Problem 1

183 DA Pset 0

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Problem 1)

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 6 & 9 & 5 \\ 2 & 4 & 6 & 9 \end{bmatrix}$$

1.a) rank of A

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 6 & 9 & 5 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad r_3 - 2r_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad r_2 - 3r_1$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad r_3 + r_2$$

Therefore Rank(A) = 2

1b) What b_3 does equation $Ax = b = \begin{bmatrix} 1 \\ 2 \\ b_3 \end{bmatrix}$ have a solution?

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 5 & 2 \\ 2 & 4 & 6 & 9 & b_3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & b_3 - 2 \end{array}\right]$$

$r_2 - 3r_1$ And $r_3 - 2r_1$

$-x_4 = -1$

$5x_4 = b_3 - 2 \rightarrow x_4 = 1 \rightarrow 5 = b_3 - 2 \rightarrow \boxed{b_3 = 7}$

1c)

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 6 & 9 & 5 \\ 2 & 4 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

1.2) needed review

$x_4 = 1$ from above

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 3 & 6 & 9 & 5 \\ 2 & 4 & 6 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$x_1 + 2x_2 + 3x_3 + 2 = 1$

$x_1 = 1 - 2x_2 - 3x_3$

so $x = \begin{bmatrix} 1 - 2x_2 - 3x_3 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$

Problem 2

Problem 2)

$\dot{y}(t) = z(t) + u(t) \qquad \dot{z}(t) = 2y(t) + z(t) + 3u(t)$

2b.) transfer function $H(s) = \frac{Y(s)}{U(s)}$

$\underline{\dot{x}} = \begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} z+u \\ 2y+z+3u \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y \\ z \end{bmatrix}}_{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$

$A\underline{x} + Bu$

$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} y \\ z \end{bmatrix}}_{x} + \underbrace{0 \cdot u}_{D} = C\underline{x} + Du$

2a) $Y(s) = \underbrace{\left(D + C(sI - A)^{-1}B\right)}_{H(s)} U(s)$

$H(s) = \left(0 + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ -2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$

$= \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s^2 - s - 2\right)^{-1} \begin{bmatrix} s-1 & 1 \\ 2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= \left(s^2 - s - 2\right)^{-1} \begin{bmatrix} s-1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= \frac{s - 1 + 3}{s^2 - s - 2} = \boxed{\frac{s+2}{s^2 - s - 2} = H(s)}$

2c) Describe normal modes of the system ? stability, given by evaluation of A

$H(s) = \frac{s+2}{(s-2)(s+1)}$

$\lambda_1 = 2 \qquad A v^{(1)} = \lambda_1 v^{(1)}$ so $v^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\lambda_2 = -1 \qquad A v^{(2)} = \lambda_2 v^{(2)}$ so $v^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

so modes are $\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$ (unstable)

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$ (stable)

2d) $U = ky$. What value of k is system stable?

$U = \begin{bmatrix} k & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} k & 0 \end{bmatrix} \underline{x}$

$\underline{\dot{x}} = A\underline{x} + B\underline{u} = Ax + B \begin{bmatrix} k & 0 \end{bmatrix} \underline{x} = \left(A + B \begin{bmatrix} k & 0 \end{bmatrix}\right) \underline{x} = \left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} k & 0 \end{bmatrix}\right) \underline{x}$ so $\dot{x} = \begin{bmatrix} k & 1 \\ 2+3k & 1 \end{bmatrix} \underline{x} = A_{cl}\, \underline{x}$

$\det\left(sI - A_{cl}\right) = \det \begin{bmatrix} s-k & -1 \\ -2-3k & s-1 \end{bmatrix} = (s-k)(s-1) - (2+3k) = s^2 + (-k-1)s + (-2-2k)$

To be stable: $\lambda(A_{cl}) < 0$ so $(-k-1) > 0$? $(-2-2k) > 0$

$k < -1$? $k < -1$

$\boxed{k < -1}$

2.2 Very Hard

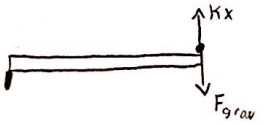
Problem 3

Problem 3) $m_d = 10\,kg$ $\ell = 2m$, $w = .4m$, thickness $= .02\,m$

Young's mod $E = 100\,GPa$. Clamped at one end

take $g \approx 10\,N/kg$ $k = E\frac{w\ell^3}{4\ell^3}$ for cantilevered board

3a)



$F_{grav} = m_d g$

$kx = k(\text{board displacement}) = (100\,GPa)\frac{w\ell^3}{4\ell^3}(\text{board disp.})$

3b)

Force on board: $\frac{1}{2}kx^2$

$m_d g h = \frac{1}{2}kx^2$

$$h = \frac{kx^2}{m_d g} = \frac{(100\,GPa)\frac{(.4)(.02)^3}{4(2)^3}(0.1)^2}{2(10\,kg)(10\,\frac{N}{kg})} = \boxed{0.5\ meter}$$

dog speed $= V_{final}$ $V_{initial} = 0$

$V_{final}^2 - V_{initial}^2 = 2gh$

$$V_{final} = \sqrt{2(10\,\frac{N}{kg})(.5\,meter)} = \sqrt{10\,\frac{m\,kg}{s^2 kg}\,m} = \boxed{\sqrt{10}\,\frac{m}{s}}$$

3c) board 1% thicker => thickness: 0.0202 meter

force board = force dog launched

$\frac{1}{2}kx^2 = m_d g h$

$$h = \frac{kx^2}{2m_d g} = \frac{(100\,GPa)(\frac{.4(.0202)^3}{4(2)^3})(.1)^2}{2(10\,kg)(10\,\frac{N}{kg})} = \boxed{0.515\ meter}$$

3.2 Easy

Problem 4

Problem 4) T T T H T T H T

4a) fair P(sequence) $= (.5)^8 \cong$ 0.003906

4b) $p(H) = .25$ $= (.75)^6 (.25)^2 =$ 0.01112

4c) $P(\text{biased}) = \frac{1}{4}$

4d) Prob biased given above sequence?

$P(b|a)$

b: biased, a: above sequence

$P(b \text{ and } a) = P(b|a)\,P(a)$

$$P(b|a) = \frac{P(b \text{ and } a)}{P(a)} = \frac{(\frac{1}{4})(0.01112)}{(\frac{1}{4})(0.01112) + (\frac{3}{4})(0.003906)} = 0.487 \quad P(b|a)$$

4.2 easy

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Problem 5 (Code in Appendix)

Part A)

5 Sequences of 40 flops for a fair coin

0	1	0	1	0	0	0	0	1	0	0	0	0	1	1	0	1	0	1	1	0	1	1	0	0	1	1	0	1	1	1	0	1	0	1	0	0	1	0	1	0
0	1	0	1	0	1	1	1	1	0	0	0	1	1	0	0	1	0	1	1	0	1	1	0	0	1	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
1	0	1	0	0	1	0	1	0	0	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	0	1	1	1	0	1	1	0	1	0	0	1	0	0	0	1
1	1	0	1	1	1	0	0	0	0	1	1	1	0	1	0	1	1	0	0	0	0	0	0	1	1	0	0	1	1	1	1	0	1	0	1	1	1	1	0	1
1	1	0	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	1	1	1	0	0	1	0	0	0	0	1	1	0	1	0	0	1	1
0	1	0	1	0	0	0	0	1	0	0	0	0	1	1	0	1	0	1	1	0	1	1	0	0	1	1	0	1	1	1	0	1	0	1	0	0	1	0	1	0

5 sequences of 40 flips for a biased coin

0	0	1	1	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	1	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	1	1	0	0	0
1	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	1	1
0	0	1	1	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0

Part B)

Likelihood coin is biased given coin is biased, over the course of each of the 5 sequences in part A

	1	2	3	4	5	6	7	8	9	10
2	0.3333	0.4286	0.2727	0.1579	0.2195	0.2967	0.3876	0.4870	0.5874	0.6811
3	0.3333	0.4286	0.5294	0.6279	0.7168	0.7915	0.8506	0.8952	0.9276	0.9505
4	0.3333	0.4286	0.2727	0.3600	0.2195	0.1233	0.0657	0.0954	0.1366	0.1918
5	0.3333	0.4286	0.5294	0.3600	0.2195	0.2967	0.3876	0.4870	0.5874	0.6811

11	12	13	14	15	16	17	18	19	20
0.7621	0.8277	0.8782	0.7828	0.8439	0.8902	0.9240	0.8588	0.9012	0.9319
0.9665	0.9351	0.8782	0.9153	0.9419	0.8902	0.9240	0.9480	0.9647	0.9762
0.1061	0.1511	0.0817	0.0426	0.0626	0.0910	0.1306	0.1838	0.2525	0.3364
0.7621	0.8277	0.7061	0.5457	0.6431	0.4740	0.3106	0.4033	0.5034	0.6032

21	22	23	24	25	26	27	28	29	30
0.9535	0.9685	0.9788	0.9585	0.9719	0.9454	0.9629	0.9750	0.9832	0.9669
0.9840	0.9893	0.9928	0.9858	0.9905	0.9936	0.9957	0.9972	0.9981	0.9987
0.4319	0.5328	0.3631	0.4610	0.5620	0.6580	0.7427	0.8124	0.8666	0.7646
0.6952	0.5328	0.3631	0.4610	0.5620	0.3908	0.4904	0.5907	0.6840	0.7646

31	32	33	34	35	36	37	38	39	40
0.9360	0.9564	0.9705	0.9427	0.9610	0.9737	0.9823	0.9881	0.9921	0.9947
0.9975	0.9983	0.9966	0.9978	0.9985	0.9990	0.9993	0.9996	0.9997	0.9994
0.6189	0.7089	0.7851	0.8457	0.8915	0.9250	0.9487	0.9024	0.8222	0.6981
0.8297	0.7089	0.7851	0.8457	0.7326	0.8043	0.8604	0.7550	0.8222	0.8740

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Likelihood coin is biased given coin is fair, over the course of each of the 5 sequences in part A

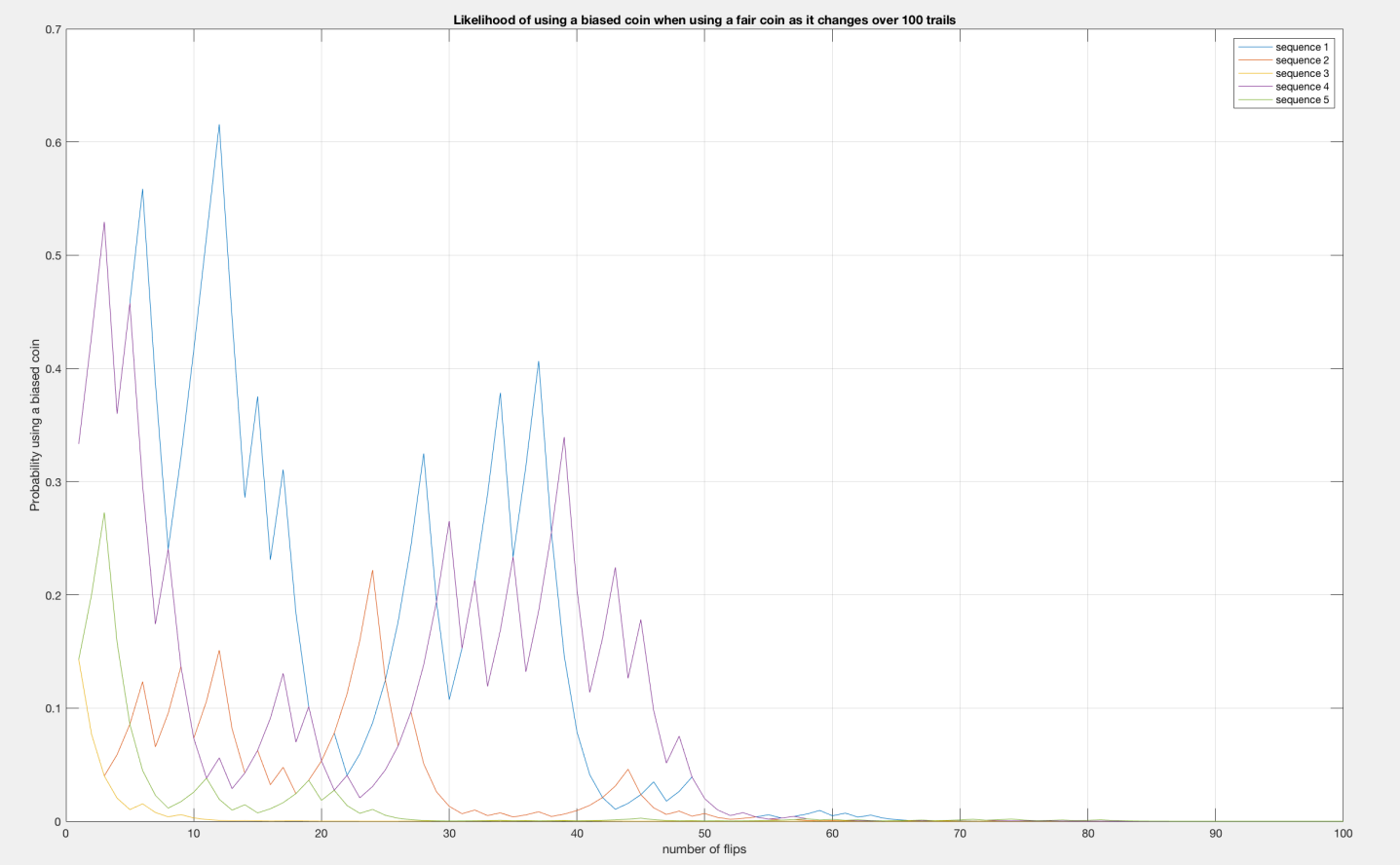
	1	2	3	4	5	6	7	8	9	10
1	0.3333	0.2000	0.2727	0.3600	0.4576	0.2967	0.3876	0.4870	0.3219	0.1918
2	0.3333	0.2000	0.2727	0.3600	0.2195	0.1233	0.0657	0.0340	0.0173	0.0087
3	0.1429	0.2000	0.1111	0.0588	0.0857	0.1233	0.0657	0.0954	0.1366	0.1918
4	0.3333	0.4286	0.5294	0.3600	0.2195	0.1233	0.1742	0.2404	0.3219	0.4159
5	0.3333	0.4286	0.5294	0.3600	0.2195	0.1233	0.0657	0.0340	0.0501	0.0257

	11	12	13	14	15	16	17	18	19	20
3	0.2625	0.3481	0.2107	0.2859	0.3753	0.4740	0.5747	0.6697	0.7525	0.8202
7	0.0130	0.0065	0.0098	0.0146	0.0218	0.0323	0.0164	0.0244	0.0124	0.0062
3	0.2625	0.3481	0.4447	0.2859	0.1668	0.2310	0.3106	0.4033	0.5034	0.6032
9	0.2625	0.3481	0.2107	0.1178	0.1668	0.2310	0.3106	0.4033	0.2525	0.3364
7	0.0130	0.0065	0.0098	0.0049	0.0074	0.0110	0.0055	0.0028	0.0042	0.0062

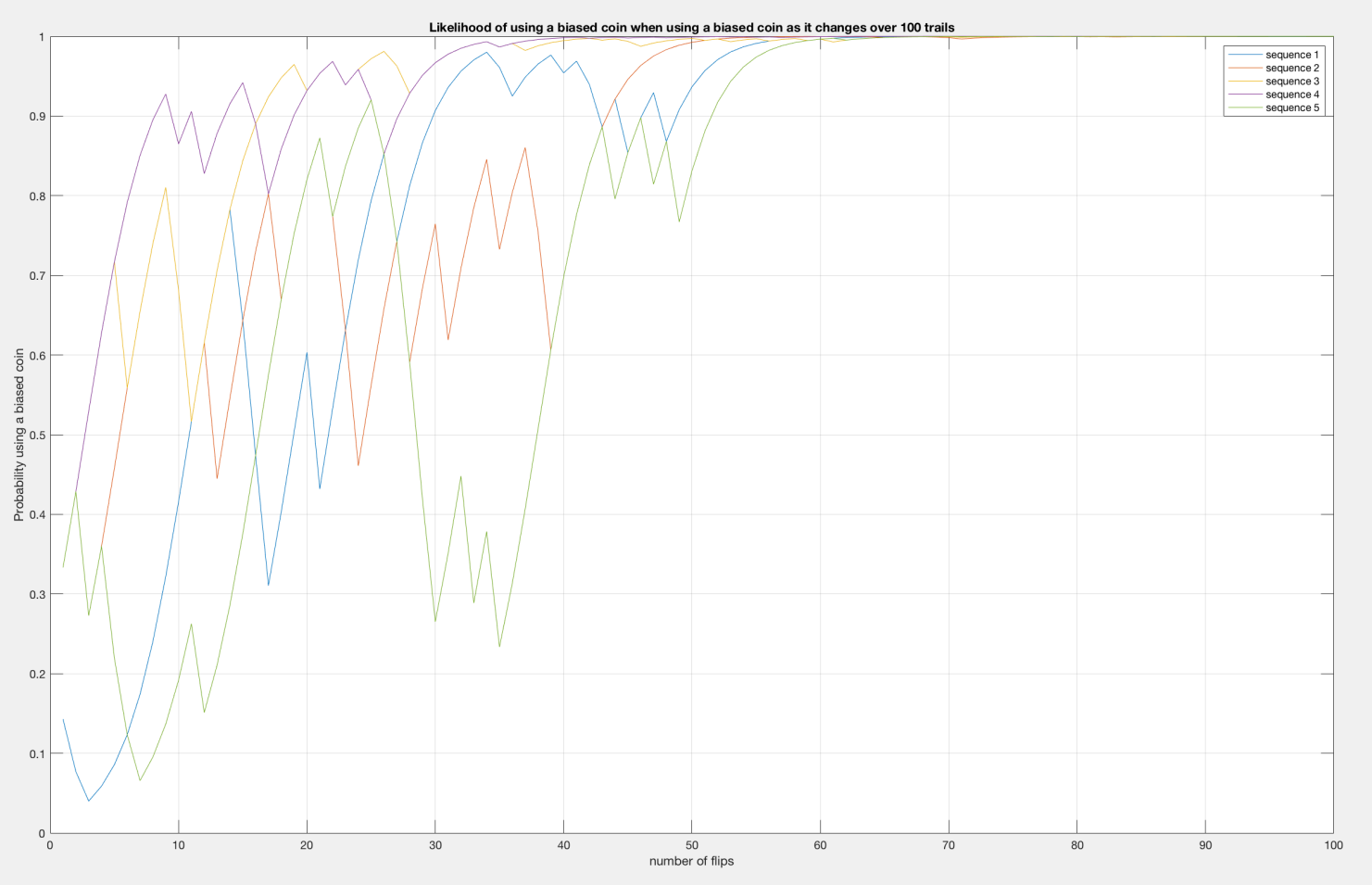
	21	22	23	24	25	26	27	28	29	30
2	0.0031	0.0047	0.0070	0.0104	0.0053	0.0026	0.0013	6.5949e-...	9.8891e-...	0.0015
2	0.6952	0.7738	0.6311	0.7196	0.5620	0.6580	0.4904	0.3248	0.1939	0.2651
4	0.4319	0.5328	0.6311	0.7196	0.5620	0.6580	0.4904	0.5907	0.6840	0.5198
2	0.0093	0.0047	0.0070	0.0035	0.0018	0.0026	0.0013	6.5949e-...	9.8891e-...	0.0015

	31	32	33	34	35	36	37	38	39	40
3	0.0022	0.0011	0.0017	8.3452e-...	4.1743e-...	2.0876e-...	1.0439e-...	5.2198e-...	2.6100e-...	1.3050e-...
	0.3512	0.4481	0.5491	0.6462	0.4773	0.3135	0.4065	0.2551	0.1462	0.0789
3	0.6189	0.7089	0.5491	0.6462	0.4773	0.3135	0.1859	0.2551	0.1462	0.2044
3	0.0022	0.0011	5.5650e-...	8.3452e-...	0.0013	6.2602e-...	9.3874e-...	0.0014	7.0422e-...	3.5223e-...

Part C)



Part D)



5.2: Hard

Problem 6

Problem 6)

Prob 1: 25 mins

Prob 2: 2.5 hours or more

Prob 3: 15 mins

Prob 4: 10 mins

Prob 5: 4 hours.

Appendix

```
clc; clear all; close all;

% 5a
% Fair
seqFair = [];
for i = 1:5
    seqFairOutcome = cf(1,40);
    seqFair(i,:) = seqFairOutcome;
end
% Biased
seqBiased = [];
for i = 1:5
    seqBiasedOutcome = cf(2,40);
    seqBiased(i,:) = seqBiasedOutcome;
end

% 5b
% Fair
seqFairB = [];
likeFairB = [];
for i = 1:5
    [sequence_fair_b,likehood_fair_b] = cfl(1,40);
    seqFairB(i,:) = sequence_fair_b;
    likeFairB(i,:) = likehood_fair_b;
end
% Biased
seqBiasedB = [];
likeBiasedB = [];
for i = 1:5
    [sequence_bias_b,likehood_bias_b] = cfl(2,40);
    seqBiasedB(i,:) = sequence_bias_b;
    likeBiasedB(i,:) = likehood_bias_b;
end

% 5c
seqFairC = [];
likeFairC = [];
for i = 1:5
    [sequence_fair_c,likehood_fair_c] = cfl(1,100);
    seqFairC(i,:) = sequence_fair_c;
    likeFairC(i,:) = likehood_fair_c;
end

figure();
for i = 1:5
    plot(1:100,likeFairC(i,:));
    hold on;
end
hold off;
xlabel('number of flips');
ylabel('Probability using a biased coin');
title('Likelihood of using a biased coin when using a fair coin as it changes over 100 trails');
legend({'sequence 1','sequence 2','sequence 3','sequence 4','sequence 5'});
grid on;

% 5d
seqBiasedD = [];
likeBiasedD = [];
for i = 1:5
    [sequence_bias_d,likehood_bias_d] = cfl(2,100);
    seqBiasedD(i,:) = sequence_bias_d;
    likeBiasedD(i,:) = likehood_bias_d;
end

figure();
for i = 1:5
    plot(1:100,likeBiasedD(i,:));
```

```
    hold on;
end
hold off;
xlabel('number of flips');
ylabel('Probability using a biased coin');
title('Likelihood of using a biased coin when using a biased coin as it changes over 100 trails');
legend({'sequence 1','sequence 2','sequence 3','sequence 4','sequence 5'});
grid on;

% 5a Fn
function seq = cf(cointype,numFlip)
    switch cointype
        % fair
        case 1
                seq = (rand(1,numFlip) < 0.5);  % heads prob =0.5
        % biased
        case 2
                seq = (rand(1,numFlip) < 0.25); % heads prob = 0.25
    end
end

% 5b Fn
function [seq,like] = cfl(type,numFlip)
    switch type
        case 1
            seq = (rand(1,numFlip) < 0.5);
            pFair = 3/4;    % 3 fair, 1 biased
            pBiased = 1/4;
                for j = 1:numFlip
                    pFair = pFair*(1/2);
                    % if get head
                    if seq(j) == 1
                        pBiased = pBiased*(1/4);
                    % if get tail
                    else
                        pBiased = pBiased*(3/4);
                    end
                    like(j) = pBiased/(pBiased+pFair);
                end
        case 2
            seq = (rand(1,numFlip) < 0.25);
            disp(seq);
                pFair = 3/4;    % 3 fair, 1 biased
                pBiased = 1/4;
                for j = 1:numFlip
                    pFair = pFair*(1/2);
                    % if get head
                    if seq(j) == 1
                        pBiased = pBiased*(1/4);
                    % if get tail
                    else
                        pBiased = pBiased*(3/4);
                    end
                    like(j) = pBiased/(pBiased+pFair);
                end
    end
end
```