Logical Foundations

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Chapter 1

Introduction

1.1 Acknowledgements

This document presents the logical foundations for a formal system based on first-order logic. The definitions and theorems presented here were designed by the human author. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed, verified and corercted by the human author who takes the full responsibility for the content of this book.

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Chapter 2

First Order Logic

2.1 Preliminaries

We will use $\mathbb{N}$ to denote a set of all nonnegative integers.

Definition 2.1.1. *Let S be an arbitrary set.*

1. $S^0 \overset{\text{def}}{=} \{\emptyset\}$.

2. $S^k \overset{\text{def}}{=} \underbrace{S \times \cdots \times S}_{k}$.

3. $S^* \overset{\text{def}}{=} \bigcup_{k=0}^{\infty} S^k$.

Note that $\emptyset$ is treated as empty sequence.

Remark 2.1.2. *For a sequence $\boldsymbol{s} \in S^k$, we denote the i-th element of the sequence by s_i where $i \in \{1, \ldots, k\}$. let also $|\boldsymbol{s}| \overset{\text{def}}{=} k$ denote a length of $\boldsymbol{s}$.*

Note that according to the above convention $|\emptyset| = 0$.

Definition 2.1.3. *Let S be an arbitrary set.*

1. $S^{\nabla 0} \overset{\text{def}}{=} \{\emptyset\}$.

2. $S^{\nabla k} \overset{\text{def}}{=} \{\boldsymbol{s} \in S^k : \boldsymbol{s}_i \neq \boldsymbol{s}_j \text{ for } i \neq j\}$.

3. $S^{\nabla} \overset{\text{def}}{=} \bigcup_{k=0}^{\infty} S^{\nabla k}$.

2.2 Well Formed Formula

2.2.1 Syntax

Let $\mathfrak{L}$ denote a set of all logic symbols " ( ) ", " $\rightarrow$ ", " . ", " , ", " $\forall$ ". Let $\mathscr{V}$ be an infinite countable set of variable symbols. And $\mathfrak{f}$ be a special predicate symbol.

Definition 2.2.1. *Let $\mathscr{P}$ be a set of predicate symbols for which $\mathfrak{f} \in \mathscr{P}$ and $\mathscr{F}$ be a set of function symbols. Let $\mathfrak{a} : \mathscr{P} \cup \mathscr{F} \rightarrow \mathbb{N}$ be an arity function. then $\mathscr{L} \overset{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ is a signature of a language.*

To write first order logic formulas for a given language signature $\mathscr{L}$, we will use symbols from

$$\mathcal{S}(\mathscr{L}) \stackrel{\text{def}}{=} \mathfrak{L} \cup \mathscr{V} \cup \mathscr{P} \cup \mathscr{F}. \tag{2.1}$$

All formulas will be then a subset of $\mathcal{S}(\mathscr{L})^*$. Elements of $\mathcal{S}(\mathscr{L})^*$ will be called strings.

Note 2.2.2 (**syntactic equality**)**.** *We use the notation $\doteq$ to denote equality of strings. We will sometimes overload it when cast of some object to string is obvious.*

When we stack symbols together from left to right, we mean that we concatenate them in the same order. E.g let p be a predicate symbol and x, y be variable symbols, then $p(x, y)$ is just a sequence of 6 symbols: " p ", "(", " x ", " , ", " y ", ") ".

For convenience we will use an abreviated style of writing, in which for any symbol $s \in \mathscr{P} \cup \mathscr{F}$ and for any seqence of strings $\boldsymbol{x} \in \mathcal{S}(\mathscr{L})^*$

$$s(\boldsymbol{x}) \doteq \begin{cases} s(x_1, \ldots, x_k) \text{ for } |\boldsymbol{x}| > 0, \\ s \text{ for } |\boldsymbol{x}| = 0. \end{cases} \tag{2.2}$$

Definition 2.2.3 (**arity**)**.** *Let $\mathscr{L} \stackrel{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

$$\mathscr{P}_k \stackrel{\text{def}}{=} \{p \in \mathscr{P} : \mathfrak{a}(p) = k\}, \tag{2.3}$$

$$\mathscr{F}_k \stackrel{\text{def}}{=} \{f \in \mathscr{F} : \mathfrak{a}(f) = k\}. \tag{2.4}$$

Definition 2.2.4 (**atomic term**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

    *1. For any $v \in \mathscr{V}$, v is an **atomic term**.*

    *2. For any $c \in \mathscr{F}_0$, c is an **atomic term**.*

Definition 2.2.5 (**recursive: term**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all terms $\mathcal{T}(\mathscr{L})$.*

Let $s \in \mathcal{S}(\mathscr{L})^*$.

    1. If $s \doteq v$ where $v \in \mathscr{V}$, then $s \in \mathcal{T}(\mathscr{L})$.

    2. If $s \doteq f(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\tau}|}$.

    3. Otherwise $s \notin \mathcal{T}(\mathscr{L})$.

Definition 2.2.6 (**atomic formula**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

$$\mathcal{A}(\mathscr{L}) = \bigcup_{k=0}^{\infty} \{p(\boldsymbol{\tau}) : p \in \mathscr{P}_k, \boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k\} \tag{2.5}$$

Definition 2.2.7 (**recursive: base formula, well formed formula**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all base formulas $\mathcal{B}(\mathscr{L})$ and a set of all well formed formulas $\mathcal{F}(\mathscr{L})$.*

Let $s \in \mathcal{S}(\mathscr{L})^*$.

    1. If $s \in \mathcal{A}(\mathscr{L})$, then $s \in \mathcal{B}(\mathscr{L})$.

    2. If $s \in \mathcal{B}(\mathscr{L})$, then $s \in \mathcal{F}(\mathscr{L})$.

3. *If $s \doteq (\phi)$ where $\phi \in \mathcal{F}(\mathscr{L})$, then $s \in \mathcal{B}(\mathscr{L})$.*

4. *If $s \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then $s \in \mathcal{F}(\mathscr{L})$.*

5. *If $s \doteq \forall v.\phi$ where $\phi \in \mathcal{B}(\mathscr{L})$, then $s \in \mathcal{F}(\mathscr{L})$.*

6. *Otherwise $s \notin \mathcal{F}(\mathscr{L})$ and $s \notin \mathcal{B}(\mathscr{L})$.*

2.2.2 Semantics

Definition 2.2.8 ($\mathscr{L}$-**structure**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, then a mapping $\mathbf{M}$ such that*

$$\mathbf{M} : \mathscr{P} \cup \mathscr{F} \to \bigcup_{k=0}^{\infty} \{0,1\}^{(A^k)} \cup \bigcup_{k=0}^{\infty} A^{(A^k)}, \tag{2.6}$$

where

$$\mathbf{M}(\mathfrak{f}) = (\emptyset \mapsto 0),$$

$$\mathbf{M}(\mathscr{P}) \subset \bigcup_{k=0}^{\infty} \{0,1\}^{(A^k)},$$

$$\mathbf{M}(\mathscr{F}) \subset \bigcup_{k=0}^{\infty} A^{(A^k)}$$

is called an $\mathscr{L}$-structure with domain A.

For convenience we will denote values of $\mathbf{M}$ as $p^{\mathbf{M}} \overset{\text{def}}{=} \mathbf{M}(p)$ for $p \in \mathscr{P}$ and $f^{\mathbf{M}} \overset{\text{def}}{=} \mathbf{M}(f)$ for $f \in \mathscr{F}$.

Definition 2.2.9 (**assignment**). *Let A be an arbitrary mathematical domain. Then $j : \mathscr{V} \to A$ is called an variables assignment in A.*

Definition 2.2.10 (**recursive: term evaluation**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure and $j : \mathscr{V} \to A$.*

Let $\tau \in \mathcal{T}(\mathscr{L})$.

1. *If $\tau \doteq v$ where $v \in \mathscr{V}$, then $\tau^{\mathbf{M},j} \overset{\text{def}}{=} j(v)$.*

2. *If $\tau \doteq f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\sigma}|}$, then $\tau^{\mathbf{M},j} \overset{\text{def}}{=} f^{\mathbf{M}}(\boldsymbol{\sigma}^{\mathbf{M},j})$ where*

$$\boldsymbol{\sigma}^{\mathbf{M},j} \overset{\text{def}}{=} \begin{cases} (\sigma_1^{\mathbf{M},j}, \dots, \sigma_k^{\mathbf{M},j}) \text{ for } k = |\boldsymbol{\sigma}| > 0, \\ \emptyset \text{ for } |\boldsymbol{\sigma}| = 0. \end{cases} \tag{2.7}$$

Definition 2.2.11. *Let $\mathscr{L}$ be a signature of a language. Let $\mathbf{M}$ be a $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$ be an assignment of variables. Then we define*

$$j\frac{a}{x}(v) \overset{\text{def}}{=} \begin{cases} a \text{ if } v = x, \\ j(v) \text{ otherwise.} \end{cases} \tag{2.8}$$

Definition 2.2.12. *Let $\mathscr{L}$ be a signature of a language. Let $\mathbf{M}$ be a $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$ be an assignment of variables. Let k be a positive integer and let $\boldsymbol{x} \in V^{\nabla k}$, and $\boldsymbol{a} \in A^k$.*
Then we define

$$j\frac{\boldsymbol{a}}{\boldsymbol{x}}(v) \overset{\text{def}}{=} \begin{cases} a_i \ \text{if } v = x_i \text{ for some } i \in \{1, \ldots, k\}, \\ j(v) \ \text{otherwise.} \end{cases} \tag{2.9}$$

For the purpouses of this work, we will define multiplication product of infinite number of 0s and/or 1s.

Definition 2.2.13. *Let S be an arbitrary set and $a_s \in \{0, 1\}$ for any $s \in S$.*

$$\prod_{s \in S} a_s = \begin{cases} 1 \ \text{if } a_s = 1 \text{ for all } s \in S, \\ 0 \ \text{otherwise} . \end{cases} \tag{2.10}$$

Definition 2.2.14 (**recursive: formula evaluation**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$.*

1. *If $\phi \doteq p$ where $p \in \mathscr{P}$, then $\phi^{\mathbf{M},j} \overset{\text{def}}{=} p^{\mathbf{M}}$.*

2. *If $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$, then $\phi^{\mathbf{M},j} \overset{\text{def}}{=} p^{\mathbf{M}}(\boldsymbol{\tau}^{\mathbf{M},j})$.*

$$\phi^{\mathbf{M},j} \overset{\text{def}}{=} 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}). \tag{2.11}$$

3. *If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, then*

$$\phi^{\mathbf{M},j} \overset{\text{def}}{=} \prod_{a \in A} \psi^{\mathbf{M},j}\frac{a}{x}. \tag{2.12}$$

Corollary 2.2.15. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*

$$(\phi \to \mathfrak{f})^{\mathbf{M},j} = 1 - \phi^{\mathbf{M},j}. \tag{2.13}$$

Definition 2.2.16 (**model**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*

$$\mathbf{M}, j \models \phi \overset{\text{def}}{\iff} \phi^{\mathbf{M},j} = 1. \tag{2.14}$$

2.3   Substitutions

2.3.1   Syntax

Definition 2.3.1 (**recursive: variable in a term**)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $var(\tau)$ of all variables in a term τ.*

Let $v \in \mathscr{V}$ and $\tau \in \mathcal{T}(\mathscr{L})$.

1. *If $\tau \doteq v$ then $v \in var(\tau)$.*

2. If $\tau \doteq f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\sigma}|}$ and $v \in var(\boldsymbol{\sigma}) \stackrel{\text{def}}{=} \bigcup_{k=1}^{|\boldsymbol{\sigma}|} var(\sigma_k)$, then $v \in var(\tau)$.

3. Otherwise $v \notin var(\tau)$.

Definition 2.3.2 (recursive: variable in a formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $var(\phi)$ of all variables in a formula ϕ.*

Let $v \in \mathscr{V}$ and $\phi \in \mathcal{F}(\mathscr{L})$.

1. If $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$ and $v \in var(\boldsymbol{\tau})$, then $v \in var(\phi)$.

2. If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, and $v \in var(\alpha)$ or $v \in var(\beta)$, then $v \in var(\phi)$.

3. If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, and either $v = x$ or $v \in var(\psi)$, then $v \in var(\phi)$.

4. Otherwise $v \notin var(\phi)$.

Definition 2.3.3 (recursive: free variable in a formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $free(\phi)$ of all free variables in a formula ϕ.*

Let $v \in \mathscr{V}$ and $\phi \in \mathcal{F}(\mathscr{L})$.

1. If $\phi \in \mathcal{A}(\mathscr{L})$ and $v \in var(\phi)$, then $v \in free(\phi)$.

2. If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, and $v \in free(\alpha)$ or $v \in free(\beta)$, then $v \in free(\phi)$.

3. If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$ and $v \neq x$ and $v \in free(\psi)$, then $v \in free(\phi)$.

4. Otherwise $v \notin free(\phi)$.

Definition 2.3.4 (recursive: substitution in a term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$.*

Let $\sigma \in \mathcal{T}(\mathscr{L})$. Let's establish that $\sigma(\emptyset/\emptyset) \stackrel{\text{def}}{=} \sigma$.

1. If $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \tau_i$.

2. If $\sigma \doteq v$ where $v \in \mathscr{V}$ and v is not x, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} v$.

3. If $\tau \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\eta}|}$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$ where

$$\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \begin{cases} (\eta_1(\boldsymbol{x}/\boldsymbol{\tau}), \dots, \eta_k(\boldsymbol{x}/\boldsymbol{\tau})) \text{ for } k = |\boldsymbol{\eta}| > 0, \\ \emptyset \text{ for } |\boldsymbol{\eta}| = 0. \end{cases} \tag{2.15}$$

Definition 2.3.5 (recursive: admissible substitution in a formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$.*

Let $\phi \in \mathcal{F}(\mathscr{L})$. Let's establish that $\phi(\emptyset/\emptyset) \stackrel{\text{def}}{=} \phi$.

1. If $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$ and is **admissable**.

2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$ and $\alpha(\boldsymbol{x}/\boldsymbol{\tau}), \beta(\boldsymbol{x}/\boldsymbol{\tau})$ are **admissible**, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \alpha(\boldsymbol{x}/\boldsymbol{\tau}) \rightarrow \beta(\boldsymbol{x}/\boldsymbol{\tau})$ and is **admissable**.

3. If $\phi \doteq \forall u.\psi$ where $u \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$:
   Let $\boldsymbol{v}$ be $\boldsymbol{x}$ with x_i removed when $x_i = u$ or $x_i \notin free(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from $\boldsymbol{x}$.

   (a) If $\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin var(\boldsymbol{\sigma})$, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ and is **admissable**.

   (b) Otherwise $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is **not admissible**.

Lemma 2.3.6. *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. Let* σ *be a term. Let* $x \in \mathscr{V}$ *and* τ *be a term. Let* $y \in \mathscr{V}$ *and* $y \notin var(\tau) \cup var(\sigma)$. *Then*

$$\sigma(x/y)(y/\tau) \doteq \sigma(x/\tau). \tag{2.16}$$

Proof. We proceed by structural induction on the term σ.
Case 1: $\sigma \doteq x$.
Then $\sigma(x/y) \doteq y$, and
$$(\sigma(x/y))(y/\tau) \doteq y(y/\tau) \doteq \tau.$$

Also, $\sigma(x/\tau) \doteq \tau$. Therefore, $\sigma(x/y)(y/\tau) \doteq \sigma(x/\tau)$.
Case 2: $\sigma \doteq v$ where $v \in \mathscr{V}$ and $v \neq x$.
Since y is not a variable in σ, we have $v \neq y$. Then $\sigma(x/y) \doteq v$, and

$$(\sigma(x/y))(y/\tau) \doteq v(y/\tau) \doteq v$$

(since $v \neq y$). Also, $\sigma(x/\tau) \doteq v$ (since $v \neq x$). Therefore, $\sigma(x/y)(y/\tau) \doteq \sigma(x/\tau)$.
Case 3: $\sigma \doteq c$ where $c \in \mathscr{F}_0$.
Then $\sigma(x/y) \doteq c$, and
$$(\sigma(x/y))(y/\tau) \doteq c(y/\tau) \doteq c.$$

Also, $\sigma(x/\tau) \doteq c$. Therefore, $\sigma(x/y)(y/\tau) \doteq \sigma(x/\tau)$.
Case 4: $\sigma \doteq f(\sigma_1, \ldots, \sigma_k)$ where $f \in \mathscr{F}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms.
Since $y \notin var(\sigma)$, $y \notin var(\sigma_i)$ for $i = 1, \ldots, k$. By the definition of substitution in term, $\sigma(x/y) \doteq f(\sigma_1(x/y), \ldots, \sigma_k(x/y))$. Then

$$(\sigma(x/y))(y/\tau) \doteq f(\sigma_1(x/y)(y/\tau), \ldots, \sigma_k(x/y)(y/\tau)).$$

By inductive hypothesis, $\sigma_i(x/y)(y/\tau) \doteq \sigma_i(x/\tau)$ for all $i = 1, \ldots, k$. Therefore,

$$(\sigma(x/y))(y/\tau) \doteq f(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau)) \doteq \sigma(x/\tau).$$

This completes the proof by structural induction. $\qquad\square$

Lemma 2.3.7. *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. Let* $\phi \in \mathcal{F}(\mathscr{L})$. *Let* $x \in \mathscr{V}$ *and* τ *be a term. Let* $y \in \mathscr{V}$ *and* $y \notin var(\tau) \cup var(\phi)$. *If* $\phi(x/\tau)$ *is admissible then*

$$\phi(x/y)(y/\tau) \doteq \phi(x/\tau). \tag{2.17}$$

Proof. We proceed by structural induction on the formula ϕ.
Case 1: $\phi \doteq p$ where $p \in \mathscr{P}_0$.
Then $\phi(x/y) \doteq p$, and
$$(\phi(x/y))(y/\tau) \doteq p(y/\tau) \doteq p.$$

Also, $\phi(x/\tau) \doteq p$. Therefore, $\phi(x/y)(y/\tau) \doteq \phi(x/\tau)$.

Case 2: $\phi \doteq p(\sigma_1, \ldots, \sigma_k)$ where $p \in \mathscr{P}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms.
Since y is not a variable in ϕ, y is not a variable in any σ_i for $i = 1, \ldots, k$. By the definition of substitution in well formed formula, $\phi(x/y) \doteq p(\sigma_1(x/y), \ldots, \sigma_k(x/y))$. Then

$$(\phi(x/y))(y/\tau) \doteq p(\sigma_1(x/y)(y/\tau), \ldots, \sigma_k(x/y)(y/\tau)).$$

By Lemma 2.3.6, $\sigma_i(x/y)(y/\tau) \doteq \sigma_i(x/\tau)$ for all $i = 1, \ldots, k$. Therefore,

$$(\phi(x/y))(y/\tau) \doteq p(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau)) \doteq \phi(x/\tau).$$

Case 3: $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.
Since $y \notin \mathrm{var}(\phi)$, $y \notin \mathrm{var}(\alpha)$ and $y \notin \mathrm{var}(\beta)$. By the definition of substitution in well formed formula, $\phi(x/y) \doteq \alpha(x/y) \rightarrow \beta(x/y)$. Then

$$(\phi(x/y))(y/\tau) \doteq \alpha(x/y)(y/\tau) \rightarrow \beta(x/y)(y/\tau).$$

By inductive hypothesis, $\alpha(x/y)(y/\tau) \doteq \alpha(x/\tau)$ and $\beta(x/y)(y/\tau) \doteq \beta(x/\tau)$. Therefore,

$$(\phi(x/y))(y/\tau) \doteq \alpha(x/\tau) \rightarrow \beta(x/\tau) \doteq \phi(x/\tau).$$

Case 4: $\phi \doteq \forall z.\psi$ where $z \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$.
Subcase 4a: If $z = x$ or $x \notin \mathrm{free}(\psi)$, then by the definition of substitution, $\phi(x/y) \doteq \forall z.\psi$. Since $y \notin \mathrm{var}(\phi)$, $y \notin \mathrm{var}(\psi)$. Then

$$(\phi(x/y))(y/\tau) \doteq (\forall z.\psi)(y/\tau) \doteq \forall z.\psi(y/\tau).$$

Since $y \notin \mathrm{var}(\psi)$, we have $\psi(y/\tau) \doteq \psi$. Also, $\phi(x/\tau) \doteq \forall z.\psi$ (since $z = x$ or $x \notin \mathrm{free}(\psi)$). Therefore, $\phi(x/y)(y/\tau) \doteq \phi(x/\tau)$.
Subcase 4b: If $z \neq x$ and $x \in \mathrm{free}(\psi)$, then since $\phi(x/\tau)$ is admissible, we have $z \notin \mathrm{var}(\tau)$ and $\psi(x/\tau)$ is admissible. Since $y \notin \mathrm{var}(\phi)$, we have $y \neq z$ and $y \notin \mathrm{var}(\psi)$ and then by the definition of substitution, $\phi(x/y) \doteq \forall z.\psi(x/y)$ ($\psi(x/y)$ is trivially admissible). Thus, we have

$$(\phi(x/y))(y/\tau) \doteq (\forall z.\psi(x/y))(y/\tau) \doteq \forall z.\psi(x/y)(y/\tau).$$

By inductive hypothesis, $\psi(x/y)(y/\tau) \doteq \psi(x/\tau)$. Therefore,

$$(\phi(x/y))(y/\tau) \doteq \forall z.\psi(x/\tau) \doteq \phi(x/\tau).$$

This completes the proof by structural induction. $\qquad\square$

Definition 2.3.8. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $x_1, \ldots, x_k \in \mathscr{V}$ be pairwise distincive variables, let $\tau_1, \ldots, \tau_k$ be terms, and $\phi \in \mathcal{F}(\mathscr{L})$. Let $\phi(x_i/\tau_i)$ be admissible for $i = 1, \ldots, k$.*

$$\phi(x_1/\tau_1, \ldots, x_k/\tau_k) \stackrel{\text{def}}{=} \phi(x_1/y_1) \ldots (x_k/y_k)(y_1/\tau_1) \ldots (y_k/\tau_k), \qquad (2.18)$$

where $y_j \notin \mathrm{var}(\phi) \cup \bigcup_{i=1}^{k} \mathrm{var}(\tau_i)$ for $j = 1, \ldots, k$.

2.3.2 Semantics

Lemma 2.3.9. *Let $\mathscr{L}$ be a signature of a language and A be an arbitrary mathematical domain. Let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \rightarrow A$ be an assignment. For any term σ, any $x \in \mathscr{V}$, and any term τ, we have*

$$\sigma^{\mathbf{M},j}\overline{\frac{\tau^{(\mathbf{M},j)}}{x}} = (\sigma(x/\tau))^{\mathbf{M},j}. \qquad (2.19)$$

Proof. We proceed by structural induction on the term σ.

Case 1: $\sigma \doteq x$.

Then $\sigma(x/\tau) \doteq \tau$ and

$$\sigma^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = (j\frac{\tau^{(\mathbf{M},j)}}{x})(x) = \tau^{\mathbf{M},j} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 2: $\sigma \doteq v$ where $v \in \mathscr{V}$ and $v \neq x$.

Then $\sigma(x/\tau) \doteq v$ and

$$\sigma^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = (j\frac{\tau^{(\mathbf{M},j)}}{x})(v) = j(v) = v^{\mathbf{M},j} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 3: $\sigma \doteq c$ where $c \in \mathscr{F}_0$.

Then $\sigma(x/\tau) \doteq c$ and

$$\sigma^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = c^{\mathbf{M}} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 4: $\sigma \doteq f(\sigma_1, \ldots, \sigma_k)$ where $f \in \mathscr{F}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms.

Then $\sigma(x/\tau) \doteq f(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau))$. By inductive hypothesis, $\sigma_i^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = (\sigma_i(x/\tau))^{\mathbf{M},j}$ for all $i = 1, \ldots, k$. Thus

$$\begin{aligned}
\sigma^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} &= f^{\mathbf{M}}(\sigma_1^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}}, \ldots, \sigma_k^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}}) \\
&= f^{\mathbf{M}}((\sigma_1(x/\tau))^{\mathbf{M},j}, \ldots, (\sigma_k(x/\tau))^{\mathbf{M},j}) \\
&= (f(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau)))^{\mathbf{M},j} \\
&= (\sigma(x/\tau))^{\mathbf{M},j}.
\end{aligned}$$

This completes the proof by structural induction. □

Theorem 2.3.10. *Let $\mathscr{L}$ be a signature of a language. Let $\phi \in \mathcal{F}(\mathscr{L})$ and let τ be an arbitrary term. If $\phi(x/\tau)$ is admissible, then for any $\mathscr{L}$-structure $\mathbf{M}$ with domain A and for any assignment of variables $j : \mathscr{V} \to A$ we have*

$$(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi \iff (\mathbf{M}, j) \models \phi(x/\tau). \tag{2.20}$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathscr{L})$.

Case 1: $\phi \doteq p$ where $p \in \mathscr{P}_0$.

By the definition of substitution in well formed formula, $\phi(x/\tau) \doteq p$. By the definition of formula evaluation, $\phi^{\mathbf{M},j} = p^{\mathbf{M}}$ for any assignment j. Thus

$$\begin{aligned}
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi &\iff \phi^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1 \\
&\iff p^{\mathbf{M}} = 1 \\
&\iff \phi^{\mathbf{M},j} = 1 \\
&\iff (\mathbf{M}, j) \models \phi \\
&\iff (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}$$

Case 2: $\phi \doteq p(\sigma_1, \ldots, \sigma_k)$ where $p \in \mathscr{P}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms.

By the definition of substitution in well formed formula, $\phi(x/\tau) \doteq p(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau))$.

By Lemma 2.3.9, for each term σ_i, we have $\sigma_i^{\mathbf{M}, j \frac{\tau^{(\mathbf{M},j)}}{x}} = (\sigma_i(x/\tau))^{\mathbf{M},j}$. Therefore,

$$
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi \iff \phi^{\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1
$$

$$
\iff p^{\mathbf{M}}(\sigma_1^{\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}}, \ldots, \sigma_k^{\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}}) = 1
$$

$$
\iff p^{\mathbf{M}}((\sigma_1(x/\tau))^{\mathbf{M},j}, \ldots, (\sigma_k(x/\tau))^{\mathbf{M},j}) = 1
$$

$$
\iff (p(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau)))^{\mathbf{M},j} = 1
$$

$$
\iff (\phi(x/\tau))^{\mathbf{M},j} = 1
$$

$$
\iff (\mathbf{M}, j) \models \phi(x/\tau).
$$

Case 3: $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.

By the definition of substitution in well formed formula, $\phi(x/\tau) \doteq \alpha(x/\tau) \to \beta(x/\tau)$. By inductive hypothesis,

$$
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \alpha \iff (\mathbf{M}, j) \models \alpha(x/\tau),
$$

$$
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \beta \iff (\mathbf{M}, j) \models \beta(x/\tau).
$$

By the definition of formula evaluation,

$$
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi \iff \phi^{\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1
$$

$$
\iff 1 - \alpha^{\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}}(1 - \beta^{\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}}) = 1
$$

$$
\iff 1 - (\alpha(x/\tau))^{\mathbf{M},j}(1 - (\beta(x/\tau))^{\mathbf{M},j}) = 1
$$

$$
\iff (\alpha(x/\tau) \to \beta(x/\tau))^{\mathbf{M},j} = 1
$$

$$
\iff (\phi(x/\tau))^{\mathbf{M},j} = 1
$$

$$
\iff (\mathbf{M}, j) \models \phi(x/\tau).
$$

Case 4: $\phi \doteq \forall y.\psi$ where $y \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$.

Subcase 4a: If $y = x$ or $x \notin \text{free}(\psi)$, then by the definition of substitution, $\phi(x/\tau) \doteq$

$\forall y.\psi$. Thus

$$(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi \iff \phi^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1$$

$$\iff \prod_{a \in A} \psi^{\mathbf{M},(j\frac{\tau^{(\mathbf{M},j)}}{x})\frac{a}{y}} = 1$$

$$\iff \prod_{a \in A} \psi^{\mathbf{M},j\frac{a}{y}} = 1 \quad (\text{since } y = x \text{ and } (j\frac{\tau^{(\mathbf{M},j)}}{x})\frac{a}{x} = j\frac{a}{x}, \text{ or } x \notin \text{free}(\psi))$$

$$\iff \phi^{\mathbf{M},j} = 1$$

$$\iff (\mathbf{M}, j) \models \phi$$

$$\iff (\mathbf{M}, j) \models \phi(x/\tau).$$

Subcase 4b: If $y \neq x$ and $x \in \text{free}(\psi)$. Since $\phi(x/\tau)$ is admissible, $y \notin \text{var}(\tau)$ and $\psi(x/\tau)$ is admissible, then by the definition of substitution, $\phi(x/\tau) \doteq \forall y.\psi(x/\tau)$. By inductive hypothesis, for any $a \in A$,

$$(\mathbf{M}, j\frac{a}{y}\frac{\tau^{(\mathbf{M},j\frac{a}{y})}}{x}) \models \psi \iff (\mathbf{M}, j\frac{a}{y}) \models \psi(x/\tau).$$

Since $y \notin \text{var}(\tau)$, the evaluation of τ does not depend on the value assigned to y.

Therefore, $\tau^{(\mathbf{M},j\frac{a}{y})} = \tau^{(\mathbf{M},j)}$ for all $a \in A$.

Moreover, since $x \neq y$, we have $j\frac{\tau^{(\mathbf{M},j)}}{x}\frac{a}{y} = j\frac{a}{y}\frac{\tau^{(\mathbf{M},j)}}{x}$ for all $a \in A$.

Thus,

$$(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi \iff \phi^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1$$

$$\iff \prod_{a \in A} \psi^{\mathbf{M},(j\frac{\tau^{(\mathbf{M},j)}}{x})\frac{a}{y}} = 1$$

$$\iff \prod_{a \in A} \psi^{\mathbf{M},j\frac{a}{y}\frac{\tau^{(\mathbf{M},j)}}{x}} = 1$$

$$\iff \prod_{a \in A} (\psi(x/\tau))^{\mathbf{M},j\frac{a}{y}} = 1 \quad (\text{by inductive hypothesis})$$

$$\iff (\forall y.\psi(x/\tau))^{\mathbf{M},j} = 1$$

$$\iff (\phi(x/\tau))^{\mathbf{M},j} = 1$$

$$\iff (\mathbf{M}, j) \models \phi(x/\tau).$$

This completes the proof by structural induction. $\square$