

Logical Foundations

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Chapter 1

Introduction

1.1 Acknowledgements

This document presents the logical foundations for a formal system based on first-order logic. The net of definitions and theorems presented here were designed by a human author - Michał Stanisław Wójcik. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed, verified and corrected by Michał Stanisław Wójcik who takes the full responsibility for the content of this work.

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Chapter 2

First Order Logic

2.1 Preliminaries

We will use \mathbb{N} to denote a set of all nonnegative integers.

Definition 2.1.1. Let S be an arbitrary set.

1. $S^0 \stackrel{\text{def}}{=} \{\emptyset\}.$
2. $S^k \stackrel{\text{def}}{=} \underbrace{S \times \cdots \times S}_{k}.$
3. $S^* \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^k.$

Note that \emptyset is treated as empty sequence.

Remark 2.1.2. For a sequence $s \in S^k$, we denote the i -th element of the sequence by s_i where $i \in \{1, \dots, k\}$. let also $|s| \stackrel{\text{def}}{=} k$ denote a length of s .

Note that according to the above convention $|\emptyset| = 0$.

Definition 2.1.3. Let S be an arbitrary set.

1. $S^{\nabla 0} \stackrel{\text{def}}{=} \{\emptyset\}.$
2. $S^{\nabla k} \stackrel{\text{def}}{=} \{s \in S^k : s_i \neq s_j \text{ for } i \neq j\}.$
3. $S^{\nabla} \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^{\nabla k}.$

2.2 First Order Logic Formula

2.2.1 Syntax

Let \mathfrak{L} denote a set of all logical symbols “(”, “)”, “ \rightarrow ”, “ \cdot ”, “ $,$ ”, “ \forall ”. Let \mathcal{V} be an infinite countable set of variable symbols. And f be a special predicate symbol.

Definition 2.2.1. Let \mathcal{P} be a set of predicate symbols for which $f \in \mathcal{P}$ and \mathcal{F} be a set of function symbols. Let $a : \mathcal{P} \cup \mathcal{F} \rightarrow \mathbb{N}$ be an arity function. then $\mathcal{L} \stackrel{\text{def}}{=} (\mathcal{P}, \mathcal{F}, a)$ is a signature of a language.

To write first order logic formulas for a given language signature \mathcal{L} , we will use symbols from

$$\mathcal{S}(\mathcal{L}) \stackrel{\text{def}}{=} \mathcal{L} \cup \mathcal{V} \cup \mathcal{P} \cup \mathcal{F}. \quad (2.1)$$

All formulas will be then a subset of $\mathcal{S}(\mathcal{L})^*$. Elements of $\mathcal{S}(\mathcal{L})^*$ will be called strings.

Note 2.2.2 (syntactic equality). We use the notation \doteq to denote equality of strings. We will sometimes overload it when cast of some object to string is obvious.

When we stack symbols together from left to right, we mean that we concatenate them in the same order. E.g let p be a predicate symbol and x, y be variable symbols, then $p(x, y)$ is just a sequence of 6 symbols: “ p ”, “(”, “ x ”, “,”, “ y ”, “)”).

For convenience we will use an abbreviated style of writing, in which for any symbol $s \in \mathcal{P} \cup \mathcal{F}$ and for any seqence of strings $\mathbf{x} \in \mathcal{S}(\mathcal{L})^*$

$$s(\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} s(x_1, \dots, x_k) & \text{for } |\mathbf{x}| > 0, \\ s & \text{for } |\mathbf{x}| = 0. \end{cases} \quad (2.2)$$

Definition 2.2.3 (arity). Let $\mathcal{L} \stackrel{\text{def}}{=} (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language.

$$\mathcal{P}_k \stackrel{\text{def}}{=} \{p \in \mathcal{P} : \mathfrak{a}(p) = k\}, \quad (2.3)$$

$$\mathcal{F}_k \stackrel{\text{def}}{=} \{f \in \mathcal{F} : \mathfrak{a}(f) = k\}. \quad (2.4)$$

Definition 2.2.4 (atomic term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language.

1. For any $v \in \mathcal{V}$, v is an **atomic term**.
2. For any $c \in \mathcal{F}_0$, c is an **atomic term**.

Definition 2.2.5 (recursive: term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all terms $\mathcal{T}(\mathcal{L})$.

Let $s \in \mathcal{S}(\mathcal{L})^*$.

1. If $s \doteq v$ where $v \in \mathcal{V}$, then $s \in \mathcal{T}(\mathcal{L})$.
2. If $s \doteq f(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\tau|}$, then $s \in \mathcal{T}(\mathcal{L})$.
3. Otherwise $s \notin \mathcal{T}(\mathcal{L})$.

Definition 2.2.6 (atomic formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define a set of all atomic formulas $\mathcal{A}(\mathcal{L})$.

$$\mathcal{A}(\mathcal{L}) \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \{p(\tau) : p \in \mathcal{P}_k, \tau \in \mathcal{T}(\mathcal{L})^k\}. \quad (2.5)$$

Definition 2.2.7 (recursive: base formula, well-formed formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all base formulas $\mathcal{B}(\mathcal{L})$ and a set of all well-formed formulas $\mathcal{F}(\mathcal{L})$.

Let $s \in \mathcal{S}(\mathcal{L})^*$.

1. If $s \in \mathcal{A}(\mathcal{L})$, then $s \in \mathcal{B}(\mathcal{L})$.

2. If $s \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.
3. If $s \doteq (\phi)$ where $\phi \in \mathcal{F}(\mathcal{L})$, then $s \in \mathcal{B}(\mathcal{L})$.
4. If $s \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.
5. If $s \doteq \forall v. \phi$ where $\phi \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.
6. Otherwise $s \notin \mathcal{F}(\mathcal{L})$ and $s \notin \mathcal{B}(\mathcal{L})$.

Definition 2.2.8 (recursive: variable in term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. We will define recursively a set $\text{var}(\tau)$ of all variables in a term τ .

Let $v \in \mathcal{V}$ and $\tau \in \mathcal{T}(\mathcal{L})$.

1. If $\tau \doteq v$ then $v \in \text{var}(\tau)$.
2. If $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$ and $v \in \text{var}(\sigma) \stackrel{\text{def}}{=} \bigcup_{k=1}^{|\sigma|} \text{var}(\sigma_k)$, then $v \in \text{var}(\tau)$.
3. Otherwise $v \notin \text{var}(\tau)$.

Definition 2.2.9 (recursive: variable in formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. We will define recursively a set $\text{var}(\phi)$ of all variables in a formula ϕ .

Let $v \in \mathcal{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$ and $v \in \text{var}(\tau)$, then $v \in \text{var}(\phi)$.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and $v \in \text{var}(\alpha)$ or $v \in \text{var}(\beta)$, then $v \in \text{var}(\phi)$.
3. If $\phi \doteq \forall x. \psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, and either $v = x$ or $v \in \text{var}(\psi)$, then $v \in \text{var}(\phi)$.
4. Otherwise $v \notin \text{var}(\phi)$.

Definition 2.2.10 (recursive: free variable in formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. We will define recursively a set $\text{free}(\phi)$ of all free variables in a formula ϕ .

Let $v \in \mathcal{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \in \mathcal{A}(\mathcal{L})$ and $v \in \text{var}(\phi)$, then $v \in \text{free}(\phi)$.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and $v \in \text{free}(\alpha)$ or $v \in \text{free}(\beta)$, then $v \in \text{free}(\phi)$.
3. If $\phi \doteq \forall x. \psi$ where $\psi \in \mathcal{B}(\mathcal{L})$, $x \in \mathcal{V}$ and $x \neq v$, and $v \in \text{free}(\psi)$, then $v \in \text{free}(\phi)$.
4. Otherwise $v \notin \text{free}(\phi)$.

2.2.2 Semantics

Definition 2.2.11 (\mathcal{L} -structure). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary mathematical domain, then a mapping \mathbf{M} such that

$$\mathbf{M} : \mathcal{P} \cup \mathcal{F} \rightarrow \bigcup_{k=0}^{\infty} \{0, 1\}^{(A^k)} \cup \bigcup_{k=0}^{\infty} A^{(A^k)}, \quad (2.6)$$

where

$$\begin{aligned}\mathbf{M}(\mathfrak{f}) &= (\emptyset \mapsto 0), \\ \mathbf{M}(\mathcal{P}_k) &\subset \{0, 1\}^{(A^k)} \text{ for } k \in \mathbb{N}, \\ \mathbf{M}(\mathcal{F}_k) &\subset A^{(A^k)} \text{ for } k \in \mathbb{N}.\end{aligned}$$

is called an \mathcal{L} -structure with domain A .

For convenience we will denote values of \mathbf{M} as $p^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(p)$ for $p \in \mathcal{P}$ and $f^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(f)$ for $f \in \mathcal{F}$. Note that $p^{\mathbf{M}} : A^k \rightarrow \{0, 1\}$ and $f^{\mathbf{M}} : A^k \rightarrow A$.

Definition 2.2.12 (assignment). Let A be an arbitrary mathematical domain. Then $j : \mathcal{V} \rightarrow A$ is called a variables assignment in A .

Definition 2.2.13 (recursive: term evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure and $j : \mathcal{V} \rightarrow A$.

Let $\tau \in \mathcal{T}(\mathcal{L})$.

1. If $\tau \doteq v$ where $v \in \mathcal{V}$, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} j(v)$.
2. If $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} f^{\mathbf{M}}(\sigma^{\mathbf{M}, j})$ where

$$\sigma^{\mathbf{M}, j} \stackrel{\text{def}}{=} \begin{cases} (\sigma_1^{\mathbf{M}, j}, \dots, \sigma_k^{\mathbf{M}, j}) & \text{for } k = |\sigma| > 0, \\ \emptyset & \text{for } |\sigma| = 0. \end{cases} \quad (2.7)$$

Definition 2.2.14. Let \mathcal{L} be a signature of a language. Let \mathbf{M} be a \mathcal{L} -structure with domain A and let $j : \mathcal{V} \rightarrow A$ be an assignment of variables. Then we define

$$j \frac{a}{x}(v) \stackrel{\text{def}}{=} \begin{cases} a & \text{if } v = x, \\ j(v) & \text{otherwise.} \end{cases} \quad (2.8)$$

Definition 2.2.15. Let \mathcal{L} be a signature of a language. Let \mathbf{M} be a \mathcal{L} -structure with domain A and let $j : \mathcal{V} \rightarrow A$ be an assignment of variables. Let k be a positive integer and let $\mathbf{x} \in V^{\nabla k}$, and $\mathbf{a} \in A^k$.

Then we define

$$j \frac{\mathbf{a}}{\mathbf{x}}(v) \stackrel{\text{def}}{=} \begin{cases} a_i & \text{if } v = x_i \text{ for some } i \in \{1, \dots, k\}, \\ j(v) & \text{otherwise.} \end{cases} \quad (2.9)$$

For the purposes of this work, we will define multiplication product of infinite number of 0s and/or 1s.

Definition 2.2.16. Let S be an arbitrary set and $a_s \in \{0, 1\}$ for any $s \in S$.

$$\prod_{s \in S} a_s = \begin{cases} 1 & \text{if } a_s = 1 \text{ for all } s \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (2.10)$$

Definition 2.2.17 (recursive: formula evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$, then $\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} p^{\mathbf{M}}(\tau^{\mathbf{M},j})$.

2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}). \quad (2.11)$$

3. If $\phi \doteq \forall x.\psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, then

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} \prod_{a \in A} \psi^{\mathbf{M},j} \frac{a}{x}. \quad (2.12)$$

Corollary 2.2.18. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$, then

$$(\phi \rightarrow \mathfrak{f})^{\mathbf{M},j} = 1 - \phi^{\mathbf{M},j}. \quad (2.13)$$

Definition 2.2.19 (model). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$, then

$$\mathbf{M}, j \models \phi \iff \phi^{\mathbf{M},j} = 1. \quad (2.14)$$

Lemma 2.2.20. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j_1, j_2 : \mathcal{V} \rightarrow A$. Let $\tau \in \mathcal{T}(\mathcal{L})$. If $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, then

$$\tau^{\mathbf{M},j_1} = \tau^{\mathbf{M},j_2}. \quad (2.15)$$

Proof. We proceed by structural induction on the term τ .

Case 1: $\tau \doteq v$ where $v \in \mathcal{V}$.

Then $\text{var}(\tau) = \{v\}$. Since $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, we have $j_1(v) = j_2(v)$. By the definition of term evaluation,

$$\tau^{\mathbf{M},j_1} = j_1(v) = j_2(v) = \tau^{\mathbf{M},j_2}.$$

Case 2: $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$.

Then $\text{var}(\tau) = \text{var}(\sigma) = \bigcup_{i=1}^{|\sigma|} \text{var}(\sigma_i)$. Since $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, we have $j_1(\text{var}(\sigma_i)) = j_2(\text{var}(\sigma_i))$ for all $i \in \{1, \dots, |\sigma|\}$. By inductive hypothesis, $\sigma_i^{\mathbf{M},j_1} = \sigma_i^{\mathbf{M},j_2}$ for all i . Therefore, $\sigma^{\mathbf{M},j_1} = \sigma^{\mathbf{M},j_2}$. By the definition of term evaluation,

$$\tau^{\mathbf{M},j_1} = f^{\mathbf{M}}(\sigma^{\mathbf{M},j_1}) = f^{\mathbf{M}}(\sigma^{\mathbf{M},j_2}) = \tau^{\mathbf{M},j_2}.$$

This completes the proof by structural induction. \square

Theorem 2.2.21. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j_1, j_2 : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$. If $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, then

$$\phi^{\mathbf{M},j_1} = \phi^{\mathbf{M},j_2}. \quad (2.16)$$

Proof. We proceed by structural induction on the formula ϕ .

Case 1: $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$.

Then $\text{free}(\phi) = \text{var}(\phi) = \text{var}(\tau) = \bigcup_{i=1}^{|\tau|} \text{var}(\tau_i)$. Since $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, we have $j_1(\text{var}(\tau_i)) = j_2(\text{var}(\tau_i))$ for all i . By Lemma 2.2.20, $\tau_i^{\mathbf{M}, j_1} = \tau_i^{\mathbf{M}, j_2}$ for all i . Therefore, $\tau^{\mathbf{M}, j_1} = \tau^{\mathbf{M}, j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = p^{\mathbf{M}}(\tau^{\mathbf{M}, j_1}) = p^{\mathbf{M}}(\tau^{\mathbf{M}, j_2}) = \phi^{\mathbf{M}, j_2}.$$

Case 2: $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$.

Then $\text{free}(\phi) = \text{free}(\alpha) \cup \text{free}(\beta)$. Since $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, we have $j_1(\text{free}(\alpha)) = j_2(\text{free}(\alpha))$ and $j_1(\text{free}(\beta)) = j_2(\text{free}(\beta))$. By inductive hypothesis, $\alpha^{\mathbf{M}, j_1} = \alpha^{\mathbf{M}, j_2}$ and $\beta^{\mathbf{M}, j_1} = \beta^{\mathbf{M}, j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = 1 - \alpha^{\mathbf{M}, j_1}(1 - \beta^{\mathbf{M}, j_1}) = 1 - \alpha^{\mathbf{M}, j_2}(1 - \beta^{\mathbf{M}, j_2}) = \phi^{\mathbf{M}, j_2}.$$

Case 3: $\phi \doteq \forall x. \psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$.

Then $\text{free}(\phi) = \text{free}(\psi) \setminus \{x\}$. For any $a \in A$, the assignments $j_1 \frac{a}{x}$ and $j_2 \frac{a}{x}$ agree on $\text{free}(\psi)$ because:

- For $v \in \text{free}(\psi) \setminus \{x\}$, we have $v \in \text{free}(\phi)$, so $(j_1 \frac{a}{x})(v) = j_1(v) = j_2(v) = (j_2 \frac{a}{x})(v)$.
- For $v = x$, we have $(j_1 \frac{a}{x})(x) = a = (j_2 \frac{a}{x})(x)$.

Thus $j_1 \frac{a}{x}(\text{free}(\psi)) = j_2 \frac{a}{x}(\text{free}(\psi))$. By inductive hypothesis, $\psi^{\mathbf{M}, j_1} \frac{a}{x} = \psi^{\mathbf{M}, j_2} \frac{a}{x}$ for all $a \in A$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = \prod_{a \in A} \psi^{\mathbf{M}, j_1} \frac{a}{x} = \prod_{a \in A} \psi^{\mathbf{M}, j_2} \frac{a}{x} = \phi^{\mathbf{M}, j_2}.$$

This completes the proof by structural induction. \square

2.3 Substitutions

2.3.1 Syntax

Definition 2.3.1 (recursive: substitution in a term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\tau \in \mathcal{T}(\mathcal{L})^k$. Let $\sigma \in \mathcal{T}(\mathcal{L})$. We will define recursively $\sigma(\mathbf{x}/\tau)$. Let's establish that $\sigma(\emptyset/\emptyset) \stackrel{\text{def}}{=} \sigma$.

1. If $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$, then $\sigma(\mathbf{x}/\tau) \stackrel{\text{def}}{=} \tau_i$.
2. If $\sigma \doteq v$ where $v \in \mathcal{V}$ and $v \neq x$, then $\sigma(\mathbf{x}/\tau) \stackrel{\text{def}}{=} v$.
3. If $\tau \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\boldsymbol{\eta}|}$, then $\sigma(\mathbf{x}/\tau) \stackrel{\text{def}}{=} f(\boldsymbol{\eta}(\mathbf{x}/\tau))$ where

$$\boldsymbol{\eta}(\mathbf{x}/\tau) \stackrel{\text{def}}{=} \begin{cases} (\eta_1(\mathbf{x}/\tau), \dots, \eta_k(\mathbf{x}/\tau)) & \text{for } k = |\boldsymbol{\eta}| > 0, \\ \emptyset & \text{for } |\boldsymbol{\eta}| = 0. \end{cases} \quad (2.17)$$

Definition 2.3.2 (recursive: admissible substitution in a formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$.

Let $\phi \in \mathcal{F}(\mathcal{L})$. We will define recursively $\sigma(\mathbf{x}/\boldsymbol{\tau})$. Let's establish that $\phi(\emptyset/\emptyset) \stackrel{\text{def}}{=} \phi$ and is **admissible**.

1. If $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} p(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))$ and is **admissible**.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(\mathbf{x}/\boldsymbol{\tau}), \beta(\mathbf{x}/\boldsymbol{\tau})$ are **admissible**, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \alpha(\mathbf{x}/\boldsymbol{\tau}) \rightarrow \beta(\mathbf{x}/\boldsymbol{\tau})$ and is **admissible**.
3. If $\phi \doteq \forall u.\psi$ where $u \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$:
Let \mathbf{v} be \mathbf{x} with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from \mathbf{x} .
 - (a) If $\psi(\mathbf{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \forall u.\psi(\mathbf{v}/\boldsymbol{\sigma})$ and is **admissible**.
 - (b) Otherwise $\phi(\mathbf{x}/\boldsymbol{\tau})$ is **not admissible**.

Definition 2.3.3. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $q \in \mathcal{P}_k$ and $\phi \in \mathcal{F}(\mathcal{L})$.

Let $\psi \in \mathcal{F}(\mathcal{L})$. We will define recursively $\psi(q(\mathbf{x})/\phi)$.

1. If $\psi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$.
 - (a) If $p = q$ and $\phi(\mathbf{x}/\boldsymbol{\eta})$ is **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \phi(\mathbf{x}/\boldsymbol{\eta})$ and is **admissible**.
 - (b) If $p \neq q$, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \psi$ and is **admissible**.
2. If $\psi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(q(\mathbf{x})/\phi), \beta(q(\mathbf{x})/\phi)$ are **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \alpha(q(\mathbf{x})/\phi) \rightarrow \beta(q(\mathbf{x})/\phi)$ and is **admissible**.
3. If $\psi \doteq \forall u.\gamma$ where $u \in \mathcal{V}$ and $\gamma \in \mathcal{B}(\mathcal{L})$:
 - (a) If q occurs in γ , $u \notin \text{free}(\phi) \setminus \mathbf{x}$, $\gamma(q(\mathbf{x})/\phi)$ is **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \forall u.\gamma(q(\mathbf{x})/\phi)$ and is **admissible**.
 - (b) If q does not occur in γ , then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \forall u.\gamma$ and is **admissible**.
 - (c) Otherwise $\psi(q(\mathbf{x})/\phi)$ is **not admissible**.

2.3.2 Semantics

Lemma 2.3.4. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain. Let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$ be an assignment.

Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$.

Then for any term σ , we have

$$\sigma^{\mathbf{M}, j} \frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}} = \sigma(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j}. \quad (2.18)$$

Proof. We proceed by structural induction on the term σ .

Case 1: $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$.

Then $\sigma(\mathbf{x}/\tau) \doteq \tau_i$ and

$$\sigma^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} = (j \frac{\tau^{\mathbf{M},j}}{\mathbf{x}})(x_i) = \tau_i^{\mathbf{M},j} = \sigma(\mathbf{x}/\tau)^{\mathbf{M},j}.$$

Case 2: $\sigma \doteq v$ where $v \in \mathcal{V}$ and $v \neq x$.

Then $\sigma(\mathbf{x}/\tau) \doteq v$ and

$$\sigma^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} = (j \frac{\tau^{\mathbf{M},j}}{\mathbf{x}})(v) = j(v) = v^{\mathbf{M},j} = (\mathbf{x}/\tau)^{\mathbf{M},j}.$$

Case 3: $\sigma \doteq f(\eta)$ where $\eta \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\eta|}$.

Then $\sigma(\mathbf{x}/\tau) \doteq f(\eta(\mathbf{x}/\tau))$. By inductive hypothesis, $\eta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} = \eta(\mathbf{x}/\tau)^{\mathbf{M},j}$. Thus

$$\begin{aligned} \sigma^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} &= f^{\mathbf{M}} \left(\eta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} \right) = f^{\mathbf{M}} \left(\eta(\mathbf{x}/\tau)^{\mathbf{M},j} \right) \\ &= f(\eta(\mathbf{x}/\tau))^{\mathbf{M},j} = \sigma(\mathbf{x}/\tau)^{\mathbf{M},j}. \end{aligned}$$

This completes the proof by structural induction. \square

Theorem 2.3.5. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $\phi \in \mathcal{F}(\mathcal{L})$. Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\tau \in \mathcal{T}(\mathcal{L})^k$. If $\phi(\mathbf{x}/\tau)$ is admissible, then for any \mathcal{L} -structure \mathbf{M} with domain A and for any assignment of variables $j : \mathcal{V} \rightarrow A$ we have

$$(\mathbf{M}, j \frac{\tau^{\mathbf{M},j}}{\mathbf{x}}) \models \phi \iff \mathbf{M}, j \models \phi(\mathbf{x}/\tau). \quad (2.19)$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathcal{L})$.

Case 1: $\phi \doteq p(\eta)$ where $\eta \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\eta|}$.

By the definition of substitution in well formed formula, $\phi(\mathbf{x}/\tau) \doteq p(\eta(\mathbf{x}/\tau))$. By

Lemma 2.3.4, we have $\eta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} = \eta(\mathbf{x}/\tau)^{\mathbf{M},j}$. Therefore,

$$\phi^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} = p^{\mathbf{M}} \left(\eta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} \right) = p^{\mathbf{M}}(\eta(\mathbf{x}/\tau)^{\mathbf{M},j}) = p(\eta(\mathbf{x}/\tau))^{\mathbf{M},j} = \phi(\mathbf{x}/\tau)^{\mathbf{M},j}.$$

Case 2: $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$.

By the definition of substitution in well formed formula, $\phi(\mathbf{x}/\tau) \doteq \alpha(\mathbf{x}/\tau) \rightarrow \beta(\mathbf{x}/\tau)$.

By inductive hypothesis,

$$\begin{aligned} (\mathbf{M}, j \frac{\tau^{\mathbf{M},j}}{\mathbf{x}}) \models \alpha &\iff \mathbf{M}, j \models \alpha(\mathbf{x}/\tau), \\ (\mathbf{M}, j \frac{\tau^{\mathbf{M},j}}{\mathbf{x}}) \models \beta &\iff \mathbf{M}, j \models \beta(\mathbf{x}/\tau). \end{aligned}$$

By the definition of formula evaluation,

$$\begin{aligned}\phi^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} &= 1 - \alpha^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} \left(1 - \beta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} \right) = 1 - (\alpha(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j} (1 - (\beta(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j}) = 1 \\ &= (\alpha(\mathbf{x}/\boldsymbol{\tau}) \rightarrow \beta(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = (\phi(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j}\end{aligned}$$

Case 3: $\phi \doteq \forall u.\psi$ where $u \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$:

Let \mathbf{v} be \mathbf{x} with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from \mathbf{x} .

Since $\phi(\mathbf{x}/\boldsymbol{\tau})$ is admissible, $\psi(\mathbf{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, and we have $\phi(\mathbf{x}/\boldsymbol{\tau}) \doteq \forall u.\psi(\mathbf{v}/\boldsymbol{\sigma})$. Thus

$$\phi^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{\mathbf{x}} = \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j} a}{\mathbf{x}} \frac{a}{u} \quad (2.20)$$

By choice of \mathbf{v} , $j \frac{\tau^{\mathbf{M},j} a}{\mathbf{x}} \frac{a}{u}$ and $j \frac{\boldsymbol{\sigma}^{\mathbf{M},j} a}{\mathbf{v}} \frac{a}{u}$ agrees on $\text{free}(\psi)$ therefore by Theorem 2.2.21 we have

$$\begin{aligned}\prod_{a \in A} \psi^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j} a}{\mathbf{x}} \frac{a}{u} &= \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\boldsymbol{\sigma}^{\mathbf{M},j} a}{\mathbf{v}} \frac{a}{u} \\ &= \prod_{a \in A} \psi^{\mathbf{M},j} u \frac{a \boldsymbol{\sigma}^{\mathbf{M},j}}{\mathbf{v}} \quad (\text{since } u \text{ not in } \mathbf{v}) \\ &= \prod_{a \in A} \psi(\mathbf{v}/\boldsymbol{\sigma})^{\mathbf{M},j} \frac{a}{u} \quad (\text{by inductive hypothesis}) \\ &= (\forall u.\psi(\mathbf{v}/\boldsymbol{\sigma}))^{\mathbf{M},j} = \phi(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M},j}.\end{aligned}$$

This completes the proof by structural induction. \square