Comparison of Option Pricing Models

The difference between classical models like **Black-Scholes**, **Heston**, **Merton**, **Bates** and **Lévy-driven models** mainly lies in how they **model asset price dynamics**, especially regarding **volatility**, **jumps**, and **heavy tails** in financial returns.

Here's a structured comparison:

1. Black-Scholes Model

- **Dynamics**:
$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$
- **Assumptions**:
  - Constant volatility (σ)
  - Log-normal price distribution
  - No jumps
- **Limitations**:
  - Cannot capture volatility smile/skew
  - Misses leptokurtosis (fat tails)

2. Heston Model (Stochastic Volatility)

- **Dynamics**:
$$\begin{cases} dS_t = \mu S_t \, dt + \sqrt{v_t} S_t \, dW_t^S \\ dv_t = \kappa(\theta - v_t) \, dt + \xi \sqrt{v_t} \, dW_t^v \end{cases}$$
  with correlation ρ between W_t^S and W_t^v
- **Features**:
  - Stochastic volatility (mean-reverting)
  - Better fit for implied volatility surface
- **Limitations**:
  - No jumps
  - Still Gaussian-based

3. Merton Jump-Diffusion Model

- **Dynamics**:
$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t + J_t S_t \, dN_t$$
where $J_t \sim$ LogNormal and N_t is a Poisson process
- **Features**:
  - Adds normally distributed jumps to Black-Scholes
  - Explains return skewness and kurtosis
- **Limitations**:
  - Constant volatility
  - Fixed jump intensity and size

4. Bates Model (Heston + Jumps)

- **Dynamics** (Heston model + jump component like Merton):
$$dS_t = \mu S_t \, dt + \sqrt{v_t} S_t \, dW_t + J_t S_t \, dN_t$$
- **Features**:
  - Stochastic volatility and jumps
  - Better fit for option pricing
- **Limitations**:
  - Calibration is complex
  - Still uses fixed jump distributions

These are Diffusion + Jump models: relatively structured and parametric.

5. Lévy-driven Models (Generalization)

Lévy Processes:

- Allow for jumps of infinite activity (e.g., many small jumps)
- Include Brownian motion as a special case
- Can model heavy tails, skewness, and non-Gaussian features

Examples of Lévy models:

Model	Description
Variance Gamma (VG)	Pure-jump, finite variation
CGMY (Carr–Geman–Madan–Yor)	Infinite activity and variation
Normal Inverse Gaussian (NIG)	Heavy-tailed, skewed returns
Tempered Stable	General class with tempered jumps

Dynamics (abstract form):

$$S_t = S_0 \exp(L_t)$$

where L_t is a Lévy process with triplet (μ, σ, ν), and ν is the Lévy measure governing the jump distribution.

Advantages:

- Very flexible: can model real-world return distributions
- Rich class of dynamics: fat tails, skew, kurtosis

Drawbacks:

- Complex to calibrate
- Analytical pricing often not available $\rightarrow$ requires Fourier methods or Monte Carlo simulation

Summary Table

Feature	Black-Scholes	Heston	Merton	Bates	Lévy Models
Stochastic Volatility	No	Yes	No	Yes	Yes (some)
Jumps	No	No	Yes	Yes	Yes (general)
Heavy Tails	No	No	Yes	Yes	Yes
Infinite Activity	No	No	No	No	Yes
Calibration Ease	Easy	Moderate	Moderate	Difficult	Difficult
Analytic Pricing	Yes	Semi-closed	Semi-closed	Semi-closed	No (usually)

Conceptual Difference

- Black-Scholes, Heston, Merton, and Bates are finite-parameter models based on Brownian motion with or without jumps.

- Lévy models generalize this by allowing non-Gaussian, infinitely divisible processes, giving better realism at the cost of tractability.

Deep Dive into Option Pricing Models

Let's dive deeply into the **Black-Scholes, Heston, Merton, Bates**, and **Lévy-driven models** (with a focus on the **Variance Gamma model**) — their mathematical structures, financial intuition, calibration aspects, and real-world applications.

1. Black-Scholes Model (1973)

Stochastic Differential Equation (SDE)

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Where:

- S_t: stock price
- μ: drift (expected return)
- σ: constant volatility
- dW_t: standard Brownian motion

Assumptions

- Log-normal returns
- No arbitrage, frictionless market
- Constant risk-free rate and volatility
- No jumps

Option Pricing Formula (Call)

$$C(S, K, T) = S_0 N(d_1) - K e^{-rT} N(d_2)$$

Where:

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Applications

- Used for vanilla options
- Serves as a benchmark for model comparison

Limitations

- Cannot capture implied volatility smile/skew
- Ignores jumps and stochastic volatility

2. Heston Model (1993) – Stochastic Volatility

Dynamics

$$\begin{cases} dS_t = \mu S_t \, dt + \sqrt{v_t} S_t \, dW_t^S \\ dv_t = \kappa(\theta - v_t) \, dt + \xi \sqrt{v_t} \, dW_t^v \end{cases}$$

With:

$$\rho = \mathrm{corr}(dW_t^S, dW_t^v)$$

Parameters

- κ: speed of mean reversion
- θ: long-term variance
- ξ: volatility of volatility
- v_0: initial variance
- ρ: correlation

Benefits

- Captures volatility smile
- More realistic dynamics

Calibration

- Fit to implied volatility surface
- Closed-form pricing via Fourier transforms (e.g., COS, FFT)

3. Merton Jump-Diffusion Model (1976)

Dynamics

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t + J_t S_t \, dN_t$$

Where:

- N_t: Poisson process with intensity λ
- $J_t \sim \ln(1 + Y)$, with $Y \sim \mathcal{N}(\mu_J, \sigma_J^2)$

Benefits

- Models sudden price changes
- Explains excess kurtosis (fat tails)

Calibration

- Estimate jump frequency and size from high-frequency data
- Fourier methods used for pricing (e.g., Carr-Madan)

4. Bates Model (1996) – Heston + Jumps

Dynamics

Same as Heston model, with jumps:

$$dS_t = \mu S_t \, dt + \sqrt{v_t} S_t \, dW_t + J_t S_t \, dN_t$$

Benefits

- Combines stochastic volatility and jumps
- Accurately fits skew and kurtosis

Calibration

- 6+ parameters; requires optimization over implied volatilities
- Fourier-based methods often used (e.g., characteristic function pricing)

5. Lévy-Driven Models (Generalization)

Lévy Process

A stochastic process with stationary and independent increments. Includes:

- Brownian motion
- Poisson jumps
- Infinite activity processes (infinitely many small jumps)

General Asset Dynamics

$$S_t = S_0 \exp(X_t), \quad X_t \sim \text{Lévy Process}$$

Characteristic Function

$$\phi_{X_t}(u) = \mathbb{E}[e^{iuX_t}] = e^{t\psi(u)}$$

where $\psi(u)$ is the Lévy exponent, which fully characterizes the process.

6. Variance Gamma (VG) Model — A Lévy Model

Construction

A Brownian motion with drift, evaluated at a Gamma process (random clock):

$$X_t = \theta G_t + \sigma W_{G_t}, \quad G_t \sim \text{Gamma}(t/\nu, \nu)$$

Where:

- θ: drift
- σ: volatility
- ν: variance of the Gamma process (controls jump activity)

Properties

- Pure-jump model (no Brownian motion in calendar time)
- Infinite activity, finite variation
- Heavy tails and skewness
- No diffusion term

Advantages

- Better fit for asset return distributions
- Captures skewness and kurtosis
- Closed-form characteristic function

Option Pricing

- Uses Fourier transform methods (Carr–Madan, COS)
- Also applicable to Monte Carlo simulation

Calibration

- Fit characteristic function to option prices
- Faster than models with stochastic volatility

Use Cases

- Equity and index options
- Exotic derivatives
- Scenario analysis with jumps

Comparison: VG vs Classical Models

Feature	Black-Scholes	Heston	Merton	Bates	VG Model
Volatility	Constant	Stochastic	Constant	Stochastic	Constant
Jumps	No	No	Yes	Yes	Yes (infinite)
Skewness/Kurtosis	No	Partially	Yes	Yes	Yes
Analytical Pricing	Yes	Semi	Semi	Semi	Semi (FFT)
Infinite Activity	No	No	No	No	Yes

Summary

- **Black-Scholes**: Baseline model, fast, but unrealistic.
- **Heston**: Stochastic volatility, good fit for volatility surface.
- **Merton**: Adds jumps to capture fat tails.
- **Bates**: Heston + Jumps = very flexible.

- **Lévy models**: Most general class — includes infinite activity and non-Gaussian returns.
- **Variance Gamma**: Popular Lévy model — realistic returns, fast pricing via FFT.

Are All Models Lévy-Based?

Great question — and a very insightful one.

Short Answer

Not all models fall under Lévy models — but some do. Let's clarify this precisely.

What is a Lévy Process?

A **Lévy process** is a stochastic process with the following properties:

- **Stationary** and **independent increments**
- **Càdlàg paths** (right-continuous with left limits)
- Starts at zero: $X_0 = 0$

The most famous example is **Brownian motion**, but Lévy processes include a much richer class:

Type	Description
Brownian motion	Continuous paths
Poisson process	Discrete jumps
Compound Poisson	Discrete jumps with size distribution
Variance Gamma	Infinite small jumps, no diffusion
CGMY, NIG, Stable	Heavy-tailed, possibly infinite activity

Are These Models Lévy-Based?

Model	Uses a Lévy Process?	Explanation
Black-Scholes	Yes	Based on Brownian motion (a Lévy process)
Merton	Yes	Brownian motion + compound Poisson (both Lévy)
Variance Gamma	Yes	Pure Lévy model (no Brownian motion in calendar time)
Heston	No	Volatility follows CIR process (not Lévy — no independent
Bates	No	Heston + jumps: still has stochastic volatility, not Lévy

Heston No Volatility follows CIR process (not Lévy — no independent increments)

Examples

Lévy Models:

- Black-Scholes (trivially)
- Merton
- Variance Gamma (VG)
- CGMY, NIG, etc.

Not Lévy Models:

- Heston
- Bates
- SABR
- Any model with stochastic volatility from mean-reverting processes like CIR

Why This Matters

- **Lévy models** offer great **mathematical tractability** and efficient **Fourier pricing** thanks to their known characteristic functions.
- **Non-Lévy models** (like Heston) provide richer dynamics for volatility but often require solving PDEs or use semi-analytical pricing.

Takeaway

- Lévy models generalize Black-Scholes by introducing jumps, skewness, and heavy tails.
- However, not all sophisticated models — such as Heston — are Lévy-based. They introduce new sources of randomness (e.g., stochastic volatility) that violate the Lévy process structure.

CIR and NIG-CIR Models in Finance

Let's now dive deep into the **CIR model**, a key stochastic process in finance, and then explore the **NIG-CIR model**, which combines heavy-tailed Lévy processes with mean-reverting stochastic volatility.

1. Cox-Ingersoll-Ross (CIR) Model

Original Use

Developed in 1985 for modeling **interest rates** in the term structure.

Stochastic Differential Equation (CIR Process)

$$dr_t = \kappa(\theta - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t$$

Where:

- r_t: instantaneous short rate (or volatility)
- $\kappa > 0$: mean reversion speed
- $\theta > 0$: long-run mean level
- $\sigma > 0$: volatility of volatility
- W_t: Brownian motion

Key Features

- **Mean-reverting**: $r_t \to \theta$ over time
- **Non-negative**: process remains positive if $2\kappa\theta \geq \sigma^2$ (Feller condition)
- **Stochastic**: randomness governs volatility/rate evolution

Applications in Finance

Use Case	How CIR is Used
Interest Rate Modeling	Foundation for Vasicek/CIR term structure models
Stochastic Volatility	Variance process in Heston model
Credit Risk	Modeling default intensity

Analytical Properties

- Closed-form solution for zero-coupon bond prices
- Distribution of r_t is **noncentral chi-squared**
- Moment-generating function exists (used in characteristic function methods)

2. NIG-CIR Model

What is it?

A hybrid model where:

- **Returns** follow a **Normal Inverse Gaussian (NIG)** distribution
- **Volatility** follows a CIR process

Structure

Let the asset price be given by:

$$S_t = S_0 \exp\left(\int_0^t \mu_s \, ds + \int_0^t \sqrt{v_s} \, dL_s\right)$$

Where:

- L_s is an NIG Lévy process
- v_s evolves via a CIR process:

$$dv_s = \kappa(\theta - v_s) \, dt + \sigma\sqrt{v_s} \, dW_t$$

Interpretation: returns are NIG-distributed, and the **scale of shocks** is stochastic via v_s.

Why NIG-CIR?

- NIG allows jumps, skewness, and heavy tails
- CIR enables time-varying volatility (volatility clustering)
- Combined model captures:
  - Heavy tails
  - Volatility clustering
  - Skewness
  - Mean-reversion in volatility

Advantages in Finance

Feature	NIG-CIR Benefit
Realistic Returns	NIG captures empirical distributions well
Volatility Clustering	CIR mimics GARCH-like patterns
Option Pricing	Captures skew/smile in implied volatility
Risk Management	Models tail risk and dynamic volatility

Calibration

- NIG parameters: estimated from return data using MLE or characteristic function fitting
- CIR parameters: calibrated from realized volatility or VIX-type proxies
- Joint calibration often required using historical + option data

Applications

- Equity options (smile and jump risk modeling)
- Commodity markets (spot returns with volatility structure)
- Credit risk (time-varying jump intensities)

Comparison with Heston and VG

Feature	Heston	Variance Gamma	NIG-CIR
Returns	Gaussian	Heavy-tailed	Heavy-tailed (NIG)
Volatility	CIR	Constant	CIR (stochastic)
Jumps	No	Yes	Yes
Infinite Activity	No	Yes	Yes
Skew & Kurtosis	Limited	Yes	Yes
Vol Clustering	Yes	No	Yes

Summary

CIR

- Models non-negative, mean-reverting rates or volatility
- Integral to term structure and stochastic volatility models

NIG-CIR

- Combines Lévy-driven returns with stochastic volatility
- Reflects real market features like jumps, clustering, skew, and kurtosis
- Suitable for pricing, risk, and dynamic modeling tasks

Do Lévy Models Have Volatility Clustering?

Short Answer

*Standard Lévy models do **not** exhibit volatility clustering.*

Why?

Lévy processes such as Variance Gamma (VG), Normal Inverse Gaussian (NIG), and CGMY have the following properties:

- **Independent and stationary increments**
- No memory: past returns do not affect future variability
- Constant volatility (dispersion over time)

Consequence

- Lévy models capture heavy tails and jumps (explaining fat-tailed return distributions)
- They **fail to capture volatility clustering**: the phenomenon where large price changes tend to be followed by more large changes, and small ones by small ones

This behavior — volatility clustering — is a **stylized fact** of financial markets.

Stylized Fact: Volatility Clustering

Empirical features observed in real markets:

- Autocorrelation of returns is approximately zero
- Autocorrelation of squared or absolute returns is significantly positive

This is what models like GARCH, Heston, or CIR-based processes are specifically designed to reproduce.

Limitations of Pure Lévy Models

Property	Pure Lévy Models
Jumps / Skew / Kurtosis	Yes — Captured
Volatility Clustering	No — Not captured
Memory / Dependence	No — Independent increments
Implied Volatility Term Structure	Limited realism

Solution: Add Stochastic Volatility

Hybrid Models (Lévy + Stochastic Volatility)

Model	Volatility Clustering	Infinite Activity	Heavy Tails
Heston	Yes	No	Partially
Variance Gamma	No	Yes	Yes
NIG-CIR	Yes (via CIR)	Yes	Yes
VGSA (VG + Stochastic Arrival)	Yes	Yes	Yes

These hybrid models combine:

- **Lévy structure** for heavy tails, jumps, and asymmetry
- **Stochastic volatility** to replicate volatility clustering

Examples include:

- NIG-CIR
- VGSA (Variance Gamma with Stochastic Arrival times)
- Stochastic Volatility Jump models (SVJ)

Takeaway

Statement	True?
"All Lévy models show heavy tails"	True
"Lévy models explain volatility clustering"	False
"Lévy models can be extended to include volatility clustering"	True

Variance Gamma with Stochastic Arrival (VGSA) Model

The **VGSA model** (Variance Gamma with Stochastic Arrival) is one of the most sophisticated and practical Lévy–stochastic hybrid models in quantitative finance.

Full Name

Variance Gamma with Stochastic Arrival process.

Core Idea

The VGSA model enhances the Variance Gamma (VG) model by letting the **time change (subordinator)** evolve **stochastically** rather than deterministically. This introduces **volatility clustering** into an otherwise memoryless process.

1. Model Construction

Step-by-Step Construction

The original VG process is:

$$X_t^{VG} = \theta G_t + \sigma W_{G_t}$$

Where:

- $G_t \sim \text{Gamma}(t/\nu, \nu)$: deterministic Gamma subordinator
- θ: drift of the Brownian component
- σ: volatility
- ν: variance parameter of the Gamma time change

In the VGSA model, we replace G_t with a **stochastic clock** Y_t governed by a **CIR process**:

$$dY_t = \kappa(\theta_Y - Y_t)dt + \eta\sqrt{Y_t}dW_t^Y$$

Then, the VGSA process becomes:

$$X_t^{VGSA} = \theta G_{Y_t} + \sigma W_{G_{Y_t}}$$

So G_{Y_t} is a Gamma process evaluated at stochastic time Y_t.

2. Model Properties

Feature	VGSA Model Explanation
Infinite Activity	Yes — from the VG core
Heavy Tails	Yes — VG handles kurtosis well
Skewness	Yes — via θ
Volatility Clustering	Yes — CIR subordinator adds persistence
Mean Reversion	Yes — CIR dynamics of Y_t
Autocorrelated Volatility	Yes — via stochastic time change
Finite Moments	Yes — all moments are finite
Diffusion Term	No — pure jump process

3. Mathematical Summary

- $X_t = \theta G_{Y_t} + \sigma W_{G_{Y_t}}$
- Y_t follows:

$$dY_t = \kappa(\theta_Y - Y_t)\,dt + \eta\sqrt{Y_t}\,dW_t^Y$$

- Gamma subordinator: $G_s \sim \text{Gamma}(s/\nu, \nu)$

The VGSA model is a **Lévy process with stochastic volatility**, combining a jump-driven return process with a CIR-controlled stochastic clock.

4. Use in Finance

Application Area	VGSA Advantage
Option Pricing	Captures implied volatility smile, skew, and term structure
Volatility Forecasting	Models volatility clustering better than VG alone
Risk Management	Captures tail risk and stochastic dynamics
Commodity Markets	Handles erratic, clustered volatility
Credit Derivatives	Useful for intensity modeling of defaults

5. Calibration

- Calibrate VG parameters: θ, σ, ν
- Calibrate CIR parameters: κ, θ_Y, η
- Estimation methods:
  - Implied volatility surfaces
  - Characteristic function fitting

– MLE or GMM from time series

Pricing methods:

- Fourier transform methods (Carr–Madan, COS)
- Monte Carlo simulation (more involved due to double stochasticity)

6. Comparison With Other Models

Model	Jumps	Heavy Tails	Vol Clustering	Skew	Infinite Activity
Black-Scholes	No	No	No	No	No
Heston	No	Partial	Yes	Partial	No
Variance Gamma	Yes	Yes	No	Yes	Yes
VGSA	Yes	Yes	Yes	Yes	Yes
NIG-CIR	Yes	Yes	Yes	Yes	Yes

7. Intuition

Think of VGSA as:

> "The market evolves in **business time**, and that business time itself is **random and mean-reverting**."

- When Y_t is high, market evolves faster $\rightarrow$ higher volatility
- When Y_t is low, market evolves slower $\rightarrow$ lower volatility

This mechanism introduces **volatility clustering**, while preserving the **heavy tails** and **jumps** from the VG model.

8. Summary

Feature	VGSA Highlights
Core Idea	VG model with stochastic clock (via CIR)
Strengths	Infinite activity, skew, clustering, realism
Volatility	Varies over time via CIR subordinator
Use Cases	Options, commodities, equity risk
Drawbacks	Calibration and simulation complexity