Fast Fourier Transform for Pricing Vanilla options

The Fast Fourier Transform (FFT) technique introduced by Carr and Madan is a powerful numerical method for option pricing that leverages the Fourier transform to efficiently compute option prices. This method is particularly useful for pricing options when the underlying asset follows a stochastic process that is not easily tractable with traditional methods but for which there is a characteristic function available.

Fourier transform basics

1. **Fourier Transform Basics**
   - **Fourier Transform**: Transforms a function $f(x)$ into its frequency domain representation.

$$\mathcal{F}\{f(x)\}(u) = \int_{-\infty}^{\infty} f(x)e^{-iux}\, dx$$

   - **Inverse Fourier Transform**: Transforms back from the frequency domain to the time domain.

$$\mathcal{F}^{-1}\{F(u)\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)e^{iux}\, du$$

2. **Option Pricing with FFT**

   To price options using the FFT, Carr and Madan introduced a technique based on the characteristic function of the log-return distribution of the underlying asset.

   - **Characteristic Function**: For a stochastic process $X(t)$ (where $X(t) = \log(S(t))$), the characteristic function $\phi(u)$ is defined as:

$$\phi(u,t) = \mathbb{E}[e^{iuX(t)}]$$

   For log-normal processes, this is often derived from the underlying stochastic differential equation. If the density function of $X(t)$ is known and given by $q(x,t)$, the characteristic function can be defined as an integral:

$$\phi(u,t) = \int_{-\infty}^{\infty} \exp(\mathrm{i}ux)q(x,t)dx$$

3. **Carr-Madan Formula**

   The key idea is to use the characteristic function to price options through the inverse Fourier transform. For a European call option, the price can be computed as:

$$EC(K,T) = \frac{\exp(-\alpha\ln(K))}{\pi}Re(\int_{0}^{\infty} \exp(-\mathrm{i}v\ln(K))\varrho(v)dv)$$

   with

$$\varrho(v) = \frac{\exp(-rT)\cdot E\left[\exp(\mathrm{i}(v-(\alpha+1)\mathrm{i})\ln(S(T))\right]}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v} = \frac{\exp(-rT)\cdot\phi(v-(\alpha+1)\mathrm{i},t)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}$$

   where α is a positive constant chosen to ensure the integrand decays rapidly as v approaches infinity.

4. **Implementation Steps**
    - **Step 1**: Define the characteristic function $\phi(u)$ for the stochastic process.
    - **Step 2**: Transform the integral into a discrete sum using FFT.
    - **Step 3**: Compute the discrete Fourier transform of the integrand.
    - **Step 4**: Use the inverse FFT to recover the option price from the transformed data.

The Carr-Madan formula

The Carr-Madan formula is a method for pricing European options using the Fourier transform. The proof of the Carr-Madan formula is not so-difficult. The derivation involves several steps, including transforming the option pricing problem into the frequency domain using the characteristic function of the underlying asset's log-returns. Here's a detailed derivation of the Carr-Madan formula:

1. Option Pricing Framework

Consider a European call option with strike price K and maturity T. Let $S(t)$ be the price of the underlying asset at time t. We want to find the price of the option at time $t = 0$.

2. Risk-Neutral Pricing Formula

In the risk-neutral measure, the price of a European call option $EC(K, T)$ is given by:

$$EC(K,T) = \exp(-rT)\mathbb{E}[(S(T) - K)^+]$$

where r is the risk-free interest rate, and $(x)^+ = \max(x, 0)$ represents the payoff of the call option.

3. Log-Return Transformation

Define $k = \log(K)$ and $X(T) = \log(S(T))$, the log-strike and log-return of the asset price respectively. We denote the density function of $X(T)$ as $q(x, T)$. The call option payoff can be rewritten in terms of $X(T)$:

$$(S(T) - K)^+ = e^{X(T)} - K = e^{X(T)} - e^k \quad \text{if } S(T) > K \quad (\text{ or equivalently if } X(T) > k)$$

So:

$$EC(K,T) = e^{-rT}\mathbb{E}[(e^{X(T)} - e^k) \cdot \mathbf{1}_{\{X(T)>k\}}]$$

4. Change of Variable

Using $\phi(u, T) = \mathbb{E}[e^{iuX(T)}]$, the characteristic function of $X(T)$, the option price can be expressed using the characteristic function as follows:

$$EC(k,T) = e^{-rT}\int_k^\infty (e^x - e^k)q(x,T)dx$$

However, the call function $EC(k, T)$ converges to a non-zero constant if $k \to \infty$ (the zero strike call price). Hence this function is not square integrable and Fourier theory would not apply.

To obtain a square integrable function, we consider the modified call price for some $\alpha > 0$:

$$c(k,T) = \exp(\alpha k)EC(k,T)$$

5. Fourier Transform

We use the Fourier transform approach to convert the problem into the frequency domain. The Fourier transform of $c(k,T)$ is:

$$\begin{aligned}
\psi(v,T) := \mathcal{F}\{c(k,t)\}(u) &= \int_{-\infty}^{\infty} \exp(\mathrm{i}vk)c(k,T)dk \\
&= \int_{-\infty}^{\infty} \exp(\mathrm{i}vk)\exp(\alpha k)EC(k,T)dk \\
&= \int_{-\infty}^{\infty} \exp(\mathrm{i}vk)\exp(\alpha k)\exp(-rT)\int_{k}^{\infty}(e^x - e^k)q(x,T)dxdk \\
&= \exp(-rT)\int_{-\infty}^{\infty} q(x,T)\int_{-\infty}^{x}\exp(\mathrm{i}vk)\exp(\alpha k)(e^x - e^k)dkdx \\
&= \exp(-rT)\int_{-\infty}^{\infty} q(x,T)\left(\frac{\exp((\alpha+1+\mathrm{i}v)x)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}\right)dx \\
&= \frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i},T)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}
\end{aligned}$$

7. Inverse Fourier Transform

The price of the call option can be obtained by taking the inverse Fourier transform. For a European call option, the pricing formula is given by:

$$\begin{aligned}
EC(k,T) &= \exp(-\alpha k)\cdot c(k,T) \\
&= \exp(-\alpha k)\cdot \mathcal{F}^{-1}\{\psi(v,T)\}(k) \\
&= \exp(-\alpha k)\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-\mathrm{i}vk)\psi(v,T)dv \\
&= \exp(-\alpha k)\frac{1}{\pi}Re(\int_{0}^{+\infty}\exp(-\mathrm{i}vk)\psi(v,T)dv)
\end{aligned}$$

8. Complete Formula: Carr-Madan

If we put all the pieces together: if we have the characteristic function $\phi(u,T)$ for a log-normal price process, we can calculate the function $\psi(v,T)$

$$\varrho(v) = \psi(v,T) = \frac{\exp(-rT)\phi(v-(\alpha+1)\mathrm{i},T)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}.$$

The European call price is then given by the Carr-Madan formula:

$$EC(k,T) = \exp(-\alpha k)\frac{1}{\pi}Re(\int_{0}^{+\infty}\exp(-\mathrm{i}vk)\varrho(v)dv)$$

And we will demonstrate that this integral can be numerically calculated *very efficiently* by using the FFT technique. By choosing the (log) strikes k in a clever way, we can calculate this integral not just for 1 value of k but for a whole range of strikes in 1 run. The rest of the strikes can then be obtained via interpolation.

The Fast-Fourier Transform (FFT)

The FFT is an efficient algorithm, developped for computing the following transform of a vector $(\alpha_n, n=1,\ldots,N)$ into a vector $(\beta_n, n=1,\ldots,N)$:

$$\beta_n = \sum_{j=1}^{N}\exp\left(-\frac{\mathrm{i}2\pi(j-1)(n-1)}{N}\right)\alpha_j$$

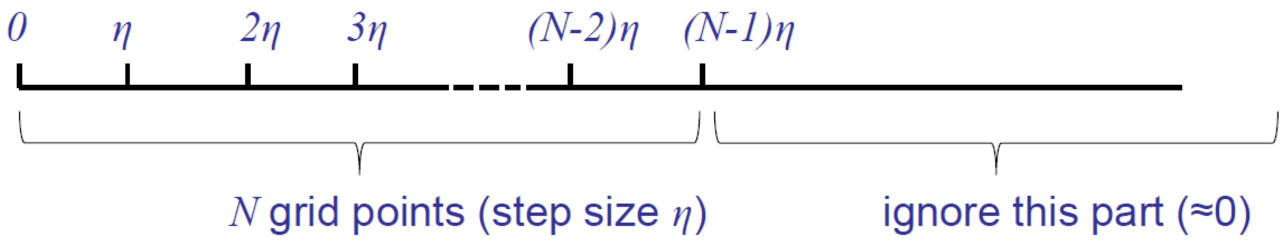
For this algorithm to be most efficient, it is useful to have N be a power of 2. To calculate each value of β_n for $n = 1, \ldots, N$ one would expect to evaluate N terms in the sum N times (one for each β_n), leading to a calculation of order $O(N^2)$.

However, due to leveraging symmetries and periodicities, the sum can be decomposed into smaller sums which can be computed recursively. This leads to a significant reduction in the computational complexity and the order of computation becomes $O(Nlog(N))$.

Application of FFT to the Carr-Madan formula

$$EC(k,T) = \exp(-\alpha k)\frac{1}{\pi}Re(\int_0^{+\infty} \exp(-\mathrm{i}vk)\varrho(v)dv)$$

We will discretize this integral into a sum. To this purpose, make a grid $v_j = \eta(j-1)$ of N points with a grid step of η:



The approximation to the integral becomes the following sum:

$$EC(k,T) \approx \exp(-\alpha k)\frac{1}{\pi}Re(\sum_{j=1}^{N} \exp(-\mathrm{i}v_j k)\varrho(v_j)\eta)$$

To be able to apply the FFT technique, we are going to calculate the option price only for a fixed set of strikes:

$$k_n = -b + \lambda(n-1), \quad n = 1, \ldots, N$$

where

$$\lambda = 2b/N$$

This means that for each such strike k_n :

$$\begin{aligned} EC(k_n,T) &\approx \exp(-\alpha k_n)\frac{1}{\pi}Re(\sum_{j=1}^{N} \exp(-\mathrm{i}v_j k_n)\varrho(v_j)\eta) \\ &= \exp(-\alpha k_n)\frac{1}{\pi}Re(\sum_{j=1}^{N} \exp(-\mathrm{i}v_j(b+\lambda(n-1)))\varrho(v_j)\eta) \\ &= \exp(-\alpha k_n)Re\left(\frac{1}{\pi}\sum_{j=1}^{N} \exp(-\mathrm{i}\eta\lambda(j-1)(n-1))\exp(\mathrm{i}v_j b)\varrho(v_j)\eta\right) \end{aligned}$$

which is almost in the right form for the FFT. We need to demand that

$$\lambda\eta = 2\pi/N$$

to obtain:

$$EC(k_n,T) \approx \exp(-\alpha k_n)Re\left(\frac{1}{\pi}\sum_{j=1}^{N} \exp\left(-\frac{\mathrm{i}2\pi(j-1)(n-1)}{N}\right)\exp(\mathrm{i}v_j b)\varrho(v_j)\eta\right).$$

Now it is clear that this sum is the FFT of the vector

$$(\exp(\mathrm{i}v_j b)\varrho(v_j)\eta, \quad j = 1, \ldots, N)$$

Numerical parameters

There are a few things to consider when implementing this method. The first is the parameter of $\alpha > 0$. There is a whole theory and various papers report different suggestions. The main role the parameter has to play is for the functions to become integrable so the Fourier theory can be applied.

Obviously we need the constraint in the grid to satisfy (1). This means that if you want a more narrow grid in the strike direction, your grid in the approximation of the integral into a sum becomes worse. This can be overcome to some extent by increasing N.

The other condition to always verify is to see if the function under the integral has become sufficiently small to make sure the cut-off in the integration does not create any errors.

Carr and Madan, in their original paper report that the following values give satisfactory results:

$$\eta = 0.25; \quad N = 4096 \quad \alpha = 1.5$$

This means that the cutoff point becomes $b = 12.57$, which remember is in log space. The inter-strike range then becomes $\lambda = 0.0061$ which is little over half a percentage point.

Simpson's rule integration

The numerical integration can actually be improved by using a more refined weighting schema. **Simpson's Rule** is a numerical integration technique used to approximate the definite integral of a function. It is a method that uses parabolic segments instead of straight lines to approximate the area under a curve. This provides a more accurate estimate than other numerical integration methods, such as the Trapezoidal Rule, especially for smooth functions.

Simpson's Rule approximates the integral of a function by dividing the integration interval into an even number of subintervals, fitting a quadratic polynomial (parabola) through the function values at these subintervals, and then calculating the area under these parabolas.

For a function $f(x)$ defined on the interval $[a, b]$, the approximate integral using Simpson's Rule is given by:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

The advantages of using Simpson's Rule are:

- **Higher Accuracy**: For smooth functions, Simpson's Rule is more accurate than the Trapezoidal Rule or other simple numerical integration methods because it uses parabolic approximations rather than linear ones.
- **Error Reduction**: The error in Simpson's Rule is proportional to the fourth power of the width of the subintervals (i.e., $O(h^4)$), making it very efficient for many applications.

If we apply this approximation instead, we obtain

$$EC(k_n, T) \approx \exp(-\alpha k_n)\frac{1}{\pi}Re\left(\sum_{j=1}^{N}\exp(-\mathrm{i}v_j k_n)\varrho(v_j)\eta\left(\frac{3 + (-1)^j - \delta_{j-1}}{3}\right)\right)$$

where $\delta_{j-1} = 1$ if $j = 1$ and zero otherwise and, as before, $v_j = \eta(j-1)$.

As can be observed, this does not drastically change the complexity of the formulas. We just have to add a factor everywhere.

The vector v_j to take the FFT from now becomes

$$\left((\exp(\mathrm{i}v_j b)\varrho(v_j)\eta\left(\frac{3+(-1)^j-\delta_{j-1}}{3}\right), \quad j=1,\dots,N\right)$$

Example of the FFT: the Black-Scholes model

For the Black-Scholes model, we know that the risk neutral stock price process at maturity $S(T)$ follows a lognormal distribution. If we denote the Brownian motion as $W(t)$, we have

$$S(T)=S(0)\exp((r-q-\sigma^2/2)T+\sigma W(T))$$

so the log process $s(T)=\ln(S(T))$ reads

$$s(T)=\ln(S(0))+(r-q-\sigma^2/2)T+\sigma W(T)$$

or $s(T)$ is normally distributed with a mean of \ln(S(0)) + (r-q-\sigma^2/2)T and a variance of $\sigma^2 T$. The characteristic function for the normal distribution is given by

$$\phi_{BS}(u,t)=\exp(\mathrm{i}u\ln(S(0))+(r-q-\sigma^2/2)T))\cdot\exp(-\frac{1}{2}\sigma^2 Tu^2)$$

The code to run to calculate the option prices over a range of strikes is then very short:

```
function y =bsfftcf2(u,p,r,t,x)
sig=x;
y=exp(i.*u*(log(p)+r.*t-(1/2).*sig.^2.*t)).*exp(-(1/2).*sig.^2.*u.^2*t);
return
```

```
N=4096; alpha=1.5; eta=0.25;
p=100; strike=90; sig=0.2; r=0.03; q=0; t=1;
lambda=2*pi/N/eta;
b=lambda*N/2;
k=[-b:lambda:b-lambda];
KK=exp(k);
v=[0:eta:(N-1)*eta];
sw=(3+(-1).^(1:1:N));
sw(1)=1;
sw=sw/3;
rho=exp(-r*t)*bsfftcf2(v-(alpha+1)*i,p,r,t,sig)./(alpha^2+alpha-v.^2+i*(2*alpha+1)*v);
A=rho.*exp(i*v*b)*eta.*sw;
Z=real(fft(A));
CallPricesBS=exp(-alpha*k)/pi.*Z;
CallPriceBSFFT=spline(KK,CallPricesBS,strike);
```

where the last line is the interpolation required to find the strikes if they are not exactly on the grid. It should also be obvious that using a different distribution is very easy as the only thing to change is the characteristic function.