HW15

Q1 [CORRECT]

The velocity (m/s) of an object at time t seconds is given by $v = \frac{2t}{\sqrt{1+t^2}}$ Using Richardson's extrapolation, find the acceleration of the particle (4th-order correct, $O(h^4)$) at time $t = 5$ using $h_1 = 0.5$ and $h_2 = 0.25$. And also find the exact solution. Rounding a decimal number to four decimal places. Acceleration (true value) = Acceleration (approx) =	Acceleration is derived from velocity by taking the first derivative with respect to time. In other words, it's the rate of change of velocity. Therefore, the derivative degree is 1 (differentiate once). The first-degree derivative was calculated by hand.
Richardson Extrapolation • As with integration, the Richardson extrapolation can be used to combine two lower-accuracy estimates of the derivative to produce a higher-accuracy estimate. • For the cases where there are two $O(h^2)$ estimates and the interval is halved ($h_2 = \frac{h_1}{2}$), an improved $O(h^4)$ estimate may be formed using $D = \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1)$ • For the cases where there are two $O(h^4)$ estimates and the interval is halved ($h_2 = \frac{h_1}{2}$), an improved $O(h^6)$ estimate may be formed using $D = \frac{16}{15}D(h_2) - \frac{1}{15}D(h_1)$ • For the cases where there are two $O(h^6)$ estimates and the interval is halved ($h_2 = \frac{h_1}{2}$), an improved $O(h^8)$ estimate may be formed using $D = \frac{64}{63}D(h_2) - \frac{1}{63}D(h_1)$	$v = \frac{2t}{\sqrt{1+t^2}}$ $\frac{dv}{dt} = 2 \times \frac{d}{dt}\left(\frac{t}{\sqrt{1+t^2}}\right)$ $= 2\left(\frac{\sqrt{1+t^2}(1) - (t)\left[\frac{1}{2}(1+t^2)^{-1/2}\right]\left[\cancel{\frac{d}{dt}(1+t^2)}\right]}{1+t^2}\right)$ $2t$ $= 2\left(\frac{\sqrt{1+t^2} - \left(\frac{t}{\cancel{2}\sqrt{1+t^2}}\right)(\cancel{2}t)}{1+t^2}\right)$ $= 2\left(\frac{1+\cancel{t^2}-\cancel{t^2}}{\cancel{\sqrt{1+t^2}}} \times \frac{1}{1+t^2}\right) = \frac{2}{(1+t^2)^{3/2}}$ $(1+t^2)^{\frac{1}{2}}$

True value

```
v = @(t) (2.*t) ./ sqrt(1 + t.^2)
dv = @(t) 2 ./ (1 + t.^2).^(3/2)
ttest = 5;
Dtrue = dv(ttest)
```

Approximation: Richardson Extrapolation, $O(h^4)$

Centered :

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

give

Use centered differences by default if not specified.

```
h1 = 0.5; h2 = 0.25;
ch1 = (v(ttest+h1) - v(ttest-h1)) / (2*h1);
ch2 = (v(ttest+h2) - v(ttest-h2)) / (2*h2);
Dapprox = (4/3)*ch2 - (1/3)*ch1
```

Q2 [CORRECT]

Use Richardson extrapolation to estimate the first derivative of $y = cos(x)$ at $x = \pi/4$ using step sizes of $h_1 = \pi/3$ and $h_2 = \pi/6$.

Employ centered differences of $O(h^2)$ for the initial estimates.

Rounding a decimal number to four decimal places.

First derivative (true value) :

First derivative (approx) :

"Employ centered differences of $O(h^2)$ for the initial estimates" = "Use the $O(h^4)$ formula of Richardson Expansion"

```
f = @(x) cos(x);
dx = @(x) -sin(x); % 1st derivative of cos(x)
xtest = pi/4;
h1 = pi/3;
h2 = pi/6;
Dtrue = dx(xtest)
ch1 = (f(xtest+h1) - f(xtest-h1)) / (2*h1);
ch2 = (f(xtest+h2) - f(xtest-h2)) / (2*h2);
```

```
Dapprox = (4/3)*ch2 - (1/3)*ch1
```

Q3 [CORRECT]

Compute forward and backward difference approximations of $O(h)$ and $O(h^2)$, and central difference approximations of $O(h^2)$ and $O(h^4)$ for the first derivative of $y = sin(x)$ at $x = \pi/4$ using a value of $h = \pi/12$.

Estimate the absolute value of the true percent relative error $|\varepsilon_t|$ for each approximation.

Rounding a decimal number to four decimal places.

First derivative (true value) =

Forward $O(h)$: $|\varepsilon_t|$ = %

Backward $O(h)$: $|\varepsilon_t|$ = %

Centered $O(h^2)$: $|\varepsilon_t|$ = %

Forward $O(h^2)$: $|\varepsilon_t|$ = %

Backward $O(h^2)$: $|\varepsilon_t|$ = %

Centered $O(h^4)$: $|\varepsilon_t|$ = %

Numerical Differentiation (1st derivative)

- Forward :

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

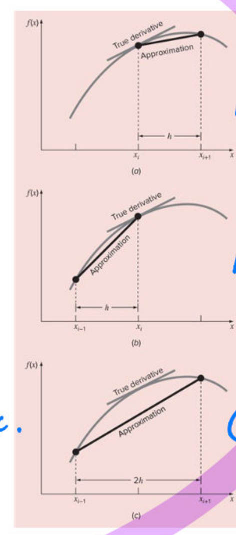
- Backward :

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$$

- Centered :

gives better approx.

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$



F

B

C

High-accuracy difference formula (1st derivative)

- Forward :

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)$$

- Backward :

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} + O(h^2)$$

- Centered :

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} + O(h^4)$$

```
f = @(x) sin(x);
dx = @(x) cos(x);
xtest = pi/4;
h = pi/12;

Dtrue = dx(xtest)

% NUMERICAL DIFFERENTIATION
Dnf = (f(xtest + h) - f(xtest)) / h                                                       % forward
Dnb = (f(xtest) - f(xtest - h)) / h                                                       % backward
Dnc = (f(xtest + h) - f(xtest - h)) / (2*h)                                               % centered

% HIGH-ACCURACY
Dhf = (-f(xtest + (2*h)) + 4*f(xtest + h) - 3*f(xtest)) / (2*h)                           % forward
Dhb = (3*f(xtest) - 4*f(xtest - h) + f(xtest - (2*h))) / (2*h)                            % backward
Dhc = (-f(xtest + (2*h)) + 8*f(xtest + h) - 8*f(xtest - h) + f(xtest - (2*h))) / (12*h) % centered

for i = [Dnf Dnb Dnc Dhf Dhb Dhc]
    err = abs(((Dtrue - i) / Dtrue) * 100);
    fprintf("%.4f | err = %.4f\n", i, err)
end
```

Q4 [CORRECT]

The following data were collected for the distance traveled versus time for a rocket:

t (sec)	0	25	50	75	100	125
y (km)	0	32	58	78	92	100

Use numerical differentiation (second order correct, $O(h^2)$) to estimate the rocket's velocity and acceleration at each time.

t	y	v	a
0	0		
25	32		
50	58		
75	78		
100	92		
125	100		

Numerical Differentiation (1st derivative)

- Forward :

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

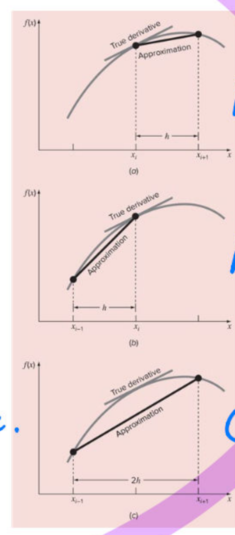
- Backward :

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$$

- Centered :

gives better approx.

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$



High-accuracy difference formula (1st derivative)

- Forward :

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2)$$

- Backward :

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} + O(h^2)$$

- Centered :

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} + O(h^4)$$

Finite-difference approximations of 2nd derivatives

- Forward :

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

- Backward :

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h)$$

- Centered :

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2)$$

```
t = [0     25     50     75     100 125];
y = [0     32     58     78     92     100];

h = 25; % compute step sizes from t (x-axis)

% CENTER
% for the inbetween data points
% do not do "±h" as it'll definitely go out of bounds
```

```
% start the loop from the second index and stop at second-last
% otherwise it'll cause vector index out of bounds
for i = 2 : length(t) - 1
    front = y(i+1);
    back = y(i-1);
    curr = y(i);
    ncv = (front - back) / (2*h); % numerical differentiation
    fca = (front - 2*curr + back) / (h^2); % finite-difference approximation
    fprintf("Center\t| t = %d~%d\t| v = %.4f | a = %.4f\n", back, front, ncv, fca)
end
```

```
Center    | t = 0~58    | v = 1.1600 | a = -0.0096
Center    | t = 32~78    | v = 0.9200 | a = -0.0096
Center    | t = 58~92    | v = 0.6800 | a = -0.0096
Center    | t = 78~100    | v = 0.4400 | a = -0.0096
```

```
% FRONT
% for the first data point
k = 1;
hfv = (-y(k+2) + 4*y(k+1) - 3*y(k)) / (2*h); % high-accuracy
ffa = (y(k+2) - 2*y(k+1) + y(k)) / (h^2); % finite-difference approximation
fprintf("Front\t| v = %.4f | a = %.4f\n", hfv, ffa)
```

```
Front    | v = 1.4000 | a = -0.0096
```

```
% BACK
% for the last data point
k = length(t);
hbv = (3*y(k) - 4*y(k-1) + y(k-2)) / (2*h); % high-accuracy
fba = (front - 2*curr + back) / (h^2); % finite-difference approximation
fprintf("Back\t| v = %.4f | a = %.4f\n", hbv, fba)
```

```
Back    | v = 0.2000 | a = -0.0096
```

Q5 [CORRECT]

Use centered difference approximations to estimate the first and second derivatives of $y = e^x$ at $x = 2$ for $h = 0.1$.

Employ both $O(h^2)$ and $O(h^4)$ formulas for your estimates.

Rounding a decimal number to four decimal places.

First derivative (true value) :

Second derivative (true value) :

First derivative $O(h^2)$:

Second derivative $O(h^2)$:

First derivative $O(h^4)$:

Second derivative $O(h^4)$:

First Derivative	Error
$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_{i-1})}{2h}$	$O(h^2)$
$f'(x_i) = \dfrac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h}$	$O(h^4)$
Second Derivative	
$f''(x_i) = \dfrac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$	$O(h^2)$
$f''(x_i) = \dfrac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2}$	$O(h^4)$

```
f = @(x) exp(x);
xtest = 2;
h = 0.1;

Dtrue = exp(xtest)
```

```
Dtrue =
7.3891
```

```
fdh2 = (f(xtest+h) - f(xtest-h)) / (2*h)
```

```
fdh2 =
7.4014
```

```
fddh2 = (f(xtest+h) - 2*f(xtest) + f(xtest-h)) / (h^2)
```

```
fddh2 =
7.3952
```

```
fdh4 = (-f(xtest+(2*h)) + 8*f(xtest+h) - 8*f(xtest-h) + f(xtest-(2*h))) / (12*h)
```

```
fdh4 =
7.3890
```

```
fddh4 = (-f(xtest+(2*h)) + 16*f(xtest+h) - 30*f(xtest) + 16*f(xtest-h) - f(xtest-(2*h))) / (12*(h^2))
```

```
fddh4 =
7.3890
```