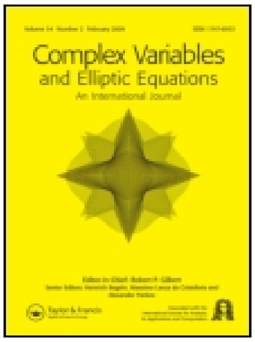

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On the Structure of the Set of Zeros of Quaternionic Polynomials

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We prove that any quaternionic polynomial (with the coefficients on the same side) has two types of zeroes: the zeroes are either isolated or spherical ones, i.e., those ones which form a whole sphere. What is more, the total quantity of the isolated zeroes and of the double number of the spheres does not outnumber the degree of the polynomial.

Keywords: Quaternionic polynomials; Fundamental theorem of algebra

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INTRODUCTION

The name "*fundamental theorem of algebra*" is normally attributed to the statement that any complex polynomial must have a complex zero (or root). It proved to be a very rich and deep object lying in the intersection of several areas of mathematics, among them algebra, topology, complex analysis. One can read the book [2] entirely dedicated to the topic.

The (real) quaternions have many properties linking them to complex numbers; at the same time, the noncommutativity of quaternionic multiplication puts in many delicate differences. Niven in [3,4] made first steps in the direction of generalizing the fundamental theorem onto quaternionic situation which led to the article by Eilenberg and Niven [1] where the fundamental theorem for quaternionic polynomials were established using strongly topological methods. Recently, Serôdio and Siu showed in [5] that if a quaternionic polynomial $f(x)$ admits factorization $f(x) = (x - x_1) \cdots (x - x_n)$ then, the equivalence class of each x_k contains a zero of $f(x)$. Some computational aspects of the problem are considered in [6].

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In this article we prove that any quaternionic polynomial (with the coefficients on the same side) has either isolated zeroes or spherical ones, that is, those ones which form a whole sphere. What is more, the total quantity of the isolated zeroes and of the double number of the spheres does not outnumber the degree of the polynomial. The proofs use quite simple means; as a matter of fact, all is based on the intimate relation between a given quaternionic polynomial of degree n and a polynomial of degree $2n$ with *real* coefficients. We denote the skew-field of real quaternions by $\mathbb{H}$, that is, any elements of $\mathbb{H}$ is of the form $a = a_0 + a_1 i + a_2 j + a_3 k$, where i, j, k are usual quaternionic imaginary units, and we employ also the denotation: $a_0 =: \mathrm{Sc}(a)$, $\boldsymbol{a} := \mathrm{Vec}(a) := a_1 i + a_2 j + a_3 k$.

$\mathbb{S}_3(a; R)$ means the sphere in $\mathbb{H} = \mathbb{R}^4$ of radius R and centered in a.

1

We shall consider the following polynomials of degree $n \geq 1$ of a quaternionic variable z:

$$\mathcal{R}_n(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{1}$$

$$\mathcal{L}_n(z) := z^n a_n + z^{n-1} a_{n-1} + \cdots + z a_1 + a_0, \tag{2}$$

where $\{a_0, a_1, \ldots a_n\} \subset \mathbb{H}$ and $a_n \neq 0$. Since $\overline{\mathcal{R}_n(\bar{z})} = z^n \cdot \overline{a_n} + z^{n-1}\overline{a_{n-1}} + \cdots + \overline{a_0}$, a polynomial of the form (2), it is enough to consider, in many occasions, one of the two polynomials only.

1.1

Let us demonstrate here that any polynomial $\mathcal{R}_n$ (or $\mathcal{L}_n$) has a special representation which we shall use in what follows.

(a) First of all, note that any quaternion h is a zero of the quadratic equation

$$z^2 - pz + q = 0 \tag{3}$$

with $p = 2\mathrm{Sc}(h)$, $q = [h]^2$, that is, such an h satisfies

$$h^2 = ph - q \tag{4}$$

with $\{p, q\} \subset \mathbb{R}$. Hence, for h^3 we have:

$$\begin{aligned} h^3 &= hh^2 = h(ph - q)ph^2 - qh \\ &= p(ph - q) - qh = (p^2 - q)h - pq. \end{aligned}$$

(b) Going along this way, we arrive at the formula

$$h^n = A_n(h) \cdot h + B_n(h), \tag{5}$$

where $A_n()$ and $B_n()$ are $\mathbb{R}$-valued functions of h defined by the recurrent formulas:

$$A_{n+1}(h) = 2\mathrm{Sc}(h) \cdot A_n(h) - |h|^2 \cdot A_{n-1}(h);$$
$$B_{n+1}(h) = -|h|^2 A_n,$$

with

$$A_1(h) = 1,$$
$$A_2(h) = 2\mathrm{Sc}(h),$$
$$B_1(h) = 0,$$
$$B_2(h) = -|h|^2.$$

(c) Finally, for the polynomial (1) we obtain:

$$\mathcal{R}_n(z) = A(z) \cdot z + B(z) \tag{6}$$

where $A(z) := \sum a_i A_i(z)$, $B(z) := \sum a_i B_i(z)$. Of course, the fine point here is that (6) is a quite special representation involving real-valued functions instead of the powers of z.

(d) As a matter of fact, given any z, the quaternions $A_i(z)$ and $B_i(z)$ depend not on z but on its scalar part and the modulus of its vector part, Thus, if two quaterions z and w are such that $z = \alpha_0 + \mathrm{Vec}(z)$, $w = \alpha_0 + \mathrm{Vec}(w)$ with $|\mathrm{Vec}(z)| = |\mathrm{Vec}(w)|$ then $A_i(z) = A_i(w)$, $B_i(z) = B_i(w)$ and hence $A(z) = A(w)$, $B(z) = B(w)$.

In particular, all this is true if $w = \bar{z}$, the conjugate of z.

1.2

We are interested in the properties of the zeroes of both polynomials (1) and (2), and it is well known that the situation differs, in general, with that of complex polynomials. Indeed, although in the case of polynomials of degree 1, $\mathcal{R}_1(z) = a_1 z + a_0$, $\mathcal{L}_1 = z a_1 + a_0$, there are no peculiarities and both polynomials have, for $a_1 \neq 0$, exactly one zero defined by the formulas $z_0 := -a_1^{-1} \cdot a_0$ and $z_0 := -a_0 \cdot a_1^{-1}$ respectively; already for $n = 2$ there exist polynomials with more than two zeroes. The simpliest of them is the polynomial $z^2 + 1$ whose zeroes are all the vectors of modulus 1.

1.3

Another example is provided by the following reasoning. Given real numbers p and $q \geq 0$, consider the quadratic equation

$$z^2 - pz + q = 0, \tag{7}$$

for which we are looking for quaternionic solutions of course. If $q - (p^2/4) > 0$ then any quaternion in the set $\{z = (p/2) + \boldsymbol{r}\colon |\boldsymbol{r}|^2 = q - (p^2/4)\}$ gives a solution to (7); indeed, for such a quaternion z there holds: $z + \bar{z} = p$, $z \cdot \bar{z} = q$, and that is all. Hence under the condition $q - (p^2/4) > 0$ there are as many solutions as the points of the sphere $\mathbb{S}(0; q - (p^2/4))$. If $q - (p^2/4) = 0$, there are no quaternionic, nonreal solutions.

1.4

Quite similar reasoning gives the following conclusion: given a quaternion h and its conjugate $\bar{h}$, both are zeroes of the equation

$$z^2 - 2\mathrm{Sc}(h)z + |h|^2 = 0$$

with real coefficients.

2 PROPOSITION

Assume that $w_0 = \alpha_0 + \mathrm{Vec}(w_0)$, *and* $w_1 = \alpha_0 + \mathrm{Vec}(w_1)$ *are two different zeroes of the polynomial* (1) *with* $|\mathrm{Vec}(w_0)| = |\mathrm{Vec}(w_1)|$; *then any quaternion*

$$w := \alpha_0 + \boldsymbol{r} = \mathrm{Sc}(w_0) + \boldsymbol{r}, \tag{8}$$

with $|\boldsymbol{r}| = |\mathrm{Vec}(w_0)|$, *is also a zero of* (1). *The same is true for* (2).

Proof

(a) For the above w_0 we have:

$$\mathcal{R}_n(w_0) = 0 = A(w_0)w_0 + B(w_0),$$
$$\mathcal{R}_n(w_1) = 0 = A(w_1)w_1 + B(w_1).$$

As is proved in Subsection 1.1, paragraph (d),

$$A(w_0) = A(w_1), \qquad B(w_0) = B(w_1),$$

hence

$$A(w_0)w_0 + B(w_0) = A(w_0)w_1 + B(w_0)$$

which implies that

$$A(w_0) \cdot (\mathrm{Vec}(w_0) - \mathrm{Vec}(w_1)) = 0.$$

Since w_0 and w_1 are different quaternions, the above means that

$$A(w_0) = 0$$

and, then,

$$B(w_0) = 0.$$

(b) Let w be a quaternion such as in (8). Then $w + \overline{w} = 2\mathrm{Sc}(w_0) = 2\alpha_0$, $|w|^2 = |w_0|^2$, hence $A(w) = A(w_0)$, $B(w) = B(w_0)$. Consider $\mathcal{R}_n(w)$, one has:

$$\mathcal{R}_n(w) = A(w)w + B(w) = A(w_0)w + B(w_0) = 0 \cdot w + 0 = 0,$$

and that is all.

2.1 Corollary

Let w_0 be a zero of the polynomial (1) *such that* $\mathrm{Vec}(w_0) \neq 0$ *and that $\overline{w_0}$ is also a zero of* (1). *Then any quaternion* (8) *is also a zero of* (1).

2.2 Definition

Given a polynomial $\mathcal{R}_n$, any of its zeroes which generates a family of zeroes (8), will be called a spherical zero.

2.3

As a matter of fact, Proposition 2 and Corollary 2.1 give two equivalent characterizations of a spherical zero of a polynomial: the latter says that the conjugate of a zero should be also a zero, and the former says that it is enough to have one more zero with special properties not necessary the conjugate of the zero.

3 PROPOSITION

If the polynomial $\mathcal{R}_n(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ has no spherical zeroes then it has only isolated zeroes and their number is less or equal to n.

Proof

(a) Let $w_1, w_2, \ldots, w_{n+1}$ be *different* quaternions which are zeroes of $\mathcal{R}_n$ and no pair of them are mutually conjugate. Consider the polynomial $\mathcal{L}_n := z^n\overline{a_n} + z^{n-1}\overline{a_{n-1}} + \cdots + z\overline{a_1} + \overline{a_0} = \overline{\mathcal{R}_n(\overline{z})}$ for which the quaternions $\overline{w_1}, \ldots, \overline{w_{n+1}}$ are the zeroes.

(b) At each point w_i the polynomial $\mathcal{R}_n(z)$ has, according to (6), the following representation:

$$\mathcal{R}_n(w_i) = A(w_i) \cdot w_i + B(w_i), \quad i \in \{1, 2, \ldots, n+1\},$$

which implies that for $\mathcal{L}_n(z)$ there holds:

$$\begin{aligned}\mathcal{L}_n(w_i) &= \overline{\mathcal{R}_n(\overline{w_i})} = \overline{A(\overline{w_i}) \cdot \overline{w_i} + B(\overline{w_i})} \\ &= w_i\overline{A(\overline{w_i})} + \overline{B(\overline{w_i})} \\ &= w_i\overline{A(w_i)} + \overline{B(w_i)},\end{aligned}$$

where we have used paragraph (d) from Subsection 1.1.

(c) Consider the product

$$\mathcal{L}_n(z)\mathcal{R}_n(z) =: \mathcal{F}_{2n}(z)$$

for which one has:

$$\begin{aligned}\mathcal{F}_{2n}(w_i) &= \mathcal{L}_n(w_i) \cdot \mathcal{R}_n(w_i)\\ &= (w_i\overline{A(w_i)} + \overline{B(w_i)}) \cdot (A(w_i)w_i + B(w_i))\\ &= |A(w_i)|^2 w_i^2 + w_i\overline{A(w_i)}B(w_i) + \overline{B(w_i)}A(w_i)w_i + |B(w_i)|^2 = 0.\end{aligned} \tag{9}$$

(d) Let us prove that the equality (9) implies that $\mathrm{Vec}(\overline{A(w_i)}B(w_i)) \parallel \mathrm{Vec}(w_i)$. Indeed, (9) is of the form:

$$\alpha^2 \cdot (u_0 + \boldsymbol{u})^2 + (u_0 + \boldsymbol{u})(v_0 + \boldsymbol{v}) + (v_0 - \boldsymbol{v})(u_0 + \boldsymbol{u}) + \beta^2 = 0, \tag{10}$$

or, equivalently:

$$\begin{cases}\alpha^2 u_0^2 + 2u_0 v_0 + \beta^2 = 0,\\ 2(\alpha^2 u_0 + v_0)\boldsymbol{u} + 2[\boldsymbol{u}, \boldsymbol{v}] = 0,\end{cases} \tag{11}$$

where $\overline{A(w_i)} \cdot B(w_i) =: v_0 + \boldsymbol{v},\ w_i =: u_0 + \boldsymbol{u},\ |A(w_i)|^2 = \alpha^2,\ B(w_i) = \beta^2$. The second equality in (11) says that $\boldsymbol{u} \parallel [\boldsymbol{u}, \boldsymbol{v}]$, and hence $\boldsymbol{u} \parallel \boldsymbol{v}$; i.e., $\mathrm{Vec}(w_i) \parallel \mathrm{Vec}(\overline{A(w_i)} \cdot B(w_i))$. The last condition means, in particular, that $w_i\overline{A(w_i)} \cdot B(w_i) = \overline{A(w_i)}B(w_i)w_i$ and we get from (10):

$$|A(w_i)|^2 w_i^2 + 2\mathrm{Sc}(\overline{A(w_i)} \cdot B(w_i))w_i + |B(w_i)|^2 = 0. \tag{12}$$

(e) Consider an equation with real coefficients:

$$|A(w_i)|^2 z^2 + 2\mathrm{Sc}(\overline{A(w_i)} \cdot B(w_i))z + |B(w_i)|^2 = 0 \tag{13}$$

Recalling that (12) originates from the product $\mathcal{F}_{2n} := \mathcal{L}_n(z)\mathcal{R}_n(z)$ which has both w_i and $\overline{w_i}$ as its zeroes, one may conclude that (13) has both w_i and $\overline{w_i}$ as its zeroes; it follows also directly from the fact that the coefficients of (13) are real. Hence, by Proposition 2, the Eq. (13) has as its zero any quaternion w such that $\mathrm{Sc}(w) = \mathrm{Sc}(w_0)$, $|\mathrm{Vec}(w)| = |\mathrm{Vec}(w_0)|$; denote by c_i and $\overline{c_i}$ the two complex numbers which are among them.

(f) Consider now the polynomial with real coefficients but of the quaternionic variable:

$$\begin{aligned}\mathcal{F}_{2n}^* :&= \overline{a_0}a_0 + (\overline{a_0}a_1 + \overline{a_1}a_0)z + +\overline{a_n}a_n z^{2n},\\ &= \sum_{k=0}^{2n}\left(\sum_{i=0}^{k}\overline{a_i}a_{k-i}\right)z^k,\end{aligned} \tag{14}$$

compare with the polynomial $\mathcal{F}_{2n}$ from which $\mathcal{F}^*_{2n}$ is obtained by formally commuting the coefficients and the variable.

Let us prove here that the procedure of the degree decreasing from Subsection 3 is, in a sense, commutative with the procedure of forming the polynomial $\mathcal{F}^*_{2n}$ from F_{2n}. More exactly, let us prove that, given a z fixed, if

$$\begin{aligned}\mathcal{F}_{2n}(z) &= (\overline{a_0} + z\overline{a_1} + \cdots + z^n\overline{a_n})(a_0 + a_1 z + \cdots + a_n z^n)\\ &= (z\overline{A^{(n)}} + \overline{B^{(n)}})(A^{(n)}z + B^{(n)})\\ &= |A^{(n)}|^2 z^2 + z\overline{A^{(n)}}B^{(n)} + \overline{B^{(n)}}A^{(n)}z + |B^{(n)}|^2,\end{aligned}$$

then for the polynomial $\mathcal{F}^*_{2n}(z)$ it holds:

$$\mathcal{F}^*_{2n}(z) = |A^{(n)}|^2 z^2 + S^{(n)}z + |B^{(n)}|^2,$$

where $S^{(n)} := \overline{A^{(n)}}B^{(n)} + \overline{B^{(n)}}A^{(n)}$ is a real number, and where we write $A^{(n)} := A(z)$, $B^{(n)} := B(z)$ omitting the variable and using a superscript in order to indicate the dependence on n of the polynomial in formula (6).

The proof goes by induction. For $n = 1$, it is obviously true which follows from (14). Assume now that we have it already for some $n > 1$ and consider $n+1$. Using (5), we can write first:

$$z^{n+1} = A_{n+1}(z)z + B_{n+1}(z); \tag{15}$$

and then:

$$\begin{aligned}\mathcal{F}_{2n}(z) &= (\overline{a_0} + z\overline{a_1} + \cdots + z^n\overline{a_n})(a_0 + a_1 z + \cdots + a_n z^n)\\ &= (z\overline{A^{(n)}} + \overline{B^{(n)}} + (zA_{n+1}(z) + B_{n+1}(z))\overline{a_{n+1}})\\ &\quad\times (A^{(n)}z + B^{(n)} + a_{n+1}(A_{n+1}(z)z + B_{n+1}(z))).\end{aligned}$$

Since it is assumed that the statement is true for n, then it is enough to show that constructing the polynomial (14) for the following one:

$$(\overline{a_0} + z\overline{a_1} + \cdots + z^n\overline{a_n})a_{n+1}z^{n+1} + z^{n+1}\overline{a_{n+1}}(a_0 + a_1 z + \cdots + a_n z^n) + |a_{n+1}|^2 z^{2n+4},$$

and reducing its degree up to the second, we obtain a quadratic trinomial which, in turn, is obtained by commuting the coefficients and the variable from the trinomial.

$$\begin{aligned}&(z\overline{A^{(n)}} + \overline{B^{(n)}})(a_{n+1}(A_{n+1}(z)z + B_{n+1}(z)))\\ &\quad + (A_{n+1}(z)z + B_{n+1}(z))\overline{a_{n+1}}(A^{(n)}z + B^{(n)})\\ &\quad + |a_{n+1}|^2(A_{n+1}(z)z + B_{n+1}(z))^2.\end{aligned}$$

By all this, take the polynomial

$$P_{2n}(z) = (\overline{a_0} + z\overline{a_1} + +z^n\overline{a_n}) + z^{n+1}\overline{a_{n+1}}(a_0 + a_1 z + \cdots + z^n a_n).$$

and consider (14) for the latter:

$$P^*_{2n}(z) = (\overline{a_0}a_{n+1} + \overline{a_{n+1}}a_0)z^{n+1} + (\overline{a_1}a_{n+1} + \overline{a_{n+1}}a_1)z^{n+2} \\ + \cdots + (\overline{a_n}a_{n+1} + \overline{a_{n+1}}a_n)z^{n+1}.$$

It follows from (15) that

$$P^*_{2n}(z) = ((\overline{a_0}a_{n+1} + \overline{a_{n+1}}a_0) + (\overline{a_1}a_{n+1} + \overline{a_{n+1}}a_1)z \\ + \cdots + (\overline{a_n}a_{n+1} + \overline{a_{n+1}}a_n)z^n)(A_{n+1}(z)z + B_{n+1}(z)). \tag{16}$$

In the first factor of (16), that is, in the polynomial

$$(\overline{a_0}a_{n+1} + \overline{a_{n+1}}a_0) + (\overline{a_1}a_{n+1} + \overline{a_{n+1}}a_1)z \\ + \cdots + (\overline{a_n}a_{n+1} + \overline{a_{n+1}}a_n)z^n,$$

we reduce the degrees of the summands up to the degree one and, taking into account that the coefficients $A_k(z)$ and $B_k(z)$ which arise under this procedure (see Section 1.1) are real and thus commute with all a_i, we conclude that

$$(\overline{a_0}a_{n+1} + \overline{a_{n+1}}a_0) + (\overline{a_1}a_{n+1} + \overline{a_{n+1}}a_1)z \\ + \cdots + (\overline{a_n}a_{n+1} + \overline{a_{n+1}}a_n)z^n \\ = \overline{A^{(n)}}a_{n+1}z + \overline{B^{(n)}}a_{n+1} + \overline{a_{n+1}}(A^{(n)}z + B^{(n)}). \tag{17}$$

Substituting (17) into (16), we have:

$$P^*_{2n}(z) = (\overline{A^n}a_{n+1}z + \overline{B_n}a_{n+1} + \overline{a_{n+1}}(A^{(n)}z + B^{(n)})) \\ \times (A_{n+1}(z)z + B_{n+1}(z))$$

which is obtained by the same procedure of commuting the coefficients and the variable from

$$(z\overline{A^{(n)}} + \overline{B^{(n)}})(a_{n+1}(A_{n+1}(z)z + B_{n+1}(z))) \\ + (A_{n+1}(z)z + B^{(n)}_{n+1}(z)))\overline{a_{n+1}}(A^{(n)}z + B^{(n)}).$$

It is enough to observe now that reducing the degree of $|a_{n+1}|^2z^{2n+2}$ to the degree two, we get simply the polynomial $|a^2_{n+1}|(A_{n+1}(z)z + B_{n+1}(z))^2$ which finally concludes this step.

(g) Applying the above reasoning to the the point c_i we get:

$$\mathcal{F}^*_{2n}(c_i) = |\overline{A(w_i)}|^2 \cdot c_i^2 + 2\mathrm{Sc}(\overline{A(w_i)}B(w_i))c_i + |B(w_i)|^2 \tag{18}$$

which, according to the previous paragraph (d), means that any c_i is a zero of the polynomial (14).

But, since (14) has real coefficients, any $\overline{c_i}$ is also its zero, hence $\mathcal{F}^*_{2n}$ seen as a complex polynomial of degree $2n$ has $2n+2$ complex zeroes $c_1, \ldots, c_n, c_{n+1}, \overline{c_1}, \ldots, \overline{c_{n+1}}$. This completes the proof.

3.1 Definition

We shall call the polynomial $\mathcal{F}^*_{2n}$, which was used in the proof, the basic polynomial for the quaternionic polynomial $\mathcal{R}_n$.

4 THEOREM

Any quaternionic polynomial of degree $n \geq 1$ has at least one quaternionic zero.

Proof

(a) The case $n=1$ is trivial, so take $n \geq 2$.

(b) Given a polynomial $\mathcal{R}_n$, we write its basic polynomial $\mathcal{F}^*_{2n}$ as

$$\mathcal{F}^*_{2n} := b_{2n}z^{2n} + \cdots + b_1 z + b_0$$

where $b_{2n}, \ldots, b_1, b_0$ are real numbers. Note that if $d \in \mathbb{R}$ is a zero of the polynomial $\mathcal{F}^*_{2n}$, it is also a zero of the polynomial $\mathcal{R}_n$; indeed, this follows from the reasonings in the proof of Proposition 3, since $\overline{d} = d$.

(c) Since $n \geq 2$, then $2n \geq 4$ and $\mathcal{F}^*_{2n}$ has at least one complex zero, denote it by $\gamma = \alpha + i\beta,\ \beta \neq 0$; then $\overline{\gamma} = \alpha - i\beta$ is also a zero of $\mathcal{F}^*_{2n}$. Repeating again the above procedures we obtain:

$$\begin{aligned}\mathcal{F}^*_{2n}(\gamma) &= M^*(\gamma)\cdot\gamma^2 + N^*(\gamma)\cdot\gamma + P^*(\gamma) = 0,\\ \mathcal{F}^*_{2n}(\overline{\gamma}) &= M^*(\gamma)\cdot\overline{\gamma}^2 + N^*(\gamma)\cdot\overline{\gamma} + P^*(\gamma) = 0.\end{aligned} \tag{19}$$

(d) Let $\mathbb{S}(\alpha; |\beta|) := \{z \in \mathbb{H} | z = \alpha + \boldsymbol{r} \text{ with } |\boldsymbol{r}| = |\beta|\}$ be a sphere centered at the point $\alpha \in \mathbb{R}$ and of radius $R = |\beta|$. As it follows from the proof of Proposition 2, if $w \in \mathbb{S}(\alpha; R)$ then w is a zero of $\mathcal{F}^*_{2n}$.

(e) Let

$$\mathcal{R}_n(\gamma) = A(\gamma)\cdot\gamma + B(\gamma)$$

be the representation (6) for $\mathcal{R}_n$; then

$$\mathcal{L}_n(\gamma) = \gamma\cdot\overline{A(\gamma)} + \overline{B(\gamma)}.$$

Note that there hold the following relations for the coefficients of (19):

$$\begin{aligned} M^*(\gamma) &= |A(\gamma)|^2; \\ N^*(\gamma) &= 2\mathrm{Sc}(\overline{A(\gamma)}B(\gamma)); \\ P^*(\gamma) &= |B(\gamma)|^2. \end{aligned}$$

(f) Take now $w_0 \in \mathbb{S}(\alpha; R)$ such that $\mathrm{Vec}(w_0) \parallel \mathrm{Vec}(\overline{A}B)$, then

$$\begin{aligned} \mathcal{L}_n(w_0) \cdot \mathcal{R}_n(w_0) &= (w_0\overline{A(\gamma)} + \overline{B(\gamma)}) \cdot (A(\gamma)w_0 + B(\gamma)) \\ &= M^*(\gamma) \cdot w_0^2 + N^*(\gamma)w_0 + P^*(\gamma) \\ &= \mathcal{F}^*_{2n}(w_0) = 0, \end{aligned}$$

and we may conclude that either w_0 or $\overline{w_0}$ is a zero of the polynomial $\mathcal{R}_n$.

5 COROLLARY

Given a polynomial $\mathcal{R}_n$ (or $\mathcal{L}_n$), there exist a one-to-one correspondence between its nonspherical zeroes and the pairs of the complex-conjugate zeroes of the basic polynomial $\mathcal{F}^*_{2n}$ as well as a one-to-one correspondence between the spherical zeroes of $\mathcal{R}_n$ (or $\mathcal{L}_n$) and the pairs of complex-conjugate zeroes of multiplicity 2 of the basic polynomial $\mathcal{F}^*_{2n}$.

6 THEOREM

Given a quaternionic polynomial $\mathcal{R}_n$ (or $\mathcal{L}_n$), it has zeroes which are either isolated points of fill in entire spheres; the number of the isolated points together with the double number of the spheres is less or equal to n.

7 EXAMPLES

7.1

Let $\mathcal{R}_2(z) := z^2 + jz + k$, then (see paragraph (a) in the proof of Proposition 3) $\mathcal{L}_2(z) := \overline{\mathcal{R}_3(\overline{z})} = (z^2 - zj - k)$ and $\mathcal{F}_4 := \mathcal{L}_2(z) \cdot \mathcal{R}_2(z) = z^4 + z^2jz + z^2k - zjz^2 + z^2 - zi - kz^2 + iz + 1$ (according to paragraph (c) of the same proof). The latter generates the basic polynomial $\mathcal{F}^*_4$ for the polynomial $\mathcal{R}_2$ by the formula (14) which gives:

$$\mathcal{F}^*_4(z) = z^4 + z^2 + 1. \tag{20}$$

As a complex polynomial, (20) has four zeroes: $z_1 = (1/2)(1 + i\sqrt{3})$, $z_2 = \overline{z_1} = (1/2)(1 - i\sqrt{3})$, $z_3 = -(1/2)(1 - i\sqrt{3})$, $z_4 = \overline{z}_3 = -(1/2)(1 + i\sqrt{3})$. Hence, all the zeroes of the polynomial $\mathcal{R}_2$ should be looked for on the spheres $\mathbb{S}_3((1/2);(\sqrt{3}/2))$ and $\mathbb{S}_3(-(1/2);(\sqrt{3}/2))$. Direct computation leads to the following ones:

$$w_1 = \tfrac{1}{2}(1+i-j-k); \qquad w_2 = -\tfrac{1}{2}(1-i+j-k).$$

Thus, in this example the polynomial $\mathcal{R}_2$ has exactly two zeroes.

It is essential to note that none of the products $(z-w_1)(z-w_2)$ and $(z-w_2)(z-w_1)$ coincides with the polynomial $\mathcal{R}_2$!

The reasoning for the polynomial $\mathcal{R}_3(z) := z^3 + iz^2 + jz + k$ is quite similar so we omit the details. The basic polynomial $\mathcal{F}_6^*$ has the form: $\mathcal{F}_6^* = z^6 + z^4 + z^2 + 1$, and it has the following six (complex) zeroes: $z_1 = e^{i(\pi/4)}$, $z_2 = e^{-i(\pi/4)}$, $z_3 = e^{i(5\pi/4)}$, $z_4 = e^{-i(5\pi/4)}$, $z_5 = i$, $z_6 = -i$.

Hence, all the zeroes of $\mathcal{R}_3$ are located on the spheres $\mathbb{S}_3((\sqrt{2}/2);\sqrt{(2/2)})$; $\mathbb{S}_3(-(\sqrt{2}/2);(\sqrt{2}/2))$, $\mathbb{S}_3(0;1)$ which give the following ones: $w_1 = (\sqrt{2}/2)(1-j)$; $w_2 = -(\sqrt{2}/2)(1-j)$, $w_3 = k$.

The next polynomial has both isolated and spherical zeros. Take $\mathcal{R}_5(z) = z^5 + iz^4 + (j+1)z^3 + (k+1)z^2 + jz + k$, then there holds:

$$\mathcal{R}_5(z) = \mathcal{R}_3(z)(z^2+1)$$

with $\mathcal{R}_3$ from the previous subsection. Hence for $\mathcal{R}_5$ the zeros are: w_1, w_2, w_3 from subsection 7.2 and the entire sphere $\mathbb{S}_3(0;1)$.

Theorem 6 states that all the considered polynomials have no other zeroes.

All the above polynomials illustrate the situation for which there is "equal" in the theorem, i.e., the number of the isolated zeroes together with the double number of the spheres coincides with the degree of the polynomial. A simple example of the polynomial $\mathcal{R}_4(z) = z^4 + 2z^2 + 1 = (z^2+1)^2$ having one sphere of zeroes while its degree if four, demonstrates the case when they do not coincide. Of course, this is related to a much more complicated question about the notion of the multiplicity of the zero.

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