Foundations of Computational Logic

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Chapter 1

Introduction

1.1 Acknowledgements

This document presents the logical foundations for a formal system based on first-order logic. The net of definitions and theorems presented here were designed by a human author - Michał Stanisław Wójcik. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed, verified and corrected by Michał Stanisław Wójcik who takes the full responsibility for the content of this work.

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Chapter 2

First Order Logic

2.1 Preliminaries

We will use $\mathbb{N}$ to denote a set of all nonnegative integers.

Definition 2.1.1. *Let S be an arbitrary set.*

1. *$S^0 \stackrel{\text{def}}{=} \{\emptyset\}$.*

2. *$S^k \stackrel{\text{def}}{=} \underbrace{S \times \cdots \times S}_{k}$.*

3. *$S^* \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^k$.*

Note that $\emptyset$ is treated as empty sequence.

Remark 2.1.2. *For a sequence $\boldsymbol{s} \in S^k$, we denote the i-th element of the sequence by s_i where $i \in \{1, \ldots, k\}$. let also $|\boldsymbol{s}| \stackrel{\text{def}}{=} k$ denote a length of $\boldsymbol{s}$.*

Note that according to the above convention $|\emptyset| = 0$.

Definition 2.1.3. *Let S be an arbitrary set.*

1. *$S^{\nabla 0} \stackrel{\text{def}}{=} \{\emptyset\}$.*

2. *$S^{\nabla k} \stackrel{\text{def}}{=} \{\boldsymbol{s} \in S^k : \boldsymbol{s}_i \neq \boldsymbol{s}_j \text{ for } i \neq j\}$.*

3. *$S^{\nabla} \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^{\nabla k}$.*

Sometimes if not disambiguous, we will treat $s \in S^{\nabla}$ notationally as set, so we will write $s_i \in s$, or for a set a, $a \setminus s$, $a \subset s$ etc.

2.2 First Order Logic Formula

2.2.1 Syntax

Let $\mathfrak{L}$ denote a set of all logical symbols " (", ") ", " $\rightarrow$ ", " . ", ", ", " $\forall$ ". Let $\mathcal{V}$ be an infinite countable set of variable symbols. And $\mathfrak{f}$ be a special predicate symbol.

Definition 2.2.1. *Let $\mathscr{P}$ be a set of predicate symbols for which $\mathfrak{f} \in \mathscr{P}$ and $\mathscr{F}$ be a set of function symbols. Let $\mathfrak{a} : \mathscr{P} \cup \mathscr{F} \to \mathbb{N}$ be an arity function. then $\mathscr{L} \stackrel{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ is a signature of a language.*

To write first order logic formulas for a given language signature $\mathscr{L}$, we will use symbols from

$$\mathcal{S}(\mathscr{L}) \stackrel{\text{def}}{=} \mathfrak{L} \cup \mathscr{V} \cup \mathscr{P} \cup \mathscr{F}. \tag{2.1}$$

All formulas will be then a subset of $\mathcal{S}(\mathscr{L})^*$. Elements of $\mathcal{S}(\mathscr{L})^*$ will be called strings.

Note 2.2.2 (**syntactic equality**). *We use the notation $\doteq$ to denote equality of strings. We will sometimes overload it when cast of some object to string is obvious.*

When we stack symbols together from left to right, we mean that we concatenate them in the same order. E.g let p be a predicate symbol and x, y be variable symbols, then $p(x, y)$ is just a sequence of 6 symbols: " p ", "(", " x ", " , ", " y ", ") ".

For convenience we will use an abreviated style of writing, in which for any symbol $s \in \mathscr{P} \cup \mathscr{F}$ and for any seqence of strings $\boldsymbol{x} \in \mathcal{S}(\mathscr{L})^*$

$$s(\boldsymbol{x}) \doteq \begin{cases} s(x_1, \ldots, x_k) \text{ for } |\boldsymbol{x}| > 0, \\ s \text{ for } |\boldsymbol{x}| = 0. \end{cases} \tag{2.2}$$

Definition 2.2.3 (**arity**). *Let $\mathscr{L} \stackrel{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

$$\mathscr{P}_k \stackrel{\text{def}}{=} \{p \in \mathscr{P} : \mathfrak{a}(p) = k\}, \tag{2.3}$$

$$\mathscr{F}_k \stackrel{\text{def}}{=} \{f \in \mathscr{F} : \mathfrak{a}(f) = k\}. \tag{2.4}$$

Definition 2.2.4 (**atomic term**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

 1. *For any $v \in \mathscr{V}$, v is an **atomic term**.*

 2. *For any $c \in \mathscr{F}_0$, c is an **atomic term**.*

Definition 2.2.5 (**recursive: term**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all terms $\mathcal{T}(\mathscr{L})$.*

Let $s \in \mathcal{S}(\mathscr{L})^*$.

 1. *If $s \doteq v$ where $v \in \mathscr{V}$, then $s \in \mathcal{T}(\mathscr{L})$.*

 2. *If $s \doteq f(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\tau}|}$, then $s \in \mathcal{T}(\mathscr{L})$.*

 3. *Otherwise $s \notin \mathcal{T}(\mathscr{L})$.*

Definition 2.2.6 (**atomic formula**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define a set of all atomic formulas $\mathcal{A}(\mathscr{L})$.*

$$\mathcal{A}(\mathscr{L}) \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \{p(\boldsymbol{\tau}) : p \in \mathscr{P}_k, \boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k\}. \tag{2.5}$$

Definition 2.2.7 (**recursive: base formula, well-formed formula**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all base formulas $\mathcal{B}(\mathscr{L})$ and a set of all well-formed formulas $\mathcal{F}(\mathscr{L})$.*

Let $s \in \mathcal{S}(\mathscr{L})^*$.

 1. *If $s \in \mathcal{A}(\mathscr{L})$, then $s \in \mathcal{B}(\mathscr{L})$.*

2. If $s \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.

3. If $s \doteq (\phi)$ where $\phi \in \mathcal{F}(\mathcal{L})$, then $s \in \mathcal{B}(\mathcal{L})$.

4. If $s \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.

5. If $s \doteq \forall v.\phi$ where $\phi \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.

6. Otherwise $s \notin \mathcal{F}(\mathcal{L})$ and $s \notin \mathcal{B}(\mathcal{L})$.

Definition 2.2.8 (recursive: variable in term). *Let $\mathcal{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $var(\tau)$ of all variables in a term τ.*

Let $v \in \mathscr{V}$ and $\tau \in \mathcal{T}(\mathcal{L})$.

1. *If $\tau \doteq v$ then $v \in var(\tau)$.*

2. *If $\tau \doteq f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\sigma}|}$ and $v \in var(\boldsymbol{\sigma}) \stackrel{\text{def}}{=} \bigcup_{k=1}^{|\boldsymbol{\sigma}|} var(\sigma_k)$, then $v \in var(\tau)$.*

3. *Otherwise $v \notin var(\tau)$.*

Definition 2.2.9 (recursive: variable in formula). *Let $\mathcal{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $var(\phi)$ of all variables in a formula ϕ.*

Let $v \in \mathscr{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. *If $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$ and $v \in var(\boldsymbol{\tau})$, then $v \in var(\phi)$.*

2. *If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and $v \in var(\alpha)$ or $v \in var(\beta)$, then $v \in var(\phi)$.*

3. *If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, and either $v = x$ or $v \in var(\psi)$, then $v \in var(\phi)$.*

4. *Otherwise $v \notin var(\phi)$.*

Definition 2.2.10 (recursive: free variable in formula). *Let $\mathcal{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $free(\phi)$ of all free variables in a formula ϕ.*

Let $v \in \mathscr{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. *If $\phi \in \mathcal{A}(\mathcal{L})$ and $v \in var(\phi)$, then $v \in free(\phi)$.*

2. *If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and $v \in free(\alpha)$ or $v \in free(\beta)$, then $v \in free(\phi)$.*

3. *If $\phi \doteq \forall x.\psi$ where $\psi \in \mathcal{B}(\mathcal{L})$, $x \in \mathscr{V}$ and $x \neq v$, and $v \in free(\psi)$, then $v \in free(\phi)$.*

4. *Otherwise $v \notin free(\phi)$.*

2.2.2 Semantics

Definition 2.2.11 ($\mathcal{L}$-structure). *Let $\mathcal{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and $A \neq \emptyset$ be an arbitrary mathematical domain, then a mapping $\mathbf{M}$ such that*

$$\mathbf{M} : \mathscr{P} \cup \mathscr{F} \to \bigcup_{k=0}^{\infty} \{0,1\}^{(A^k)} \cup \bigcup_{k=0}^{\infty} A^{(A^k)}, \tag{2.6}$$

where

$$\mathbf{M}(\mathfrak{f}) = (\emptyset \mapsto 0),$$
$$\mathbf{M}(\mathscr{P}_k) \subset \{0,1\}^{(A^k)} \text{ for } k \in \mathbb{N},$$
$$\mathbf{M}(\mathscr{F}_k) \subset A^{(A^k)} \text{ for } k \in \mathbb{N}.$$

is called an $\mathscr{L}$-structure with domain A.

For convenience we will denote values of $\mathbf{M}$ as $p^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(p)$ for $p \in \mathscr{P}$ and $f^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(f)$ for $f \in \mathscr{F}$. Note that $p^{\mathbf{M}} : A^k \to \{0,1\}$ and $f^{\mathbf{M}} : A^k \to A$.

Definition 2.2.12 (**assignment**)**.** *Let A be an arbitrary mathematical domain. Then $j : \mathscr{V} \to A$ is called a variables assignment in A.*

Definition 2.2.13 (**recursive: term evaluation**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure and $j : \mathscr{V} \to A$.*

Let $\tau \in \mathcal{T}(\mathscr{L})$.

1. *If $\tau \doteq v$ where $v \in \mathscr{V}$, then $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} j(v)$.*

2. *If $\tau \doteq f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\sigma}|}$, then $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} f^{\mathbf{M}}(\boldsymbol{\sigma}^{\mathbf{M},j})$ where*

$$\boldsymbol{\sigma}^{\mathbf{M},j} \stackrel{\text{def}}{=} \begin{cases} (\sigma_1^{\mathbf{M},j}, \ldots, \sigma_k^{\mathbf{M},j}) \text{ for } k = |\boldsymbol{\sigma}| > 0, \\ \emptyset \text{ for } |\boldsymbol{\sigma}| = 0. \end{cases} \tag{2.7}$$

Definition 2.2.14. *Let $\mathscr{L}$ be a signature of a language. Let $\mathbf{M}$ be a $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$ be an assignment of variables. Then we define*

$$j\frac{a}{x}(v) \stackrel{\text{def}}{=} \begin{cases} a \text{ if } v = x, \\ j(v) \text{ otherwise.} \end{cases} \tag{2.8}$$

Definition 2.2.15. *Let $\mathscr{L}$ be a signature of a language. Let $\mathbf{M}$ be a $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$ be an assignment of variables. Let k be a positive integer and let $\boldsymbol{x} \in V^{\nabla k}$, and $\boldsymbol{a} \in A^k$.*
Then we define

$$j\frac{\boldsymbol{a}}{\boldsymbol{x}}(v) \stackrel{\text{def}}{=} \begin{cases} a_i \text{ if } v = x_i \text{ for some } i \in \{1, \ldots, k\}, \\ j(v) \text{ otherwise.} \end{cases} \tag{2.9}$$

For the purpouses of this work, we will define multiplication product of infinite number of 0s and/or 1s.

Definition 2.2.16. *Let S be an arbitrary set and $a_s \in \{0,1\}$ for any $s \in S$.*

$$\prod_{s \in S} a_s = \begin{cases} 1 \text{ if } a_s = 1 \text{ for all } s \in S, \\ 0 \text{ otherwise .} \end{cases} \tag{2.10}$$

Definition 2.2.17 (**recursive: formula evaluation**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$.*

1. If $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$, then $\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} p^{\mathbf{M}}(\boldsymbol{\tau}^{\mathbf{M},j})$.

2. If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}). \tag{2.11}$$

3. If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, then

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} \prod_{a \in A} \psi^{\mathbf{M},j\frac{a}{x}}. \tag{2.12}$$

Corollary 2.2.18. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*

$$(\phi \to \mathfrak{f})^{\mathbf{M},j} = 1 - \phi^{\mathbf{M},j}. \tag{2.13}$$

Definition 2.2.19 (**interpretation**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*

$$\mathbf{M}, j \models \phi \stackrel{\text{def}}{\iff} \phi^{\mathbf{M},j} = 1. \tag{2.14}$$

Definition 2.2.20 (**model**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ and let $\mathbf{M}$ be an $\mathscr{L}$-structure. Let $\phi \in \mathcal{F}(\mathscr{L})$. We define*

$$\mathbf{M} \models \phi \tag{2.15}$$

iff for any assignment $j : \mathscr{V} \to A$, we have $M, j \models \phi$.

Definition 2.2.21 (**semantic consequence**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\Gamma \subset \mathcal{F}(\mathscr{L})$ and $\phi \in \mathcal{F}(\mathscr{L})$. We define that $\Gamma \models \phi$ iff for any $\mathscr{L}$-structure $\mathbf{M}$, if $\mathbf{M} \models \psi$ for all $\psi \in \Gamma$, then $\mathbf{M} \models \phi$.*

Definition 2.2.22 (**tautology**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\phi \in \mathcal{F}(\mathscr{L})$. We define that $\models \phi$ iff $\emptyset \models \phi$.*

Lemma 2.2.23. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j_1, j_2 : \mathscr{V} \to A$. Let $\tau \in \mathcal{T}(\mathscr{L})$. If $j_1(var(\tau)) = j_2(var(\tau))$, then*

$$\tau^{\mathbf{M},j_1} = \tau^{\mathbf{M},j_2}. \tag{2.16}$$

Proof. We proceed by structural induction on the term τ.
Case 1: $\tau \doteq v$ where $v \in \mathscr{V}$.
Then $\text{var}(\tau) = \{v\}$. Since $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, we have $j_1(v) = j_2(v)$. By the definition of term evaluation,

$$\tau^{\mathbf{M},j_1} = j_1(v) = j_2(v) = \tau^{\mathbf{M},j_2}.$$

Case 2: $\tau \doteq f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\sigma}|}$.
Then $\text{var}(\tau) = \text{var}(\boldsymbol{\sigma}) = \bigcup_{i=1}^{|\boldsymbol{\sigma}|} \text{var}(\sigma_i)$. Since $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, we have $j_1(\text{var}(\sigma_i)) = j_2(\text{var}(\sigma_i))$ for all $i \in \{1, \ldots, |\boldsymbol{\sigma}|\}$. By inductive hypothesis, $\sigma_i^{\mathbf{M},j_1} = \sigma_i^{\mathbf{M},j_2}$ for all i. Therefore, $\boldsymbol{\sigma}^{\mathbf{M},j_1} = \boldsymbol{\sigma}^{\mathbf{M},j_2}$. By the definition of term evaluation,

$$\tau^{\mathbf{M},j_1} = f^{\mathbf{M}}(\boldsymbol{\sigma}^{\mathbf{M},j_1}) = f^{\mathbf{M}}(\boldsymbol{\sigma}^{\mathbf{M},j_2}) = \tau^{\mathbf{M},j_2}.$$

This completes the proof by structural induction. $\square$

Theorem 2.2.24. *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j_1, j_2 : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$. If $j_1\big(\mathrm{free}(\phi)\big) = j_2\big(\mathrm{free}(\phi)\big)$, then*

$$\phi^{\mathbf{M}, j_1} = \phi^{\mathbf{M}, j_2}. \tag{2.17}$$

Proof. We proceed by structural induction on the formula ϕ.

Case 1: $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$.

Then $\mathrm{free}(\phi) = \mathrm{var}(\phi) = \mathrm{var}(\boldsymbol{\tau}) = \bigcup_{i=1}^{|\boldsymbol{\tau}|} \mathrm{var}(\tau_i)$. Since $j_1(\mathrm{free}(\phi)) = j_2(\mathrm{free}(\phi))$, we have $j_1(\mathrm{var}(\tau_i)) = j_2(\mathrm{var}(\tau_i))$ for all i. By Lemma 2.2.23, $\tau_i^{\mathbf{M}, j_1} = \tau_i^{\mathbf{M}, j_2}$ for all i. Therefore, $\boldsymbol{\tau}^{\mathbf{M}, j_1} = \boldsymbol{\tau}^{\mathbf{M}, j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = p^{\mathbf{M}}(\boldsymbol{\tau}^{\mathbf{M}, j_1}) = p^{\mathbf{M}}(\boldsymbol{\tau}^{\mathbf{M}, j_2}) = \phi^{\mathbf{M}, j_2}.$$

Case 2: $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.

Then $\mathrm{free}(\phi) = \mathrm{free}(\alpha) \cup \mathrm{free}(\beta)$. Since $j_1(\mathrm{free}(\phi)) = j_2(\mathrm{free}(\phi))$, we have $j_1(\mathrm{free}(\alpha)) = j_2(\mathrm{free}(\alpha))$ and $j_1(\mathrm{free}(\beta)) = j_2(\mathrm{free}(\beta))$. By inductive hypothesis, $\alpha^{\mathbf{M}, j_1} = \alpha^{\mathbf{M}, j_2}$ and $\beta^{\mathbf{M}, j_1} = \beta^{\mathbf{M}, j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = 1 - \alpha^{\mathbf{M}, j_1}(1 - \beta^{\mathbf{M}, j_1}) = 1 - \alpha^{\mathbf{M}, j_2}(1 - \beta^{\mathbf{M}, j_2}) = \phi^{\mathbf{M}, j_2}.$$

Case 3: $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$.

Then $\mathrm{free}(\phi) = \mathrm{free}(\psi) \setminus \{x\}$. For any $a \in A$, the assignments $j_1 \frac{a}{x}$ and $j_2 \frac{a}{x}$ agree on $\mathrm{free}(\psi)$ because:

- For $v \in \mathrm{free}(\psi) \setminus \{x\}$, we have $v \in \mathrm{free}(\phi)$, so $(j_1 \frac{a}{x})(v) = j_1(v) = j_2(v) = (j_2 \frac{a}{x})(v)$.

- For $v = x$, we have $(j_1 \frac{a}{x})(x) = a = (j_2 \frac{a}{x})(x)$.

Thus $j_1 \frac{a}{x}(\mathrm{free}(\psi)) = j_2 \frac{a}{x}(\mathrm{free}(\psi))$. By inductive hypothesis, $\psi^{\mathbf{M}, j_1 \frac{a}{x}} = \psi^{\mathbf{M}, j_2 \frac{a}{x}}$ for all $a \in A$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = \prod_{a \in A} \psi^{\mathbf{M}, j_1 \frac{a}{x}} = \prod_{a \in A} \psi^{\mathbf{M}, j_2 \frac{a}{x}} = \phi^{\mathbf{M}, j_2}.$$

This completes the proof by structural induction. □

2.3   Syntactic Sugar

In this section, we introduce common logical connectives as syntactic abbreviations of formulas using implication and the false predicate $\mathfrak{f}$.

Definition 2.3.1 (**logical connectives as syntactic sugar**). *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language and let $\alpha, \beta \in \mathcal{B}(\mathscr{L})$. We define the following syntactic*

abbreviations:

$$\mathfrak{t} \overset{\text{def}}{=} \mathfrak{f} \to \mathfrak{f} \tag{2.18}$$

$$\sim \alpha \overset{\text{def}}{=} \alpha \to \mathfrak{f}, \tag{2.19}$$

$$\alpha \wedge \beta \overset{\text{def}}{=} (\alpha \to (\beta \to \mathfrak{f})) \to \mathfrak{f}, \tag{2.20}$$

$$\alpha \vee \beta \overset{\text{def}}{=} (\alpha \to \mathfrak{f}) \to \beta, \tag{2.21}$$

$$\alpha \leftrightarrow \beta \overset{\text{def}}{=} (\alpha \to \beta) \wedge (\beta \to \alpha). \tag{2.22}$$

For any $v \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, we also define:

$$\exists v.\psi \overset{\text{def}}{=} \sim (\forall v. \sim \psi). \tag{2.23}$$

The following theorems characterize the semantic behavior of these connectives.

Theorem 2.3.2 (negation evaluation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\alpha \in \mathcal{B}(\mathscr{L})$, then*

$$(\sim \alpha)^{\mathbf{M},j} = 1 - \alpha^{\mathbf{M},j}. \tag{2.24}$$

Proof. By definition, $\sim \alpha \doteq \alpha \to \mathfrak{f}$. By the definition of formula evaluation for implication and the fact that $\mathfrak{f}^{\mathbf{M},j} = 0$,

$$(\sim \alpha)^{\mathbf{M},j} = (\alpha \to \mathfrak{f})^{\mathbf{M},j} = 1 - \alpha^{\mathbf{M},j}(1 - \mathfrak{f}^{\mathbf{M},j}) = 1 - \alpha^{\mathbf{M},j}(1 - 0) = 1 - \alpha^{\mathbf{M},j}. \tag{2.25}$$

$\square$

Theorem 2.3.3 (conjunction evaluation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then*

$$(\alpha \wedge \beta)^{\mathbf{M},j} = \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}. \tag{2.26}$$

Proof. By definition, $\alpha \wedge \beta \doteq (\alpha \to (\beta \to \mathfrak{f})) \to \mathfrak{f}$. By the definition of formula evaluation for implication and the fact that $\mathfrak{f}^{\mathbf{M},j} = 0$,

$$\begin{aligned}
(\alpha \wedge \beta)^{\mathbf{M},j} &= ((\alpha \to (\beta \to \mathfrak{f})) \to \mathfrak{f})^{\mathbf{M},j} \\
&= 1 - (\alpha \to (\beta \to \mathfrak{f}))^{\mathbf{M},j}(1 - 0) \\
&= 1 - (\alpha \to (\beta \to \mathfrak{f}))^{\mathbf{M},j}.
\end{aligned}$$

Now, expanding the inner implication:

$$\begin{aligned}
(\alpha \to (\beta \to \mathfrak{f}))^{\mathbf{M},j} &= 1 - \alpha^{\mathbf{M},j}(1 - (\beta \to \mathfrak{f})^{\mathbf{M},j}) \\
&= 1 - \alpha^{\mathbf{M},j}(1 - (1 - \beta^{\mathbf{M},j}(1 - 0))) \\
&= 1 - \alpha^{\mathbf{M},j}(1 - (1 - \beta^{\mathbf{M},j})) \\
&= 1 - \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}.
\end{aligned}$$

Therefore,

$$(\alpha \wedge \beta)^{\mathbf{M},j} = 1 - (1 - \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}) = \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}. \tag{2.27}$$

$\square$

Theorem 2.3.4 (**disjunction evaluation**). *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language and A be an arbitrary mathematical domain, let* $\mathbf{M}$ *be an* $\mathscr{L}$-*structure with domain A and $j : \mathscr{V} \to A$. Let $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then*

$$(\alpha \vee \beta)^{\mathbf{M},j} = 1 - (1 - \alpha^{\mathbf{M},j})(1 - \beta^{\mathbf{M},j}). \tag{2.28}$$

Proof. By definition, $\alpha \vee \beta \doteq (\alpha \to \mathfrak{f}) \to \beta$. By the definition of formula evaluation for implication:

$$\begin{aligned}
(\alpha \vee \beta)^{\mathbf{M},j} &= ((\alpha \to \mathfrak{f}) \to \beta)^{\mathbf{M},j} \\
&= 1 - (\alpha \to \mathfrak{f})^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) \\
&= 1 - (1 - \alpha^{\mathbf{M},j})(1 - \beta^{\mathbf{M},j}).
\end{aligned}$$

$\square$

Theorem 2.3.5 (**biconditional evaluation**). *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language and A be an arbitrary mathematical domain, let* $\mathbf{M}$ *be an* $\mathscr{L}$-*structure with domain A and $j : \mathscr{V} \to A$. Let $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then*

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = 1 \text{ if and only if } \alpha^{\mathbf{M},j} = \beta^{\mathbf{M},j}. \tag{2.29}$$

Proof. By definition, $\alpha \leftrightarrow \beta \doteq (\alpha \to \beta) \wedge (\beta \to \alpha)$. By Theorem 2.3.3,

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = (\alpha \to \beta)^{\mathbf{M},j} \cdot (\beta \to \alpha)^{\mathbf{M},j}. \tag{2.30}$$

Expanding using the definition of implication:

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = (1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}))(1 - \beta^{\mathbf{M},j}(1 - \alpha^{\mathbf{M},j})).$$

This product equals 1 if and only if both factors equal 1, which happens if and only if:

- $\alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) = 0$, and

- $\beta^{\mathbf{M},j}(1 - \alpha^{\mathbf{M},j}) = 0$.

These conditions hold if and only if $\alpha^{\mathbf{M},j} = \beta^{\mathbf{M},j}$. $\square$

Theorem 2.3.6 (**existential quantifier evaluation**). *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language and A be an arbitrary mathematical domain, let* $\mathbf{M}$ *be an* $\mathscr{L}$-*structure with domain A and $j : \mathscr{V} \to A$. Let $v \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, then*

$$(\exists v.\psi)^{\mathbf{M},j} = 1 - \prod_{a \in A} (1 - \psi^{\mathbf{M},j\frac{a}{v}}). \tag{2.31}$$

Proof. By definition, $\exists v.\psi \doteq \sim (\forall v. \sim \psi)$. By Theorem 2.3.2,

$$(\exists v.\psi)^{\mathbf{M},j} = 1 - (\forall v. \sim \psi)^{\mathbf{M},j}. \tag{2.32}$$

By the definition of formula evaluation for universal quantification,

$$(\forall v. \sim \psi)^{\mathbf{M},j} = \prod_{a \in A} (\sim \psi)^{\mathbf{M},j\frac{a}{v}}. \tag{2.33}$$

By Theorem 2.3.2, for each $a \in A$,

$$(\sim \psi)^{\mathbf{M},j}\frac{a}{v} = 1 - \psi^{\mathbf{M},j}\frac{a}{v}. \tag{2.34}$$

Therefore,

$$(\exists v.\psi)^{\mathbf{M},j} = 1 - \prod_{a \in A}(1 - \psi^{\mathbf{M},j}\frac{a}{v}). \tag{2.35}$$

$\square$

2.4 Substitutions

2.4.1 Syntax

Definition 2.4.1 (recursive: substitution in a term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$. Let $\sigma \in \mathcal{T}(\mathscr{L})$. We will define recursively $\sigma(\boldsymbol{x}/\boldsymbol{\tau})$. Let's establish that $\sigma(\emptyset/\emptyset) \overset{\text{def}}{=} \sigma$.*

1. *If $\sigma \doteq x_i$ for some $i \in \{1, \ldots, k\}$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} \tau_i$.*

2. *If $\sigma \doteq v$ where $v \in \mathscr{V}$ and $v \neq x$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} v$.*

3. *If $\tau \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\eta}|}$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$ where*

$$\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} \begin{cases} (\eta_1(\boldsymbol{x}/\boldsymbol{\tau}), \ldots, \eta_k(\boldsymbol{x}/\boldsymbol{\tau})) \text{ for } k = |\boldsymbol{\eta}| > 0, \\ \emptyset \text{ for } |\boldsymbol{\eta}| = 0. \end{cases} \tag{2.36}$$

Definition 2.4.2 (recursive: admissible substitution in a formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$.*

*Let $\phi \in \mathcal{F}(\mathscr{L})$. We will define recursively $\sigma(\boldsymbol{x}/\boldsymbol{\tau})$. Let's establish that $\phi(\emptyset/\emptyset) \overset{\text{def}}{=} \phi$ and is **admissible**.*

1. *If $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$ and is **admissible**.*

2. *If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$ and $\alpha(\boldsymbol{x}/\boldsymbol{\tau}), \beta(\boldsymbol{x}/\boldsymbol{\tau})$ are **admissible**, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} \alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau})$ and is **admissible**.*

3. *If $\phi \doteq \forall u.\psi$ where $u \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$:*
   Let $\boldsymbol{v}$ be $\boldsymbol{x}$ with x_i removed when $x_i = u$ or $x_i \notin free(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from $\boldsymbol{x}$.

   (a) *If $\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin var(\boldsymbol{\sigma})$, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} \forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ and is **admissable**.*

   (b) *Otherwise $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is **not admissible**.*

Definition 2.4.3 (recursive: admissible predicate substitution). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Let $q \in \mathscr{P}_k$ and $\phi \in \mathcal{F}(\mathscr{L})$.*

Let $\psi \in \mathcal{F}(\mathscr{L})$. We will define recursively $\psi(q(\boldsymbol{x})/\phi)$.

1. If $\psi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.

    (a) If $p = q$ and $\phi(\boldsymbol{x}/\boldsymbol{\eta})$ is admissible, then $\psi(q(\boldsymbol{x})/\phi) \overset{\text{def}}{=} \phi(\boldsymbol{x}/\boldsymbol{\eta})$ and is **admissible**.

    (b) If $p \neq q$, then $\psi(q(\boldsymbol{x})/\phi) \overset{\text{def}}{=} \psi$ and is **admissible**.

2. If $\psi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(q(\boldsymbol{x})/\phi)$, $\beta(q(\boldsymbol{x})/\phi)$ are **admissible**, then $\psi(q(\boldsymbol{x})/\phi) \overset{\text{def}}{=} \alpha(q(\boldsymbol{x})/\phi) \to \beta(q(\boldsymbol{x})/\phi)$ and is **admissible**.

3. If $\psi \doteq \forall u.\gamma$ where $u \in \mathcal{V}$ and $\gamma \in \mathcal{B}(\mathcal{L})$:

    (a) If q occurs in γ, $u \notin \text{free}(\phi) \setminus \boldsymbol{x}$, $\gamma(q(\boldsymbol{x})/\phi)$ is **admissible**, then $\psi(q(\boldsymbol{x})/\phi) \overset{\text{def}}{=} \forall u.\gamma(q(\boldsymbol{x})/\phi)$ and is **admissible**.

    (b) If q does not occur in γ, then $\psi(q(\boldsymbol{x})/\phi) \overset{\text{def}}{=} \forall u.\gamma$ and is **admissible**.

    (c) Otherwise $\psi(q(\boldsymbol{x})/\phi)$ is **not admissible**.

2.4.2   Semantics

Lemma 2.4.4. *Let $\mathcal{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain. Let $\mathbf{M}$ be an $\mathcal{L}$-structure with domain A and $j : \mathcal{V} \to A$ be an assignment.*
Let k be a positive integer and let $\boldsymbol{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$.

Then for any term σ, we have

$$\sigma^{\mathbf{M},j \dfrac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}. \tag{2.37}$$

Proof. We proceed by structural induction on the term σ.
Case 1: $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$.
Then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \tau_i$ and

$$\sigma^{\mathbf{M},j \dfrac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = (j \dfrac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}})(x_i) = \tau_i^{\mathbf{M},j} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 2: $\sigma \doteq v$ where $v \in \mathcal{V}$ and $v \neq x$.
Then $\sigma(x/\tau) \doteq v$ and

$$\sigma^{\mathbf{M},j \dfrac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = (j \dfrac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}})(v) = j(v) = v^{\mathbf{M},j} = (\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 3: $\sigma \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\eta}|}$.
Then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \doteq f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$. By inductive hypothesis, $\boldsymbol{\eta}^{\mathbf{M},j \dfrac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} = \boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}$. Thus

$$\sigma^{\mathbf{M},j \dfrac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = f^{\mathbf{M}}\left(\boldsymbol{\eta}^{\mathbf{M},j \dfrac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}}\right) = f^{\mathbf{M}}\left(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}\right)$$
$$= f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

This completes the proof by structural induction. $\square$

Theorem 2.4.5. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $\phi \in \mathcal{F}(\mathscr{L})$. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$. If $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is admissible, then for any $\mathscr{L}$-structure $\mathbf{M}$ with domain A and for any assignment of variables $j : \mathscr{V} \to A$ we have*

$$(\mathbf{M}, j \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}) \models \phi \iff \mathbf{M}, j \models \phi(\boldsymbol{x}/\boldsymbol{\tau}). \tag{2.38}$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathscr{L})$.

Case 1: $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.

By the definition of substitution in well formed formula, $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$. By Lemma 2.4.4, we have $\boldsymbol{\eta}^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}} = \boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}$. Therefore,

$$\phi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{x} = p^{\mathbf{M}}\left(\boldsymbol{\eta}^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\right) = p^{\mathbf{M}}(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}) = p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = \phi(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 2: $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.

By the definition of substitution in well formed formula, $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau})$. By inductive hypothesis,

$$(\mathbf{M}, j \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}) \models \alpha \iff \mathbf{M}, j \models \alpha(\boldsymbol{x}/\boldsymbol{\tau}),$$

$$(\mathbf{M}, j \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}) \models \beta \iff \mathbf{M}, j \models \beta(\boldsymbol{x}/\boldsymbol{\tau}).$$

By the definition of formula evaluation,

$$\phi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}} = 1 - \alpha^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\left(1 - \beta^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\right) = 1 - (\alpha(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}(1 - (\beta(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}) = 1$$
$$= (\alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = (\phi(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}$$

Case 3: $\phi \doteq \forall u.\psi$ where $u \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$:

Let $\boldsymbol{v}$ be $\boldsymbol{x}$ with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from $\boldsymbol{x}$.

Since $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is admissible, $\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, and we have $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma})$. Thus

$$\phi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}} = \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}} \frac{a}{u} \tag{2.39}$$

By choice of $\boldsymbol{v}$, $j \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}} \frac{a}{u}$ and $j \frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}} \frac{a}{u}$ agrees on $\text{free}(\psi)$ therefore by Theorem 2.2.24 we have

$$\prod_{a\in A}\psi^{\mathbf{M},j}\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\frac{a}{u}=\prod_{a\in A}\psi^{\mathbf{M},j}\frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}}\frac{a}{u}$$

$$=\prod_{a\in A}\psi^{\mathbf{M},j}\frac{a}{u}\frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}}\quad\left(\text{since } u \text{ not in } \boldsymbol{v}\right)$$

$$=\prod_{a\in A}\psi(\boldsymbol{v}/\boldsymbol{\sigma})^{\mathbf{M},j}\frac{a}{u}\quad\left(\text{by inductive hypothesis}\right)$$

$$=(\forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma}))^{\mathbf{M},j}=\phi(\boldsymbol{x}/\boldsymbol{\tau})^{M,j}.$$

This completes the proof by structural induction. $\square$

Lemma 2.4.6. *Let $\mathscr{L} = (\mathscr{P},\mathscr{F},\mathfrak{a})$ be a signature of a language and let $q \in \mathscr{P}$. Let $\phi \in \mathcal{F}(\mathscr{L})$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Let $j_0 : free(\phi) \setminus \boldsymbol{x} \to A$, where A is a domain of an $\mathscr{L}$-structure M for which*

$$M, j \models q(\boldsymbol{x}) \text{ iff } M, j \models \phi \tag{2.40}$$

for any j that is an extention of j_0. If $\psi \in \mathcal{F}(\mathscr{L})$ and $\psi(q(\boldsymbol{x})/\phi)$ is admissible, then

$$M, j \models \psi \text{ iff } M, j \models \psi(q(\boldsymbol{x})/\phi) \tag{2.41}$$

for any j that is an extention of j_0.

Proof. We proceed by structural induction on $\psi \in \mathcal{F}(\mathscr{L})$ following Definition 2.4.3.
Take any $j : \mathscr{V} \to A$ which is an extension of j_0.
Case 1: $\psi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.
Subcase 1a: $p = q$ and $\phi(\boldsymbol{x}/\boldsymbol{\eta})$ is admissible.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \phi(\boldsymbol{x}/\boldsymbol{\eta})$. By the definition of formula evaluation,

$$\psi^{M,j} = q^M(\boldsymbol{\eta}^{M,j}) = q(\boldsymbol{x})^{M,j}\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}}.$$

Since $j\dfrac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}}$ is an extension of j_0, by (2.40), we have

$$q(\boldsymbol{x})^{M,j}\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}} = \phi^{M,j}\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}}.$$

By Theorem 2.4.5, since $\phi(\boldsymbol{x}/\boldsymbol{\eta})$ is admissible,

$$\phi^{M,j}\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}} = \phi(\boldsymbol{x}/\boldsymbol{\eta})^{M,j}.$$

Therefore, $\psi^{M,j} = \phi(\boldsymbol{x}/\boldsymbol{\eta})^{M,j} = \psi(q(\boldsymbol{x})/\phi)^{M,j}$.
Subcase 1b: $p \neq q$.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \psi$, so $\psi^{M,j} = \psi(q(\boldsymbol{x})/\phi)^{M,j}$ holds trivially.
Case 2: $\psi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$ and $\alpha(q(\boldsymbol{x})/\phi), \beta(q(\boldsymbol{x})/\phi)$ are admissible.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \alpha(q(\boldsymbol{x})/\phi) \to \beta(q(\boldsymbol{x})/\phi)$. By inductive hypothesis, $\alpha^{M,j} = \alpha(q(\boldsymbol{x})/\phi)^{M,j}$ and $\beta^{M,j} = \beta(q(\boldsymbol{x})/\phi)^{M,j}$. By the definition of formula

evaluation,

$$
\begin{aligned}
\psi^{M,j} &= 1 - \alpha^{M,j}(1 - \beta^{M,j}) \\
&= 1 - \alpha(q(\boldsymbol{x})/\phi)^{M,j}(1 - \beta(q(\boldsymbol{x})/\phi)^{M,j}) \\
&= (\alpha(q(\boldsymbol{x})/\phi) \to \beta(q(\boldsymbol{x})/\phi))^{M,j} \\
&= \psi(q(\boldsymbol{x})/\phi)^{M,j}.
\end{aligned}
$$

Case 3: $\psi \doteq \forall u.\gamma$ where $u \in \mathscr{V}$ and $\gamma \in \mathcal{B}(\mathscr{L})$.
Subcase 3a: q does not occur in γ.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \forall u.\gamma = \psi$, so $\psi^{M,j} = \psi(q(\boldsymbol{x})/\phi)^{M,j}$ holds trivially.
Subcase 3b: q occurs in γ. Since $\psi(q(\boldsymbol{x})/\phi)$ is admissible, $u \notin \text{free}(\phi) \setminus \boldsymbol{x}$ and $\gamma(q(\boldsymbol{x})/\phi)$ is admissible.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \forall u.\gamma(q(\boldsymbol{x})/\phi)$.

For any $a \in A$, let $j' = j\dfrac{a}{u}$. Since $u \notin \text{free}(\phi) \setminus \boldsymbol{x}$, the assignment j' is also an extension of j_0. By inductive hypothesis, $\gamma^{M,j'} = \gamma(q(\boldsymbol{x})/\phi)^{M,j'}$.
This holds for all $a \in A$, therefore by the definition of formula evaluation,

$$
\begin{aligned}
\psi^{M,j} &= \prod_{a \in A} \gamma^{M,j\frac{a}{u}} \\
&= \prod_{a \in A} \gamma(q(\boldsymbol{x})/\phi)^{M,j\frac{a}{u}} \\
&= (\forall u.\gamma(q(\boldsymbol{x})/\phi))^{M,j} \\
&= \psi(q(\boldsymbol{x})/\phi)^{M,j}.
\end{aligned}
$$

This completes the proof by structural induction. $\qquad\square$

Theorem 2.4.7. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $q \in \mathscr{P}$. Let $\phi \in \mathcal{F}(\mathscr{L})$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. If $\models \psi$ and $\psi(q(\boldsymbol{x})/\phi)$ is admissible, then $\models \psi(q(\boldsymbol{x})/\phi)$.*

Proof. Let's introduce a new predicate symbol q' such that $q' \notin \mathscr{P}$. Let $\mathscr{L}' = (\mathscr{P} \cup \{q'\}, \mathscr{F}, \mathfrak{a}')$ such that $\mathfrak{a}'$ is an extension of $\mathfrak{a}$ such that $\mathfrak{a}'(q') = \mathfrak{a}(q)$. Let $k = \mathfrak{a}(q)$. Let ψ' be ψ with each q symbol replaced by q'. Since $\models \psi$, also $\models \psi'$. Take any non-empty domain A. Take any $\mathscr{L}$-structure $\mathbf{M}$ with domain A. Take any $j_0 : \text{free}(\phi) \setminus \boldsymbol{x} \to A$. We will define an $\mathscr{L}'$ structure.

$$
\mathbf{M}'_{j_0}(s) = \begin{cases} \mathbf{M}'(s) \text{ for } s \in \mathscr{P} \cup \mathscr{F}, \\ \left(A^k \ni \boldsymbol{a} \mapsto \phi^{M,l\frac{\boldsymbol{a}}{\boldsymbol{x}}}\right) \text{ for } s = q'. \end{cases} \tag{2.42}
$$

Where $l : \mathscr{V} \to A$ is some fixed extension of j_0 (note that choice of l does not matter for the above definition). Take any $j : \mathscr{V} \to A$ which is an extension of j_0. Since $\models \psi'$, we have $M'_{j_0}, j \models \psi'$. Note that

$$
q'(\boldsymbol{x})^{M'_{j_0},j} = (q')^{M'_{j_0}}(\boldsymbol{x}^{M'_{j_0},j}) = \phi^{M'_{j_0},l\frac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}}}. \tag{2.43}
$$

Since $l\dfrac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}}$ and $j\dfrac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}}$ agrees on $\text{free}(\phi)$, by Theorem 2.2.24 we have

$$\phi^{M'_{j_0},l}\frac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}} = \phi^{M'_{j_0},j}\frac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}} = \phi^{M'_{j_0},j}. \tag{2.44}$$

We have shown that

$$q'(\boldsymbol{x})^{M'_{j_0},j} = \phi^{M'_{j_0},j}. \tag{2.45}$$

Thus by Theorem 2.4.7, we have $M'_{j_0}, j \models \psi'(q(\boldsymbol{x})/\phi)$. Note that $\psi'(q'(\boldsymbol{x})/\phi) \doteq \psi(q(\boldsymbol{x})/\phi)$ and therefore $\psi'(q'(\boldsymbol{x})/\phi) \in \mathcal{F}(\mathscr{L})$. Then by construction of M'_{j_0}, we have $M, j \models \psi'(q'(\boldsymbol{x})/\phi)$, and consequently

$$M, j \models \psi(q(\boldsymbol{x})/\phi). \tag{2.46}$$

Since $j_0 : \text{free}(\phi) \setminus \boldsymbol{x} \to A$ was arbitrarly chosen, (2.46) holds for arbirary $j : \mathscr{V} \to A$. This completes the proof. $\qquad\square$

2.5   Preliminaries for Rules

Theorem 2.5.1 (modus ponens). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$ be an assignment Let $\alpha, \beta \in \mathcal{F}(\mathscr{L})$. If $\mathbf{M}, j \models \alpha$ and $\mathbf{M}, j \models \alpha \to \beta$, then $\mathbf{M}, j \models \beta$.*

Proof. We have $\mathbf{M}, j \models \alpha$, which means $\alpha^{\mathbf{M},j} = 1$.
Since $\Gamma \models \alpha \to \beta$, we have $\mathbf{M}, j \models \alpha \to \beta$, which means $(\alpha \to \beta)^{\mathbf{M},j} = 1$.
By the definition of formula evaluation for implication,

$$(\alpha \to \beta)^{\mathbf{M},j} = 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) = 1. \tag{2.47}$$

Since $\alpha^{\mathbf{M},j} = 1$, we have

$$1 - 1 \cdot (1 - \beta^{\mathbf{M},j}) = 1, \tag{2.48}$$

which simplifies to $1 - (1 - \beta^{\mathbf{M},j}) = 1$, hence $\beta^{\mathbf{M},j} = 1$.
Therefore, $\mathbf{M}, j \models \beta$. $\qquad\square$

Theorem 2.5.2 (generalization). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\mathbf{M}$ be an $\mathscr{L}$-structure. Let $v \in \mathscr{V}$ and $\phi \in \mathcal{B}(\mathscr{L})$. If $\mathbf{M} \models \phi$, then $\mathbf{M} \models \forall v.\phi$.*

Proof. Take any non-empty domain A and any $\mathscr{L}$-structure $\mathbf{M}$ with domain A, such that $\mathbf{M} \models \phi$. Take any $j : \mathscr{V} \to A$. We need to show that $\mathbf{M}, j \models \forall v.\phi$.
By the definition of formula evaluation for universal quantification,

$$(\forall v.\phi)^{\mathbf{M},j} = \prod_{a \in A} \phi^{\mathbf{M},j\frac{a}{v}}. \tag{2.49}$$

Since $\mathbf{M} \models \phi$, we have $\mathbf{M}, j\frac{a}{v} \models \phi$, which means $\phi^{\mathbf{M},j\frac{a}{v}} = 1$.
This holds for all $a \in A$, therefore

$$(\forall v.\phi)^{\mathbf{M},j} = \prod_{a \in A} \phi^{\mathbf{M},j\frac{a}{v}} = \prod_{a \in A} 1 = 1. \tag{2.50}$$

Hence $\mathbf{M}, j \models \forall v.\phi$. And since $j : \mathscr{V} \to A$ was arbitrary, then $\mathbf{M} \models \forall v.\phi$. $\qquad\square$

Definition 2.5.3. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A. For any $z \in \mathscr{P} \cup \mathscr{F}$,*

$$\mathscr{M}_z(\mathbf{M}) \overset{\text{def}}{=} \{\mathbf{M}' : \mathbf{M}' \text{ is } \mathscr{L}\text{-structure with domain } A$$
$$\text{and } \mathbf{M}'(s) = \mathbf{M}(s) \text{ for all } s \in (\mathscr{P} \cup \mathscr{F}) \setminus \{z\} \}. \tag{2.51}$$

Lemma 2.5.4. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathscr{P}_k$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Let $\phi, \psi \in \mathcal{F}(\mathscr{L})$ such that q does not occure in both ϕ and ψ.*
Let A be an non-empty mathematical domain and $j : \mathscr{V} \to A$. Let $\mathbf{M}$ be $\mathscr{L}$-structure with domain A. Then

$$\mathbf{M}', j \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi \text{ for any } \mathbf{M}' \in \mathscr{M}_q(\mathbf{M}) \implies \mathbf{M}, j \models \psi. \tag{2.52}$$

Proof. Assume that assumption of implication (2.52) holds. We will construct a new structure $\mathbf{M}'$ that differs from $\mathbf{M}$ only in the interpretation of q. Namely,

$$(q)^{\mathbf{M}'}(\boldsymbol{a}) = \phi^{\mathbf{M}, j \frac{\boldsymbol{a}}{\boldsymbol{x}}}. \tag{2.53}$$

For all other predicates and functions, $\mathbf{M}'$ agrees with $\mathbf{M}$. Note that $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$. With the structure $\mathbf{M}'$, we have

$$q(\boldsymbol{x})^{\mathbf{M}', j} = (q)^{\mathbf{M}'}(\boldsymbol{x}^{\mathbf{M}', j}) = \phi^{\mathbf{M}, j \frac{\boldsymbol{x}^{\mathbf{M}', j}}{\boldsymbol{x}}}. \tag{2.54}$$

Since by Definition 2.2.13 (term evaluation), we have $\boldsymbol{x}^{\mathbf{M}', j} = \boldsymbol{x}^{\mathbf{M}, j}$, thereforre

$$\phi^{\mathbf{M}, j \frac{\boldsymbol{x}^{\mathbf{M}', j}}{\boldsymbol{x}}} = \phi^{\mathbf{M}, j \frac{\boldsymbol{x}^{\mathbf{M}, j}}{\boldsymbol{x}}} = \phi^{M, j}. \tag{2.55}$$

Hence $q(\boldsymbol{x})^{\mathbf{M}', j} = \phi^{\mathbf{M}, j}$. Since q does not occur in ϕ, we have $\phi^{\mathbf{M}, j} = \phi^{\mathbf{M}', j}$ and consequently $q(\boldsymbol{x})^{\mathbf{M}', j} = \phi^{\mathbf{M}', j}$. By Theorem 2.3.5 this is the same as $(q(\boldsymbol{x}) \leftrightarrow \phi)^{\mathbf{M}', j} = 1$, and by Definition 2.2.19, we have $\mathbf{M}', j \models q(\boldsymbol{x}) \leftrightarrow \phi$.
By assumption of implication (2.52), since $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$, we have $\mathbf{M}', j \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$. And then, since $\mathbf{M}', j \models q(\boldsymbol{x}) \leftrightarrow \phi$, by Theorem 2.5.1 we have $\mathbf{M}', j \models \psi$.
And since q does not occur in ψ, we have $\psi^{\mathbf{M}', j} = \psi^{\mathbf{M}, j}$, therefore $\mathbf{M}, j \models \psi$. $\square$

Lemma 2.5.5. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathscr{P}_k$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Let $\phi, \psi \in \mathcal{F}(\mathscr{L})$ such that q does not occur in both ϕ and ψ.*
Let $\mathbf{M}$ be $\mathscr{L}$-structure. Then

$$\mathbf{M}' \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi \text{ for any } \mathbf{M}' \in \mathscr{M}_q(\mathbf{M}) \implies \mathbf{M} \models \psi. \tag{2.56}$$

Proof. Assume that assumption of implication (2.56) holds. Take an arbitrary $\mathscr{L}$-structure $\mathbf{M}$ with domain A. Take an arbitrary assignment $j : \mathscr{V} \to A$. Take an arbitrary $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$. By assumption from implication (2.56), we have $\mathbf{M}' \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$. Thus in particular $\mathbf{M}', j \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$, but since $\mathbf{M}'$ is arbitrary chosen, by Theorem 2.5.4, we have $\mathbf{M}, j \models \psi$. And since j was arbitrary chosen, we have $\mathbf{M} \models \psi$. $\square$

Next, we will define a relation $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$. The intuition behind is that $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ holds when ψ_2 is created from ψ_1 by replacing some or none ϕ_1 which occurs in ψ_1 by ϕ_2.

Definition 2.5.6 (replace relation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $\phi_1, \phi_2 \in \mathcal{F}(\mathscr{L})$.*

Let $\psi_1, \psi_2 \in \mathcal{F}(\mathscr{L})$. We will define recursively $\text{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$.

  1. *If $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$, then $\text{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$ holds.*

  2. *If $\psi_1 \doteq p(\boldsymbol{\eta})$ and $\psi_2 \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$, then $\text{rep}_{\psi_1,\phi_2}(\psi_1, \phi_2)$ holds.*

  3. *If $\psi_1 \doteq \alpha_1 \to \beta_1$ and $\psi_2 \doteq \alpha_2 \to \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathscr{L})$, then $\text{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$ iff $\text{rep}_{\phi_1,\phi_2}(\alpha_1, \alpha_2)$ and $\text{rep}_{\phi_1,\phi_2}(\beta_1, \beta_2)$.*

  4. *If $\psi_1 \doteq \forall u.\gamma_1$ and $\psi_2 \doteq \forall u.\gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathscr{L})$, then $\text{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$ iff $\text{rep}_{\phi_1,\phi_2}(\gamma_1, \gamma_2)$.*

  5. *Otherwise $\text{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$ does not hold.*

Theorem 2.5.7. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{F}(\mathscr{L})$ such that $\text{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$. For any $\mathscr{L}$-structure $\mathbf{M}$,*

$$\mathbf{M} \models \phi_1 \leftrightarrow \phi_2 \implies \mathbf{M} \models \psi_1 \leftrightarrow \psi_2. \tag{2.57}$$

Proof. We proceed by structural induction on the relation $\text{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$.

Assume that $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathscr{V} \to A$. By Theorem 2.3.5, this means $\phi_1^{\mathbf{M},j} = \phi_2^{\mathbf{M},j}$ for any $j : \mathscr{V} \to A$.

Take an arbitrary $j : \mathscr{V} \to A$. We will show that $\psi_1^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$.

Case 1: $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$.

Then $\psi_1^{\mathbf{M},j} = \phi_1^{\mathbf{M},j} = \phi_2^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$.

By Theorem 2.3.5, $\psi_1^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$ implies $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 2: $\psi_1 \doteq p(\boldsymbol{\eta})$ and $\psi_2 \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.

Then $\psi_1 \doteq \psi_2$, so trivially $\psi_1^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$ and $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 3: $\psi_1 \doteq \alpha_1 \to \beta_1$ and $\psi_2 \doteq \alpha_2 \to \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathscr{L})$, and $\text{rep}_{\phi_1,\phi_2}(\alpha_1, \alpha_2)$ and $\text{rep}_{\phi_1,\phi_2}(\beta_1, \beta_2)$.

By the inductive hypothesis, since $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathscr{V} \to A$, we have:

  - $M, j \models \alpha_1 \leftrightarrow \alpha_2$ for any $j : \mathscr{V} \to A$, which means $\alpha_1^{\mathbf{M},j} = \alpha_2^{\mathbf{M},j}$ for any $j : \mathscr{V} \to A$.

  - $M, j \models \beta_1 \leftrightarrow \beta_2$ for any $j : \mathscr{V} \to A$, which means $\beta_1^{\mathbf{M},j} = \beta_2^{\mathbf{M},j}$ for any $j : \mathscr{V} \to A$.

By the definition of formula evaluation for implication:

$$\psi_1^{\mathbf{M},j} = (\alpha_1 \to \beta_1)^{\mathbf{M},j} = 1 - \alpha_1^{\mathbf{M},j}(1 - \beta_1^{\mathbf{M},j})$$
$$= 1 - \alpha_2^{\mathbf{M},j}(1 - \beta_2^{\mathbf{M},j}) = (\alpha_2 \to \beta_2)^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}.$$

Therefore, $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 4: $\psi_1 \doteq \forall u.\gamma_1$ and $\psi_2 \doteq \forall u.\gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathscr{L})$, and $\text{rep}_{\phi_1,\phi_2}(\gamma_1, \gamma_2)$.

By the inductive hypothesis, since $M, j' \models \phi_1 \leftrightarrow \phi_2$ for any $j' : \mathscr{V} \to A$, we have $M, j' \models \gamma_1 \leftrightarrow \gamma_2$ for any $j' : \mathscr{V} \to A$.

In particular, for any $a \in A$, taking $j' = j\dfrac{a}{u}$, we have $M, j\dfrac{a}{u} \models \gamma_1 \leftrightarrow \gamma_2$, which means

$$\gamma_1^{\mathbf{M},j\frac{a}{u}} = \gamma_2^{\mathbf{M},j\frac{a}{u}}.$$

By the definition of formula evaluation for universal quantification:

$$\psi_1^{\mathbf{M},j} = (\forall u.\gamma_1)^{\mathbf{M},j} = \prod_{a \in A} \gamma_1^{\mathbf{M},j\frac{a}{u}}$$

$$= \prod_{a \in A} \gamma_2^{\mathbf{M},j\frac{a}{u}} = (\forall u.\gamma_2)^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}.$$

Therefore, $M, j \models \psi_1 \leftrightarrow \psi_2$.

Since this holds for arbitrary $j : \mathscr{V} \to A$, we conclude that $M, j \models \psi_1 \leftrightarrow \psi_2$ for any $j : \mathscr{V} \to A$.

This completes the proof by structural induction. $\square$

Lemma 2.5.8. *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. Let* $c \in \mathscr{F}_0$, *and* $x \in \mathscr{V}$. *Let* $\psi \in \mathcal{F}(\mathscr{L})$ *such that* c *does not occur in* ψ. *Let* $\mathbf{M}$ *be* $\mathscr{L}$-*structure. Then*

$$\mathbf{M}' \models \psi(x/c) \text{ for any } \mathbf{M}' \in \mathscr{M}_q(\mathbf{M}) \implies \mathbf{M} \models \psi. \tag{2.58}$$

Proof. Since $c \in \mathcal{F}_0$, $\psi(x/c)$ is always an admissible substitution.

Take any $\mathscr{L}$-structure $\mathbf{M}$ with domain A. Assume that assumption of implication (2.58) holds. We need to show that $\mathbf{M} \models \psi$. Take any assignment $j : \mathscr{V} \to A$. We need to show that $\mathbf{M}, j \models \psi$.

We construct $\mathbf{M}'$ such that:
$$c^{\mathbf{M}'} = j(x), \tag{2.59}$$

and for all other predicates and functions, $\mathbf{M}'$ agrees with $\mathbf{M}$. Note that $\mathbf{M}' \in \mathscr{M}_c(\mathbf{M})$. By assumption of implication (2.58), we have $\mathbf{M}' \models \psi(x/c)$. In particular, $\mathbf{M}', j \models \psi(x/c)$.

By Theorem 2.4.5, since $\psi(x/c)$ is an admissible substitution, we have

$$\mathbf{M}', j \models \psi(x/c) \iff \mathbf{M}', j\frac{c^{\mathbf{M}',j}}{x} \models \psi. \tag{2.60}$$

Since $c^{\mathbf{M}',j} = c^{\mathbf{M}'} = j(x)$, we have

$$j\frac{j(x)}{x} = j. \tag{2.61}$$

Therefore, $\mathbf{M}', j \models \psi$. Since c does not occur in ψ, and $\mathbf{M}'$ differs from $\mathbf{M}$ only in the interpretation of c, we have $\psi^{\mathbf{M}',j} = \psi^{\mathbf{M},j}$. Therefore, $\mathbf{M}, j \models \psi$. Since this holds for any assignment $j : \mathscr{V} \to A$, we have $\mathbf{M} \models \psi$. $\square$

Chapter 3

Inference Rules

Let $\mathscr{F}_k^g$ be an ininite and countable set of special function symbols of arity k for $k = 0, \ldots$.

Let $\mathscr{P}_k^g$ be an ininite and countable set of special predicate symbols of arity k for $k = 0, \ldots$.

$$\mathscr{F}^g \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \mathscr{F}_k^g \tag{3.1}$$

$$\mathscr{P}^g \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \mathscr{P}_k^g \tag{3.2}$$

Definition 3.0.1.
$$\mathscr{L}^g \stackrel{\text{def}}{=} (\mathscr{F}^g, \mathscr{P}^g, \mathfrak{a}^g), \tag{3.3}$$

where $\mathfrak{a}^g(p) = k$ for any $p \in \mathscr{P}_k^g$ and $\mathfrak{a}^g(f) = k$ for any $f \in \mathscr{F}_k^g$ for any integer $k \geq 0$.

Definition 3.0.2. Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be signature of a language and $\mathscr{F}^g \cap \mathscr{F} = \emptyset$ and $\mathscr{P}^g \cap \mathscr{P} = \emptyset$. We will call such $\mathscr{L}$ a sub-signature and consider extention of the language with signature

$$\mathscr{L}^G \stackrel{\text{def}}{=} (\mathscr{F}^g \cup \mathscr{F}, \mathscr{P}^g \cup \mathscr{P}, \mathfrak{a} \cup \mathfrak{a}^g). \tag{3.4}$$

When we consider $\mathscr{L}^G$-structures $\mathbf{M}_1, \mathbf{M}_2$ and $\mathscr{L}^G$-structure M', we say that they agree on $\mathscr{L}$ iff $\mathbf{M}_1(\mathscr{M} \cup \mathscr{F}) = \mathbf{M}_2(\mathscr{M} \cup \mathscr{F})$. We will denote it by $M'|_{\mathscr{L}} = M|_{\mathscr{L}}$.

Definition 3.0.3 (modus ponens). Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language. We will define a function $Mod_{\mathscr{L}} : \mathcal{F}(\mathscr{L}) \times \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$.

$$Mod_{\mathscr{L}}(\psi, \alpha) \stackrel{\text{def}}{=} \begin{cases} \beta \text{ for } \psi \doteq \alpha \to \beta \text{ and } \alpha, \beta \in \mathcal{B}(\mathscr{L}), \\ \mathfrak{t} \text{ otherwise.} \end{cases} \tag{3.5}$$

Definition 3.0.4 (generalisation). Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language. We will define a family of functions $Gen_{\mathscr{L}}^x : \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$ for $x \in \mathscr{V}$.

$$Gen_{\mathscr{L}}^x(\psi) \stackrel{\text{def}}{=} \begin{cases} \forall x.\psi \text{ for } \psi \in \mathcal{B}(\mathscr{L}), \\ \mathfrak{t} \text{ otherwise.} \end{cases} \tag{3.6}$$

Definition 3.0.5 (replacement). *Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language. We will define a family of functions $Rep_{\mathscr{L}}^{\psi_1, \psi_2} : \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$ for $\psi_1, \psi_2 \in \mathcal{B}(\mathscr{L})$.*

$$
Rep_{\mathscr{L}}^{\psi_1, \psi_2}(\phi) \stackrel{\text{def}}{=} \begin{cases} \psi_1 \leftrightarrow \psi_2 \text{ for } \phi \doteq \phi_1 \leftrightarrow \phi_2 \text{ where } \phi_1, \phi_2 \in \mathcal{B}(\mathcal{F}) \text{ and } rep_{\phi_1, \phi_2}(\psi_1, \psi_2), \\ \mathfrak{t} \text{ otherwise.} \end{cases}
$$

$$(3.7)$$

Definition 3.0.6 (definition reduction). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer.*
We will define a family of functions $Red_{\mathscr{L}}^{q(\boldsymbol{x})} : \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$ for $q \in \mathscr{P}_k$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$.

$$
Red_{\mathscr{L}}^{q(\boldsymbol{x})}(\gamma) \stackrel{\text{def}}{=} \begin{cases} \psi \text{ for } \gamma \doteq (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi \text{ where } \psi, \phi \in \mathcal{B}(\mathscr{L}) \\ \qquad\qquad\qquad\qquad \text{and } q \text{ does not occur in } \psi \text{ and } \phi, \\ \mathfrak{t} \text{ otherwise.} \end{cases} \quad (3.8)
$$

Chapter 4

Abstract Rules Theory

Definition 4.0.1. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathscr{L})^{k_1} \to \mathcal{F}(\mathscr{L})^{k_2}$, we will say that r is **strictly truth-preserving** iff for any $\mathscr{L}$-structure $\mathbf{M}$ and any $\phi \in \mathcal{F}(\mathscr{L})^{k_1}$*

$$\mathbf{M} \models \phi \implies \mathbf{M} \models r(\phi). \tag{4.1}$$

Theorem 4.0.2. *Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language. Then $\mathrm{Mod}_{\mathscr{L}}$ is strictly truth preserving.*

Proof. Take any $\mathscr{L}$-structure $\mathbf{M}$ and any $(\psi, \alpha) \in \mathcal{F}(\mathscr{L})^2$ such that $\mathbf{M} \models (\psi, \alpha)$.
By definition, $\mathbf{M} \models (\psi, \alpha)$ means $\mathbf{M} \models \psi$ and $\mathbf{M} \models \alpha$.
We need to show that $\mathbf{M} \models \mathrm{Mod}_{\mathscr{L}}(\psi, \alpha)$.
Case 1: $\psi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.
Then $\mathrm{Mod}_{\mathscr{L}}(\psi, \alpha) = \beta$.
Take any assignment $j : \mathscr{V} \to A$. We need to show that $\mathbf{M}, j \models \beta$.
Since $\mathbf{M} \models \alpha$, we have $\mathbf{M}, j \models \alpha$.
Since $\mathbf{M} \models \psi$, we have $\mathbf{M}, j \models \alpha \to \beta$.
By Theorem 2.5.1, since $\mathbf{M}, j \models \alpha$ and $\mathbf{M}, j \models \alpha \to \beta$, we have $\mathbf{M}, j \models \beta$.
Since this holds for any assignment $j : \mathscr{V} \to A$, we have $\mathbf{M} \models \beta = \mathrm{Mod}_{\mathscr{L}}(\psi, \alpha)$.
Case 2: Otherwise.
Then $\mathrm{Mod}_{\mathscr{L}}(\psi, \alpha) = \mathfrak{t}$, which is trivially satisfied by any structure. $\square$

Theorem 4.0.3. *Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language and let $x \in \mathscr{V}$. Then $\mathrm{Gen}_{\mathscr{L}}^x$ is strictly truth preserving.*

Proof. Take any $\mathscr{L}$-structure $\mathbf{M}$ and any $\psi \in \mathcal{F}(\mathscr{L})$ such that $\mathbf{M} \models \psi$.
We need to show that $\mathbf{M} \models \mathrm{Gen}_{\mathscr{L}}^x(\psi)$.
Case 1: $\psi \in \mathcal{B}(\mathscr{L})$.
Then $\mathrm{Gen}_{\mathscr{L}}^x(\psi) = \forall x.\psi$.
By Theorem 2.5.2, since $\mathbf{M} \models \psi$, we have $\mathbf{M} \models \forall x.\psi = \mathrm{Gen}_{\mathscr{L}}^x(\psi)$.
Case 2: $\psi \notin \mathcal{B}(\mathscr{L})$.
Then $\mathrm{Gen}_{\mathscr{L}}^x(\psi) = \mathfrak{t}$, which is trivially satisfied by any structure. $\square$

Theorem 4.0.4. *Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language and let $\psi_1, \psi_2 \in \mathcal{B}(\mathscr{L})$. Then $\mathrm{Rep}_{\mathscr{L}}^{\psi_1, \psi_2}$ is strictly truth preserving.*

Proof. Take any $\mathscr{L}$-structure $\mathbf{M}$ and any $\phi \in \mathcal{F}(\mathscr{L})$ such that $\mathbf{M} \models \phi$.
We need to show that $\mathbf{M} \models \mathrm{Rep}_{\mathscr{L}}^{\psi_1, \psi_2}(\phi)$.
Case 1: $\phi \doteq \phi_1 \leftrightarrow \phi_2$ where $\phi_1, \phi_2 \in \mathcal{B}(\mathscr{L})$ and $\mathrm{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$.

Then $\mathrm{Rep}_{\mathscr{L}}^{\psi_1,\psi_2}(\phi) = \psi_1 \leftrightarrow \psi_2$.

Since $\mathbf{M} \models \phi_1 \leftrightarrow \phi_2$, by Theorem 2.5.7, we have $\mathbf{M} \models \psi_1 \leftrightarrow \psi_2 = \mathrm{Rep}_{\mathscr{L}}^{\psi_1,\psi_2}(\phi)$.

Case 2: Otherwise.

Then $\mathrm{Rep}_{\mathscr{L}}^{\psi_1,\psi_2}(\phi) = \mathfrak{t}$, which is trivially satisfied by any structure.   □

Definition 4.0.5. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a sub-signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathscr{L})^{k_1} \to \mathcal{F}(\mathscr{L})^{k_2}$, we will say that r is **weakly truth-preserving** iff for any $\mathscr{L}^G$-structure $\mathbf{M}$ and any $\phi \in \mathcal{F}(\mathscr{L}^G)^{k_1}$*

$$\mathbf{M}' \models \phi \text{ for any } \mathscr{L}^G\text{-structure } \mathbf{M}' \text{ for which } \mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}} \implies \mathbf{M} \models r(\phi). \quad (4.2)$$

Corollary 4.0.6. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a sub-signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathscr{L})^{k_1} \to \mathcal{F}(\mathscr{L})^{k_2}$. If r is strictly truth-preserving, then r is weakly truth-preserving.*

Proof. Assume r is strictly truth-preserving.

Take any $\mathscr{L}^G$-structure $\mathbf{M}$ and any $\phi \in \mathcal{F}(\mathscr{L})^{k_1}$ such that $\mathbf{M}' \models \phi$ for all $\mathscr{L}^G$-structure $\mathbf{M}'$ for which $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$.

In particular, taking $\mathbf{M}' = \mathbf{M}$, we have $\mathbf{M} \models \phi$.

Since r is strictly truth-preserving and $\mathbf{M} \models \phi$, we have $\mathbf{M} \models r(\phi)$.

Therefore, r is weakly truth-preserving.   □

Theorem 4.0.7. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a sub-signature of a language. Let k be a non-negative integer, let $q \in \mathscr{P}_k^g$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Then $Red_{\mathscr{L}}^{q(\boldsymbol{x})}$ is weakly truth-preserving.*

Proof. Take any $\mathscr{L}^G$-structure $\mathbf{M}$ and any $\gamma \in \mathcal{F}(\mathscr{L}^G)$ such that $\mathbf{M}' \models \gamma$ for all $\mathscr{L}^G$-structure $\mathbf{M}'$ for which $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$.

We need to show that $\mathbf{M} \models \mathrm{Red}_{\mathscr{L}}^{q(\boldsymbol{x})}(\gamma)$.

Case 1: $\gamma \doteq (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$ where $\psi, \phi \in \mathcal{B}(\mathscr{L})$ and q does not occur in ψ and ϕ.

Then $\mathrm{Red}_{\mathscr{L}}^{q(\boldsymbol{x})}(\gamma) = \psi$.

We need to show that $\mathbf{M} \models \psi$.

Since $q \in \mathscr{P}_k^g$ and $q \notin \mathscr{P}$, for any $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$ (where $\mathbf{M}'$ is an $\mathscr{L}^G$-structure), we have $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$, so by assumption $\mathbf{M}' \models \gamma$, which means $\mathbf{M}' \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$.

By Lemma 2.5.5, since $\mathbf{M}' \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$, we have $\mathbf{M} \models \psi$.

Therefore, $\mathbf{M} \models \mathrm{Red}_{\mathscr{L}}^{q(\boldsymbol{x})}(\gamma)$.

Case 2: Otherwise.

Then $\mathrm{Red}_{\mathscr{L}}^{q(\boldsymbol{x})}(\gamma) = \mathfrak{t}$, which is trivially satisfied by any structure.

Therefore, $\mathrm{Red}_{\mathscr{L}}^{q(\boldsymbol{x})}$ is weakly truth-preserving.   □