

# Limitations of adversarial robustness: strong No Free Lunch Theorem

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## Abstract

This manuscript presents some new results on adversarial robustness in machine learning, a very important yet largely open problem. We show that if conditioned on a class label the data distribution satisfies the  $W_2$  Talagrand transportation-cost inequality (for example, this condition is satisfied if the conditional distribution has density which is log-concave; is the uniform measure on a compact homogeneous Riemannian manifold like a sphere, torus, or in fact any compact Lie group; etc.), any classifier can be adversarially fooled with high probability once the perturbations are slightly greater than the natural noise level in the problem. We call this result The Strong "No Free Lunch" Theorem as some recent results (Tsipras et al. 2018, Fawzi et al. 2018, etc.) on the subject can be immediately recovered as very particular cases. Our theoretical bounds are demonstrated on both simulated and real data (MNIST). These bounds readily extend to distributional robustness (with 0/1 loss). We conclude the manuscript with some speculation on possible future research directions.

## 1 Introduction

An adversarial attack operates as follows:

- A classifier is trained and deployed (e.g the road traffic sign recognition system on a self-driving car).
- At test / inference time, an attacker may submit queries to the classifier by sampling a real data

point  $x$  with true label  $k$ , and modifying it  $x \rightarrow x^{\text{adv}}$  according to a prescribed threat model. For example, modifying a few pixels on a road traffic sign Su et al. (2017), modifying intensity of pixels by a limited amount determined by a prescribed tolerance level  $\epsilon$  Tsipras et al. (2018), etc.  $\epsilon$ , on it.

- The goal of the attacker is to fool the classifier into classifying  $x^{\text{adv}}$  as label different from  $k$ .
- A robust classifier tries to limit this failure mode, at a prescribed tolerance  $\epsilon$ .

### 1.1 Notation and terminology

$\mathcal{X}$  will denote the feature space and  $\mathcal{Y} := \{1, 2, \dots, K\}$  will be the set of class labels, where  $K \geq 2$  is the number of classes, with  $K = 2$  for binary classification.  $P$  will be the (unknown) joint probability distribution over  $\mathcal{X} \times \mathcal{Y}$ , of two prototypical random variables  $X$  and  $Y$  referred to the features and the target variable, which take values in  $\mathcal{X}$  and  $\mathcal{Y}$  respectively. Random variables will be denoted by capital letters  $X, Y, Z$ , etc., and realizations thereof will be denoted  $x, y, z$ , etc. respectively.

For a given class label  $k \in \mathcal{Y}$ ,  $\mathcal{X}_k \subseteq \mathcal{X}$  will denote the set of all samples whose label is  $k$  with positive probability under  $P$ . It is the support of the restriction of  $P$  onto the plane  $\mathcal{X} \times \{k\}$ . This restriction is denoted  $P_{X|Y=k}$  or just  $P_{X|k}$ , and defines the conditional distribution of the features  $X$  given that the class label has the value  $k$ . We will assume that all the  $\mathcal{X}_k$ 's are finite-dimensional smooth Riemannian manifolds. This is the so-called *manifold assumption*, and is not unpopular in machine learning literature. A classifier is just a mapping  $h : \mathcal{X} \rightarrow \mathcal{Y}$ , from features to class labels.

In our bounds,  $\tilde{O}(\dots)$  will be used to mean the usual *big-O notation*, except for multiplicative constants which only depend on the performance of the classifier.

**Threat models.** Let  $d_{\mathcal{X}}$  be a distance / metric on the input space  $\mathcal{X}$  and  $\epsilon \geq 0$  be a tolerance level. The

$d_{\mathcal{X}}$  threat model at tolerance  $\epsilon$  is a scenario where the attacker is allowed to perturb any input point  $x \mapsto x^{\text{adv}}$ , with the constraint that  $d_{\mathcal{X}}(x^{\text{adv}}, x) \leq \epsilon$ . When  $\mathcal{X}$  is a manifold, the threat model considered will be that induced by the *geodesic distance*, and will be naturally referred to as the *geodesic threat model*.

**Flat threat models.** In the special case of euclidean space  $\mathcal{X} = \mathbb{R}^n$ , we will always consider the distances defined for  $q \in [1, \infty]$  by  $d(x, z) = \|x - z\|_q$ , where

$$\|a\|_q := \begin{cases} \left( \sum_{j=1}^p |a^j|^q \right)^{1/q}, & \text{if } 1 \leq q < \infty, \\ \max\{|a^1|, \dots, |a^p|\}, & \text{if } q = \infty. \end{cases} \quad (1)$$

The  $\ell_\infty$  / *sup* case where  $q = \infty$  Tsipras et al. (2018) is particularly important: the corresponding threat model allows the adversary to separately increase or decrease each feature by an amount at most  $\epsilon$ . The *sparse* case  $q = 1$  is a convex proxy for so-called ‘‘few-pixel’’ attacks Su et al. (2017) wherein the total number of features that can be tampered-with by the adversary is limited.

#### Adversarial robustness accuracy and test error.

The *adversarial robustness accuracy* of  $h$  at tolerance  $\epsilon$  for a class label  $k \in \mathcal{Y}$  and w.r.t the  $d_{\mathcal{X}}$  threat model, denoted  $\text{acc}_{d_{\mathcal{X}}, \epsilon}(h|k)$ , is defined by

$$\text{acc}_{d_{\mathcal{X}}, \epsilon}(h|k) := P_{X|k}(h(x') = k \ \forall x' \in \text{Ball}_{\mathcal{X}}(X; \epsilon)). \quad (2)$$

This is simply the probability that a sample point  $x$  with true class label  $k$  can be perturbed by an amount  $\leq \epsilon$  measured by the distance  $d_{\mathcal{X}}$ , so that it get misclassified by  $h$ . This is an adversarial version of the standard class-conditional accuracy  $\text{acc}(h) = P_{(X, Y)}(h(X) = Y)$  corresponding to  $\epsilon = 0$ . The corresponding *adversarial robustness error* is then  $\text{err}_{\epsilon}(h|k) := 1 - \text{acc}_{\epsilon}(h|k)$ . This is the adversarial analogue of the standard notion of the class-conditional *generalization* / *test error*, corresponding to  $\epsilon = 0$ .

Similarly, one defines the *unconditional adversarial accuracy*

$$\text{acc}_{\epsilon}(h) = P_{(X, Y)}(h(x') = Y \ \forall x' \in \text{Ball}_{\mathcal{X}}(X; \epsilon)), \quad (3)$$

which is an adversarial version of the standard accuracy  $\text{acc}(h) = P_{(X, Y)}(h(X) = Y)$ . Finally, *adversarial robustness radius* of  $h$  on class  $k$

$$d(h|k) := \mathbb{E}_{X|k}[d(X, B(h, k))]. \quad (4)$$

This is the average distance of a sample point  $x \in \mathcal{X}$  with true label  $k$ , from the set of samples classified by  $h$  as being of another label. The smaller the value of  $d(h|k)$ , the less robust the classifier  $h$  is to adversarial attacks on samples of class  $k$ .

## 1.2 Highlight of main contributions

In this manuscript, we prove that under some ‘‘curvature conditions’’ (to be precised later) on the conditional density of the data, it holds that

- For geodesic / faithful attacks:
  - Every (non-perfect) classifier can be adversarially fooled with high probability by moving sample points an amount  $\epsilon \geq \epsilon(h|k) := \sigma_k \sqrt{2 \log(1/\text{err}(h|k))} = \tilde{O}(\sigma_k)$  along the data manifold, where  $\sigma_k$  is the ‘‘natural noise level’’ in the data points with class label  $k$ .
  - Moreover, the average distance of a sample point of true label  $k$  to the error set is upper-bounded:

$$d(h|k) \leq \epsilon(h|k) + \sigma_k \sqrt{\pi/2} = \tilde{O}(\sigma_k).$$

- For attacks in flat space  $\mathbb{R}^p$ :
  - In particular, if the data points live in  $\mathbb{R}^p$ , where  $p$  is the number of features), then every classifier can be adversarially fooled with high probability, by changing each feature by an amount  $\epsilon = O(\sigma_k/\sqrt{p})$ , or more precisely, once

$$\begin{aligned} \epsilon &\geq \epsilon_{\infty}(h|k) := \sigma_k \sqrt{2 \log(1/\text{err}(h|k))}/p \\ &= \tilde{O}(\sigma_k/\sqrt{p}). \end{aligned}$$

- Moreover, we have the bound

$$d(h|k) \leq \epsilon_{\infty}(h|k) + \frac{\sigma_k}{\sqrt{p}} \sqrt{\pi/2} = \tilde{O}(\sigma_k/\sqrt{p}).$$

In fact, we prove similar results for  $\ell_1$  (reminiscent of few-pixel attacks Su et al. (2017)) and even any  $\ell_s$  norm on  $\mathbb{R}^p$ . We call these results The Strong ‘‘No Free Lunch’’ Theorem as some recent results (e.g Fawzi et al. (2018a), Gilmer et al. (2018b), Tsipras et al. (2018)), etc.) on the subject can be immediately recovered as very particular cases. Thus adversarial (non-)robustness should really be thought of as a measure of complexity of a problem. A similar remark has been recently made in Bubeck et al. (2018).

The sufficient ‘‘curvature conditions’’ alluded to above imply *concentration of measure phenomena*, which in turn imply our impossibility bounds. These conditions are satisfied in a large number of situations, including cases where the class-conditional distribution is the volume element of a compact *homogeneous* Riemannian manifold; the class-conditional data distribution is supported on a smooth manifold and has log-concave density w.r.t the curvature of the manifold; or the manifold is compact; is the *pushforward* via a Lipschitz continuous map, of another distribution which verifies these curvature conditions; etc.

**Remark 1.** By the properties of expectation and conditioning, it holds that  $\min_k \text{acc}_\epsilon(h|k) \leq \text{acc}_\epsilon(h) = \mathbb{E}_Y[\text{acc}_\epsilon(h|Y)] = \sum_{k=1}^K \pi_k \text{acc}_\epsilon(h|k) \leq \max_k \text{acc}_\epsilon(h|k)$ , where  $\pi_k := P(k)$ . Thus, bounds on the  $\text{acc}_\epsilon(h|k)$ 's imply bounds on  $\text{acc}_\epsilon(h)$ .

### 1.3 High-level overview of the manuscript

In section 1.4, we start off by presenting a simple motivating classification problem from Tsipras et al. (2018), which as shown by the authors, already exhibits the “No Free Lunch” issue. In section 2.1 we present some relevant notions from geometric probability theory which will be relevant for our work, especially Talagrand’s *transportation-cost* inequality and also Marton’s blowup Lemma. Then in section 2.3, we present the main result of this manuscript, namely, that on a rich set of distributions no classifier can be robust even to modest perturbations (comparable to the natural noise level in the problem). This generalizes the results of Gilmer et al. (2018b), Tsipras et al. (2018) and to some extent, Fawzi et al. (2018a). Section 2.5 extends the results to distributional robustness, a much more difficult setting. All proofs are presented in Appendix A. An in-depth presentation of related works is given in section 3.

Section 4 presents experiments on both simulated and real data that confirm our theoretical results. Finally, section 5 concludes the manuscript with possible future research directions.

### 1.4 A toy example illustrating the fundamental issue

To motivate things, consider the following “toy” problem from Tsipras et al. (2018), which consists of classifying a target  $Y \sim \text{Bern}(1/2, \{\pm 1\})$  based on  $p \geq 2$  explanatory variables  $X := (X^1, X^2, \dots, X^p)$  given by

$$X^1|Y = \begin{cases} +Y, & \text{w.p } 70\%, \\ -Y, & \text{w.p } 30\%, \end{cases}$$

and  $X^j|Y \sim \mathcal{N}(\eta Y, 1)$ , for  $j = 2, \dots, p$ , where  $\eta \sim p^{-1/2}$  is a fixed scalar which (as we will see) controls the difficulty of the problem. Now, as was shown in Tsipras et al. (2018), the above problem can be solved perfectly with generalization accuracy  $\approx 100\%$ , but the “champion” estimator can also be fooled, perfectly! Indeed, the linear estimator given by  $h_{\text{avg}}(x) := \text{sign}(w^T x)$  with  $w = (0, 1/(p-1), \dots, 1/(p-1)) \in \mathbb{R}^p$ , where we allow  $\ell_\infty$ -perturbations of maximum size  $\epsilon \geq 2\eta$ , has

the afore-mentioned properties. Indeed,

$$\begin{aligned} \text{acc}(h_{\text{avg}}) &:= \mathbb{P}_{(X,Y)}(h_{\text{avg}}(X) = Y) = \mathbb{P}(Y w^T X \geq 0) \\ &= \mathbb{P}_Y \left( (Y/(p-1)) \sum_{j \geq 2} \mathcal{N}(\eta Y, 1) \geq 0 \right) \\ &= \mathbb{P}(\mathcal{N}(\eta, 1/(p-1)) \geq 0) = \mathbb{P}(\mathcal{N}(0, 1/(p-1)) \geq -\eta) \\ &= \mathbb{P}(\mathcal{N}(0, 1/(p-1)) \leq \eta) \geq 1 - e^{-(p-1)\eta^2/2}, \end{aligned}$$

which is  $\geq 1 - \delta$  if  $\eta \geq \sqrt{2 \log(1/\delta)/(p-1)}$ . Likewise, for  $\epsilon \geq \eta$ , it was shown in Tsipras et al. (2018) that the adversarial robustness accuracy of  $h_{\text{avg}}$  writes

$$\begin{aligned} \text{acc}_\epsilon(h_{\text{avg}}) &:= \mathbb{P}_{(X,Y)}(Y h_{\text{avg}}(X + \Delta x) \geq 0 \ \forall \|\Delta x\|_\infty \leq \epsilon) \\ &= \mathbb{P}_{(X,Y)} \left( \inf_{\|\Delta x\|_\infty \leq \epsilon} Y w^T (X + \Delta x) \geq 0 \right) \\ &= \mathbb{P}_{(X,Y)}(Y w^T X - \epsilon \|Y w\|_1 \geq 0) \\ &= \mathbb{P}_{(X,Y)}(Y w^T X - \epsilon \geq 0) \\ &= \mathbb{P}(\mathcal{N}(0, 1/(p-1)) \geq \epsilon - \eta) \leq e^{-(p-1)(\epsilon - \eta)^2/2}. \end{aligned}$$

Thus  $\text{acc}_\epsilon(h_{\text{avg}}) \leq \delta$  for  $\epsilon \geq \eta + \sqrt{2 \log(1/\delta)/(p-1)}$ .

By the way, we note that an optimal adversarial attack can be done by taking  $\Delta x^1 = 0$  and  $\Delta x^j = -\epsilon y$  for all  $j = 2, \dots, p$ .

**An autopsy of what is going on.** Recall that the entropy of a univariate Gaussian is  $\text{Ent}(\mathcal{N}(\mu, \sigma)) = \ln(\sqrt{2\pi\sigma}e)$  nats. Now, for  $j = 2, 3, \dots, p$ , the distribution of feature  $X^j$  is a Gaussian mixture  $\frac{1}{2} \sum_{Y=\pm 1} \mathcal{N}(\eta Y, 1)$  and so one computes the mutual information between  $X^j$  and the class label  $Y$  as

$$\begin{aligned} \text{MI}(X^j; Y) &:= \text{Ent}(X^j) - \text{Ent}(X^j|Y) \\ &= \text{Ent} \left( \frac{1}{2} \sum_{y=\pm 1} \mathcal{N}(\eta y, 1) \right) - \frac{1}{2} \sum_{y=\pm 1} \text{Ent}(\mathcal{N}(\eta y, 1)) \\ &= \ln(\sqrt{2\pi}e) + \eta^2 - r - 2(1/2) \ln(\sqrt{2\pi}e) = \eta^2 - r \leq \eta^2, \end{aligned}$$

where (see Michalowicz et al. (2008) for the details)

$$r := \frac{2}{\sqrt{2\pi}\eta} e^{-\eta^2/2} \int_0^\infty e^{-\frac{z^2}{2\eta^2}} \cosh(z) \ln(\cosh(z)) dz \geq 0.$$

Thus  $\text{MI}(X^j; Y) \leq \eta^2$ . Since  $\eta^2 \sim 1/p$ , we conclude that these features barely share any information with the target variable  $Y$ . Indeed, Tsipras et al. (2018) showed improved robustness on the above problem, with feature-selection based on mutual information.

**Basic “No Free Lunch” Theorem.** Reading the information calculations above, a skeptic could point out that the underlying issue here is that the estimator  $h_{\text{avg}}$  over-exploits the fragile / non-robust variables

$X^2, \dots, X^p$  to boost ordinary generalization accuracy, at the expense of adversarial robustness. However, it was rigorously shown in Tsipras et al. (2018) that on this particular problem, every estimator is vulnerable. Precisely, the authors proved the following basic “No Free Lunch” theorem.

**Theorem 1** (Basic No Free Lunch, Tsipras et al. (2018)). *For the problem above, **any** estimator which has ordinary accuracy **at least**  $1 - \delta$  must have robust adversarial robustness accuracy **at most**  $7\delta/3$  against  $\ell_\infty$ -perturbations of maximum size  $\epsilon \geq 2\eta$ .*

## 2 Strong “No Free Lunch” Theorem for adversarial robustness

### 2.1 Terminology and background

**Neighborhood of a set in a metric space.** The  $\epsilon$ -blowup (aka  $\epsilon$ -neighborhood, aka  $\epsilon$ -fattening, aka  $\epsilon$ -enlargement) of a subset  $B$  of a metric space  $\mathcal{X} = (\mathcal{X}, d_{\mathcal{X}})$ , denoted  $B_{\mathcal{X}}^{\epsilon}$ , is defined by  $B_{\mathcal{X}}^{\epsilon} := \{x \in \mathcal{X} | d_{\mathcal{X}}(x, B) \leq \epsilon\}$ , where  $d_{\mathcal{X}}(x, B) := \inf\{d_{\mathcal{X}}(x, y) | y \in B\}$  is the distance of  $x$  from  $B$ . Note that  $B_{\mathcal{X}}^{\epsilon}$  is an increasing function of both  $B$  and  $\epsilon$ ; that is, if  $A \subseteq B \subseteq \mathcal{X}$  and  $0 \leq \epsilon_1 \leq \epsilon_2$ , then  $A \subseteq A_{\mathcal{X}}^{\epsilon_1} \subseteq B_{\mathcal{X}}^{\epsilon_1} \subseteq B_{\mathcal{X}}^{\epsilon_2}$ . In particular,  $B_{\mathcal{X}}^0 = B$  and  $B_{\mathcal{X}}^{\infty} = \mathcal{X}$ . Also observe that each  $B_{\mathcal{X}}^{\epsilon}$  can be rewritten in the form

$$\begin{aligned} B_{\mathcal{X}}^{\epsilon} &= \{x' \in \mathcal{X} | \exists x \in B \text{ with } d_{\mathcal{X}}(x', x) \leq \epsilon\} \\ &= \bigcup_{x \in B} \text{Ball}_{\mathcal{X}}(x; \epsilon), \end{aligned} \quad (5)$$

where  $\text{Ball}_{\mathcal{X}}(x; \epsilon) := \{x' \in \mathcal{X} | d_{\mathcal{X}}(x', x) \leq \epsilon\}$  the closed ball in  $\mathcal{X}$  with center  $x$  and radius  $\epsilon$ . Refer to Fig. 1. In

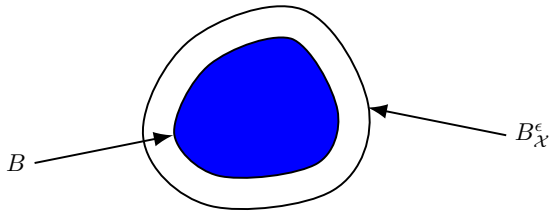


Figure 1:  $\epsilon$ -blowup of a subset  $B$  of a metric space  $\mathcal{X}$ .

a bid to simplify notation, when there is no confusion about the underlying metric space, we will simply write  $B^{\epsilon}$  for  $B_{\mathcal{X}}^{\epsilon}$ . When there is no confusion about the underlying set  $\mathcal{X}$  but not the metric thereupon, we will write  $B_{d_{\mathcal{X}}}^{\epsilon}$ . For example, in the metric space  $(\mathbb{R}^p, \ell_q)$ , we will write  $B_{\ell_q}^{\epsilon}$  instead of  $B_{(\mathbb{R}^p, \ell_q)}^{\epsilon}$  for the  $\epsilon$ -blowup of  $B \subseteq \mathbb{R}^p$ .

An example which will be central to us is when  $h : \mathcal{X} \rightarrow \mathcal{Y}$  is a classifier,  $k \in \mathcal{Y}$  is a class label, and we take  $B$  to be the “bad set”  $B(h, k)$  of inputs which are classified which are assigned a label different from  $k$ , i.e.

$$\begin{aligned} B(h, k) &:= \{x \in \mathcal{X} | h(x) \neq k\} \\ &= \bigcup_{k' \neq k} \{x \in \mathcal{X} | h(x) = k'\}. \end{aligned} \quad (6)$$

$B_{\mathcal{X}}^{\epsilon} = B(h, k)^{\epsilon}$  is then nothing but the event that there is data point with a “bad  $\epsilon$ -neighbor”, i.e the example can be misclassified by applying a small perturbation of size  $\leq \epsilon$ . This interpretation of blowups will be central in the sequel, and we will be concerned with lower-bounding the probability of the event  $B(h, k)^{\epsilon}$  under the conditional measure  $P_{X|k}$ . This is the proportion of points  $x \in \mathcal{X}$  with true class label  $k$ , such that  $h$  assigns a label  $\neq k$  to some  $\epsilon$ -neighbor  $x' \in \mathcal{X}$  of  $x$ . Alternatively, one could study the local *robustness radii*  $r_h(x, k) := \inf\{d(x', x) | x' \in \mathcal{X}, h(x') \neq k\} =: d(x, B(h, k))$ , for  $(x, k) \sim P$ , as was done in Fawzi et al. (2018a), albeit for a very specific problem setting (generative models with Gaussian noise). More on this in section 3. Indeed  $r_h(x, k) \leq \epsilon \iff x \in B(h, k)^{\epsilon}$ .

### 2.2 Measure concentration on metric spaces

For our main results, we will need some classical inequalities from *optimal transport* theory, mainly the Talagrand transportation-cost inequality and Marton’s Blowup inequality (see definitions below). Let  $\mu$  be a probability distribution on a metric space  $\mathcal{X} = (\mathcal{X}, d_{\mathcal{X}})$  and let  $c \geq 0$ .

**Definition 1** ( $T_2(c)$  property –a.k.a Talagrand  $W_2$  transportation-cost inequality).  $\mu$  is said to satisfy  $T_2(c)$  if for every other distribution  $\nu$  on  $\mathcal{X}$ , which is absolutely continuous w.r.t  $\mu$  (written  $\nu \ll \mu$ ), one has

$$W_2(\nu, \mu) \leq \sqrt{2c \text{kl}(\nu \| \mu)}, \quad (7)$$

where for  $s \in [1, \infty)$ ,  $W_s(\nu, \mu)$  is the Wasserstein  $s$ -distance between  $\nu$  and  $\mu$  defined by

$$W_s(\nu, \mu) := \left( \inf_{\pi \in \Pi(\nu, \mu)} \mathbb{E}_{\pi}[d_{\mathcal{X}}(X', X)^s] \right)^{1/s}, \quad (8)$$

and  $\Pi(\nu, \mu)$  is the set of all couplings of  $\nu$  and  $\mu$ .

Note that if  $0 \leq c \leq c'$ , then  $T_2(c) \subseteq T_2(c')$ . The inequality (7) in the above definition is a generalization of the well-known *Pinsker’s inequality* for the total variation distance between probability measures. Unlike Pinsker’s inequality which holds unconditionally, (7) is a privilege only enjoyed by special classes of reference distributions  $\mu$ . These include: log-concave

distributions on manifolds (e.g multi-variate Gaussian), distributions on compact *homogeneous* Riemannian manifolds (e.g hyper-spheres, tori, or more generally, any compact Lie groups), pushforwards of distributions that satisfy some  $T_2$  inequality, etc. In section 2.4, these classes of distributions will be discussed in detail as sufficient conditions for our impossibility theorems.

**Definition 2** (BLOWUP( $c$ ) property).  $\mu$  is said to satisfy BLOWUP( $c$ ) if for every Borel  $B \subseteq \mathcal{X}$  with  $\mu(B) > 0$  and for every  $\epsilon \geq \sqrt{2c \log(1/\mu(B))}$ , it holds that

$$\mu(B^\epsilon) \geq 1 - e^{-\frac{1}{2c}(\epsilon - \sqrt{2c \log(1/\mu(B))})^2}. \quad (9)$$

It is a classical result that the Gaussian distribution on  $\mathbb{R}^p$  has BLOWUP(1) and  $T_2(1)$ , a phenomenon known as *Gaussian isoperimetry*. These results date back to at least works of E. Borel, P. Lévy, M. Talagrand and of K. Marton [Boucheron et al. \(2013\)](#).

The following lemma is the most important tool we will use to derive our bounds.

**Lemma 1** (Marton’s Blowup lemma). *On a fixed metric space, it holds that  $T_2(c) \subseteq \text{BLOWUP}(c)$ .*

*Proof.* The proof is classical, and is a variation of original arguments by Marton. We provide it in Appendix A, for the sake of completeness.  $\square$

### 2.3 Strong “No Free Lunch” Theorem

It is now ripe to present the main results of this manuscript.

**Theorem 2** (Strong “No Free Lunch” on curved space). *Suppose that for some  $\sigma_k > 0$ ,  $P_{X|k}$  has the  $T_2(\sigma_k^2)$  property on the conditional manifold  $\mathcal{X}_k := \text{supp}(P_{X|k}) \subseteq \mathcal{X}$ . Given a classifier  $h : \mathcal{X} \rightarrow \{1, 2, \dots, K\}$  for which  $\text{acc}(h|k) < 1$  (i.e the classifier is not perfect on the class  $k$ ), define*

$$\epsilon(h|k) := \sigma_k \sqrt{2 \log(1/\text{err}(h|k))} = \tilde{\mathcal{O}}(\sigma_k). \quad (10)$$

*Then for the geodesic threat model, we have*

(A) *Bound on adversarial robustness accuracy:*

$$\text{acc}_\epsilon(h|k) \leq \min(\text{acc}(h|k), e^{-\frac{1}{2\sigma_k^2}(\epsilon - \epsilon(h|k))^2_+}). \quad (11)$$

*Furthermore, if  $\epsilon \geq 2\epsilon(h|k)$ , then*

$$\text{acc}_\epsilon(h|k) \leq \min(\text{acc}(h|k), \text{err}(h|k)). \quad (12)$$

(B) *Bound on average distance to error set:*

$$\begin{aligned} d(h|k) &\leq \sigma_k \left( \sqrt{\log(1/\text{err}(h|k))} + \sqrt{\pi/2} \right) \\ &= \tilde{\mathcal{O}}(\sigma_k). \end{aligned} \quad (13)$$

*Proof.* The main idea is to invoke Lemma 1, and then apply the bound (9) with  $B = B(h, k) := \{x \in \mathcal{X} | h(x) \neq k\}$ ,  $\mu = P_{X|k}$ , and  $c = \sigma_k^2$ . See Appendix A for details.  $\square$

In the particular case of attacks happening in euclidean space (this is the default setting in the literature), the above theorem has the following corollary.

**Corollary 1** (Strong “No Free Lunch” Theorem on flat space). *Let  $1 \leq q \leq \infty$ . If in addition to the assumptions of Theorem 2 the conditional data manifold  $\mathcal{X}_k$  is flat, i.e  $\text{Ric}_{\mathcal{X}_k} = 0$ , then for the  $\ell_q$  threat model*

(A1) *Bound on adversarial robustness accuracy:*

$$\text{acc}_\epsilon(h|k) \leq \min(\text{acc}(h|k), e^{-\frac{p^{1-2/q}}{2\sigma_k^2}(\epsilon - \epsilon_q(h|k))^2_+}), \quad (14)$$

where  $\epsilon_q(h|k) := \epsilon(h|k)p^{1/q-1/2} = \mathcal{O}(\sigma_k p^{1/q-1/2})$ . Furthermore, if  $\epsilon \geq 2\epsilon_q(h|k)$ , then

$$\text{acc}_\epsilon(h|k) \leq \min(\text{acc}(h|k), \text{err}(h|k)). \quad (15)$$

(A2) *Bound on average distance to error set:*

$$\begin{aligned} d(h|k) &\leq \frac{\sigma_k}{p^{1/2-1/q}} \left( \sqrt{\log(1/\text{err}(h|k))} + \sqrt{\pi/2} \right) \\ &= \tilde{\mathcal{O}}(\sigma_k/p^{1/2-1/q}). \end{aligned} \quad (16)$$

In particular, for the  $\ell_\infty$  threat model, we have

(B1) *Bound on adversarial robustness accuracy:*

$$\text{acc}_\epsilon(h|k) \leq \min(\text{acc}(h|k), e^{-\frac{p}{2\sigma_k^2}(\epsilon - \epsilon(h|k)/\sqrt{p})^2_+}). \quad (17)$$

Furthermore, if  $\epsilon \geq 2\epsilon(h|k)/\sqrt{p}$ , then

$$\text{acc}_\epsilon(h|k) \leq \min(\text{acc}(h|k), \text{err}(h|k)). \quad (18)$$

(B2) *Bound on average distance to error set:*

$$\begin{aligned} d(h|k) &\leq \frac{\sigma_k}{\sqrt{p}} \left( \sqrt{\log(1/\text{err}(h|k))} + \sqrt{\pi/2} \right) \\ &= \tilde{\mathcal{O}}(\sigma_k/\sqrt{p}). \end{aligned} \quad (19)$$

*Proof.* See Appendix A.  $\square$

**Making sense of the theorems.** Fig. 2 gives an instructive illustration of bounds in the above theorems. For perfect classifiers, the test error  $\text{err}(h|k) := 1 - \text{acc}(h|k)$  is zero and so the factor  $\sqrt{\log(1/\text{err}(h|k))}$  appearing in definitions for  $\epsilon(h|k)$  and  $\epsilon_q(h|k)$  is  $\infty$ ; else this classifier-specific factor grows only very slowly (the log function grows very slowly) as  $\text{acc}(h|k)$  increases



towards the perfect limit where  $\text{acc}(h|k) = 1$ . As predicted by Corollary 1, we observe in Fig. 2 that beyond the critical value  $\epsilon = \epsilon_\infty(h) := \sigma \sqrt{2 \log(1/\text{err}(h))/p} = \tilde{O}(\sigma/\sqrt{p})$ , the adversarial accuracy  $\text{acc}_\epsilon(h)$  decays at a Gaussian rate, and eventually passes below the  $1 - \text{acc}(h)$  as soon as  $\epsilon \geq 2\epsilon_\infty(h)$ .

Comparing to the Gaussian special case below, we see that the curvature parameter  $\sigma_k$  appearing in the theorems is an analogue to the natural noise-level in the problem. The flat case  $\mathcal{X}_k = \mathbb{R}^p$  with an  $\ell_\infty$  threat model is particularly instructive. The critical values of  $\epsilon$ , namely  $\epsilon_\infty(h|k)$  and  $2\epsilon_\infty(h|k)$  beyond which the compromising conclusions of the Corollary 1 come into play is proportional to  $\sigma_k/\sqrt{p}$ .

Finally note that the  $\ell_1$  threat model corresponding to  $q = 1$  in Corollary 1, is a convex proxy for the “few-pixel” threat model which was investigated in Su et al. (2017).

## 2.4 Some applications of the bounds

It turns out that the general “No Free Lunch” Theorem 2 and Corollary 1 apply to a broad range of problems. We discuss some of them hereunder.

**Conditional log-concave data distributions on manifolds.** Consider a conditional data model of the form  $P_{X|k} \propto e^{-v_k(x)} dx$  on a  $d$ -dimensional Riemannian manifold  $\mathcal{X}_k \subseteq \mathcal{X}$  satisfying the Bakry-Émery curvature condition Bakry and Émery (1985)

$$\text{Hess}_x(v_k) + \text{Ric}_x(\mathcal{X}) \succeq (1/\sigma_k^2)I_p, \quad (20)$$

for some  $\sigma_k > 0$ . Such a distribution is called *log-concave*. By (Otto and Villani, 2000, Corollary 1.1), (Bobkov and Goetze, 1999, Corollary 3.2),  $P_{X|k}$  has the  $T_2(\sigma_k^2)$  property and therefore by Lemma 1, the BLOWUP( $\sigma_k^2$ ) property, and Theorem 2 (and Corollary 1 for flat space) applies.

**Elliptical Gaussian conditional data distributions.** Consider the flat manifold  $\mathcal{X}_k = \mathbb{R}^p$  and multivariate Gaussian distribution  $P_{X|k} \propto e^{-v_k(x)} dx$  thereupon, where  $v_k(x) = \frac{1}{2}(x - m_k)^T \Sigma_k^{-1}(x - m_k)$ , for some vector  $m_k \in \mathbb{R}^p$  (called the mean) and positive-definite matrix  $\Sigma_k$  (called the covariance matrix) all of whose eigenvalues are  $\leq \sigma_k^2$ . A direct computation gives  $\text{Hess}_x(v_k) + \text{Ric}_x \succeq 1/\sigma_k^2 + 0 = 1/\sigma_k^2$  for all  $x \in \mathbb{R}^p$ . So this is an instance of the above log-concave example, and so the same bounds hold. Thus we get an elliptical version (and therefore a strict generalization) of the basic “No Free Lunch” theorem in Tsipras et al. (2018), with exactly the same constants in the bounds.

**Perturbed log-concave distributions.** The Holley-Stroock perturbation Theorem ensures that if

$P_{X|k} \propto e^{-v_k(x) - u_k(x)} dx$  where  $u_k$  is bounded, then Theorem 2 (and Corollary 1 for flat space) holds with the noise parameter  $\sigma_k$  degraded to  $\tilde{\sigma}_k := \sigma_k e^{\text{osc}(u_k)}$ , where  $\text{osc}(u_k) := \sup_x u_k(x) - \inf_x u_k(x) \geq 0$ .

**Uniform measure on compact homogeneous Riemannian manifolds.** By Rothaus (1998), such distributions satisfy *Log-Sobolev Inequalities (LSI)* which imply  $T_2(\rho)$ . The constant  $\rho$  can be taken to be any positive scalar less than the *hyper-contractivity* constant of the manifold. A prime example of such a manifold is a hyper-sphere of radius  $R > 0$ . For this example, one can take  $\rho = R^2$ . The “concentric spheres” dataset considered in Gilmer et al. (2018b) is an instance (more on this in section 3).

**Lipschitzian pushforward of distributions having  $T_2$  property.** Lemma 2.1 of Djellout et al. (2004) ensures that if  $P_{X|k}$  is the *pushforward* via an  $L_k$ -Lipschitz map ( $0 \leq L_k < \infty$ )  $\mathcal{Z}_k \rightarrow \mathcal{X}_k$  between metric spaces (an assumption which is implicitly made when machine learning practitioners model images using generative neural networks<sup>1</sup>, for example), of a distribution  $\mu_k$  which satisfies  $T_2(\tilde{\sigma}_k^2)$  on  $\mathcal{Z}_k$  for some  $\tilde{\sigma}_k > 0$ , then  $P_{X|k}$  satisfies  $T_2(L_k^2 \tilde{\sigma}_k^2)$  on  $\mathcal{X}_k$ , and so Theorem 2 (and Corollary 1 for flat space) holds with  $\sigma_k = L_k \tilde{\sigma}_k$ . This is precisely the data model assumed by Fawzi et al. (2018a), with  $\mathcal{Z}_k := \mathbb{R}^{p'}$  and  $\mu_k = \mathcal{N}(0, \sigma I_{p'})$  for all  $k$ .

## 2.5 Distributional No “Free Lunch” Theorem

As before, let  $h : \mathcal{X} \rightarrow \mathcal{Y}$  be a classifier and  $\epsilon \geq 0$  be a tolerance level. Let  $\widetilde{\text{acc}}_\epsilon(h)$  denote the *distributional robustness accuracy* of  $h$  at tolerance  $\epsilon$ , that is the worst possible classification accuracy at test time, when the conditional distribution  $P$  is changed by at most  $\epsilon$  in the Wasserstein-1 sense. More precisely,

$$\widetilde{\text{acc}}_\epsilon(h) := \inf_{Q \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}), W_1(Q, P) \leq \epsilon} Q(h(x) = y), \quad (21)$$

where the Wasserstein 1-distance  $W_1(Q, P)$  (see equation (8) for definition) in the constraint is with respect to the pseudo-metric  $\tilde{d}$  on  $\mathcal{X} \times \mathcal{Y}$  defined by

$$\tilde{d}((x', y'), (x, y)) := \begin{cases} d(x', x), & \text{if } y' = y, \\ \infty, & \text{else.} \end{cases}$$

The choice of  $\tilde{d}$  ensures that we only consider alternative distributions that conserve the marginals  $\pi_y$ ; robustness is only considered w.r.t to changes in the class-conditional distributions  $P_{X|k}$ .

<sup>1</sup>The Lipschitz constant of a feed-forward neural network with 1-Lipschitz activation function, e.g ReLU, sigmoid, etc., is bounded by the product of operator norms of the layer-to-layer parameter matrices.

Note that we can rewrite  $\widetilde{\text{acc}}_\epsilon(h) = 1 - \widetilde{\text{err}}_\epsilon(h)$ ,

$$\widetilde{\text{err}}_\epsilon(h) := \sup_{Q \in \mathcal{P}(\mathcal{X} \times \mathcal{Y}), W_1(Q, P) \leq \epsilon} Q(X \in B(h, Y)), \quad (22)$$

where is the distributional robustness test error and  $B(h, y) := \{x \in \mathcal{X} | h(x) \neq y\}$  as before. Of course, the goal of a machine learning algorithm is to select a classifier (perhaps from a restricted family) for which the average adversarial accuracy  $\text{acc}_\epsilon(h)$  is maximized. This can be seen as a two player game: the machine learner chooses a strategy  $h$ , to which an adversary replies by choosing a perturbed version  $Q \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$  of the data distribution, used to measure the bad event “ $h(X) \neq Y$ ”.

It turns out that the lower bounds on adversarial accuracy obtained in Theorem 2 apply to distributional robustness as well.

**Corollary 2** (No “Free Lunch” for distributional robustness). Theorem 2 holds for distributional robustness, i.e with  $\text{acc}_\epsilon(h|k)$  replaced with  $\widetilde{\text{acc}}_\epsilon(h|k)$ .

*Proof.* See Appendix A.  $\square$

### 3 Related works

There is now a rich literature trying to understand adversarial robustness. Just to name a few, let us mention Blanchet and Murthy (2016), Bubeck et al. (2018), Fawzi et al. (2018a), Gilmer et al. (2018b), Mahloujifar et al. (2018), Mohajerin Esfahani and Kuhn (2017), Schmidt et al. (2018), Sinha et al. (2017), Tsipras et al. (2018). Below, we discuss a representative subset of these works, which is most relevant to our own contributions presented in this manuscript. These all use some kind of Gaussian isoperimetric inequality Boucheron et al. (2013), and turn out to be very special cases of the general bounds presented in Theorem 2 and Corollary 1.

**Gaussian and Bernoulli models.** We have already mentioned the work Tsipras et al. (2018), which first showed that motivating problem presented in section 1.4, every classifier can be fooled with high probability. In a followup paper Schmidt et al. (2018), the authors have also suggested that the sample complexity for robust generalization is much higher than for standard generalization. These observations are also strengthened by independent works of Bubeck et al. (2018).

**Generative models.** The authors posit a model in which data is generated by pushing-forward a multivariate Gaussian distribution through a (surjective) Lips-

chitz continuous mapping<sup>2</sup>  $g : \mathbb{R}^m \rightarrow \mathcal{X}$ , called the generator. The authors then studied the per-sample robustness radius defined by  $r_{\mathcal{X}}(x, k) := \inf\{\|x' - x\|_2 \text{ s.t. } x' \in \mathcal{X}, h(x') \neq k\}$ . In the notation of our manuscript, this can be rewritten as  $r_{\mathcal{X}}(x, k) := d_{\mathcal{X}}(x, B(h, k))$ , from which it is clear that  $r_{\mathcal{X}}(x, k) \leq \epsilon$  iff  $x \in B(h, k)^\epsilon$ . Using the basic Gaussian isoperimetric inequality Boucheron et al. (2013), the authors then proceed to obtain bounds on the probability that the classifier changes its output on an  $\epsilon$ -perturbation of some point on manifold the data manifold, namely  $\text{acc}_\epsilon^{\text{switch}}(h) := 1 - \sum_k \pi_k \text{err}_\epsilon^{\text{switch}}(h|k)$ , where  $\text{err}_\epsilon^{\text{switch}}(h|k) := P_{X|k}(C_{k \rightarrow}(\epsilon)) = \text{acc}(h|k) \text{err}_\epsilon(h|k)$  and  $C_{k \rightarrow}(\epsilon) := B(h, k)^\epsilon - B(h, k)$  is the annulus in Fig. 1. Our bounds in Theorem 2 and Corollary 1 can then be seen as generalizing the methods and bounds in Fawzi et al. (2018a) to more general data distributions satisfying  $W_2$  transportation-cost inequalities  $T_2(c)$ , with  $c > 0$ .

**Adversarial spheres.** The work which is most similar in flavor to ours is the recent “Adversarial Spheres” paper Gilmer et al. (2018b), wherein the authors consider a 2-class problem on a so-called “concentric spheres” dataset. This problem can be described in our notation as:  $P_{X|+}$  = uniform distribution on  $p$ -dimensional sphere of unit radius and  $P_{X|-}$  = uniform distribution on  $p$ -dimensional sphere of radius  $R > 1$ . Thus, the classification problem is to decide which of the two concentric spheres a sampled point came from. One first observes that these two class-conditional distributions are constant (and therefore log-concave) over manifolds of constant curvature, namely 1 and  $R^{-2}$  respectively. The situation is therefore an instance of the Bakry-Emery curvature condition (20), with potentials  $v_k \equiv 0$ . Whence, these distributions satisfy  $T_2(1)$  and  $T_2(R^2)$  respectively. Consequently, Theorem 2 and Corollary 1 kick-in and bound the average distance of sample points with true label  $k \in \{+, -\}$ , to the error set (set of misclassified samples):  $d(h|k) \leq R(\sqrt{2 \log(1/\text{err}(h|k))}) + \sqrt{\pi/2} = \tilde{O}(R)$  for the  $\ell_2$  threat model, and  $d(h|k) \leq \frac{R}{\sqrt{p}}(\sqrt{2 \log(1/\text{err}(h|k))}) + \sqrt{\pi/2} = \tilde{O}(R/\sqrt{p})$  for the  $\ell_\infty$  threat model (spheres are locally flat, so this makes sense). To link more explicitly with the bound  $d(h|k) = \mathcal{O}(\Phi^{-1}(\text{acc}(h|k))/\sqrt{p}) = \mathcal{O}(\Phi^{-1}(1 - \text{err}(h|k))/\sqrt{p})$  proposed in (Gilmer et al., 2018b, Theorem 5.1), one notes the following elementary (and very crude) approximation of Gaussian quantile function

<sup>2</sup>Strictly speaking, Fawzi et al. (2018a) imposes a condition on the pushforward map  $g$  which is slightly weaker than Lipschitz continuity.

<sup>3</sup>:  $\Phi^{-1}(a) \approx \sqrt{2 \log(1/(1-a))}$  for  $0 \leq a < 1$ . Thus,  $\Phi^{-1}(\text{acc}(h|k))/\sqrt{p}$  and  $\sqrt{2 \log(1/\text{err}(h|k))/p}$  are of the same order, for large  $p$ . Consequently, our bounds can be seen as a strict generalization of the bounds in Gilmer et al. (2018b).

**Distributional robustness and regularization.** On a completely different footing, Blanchet and Murthy (2016), Mohajerin Esfahani and Kuhn (2017), Sinha et al. (2017) have linked distributional robustness to robust estimation theory from classical statistics and regularization. An interesting bi-product of these developments is that penalized regression problems like the square-root Lasso and sparse logistic regression have been recovered as distributional robust counterparts of the unregularized problems.

## 4 Experimental evaluation

### 4.1 Simulated data

The simulated data are discussed in section 1.4:  $Y \sim \text{Bern}(\{\pm 1\})$ ,  $X|Y \sim \mathcal{N}(Y\eta, 1)^{\times p}$ , with  $p = 1000$  where  $\eta$  is an SNR parameter which controls the difficulty of the problem. The results are shown in Fig. 2. Here the classifier  $h$  is the linear classifier presented in section 1.4. As predicted by the theorem, we observe that beyond the critical value  $\epsilon = \epsilon_{\infty}(h) := \sigma \sqrt{2 \log(1/\text{err}(h))/p} = \tilde{O}(\sigma/\sqrt{p})$ , where  $\text{err}(h) := 1 - \text{acc}(h)$ , the adversarial accuracy  $\text{acc}_{\epsilon}(h)$  decays exponential fast, and passes below the horizontal line  $\text{err}(h)$  as soon as  $\epsilon \geq 2\epsilon_{\infty}(h)$ .

### 4.2 Real data

Wondering whether the phase transition and bounds predicted by Theorem 2 and Corollary 2 holds for real data, we trained a deep feed-forward CNN for classification on the MNIST dataset LeCun and Cortes (2010), a standard benchmark problem in supervised machine-learning. The results are shown in Fig. 3. This model attains a classification accuracy of 98% on held-out data. We consider the performance of the model on adversarially modified images according to the  $\ell_{\infty}$  threat model, at a given tolerance level (maximum allowed modification per pixel)  $\epsilon$ . As  $\epsilon$  is increased, the performance degrades slowly and then eventually hits a phase-transition point; it then decays exponentially fast and the performance is eventually reduced to chance level. This behavior is in accordance with Corollary 1, and suggests that the range of applicability of our results may be much larger than what we have been able

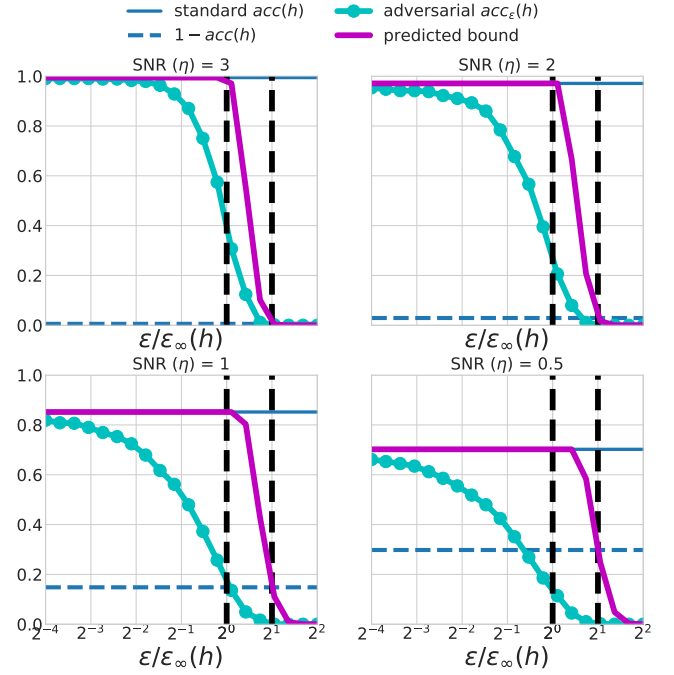


Figure 2: Illustrating The Extended “No Free Lunch” Theorem 1 for the  $\ell_{\infty}$  threat model on the classification problem from Tsipras et al. (2018) (discussed in section 1.4)  $Y \sim \text{Bern}(\{\pm 1\})$ ,  $X|Y \sim \mathcal{N}(Y\eta, 1)^{\times p}$ , with  $p = 1000$  and different values of the SNR  $\eta$ , which controls the difficulty of the problem.

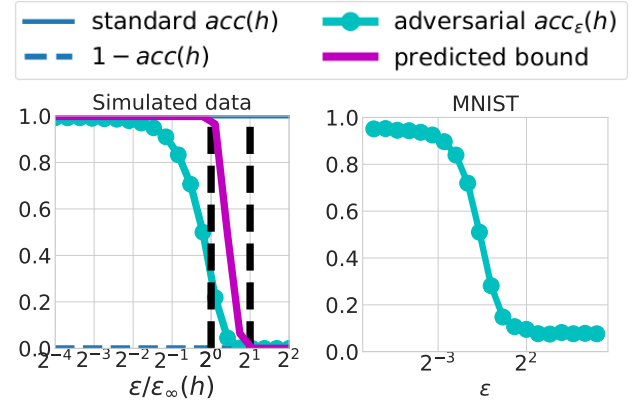


Figure 3: **Left:** Simulated data from section 1.4, shown here as a reference (refer to Fig. 2 for more details). **Right:** MNIST dataset: A deep feed-forward CNN is trained using PyTorch <https://pytorch.org/> to predict MNIST classification problem. We consider the performance of the model on adversarially modified images according to the  $\ell_{\infty}$  threat model, at a given tolerance level (maximum allowed modification per pixel)  $\epsilon$ . As  $\epsilon$  is increased, the performance degrades slowly and then eventually hits a phase-transition point; it then decays exponentially fast, and the performance is eventually reduced to chance level.

<sup>3</sup><https://stats.stackexchange.com/questions/245527/standard-normal-quantile-approximation>



to theoretically establish in Theorem 2 and Corollary 1.

Of course, a more extensive experimental study would be required to strengthen this empirical observation.

## 5 Concluding remarks

We have shown that on a very broad class of data distributions, any classifier with even a bit of accuracy is vulnerable to adversarial attacks. Our work uses powerful tools from geometric probability theory to generalize all the main impossibility results that have appeared in adversarial robustness literature. Moreover, our results would encourage one to conjecture that the modulus of concentration of probability distribution (e.g in  $T_2$  inequalities) on a manifold completely characterizes the adversarial or distributional robust accuracy in classification problems.

### 5.1 Redefine the rules of the game ?

A limitation for adversarial robustness, as universal our strong No Free Lunch Theorem we have developed in this paper could indicate that the attack models currently being considered in the literature, namely additive perturbations measured in the  $\ell_0$ ,  $\ell_1$ ,  $\ell_2$ ,  $\ell_\infty$ , etc. norms, and in which the attacker can make as many queries as they wish, may be too lax. Just like in coding theory where a rethinking of the constraints on a channel leads to the Shannon limit to be improved, one could hope that a careful rethink of the constraints put on the adversarial attacker might alleviate the pessimistic effect of our impossibility results. As remarked in Gilmer et al. (2018a), it is not even clear if the current existing attack models are most plausible.

### 5.2 Future directions

One could consider the following open questions, as natural continuation of our work:

- Extend Theorem 2 and Corollary 1 to more general data distributions.
- Study more complex threat models, e.g small deformations.
- Fine grained analysis of sample complexity and complexity of hypotheses class, with respect to adversarial and distributional robustness. This question has been partially studied in Bubeck et al. (2018), Schmidt et al. (2018) in the adversarial case, and Sinha et al. (2017) in the distributional robust scenario.
- Study more general threat models. Gilmer et al. (2018a) has argued that most of the proof-of-concept problems studied in theory papers might not be completely aligned with real security concerns faced by machine learning applications. It would be interesting to see how the theoretical bounds presented in our manuscript translate on real-world datasets, beyond the MNIST on which we showed some preliminary experimental results.
- Develop more geometric insights linking adversarial robustness and curvature of decision boundaries. This view was first introduced in Fawzi et al. (2018b).

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## A Proofs

*Proof of Theorem 2.* Let  $h : \mathcal{X} \rightarrow \{1, \dots, K\}$  be a classifier, and for a fixed class label  $k \in \{1, 2, \dots, K\}$ , define the set  $B(h, k) := \{x \in \mathcal{X} | h(x) \neq k\}$ . Because we only consider  $P_{X|Y}$ -a.e continuous classifiers, each  $B(h, k)$  is Borel. Conditioned on the event “ $y = k$ ”, the probability of  $B(h, k)$  is precisely the

average error made by the classifier  $h$  on the class label  $k$ . That is,  $\text{acc}(h|k) = 1 - P_{X|k}(B(h, k))$ . Now, the assumptions imply by virtue of Lemma 1, that  $P_{X|k}$  has the BLOWUP( $c$ ) property. Thus, if  $\epsilon \geq \sigma_k \sqrt{2 \log(1/(P_{X|k}(B(h, k))))} = \sigma_k \sqrt{2 \log(1/\text{err}(h|k))} =: \epsilon(h|k)$ , then one has

$$\begin{aligned} \text{acc}_\epsilon(h|k) &= 1 - P_{X|k}(B(h, k)^\epsilon_{d_{\text{geo}}}) \\ &\leq e^{-\frac{1}{2\sigma_k^2}(\epsilon - \sigma_k \sqrt{2 \log(1/(P_{X|k}(B(h, k))))})^2} \\ &= e^{-\frac{1}{2\sigma_k^2}(\epsilon - \sigma_k \sqrt{2 \log(1/\text{err}(h|k))})^2} = e^{-\frac{1}{2\sigma_k^2}(\epsilon - \epsilon(h|k))^2} \\ &\leq e^{-\frac{1}{2\sigma_k^2}\epsilon(h|k)^2} = \text{err}(h|k), \text{ if } \epsilon \geq 2\epsilon(h|k). \end{aligned}$$

On the other hand, it is clear that  $\text{acc}_\epsilon(h|k) \leq \text{acc}(h|k)$  for any  $\epsilon \geq 0$  since  $B(h, k) \subseteq B(h, k)^\epsilon$  for any threat model. This concludes the proof of part (A). For part (B), define the random variable  $Z := d(X, B(h, k))$  and note that

$$\begin{aligned} d(h|k) &:= \mathbb{E}_{X|k}[d(X, B(h, k))] = \int_0^\infty P_{X|k}(Z \geq \epsilon) d\epsilon \\ &= \int_0^{\epsilon(h|k)} P_{X|k}(Z \geq \epsilon) d\epsilon + \int_{\epsilon(h|k)}^\infty P_{X|k}(Z \geq \epsilon) d\epsilon \\ &\leq \epsilon(h|k) + \int_{\epsilon(h|k)}^\infty P_{X|k}(Z \geq \epsilon) d\epsilon, \text{ since } P_{X|k}(Z \geq \epsilon) \leq 1 \\ &\leq \epsilon(h|k) + \int_{\epsilon(h|k)}^\infty e^{-\frac{1}{2\sigma_k^2}(\epsilon - \epsilon(h|k))^2} d\epsilon, \text{ by inequality (11)} \\ &= \epsilon(h|k) + \frac{\sigma_k \sqrt{2\pi}}{2} \left( \int_{-\infty}^\infty \frac{1}{\sigma_k \sqrt{2\pi}} e^{-\frac{1}{2\sigma_k^2}\epsilon^2} d\epsilon \right), \text{ symmetry} \\ &= \epsilon(h|k) + \frac{\sigma_k \sqrt{2\pi}}{2} (1) = \sigma_k \left( \sqrt{\log(1/\text{err}(h|k))} + \sqrt{\pi/2} \right), \end{aligned}$$

which is the desired inequality.  $\square$

*Proof of Corollary 1.* For flat geometry  $\mathcal{X}_k = \mathbb{R}^p$ ; part (A1) of Corollary 1 then follows from Theorem 2 and the equivalence of  $\ell_q$  norms, in particular

$$\|x\|_2 \leq p^{1/2-1/q} \|x\|_q, \quad (23)$$

for all  $x \in \mathbb{R}^p$  and for all  $q \in [1, \infty]$ . Thus we have the blowup inclusion  $B(h, k)_{\ell_2}^{\epsilon p^{1/2-1/q}} \subseteq B(h, k)_{\ell_q}^\epsilon$ . Part (B1) is just the result restated for  $q = \infty$ . The proofs of parts (A2) and (B2) trivially follow from the inequality (23).  $\square$

**Remark 2.** Note that the particular structure of the error set  $B(h, k)$  did not play any part in the proof of Theorem 2 or of Corollary 1, beyond the requirement that the set be Borel. This means that we can obtain and prove analogous bounds for much broader class of losses. For example, it is trivial to extend the theorem to targeted attacks, wherein the attacker can aim to change an images label from  $k$  to a particular  $k'$ .

*Proof of Corollary 2.* We will use a dual representation of  $\widetilde{\text{acc}}_\epsilon(h|k)$  to establish that  $\widetilde{\text{acc}}_\epsilon(h|k) \leq \text{acc}_\epsilon(h|k)$ . That is, distributional robustness is harder than adversarial robustness. In particular, this will allow us apply the lower bounds on adversarial accuracy obtained in Theorem 2 to distributional robustness as well!

So, for  $\lambda \geq 0$ , consider the convex-conjugate of  $(x, y) \mapsto 1_{x \in B(h, y)}$  with respect to the pseudo-metric  $\tilde{d}$ , namely  $1_{x \in B(h, y)}^{\lambda \tilde{d}} := \sup_{(x', y') \in \mathcal{X} \times \mathcal{Y}} 1_{x' \in B(h, y)} - \lambda \tilde{d}((x', y'), (x, y))$ .

A straightforward computation gives

$$\begin{aligned} 1_{x \in B(h, y)}^{\lambda \tilde{d}} &:= \sup_{(x', y') \in \mathcal{X} \times \mathcal{Y}} 1_{x' \in B(h, y)} - \lambda \tilde{d}((x', y'), (x, y)) \\ &= \max_{B \in \{B(h, y), \mathcal{X} \setminus B(h, y)\}} \sup_{x' \in B} 1_{x' \in B(h, y)} - \lambda d(x', x) \\ &= \max(1 - \lambda d(x, B(h, y)), -\lambda d(x, \mathcal{X} \setminus B(h, y))) \\ &= (1 - \lambda d(x, B(h, y)))_+. \end{aligned}$$

Now, since the transport cost function  $\tilde{d}$  is non-negative and lower-semicontinuous, strong-duality holds [Blanchet and Murthy \(2016\)](#), [Villani \(2008\)](#) and one has

$$\begin{aligned} &\sup_{W_1(Q, P) \leq \epsilon} Q(h(X) \neq Y) \\ &= \inf_{\lambda \geq 0} \sup_Q (Q(X \in B(h, Y)) + \lambda(\epsilon - W_1(Q, P))) \\ &= \inf_{\lambda \geq 0} \left( \sup_Q (Q(X \in B(h, Y)) - \lambda W_1(Q, P)) + \lambda \epsilon \right) \\ &= \inf_{\lambda \geq 0} (\mathbb{E}_{(x, y) \sim P} [1_{x \in B(h, y)}^{\lambda \tilde{d}}] + \lambda \epsilon) \\ &= \inf_{\lambda \geq 0} (\mathbb{E}_{(x, y) \sim P} [(1 - \lambda d(x, B(h, y)))_+] + \lambda \epsilon) \\ &= P(X \in B(h, Y)^{\lambda_*^{-1}}), \end{aligned}$$

where  $\lambda_* = \lambda_*(h) \geq 0$  is the (unique!) value of  $\lambda$  at which the infimum is attained and we have used the previous computations and the handy formula

$$\sup_Q (Q(X \in B(h, Y)) - \lambda W_1(Q, P)) = \mathbb{E}_P [1_{X \in B(h, Y)}^{\lambda \tilde{d}}],$$

which is a direct consequence of [\(Blanchet and Murthy, 2016, Remark 1\)](#). Furthermore, by Lemma 2 of [Blanchet and Murthy \(2016\)](#), one has

$$\begin{aligned} \epsilon &\leq \sum_k \pi_k \int_{B(h, k)^{\lambda_*^{-1}}} d(x, B(h, k)) dP_{X|k}(x) \\ &\leq \sum_k \pi_k \lambda_*^{-1} P_{X|k}(X \in B(h, k)^{\lambda_*^{-1}}) \\ &= \lambda_*^{-1} P(X \in B(h, Y)^{\lambda_*^{-1}}) \leq \lambda_*^{-1}. \end{aligned}$$

Thus  $\lambda_*^{-1} \geq \epsilon$  and combining with the previous inequalities gives

$$\begin{aligned} \sup_{Q \in \mathcal{P}(\mathcal{X}), W_1(Q, P) \leq \epsilon} Q(h(X) \neq Y) &\geq P(X \in B(h, Y)^{\lambda_*^{-1}}) \\ &\geq P(X \in B(h, Y)^\epsilon). \end{aligned}$$

Finally, noting that  $\text{acc}_\epsilon(h) = 1 - P(X \in B(h, Y)^\epsilon)$ , one gets the claimed inequality  $\widetilde{\text{acc}}_\epsilon(h) \leq \text{acc}_\epsilon(h)$ .  $\square$

*Proof of Lemma 1.* Let  $B$  be a Borel subset of  $\mathcal{X} = (\mathcal{X}, d)$  with  $\mu(B) > 0$ , and let  $\mu|_B$  be the restriction of  $\mu$  onto  $B$  defined by  $\mu|_B(A) := \mu(A \cap B)/\mu(B)$  for every Borel  $A \subseteq \mathcal{X}$ . Note that  $\mu|_B \ll \mu$  with *Radon-Nikodym derivative*  $\frac{d\mu|_B}{d\mu} = \frac{1}{\mu(B)} 1_B$ . A direct computation then reveals that

$$\begin{aligned} \text{kl}(\mu|_B \| \mu) &= \int \log \left( \frac{d\mu|_B}{d\mu} \right) d\mu|_B = \int_B \log(1/\mu(B)) d\mu|_B \\ &= \log(1/\mu(B)) \mu|_B(B) = \log(1/\mu(B)). \end{aligned}$$

On the other hand, if  $X$  is a random variable with law  $\mu|_B$  and  $X'$  is a random variable with law  $\mu|_{\mathcal{X} \setminus B^\epsilon}$ , then the definition of  $B^\epsilon$  ensures that  $d(X, X') \geq \epsilon$   $\mu$ -a.s., and so by definition (8), one has  $W_2(\mu|_B, \mu|_{\mathcal{X} \setminus B^\epsilon}) \geq \epsilon$ . Putting things together yields

$$\begin{aligned} \epsilon &\leq W_2(\mu|_B, \mu|_{\mathcal{X} \setminus B^\epsilon}) \leq W_2(\mu|_B, \mu) + W_2(\mu|_{\mathcal{X} \setminus B^\epsilon}, \mu) \\ &\leq \sqrt{2c \text{kl}(\mu|_B \| \mu)} + \sqrt{2c \text{kl}(\mu|_{\mathcal{X} \setminus B^\epsilon} \| \mu)} \\ &\leq \sqrt{2c \log(1/\mu(B))} + \sqrt{2c \log(1/\mu(\mathcal{X} \setminus B^\epsilon))} \\ &= \sqrt{2c \log(1/\mu(B))} + \sqrt{2c \log(1/(1 - \mu(B^\epsilon)))}, \end{aligned}$$

where the first inequality is the triangle inequality for  $W_2$  and the second is the  $T_2(c)$  property assumed in the Lemma. Rearranging the above inequality gives

$$\sqrt{2c \log(1/(1 - \mu(B^\epsilon)))} \geq \epsilon - \sqrt{2c \log(1/\mu(B))},$$

Thus, if  $\epsilon \geq \sqrt{2c \log(1/\mu(B))}$ , we can square both sides, multiply by  $c/2$  and apply the increasing function  $t \mapsto e^t$ , to get the claimed inequality.  $\square$