Logical Foundations

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Chapter 1

Introduction

1.1 Acknowledgements

This document presents the logical foundations for a formal system based on first-order logic. The definitions and theorems presented here were designed by the human author. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed, verified and corercted by the human author who takes the full responsibility for the content of this book.

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Chapter 2

First Order Logic

2.1 Preliminaries

We will use $\mathbb{N}$ to denote a set of all nonnegative integers.

Definition 2.1.1. *Let S be an arbitrary set.*

1. $S^0 \overset{\text{def}}{=} \{\emptyset\}$.
2. $S^k \overset{\text{def}}{=} \underbrace{S \times \cdots \times S}_{k}$.
3. $S^* \overset{\text{def}}{=} \bigcup_{k=0}^{\infty} S^k$.

Note that $\emptyset$ is treated as empty sequence.

Remark 2.1.2. *For a sequence $\boldsymbol{s} \in S^k$, we denote the i-th element of the sequence by s_i where $i \in \{1, \ldots, k\}$. let also $|\boldsymbol{s}| \overset{\text{def}}{=} k$ denote a length of $\boldsymbol{s}$.*

Note that according to the above convention $|\emptyset| = 0$.

Definition 2.1.3. *Let S be an arbitrary set.*

1. $S^{\nabla 0} \overset{\text{def}}{=} \{\emptyset\}$.
2. $S^{\nabla k} \overset{\text{def}}{=} \{\boldsymbol{s} \in S^k : \boldsymbol{s}_i \neq \boldsymbol{s}_j \text{ for } i \neq j\}$.
3. $S^{\nabla} \overset{\text{def}}{=} \bigcup_{k=0}^{\infty} S^{\nabla k}$.

2.2 Well Formed Formula

2.2.1 Syntax

Let $\mathfrak{L}$ denote a set of all logic symbols " ( ) ", " $\to$ ", " . ", ", ", " $\forall$ ". Let $\mathscr{V}$ be an infinite countable set of variable symbols. And $\mathfrak{f}$ be a special predicate symbol.

Definition 2.2.1. *Let $\mathscr{P}$ be a set of predicate symbols for which $\mathfrak{f} \in \mathscr{P}$ and $\mathscr{F}$ be a set of function symbols. Let $\mathfrak{a} : \mathscr{P} \cup \mathscr{F} \to \mathbb{N}$ be an arity function. then $\mathscr{L} \overset{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ is a signature of a language.*

To write first order logic formulas for a given language signature $\mathscr{L}$, we will use symbols from

$$\mathcal{S}(\mathscr{L}) \stackrel{\text{def}}{=} \mathfrak{L} \cup \mathscr{V} \cup \mathscr{P} \cup \mathscr{F}. \tag{2.1}$$

All formulas will be then a subset of $\mathcal{S}(\mathscr{L})^*$. Elements of $\mathcal{S}(\mathscr{L})^*$ will be called strings.

Note 2.2.2 (syntactic equality). *We use the notation $\doteq$ to denote equality of strings. We will sometimes overload it when cast of some object to string is obvious.*

When we stack symbols together from left to right, we mean that we concatenate them in the same order. E.g let p be a predicate symbol and x, y be variable symbols, then $p(x, y)$ is just a sequence of 6 symbols: " p ", "(", " x ", " , ", " y ", ") ".

For convenience we will use an abreviated style of writing, in which for any symbol $s \in \mathscr{P} \cup \mathscr{F}$ and for any seqence of strings $\boldsymbol{x} \in \mathcal{S}(\mathscr{L})^*$

$$s(\boldsymbol{x}) \doteq \begin{cases} s(x_1, \ldots, x_k) \text{ for } |\boldsymbol{x}| > 0, \\ s \text{ for } |\boldsymbol{x}| = 0. \end{cases} \tag{2.2}$$

Definition 2.2.3 (arity). *Let $\mathscr{L} \stackrel{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

$$\mathscr{P}_k \stackrel{\text{def}}{=} \{p \in \mathscr{P} : \mathfrak{a}(p) = k\}, \tag{2.3}$$

$$\mathscr{F}_k \stackrel{\text{def}}{=} \{f \in \mathscr{F} : \mathfrak{a}(f) = k\}. \tag{2.4}$$

Definition 2.2.4 (atomic term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

1. *For any $v \in \mathscr{V}$, v is an* ***atomic term****.*
2. *For any $c \in \mathscr{F}_0$, c is an* ***atomic term****.*

Definition 2.2.5 (recursive: term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all terms $\mathcal{T}(\mathscr{L})$.*

Let $s \in \mathcal{S}(\mathscr{L})^*$.

1. *If $s \doteq v$ where $v \in \mathscr{V}$, then $s \in \mathcal{T}(\mathscr{L})$.*
2. *If $s \doteq f(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\tau}|}$.*
3. *Otherwise $s \notin \mathcal{T}(\mathscr{L})$.*

Definition 2.2.6 (atomic formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

$$\mathcal{A}(\mathscr{L}) = \bigcup_{k=0}^{\infty} \{p(\boldsymbol{\tau}) : p \in \mathscr{P}_k, \boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k\} \tag{2.5}$$

Definition 2.2.7 (recursive: base formula, well formed formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all base formulas $\mathcal{B}(\mathscr{L})$ and a set of all well formed formulas $\mathcal{F}(\mathscr{L})$.*

Let $s \in \mathcal{S}(\mathscr{L})^*$.

1. *If $s \in \mathcal{A}(\mathscr{L})$, then $s \in \mathcal{B}(\mathscr{L})$.*
2. *If $s \in \mathcal{B}(\mathscr{L})$, then $s \in \mathcal{F}(\mathscr{L})$.*

3. *If* $s \doteq (\phi)$ *where* $\phi \in \mathcal{F}(\mathscr{L})$*, then* $s \in \mathcal{B}(\mathscr{L})$*.*

4. *If* $s \doteq \alpha \to \beta$ *where* $\alpha, \beta \in \mathcal{B}(\mathscr{L})$*, then* $s \in \mathcal{F}(\mathscr{L})$*.*

5. *If* $s \doteq \forall v.\phi$ *where* $\phi \in \mathcal{B}(\mathscr{L})$*, then* $s \in \mathcal{F}(\mathscr{L})$*.*

6. *Otherwise* $s \notin \mathcal{F}(\mathscr{L})$ *and* $s \notin \mathcal{B}(\mathscr{L})$*.*

2.2.2 Semantics

Definition 2.2.8 ($\mathscr{L}$**-structure**)**.** *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language and* A *be an arbitrary mathematical domain, then a mapping* $\mathbf{M}$ *such that*

$$\mathbf{M} : \mathscr{P} \cup \mathscr{F} \to \bigcup_{k=0}^{\infty} \{0,1\}^{(A^k)} \cup \bigcup_{k=0}^{\infty} A^{(A^k)}, \tag{2.6}$$

where

$$\begin{aligned} &\mathbf{M}(\mathfrak{f}) = (\emptyset \mapsto 0), \\ &\mathbf{M}(\mathscr{P}) \subset \bigcup_{k=0}^{\infty} \{0,1\}^{(A^k)}, \\ &\mathbf{M}(\mathscr{F}) \subset \bigcup_{k=0}^{\infty} A^{(A^k)} \end{aligned}$$

is called an $\mathscr{L}$*-structure with domain* A*.*

For convenience we will denote values of $\mathbf{M}$ *as* $p^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(p)$ *for* $p \in \mathscr{P}$ *and* $f^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(f)$ *for* $f \in \mathscr{F}$*.*

Definition 2.2.9 (**assignment**)**.** *Let* A *be an arbitrary mathematical domain. Then* $j : \mathscr{V} \to A$ *is called an variables assignment in* A*.*

Definition 2.2.10 (**recursive: term evaluation**)**.** *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language and* A *be an arbitrary mathematical domain, let* $\mathbf{M}$ *be an* $\mathscr{L}$*-structure and* $j : \mathscr{V} \to A$*.*

Let $\tau \in \mathcal{T}(\mathscr{L})$*.*

1. *If* $\tau \doteq v$ *where* $v \in \mathscr{V}$*, then* $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} j(v)$*.*

2. *If* $\tau \doteq f(\boldsymbol{\sigma})$ *where* $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ *and* $f \in \mathcal{F}_{|\boldsymbol{\sigma}|}$*, then* $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} f^{\mathbf{M}}(\boldsymbol{\sigma}^{\mathbf{M},j})$ *where*

$$\boldsymbol{\sigma}^{\mathbf{M},j} \stackrel{\text{def}}{=} \begin{cases} (\sigma_1^{\mathbf{M},j}, \ldots, \sigma_k^{\mathbf{M},j}) \text{ for } k = |\boldsymbol{\sigma}| > 0, \\ \emptyset \text{ for } |\boldsymbol{\sigma}| = 0. \end{cases} \tag{2.7}$$

Definition 2.2.11. *Let* $\mathscr{L}$ *be a signature of a language. Let* $\mathbf{M}$ *be a* $\mathscr{L}$*-structure with domain* A *and let* $j : \mathscr{V} \to A$ *be an assignment of variables. Then we define*

$$j\frac{a}{x}(v) \stackrel{\text{def}}{=} \begin{cases} a \text{ if } v = x, \\ j(v) \text{ otherwise.} \end{cases} \tag{2.8}$$

Definition 2.2.12. *Let $\mathscr{L}$ be a signature of a language. Let $\mathbf{M}$ be a $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$ be an assignment of variables. Let k be a positive integer and let $\boldsymbol{x} \in V^{\nabla k}$, and $\boldsymbol{a} \in A^k$.*
Then we define

$$j\frac{\boldsymbol{a}}{\boldsymbol{x}}(v) \overset{\text{def}}{=} \begin{cases} a_i \text{ if } v = x_i \text{ for some } i \in \{1, \dots, k\}, \\ j(v) \text{ otherwise.} \end{cases} \tag{2.9}$$

For the purpouses of this work, we will define multiplication product of infinite number of 0s and/or 1s.

Definition 2.2.13. *Let S be an arbitrary set and $a_s \in \{0, 1\}$ for any $s \in S$.*

$$\prod_{s \in S} a_s = \begin{cases} 1 \text{ if } a_s = 1 \text{ for all } s \in S, \\ 0 \text{ otherwise .} \end{cases} \tag{2.10}$$

Definition 2.2.14 (recursive: formula evaluation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$.*

1. *If $\phi \doteq p$ where $p \in \mathscr{P}$, then $\phi^{\mathbf{M},j} \overset{\text{def}}{=} p^{\mathbf{M}}$.*

2. *If $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$, then $\phi^{\mathbf{M},j} \overset{\text{def}}{=} p^{\mathbf{M}}(\boldsymbol{\tau}^{\mathbf{M},j})$.*

$$\phi^{\mathbf{M},j} \overset{\text{def}}{=} 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}). \tag{2.11}$$

3. *If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, then*

$$\phi^{\mathbf{M},j} \overset{\text{def}}{=} \prod_{a \in A} \psi^{\mathbf{M},j\frac{a}{x}}. \tag{2.12}$$

Corollary 2.2.15. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*

$$(\phi \to \mathfrak{f})^{\mathbf{M},j} = 1 - \phi^{\mathbf{M},j}. \tag{2.13}$$

Definition 2.2.16 (model). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*

$$\mathbf{M}, j \models \phi \overset{\text{def}}{\iff} \phi^{\mathbf{M},j} = 1. \tag{2.14}$$

Theorem 2.2.17. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j_1, j_2 : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$. If $j_1\big(\mathit{free}(\phi)\big) = j_2\big(\mathit{free}(\phi)\big)$, then*

$$\phi^{M,j_1} = \phi^{M,j_2}. \tag{2.15}$$

2.3 Substitutions

2.3.1 Syntax

Definition 2.3.1 (recursive: variable in a term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $var(\tau)$ of all variables in a term τ.*

Let $v \in \mathscr{V}$ and $\tau \in \mathcal{T}(\mathscr{L})$.

1. *If $\tau \doteq v$ then $v \in var(\tau)$.*
2. *If $\tau \doteq f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\sigma}|}$ and $v \in var(\boldsymbol{\sigma}) \overset{\text{def}}{=} \bigcup_{k=1}^{|\boldsymbol{\sigma}|} var(\sigma_k)$, then $v \in var(\tau)$.*
3. *Otherwise $v \notin var(\tau)$.*

Definition 2.3.2 (recursive: variable in a formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $var(\phi)$ of all variables in a formula ϕ.*

Let $v \in \mathscr{V}$ and $\phi \in \mathcal{F}(\mathscr{L})$.

1. *If $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$ and $v \in var(\boldsymbol{\tau})$, then $v \in var(\phi)$.*
2. *If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, and $v \in var(\alpha)$ or $v \in var(\beta)$, then $v \in var(\phi)$.*
3. *If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, and either $v = x$ or $v \in var(\psi)$, then $v \in var(\phi)$.*
4. *Otherwise $v \notin var(\phi)$.*

Definition 2.3.3 (recursive: free variable in a formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $free(\phi)$ of all free variables in a formula ϕ.*

Let $v \in \mathscr{V}$ and $\phi \in \mathcal{F}(\mathscr{L})$.

1. *If $\phi \in \mathcal{A}(\mathscr{L})$ and $v \in var(\phi)$, then $v \in free(\phi)$.*
2. *If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, and $v \in free(\alpha)$ or $v \in free(\beta)$, then $v \in free(\phi)$.*
3. *If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$ and $v \neq x$ and $v \in free(\psi)$, then $v \in free(\phi)$.*
4. *Otherwise $v \notin free(\phi)$.*

Definition 2.3.4 (recursive: substitution in a term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$.*

Let $\sigma \in \mathcal{T}(\mathscr{L})$. Let's establish that $\sigma(\emptyset/\emptyset) \overset{\text{def}}{=} \sigma$.

1. *If $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} \tau_i$.*
2. *If $\sigma \doteq v$ where $v \in \mathscr{V}$ and $v \neq x$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} v$.*
3. *If $\tau \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\eta}|}$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$ where*

$$\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} \begin{cases} (\eta_1(\boldsymbol{x}/\boldsymbol{\tau}), \dots, \eta_k(\boldsymbol{x}/\boldsymbol{\tau})) \text{ for } k = |\boldsymbol{\eta}| > 0, \\ \emptyset \text{ for } |\boldsymbol{\eta}| = 0. \end{cases} \tag{2.16}$$

Definition 2.3.5 (recursive: admissible substitution in a formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$.*

Let $\phi \in \mathcal{F}(\mathscr{L})$. Let's establish that $\phi(\emptyset/\emptyset) \overset{\text{def}}{=} \phi$.

1. *If $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$ and is **admissable**.*
2. *If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$ and $\alpha(\boldsymbol{x}/\boldsymbol{\tau}), \beta(\boldsymbol{x}/\boldsymbol{\tau})$ are **admissible**, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} \alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau})$ and is **admissable**.*
3. *If $\phi \doteq \forall u.\psi$ where $u \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$:*
   Let $\boldsymbol{v}$ be $\boldsymbol{x}$ with x_i removed when $x_i = u$ or $x_i \notin free(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from $\boldsymbol{x}$.
   (a) *If $\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin var(\boldsymbol{\sigma})$, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \overset{\text{def}}{=} \forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ and is **admissable**.*
   (b) *Otherwise $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is **not admissible**.*

2.3.2 Semantics

Lemma 2.3.6. *Let $\mathscr{L}$ be a signature of a language and A be an arbitrary mathematical domain. Let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$ be an assignment. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$.*

Then for any term σ, we have

$$\sigma^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}. \tag{2.17}$$

Proof. We proceed by structural induction on the term σ.
Case 1: $\sigma \doteq x_i$ for some $i \in \{1, \ldots, k\}$.
Then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \tau_i$ and

$$\sigma^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} = (j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}})(x_i) = \tau_i^{\mathbf{M},j} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 2: $\sigma \doteq v$ where $v \in \mathscr{V}$ and $v \neq x$.
Then $\sigma(x/\tau) \doteq v$ and

$$\sigma^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} = (j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}})(v) = j(v) = v^{\mathbf{M},j} = (\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 3: $\sigma \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\eta}|}$.

Then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \doteq f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$. By inductive hypothesis, $\boldsymbol{\eta}^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} = \boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}$. Thus

$$\begin{aligned}\sigma^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} &= f^{\mathbf{M}}\left(\boldsymbol{\eta}^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}}\right) = f^{\mathbf{M}}\left(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}\right) \\ &= f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.\end{aligned}$$

This completes the proof by structural induction. $\square$

Theorem 2.3.7. *Let $\mathscr{L}$ be a signature of a language. Let $\phi \in \mathcal{F}(\mathscr{L})$. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$. If $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is admissible, then for any $\mathscr{L}$-structure $\mathbf{M}$ with domain A and for any assignment of variables $j : \mathscr{V} \to A$ we have*

$$(\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}) \models \phi \iff (\mathbf{M}, j) \models \phi(\boldsymbol{x}/\boldsymbol{\tau}). \tag{2.18}$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathscr{L})$.
Case 1: $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.
By the definition of substitution in well formed formula, $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$. By Lemma 2.3.6, we have $\boldsymbol{\eta}^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} = \boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}$. Therefore,

$$\phi^{\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}} = p^{\mathbf{M}}\left(\boldsymbol{\eta}^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}}\right) = p^{\mathbf{M}}(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}) = p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = \phi(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 2: $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.
By the definition of substitution in well formed formula, $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau})$.
By inductive hypothesis,

$$(\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}) \models \alpha \iff (\mathbf{M}, j) \models \alpha(\boldsymbol{x}/\boldsymbol{\tau}),$$
$$(\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}) \models \beta \iff (\mathbf{M}, j) \models \beta(\boldsymbol{x}/\boldsymbol{\tau}).$$

By the definition of formula evaluation,

$$\begin{aligned}\phi^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} &= 1 - \alpha^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}}\left(1 - \beta^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}}\right) = 1 - (\alpha(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}(1 - (\beta(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}) = 1 \\ &= (\alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = (\phi(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}\end{aligned}$$

Case 3: $\phi \doteq \forall u.\psi$ where $u \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$:
Let $\boldsymbol{v}$ be $\boldsymbol{x}$ with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from $\boldsymbol{x}$.
Since $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is admissable, $\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, and we have $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma})$. Thus

$$\begin{aligned}\phi^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} &= \prod_{a \in A} \psi^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\frac{a}{u}} \\ &= \prod_{a \in A} \psi^{\mathbf{M}, j\frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}}\frac{a}{u}} \quad \Big(\text{by choice of } \boldsymbol{v},\ j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\frac{a}{u} \text{ and } j\frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}}\frac{a}{u} \text{ agrees on free}(\psi)\Big) \\ &= \prod_{a \in A} \psi^{\mathbf{M}, j\frac{a}{u}\frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}}} \quad \big(\text{since } u \text{ not in } \boldsymbol{v}\big) \\ &= \prod_{a \in A} \psi(\boldsymbol{v}/\boldsymbol{\sigma})^{\mathbf{M}, j\frac{a}{u}} \quad \big(\text{by inductive hypothesis}\big) \\ &= (\forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma}))^{\mathbf{M},j} = \phi(\boldsymbol{x}/\boldsymbol{\tau})^{M,j}.\end{aligned}$$

This completes the proof by structural induction. ∎