

Logical Foundations

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1 Introduction

This document presents the logical foundations for a formal system based on first-order logic. The definitions and theorems presented here were designed by the human author. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed and verified by the human author.

2 Well Formed Formula

2.1 Syntax

Let \mathcal{V} be an infinite countable set of variable symbols. And f be a special predicate symbol.

Definition 2.1. Let \mathcal{P} be a set of predicate symbols for which $f \in \mathcal{P}$ and \mathcal{F} be a set of function symbols. Let $a : \mathcal{P} \cup \mathcal{F} \rightarrow \mathbb{N}$ be an arity function. then $\mathcal{L} \stackrel{\text{def}}{=} (\mathcal{P}, \mathcal{F}, a)$ is a signature of a language.

Definition 2.2 (arity). Let $\mathcal{L} \stackrel{\text{def}}{=} (\mathcal{P}, \mathcal{F}, a)$ be a signature of a language.

$$\mathcal{P}_k \stackrel{\text{def}}{=} \{p \in \mathcal{P} : a(p) = k\}, \quad (1)$$

$$\mathcal{F}_k \stackrel{\text{def}}{=} \{f \in \mathcal{F} : a(f) = k\}. \quad (2)$$

Definition 2.3 (recursive: term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, a)$ be a signature of a language.

1. For any $v \in \mathcal{V}$, v is a **term**.
2. For any $c \in \mathcal{F}_0$, c is a **term**.
3. For any $f \in \mathcal{F}_k$ and τ_1, \dots, τ_k are terms, $f(\tau_1, \dots, \tau_k)$ is a **term**.

Definition 2.4 (recursive: atomic formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, a)$ be a signature of a language.

1. For any $p \in \mathcal{P}_0$, p is an **atomic formula**.
2. For any $p \in \mathcal{P}_k$ and τ_1, \dots, τ_k are terms, $p(\tau_1, \dots, \tau_k)$ is an **atomic formula**.

We will denote a set of all atomic formulas as $\mathcal{A}(\mathcal{L})$.

Definition 2.5 (recursive: base formula, well formed formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language.

1. For any atomic formula ϕ , ϕ is a **base formula**.
2. For any **base formula** ϕ , ϕ is a **well formed formula**.
3. For any **well formed formula** ϕ , (ϕ) is a **base formula**.
4. For any **base formulas** α, β , $\alpha \rightarrow \beta$ is a **well formed formula**.
5. For any $v \in \mathcal{V}$ and a **base formula** ϕ , $\forall v. \phi$ is a **well formed formula**.

We will denote set of all base formulas as $\mathcal{B}(\mathcal{L})$ and set of all well formed formulas as $\mathcal{F}(\mathcal{L})$.

Definition 2.6 (recursive: variable in a term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $v \in \mathcal{V}$ and τ be a term.

1. If τ is v then v is a **variable in term** τ .
2. If τ is $f(\tau_1, \dots, \tau_k)$ where $f \in \mathcal{F}_k$ and τ_1, \dots, τ_k are terms and v is a **variable in some term** τ_i where i is an integer with $1 \leq i \leq k$, then v is a **variable in term** τ .
3. Otherwise v is not a **variable in a term** τ .

Definition 2.7 (recursive: variable in a well formed formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $v \in \mathcal{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If ϕ is $p(\tau_1, \dots, \tau_k)$ where $p \in \mathcal{P}_k$ and τ_1, \dots, τ_k are terms, and v is a **variable in some term** τ_i where i is an integer with $1 \leq i \leq k$, then v is a **variable in** ϕ .
2. If ϕ is $\alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and v is a **variable in** α or v is a **variable in** β , then v is a **variable in** ϕ .
3. If ϕ is $\forall x. \psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, and either v is x or v is a **variable in** ψ , then v is a **variable in** ϕ .
4. Otherwise v is not a **variable in** ϕ .

Definition 2.8 (recursive: free variable in a well formed formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $v \in \mathcal{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \in \mathcal{A}(\mathcal{L})$ and v is a variable in ϕ , then v is a **free variable in** ϕ .

2. If ϕ is $\alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and v is a **free variable in** α or v is a **free variable in** β , then v is a **free variable in** ϕ .
3. If ϕ is $\forall x.\psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$ and v is not x and v is a **free variable in** ψ , then v is a **free variable in** ϕ .
4. Otherwise v is not a **free variable in** ϕ .

Definition 2.9 (recursive: substitution in term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $v, x \in \mathcal{V}$ and τ, σ be terms.

1. If σ is x , then $\sigma(x/\tau) \stackrel{\text{def}}{=} \tau$.
2. If σ is v and v is not x , then $\sigma(x/\tau) \stackrel{\text{def}}{=} v$.
3. If σ is c where $c \in \mathcal{F}_0$, then $\sigma(x/\tau) \stackrel{\text{def}}{=} c$.
4. If σ is $f(\sigma_1, \dots, \sigma_k)$ where $f \in \mathcal{F}_k$ and $\sigma_1, \dots, \sigma_k$ are terms, then $\sigma(x/\tau) \stackrel{\text{def}}{=} f(\sigma_1(x/\tau), \dots, \sigma_k(x/\tau))$.

Definition 2.10 (recursive: substitution in well formed formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $x \in \mathcal{V}$, τ be a term, and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If ϕ is p where $p \in \mathcal{P}_0$, then $\phi(x/\tau) \stackrel{\text{def}}{=} p$.
2. If ϕ is $p(\sigma_1, \dots, \sigma_k)$ where $p \in \mathcal{P}_k$ and $\sigma_1, \dots, \sigma_k$ are terms, then $\phi(x/\tau) \stackrel{\text{def}}{=} p(\sigma_1(x/\tau), \dots, \sigma_k(x/\tau))$.
3. If ϕ is $\alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then $\phi(x/\tau) \stackrel{\text{def}}{=} \alpha(x/\tau) \rightarrow \beta(x/\tau)$.
4. If ϕ is $\forall y.\psi$ where $y \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$ and y is x , then $\phi(x/\tau) \stackrel{\text{def}}{=} \forall y.\psi$.
5. If ϕ is $\forall y.\psi$ where $y \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$ and y is not x and y is not a variable in τ , then $\phi(x/\tau) \stackrel{\text{def}}{=} \forall y.\psi(x/\tau)$.
6. Otherwise $\phi(x/\tau)$ is **not admissible**.

3 Semantics

Definition 3.1 (\mathcal{L} -structure). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, then a mapping \mathbf{M} such that

$$\mathbf{M} : \mathcal{P} \cup \mathcal{F} \rightarrow \{0, 1\} \cup \bigcup_{k=1}^{\infty} \{0, 1\}^{(A^k)} \cup A \cup \bigcup_{k=1}^{\infty} A^{(A^k)}, \quad (3)$$

where $\mathbf{M}(\mathfrak{f}) = 0$,

$$\mathbf{M}(\mathcal{P}_0) \subset \{0, 1\} \text{ and } \mathbf{M}(\mathcal{P}_k) \subset \{0, 1\}^{(A^k)} \text{ for } k = 1, \dots, \quad (4)$$

and

$$\mathbf{M}(\mathcal{F}_0) \subset A \text{ and } \mathbf{M}(\mathcal{F}_k) \subset A^{(A^k)} \text{ for } k = 1, \dots \quad (5)$$

is called an \mathcal{L} -structure with domain A .

For convenience we will denote values of \mathbf{M} as $p^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(p)$ for $p \in \mathcal{P}$ and $f^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(f)$ for $f \in \mathcal{F}$.

Definition 3.2 (assignment). Let A be an arbitrary mathematical domain. Then $j : \mathcal{V} \rightarrow A$ is called an variables assignment in A .

Definition 3.3 (recursive: term evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure and $j : \mathcal{V} \rightarrow A$. Let τ be an arbitrary term.

1. If τ is v where $v \in \mathcal{V}$, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} j(v)$.
2. If τ is f where $f \in \mathcal{F}_0$, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} f^{\mathbf{M}}$.
3. If τ is $f(\tau_1, \dots, \tau_k)$ where $f \in \mathcal{F}_k$ and τ_1, \dots, τ_k are terms, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} f^{\mathbf{M}}(\tau_1^{\mathbf{M}, j}, \dots, \tau_k^{\mathbf{M}, j})$.

Definition 3.4. Let \mathcal{L} be a signature of a language. Let \mathbf{M} be a \mathcal{L} -structure with domain A and let $j : \mathcal{V} \rightarrow A$ be an assignment of variables. Then we define

$$j \frac{a}{x}(v) \stackrel{\text{def}}{=} \begin{cases} a & \text{if } v \text{ is } x, \\ j(v) & \text{otherwise.} \end{cases} \quad (6)$$

For the purposes of this work, we will define product of infinite number of 0s and 1s.

Definition 3.5. Let S be an arbitrary set and $a_s \in \{0, 1\}$ for any $s \in S$.

$$\prod_{s \in S} a_s = \begin{cases} 1 & \text{if } a_s = 1 \text{ for all } s \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Definition 3.6 (recursive: formula evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$.

1. If ϕ is p where $p \in \mathcal{P}$, then $\phi^{\mathbf{M}, j} \stackrel{\text{def}}{=} p^{\mathbf{M}}$.
2. If ϕ is $p(\tau_1, \dots, \tau_k)$ where $p \in \mathcal{P}_k$ and τ_1, \dots, τ_k are terms, then $\phi^{\mathbf{M}, j} \stackrel{\text{def}}{=} p^{\mathbf{M}}(\tau_1^{\mathbf{M}, j}, \dots, \tau_k^{\mathbf{M}, j})$.
3. If ϕ is $\alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$\phi^{\mathbf{M}, j} \stackrel{\text{def}}{=} 1 - \alpha^{\mathbf{M}, j}(1 - \beta^{\mathbf{M}, j}). \quad (8)$$

4. If ϕ is $\forall x.\psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, then

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} \prod_{a \in A} \psi^{\mathbf{M},j} \frac{a}{x}. \quad (9)$$

Corollary 3.7. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$, then

$$(\phi \rightarrow \mathfrak{f})^{\mathbf{M},j} = 1 - \phi^{\mathbf{M},j}. \quad (10)$$

Definition 3.8 (model). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$, then

$$\mathbf{M}, j \models \phi \text{ iff } \phi^{\mathbf{M},j} = 1. \quad (11)$$

Lemma 3.9. Let \mathcal{L} be a signature of a language and A be an arbitrary mathematical domain. Let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$ be an assignment. For any term σ , any $x \in \mathcal{V}$, and any term τ , we have

$$\sigma^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} = (\sigma(x/\tau))^{\mathbf{M},j}. \quad (12)$$

Proof. We proceed by structural induction on the term σ .

Case 1: $\sigma = x$.

Then $\sigma(x/\tau) = \tau$ and

$$\sigma^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} = (j \frac{\tau^{(\mathbf{M},j)}}{x})(x) = \tau^{\mathbf{M},j} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 2: $\sigma = v$ where $v \in \mathcal{V}$ and $v \neq x$.

Then $\sigma(x/\tau) = v$ and

$$\sigma^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} = (j \frac{\tau^{(\mathbf{M},j)}}{x})(v) = j(v) = v^{\mathbf{M},j} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 3: $\sigma = c$ where $c \in \mathcal{F}_0$.

Then $\sigma(x/\tau) = c$ and

$$\sigma^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} = c^{\mathbf{M}} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 4: $\sigma = f(\sigma_1, \dots, \sigma_k)$ where $f \in \mathcal{F}_k$ and $\sigma_1, \dots, \sigma_k$ are terms.

Then $\sigma(x/\tau) = f(\sigma_1(x/\tau), \dots, \sigma_k(x/\tau))$. By inductive hypothesis, $\sigma_i \xrightarrow{\mathbf{M}, j} \frac{\tau^{(\mathbf{M}, j)}}{x} = (\sigma_i(x/\tau))^{\mathbf{M}, j}$ for all $i = 1, \dots, k$. Thus

$$\begin{aligned} \sigma \xrightarrow{\mathbf{M}, j} \frac{\tau^{(\mathbf{M}, j)}}{x} &= f^{\mathbf{M}}(\sigma_1 \xrightarrow{\mathbf{M}, j} \frac{\tau^{(\mathbf{M}, j)}}{x}, \dots, \sigma_k \xrightarrow{\mathbf{M}, j} \frac{\tau^{(\mathbf{M}, j)}}{x}) \\ &= f^{\mathbf{M}}((\sigma_1(x/\tau))^{\mathbf{M}, j}, \dots, (\sigma_k(x/\tau))^{\mathbf{M}, j}) \\ &= (f(\sigma_1(x/\tau), \dots, \sigma_k(x/\tau)))^{\mathbf{M}, j} \\ &= (\sigma(x/\tau))^{\mathbf{M}, j}. \end{aligned}$$

This completes the proof by structural induction. \square

Theorem 3.10. Let \mathcal{L} be a signature of a language. Let $\phi \in \mathcal{F}(\mathcal{L})$ and let τ be an arbitrary term. If $\phi(x/\tau)$ is admissible, then for any \mathcal{L} -structure \mathbf{M} with domain A and for any assignment of variables $j : \mathcal{V} \rightarrow A$ we have

$$(\mathbf{M}, j \xrightarrow{x} \frac{\tau^{(\mathbf{M}, j)}}{x}) \models \phi \text{ iff } (\mathbf{M}, j) \models \phi(x/\tau). \quad (13)$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathcal{L})$.

Case 1: $\phi = p$ where $p \in \mathcal{P}_0$.

By the definition of substitution in well formed formula, $\phi(x/\tau) = p$. By the definition of formula evaluation, $\phi^{\mathbf{M}, j} = p^{\mathbf{M}}$ for any assignment j . Thus

$$\begin{aligned} (\mathbf{M}, j \xrightarrow{x} \frac{\tau^{(\mathbf{M}, j)}}{x}) \models \phi &\iff \phi \xrightarrow{\mathbf{M}, j} \frac{\tau^{(\mathbf{M}, j)}}{x} = 1 \\ &\iff p^{\mathbf{M}} = 1 \\ &\iff \phi^{\mathbf{M}, j} = 1 \\ &\iff (\mathbf{M}, j) \models \phi \\ &\iff (\mathbf{M}, j) \models \phi(x/\tau). \end{aligned}$$

Case 2: $\phi = p(\sigma_1, \dots, \sigma_k)$ where $p \in \mathcal{P}_k$ and $\sigma_1, \dots, \sigma_k$ are terms.

By the definition of substitution in well formed formula, $\phi(x/\tau) = p(\sigma_1(x/\tau), \dots, \sigma_k(x/\tau))$.

By Lemma 2.9, for each term σ_i , we have $\sigma_i \xrightarrow{\mathbf{M}, j} \frac{\tau^{(\mathbf{M}, j)}}{x} = (\sigma_i(x/\tau))^{\mathbf{M}, j}$. There-

fore,

$$\begin{aligned}
(\mathbf{M}, j \frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi &\iff \phi^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} = 1 \\
&\iff p^{\mathbf{M}}(\sigma_1^{\mathbf{M},j} \frac{x}{x}, \dots, \sigma_k^{\mathbf{M},j} \frac{x}{x}) = 1 \\
&\iff p^{\mathbf{M}}((\sigma_1(x/\tau))^{\mathbf{M},j}, \dots, (\sigma_k(x/\tau))^{\mathbf{M},j}) = 1 \\
&\iff (p(\sigma_1(x/\tau), \dots, \sigma_k(x/\tau)))^{\mathbf{M},j} = 1 \\
&\iff (\phi(x/\tau))^{\mathbf{M},j} = 1 \\
&\iff (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}$$

Case 3: $\phi = \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$.

By the definition of substitution in well formed formula, $\phi(x/\tau) = \alpha(x/\tau) \rightarrow \beta(x/\tau)$. By inductive hypothesis,

$$\begin{aligned}
(\mathbf{M}, j \frac{\tau^{(\mathbf{M},j)}}{x}) \models \alpha &\iff (\mathbf{M}, j) \models \alpha(x/\tau), \\
(\mathbf{M}, j \frac{\tau^{(\mathbf{M},j)}}{x}) \models \beta &\iff (\mathbf{M}, j) \models \beta(x/\tau).
\end{aligned}$$

By the definition of formula evaluation,

$$\begin{aligned}
(\mathbf{M}, j \frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi &\iff \phi^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} = 1 \\
&\iff 1 - \alpha^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} (1 - \beta^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x}) = 1 \\
&\iff 1 - (\alpha(x/\tau))^{\mathbf{M},j} (1 - (\beta(x/\tau))^{\mathbf{M},j}) = 1 \\
&\iff (\alpha(x/\tau) \rightarrow \beta(x/\tau))^{\mathbf{M},j} = 1 \\
&\iff (\phi(x/\tau))^{\mathbf{M},j} = 1 \\
&\iff (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}$$

Case 4: $\phi = \forall y. \psi$ where $y \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$.

Subcase 4a: If $y = x$, then by the definition of substitution, $\phi(x/\tau) = \forall y. \psi$.

Thus

$$\begin{aligned}
(\mathbf{M}, j \frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi &\iff \phi^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} = 1 \\
&\iff \prod_{a \in A} \psi^{\mathbf{M},(j \frac{\tau^{(\mathbf{M},j)}}{x})} \frac{a}{y} = 1 \\
&\iff \prod_{a \in A} \psi^{\mathbf{M},j} \frac{a}{y} = 1 \quad (\text{since } y = x \text{ and } (j \frac{\tau^{(\mathbf{M},j)}}{x}) \frac{a}{x} = j \frac{a}{x}) \\
&\iff \phi^{\mathbf{M},j} = 1 \\
&\iff (\mathbf{M}, j) \models \phi \\
&\iff (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}$$

Subcase 4b: If $y \neq x$ and y is not a **variable in** τ , then by the definition of substitution, $\phi(x/\tau) = \forall y. \psi(x/\tau)$. By inductive hypothesis, for any $a \in A$,

$$(\mathbf{M}, j \frac{a \tau}{y} \frac{a}{x}) \models \psi \iff (\mathbf{M}, j \frac{a}{y}) \models \psi(x/\tau).$$

Since y is not a **variable in** τ , the evaluation of τ does not depend on the value

assigned to y . Therefore, $\tau^{\mathbf{M},j} \frac{a}{y} = \tau^{(\mathbf{M},j)}$ for all $a \in A$.

Moreover, since $x \neq y$, we have $j \frac{\tau^{(\mathbf{M},j)} a}{x} = j \frac{a \tau^{(\mathbf{M},j)}}{y}$ for all $a \in A$.

Thus,

$$\begin{aligned}
(\mathbf{M}, j \frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi &\iff \phi^{\mathbf{M},j} \frac{\tau^{(\mathbf{M},j)}}{x} = 1 \\
&\iff \prod_{a \in A} \psi^{\mathbf{M},(j \frac{\tau^{(\mathbf{M},j)}}{x})} \frac{a}{y} = 1 \\
&\iff \prod_{a \in A} \psi^{\mathbf{M},j} \frac{a \tau^{(\mathbf{M},j)}}{y} = 1 \\
&\iff \prod_{a \in A} (\psi(x/\tau))^{\mathbf{M},j} \frac{a}{y} = 1 \quad (\text{by inductive hypothesis}) \\
&\iff (\forall y. \psi(x/\tau))^{\mathbf{M},j} = 1 \\
&\iff (\phi(x/\tau))^{\mathbf{M},j} = 1 \\
&\iff (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}$$

This completes the proof by structural induction. \square