

Logical Foundations

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Chapter 1

Introduction

1.1 Acknowledgements

This document presents the logical foundations for a formal system based on first-order logic. The net of definitions and theorems presented here were designed by a human author - Michał Stanisław Wójcik. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed, verified and corrected by Michał Stanisław Wójcik who takes the full responsibility for the content of this work.

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Chapter 2

First Order Logic

2.1 Preliminaries

We will use \mathbb{N} to denote a set of all nonnegative integers.

Definition 2.1.1. Let S be an arbitrary set.

1. $S^0 \stackrel{\text{def}}{=} \{\emptyset\}.$
2. $S^k \stackrel{\text{def}}{=} \underbrace{S \times \cdots \times S}_{k}.$
3. $S^* \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^k.$

Note that \emptyset is treated as empty sequence.

Remark 2.1.2. For a sequence $s \in S^k$, we denote the i -th element of the sequence by s_i where $i \in \{1, \dots, k\}$. let also $|s| \stackrel{\text{def}}{=} k$ denote a length of s .

Note that according to the above convention $|\emptyset| = 0$.

Definition 2.1.3. Let S be an arbitrary set.

1. $S^{\nabla 0} \stackrel{\text{def}}{=} \{\emptyset\}.$
2. $S^{\nabla k} \stackrel{\text{def}}{=} \{s \in S^k : s_i \neq s_j \text{ for } i \neq j\}.$
3. $S^{\nabla} \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^{\nabla k}.$

2.2 First Order Logic Formula

2.2.1 Syntax

Let \mathfrak{L} denote a set of all logical symbols “(”, “)”, “ \rightarrow ”, “ \cdot ”, “ $,$ ”, “ \forall ”. Let \mathcal{V} be an infinite countable set of variable symbols. And f be a special predicate symbol.

Definition 2.2.1. Let \mathcal{P} be a set of predicate symbols for which $f \in \mathcal{P}$ and \mathcal{F} be a set of function symbols. Let $a : \mathcal{P} \cup \mathcal{F} \rightarrow \mathbb{N}$ be an arity function. then $\mathcal{L} \stackrel{\text{def}}{=} (\mathcal{P}, \mathcal{F}, a)$ is a signature of a language.

To write first order logic formulas for a given language signature \mathcal{L} , we will use symbols from

$$\mathcal{S}(\mathcal{L}) \stackrel{\text{def}}{=} \mathcal{L} \cup \mathcal{V} \cup \mathcal{P} \cup \mathcal{F}. \quad (2.1)$$

All formulas will be then a subset of $\mathcal{S}(\mathcal{L})^*$. Elements of $\mathcal{S}(\mathcal{L})^*$ will be called strings.

Note 2.2.2 (syntactic equality). We use the notation \doteq to denote equality of strings. We will sometimes overload it when cast of some object to string is obvious.

When we stack symbols together from left to right, we mean that we concatenate them in the same order. E.g let p be a predicate symbol and x, y be variable symbols, then $p(x, y)$ is just a sequence of 6 symbols: “ p ”, “(”, “ x ”, “,”, “ y ”, “)”).

For convenience we will use an abbreviated style of writing, in which for any symbol $s \in \mathcal{P} \cup \mathcal{F}$ and for any seqence of strings $\mathbf{x} \in \mathcal{S}(\mathcal{L})^*$

$$s(\mathbf{x}) \stackrel{\text{def}}{=} \begin{cases} s(x_1, \dots, x_k) & \text{for } |\mathbf{x}| > 0, \\ s & \text{for } |\mathbf{x}| = 0. \end{cases} \quad (2.2)$$

Definition 2.2.3 (arity). Let $\mathcal{L} \stackrel{\text{def}}{=} (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language.

$$\mathcal{P}_k \stackrel{\text{def}}{=} \{p \in \mathcal{P} : \mathfrak{a}(p) = k\}, \quad (2.3)$$

$$\mathcal{F}_k \stackrel{\text{def}}{=} \{f \in \mathcal{F} : \mathfrak{a}(f) = k\}. \quad (2.4)$$

Definition 2.2.4 (atomic term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language.

1. For any $v \in \mathcal{V}$, v is an **atomic term**.
2. For any $c \in \mathcal{F}_0$, c is an **atomic term**.

Definition 2.2.5 (recursive: term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all terms $\mathcal{T}(\mathcal{L})$.

Let $s \in \mathcal{S}(\mathcal{L})^*$.

1. If $s \doteq v$ where $v \in \mathcal{V}$, then $s \in \mathcal{T}(\mathcal{L})$.
2. If $s \doteq f(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\tau|}$, then $s \in \mathcal{T}(\mathcal{L})$.
3. Otherwise $s \notin \mathcal{T}(\mathcal{L})$.

Definition 2.2.6 (atomic formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define a set of all atomic formulas $\mathcal{A}(\mathcal{L})$.

$$\mathcal{A}(\mathcal{L}) \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \{p(\tau) : p \in \mathcal{P}_k, \tau \in \mathcal{T}(\mathcal{L})^k\}. \quad (2.5)$$

Definition 2.2.7 (recursive: base formula, well-formed formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all base formulas $\mathcal{B}(\mathcal{L})$ and a set of all well-formed formulas $\mathcal{F}(\mathcal{L})$.

Let $s \in \mathcal{S}(\mathcal{L})^*$.

1. If $s \in \mathcal{A}(\mathcal{L})$, then $s \in \mathcal{B}(\mathcal{L})$.

2. If $s \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.
3. If $s \doteq (\phi)$ where $\phi \in \mathcal{F}(\mathcal{L})$, then $s \in \mathcal{B}(\mathcal{L})$.
4. If $s \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.
5. If $s \doteq \forall v. \phi$ where $\phi \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.
6. Otherwise $s \notin \mathcal{F}(\mathcal{L})$ and $s \notin \mathcal{B}(\mathcal{L})$.

Definition 2.2.8 (recursive: variable in term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. We will define recursively a set $\text{var}(\tau)$ of all variables in a term τ .

Let $v \in \mathcal{V}$ and $\tau \in \mathcal{T}(\mathcal{L})$.

1. If $\tau \doteq v$ then $v \in \text{var}(\tau)$.
2. If $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$ and $v \in \text{var}(\sigma) \stackrel{\text{def}}{=} \bigcup_{k=1}^{|\sigma|} \text{var}(\sigma_k)$, then $v \in \text{var}(\tau)$.
3. Otherwise $v \notin \text{var}(\tau)$.

Definition 2.2.9 (recursive: variable in formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. We will define recursively a set $\text{var}(\phi)$ of all variables in a formula ϕ .

Let $v \in \mathcal{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$ and $v \in \text{var}(\tau)$, then $v \in \text{var}(\phi)$.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and $v \in \text{var}(\alpha)$ or $v \in \text{var}(\beta)$, then $v \in \text{var}(\phi)$.
3. If $\phi \doteq \forall x. \psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, and either $v = x$ or $v \in \text{var}(\psi)$, then $v \in \text{var}(\phi)$.
4. Otherwise $v \notin \text{var}(\phi)$.

Definition 2.2.10 (recursive: free variable in formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. We will define recursively a set $\text{free}(\phi)$ of all free variables in a formula ϕ .

Let $v \in \mathcal{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \in \mathcal{A}(\mathcal{L})$ and $v \in \text{var}(\phi)$, then $v \in \text{free}(\phi)$.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and $v \in \text{free}(\alpha)$ or $v \in \text{free}(\beta)$, then $v \in \text{free}(\phi)$.
3. If $\phi \doteq \forall x. \psi$ where $\psi \in \mathcal{B}(\mathcal{L})$, $x \in \mathcal{V}$ and $x \neq v$, and $v \in \text{free}(\psi)$, then $v \in \text{free}(\phi)$.
4. Otherwise $v \notin \text{free}(\phi)$.

2.2.2 Semantics

Definition 2.2.11 (\mathcal{L} -structure). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and $A \neq \emptyset$ be an arbitrary mathematical domain, then a mapping \mathbf{M} such that

$$\mathbf{M} : \mathcal{P} \cup \mathcal{F} \rightarrow \bigcup_{k=0}^{\infty} \{0, 1\}^{(A^k)} \cup \bigcup_{k=0}^{\infty} A^{(A^k)}, \quad (2.6)$$

where

$$\begin{aligned}\mathbf{M}(\mathfrak{f}) &= (\emptyset \mapsto 0), \\ \mathbf{M}(\mathcal{P}_k) &\subset \{0, 1\}^{(A^k)} \text{ for } k \in \mathbb{N}, \\ \mathbf{M}(\mathcal{F}_k) &\subset A^{(A^k)} \text{ for } k \in \mathbb{N}.\end{aligned}$$

is called an \mathcal{L} -structure with domain A .

For convenience we will denote values of \mathbf{M} as $p^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(p)$ for $p \in \mathcal{P}$ and $f^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(f)$ for $f \in \mathcal{F}$. Note that $p^{\mathbf{M}} : A^k \rightarrow \{0, 1\}$ and $f^{\mathbf{M}} : A^k \rightarrow A$.

Definition 2.2.12 (assignment). Let A be an arbitrary mathematical domain. Then $j : \mathcal{V} \rightarrow A$ is called a variables assignment in A .

Definition 2.2.13 (recursive: term evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure and $j : \mathcal{V} \rightarrow A$.

Let $\tau \in \mathcal{T}(\mathcal{L})$.

1. If $\tau \doteq v$ where $v \in \mathcal{V}$, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} j(v)$.
2. If $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} f^{\mathbf{M}}(\sigma^{\mathbf{M}, j})$ where

$$\sigma^{\mathbf{M}, j} \stackrel{\text{def}}{=} \begin{cases} (\sigma_1^{\mathbf{M}, j}, \dots, \sigma_k^{\mathbf{M}, j}) & \text{for } k = |\sigma| > 0, \\ \emptyset & \text{for } |\sigma| = 0. \end{cases} \quad (2.7)$$

Definition 2.2.14. Let \mathcal{L} be a signature of a language. Let \mathbf{M} be a \mathcal{L} -structure with domain A and let $j : \mathcal{V} \rightarrow A$ be an assignment of variables. Then we define

$$j \frac{a}{x}(v) \stackrel{\text{def}}{=} \begin{cases} a & \text{if } v = x, \\ j(v) & \text{otherwise.} \end{cases} \quad (2.8)$$

Definition 2.2.15. Let \mathcal{L} be a signature of a language. Let \mathbf{M} be a \mathcal{L} -structure with domain A and let $j : \mathcal{V} \rightarrow A$ be an assignment of variables. Let k be a positive integer and let $\mathbf{x} \in V^{\nabla k}$, and $\mathbf{a} \in A^k$.

Then we define

$$j \frac{\mathbf{a}}{\mathbf{x}}(v) \stackrel{\text{def}}{=} \begin{cases} a_i & \text{if } v = x_i \text{ for some } i \in \{1, \dots, k\}, \\ j(v) & \text{otherwise.} \end{cases} \quad (2.9)$$

For the purposes of this work, we will define multiplication product of infinite number of 0s and/or 1s.

Definition 2.2.16. Let S be an arbitrary set and $a_s \in \{0, 1\}$ for any $s \in S$.

$$\prod_{s \in S} a_s = \begin{cases} 1 & \text{if } a_s = 1 \text{ for all } s \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (2.10)$$

Definition 2.2.17 (recursive: formula evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$, then $\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} p^{\mathbf{M}}(\tau^{\mathbf{M},j})$.

2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}). \quad (2.11)$$

3. If $\phi \doteq \forall x.\psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, then

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} \prod_{a \in A} \psi^{\mathbf{M},j} \frac{a}{x}. \quad (2.12)$$

Corollary 2.2.18. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$, then

$$(\phi \rightarrow \mathfrak{f})^{\mathbf{M},j} = 1 - \phi^{\mathbf{M},j}. \quad (2.13)$$

Definition 2.2.19 (interpretation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$, then

$$\mathbf{M}, j \models \phi \iff \phi^{\mathbf{M},j} = 1. \quad (2.14)$$

Definition 2.2.20 (model). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ and let \mathbf{M} be an \mathcal{L} -structure. Let $\phi \in \mathcal{F}(\mathcal{L})$. We define

$$\mathbf{M} \models \phi \quad (2.15)$$

iff for any assignment $j : \mathcal{V} \rightarrow A$, we have $\mathbf{M}, j \models \phi$.

Definition 2.2.21 (semantic consequence). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $\Gamma \subset \mathcal{F}(\mathcal{L})$ and $\phi \in \mathcal{F}(\mathcal{L})$. We define that $\Gamma \models \phi$ iff for any \mathcal{L} -structure \mathbf{M} , if $\mathbf{M} \models \psi$ for all $\psi \in \Gamma$, then $\mathbf{M} \models \phi$.

Definition 2.2.22 (tautology). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $\phi \in \mathcal{F}(\mathcal{L})$. We define that $\models \phi$ iff $\emptyset \models \phi$.

Lemma 2.2.23. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j_1, j_2 : \mathcal{V} \rightarrow A$. Let $\tau \in \mathcal{T}(\mathcal{L})$. If $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, then

$$\tau^{\mathbf{M},j_1} = \tau^{\mathbf{M},j_2}. \quad (2.16)$$

Proof. We proceed by structural induction on the term τ .

Case 1: $\tau \doteq v$ where $v \in \mathcal{V}$.

Then $\text{var}(\tau) = \{v\}$. Since $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, we have $j_1(v) = j_2(v)$. By the definition of term evaluation,

$$\tau^{\mathbf{M},j_1} = j_1(v) = j_2(v) = \tau^{\mathbf{M},j_2}.$$

Case 2: $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$.

Then $\text{var}(\tau) = \text{var}(\sigma) = \bigcup_{i=1}^{|\sigma|} \text{var}(\sigma_i)$. Since $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, we have $j_1(\text{var}(\sigma_i)) = j_2(\text{var}(\sigma_i))$ for all $i \in \{1, \dots, |\sigma|\}$. By inductive hypothesis, $\sigma_i^{\mathbf{M},j_1} = \sigma_i^{\mathbf{M},j_2}$ for all i . Therefore, $\sigma^{\mathbf{M},j_1} = \sigma^{\mathbf{M},j_2}$. By the definition of term evaluation,

$$\tau^{\mathbf{M},j_1} = f^{\mathbf{M}}(\sigma^{\mathbf{M},j_1}) = f^{\mathbf{M}}(\sigma^{\mathbf{M},j_2}) = \tau^{\mathbf{M},j_2}.$$

This completes the proof by structural induction. \square

Theorem 2.2.24. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j_1, j_2 : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$. If $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, then

$$\phi^{\mathbf{M}, j_1} = \phi^{\mathbf{M}, j_2}. \quad (2.17)$$

Proof. We proceed by structural induction on the formula ϕ .

Case 1: $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$.

Then $\text{free}(\phi) = \text{var}(\phi) = \text{var}(\tau) = \bigcup_{i=1}^{|\tau|} \text{var}(\tau_i)$. Since $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, we have $j_1(\text{var}(\tau_i)) = j_2(\text{var}(\tau_i))$ for all i . By Lemma 2.2.23, $\tau_i^{\mathbf{M}, j_1} = \tau_i^{\mathbf{M}, j_2}$ for all i . Therefore, $\tau^{\mathbf{M}, j_1} = \tau^{\mathbf{M}, j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = p^{\mathbf{M}}(\tau^{\mathbf{M}, j_1}) = p^{\mathbf{M}}(\tau^{\mathbf{M}, j_2}) = \phi^{\mathbf{M}, j_2}.$$

Case 2: $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$.

Then $\text{free}(\phi) = \text{free}(\alpha) \cup \text{free}(\beta)$. Since $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, we have $j_1(\text{free}(\alpha)) = j_2(\text{free}(\alpha))$ and $j_1(\text{free}(\beta)) = j_2(\text{free}(\beta))$. By inductive hypothesis, $\alpha^{\mathbf{M}, j_1} = \alpha^{\mathbf{M}, j_2}$ and $\beta^{\mathbf{M}, j_1} = \beta^{\mathbf{M}, j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = 1 - \alpha^{\mathbf{M}, j_1}(1 - \beta^{\mathbf{M}, j_1}) = 1 - \alpha^{\mathbf{M}, j_2}(1 - \beta^{\mathbf{M}, j_2}) = \phi^{\mathbf{M}, j_2}.$$

Case 3: $\phi \doteq \forall x. \psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$.

Then $\text{free}(\phi) = \text{free}(\psi) \setminus \{x\}$. For any $a \in A$, the assignments $j_1 \frac{a}{x}$ and $j_2 \frac{a}{x}$ agree on $\text{free}(\psi)$ because:

- For $v \in \text{free}(\psi) \setminus \{x\}$, we have $v \in \text{free}(\phi)$, so $(j_1 \frac{a}{x})(v) = j_1(v) = j_2(v) = (j_2 \frac{a}{x})(v)$.
- For $v = x$, we have $(j_1 \frac{a}{x})(x) = a = (j_2 \frac{a}{x})(x)$.

Thus $j_1 \frac{a}{x}(\text{free}(\psi)) = j_2 \frac{a}{x}(\text{free}(\psi))$. By inductive hypothesis, $\psi^{\mathbf{M}, j_1} \frac{a}{x} = \psi^{\mathbf{M}, j_2} \frac{a}{x}$ for all $a \in A$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = \prod_{a \in A} \psi^{\mathbf{M}, j_1} \frac{a}{x} = \prod_{a \in A} \psi^{\mathbf{M}, j_2} \frac{a}{x} = \phi^{\mathbf{M}, j_2}.$$

This completes the proof by structural induction. \square

2.3 Syntactic Sugar

In this section, we introduce common logical connectives as syntactic abbreviations of formulas using implication and the false predicate \mathfrak{f} .

Definition 2.3.1 (logical connectives as syntactic sugar). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$. We define the following syntactic abbreviations:

$$\sim \alpha \stackrel{\text{def}}{=} \alpha \rightarrow \mathfrak{f}, \quad (2.18)$$

$$\alpha \wedge \beta \stackrel{\text{def}}{=} (\alpha \rightarrow (\beta \rightarrow \mathfrak{f})) \rightarrow \mathfrak{f}, \quad (2.19)$$

$$\alpha \vee \beta \stackrel{\text{def}}{=} (\alpha \rightarrow \mathfrak{f}) \rightarrow \beta, \quad (2.20)$$

$$\alpha \leftrightarrow \beta \stackrel{\text{def}}{=} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha). \quad (2.21)$$

For any $v \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, we also define:

$$\exists v. \psi \stackrel{\text{def}}{=} \sim (\forall v. \sim \psi). \quad (2.22)$$

The following theorems characterize the semantic behavior of these connectives.

Theorem 2.3.2 (negation evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\alpha \in \mathcal{B}(\mathcal{L})$, then

$$(\sim \alpha)^{\mathbf{M}, j} = 1 - \alpha^{\mathbf{M}, j}. \quad (2.23)$$

Proof. By definition, $\sim \alpha \doteq \alpha \rightarrow \mathfrak{f}$. By the definition of formula evaluation for implication and the fact that $\mathfrak{f}^{\mathbf{M}, j} = 0$,

$$(\sim \alpha)^{\mathbf{M}, j} = (\alpha \rightarrow \mathfrak{f})^{\mathbf{M}, j} = 1 - \alpha^{\mathbf{M}, j}(1 - \mathfrak{f}^{\mathbf{M}, j}) = 1 - \alpha^{\mathbf{M}, j}(1 - 0) = 1 - \alpha^{\mathbf{M}, j}. \quad (2.24)$$

□

Theorem 2.3.3 (conjunction evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$(\alpha \wedge \beta)^{\mathbf{M}, j} = \alpha^{\mathbf{M}, j} \cdot \beta^{\mathbf{M}, j}. \quad (2.25)$$

Proof. By definition, $\alpha \wedge \beta \doteq (\alpha \rightarrow (\beta \rightarrow \mathfrak{f})) \rightarrow \mathfrak{f}$. By the definition of formula evaluation for implication and the fact that $\mathfrak{f}^{\mathbf{M}, j} = 0$,

$$\begin{aligned} (\alpha \wedge \beta)^{\mathbf{M}, j} &= ((\alpha \rightarrow (\beta \rightarrow \mathfrak{f})) \rightarrow \mathfrak{f})^{\mathbf{M}, j} \\ &= 1 - (\alpha \rightarrow (\beta \rightarrow \mathfrak{f}))^{\mathbf{M}, j}(1 - 0) \\ &= 1 - (\alpha \rightarrow (\beta \rightarrow \mathfrak{f}))^{\mathbf{M}, j}. \end{aligned}$$

Now, expanding the inner implication:

$$\begin{aligned} (\alpha \rightarrow (\beta \rightarrow \mathfrak{f}))^{\mathbf{M}, j} &= 1 - \alpha^{\mathbf{M}, j}(1 - (\beta \rightarrow \mathfrak{f})^{\mathbf{M}, j}) \\ &= 1 - \alpha^{\mathbf{M}, j}(1 - (1 - \beta^{\mathbf{M}, j}(1 - 0))) \\ &= 1 - \alpha^{\mathbf{M}, j}(1 - (1 - \beta^{\mathbf{M}, j})) \\ &= 1 - \alpha^{\mathbf{M}, j} \cdot \beta^{\mathbf{M}, j}. \end{aligned}$$

Therefore,

$$(\alpha \wedge \beta)^{\mathbf{M}, j} = 1 - (1 - \alpha^{\mathbf{M}, j} \cdot \beta^{\mathbf{M}, j}) = \alpha^{\mathbf{M}, j} \cdot \beta^{\mathbf{M}, j}. \quad (2.26)$$

□

Theorem 2.3.4 (disjunction evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$(\alpha \vee \beta)^{\mathbf{M}, j} = 1 - (1 - \alpha^{\mathbf{M}, j})(1 - \beta^{\mathbf{M}, j}). \quad (2.27)$$

Proof. By definition, $\alpha \vee \beta \doteq (\alpha \rightarrow \mathfrak{f}) \rightarrow \beta$. By the definition of formula evaluation for implication:

$$\begin{aligned} (\alpha \vee \beta)^{\mathbf{M}, j} &= ((\alpha \rightarrow \mathfrak{f}) \rightarrow \beta)^{\mathbf{M}, j} \\ &= 1 - (\alpha \rightarrow \mathfrak{f})^{\mathbf{M}, j}(1 - \beta^{\mathbf{M}, j}) \\ &= 1 - (1 - \alpha^{\mathbf{M}, j})(1 - \beta^{\mathbf{M}, j}). \end{aligned}$$

□

Theorem 2.3.5 (biconditional evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$(\alpha \leftrightarrow \beta)^{\mathbf{M}, j} = 1 \text{ if and only if } \alpha^{\mathbf{M}, j} = \beta^{\mathbf{M}, j}. \quad (2.28)$$

Proof. By definition, $\alpha \leftrightarrow \beta \doteq (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. By Theorem 2.3.3,

$$(\alpha \leftrightarrow \beta)^{\mathbf{M}, j} = (\alpha \rightarrow \beta)^{\mathbf{M}, j} \cdot (\beta \rightarrow \alpha)^{\mathbf{M}, j}. \quad (2.29)$$

Expanding using the definition of implication:

$$(\alpha \leftrightarrow \beta)^{\mathbf{M}, j} = (1 - \alpha^{\mathbf{M}, j}(1 - \beta^{\mathbf{M}, j}))(1 - \beta^{\mathbf{M}, j}(1 - \alpha^{\mathbf{M}, j})).$$

This product equals 1 if and only if both factors equal 1, which happens if and only if:

- $\alpha^{\mathbf{M}, j}(1 - \beta^{\mathbf{M}, j}) = 0$, and
- $\beta^{\mathbf{M}, j}(1 - \alpha^{\mathbf{M}, j}) = 0$.

These conditions hold if and only if $\alpha^{\mathbf{M}, j} = \beta^{\mathbf{M}, j}$. □

Theorem 2.3.6 (existential quantifier evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $v \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, then

$$(\exists v. \psi)^{\mathbf{M}, j} = 1 - \prod_{a \in A} (1 - \psi^{\mathbf{M}, j} \frac{a}{v}). \quad (2.30)$$

Proof. By definition, $\exists v. \psi \doteq \sim (\forall v. \sim \psi)$. By Theorem 2.3.2,

$$(\exists v. \psi)^{\mathbf{M}, j} = 1 - (\forall v. \sim \psi)^{\mathbf{M}, j}. \quad (2.31)$$

By the definition of formula evaluation for universal quantification,

$$(\forall v. \sim \psi)^{\mathbf{M}, j} = \prod_{a \in A} (\sim \psi)^{\mathbf{M}, j} \frac{a}{v}. \quad (2.32)$$

By Theorem 2.3.2, for each $a \in A$,

$$(\sim \psi)^{\mathbf{M}, j} \frac{a}{v} = 1 - \psi^{\mathbf{M}, j} \frac{a}{v}. \quad (2.33)$$

Therefore,

$$(\exists v. \psi)^{\mathbf{M}, j} = 1 - \prod_{a \in A} (1 - \psi^{\mathbf{M}, j} \frac{a}{v}). \quad (2.34)$$

□

2.4 Substitutions

2.4.1 Syntax

Definition 2.4.1 (recursive: substitution in a term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$. Let $\sigma \in \mathcal{T}(\mathcal{L})$. We will define recursively $\sigma(\mathbf{x}/\boldsymbol{\tau})$. Let's establish that $\sigma(\emptyset/\emptyset) \stackrel{\text{def}}{=} \sigma$.

1. If $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$, then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \tau_i$.
2. If $\sigma \doteq v$ where $v \in \mathcal{V}$ and $v \neq x$, then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} v$.
3. If $\tau \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\boldsymbol{\eta}|}$, then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} f(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))$ where
$$\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \begin{cases} (\eta_1(\mathbf{x}/\boldsymbol{\tau}), \dots, \eta_k(\mathbf{x}/\boldsymbol{\tau})) & \text{for } k = |\boldsymbol{\eta}| > 0, \\ \emptyset & \text{for } |\boldsymbol{\eta}| = 0. \end{cases} \quad (2.35)$$

Definition 2.4.2 (recursive: admissible substitution in a formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$.

Let $\phi \in \mathcal{F}(\mathcal{L})$. We will define recursively $\sigma(\mathbf{x}/\boldsymbol{\tau})$. Let's establish that $\phi(\emptyset/\emptyset) \stackrel{\text{def}}{=} \phi$ and is **admissible**.

1. If $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} p(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))$ and is **admissible**.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(\mathbf{x}/\boldsymbol{\tau}), \beta(\mathbf{x}/\boldsymbol{\tau})$ are **admissible**, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \alpha(\mathbf{x}/\boldsymbol{\tau}) \rightarrow \beta(\mathbf{x}/\boldsymbol{\tau})$ and is **admissible**.
3. If $\phi \doteq \forall u.\psi$ where $u \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$:
Let \mathbf{v} be \mathbf{x} with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from \mathbf{x} .
 - (a) If $\psi(\mathbf{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \forall u.\psi(\mathbf{v}/\boldsymbol{\sigma})$ and is **admissible**.
 - (b) Otherwise $\phi(\mathbf{x}/\boldsymbol{\tau})$ is **not admissible**.

Definition 2.4.3 (recursive: admissible predicate substitution). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a non-negative integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $q \in \mathcal{P}_k$ and $\phi \in \mathcal{F}(\mathcal{L})$.

Let $\psi \in \mathcal{F}(\mathcal{L})$. We will define recursively $\psi(q(\mathbf{x})/\phi)$.

1. If $\psi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$.
 - (a) If $p = q$ and $\phi(\mathbf{x}/\boldsymbol{\eta})$ is **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \phi(\mathbf{x}/\boldsymbol{\eta})$ and is **admissible**.
 - (b) If $p \neq q$, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \psi$ and is **admissible**.
2. If $\psi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(q(\mathbf{x})/\phi), \beta(q(\mathbf{x})/\phi)$ are **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \alpha(q(\mathbf{x})/\phi) \rightarrow \beta(q(\mathbf{x})/\phi)$ and is **admissible**.

3. If $\psi \doteq \forall u.\gamma$ where $u \in \mathcal{V}$ and $\gamma \in \mathcal{B}(\mathcal{L})$:

- (a) If q occurs in γ , $u \notin \text{free}(\phi) \setminus \mathbf{x}$, $\gamma(q(\mathbf{x})/\phi)$ is **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \forall u.\gamma(q(\mathbf{x})/\phi)$ and is **admissible**.
- (b) If q does not occur in γ , then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \forall u.\gamma$ and is **admissible**.
- (c) Otherwise $\psi(q(\mathbf{x})/\phi)$ is **not admissible**.

2.4.2 Semantics

Lemma 2.4.4. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary mathematical domain. Let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$ be an assignment.

Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$.

Then for any term σ , we have

$$\sigma \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}}} = \sigma(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j}. \quad (2.36)$$

Proof. We proceed by structural induction on the term σ .

Case 1: $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$.

Then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \doteq \tau_i$ and

$$\sigma \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}}} = (j \frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}})(x_i) = \tau_i^{\mathbf{M}, j} = \sigma(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j}.$$

Case 2: $\sigma \doteq v$ where $v \in \mathcal{V}$ and $v \neq x$.

Then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \doteq v$ and

$$\sigma \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}}} = (j \frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}})(v) = j(v) = v^{\mathbf{M}, j} = (\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j}.$$

Case 3: $\sigma \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\boldsymbol{\eta}|}$.

Then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \doteq f(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))$. By inductive hypothesis, $\boldsymbol{\eta} \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{(\mathbf{M}, j)}}{\mathbf{x}}} = \boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j}$. Thus

$$\begin{aligned} \sigma \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}}} &= f^{\mathbf{M}} \left(\boldsymbol{\eta} \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}}} \right) = f^{\mathbf{M}} \left(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j} \right) \\ &= f(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M}, j} = \sigma(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j}. \end{aligned}$$

This completes the proof by structural induction. \square

Theorem 2.4.5. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let $\phi \in \mathcal{F}(\mathcal{L})$. Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$. If $\phi(\mathbf{x}/\boldsymbol{\tau})$ is admissible, then for any \mathcal{L} -structure \mathbf{M} with domain A and for any assignment of variables $j : \mathcal{V} \rightarrow A$ we have

$$(\mathbf{M}, j \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}}}) \models \phi \iff \mathbf{M}, j \models \phi(\mathbf{x}/\boldsymbol{\tau}). \quad (2.37)$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathcal{L})$.

Case 1: $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$.

By the definition of substitution in well formed formula, $\phi(\mathbf{x}/\boldsymbol{\tau}) \doteq p(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))$. By

$$\text{Lemma 2.4.4, we have } \boldsymbol{\eta}^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}} = \boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M},j}. \text{ Therefore,}$$

$$\phi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}} = p^{\mathbf{M}} \left(\boldsymbol{\eta}^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}} \right) = p^{\mathbf{M}}(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M},j}) = p(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = \phi(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 2: $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$.

By the definition of substitution in well formed formula, $\phi(\mathbf{x}/\boldsymbol{\tau}) \doteq \alpha(\mathbf{x}/\boldsymbol{\tau}) \rightarrow \beta(\mathbf{x}/\boldsymbol{\tau})$.

By inductive hypothesis,

$$\begin{aligned} (\mathbf{M}, j \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}}) \models \alpha &\iff \mathbf{M}, j \models \alpha(\mathbf{x}/\boldsymbol{\tau}), \\ (\mathbf{M}, j \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}}) \models \beta &\iff \mathbf{M}, j \models \beta(\mathbf{x}/\boldsymbol{\tau}). \end{aligned}$$

By the definition of formula evaluation,

$$\begin{aligned} \phi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}} &= 1 - \alpha^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}} \left(1 - \beta^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}} \right) = 1 - (\alpha(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j} (1 - (\beta(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j}) = 1 \\ &= (\alpha(\mathbf{x}/\boldsymbol{\tau}) \rightarrow \beta(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = (\phi(\mathbf{x}/\boldsymbol{\tau}))^{\mathbf{M},j} \end{aligned}$$

Case 3: $\phi \doteq \forall u. \psi$ where $u \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$:

Let \mathbf{v} be \mathbf{x} with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from \mathbf{x} .

Since $\phi(\mathbf{x}/\boldsymbol{\tau})$ is admissible, $\psi(\mathbf{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, and we have $\phi(\mathbf{x}/\boldsymbol{\tau}) \doteq \forall u. \psi(\mathbf{v}/\boldsymbol{\sigma})$. Thus

$$\phi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\mathbf{x}} = \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j} a}{\mathbf{x} u} \quad (2.38)$$

By choice of \mathbf{v} , $j \frac{\boldsymbol{\tau}^{\mathbf{M},j} a}{\mathbf{x} u}$ and $j \frac{\boldsymbol{\sigma}^{\mathbf{M},j} a}{\mathbf{v} u}$ agrees on $\text{free}(\psi)$ therefore by Theorem 2.2.24 we have

$$\begin{aligned} \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j} a}{\mathbf{x} u} &= \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\boldsymbol{\sigma}^{\mathbf{M},j} a}{\mathbf{v} u} \\ &= \prod_{a \in A} \psi^{\mathbf{M},j} \frac{a \boldsymbol{\sigma}^{\mathbf{M},j}}{u \mathbf{v}} \quad (\text{since } u \text{ not in } \mathbf{v}) \\ &= \prod_{a \in A} \psi(\mathbf{v}/\boldsymbol{\sigma})^{\mathbf{M},j} \frac{a}{u} \quad (\text{by inductive hypothesis}) \\ &= (\forall u. \psi(\mathbf{v}/\boldsymbol{\sigma}))^{\mathbf{M},j} = \phi(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M},j}. \end{aligned}$$

This completes the proof by structural induction. \square

Lemma 2.4.6. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $q \in \mathcal{P}$. Let $\phi \in \mathcal{F}(\mathcal{L})$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $j_0 : \text{free}(\phi) \setminus \mathbf{x} \rightarrow A$, where A is a domain of an \mathcal{L} -structure M for which

$$M, j \models q(\mathbf{x}) \text{ iff } M, j \models \phi \quad (2.39)$$

for any j that is an extention of j_0 . If $\psi \in \mathcal{F}(\mathcal{L})$ and $\psi(q(\mathbf{x})/\phi)$ is admissible, then

$$M, j \models \psi \text{ iff } M, j \models \psi(q(\mathbf{x})/\phi) \quad (2.40)$$

for any j that is an extention of j_0 .

Proof. We proceed by structural induction on $\psi \in \mathcal{F}(\mathcal{L})$ following Definition 2.4.3.

Take any $j : \mathcal{V} \rightarrow A$ which is an extension of j_0 .

Case 1: $\psi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$.

Subcase 1a: $p = q$ and $\phi(\mathbf{x}/\boldsymbol{\eta})$ is admissible.

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \phi(\mathbf{x}/\boldsymbol{\eta})$. By the definition of formula evaluation,

$$\psi^{M,j} = q^M(\boldsymbol{\eta}^{M,j}) = q(\mathbf{x})^{M,j} \frac{\boldsymbol{\eta}^{M,j}}{\mathbf{x}}.$$

Since $j \frac{\boldsymbol{\eta}^{M,j}}{\mathbf{x}}$ is an extension of j_0 , by (2.39), we have

$$q(\mathbf{x})^{M,j} \frac{\boldsymbol{\eta}^{M,j}}{\mathbf{x}} = \phi^{M,j} \frac{\boldsymbol{\eta}^{M,j}}{\mathbf{x}}.$$

By Theorem 2.4.5, since $\phi(\mathbf{x}/\boldsymbol{\eta})$ is admissible,

$$\phi^{M,j} \frac{\boldsymbol{\eta}^{M,j}}{\mathbf{x}} = \phi(\mathbf{x}/\boldsymbol{\eta})^{M,j}.$$

Therefore, $\psi^{M,j} = \phi(\mathbf{x}/\boldsymbol{\eta})^{M,j} = \psi(q(\mathbf{x})/\phi)^{M,j}$.

Subcase 1b: $p \neq q$.

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \psi$, so $\psi^{M,j} = \psi(q(\mathbf{x})/\phi)^{M,j}$ holds trivially.

Case 2: $\psi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(q(\mathbf{x})/\phi), \beta(q(\mathbf{x})/\phi)$ are admissible.

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \alpha(q(\mathbf{x})/\phi) \rightarrow \beta(q(\mathbf{x})/\phi)$. By inductive hypothesis, $\alpha^{M,j} = \alpha(q(\mathbf{x})/\phi)^{M,j}$ and $\beta^{M,j} = \beta(q(\mathbf{x})/\phi)^{M,j}$. By the definition of formula evaluation,

$$\begin{aligned} \psi^{M,j} &= 1 - \alpha^{M,j}(1 - \beta^{M,j}) \\ &= 1 - \alpha(q(\mathbf{x})/\phi)^{M,j}(1 - \beta(q(\mathbf{x})/\phi)^{M,j}) \\ &= (\alpha(q(\mathbf{x})/\phi) \rightarrow \beta(q(\mathbf{x})/\phi))^{M,j} \\ &= \psi(q(\mathbf{x})/\phi)^{M,j}. \end{aligned}$$

Case 3: $\psi \doteq \forall u. \gamma$ where $u \in \mathcal{V}$ and $\gamma \in \mathcal{B}(\mathcal{L})$.

Subcase 3a: q does not occur in γ .

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \forall u. \gamma = \psi$, so $\psi^{M,j} = \psi(q(\mathbf{x})/\phi)^{M,j}$ holds trivially.

Subcase 3b: q occurs in γ . Since $\psi(q(\mathbf{x})/\phi)$ is admissible, $u \notin \text{free}(\phi) \setminus \mathbf{x}$ and $\gamma(q(\mathbf{x})/\phi)$ is admissible.

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \forall u. \gamma(q(\mathbf{x})/\phi)$.

For any $a \in A$, let $j' = j \frac{a}{u}$. Since $u \notin \text{free}(\phi) \setminus \mathbf{x}$, the assignment j' is also an extension of j_0 . By inductive hypothesis, $\gamma^{M,j'} = \gamma(q(\mathbf{x})/\phi)^{M,j'}$.

This holds for all $a \in A$, therefore by the definition of formula evaluation,

$$\begin{aligned}\psi^{M,j} &= \prod_{a \in A} \gamma^{M,j} \frac{a}{u} \\ &= \prod_{a \in A} \gamma(q(\mathbf{x})/\phi)^{M,j} \frac{a}{u} \\ &= (\forall u. \gamma(q(\mathbf{x})/\phi))^{M,j} \\ &= \psi(q(\mathbf{x})/\phi)^{M,j}.\end{aligned}$$

This completes the proof by structural induction. \square

Theorem 2.4.7. *Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $q \in \mathcal{P}$. Let $\phi \in \mathcal{F}(\mathcal{L})$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. If $\models \psi$ and $\psi(q(\mathbf{x})/\phi)$ is admissible, then $\models \psi(q(\mathbf{x})/\phi)$.*

Proof. Let's introduce a new predicate symbol q' such that $q' \notin \mathcal{P}$. Let $\mathcal{L}' = (\mathcal{P} \cup \{q'\}, \mathcal{F}, \mathfrak{a}')$ such that \mathfrak{a}' is an extension of \mathfrak{a} such that $\mathfrak{a}'(q') = \mathfrak{a}(q)$. Let $k = \mathfrak{a}(q)$. Let ψ' be ψ with each q symbol replaced by q' . Since $\models \psi$, also $\models \psi'$. Take any non-empty domain A . Take any \mathcal{L} -structure \mathbf{M} with domain A . Take any $j_0 : \text{free}(\phi) \setminus \mathbf{x} \rightarrow A$. We will define an \mathcal{L}' structure.

$$\mathbf{M}'_{j_0}(s) = \begin{cases} \mathbf{M}'(s) & \text{for } s \in \mathcal{P} \cup \mathcal{F}, \\ (A^k \ni \mathbf{a} \mapsto \phi^{M,l} \frac{\mathbf{a}}{\mathbf{x}}) & \text{for } s = q'. \end{cases} \quad (2.41)$$

Where $l : \mathcal{V} \rightarrow A$ is some fixed extension of j_0 (note that choice of l does not matter for the above definition). Take any $j : \mathcal{V} \rightarrow A$ which is an extension of j_0 . Since $\models \psi'$, we have $M'_{j_0}, j \models \psi'$. Note that

$$q'(\mathbf{x})^{M'_{j_0}, j} = (q')^{M'_{j_0}}(\mathbf{x}^{M'_{j_0}, j}) = \phi^{M'_{j_0}, l} \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}}. \quad (2.42)$$

Since $l \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}}$ and $j \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}}$ agrees on $\text{free}(\phi)$, by Theorem 2.2.24 we have

$$\phi^{M'_{j_0}, l} \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}} = \phi^{M'_{j_0}, j} \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}} = \phi^{M'_{j_0}, j}. \quad (2.43)$$

We have shown that

$$q'(\mathbf{x})^{M'_{j_0}, j} = \phi^{M'_{j_0}, j}. \quad (2.44)$$

Thus by Theorem 2.4.7, we have $M'_{j_0}, j \models \psi'(q(\mathbf{x})/\phi)$. Note that $\psi'(q'(\mathbf{x})/\phi) \doteq \psi(q(\mathbf{x})/\phi)$ and therefore $\psi'(q'(\mathbf{x})/\phi) \in \mathcal{F}(\mathcal{L})$. Then by construction of M'_{j_0} , we have $M, j \models \psi'(q'(\mathbf{x})/\phi)$, and consequently

$$M, j \models \psi(q(\mathbf{x})/\phi). \quad (2.45)$$

Since $j_0 : \text{free}(\phi) \setminus \mathbf{x} \rightarrow A$ was arbitrarily chosen, (2.45) holds for arbitrary $j : \mathcal{V} \rightarrow A$. This completes the proof. \square

2.5 Preliminaries for Rules

Theorem 2.5.1 (modus ponens). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$ be an assignment. Let $\alpha, \beta \in \mathcal{F}(\mathcal{L})$. If $\mathbf{M}, j \models \alpha$ and $\mathbf{M}, j \models \alpha \rightarrow \beta$, then $\mathbf{M}, j \models \beta$.

Proof. We have $\mathbf{M}, j \models \alpha$, which means $\alpha^{\mathbf{M}, j} = 1$.

Since $\Gamma \models \alpha \rightarrow \beta$, we have $\mathbf{M}, j \models \alpha \rightarrow \beta$, which means $(\alpha \rightarrow \beta)^{\mathbf{M}, j} = 1$.

By the definition of formula evaluation for implication,

$$(\alpha \rightarrow \beta)^{\mathbf{M}, j} = 1 - \alpha^{\mathbf{M}, j}(1 - \beta^{\mathbf{M}, j}) = 1. \quad (2.46)$$

Since $\alpha^{\mathbf{M}, j} = 1$, we have

$$1 - 1 \cdot (1 - \beta^{\mathbf{M}, j}) = 1, \quad (2.47)$$

which simplifies to $1 - (1 - \beta^{\mathbf{M}, j}) = 1$, hence $\beta^{\mathbf{M}, j} = 1$.

Therefore, $\mathbf{M}, j \models \beta$. \square

Theorem 2.5.2 (generalization). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let \mathbf{M} be an \mathcal{L} -structure. Let $v \in \mathcal{V}$ and $\phi \in \mathcal{B}(\mathcal{L})$. If $\mathbf{M} \models \phi$, then $\mathbf{M} \models \forall v. \phi$.

Proof. Take any non-empty domain A and any \mathcal{L} -structure \mathbf{M} with domain A , such that $\mathbf{M} \models \phi$. Take any $j : \mathcal{V} \rightarrow A$. We need to show that $\mathbf{M}, j \models \forall v. \phi$.

By the definition of formula evaluation for universal quantification,

$$(\forall v. \phi)^{\mathbf{M}, j} = \prod_{a \in A} \phi^{\mathbf{M}, j} \frac{a}{v}. \quad (2.48)$$

Since $\mathbf{M} \models \phi$, we have $\mathbf{M}, j \frac{a}{v} \models \phi$, which means $\phi^{\mathbf{M}, j} \frac{a}{v} = 1$.

This holds for all $a \in A$, therefore

$$(\forall v. \phi)^{\mathbf{M}, j} = \prod_{a \in A} \phi^{\mathbf{M}, j} \frac{a}{v} = \prod_{a \in A} 1 = 1. \quad (2.49)$$

Hence $\mathbf{M}, j \models \forall v. \phi$. And since $j : \mathcal{V} \rightarrow A$ was arbitrary, then $\mathbf{M} \models \forall v. \phi$. \square

Definition 2.5.3. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let \mathbf{M} be an \mathcal{L} -structure with domain A . For any $z \in \mathcal{P} \cup \mathcal{F}$,

$$\begin{aligned} \mathcal{M}_z(\mathbf{M}) &\stackrel{\text{def}}{=} \{ \mathbf{M}' : \mathbf{M}' \text{ is } \mathcal{L}\text{-structure with domain } A \\ &\quad \text{and } \mathbf{M}'(s) = \mathbf{M}(s) \text{ for all } s \in (\mathcal{P} \cup \mathcal{F}) \setminus \{z\} \}. \end{aligned} \quad (2.50)$$

Theorem 2.5.4. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathcal{P}_k$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $\phi, \psi \in \mathcal{F}(\mathcal{L})$ such that q does not occur in both ϕ and ψ .

Let A be an non-empty mathematical domain and $j : \mathcal{V} \rightarrow A$.

Let $\Gamma \subset \mathcal{F}(\mathcal{L})$ such that for any $\gamma \in \Gamma$ and for any \mathcal{L} -structure \mathbf{K} with domain A

$$\mathbf{K}, j \models \gamma \implies \mathbf{K}', j \models \gamma \text{ for all } \mathbf{K}' \in \mathcal{M}_q(\mathbf{K}). \quad (2.51)$$

If for any \mathcal{L} -structure \mathbf{M} with domain A such that $\mathbf{M}, j \models \Gamma$, we have $\mathbf{M}, j \models (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi$, then for any \mathcal{L} -structure \mathbf{M} with domain A such that $\mathbf{M}, j \models \Gamma$, we have $\mathbf{M}, j \models \psi$.

Proof. Assume that for any \mathcal{L} -structure \mathbf{M} with domain A such that $\mathbf{M}, j \models \Gamma$, we have

$$\mathbf{M}, j \models (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi. \quad (2.52)$$

Take an arbitrary \mathcal{L} -structure \mathbf{M} with domain A such that $\mathbf{M}, j \models \Gamma$. We will modify the structure \mathbf{M} to create a new structure \mathbf{M}' that differs only in the interpretation of q . Namely,

$$(q)^{\mathbf{M}'}(\mathbf{a}) = \phi^{\mathbf{M},j} \frac{\mathbf{a}}{\mathbf{x}}. \quad (2.53)$$

For all other predicates and functions, \mathbf{M}' agrees with \mathbf{M} . Note that $\mathbf{M}' \in \mathcal{M}_q(M)$. With the structure \mathbf{M}' , we have

$$q(\mathbf{x})^{\mathbf{M}',j} = (q)^{\mathbf{M}'}(\mathbf{x}^{\mathbf{M}',j}) = \phi^{\mathbf{M},j} \frac{\mathbf{x}^{\mathbf{M}',j}}{\mathbf{x}}. \quad (2.54)$$

Since by Definition 2.2.13 (term evaluation), we have $\mathbf{x}^{\mathbf{M}',j} = \mathbf{x}^{\mathbf{M},j}$, therefore

$$\phi^{\mathbf{M},j} \frac{\mathbf{x}^{\mathbf{M}',j}}{\mathbf{x}} = \phi^{\mathbf{M},j} \frac{\mathbf{x}^{\mathbf{M},j}}{\mathbf{x}} = \phi^{M,j}. \quad (2.55)$$

Hence $q(\mathbf{x})^{\mathbf{M}',j} = \phi^{\mathbf{M},j}$. Since q does not occur in ϕ , we have $\phi^{\mathbf{M},j} = \phi^{\mathbf{M}',j}$ and consequently $q(\mathbf{x})^{\mathbf{M}',j} = \phi^{\mathbf{M}',j}$. By Theorem 2.3.5 this is the same as $(q(\mathbf{x}) \leftrightarrow \phi)^{\mathbf{M}',j} = 1$.

Since $\mathbf{M}' \in \mathcal{M}_q(M)$ and by (2.51), we have $\mathbf{M}', j \models \Gamma$ and thus by assumption (2.52), we have $\mathbf{M}', j \models (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi$, which is the same as

$$((q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi)^{\mathbf{M}',j} = 1. \quad (2.56)$$

This means

$$1 - (q(\mathbf{x}) \leftrightarrow \phi)^{\mathbf{M}',j} (1 - \psi^{\mathbf{M}',j}) = 1. \quad (2.57)$$

But since $(q(\mathbf{x}) \leftrightarrow \phi)^{\mathbf{M}',j} = 1$, we have

$$1 - 1 \cdot (1 - \psi^{\mathbf{M}',j}) = 1, \quad (2.58)$$

which implies $\psi^{\mathbf{M}',j} = 1$. And since q does not occur in ψ , $\psi^{\mathbf{M},j} = 1$, so $\mathbf{M}, j \models \psi$. \square

Next, we will define a relation $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$. The intuition behind is that $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ holds when ψ_2 is created from ψ_1 by replacing some or none ϕ_1 which occurs in ψ_1 by ϕ_2 .

Definition 2.5.5 (replace relation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $\phi_1, \phi_2 \in \mathcal{F}(\mathcal{L})$.

Let $\psi_1, \psi_2 \in \mathcal{F}(\mathcal{L})$. We will define recursively $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$.

1. If $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$, then $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ holds.
2. If $\psi_1 \doteq p(\boldsymbol{\eta})$ and $\psi_2 \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$, then $\text{rep}_{\psi_1, \phi_2}(\psi_1, \phi_2)$ holds.
3. If $\psi_1 \doteq \alpha_1 \rightarrow \beta_1$ and $\psi_2 \doteq \alpha_2 \rightarrow \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathcal{L})$, then $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ iff $\text{rep}_{\phi_1, \phi_2}(\alpha_1, \alpha_2)$ and $\text{rep}_{\phi_1, \phi_2}(\beta_1, \beta_2)$.
4. If $\psi_1 \doteq \forall u. \gamma_1$ and $\psi_2 \doteq \forall u. \gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathcal{L})$, then $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ iff $\text{rep}_{\phi_1, \phi_2}(\gamma_1, \gamma_2)$.

5. Otherwise $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ does not hold.

Theorem 2.5.6. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{F}(\mathcal{L})$ such that $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$. Let \mathbf{M} be an \mathcal{L} -structure with domain A . If $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathcal{V} \rightarrow A$, then $M, j \models \psi_1 \leftrightarrow \psi_2$ for any $j : \mathcal{V} \rightarrow A$.

Proof. We proceed by structural induction on the relation $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$.

Assume that $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathcal{V} \rightarrow A$. By Theorem 2.3.5, this means $\phi_1^{\mathbf{M}, j} = \phi_2^{\mathbf{M}, j}$ for any $j : \mathcal{V} \rightarrow A$.

Take an arbitrary $j : \mathcal{V} \rightarrow A$. We will show that $\psi_1^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$.

Case 1: $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$.

Then $\psi_1^{\mathbf{M}, j} = \phi_1^{\mathbf{M}, j} = \phi_2^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$.

By Theorem 2.3.5, $\psi_1^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$ implies $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 2: $\psi_1 \doteq p(\eta)$ and $\psi_2 \doteq p(\eta)$ where $\eta \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\eta|}$.

Then $\psi_1 \doteq \psi_2$, so trivially $\psi_1^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$ and $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 3: $\psi_1 \doteq \alpha_1 \rightarrow \beta_1$ and $\psi_2 \doteq \alpha_2 \rightarrow \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathcal{L})$, and $\text{rep}_{\phi_1, \phi_2}(\alpha_1, \alpha_2)$ and $\text{rep}_{\phi_1, \phi_2}(\beta_1, \beta_2)$.

By the inductive hypothesis, since $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathcal{V} \rightarrow A$, we have:

- $M, j \models \alpha_1 \leftrightarrow \alpha_2$ for any $j : \mathcal{V} \rightarrow A$, which means $\alpha_1^{\mathbf{M}, j} = \alpha_2^{\mathbf{M}, j}$ for any $j : \mathcal{V} \rightarrow A$.
- $M, j \models \beta_1 \leftrightarrow \beta_2$ for any $j : \mathcal{V} \rightarrow A$, which means $\beta_1^{\mathbf{M}, j} = \beta_2^{\mathbf{M}, j}$ for any $j : \mathcal{V} \rightarrow A$.

By the definition of formula evaluation for implication:

$$\begin{aligned}\psi_1^{\mathbf{M}, j} &= (\alpha_1 \rightarrow \beta_1)^{\mathbf{M}, j} = 1 - \alpha_1^{\mathbf{M}, j}(1 - \beta_1^{\mathbf{M}, j}) \\ &= 1 - \alpha_2^{\mathbf{M}, j}(1 - \beta_2^{\mathbf{M}, j}) = (\alpha_2 \rightarrow \beta_2)^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}.\end{aligned}$$

Therefore, $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 4: $\psi_1 \doteq \forall u. \gamma_1$ and $\psi_2 \doteq \forall u. \gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathcal{L})$, and $\text{rep}_{\phi_1, \phi_2}(\gamma_1, \gamma_2)$.

By the inductive hypothesis, since $M, j' \models \phi_1 \leftrightarrow \phi_2$ for any $j' : \mathcal{V} \rightarrow A$, we have $M, j' \models \gamma_1 \leftrightarrow \gamma_2$ for any $j' : \mathcal{V} \rightarrow A$.

In particular, for any $a \in A$, taking $j' = j \frac{a}{u}$, we have $M, j \frac{a}{u} \models \gamma_1 \leftrightarrow \gamma_2$, which means

$$\gamma_1^{\mathbf{M}, j} \frac{a}{u} = \gamma_2^{\mathbf{M}, j} \frac{a}{u}.$$

By the definition of formula evaluation for universal quantification:

$$\begin{aligned}\psi_1^{\mathbf{M}, j} &= (\forall u. \gamma_1)^{\mathbf{M}, j} = \prod_{a \in A} \gamma_1^{\mathbf{M}, j} \frac{a}{u} \\ &= \prod_{a \in A} \gamma_2^{\mathbf{M}, j} \frac{a}{u} = (\forall u. \gamma_2)^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}.\end{aligned}$$

Therefore, $M, j \models \psi_1 \leftrightarrow \psi_2$.

Since this holds for arbitrary $j : \mathcal{V} \rightarrow A$, we conclude that $M, j \models \psi_1 \leftrightarrow \psi_2$ for any $j : \mathcal{V} \rightarrow A$.

This completes the proof by structural induction. \square