{S} = (r-q)\Delta t + \sigma\epsilon\sqrt{\Delta t}$$



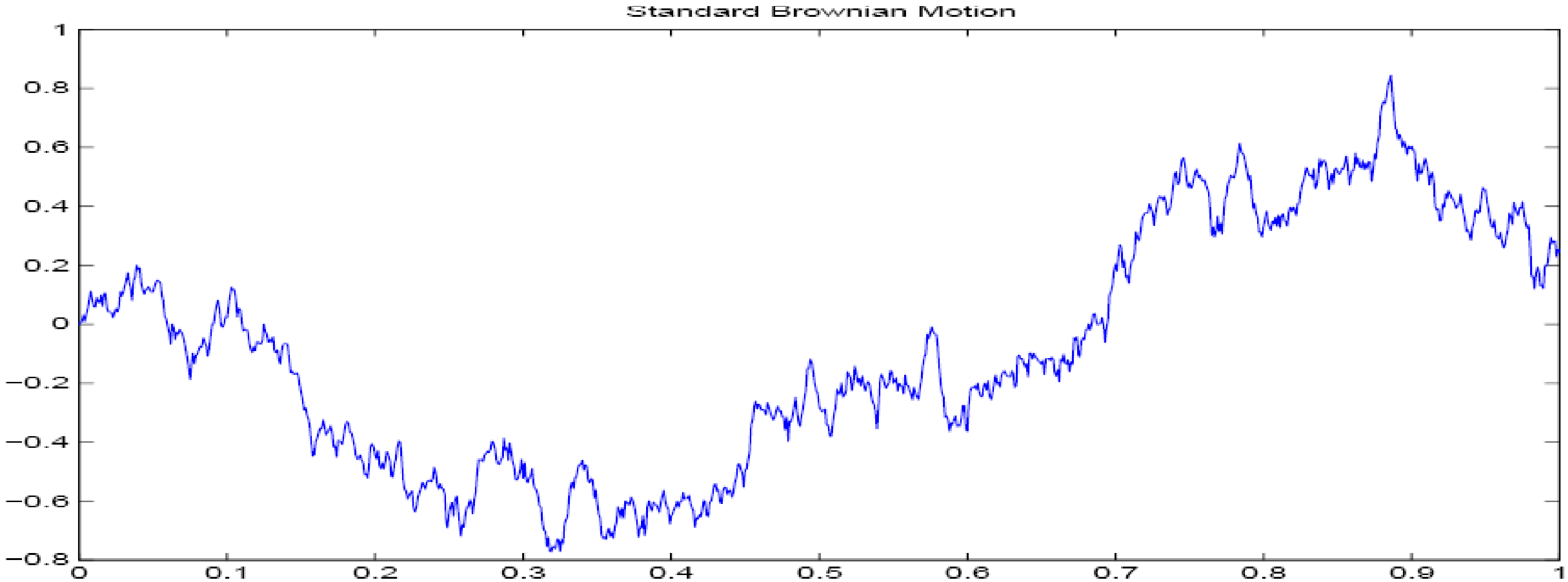
  - –Find the value of the asset or derivative at T and discount it back to today.

  - –Repeat many times, take the average and discount.

Standard Normal random number

- Simulation of a standard Brownian motion at the time points *{n∆t , n=0,1,2, …}*
- Euler Scheme:

$$W_0 = 0, W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{\Delta t}\varepsilon$$



- Simulation of a geometric Brownian motion at the time points *{$n\Delta t$, $n=0,1,2, \ldots$}*

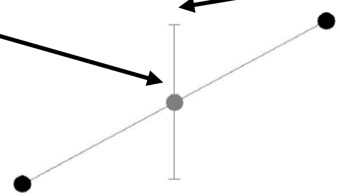
$$S_{n\Delta t} = S_{(n-1)\Delta t}\left(1 + \mu\Delta t + \sigma\varepsilon\sqrt{\Delta t}\,\right)$$

- Brownian Bridge:
  - important for path dependent options (barriers)
  - S(T) generated first (largest variance)
  - intermediate values dependent on past (smaller variance)

$$(X(t)|X(s)=x, X(u)=z) \sim N\left(\frac{u-t}{u-s}x+\frac{t-s}{u-s}z, \sqrt{\frac{(t-s)(u-t)}{u-s}}\sigma\right)$$

Conditional Mean

Conditional standard deviation



- Heston Stock : $\frac{dS_t}{S_t} = (r-q)dt + \sqrt{v_t}dW_t, \quad S_0 \geq 0$
- Variance: $dv_t = \kappa(\eta - v_t)\,dt + \lambda\sqrt{v_t}d\tilde{W}_t, \quad v_0 = \sigma_0^2 \geq 0,$
- Euler Scheme:

$$
\begin{aligned}
S_{t_{i+1}} &= S_{t_i}\left(1 + (r-q)\Delta t + \sqrt{v_{t_i}}\sqrt{\Delta t}\varepsilon_1\right) \\
v_{t_{i+1}} &= v_{t_i} + \kappa(\eta - v_{t_i})\Delta t + \lambda\sqrt{v_{t_i}}\sqrt{\Delta t}\varepsilon_2
\end{aligned}
$$

Correlated Standard Normal random numbers

- Stock : $\frac{dS_t}{S_t} = (r-q)dt + \sqrt{v_t}dW_t, \quad S_0 \geq 0$
- Variance: $dv_t = \kappa\left(\eta - v_t\right)dt + \lambda\sqrt{v_t}d\tilde{W}_t, \quad v_0 = \sigma_0^2 \geq 0,$
- Milstein Scheme:

$$S_{t_{i+1}} = S_{t_i}\left(1 + (r-q)\Delta t + \sqrt{v_{t_i}}\sqrt{\Delta t}\varepsilon_1\right)$$

$$v_{t_{i+1}} = v_{t_i} + (\kappa(\eta - v_{t_i}) - \lambda^2/4)\Delta t + \lambda\sqrt{v_{t_i}}\sqrt{\Delta t}\varepsilon_2 + \lambda^2\Delta t(\varepsilon_2)^2/4$$

Correlated N(0,1) Numbers

- The Brownian motion driving the vol and the stock in the Heston model are correlated.

- For the simulation we thus need correlated standard Normals (correlation ρ).

$$\varepsilon_1 = \varepsilon$$
$$\varepsilon_2 = \rho\varepsilon + \sqrt{1-\rho^2}\varepsilon^*$$

Independent Standard Normal random numbers

Correlated Standard Normal random numbers

Avoiding negative variance

• because of the discritization (under Euler as well as under Milstein scheme), the variance can become negative under a simulation.

Possible Solutions (all wrong, but that's life):

- **absorption:** if negative set equal to zero
- **reflection:** if negative, take absolute value
- **trunctation:**

Partial trunctation

$$
\begin{aligned}
v^*_{t_{i+1}} &= v^*_{t_i} + \kappa(\eta - v^*_{t_i})\Delta t + \lambda\sqrt{v_{t_i}}\sqrt{\Delta t}\varepsilon_2 \\
v_{t_{i+1}} &= \max(0, v^*_{t_{i+1}})
\end{aligned}
$$

Full trunctation

$$
\begin{aligned}
v^*_{t_{i+1}} &= v^*_{t_i} + \kappa(\eta - v_{t_i})\Delta t + \lambda\sqrt{v_{t_i}}\sqrt{\Delta t}\varepsilon_2 \\
v_{t_{i+1}} &= \max(0, v^*_{t_{i+1}})
\end{aligned}
$$

Model and Calibartion Risk

Model Risk

- We show that several advanced equity option models incorporating stochastic volatility can be calibrated very nicely to a realistic option surface.
- All these models are hence capable of accurately describing the marginal distribution of stock prices or indices and hence lead to almost identical European vanilla option prices.
- As such, we can hardly discriminate between the different processes on the basis of their smile-conform pricing characteristics.
- However, due to the different structure in path-behaviour between these models, the exotics prices can vary significantly.

Model Risk

- 7 models :
  - Heston Stochastic Volatility Model (HEST)
  - Heston Stochastic Volatility Model with Jumps (HESJ)
  - The Barndorff-Nielsen-Shephard Model (BNS)
  - Lévy Jump Models with Stochastic Volatility
    - VG with Integrated Gamma-OU volatility
    - NIG with Integrated Gamma-OU volatility
    - VG with Integrated CIR volatility
    - NIG with Integrated CIR volatility
- We can include many more …

Other Models

- **The other models in a nutshell:**
  - **Heston with Jumps (Bates)**
    - Heston + jumps arriving at a constant intensity
    - Jump sizes : log-normally distributed
    - 3 new parameters : jump intensity, mean jump size and variance of jump size

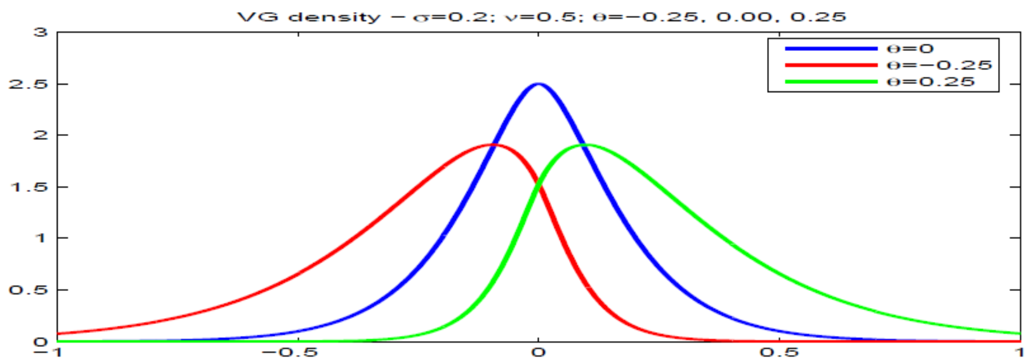
$$\log(1 + J_t) \sim \text{Normal}\left(\log(1 + \mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2\right);$$

  - **Barndorff-Nielsen-Model**
    - Similar to Heston, but different variance process
    - In total 5 parameters
    - Vol process has only up-jumps and then decays

$$\mathrm{d}\sigma_t^2 = -\lambda\sigma_t^2\mathrm{d}t + \mathrm{d}z_{\lambda t},$$

Other Models

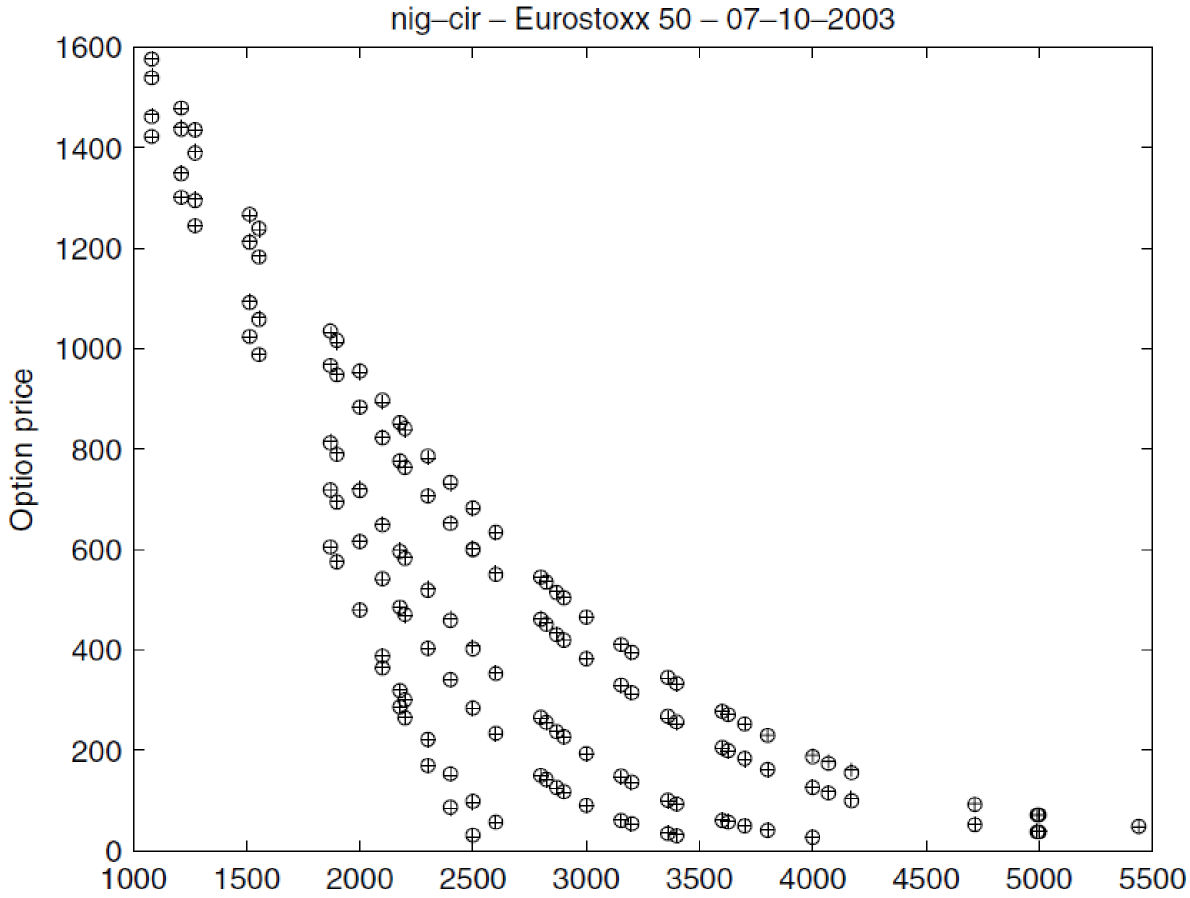
- **The other models in a nutshell:**
  - **Lévy Models**
    - Generalizations of the Brownian motion for other return distributions
    - More fat - tail behavior
    - Turn out to be pure jump models
    - NIG and VG have 3 parameters
    - Used in combination with CIR or OU vol model
    - CIR vol model is used in Heston
    - Gamma-OU vol model is used in BNS
    - Vol is incorporated via technique of time change at stochastic business time.



100

Perfect Calibration

- Calibration is seemingly almost perfect (see also next part).



Perfect Calibration

- Calibration is seemingly almost perfect (see also next part).

Model:	rmse	ape	aae	arpe
HEST	3.0281	0.0048	2.4264	0.0174
HESJ	2.8101	0.0045	2.2469	0.0126
BN-S	3.5156	0.0056	2.8194	0.0221
VG-CIR	2.3823	0.0038	1.9337	0.0106
VG-OUΓ	3.4351	0.0056	2.8238	0.0190
NIG-CIR	2.3485	0.0038	1.9194	0.0099
NIG-OUΓ	3.2737	0.0054	2.7385	0.0175

$$\mathrm{RMSE} = \sqrt{\sum_{j=1}^{N} \frac{(P_j - \hat{P}_j)^2}{N}} \qquad \mathrm{APE} = \frac{1}{\mathrm{mean}_j \hat{P}_j} \sum_{j=1}^{N} \frac{\left|P_j - \hat{P}_j\right|}{N} \qquad \mathrm{ARPE} = \frac{1}{N} \sum_{j=1}^{N} \frac{\left|P_j - \hat{P}_j\right|}{\hat{P}_j}$$

Market Price

Model Price

Optimal Parameters

- Parameters seem to make more or less sense:

HEST

$\sigma_0^2 = 0.0654,\ \kappa = 0.6067,\ \eta = 0.0707,\ \theta = 0.2928,\ \rho = -0.7571$

HESJ

$\sigma_0^2 = 0.0576,\ \kappa = 0.4963,\ \eta = 0.0650,\ \theta = 0.2286,\ \rho = -0.9900,$
$\mu_j = 0.1791,\ \sigma_j = 0.1346,\ \lambda = 0.1382$

BN-S

$\rho = -4.6750,\ \lambda = 0.5474,\ b = 18.6075,\ a = 0.6069,\ \sigma_0^2 = 0.0433$

VG-CIR

$C = 18.0968,\ G = 20.0276,\ M = 26.3971,\ \kappa = 1.2145,\ \eta = 0.5501,$
$\lambda = 1.7913,\ y_0 = 1$

VG-OUΓ

$C = 6.1610,\ G = 9.6443,\ M = 16.0260,\ \lambda = 1.6790,\ a = 0.3484,$
$b = 0.7664,\ y_0 = 1$

NIG-CIR

$\alpha = 16.1975,\ \beta = -3.1804,\ \delta = 1.0867,\ \kappa = 1.2101,\ \eta = 0.5507,$
$\lambda = 1.7864,\ y_0 = 1$

NIG-OUΓ

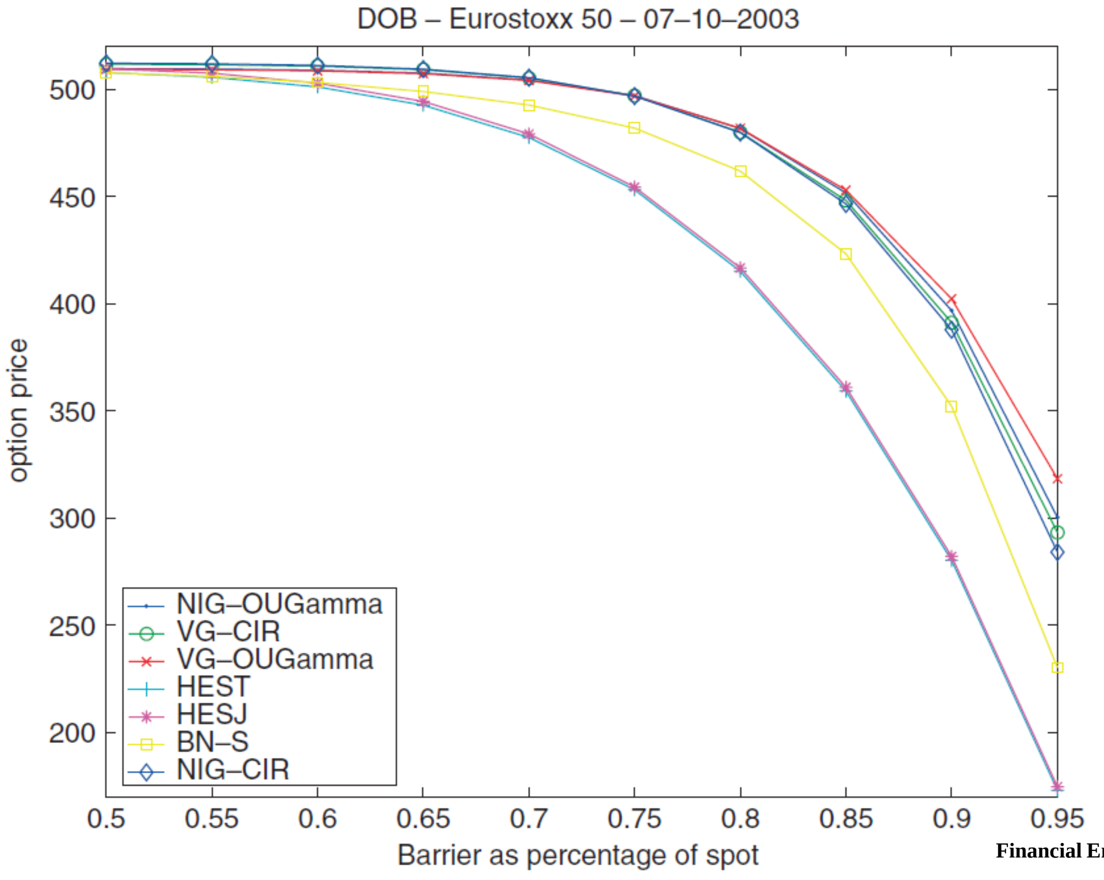
$\alpha = 8.8914,\ \beta = -3.1634,\ \delta = 0.6728,\ \lambda = 1.7478,\ a = 0.3442,$
$b = 0.7628,\ y_0 = 1$

Exotic Option Pricing

- Next, we are tempted to price (via Monte Carlo) a battery of exotic options.
  - Barriers : $DOB = e^{-rT} E_Q[(S_T - K)^+ 1(m_T^S > H)]$
  - Lookbacks : $LC = e^{-rT} E_Q[S_T - m_T^S].$
  - Cliquets : $\min\left(cap_{glob}, \max\left(floor_{glob}, \sum_{i=1}^{N} \min\left(cap_{loc}, \max\left(floor_{loc}, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)\right)\right)\right)$
- These exotics play a prominent role in Structured Products and their pricing is key.

Exotic Option Pricing

- We see huge difference for the exotics over the models.
- Note the vanillas are priced almost exactly the same.

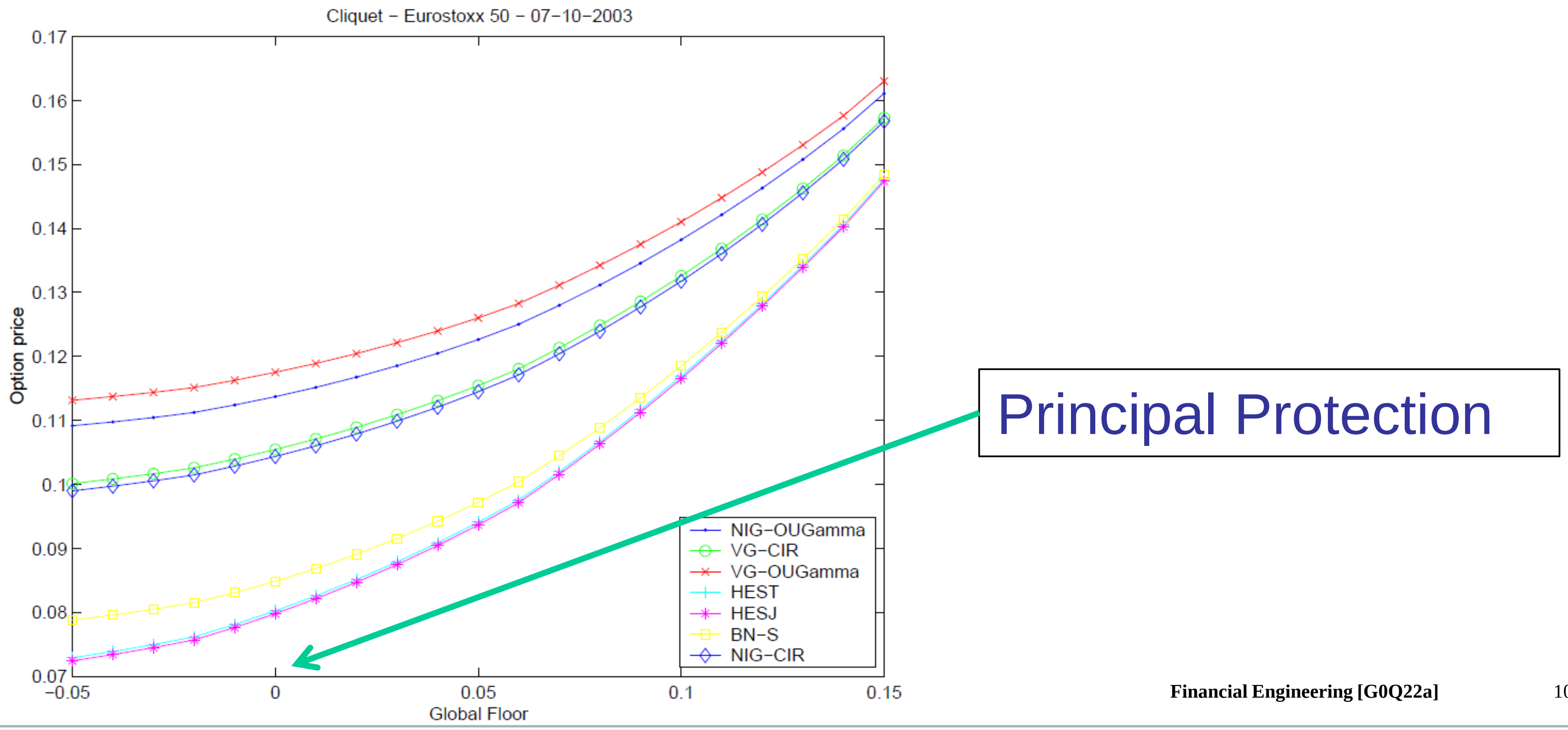


Exotic Option Pricing

- Lookback prices :

HEST	HESJ	BN-S	VG-CIR	VG-OUΓ	NIG-CIR	NIG-OUΓ
838.48	845.19	771.28	724.80	713.49	730.84	722.34

- Cliquets prices:



Model Risk Conclusions

- We have look at several models all reflecting stochastic vol and non-Gaussian returns, properties that are generally supported by empirical research.
- Although all models are almost perfectly calibrated and hence vanillas have the same prices under all models, exotic prices can differ dramatically.
- Vanillas determine the marginal distributions not the process.
- The underlying fine-grain properties of the process have an important impact on the path-dependent option prices.
- The impact of exotic price ranges is important for price setting of Structured Products.

Calibration Risk

- Note there are also choices and associated risks in the calibration exercise.
- Sometimes certain parameters can be fixed upfront on the basis of historical or other market data, or estimated via different statistical techniques.
- We can minimizing different objective functions.
- Again this can have an impact on exotic options.

Calibration Risk

- We perform our study under the HESTON model.

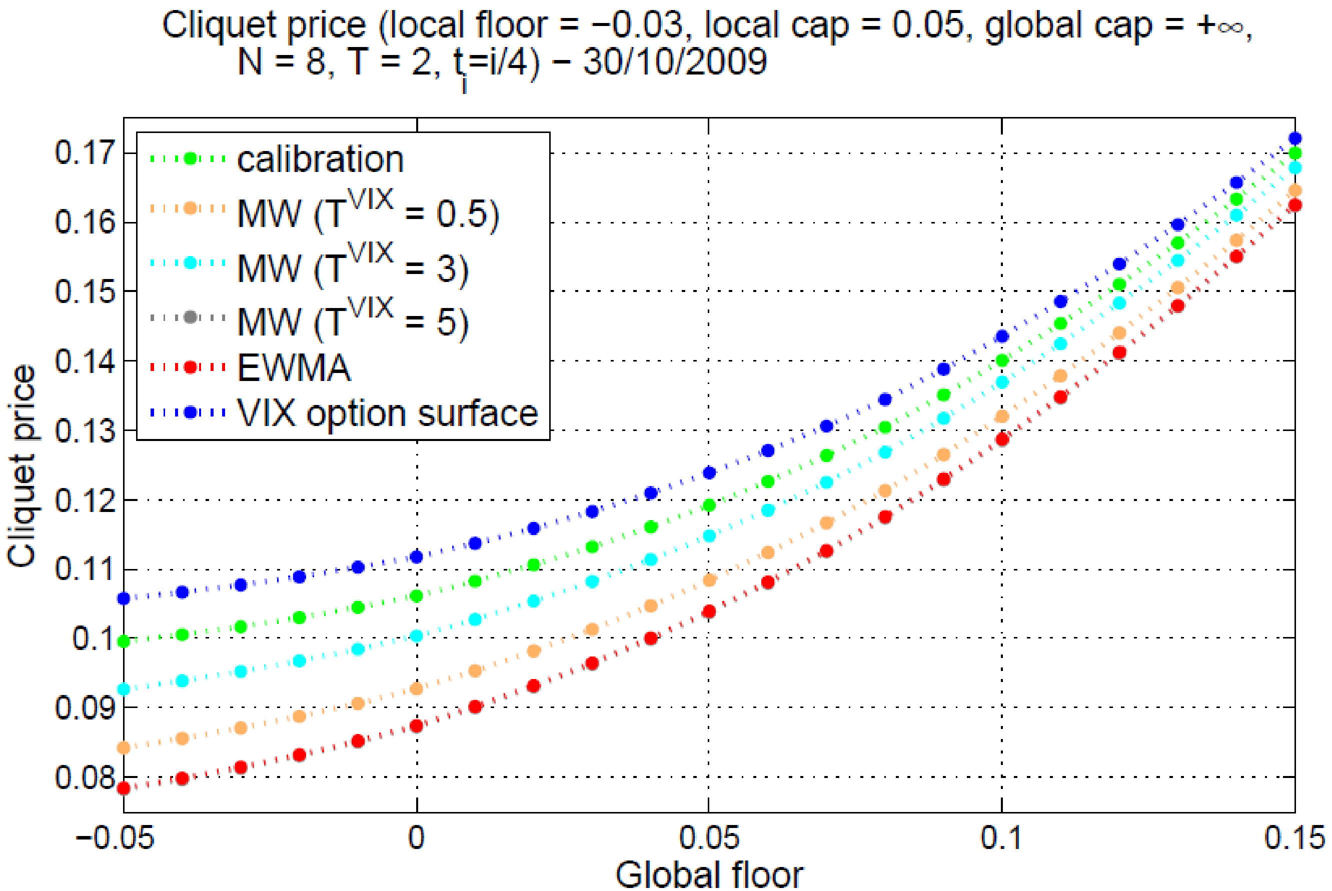
$$\frac{dS_t}{S_t} = (r-q)dt + \sqrt{v_t}dW_t, \quad S_0 \geq 0$$

$$dv_t = \kappa\left(\eta - v_t\right)dt + \lambda\sqrt{v_t}d\tilde{W}_t, \quad v_0 = \sigma_0^2 \geq 0,$$

- We make use of historical data that tries to identify the long run (historically implied) variance $\eta > 0$.
- Similarly, we try to identify using options on VIX the long run (market implied) variance $\eta > 0$.
- We also make us of market data (VIX) to fix the initial variance $v_0 > 0$.

Calibration Risk

- Impact of different models on Exotic Prices



Bid Ask Pricing

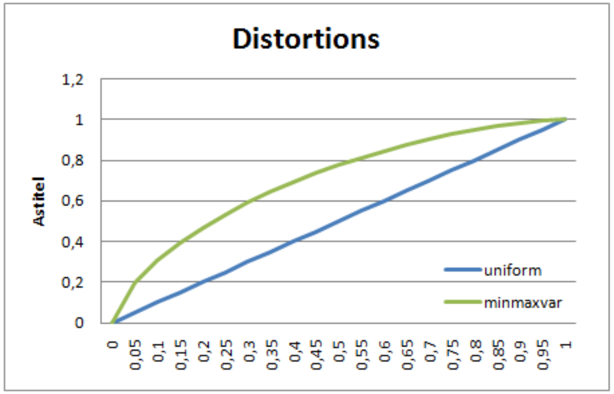
Bid-Ask Pricing

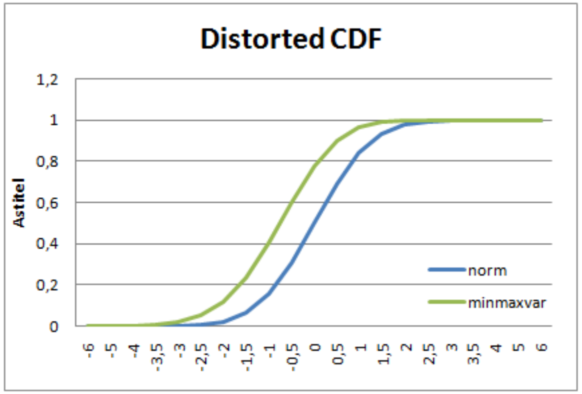
- We will make use of the minmaxvar distortion function:

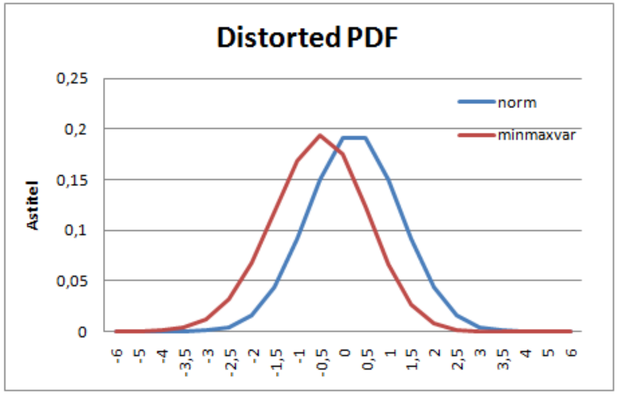
$$\Phi(u;\lambda) = 1 - \left(1 - u^{\frac{1}{1+\lambda}}\right)^{1+\lambda}$$

- We use non linear expectation to calculate (bid and ask) prices.
- The distorted expectation of a random variable with distribution function *F(x)* is defined

$$de(X;\lambda) = \mathrm{E}^{\lambda}[X] = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(G(x);\lambda).$$







Conic Finance

- The ask price of payoff *X* is determined as

$$ask(X) = -\exp(-rT)\mathrm{E}^{\lambda}[-X].$$

- The bid price of payoff X is determined as

$$bid(X) = \exp(-rT)\mathrm{E}^{\lambda}[X].$$

- Hence for the BID price we have put more weight on the down-side. For the ASK the upside has been receiving more weighted.

Conic Finance

- These formulas are derived by noting that the cash-flow of selling X at its ask price and buying *X* at its bid price is acceptable in the relevant market :

- We say that a risk *X* is acceptable if

$$E_Q[X] \geq 0 \text{ for all measures } Q \text{ in a convex set } \mathcal{M}.$$

Actually we test whether for many test-measures our cash-flow has a positive expectation.

- Operational cones were defined by Cherney and Madan and depend solely on the distribution function *G(x)* of *X* and a distortion function *Ψ*. One can show that we need to have that the distorted expectation is positive:

$$de(X;\lambda) = \mathrm{E}^{\lambda}[X] = \int_{-\infty}^{+\infty} x \mathrm{d}\Phi(G(x);\lambda).$$

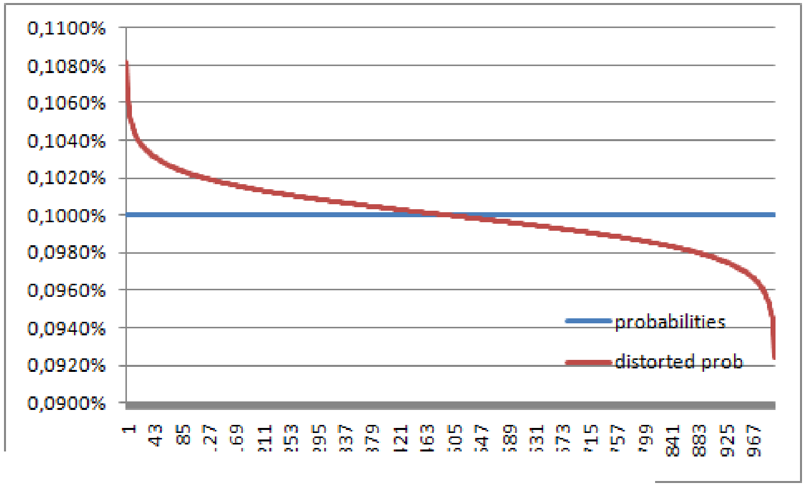
BID-ASK PRICING under Monte Carlo Simulation

MC:

$$\begin{aligned}\text{Option Price} &= \exp(-rT)E_Q[\text{Payoff}]\\ &\approx \exp(-rT)\frac{1}{N}\sum_{i=1}^{N} ith \text{ simulated payoff}\\ &= \exp(-rT)\frac{1}{N}\sum_{i=1}^{N} ith \text{ sorted simulated payoff}\end{aligned}$$

Equally weighted

DISTORTED UNIFORM (1/N) WEIGHTS:



Distorted weights

BID PRICE: $$\text{Bid Price} = \exp(-rT)\sum_{i=1}^{N} w_i \;\; ith \text{ sorted simulated payoff}$$

How does everything relate to each other ?

- **Model Risk** : different models – different Q measures – different exotics
- **Calibration Risk** : same model – different parameters/Q – different exotics

$$E_Q[X] \geq 0 \text{ for all measures } Q \text{ in a convex set } \mathcal{M}.$$

- **Acceptability** :

- Selling *X* at its ask price is acceptable : $ask(X) - X \in \mathcal{A}$

- Buying *X* at its bid price is acceptable : $X - bid(X) \in \mathcal{A}$

- Ask price is supremum of test-measure prices : $ask(X) = \sup_{Q \in \mathcal{M}} E_Q[X]$

- Bid price is infimum of test-measure prices: $bid(X) = \inf_{Q \in \mathcal{M}} E_Q[X]$

Conclusion Part I

- We have reviewed some exotic options and structured products.
- We have review some basic and more advanced models.
- We have discussed implementation details of FFT pricing.
- We have discussed calibration of a model on market data.
- We have reviewed some basics of Monte Carlo pricing.
- Model risk is omnipresent. Calibration risk within a model is also significant. Impact on exotic prices can be quite severe.
- All the above is related to the conic finance theory.