

Foundations of Computational Logic

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Chapter 1

Introduction

1.1 Acknowledgements

This document presents the logical foundations for a formal system based on first-order logic. The net of definitions and theorems presented here were designed by a human author - Michał Stanisław Wójcik. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed, verified and corrected by Michał Stanisław Wójcik who takes the full responsibility for the content of this work.

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Chapter 2

First Order Logic

2.1 Preliminaries

We will use \mathbb{N} to denote a set of all nonnegative integers.

Definition 2.1.1. Let S be an arbitrary set.

1. $S^0 \stackrel{\text{def}}{=} \{\emptyset\}.$
2. $S^k \stackrel{\text{def}}{=} \underbrace{S \times \cdots \times S}_{k}.$
3. $S^* \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^k.$

Note that \emptyset is treated as empty sequence.

Remark 2.1.2. For a sequence $s \in S^k$, we denote the i -th element of the sequence by s_i where $i \in \{1, \dots, k\}$. let also $|s| \stackrel{\text{def}}{=} k$ denote a length of s .

Note that according to the above convention $|\emptyset| = 0$.

Definition 2.1.3. Let S be an arbitrary set.

1. $S^{\nabla 0} \stackrel{\text{def}}{=} \{\emptyset\}.$
2. $S^{\nabla k} \stackrel{\text{def}}{=} \{s \in S^k : s_i \neq s_j \text{ for } i \neq j\}.$
3. $S^{\nabla} \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^{\nabla k}.$

Sometimes if not disambiguous, we will treat $s \in S^{\nabla}$ notationally as set, so we will write $s_i \in s$, or for a set a , $a \setminus s$, $a \subset s$ etc.

2.2 First Order Logic Formula

2.2.1 Syntax

Let \mathfrak{L} denote a set of all logical symbols “(”, “)”, “ \rightarrow ”, “ \cdot ”, “,”, “ \forall ”. Let \mathcal{V} be an infinite countable set of variable symbols such that $\mathcal{V} \cap \mathfrak{L} = \emptyset$. And let f be a special predicate symbol.

Definition 2.2.1. Let \mathcal{P}, \mathcal{F} be arbitrary sets of symbols such that

$$\mathfrak{f} \in \mathcal{P}, \quad (2.1)$$

$$\mathcal{P} \cap \mathcal{F} = \emptyset, \quad (2.2)$$

$$\mathcal{P} \cap \mathcal{V} = \emptyset, \quad (2.3)$$

$$\mathcal{F} \cap \mathcal{V} = \emptyset, \quad (2.4)$$

$$\mathcal{P} \cap \mathcal{L} = \emptyset, \quad (2.5)$$

$$\mathcal{F} \cap \mathcal{L} = \emptyset, \quad (2.6)$$

and let $\mathbf{a} : \mathcal{P} \cup \mathcal{F} \rightarrow \mathbb{N}$ be an arity function such that $\mathbf{a}(\mathfrak{f}) = 0$ then $\mathcal{L} \stackrel{\text{def}}{=} (\mathcal{P}, \mathcal{F}, \mathbf{a})$ is a signature of a language.

Moreover, we call \mathcal{P} a set of predicate symbols and \mathcal{F} a set of function symbols, and we call \mathfrak{f} a false predicate.

If $\mathcal{L}' = (\mathcal{P}', \mathcal{F}', \mathbf{a}')$ is a language signature and $\mathcal{P} \subset \mathcal{P}'$ and $\mathcal{F} \subset \mathcal{F}'$ and $\mathbf{a} \subset \mathbf{a}'$, we will slightly abuse notation by writing $\mathcal{L} \subset \mathcal{L}'$ meaning that \mathcal{L}' is an extention of \mathcal{L} or \mathcal{L} is a restriction of \mathcal{L}' .

To write first order logic formulas for a given language signature \mathcal{L} , we will use symbols from

$$\mathcal{S}(\mathcal{L}) \stackrel{\text{def}}{=} \mathcal{L} \cup \mathcal{V} \cup \mathcal{P} \cup \mathcal{F}. \quad (2.7)$$

All formulas will be then a subset of $\mathcal{S}(\mathcal{L})^*$. Elements of $\mathcal{S}(\mathcal{L})^*$ will be called strings.

Note 2.2.2 (syntactic equality). We use the notation \doteq to denote equality of strings. We will sometimes overload it when cast of some object to string is obvious.

When we stack symbols together from left to right, we mean that we concatenate them in the same order. E.g let p be a predicate symbol and x, y be variable symbols, then $p(x, y)$ is just a sequence of 6 symbols: “ p ”, “($,$ “ x ”, “ $,$ ”, “ y ”, “ $)$ ”.

For convenience we will use an abbreviated style of writing, in which for any symbol $s \in \mathcal{P} \cup \mathcal{F}$ and for any seqence of strings $\mathbf{x} \in \mathcal{S}(\mathcal{L})^*$

$$s(\mathbf{x}) \doteq \begin{cases} s(x_1, \dots, x_k) & \text{for } |\mathbf{x}| > 0, \\ s & \text{for } |\mathbf{x}| = 0. \end{cases} \quad (2.8)$$

Definition 2.2.3 (arity). Let $\mathcal{L} \stackrel{\text{def}}{=} (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language.

$$\mathcal{P}_k \stackrel{\text{def}}{=} \{p \in \mathcal{P} : \mathbf{a}(p) = k\}, \quad (2.9)$$

$$\mathcal{F}_k \stackrel{\text{def}}{=} \{f \in \mathcal{F} : \mathbf{a}(f) = k\}. \quad (2.10)$$

Definition 2.2.4 (atomic term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language.

1. For any $v \in \mathcal{V}$, v is an **atomic term**.
2. For any $c \in \mathcal{F}_0$, c is an **atomic term**.

Definition 2.2.5 (recursive: term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. We will define recursively a set of all terms $\mathcal{T}(\mathcal{L})$.

Let $s \in \mathcal{S}(\mathcal{L})^*$.

1. If $s \doteq v$ where $v \in \mathcal{V}$, then $s \in \mathcal{T}(\mathcal{L})$.

2. If $s \doteq f(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\tau|}$, then $s \in \mathcal{T}(\mathcal{L})$.

3. Otherwise $s \notin \mathcal{T}(\mathcal{L})$.

Definition 2.2.6 (atomic formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define a set of all atomic formulas $\mathcal{A}(\mathcal{L})$.

$$\mathcal{A}(\mathcal{L}) \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \{p(\tau) : p \in \mathcal{P}_k, \tau \in \mathcal{T}(\mathcal{L})^k\}. \quad (2.11)$$

Definition 2.2.7 (recursive: base formula, well-formed formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all base formulas $\mathcal{B}(\mathcal{L})$ and a set of all well-formed formulas $\mathcal{F}(\mathcal{L})$.

Let $s \in \mathcal{S}(\mathcal{L})^*$.

1. If $s \in \mathcal{A}(\mathcal{L})$, then $s \in \mathcal{B}(\mathcal{L})$.
2. If $s \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.
3. If $s \doteq (\phi)$ where $\phi \in \mathcal{F}(\mathcal{L})$, then $s \in \mathcal{B}(\mathcal{L})$.
4. If $s \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{F}(\mathcal{L})$.
5. If $s \doteq \forall v. \phi$ where $\phi \in \mathcal{B}(\mathcal{L})$, then $s \in \mathcal{B}(\mathcal{L})$.
6. Otherwise $s \notin \mathcal{F}(\mathcal{L})$ and $s \notin \mathcal{B}(\mathcal{L})$.

Definition 2.2.8 (recursive: variable in term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $\text{var}(\tau)$ of all variables in a term τ .

Let $v \in \mathcal{V}$ and $\tau \in \mathcal{T}(\mathcal{L})$.

1. If $\tau \doteq v$ then $v \in \text{var}(\tau)$.
2. If $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$ and $v \in \text{var}(\sigma) \stackrel{\text{def}}{=} \bigcup_{k=1}^{|\sigma|} \text{var}(\sigma_k)$, then $v \in \text{var}(\tau)$.
3. Otherwise $v \notin \text{var}(\tau)$.

Definition 2.2.9 (recursive: variable in formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $\text{var}(\phi)$ of all variables in a formula ϕ .

Let $v \in \mathcal{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$ and $v \in \text{var}(\tau)$, then $v \in \text{var}(\phi)$.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and $v \in \text{var}(\alpha)$ or $v \in \text{var}(\beta)$, then $v \in \text{var}(\phi)$.
3. If $\phi \doteq \forall x. \psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, and either $v = x$ or $v \in \text{var}(\psi)$, then $v \in \text{var}(\phi)$.
4. Otherwise $v \notin \text{var}(\phi)$.

Definition 2.2.10 (recursive: free variable in formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set $\text{free}(\phi)$ of all free variables in a formula ϕ .

Let $v \in \mathcal{V}$ and $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \in \mathcal{A}(\mathcal{L})$ and $v \in \text{var}(\phi)$, then $v \in \text{free}(\phi)$.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, and $v \in \text{free}(\alpha)$ or $v \in \text{free}(\beta)$, then $v \in \text{free}(\phi)$.
3. If $\phi \doteq \forall x.\psi$ where $\psi \in \mathcal{B}(\mathcal{L})$, $x \in \mathcal{V}$ and $x \neq v$, and $v \in \text{free}(\psi)$, then $v \in \text{free}(\phi)$.
4. Otherwise $v \notin \text{free}(\phi)$.

2.2.2 Semantics

Definition 2.2.11 (\mathcal{L} -structure). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and $A \neq \emptyset$ be an arbitrary mathematical domain, then a mapping \mathbf{M} such that

$$\mathbf{M} : \mathcal{P} \cup \mathcal{F} \rightarrow \bigcup_{k=0}^{\infty} \{0, 1\}^{(A^k)} \cup \bigcup_{k=0}^{\infty} A^{(A^k)}, \quad (2.12)$$

where

$$\begin{aligned} \mathbf{M}(\mathbf{f}) &= (\emptyset \mapsto 0), \\ \mathbf{M}(\mathcal{P}_k) &\subset \{0, 1\}^{(A^k)} \text{ for } k \in \mathbb{N}, \\ \mathbf{M}(\mathcal{F}_k) &\subset A^{(A^k)} \text{ for } k \in \mathbb{N}. \end{aligned}$$

is called an \mathcal{L} -structure with domain A .

For convenience we will denote values of \mathbf{M} as $p^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(p)$ for $p \in \mathcal{P}$ and $f^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(f)$ for $f \in \mathcal{F}$. Note that $p^{\mathbf{M}} : A^k \rightarrow \{0, 1\}$ and $f^{\mathbf{M}} : A^k \rightarrow A$.

For a \mathcal{L} -structure \mathbf{M} , we will use a standard notation of restriction of a function $\mathbf{M}|_S$, where $S \subset \mathcal{P} \cup \mathcal{F}$. Also if we have a signatures of language $\mathcal{L}, \mathcal{L}'$ for which $\mathcal{L} \subset \mathcal{L}'$ (i.e. \mathcal{L}' is an extenion of \mathcal{L}), for a \mathcal{L}' -signature \mathbf{M}' , $\mathbf{M}'|_{\mathcal{L}} \stackrel{\text{def}}{=} \mathbf{M}'|_{\mathcal{P} \cup \mathcal{F}}$. Note that $\mathbf{M}'|_{\mathcal{L}}$ is an \mathcal{L} -structure.

Definition 2.2.12 (assignment). Let A be an arbitrary mathematical domain. Then $j : \mathcal{V} \rightarrow A$ is called a variables assignment in A .

Definition 2.2.13 (recursive: term evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure and $j : \mathcal{V} \rightarrow A$.

Let $\tau \in \mathcal{T}(\mathcal{L})$.

1. If $\tau \doteq v$ where $v \in \mathcal{V}$, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} j(v)$.
2. If $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$, then $\tau^{\mathbf{M}, j} \stackrel{\text{def}}{=} f^{\mathbf{M}}(\sigma^{\mathbf{M}, j})$ where

$$\sigma^{\mathbf{M}, j} \stackrel{\text{def}}{=} \begin{cases} (\sigma_1^{\mathbf{M}, j}, \dots, \sigma_k^{\mathbf{M}, j}) & \text{for } k = |\sigma| > 0, \\ \emptyset & \text{for } |\sigma| = 0. \end{cases} \quad (2.13)$$

Definition 2.2.14. Let \mathcal{L} be a signature of a language. Let \mathbf{M} be a \mathcal{L} -structure with domain A and let $j : \mathcal{V} \rightarrow A$ be an assignment of variables. Then we define

$$j \frac{a}{x}(v) \stackrel{\text{def}}{=} \begin{cases} a & \text{if } v = x, \\ j(v) & \text{otherwise.} \end{cases} \quad (2.14)$$

Definition 2.2.15. Let \mathcal{L} be a signature of a language. Let \mathbf{M} be a \mathcal{L} -structure with domain A and let $j : \mathcal{V} \rightarrow A$ be an assignment of variables. Let k be a non-negative integer and let $\mathbf{x} \in V^{\nabla^k}$, and $\mathbf{a} \in A^k$.

Then we define

$$j_{\mathbf{x}}(\mathbf{a})(v) \stackrel{\text{def}}{=} \begin{cases} a_i & \text{if } v = x_i \text{ for some } i \in \{1, \dots, k\}, \\ j(v) & \text{otherwise.} \end{cases} \quad (2.15)$$

For the purposes of this work, we will define multiplication product of infinite number of 0s and/or 1s.

Definition 2.2.16. Let S be an arbitrary set and $a_s \in \{0, 1\}$ for any $s \in S$.

$$\prod_{s \in S} a_s = \begin{cases} 1 & \text{if } a_s = 1 \text{ for all } s \in S, \\ 0 & \text{otherwise.} \end{cases} \quad (2.16)$$

Definition 2.2.17 (recursive: formula evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$.

1. If $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$, then $\phi^{\mathbf{M}, j} \stackrel{\text{def}}{=} p^{\mathbf{M}}(\tau^{\mathbf{M}, j})$.
2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$\phi^{\mathbf{M}, j} \stackrel{\text{def}}{=} 1 - \alpha^{\mathbf{M}, j}(1 - \beta^{\mathbf{M}, j}). \quad (2.17)$$

3. If $\phi \doteq \forall x. \psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, then

$$\phi^{\mathbf{M}, j} \stackrel{\text{def}}{=} \prod_{a \in A} \psi^{\mathbf{M}, j} \frac{a}{x}. \quad (2.18)$$

Corollary 2.2.18. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$, then

$$(\phi \rightarrow \mathbf{f})^{\mathbf{M}, j} = 1 - \phi^{\mathbf{M}, j}. \quad (2.19)$$

Definition 2.2.19 (interpretation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$, then

$$\mathbf{M}, j \models \phi \iff \phi^{\mathbf{M}, j} = 1. \quad (2.20)$$

Definition 2.2.20 (model). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ and let \mathbf{M} be an \mathcal{L} -structure. Let $\phi \in \mathcal{F}(\mathcal{L})$. We define

$$\mathbf{M} \models \phi \quad (2.21)$$

iff for any assignment $j : \mathcal{V} \rightarrow A$, we have $M, j \models \phi$.

Definition 2.2.21 (semantic consequence). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $\Gamma \subset \mathcal{F}(\mathcal{L})$ and $\phi \in \mathcal{F}(\mathcal{L})$. We define that $\Gamma \models \phi$ iff for any \mathcal{L} -structure \mathbf{M} , if $\mathbf{M} \models \psi$ for all $\psi \in \Gamma$, then $\mathbf{M} \models \phi$.

Definition 2.2.22 (tautology). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $\phi \in \mathcal{F}(\mathcal{L})$. We define that $\models \phi$ iff $\emptyset \models \phi$.

Lemma 2.2.23. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j_1, j_2 : \mathcal{V} \rightarrow A$. Let $\tau \in \mathcal{T}(\mathcal{L})$. If $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, then

$$\tau^{\mathbf{M}, j_1} = \tau^{\mathbf{M}, j_2}. \quad (2.22)$$

Proof. We proceed by structural induction on the term τ .

Case 1: $\tau \doteq v$ where $v \in \mathcal{V}$.

Then $\text{var}(\tau) = \{v\}$. Since $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, we have $j_1(v) = j_2(v)$. By the definition of term evaluation,

$$\tau^{\mathbf{M}, j_1} = j_1(v) = j_2(v) = \tau^{\mathbf{M}, j_2}.$$

Case 2: $\tau \doteq f(\sigma)$ where $\sigma \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\sigma|}$.

Then $\text{var}(\tau) = \text{var}(\sigma) = \bigcup_{i=1}^{|\sigma|} \text{var}(\sigma_i)$. Since $j_1(\text{var}(\tau)) = j_2(\text{var}(\tau))$, we have $j_1(\text{var}(\sigma_i)) = j_2(\text{var}(\sigma_i))$ for all $i \in \{1, \dots, |\sigma|\}$. By inductive hypothesis, $\sigma_i^{\mathbf{M}, j_1} = \sigma_i^{\mathbf{M}, j_2}$ for all i . Therefore, $\sigma^{\mathbf{M}, j_1} = \sigma^{\mathbf{M}, j_2}$. By the definition of term evaluation,

$$\tau^{\mathbf{M}, j_1} = f^{\mathbf{M}}(\sigma^{\mathbf{M}, j_1}) = f^{\mathbf{M}}(\sigma^{\mathbf{M}, j_2}) = \tau^{\mathbf{M}, j_2}.$$

This completes the proof by structural induction. \square

Theorem 2.2.24. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j_1, j_2 : \mathcal{V} \rightarrow A$. Let $\phi \in \mathcal{F}(\mathcal{L})$. If $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, then

$$\phi^{\mathbf{M}, j_1} = \phi^{\mathbf{M}, j_2}. \quad (2.23)$$

Proof. We proceed by structural induction on the formula ϕ .

Case 1: $\phi \doteq p(\tau)$ where $\tau \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\tau|}$.

Then $\text{free}(\phi) = \text{var}(\phi) = \text{var}(\tau) = \bigcup_{i=1}^{|\tau|} \text{var}(\tau_i)$. Since $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, we have $j_1(\text{var}(\tau_i)) = j_2(\text{var}(\tau_i))$ for all i . By Lemma 2.2.23, $\tau_i^{\mathbf{M}, j_1} = \tau_i^{\mathbf{M}, j_2}$ for all i . Therefore, $\tau^{\mathbf{M}, j_1} = \tau^{\mathbf{M}, j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = p^{\mathbf{M}}(\tau^{\mathbf{M}, j_1}) = p^{\mathbf{M}}(\tau^{\mathbf{M}, j_2}) = \phi^{\mathbf{M}, j_2}.$$

Case 2: $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$.

Then $\text{free}(\phi) = \text{free}(\alpha) \cup \text{free}(\beta)$. Since $j_1(\text{free}(\phi)) = j_2(\text{free}(\phi))$, we have $j_1(\text{free}(\alpha)) = j_2(\text{free}(\alpha))$ and $j_1(\text{free}(\beta)) = j_2(\text{free}(\beta))$. By inductive hypothesis, $\alpha^{\mathbf{M}, j_1} = \alpha^{\mathbf{M}, j_2}$ and $\beta^{\mathbf{M}, j_1} = \beta^{\mathbf{M}, j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = 1 - \alpha^{\mathbf{M}, j_1}(1 - \beta^{\mathbf{M}, j_1}) = 1 - \alpha^{\mathbf{M}, j_2}(1 - \beta^{\mathbf{M}, j_2}) = \phi^{\mathbf{M}, j_2}.$$

Case 3: $\phi \doteq \forall x. \psi$ where $x \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$.

Then $\text{free}(\phi) = \text{free}(\psi) \setminus \{x\}$. For any $a \in A$, the assignments $j_1 \frac{a}{x}$ and $j_2 \frac{a}{x}$ agree on $\text{free}(\psi)$ because:

- For $v \in \text{free}(\psi) \setminus \{x\}$, we have $v \in \text{free}(\phi)$, so $(j_1 \frac{a}{x})(v) = j_1(v) = j_2(v) = (j_2 \frac{a}{x})(v)$.
- For $v = x$, we have $(j_1 \frac{a}{x})(x) = a = (j_2 \frac{a}{x})(x)$.

Thus $j_1 \frac{a}{x}(\text{free}(\psi)) = j_2 \frac{a}{x}(\text{free}(\psi))$. By inductive hypothesis, $\psi^{\mathbf{M}, j_1} \frac{a}{x} = \psi^{\mathbf{M}, j_2} \frac{a}{x}$ for all $a \in A$. By the definition of formula evaluation,

$$\phi^{\mathbf{M}, j_1} = \prod_{a \in A} \psi^{\mathbf{M}, j_1} \frac{a}{x} = \prod_{a \in A} \psi^{\mathbf{M}, j_2} \frac{a}{x} = \phi^{\mathbf{M}, j_2}.$$

This completes the proof by structural induction. \square

2.3 Syntactic Sugar

In this section, we introduce common logical connectives as syntactic abbreviations of formulas using implication and the false predicate \mathfrak{f} .

Definition 2.3.1 (logical connectives as syntactic sugar). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$. We define the following syntactic abbreviations:

$$\mathbf{t} \stackrel{\text{def}}{=} \mathfrak{f} \rightarrow \mathfrak{f} \quad (2.24)$$

$$\sim \alpha \stackrel{\text{def}}{=} \alpha \rightarrow \mathfrak{f}, \quad (2.25)$$

$$\alpha \wedge \beta \stackrel{\text{def}}{=} (\alpha \rightarrow (\beta \rightarrow \mathfrak{f})) \rightarrow \mathfrak{f}, \quad (2.26)$$

$$\alpha \vee \beta \stackrel{\text{def}}{=} (\alpha \rightarrow \mathfrak{f}) \rightarrow \beta, \quad (2.27)$$

$$\alpha \leftrightarrow \beta \stackrel{\text{def}}{=} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha). \quad (2.28)$$

For any $v \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, we also define:

$$\exists v. \psi \stackrel{\text{def}}{=} \sim (\forall v. \sim \psi). \quad (2.29)$$

The following theorems characterize the semantic behavior of these connectives.

Theorem 2.3.2 (negation evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\alpha \in \mathcal{B}(\mathcal{L})$, then

$$(\sim \alpha)^{\mathbf{M}, j} = 1 - \alpha^{\mathbf{M}, j}. \quad (2.30)$$

Proof. By definition, $\sim \alpha \doteq \alpha \rightarrow \mathfrak{f}$. By the definition of formula evaluation for implication and the fact that $\mathfrak{f}^{\mathbf{M}, j} = 0$,

$$(\sim \alpha)^{\mathbf{M}, j} = (\alpha \rightarrow \mathfrak{f})^{\mathbf{M}, j} = 1 - \alpha^{\mathbf{M}, j}(1 - \mathfrak{f}^{\mathbf{M}, j}) = 1 - \alpha^{\mathbf{M}, j}(1 - 0) = 1 - \alpha^{\mathbf{M}, j}. \quad (2.31)$$

\square

Theorem 2.3.3 (conjunction evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$(\alpha \wedge \beta)^{\mathbf{M}, j} = \alpha^{\mathbf{M}, j} \cdot \beta^{\mathbf{M}, j}. \quad (2.32)$$

Proof. By definition, $\alpha \wedge \beta \doteq (\alpha \rightarrow (\beta \rightarrow \mathbf{f})) \rightarrow \mathbf{f}$. By the definition of formula evaluation for implication and the fact that $\mathbf{f}^{\mathbf{M},j} = 0$,

$$\begin{aligned} (\alpha \wedge \beta)^{\mathbf{M},j} &= ((\alpha \rightarrow (\beta \rightarrow \mathbf{f})) \rightarrow \mathbf{f})^{\mathbf{M},j} \\ &= 1 - (\alpha \rightarrow (\beta \rightarrow \mathbf{f}))^{\mathbf{M},j}(1 - 0) \\ &= 1 - (\alpha \rightarrow (\beta \rightarrow \mathbf{f}))^{\mathbf{M},j}. \end{aligned}$$

Now, expanding the inner implication:

$$\begin{aligned} (\alpha \rightarrow (\beta \rightarrow \mathbf{f}))^{\mathbf{M},j} &= 1 - \alpha^{\mathbf{M},j}(1 - (\beta \rightarrow \mathbf{f})^{\mathbf{M},j}) \\ &= 1 - \alpha^{\mathbf{M},j}(1 - (1 - \beta^{\mathbf{M},j}(1 - 0))) \\ &= 1 - \alpha^{\mathbf{M},j}(1 - (1 - \beta^{\mathbf{M},j})) \\ &= 1 - \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}. \end{aligned}$$

Therefore,

$$(\alpha \wedge \beta)^{\mathbf{M},j} = 1 - (1 - \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}) = \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}. \quad (2.33)$$

□

Theorem 2.3.4 (disjunction evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$(\alpha \vee \beta)^{\mathbf{M},j} = 1 - (1 - \alpha^{\mathbf{M},j})(1 - \beta^{\mathbf{M},j}). \quad (2.34)$$

Proof. By definition, $\alpha \vee \beta \doteq (\alpha \rightarrow \mathbf{f}) \rightarrow \beta$. By the definition of formula evaluation for implication:

$$\begin{aligned} (\alpha \vee \beta)^{\mathbf{M},j} &= ((\alpha \rightarrow \mathbf{f}) \rightarrow \beta)^{\mathbf{M},j} \\ &= 1 - (\alpha \rightarrow \mathbf{f})^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) \\ &= 1 - (1 - \alpha^{\mathbf{M},j})(1 - \beta^{\mathbf{M},j}). \end{aligned}$$

□

Theorem 2.3.5 (biconditional evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$, then

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = 1 \text{ if and only if } \alpha^{\mathbf{M},j} = \beta^{\mathbf{M},j}. \quad (2.35)$$

Proof. By definition, $\alpha \leftrightarrow \beta \doteq (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. By Theorem 2.3.3,

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = (\alpha \rightarrow \beta)^{\mathbf{M},j} \cdot (\beta \rightarrow \alpha)^{\mathbf{M},j}. \quad (2.36)$$

Expanding using the definition of implication:

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = (1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}))(1 - \beta^{\mathbf{M},j}(1 - \alpha^{\mathbf{M},j})).$$

This product equals 1 if and only if both factors equal 1, which happens if and only if:

- $\alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) = 0$, and
- $\beta^{\mathbf{M},j}(1 - \alpha^{\mathbf{M},j}) = 0$.

These conditions hold if and only if $\alpha^{\mathbf{M},j} = \beta^{\mathbf{M},j}$. □

Theorem 2.3.6 (existential quantifier evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let $v \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$, then

$$(\exists v. \psi)^{\mathbf{M}, j} = 1 - \prod_{a \in A} (1 - \psi^{\mathbf{M}, j} \frac{a}{v}). \quad (2.37)$$

Proof. By definition, $\exists v. \psi \doteq \sim (\forall v. \sim \psi)$. By Theorem 2.3.2,

$$(\exists v. \psi)^{\mathbf{M}, j} = 1 - (\forall v. \sim \psi)^{\mathbf{M}, j}. \quad (2.38)$$

By the definition of formula evaluation for universal quantification,

$$(\forall v. \sim \psi)^{\mathbf{M}, j} = \prod_{a \in A} (\sim \psi)^{\mathbf{M}, j} \frac{a}{v}. \quad (2.39)$$

By Theorem 2.3.2, for each $a \in A$,

$$(\sim \psi)^{\mathbf{M}, j} \frac{a}{v} = 1 - \psi^{\mathbf{M}, j} \frac{a}{v}. \quad (2.40)$$

Therefore,

$$(\exists v. \psi)^{\mathbf{M}, j} = 1 - \prod_{a \in A} (1 - \psi^{\mathbf{M}, j} \frac{a}{v}). \quad (2.41)$$

□

Definition 2.3.7. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $\phi \in \mathcal{B}(\mathcal{L})$. We will define the following syntactic abbreviation

$$\forall \mathbf{x}. \phi \stackrel{\text{def}}{=} \begin{cases} \forall x_1 \dots \forall x_k. \phi & \text{for } k > 0, \\ \phi & \text{for } k = 0. \end{cases} \quad (2.42)$$

Theorem 2.3.8 (generalized quantifier evaluation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Let k be a non-negative integer, let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\phi \in \mathcal{B}(\mathcal{L})$, then

$$(\forall \mathbf{x}. \phi)^{\mathbf{M}, j} = \prod_{\mathbf{a} \in A^k} \phi^{\mathbf{M}, j} \frac{\mathbf{a}}{\mathbf{x}}. \quad (2.43)$$

Proof. We proceed by induction on k .

Base case: $k = 0$.

By definition, $\forall \mathbf{x}. \phi = \phi$ when $k = 0$.

Recall that, $A^0 = \{\emptyset\}$ and $\mathcal{V}^{\nabla k} = \{\emptyset\}$.

Therefore,

$$\prod_{\mathbf{a} \in A^0} \phi^{\mathbf{M}, j} \frac{\mathbf{a}}{\mathbf{x}} = \phi^{\mathbf{M}, j} \frac{\emptyset}{\emptyset} = \phi^{\mathbf{M}, j}. \quad (2.44)$$

Inductive step: Assume the result holds for k , we prove it for $k + 1$.

Let $\mathbf{x} = (x_1, \dots, x_k, x_{k+1}) \in \mathcal{V}^{\nabla k+1}$.

By definition,

$$\forall \mathbf{x}.\phi = \forall x_1. \dots \forall x_k. \forall x_{k+1}.\phi. \quad (2.45)$$

By the definition of formula evaluation for universal quantification,

$$(\forall x_1. \dots \forall x_k. \forall x_{k+1}.\phi)^{\mathbf{M},j} = \prod_{a_1 \in A} (\forall x_2. \dots \forall x_k. \forall x_{k+1}.\phi)^{\mathbf{M},j} \frac{a_1}{x_1}. \quad (2.46)$$

Let $\mathbf{x}' = (x_2, \dots, x_k, x_{k+1})$. By the induction hypothesis applied to \mathbf{x}' with assignment $j \frac{a_1}{x_1}$,

$$(\forall x_2. \dots \forall x_k. \forall x_{k+1}.\phi)^{\mathbf{M},j} \frac{a_1}{x_1} = \prod_{\mathbf{a}' \in A^k} \phi^{\mathbf{M},(j \frac{a_1}{x_1})} \frac{\mathbf{a}'}{\mathbf{x}'} \quad (2.47)$$

Since $x_1 \notin \mathbf{x}'$, for any $\mathbf{a}' = (a_2, \dots, a_{k+1}) \in A^k$, we have

$$(j \frac{a_1}{x_1}) \frac{\mathbf{a}'}{\mathbf{x}'} = j \frac{(a_1, a_2, \dots, a_{k+1})}{\mathbf{x}}. \quad (2.48)$$

Therefore,

$$(\forall \mathbf{x}.\phi)^{\mathbf{M},j} = \prod_{a_1 \in A} \prod_{\mathbf{a}' \in A^k} \phi^{\mathbf{M},j} \frac{(a_1, \mathbf{a}')}{\mathbf{x}} = \prod_{\mathbf{a} \in A^{k+1}} \phi^{\mathbf{M},j} \frac{\mathbf{a}}{\mathbf{x}}. \quad (2.49)$$

□

2.4 Substitutions

2.4.1 Syntax

Definition 2.4.1 (recursive: substitution in a term). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$. Let $\sigma \in \mathcal{T}(\mathcal{L})$. We will define recursively $\sigma(\mathbf{x}/\boldsymbol{\tau})$. Let's establish that $\sigma(\emptyset/\emptyset) \stackrel{\text{def}}{=} \sigma$.

1. If $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$, then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \tau_i$.
2. If $\sigma \doteq v$ where $v \in \mathcal{V}$ and $v \neq x$, then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} v$.
3. If $\tau \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\boldsymbol{\eta}|}$, then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} f(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))$ where

$$\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \begin{cases} (\eta_1(\mathbf{x}/\boldsymbol{\tau}), \dots, \eta_k(\mathbf{x}/\boldsymbol{\tau})) & \text{for } k = |\boldsymbol{\eta}| > 0, \\ \emptyset & \text{for } |\boldsymbol{\eta}| = 0. \end{cases} \quad (2.50)$$

Definition 2.4.2 (recursive: admissible substitution in a formula). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$.

Let $\phi \in \mathcal{F}(\mathcal{L})$. We will define recursively $\sigma(\mathbf{x}/\boldsymbol{\tau})$. Let's establish that $\phi(\emptyset/\emptyset) \stackrel{\text{def}}{=} \phi$ and is **admissible**.

1. If $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} p(\boldsymbol{\eta}(\mathbf{x}/\boldsymbol{\tau}))$ and is **admissible**.

2. If $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(\mathbf{x}/\boldsymbol{\tau}), \beta(\mathbf{x}/\boldsymbol{\tau})$ are **admissible**, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \alpha(\mathbf{x}/\boldsymbol{\tau}) \rightarrow \beta(\mathbf{x}/\boldsymbol{\tau})$ and is **admissible**.
3. If $\phi \doteq \forall u.\psi$ where $u \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$:
Let \mathbf{v} be \mathbf{x} with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from \mathbf{x} .
 - (a) If $\psi(\mathbf{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, then $\phi(\mathbf{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \forall u.\psi(\mathbf{v}/\boldsymbol{\sigma})$ and is **admissible**.
 - (b) Otherwise $\phi(\mathbf{x}/\boldsymbol{\tau})$ is **not admissible**.

Definition 2.4.3 (recursive: admissible predicate substitution). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $q \in \mathcal{P}_k$ and $\phi \in \mathcal{F}(\mathcal{L})$.

Let $\psi \in \mathcal{F}(\mathcal{L})$. We will define recursively $\psi(q(\mathbf{x})/\phi)$.

1. If $\psi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$.
 - (a) If $p = q$ and $\phi(\mathbf{x}/\boldsymbol{\eta})$ is **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \phi(\mathbf{x}/\boldsymbol{\eta})$ and is **admissible**.
 - (b) If $p \neq q$, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \psi$ and is **admissible**.
2. If $\psi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(q(\mathbf{x})/\phi), \beta(q(\mathbf{x})/\phi)$ are **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \alpha(q(\mathbf{x})/\phi) \rightarrow \beta(q(\mathbf{x})/\phi)$ and is **admissible**.
3. If $\psi \doteq \forall u.\gamma$ where $u \in \mathcal{V}$ and $\gamma \in \mathcal{B}(\mathcal{L})$:
 - (a) If q occurs in γ , $u \notin \text{free}(\phi) \setminus \mathbf{x}$, $\gamma(q(\mathbf{x})/\phi)$ is **admissible**, then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \forall u.\gamma(q(\mathbf{x})/\phi)$ and is **admissible**.
 - (b) If q does not occur in γ , then $\psi(q(\mathbf{x})/\phi) \stackrel{\text{def}}{=} \forall u.\gamma$ and is **admissible**.
 - (c) Otherwise $\psi(q(\mathbf{x})/\phi)$ is **not admissible**.

2.4.2 Semantics

Lemma 2.4.4. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain. Let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$ be an assignment.

Let k be a positive integer and let $\mathbf{x} \in \mathcal{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathcal{L})^k$.

Then for any term σ , we have

$$\sigma \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}}} = \sigma(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j}. \quad (2.51)$$

Proof. We proceed by structural induction on the term σ .

Case 1: $\sigma \doteq x_i$ for some $i \in \{1, \dots, k\}$.

Then $\sigma(\mathbf{x}/\boldsymbol{\tau}) \doteq \tau_i$ and

$$\sigma \stackrel{\mathbf{M}, j}{\frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}}} = (j \frac{\boldsymbol{\tau}^{\mathbf{M}, j}}{\mathbf{x}})(x_i) = \tau_i^{\mathbf{M}, j} = \sigma(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M}, j}.$$

Case 2: $\sigma \doteq v$ where $v \in \mathcal{V}$ and $v \neq x$.

Then $\sigma(x/\tau) \doteq v$ and

$$\sigma^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x} = (j \frac{\tau^{\mathbf{M},j}}{x})(v) = j(v) = v^{\mathbf{M},j} = (x/\tau)^{\mathbf{M},j}.$$

Case 3: $\sigma \doteq f(\eta)$ where $\eta \in \mathcal{T}(\mathcal{L})^*$ and $f \in \mathcal{F}_{|\eta|}$.

Then $\sigma(x/\tau) \doteq f(\eta(x/\tau))$. By inductive hypothesis, $\eta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x} = \eta(x/\tau)^{\mathbf{M},j}$. Thus

$$\begin{aligned} \sigma^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x} &= f^{\mathbf{M}}\left(\eta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x}\right) = f^{\mathbf{M}}\left(\eta(x/\tau)^{\mathbf{M},j}\right) \\ &= f(\eta(x/\tau))^{\mathbf{M},j} = \sigma(x/\tau)^{\mathbf{M},j}. \end{aligned}$$

This completes the proof by structural induction. \square

Theorem 2.4.5. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $\phi \in \mathcal{F}(\mathcal{L})$. Let k be a positive integer and let $x \in \mathcal{V}^{\nabla k}$ and $\tau \in \mathcal{T}(\mathcal{L})^k$. If $\phi(x/\tau)$ is admissible, then for any \mathcal{L} -structure \mathbf{M} with domain A and for any assignment of variables $j : \mathcal{V} \rightarrow A$ we have

$$(\mathbf{M}, j \frac{\tau^{\mathbf{M},j}}{x}) \models \phi \iff \mathbf{M}, j \models \phi(x/\tau). \quad (2.52)$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathcal{L})$.

Case 1: $\phi \doteq p(\eta)$ where $\eta \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\eta|}$.

By the definition of substitution in well formed formula, $\phi(x/\tau) \doteq p(\eta(x/\tau))$. By

Lemma 2.4.4, we have $\eta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x} = \eta(x/\tau)^{\mathbf{M},j}$. Therefore,

$$\phi^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x} = p^{\mathbf{M}}\left(\eta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x}\right) = p^{\mathbf{M}}(\eta(x/\tau)^{\mathbf{M},j}) = p(\eta(x/\tau))^{\mathbf{M},j} = \phi(x/\tau)^{\mathbf{M},j}.$$

Case 2: $\phi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$.

By the definition of substitution in well formed formula, $\phi(x/\tau) \doteq \alpha(x/\tau) \rightarrow \beta(x/\tau)$. By inductive hypothesis,

$$\begin{aligned} (\mathbf{M}, j \frac{\tau^{\mathbf{M},j}}{x}) \models \alpha &\iff \mathbf{M}, j \models \alpha(x/\tau), \\ (\mathbf{M}, j \frac{\tau^{\mathbf{M},j}}{x}) \models \beta &\iff \mathbf{M}, j \models \beta(x/\tau). \end{aligned}$$

By the definition of formula evaluation,

$$\begin{aligned} \phi^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x} &= 1 - \alpha^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x} \left(1 - \beta^{\mathbf{M},j} \frac{\tau^{\mathbf{M},j}}{x}\right) = 1 - (\alpha(x/\tau))^{\mathbf{M},j} (1 - (\beta(x/\tau))^{\mathbf{M},j}) = 1 \\ &= (\alpha(x/\tau) \rightarrow \beta(x/\tau))^{\mathbf{M},j} = (\phi(x/\tau))^{\mathbf{M},j} \end{aligned}$$

Case 3: $\phi \doteq \forall u. \psi$ where $u \in \mathcal{V}$ and $\psi \in \mathcal{B}(\mathcal{L})$:

Let \mathbf{x} be x with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let σ be τ with τ_i removed for which x_i was removed from x .

Since $\phi(\mathbf{x}/\boldsymbol{\tau})$ is admissible, $\psi(\mathbf{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, and we have $\phi(\mathbf{x}/\boldsymbol{\tau}) \doteq \forall u.\psi(\mathbf{v}/\boldsymbol{\sigma})$. Thus

$$\phi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j} a}{\mathbf{x}} = \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j} a}{\mathbf{x}} \frac{a}{u} \quad (2.53)$$

By choice of \mathbf{v} , $j \frac{\boldsymbol{\tau}^{\mathbf{M},j} a}{\mathbf{x}} \frac{a}{u}$ and $j \frac{\boldsymbol{\sigma}^{\mathbf{M},j} a}{\mathbf{v}} \frac{a}{u}$ agrees on $\text{free}(\psi)$ therefore by Theorem 2.2.24 we have

$$\begin{aligned} \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\boldsymbol{\tau}^{\mathbf{M},j} a}{\mathbf{x}} \frac{a}{u} &= \prod_{a \in A} \psi^{\mathbf{M},j} \frac{\boldsymbol{\sigma}^{\mathbf{M},j} a}{\mathbf{v}} \frac{a}{u} \\ &= \prod_{a \in A} \psi^{\mathbf{M},j} \frac{a \boldsymbol{\sigma}^{\mathbf{M},j}}{u} \frac{a}{\mathbf{v}} \quad (\text{since } u \text{ not in } \mathbf{v}) \\ &= \prod_{a \in A} \psi(\mathbf{v}/\boldsymbol{\sigma})^{\mathbf{M},j} \frac{a}{u} \quad (\text{by inductive hypothesis}) \\ &= (\forall u.\psi(\mathbf{v}/\boldsymbol{\sigma}))^{\mathbf{M},j} = \phi(\mathbf{x}/\boldsymbol{\tau})^{\mathbf{M},j}. \end{aligned}$$

This completes the proof by structural induction. \square

Lemma 2.4.6. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $q \in \mathcal{P}$. Let $\phi \in \mathcal{F}(\mathcal{L})$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $j_0 : \text{free}(\phi) \setminus \mathbf{x} \rightarrow A$, where A is a domain of an \mathcal{L} -structure M for which

$$M, j \models q(\mathbf{x}) \text{ iff } M, j \models \phi \quad (2.54)$$

for any j that is an extention of j_0 . If $\psi \in \mathcal{F}(\mathcal{L})$ and $\psi(q(\mathbf{x})/\phi)$ is admissible, then

$$M, j \models \psi \text{ iff } M, j \models \psi(q(\mathbf{x})/\phi) \quad (2.55)$$

for any j that is an extention of j_0 .

Proof. We proceed by structural induction on $\psi \in \mathcal{F}(\mathcal{L})$ following Definition 2.4.3.

Take any $j : \mathcal{V} \rightarrow A$ which is an extention of j_0 .

Case 1: $\psi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$.

Subcase 1a: $p = q$ and $\phi(\mathbf{x}/\boldsymbol{\eta})$ is admissible.

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \phi(\mathbf{x}/\boldsymbol{\eta})$. By the definition of formula evaluation,

$$\psi^M = q^M(\boldsymbol{\eta}^M) = q(\mathbf{x})^M \frac{\boldsymbol{\eta}^M}{\mathbf{x}}.$$

Since $j \frac{\boldsymbol{\eta}^M}{\mathbf{x}}$ is an extention of j_0 , by (2.54), we have

$$q(\mathbf{x})^M \frac{\boldsymbol{\eta}^M}{\mathbf{x}} = \phi^M \frac{\boldsymbol{\eta}^M}{\mathbf{x}}.$$

By Theorem 2.4.5, since $\phi(\mathbf{x}/\boldsymbol{\eta})$ is admissible,

$$\phi^M \frac{\boldsymbol{\eta}^M}{\mathbf{x}} = \phi(\mathbf{x}/\boldsymbol{\eta})^M.$$

Therefore, $\psi^{M,j} = \phi(\mathbf{x}/\eta)^{M,j} = \psi(q(\mathbf{x})/\phi)^{M,j}$.

Subcase 1b: $p \neq q$.

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \psi$, so $\psi^{M,j} = \psi(q(\mathbf{x})/\phi)^{M,j}$ holds trivially.

Case 2: $\psi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$ and $\alpha(q(\mathbf{x})/\phi), \beta(q(\mathbf{x})/\phi)$ are admissible.

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \alpha(q(\mathbf{x})/\phi) \rightarrow \beta(q(\mathbf{x})/\phi)$. By inductive hypothesis, $\alpha^{M,j} = \alpha(q(\mathbf{x})/\phi)^{M,j}$ and $\beta^{M,j} = \beta(q(\mathbf{x})/\phi)^{M,j}$. By the definition of formula evaluation,

$$\begin{aligned}\psi^{M,j} &= 1 - \alpha^{M,j}(1 - \beta^{M,j}) \\ &= 1 - \alpha(q(\mathbf{x})/\phi)^{M,j}(1 - \beta(q(\mathbf{x})/\phi)^{M,j}) \\ &= (\alpha(q(\mathbf{x})/\phi) \rightarrow \beta(q(\mathbf{x})/\phi))^{M,j} \\ &= \psi(q(\mathbf{x})/\phi)^{M,j}.\end{aligned}$$

Case 3: $\psi \doteq \forall u. \gamma$ where $u \in \mathcal{V}$ and $\gamma \in \mathcal{B}(\mathcal{L})$.

Subcase 3a: q does not occur in γ .

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \forall u. \gamma = \psi$, so $\psi^{M,j} = \psi(q(\mathbf{x})/\phi)^{M,j}$ holds trivially.

Subcase 3b: q occurs in γ . Since $\psi(q(\mathbf{x})/\phi)$ is admissible, $u \notin \text{free}(\phi) \setminus \mathbf{x}$ and $\gamma(q(\mathbf{x})/\phi)$ is admissible.

Then by Definition 2.4.3, $\psi(q(\mathbf{x})/\phi) \doteq \forall u. \gamma(q(\mathbf{x})/\phi)$.

For any $a \in A$, let $j' = j \frac{a}{u}$. Since $u \notin \text{free}(\phi) \setminus \mathbf{x}$, the assignment j' is also an extension of j_0 . By inductive hypothesis, $\gamma^{M,j'} = \gamma(q(\mathbf{x})/\phi)^{M,j'}$.

This holds for all $a \in A$, therefore by the definition of formula evaluation,

$$\begin{aligned}\psi^{M,j} &= \prod_{a \in A} \gamma^{M,j} \frac{a}{u} \\ &= \prod_{a \in A} \gamma(q(\mathbf{x})/\phi)^{M,j} \frac{a}{u} \\ &= (\forall u. \gamma(q(\mathbf{x})/\phi))^{M,j} \\ &= \psi(q(\mathbf{x})/\phi)^{M,j}.\end{aligned}$$

This completes the proof by structural induction. \square

Theorem 2.4.7. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let $q \in \mathcal{P}$. Let $\phi \in \mathcal{F}(\mathcal{L})$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. If $\models \psi$ and $\psi(q(\mathbf{x})/\phi)$ is admissible, then $\models \psi(q(\mathbf{x})/\phi)$.

Proof. Let's introduce a new predicate symbol q' such that $q' \notin \mathcal{P}$. Let $\mathcal{L}' = (\mathcal{P} \cup \{q'\}, \mathcal{F}, \mathfrak{a}')$ such that \mathfrak{a}' is an extension of \mathfrak{a} such that $\mathfrak{a}'(q') = \mathfrak{a}(q)$. Let $k = \mathfrak{a}(q)$. Let ψ' be ψ with each q symbol replaced by q' . Since $\models \psi$, also $\models \psi'$. Take any non-empty domain A . Take any \mathcal{L} -structure \mathbf{M} with domain A . Take any $j_0 : \text{free}(\phi) \setminus \mathbf{x} \rightarrow A$. We will define an \mathcal{L}' structure.

$$\mathbf{M}'_{j_0}(s) = \begin{cases} \mathbf{M}'(s) & \text{for } s \in \mathcal{P} \cup \mathcal{F}, \\ (A^k \ni \mathbf{a} \mapsto \phi^{M,l} \frac{\mathbf{a}}{\mathbf{x}}) & \text{for } s = q'. \end{cases} \quad (2.56)$$

Where $l : \mathcal{V} \rightarrow A$ is some fixed extension of j_0 (note that choice of l does not matter for the above definition). Take any $j : \mathcal{V} \rightarrow A$ which is an extension of j_0 . Since $\models \psi'$,

we have $M'_{j_0}, j \models \psi'$. Note that

$$q'(\mathbf{x})^{M'_{j_0}, j} = (q')^{M'_{j_0}}(\mathbf{x}^{M'_{j_0}, j}) = \phi^{M'_{j_0}, l} \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}}. \quad (2.57)$$

Since $l \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}}$ and $j \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}}$ agrees on $\text{free}(\phi)$, by Theorem 2.2.24 we have

$$\phi^{M'_{j_0}, l} \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}} = \phi^{M'_{j_0}, j} \frac{\mathbf{x}^{M'_{j_0}, j}}{\mathbf{x}} = \phi^{M'_{j_0}, j}. \quad (2.58)$$

We have shown that

$$q'(\mathbf{x})^{M'_{j_0}, j} = \phi^{M'_{j_0}, j}. \quad (2.59)$$

Thus by Theorem 2.4.7, we have $M'_{j_0}, j \models \psi'(q(\mathbf{x})/\phi)$. Note that $\psi'(q'(\mathbf{x})/\phi) \doteq \psi(q(\mathbf{x})/\phi)$ and therefore $\psi'(q'(\mathbf{x})/\phi) \in \mathcal{F}(\mathcal{L})$. Then by construction of M'_{j_0} , we have $M, j \models \psi'(q'(\mathbf{x})/\phi)$, and consequently

$$M, j \models \psi(q(\mathbf{x})/\phi). \quad (2.60)$$

Since $j_0 : \text{free}(\phi) \setminus \mathbf{x} \rightarrow A$ was arbitrarily chosen, (2.60) holds for arbitrary $j : \mathcal{V} \rightarrow A$. This completes the proof. \square

2.5 Preliminaries for Rules

2.6 Generic Rules

Theorem 2.6.1 (modus ponens). *Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$ be an assignment. Let $\alpha, \beta \in \mathcal{B}(\mathcal{L})$. If $\mathbf{M}, j \models \alpha$ and $\mathbf{M}, j \models \alpha \rightarrow \beta$, then $\mathbf{M}, j \models \beta$.*

Proof. We have $\mathbf{M}, j \models \alpha$, which means $\alpha^{\mathbf{M}, j} = 1$.

Since $\Gamma \models \alpha \rightarrow \beta$, we have $\mathbf{M}, j \models \alpha \rightarrow \beta$, which means $(\alpha \rightarrow \beta)^{\mathbf{M}, j} = 1$.

By the definition of formula evaluation for implication,

$$(\alpha \rightarrow \beta)^{\mathbf{M}, j} = 1 - \alpha^{\mathbf{M}, j}(1 - \beta^{\mathbf{M}, j}) = 1. \quad (2.61)$$

Since $\alpha^{\mathbf{M}, j} = 1$, we have

$$1 - 1 \cdot (1 - \beta^{\mathbf{M}, j}) = 1, \quad (2.62)$$

which simplifies to $1 - (1 - \beta^{\mathbf{M}, j}) = 1$, hence $\beta^{\mathbf{M}, j} = 1$.

Therefore, $\mathbf{M}, j \models \beta$. \square

Theorem 2.6.2 (generalization). *Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and let \mathbf{M} be an \mathcal{L} -structure. Let $v \in \mathcal{V}$ and $\phi \in \mathcal{B}(\mathcal{L})$. If $\mathbf{M} \models \phi$, then $\mathbf{M} \models \forall v. \phi$.*

Proof. Take any non-empty domain A and any \mathcal{L} -structure \mathbf{M} with domain A , such that $\mathbf{M} \models \phi$. Take any $j : \mathcal{V} \rightarrow A$. We need to show that $\mathbf{M}, j \models \forall v. \phi$.

By the definition of formula evaluation for universal quantification,

$$(\forall v. \phi)^{\mathbf{M}, j} = \prod_{a \in A} \phi^{\mathbf{M}, j} \frac{a}{v}. \quad (2.63)$$

Since $\mathbf{M} \models \phi$, we have $\mathbf{M}, j \frac{a}{v} \models \phi$, which means $\phi^{\mathbf{M}, j} \frac{a}{v} = 1$.
This holds for all $a \in A$, therefore

$$(\forall v. \phi)^{\mathbf{M}, j} = \prod_{a \in A} \phi^{\mathbf{M}, j} \frac{a}{v} = \prod_{a \in A} 1 = 1. \quad (2.64)$$

Hence $\mathbf{M}, j \models \forall v. \phi$. And since $j : \mathcal{V} \rightarrow A$ was arbitrary, then $\mathbf{M} \models \forall v. \phi$. \square

Definition 2.6.3. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and let \mathbf{M} be an \mathcal{L} -structure with domain A . For any $S \subset \mathcal{P} \cup \mathcal{F}$,

$$\begin{aligned} \mathcal{M}_S(\mathbf{M}) &\stackrel{\text{def}}{=} \{ \mathbf{M}' : \mathbf{M}' \text{ is } \mathcal{L}\text{-structure with domain } A \\ &\quad \text{and } \mathbf{M}'(s) = \mathbf{M}(s) \text{ for all } s \in (\mathcal{P} \cup \mathcal{F}) \setminus S \}. \end{aligned} \quad (2.65)$$

For $S = \{s\}$, we will write \mathcal{M}_s instead of $\mathcal{M}_{\{s\}}$ as long as not ambiguous.

Definition 2.6.4. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ and $S \subset \mathcal{P} \cup \mathcal{F}$. Let $\phi \in \mathcal{F}(\mathcal{L})$. We define that ϕ is a **weak tautology with respect to S** iff for any \mathcal{L} -structure \mathbf{M} with domain A and any assignment $j : \mathcal{V} \rightarrow A$, there exists $\mathbf{M}' \in \mathcal{M}_S(\mathbf{M})$ such that $\mathbf{M}', j \models \phi$.

Lemma 2.6.5. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and let $S \subset \mathcal{P} \cup \mathcal{F}$. Let $\phi, \psi \in \mathcal{B}(\mathcal{L})$ be such that ϕ is a weak tautology with respect to S and no symbol from S occurs in ψ . Let \mathbf{M} be an \mathcal{L} -structure with domain A and let $j : \mathcal{V} \rightarrow A$. Then

$$\mathbf{M}', j \models \phi \rightarrow \psi \text{ for any } \mathbf{M}' \in \mathcal{M}_S(\mathbf{M}) \implies \mathbf{M}, j \models \psi. \quad (2.66)$$

Proof. Assume $\mathbf{M}', j \models \phi \rightarrow \psi$ for any $\mathbf{M}' \in \mathcal{M}_S(\mathbf{M})$.

Since ϕ is a weak tautology with respect to S , for the given structure \mathbf{M} with domain A and assignment $j : \mathcal{V} \rightarrow A$, there exists $\mathbf{M}'' \in \mathcal{M}_S(\mathbf{M})$ such that $\mathbf{M}'', j \models \phi$.

By assumption, since $\mathbf{M}'' \in \mathcal{M}_S(\mathbf{M})$, we have $\mathbf{M}'', j \models \phi \rightarrow \psi$.

Since $\mathbf{M}'', j \models \phi$ and $\mathbf{M}'', j \models \phi \rightarrow \psi$, by Theorem 2.6.1, we have $\mathbf{M}'', j \models \psi$.

Since no symbol from S occurs in ψ , and $\mathbf{M}'' \in \mathcal{M}_S(\mathbf{M})$ means \mathbf{M}'' agrees with \mathbf{M} on all symbols outside S , we have $\psi^{\mathbf{M}'', j} = \psi^{\mathbf{M}, j}$.

Therefore $\mathbf{M}, j \models \psi$. \square

Lemma 2.6.6. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and let $S \subset \mathcal{P} \cup \mathcal{F}$. Let $\phi, \psi \in \mathcal{B}(\mathcal{L})$ be such that ϕ is a weak tautology with respect to S and no symbol from S occurs in ψ . Let \mathbf{M} be an \mathcal{L} -structure. Then

$$\mathbf{M}' \models \phi \rightarrow \psi \text{ for any } \mathbf{M}' \in \mathcal{M}_S(\mathbf{M}) \implies \mathbf{M} \models \psi. \quad (2.67)$$

Proof. Assume $\mathbf{M}' \models \phi \rightarrow \psi$ for any $\mathbf{M}' \in \mathcal{M}_S(\mathbf{M})$.

Let A be the domain of \mathbf{M} . Take an arbitrary assignment $j : \mathcal{V} \rightarrow A$.

By assumption, for any $\mathbf{M}' \in \mathcal{M}_S(\mathbf{M})$, we have $\mathbf{M}' \models \phi \rightarrow \psi$. In particular, $\mathbf{M}', j \models \phi \rightarrow \psi$ for any $\mathbf{M}' \in \mathcal{M}_S(\mathbf{M})$.

By Lemma 2.6.5, since ϕ is a weak tautology with respect to S , no symbol from S occurs in ψ , and $\mathbf{M}', j \models \phi \rightarrow \psi$ for any $\mathbf{M}' \in \mathcal{M}_S(\mathbf{M})$, we have $\mathbf{M}, j \models \psi$.

Since j was arbitrary, we conclude $\mathbf{M} \models \psi$. \square

Lemma 2.6.7. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathcal{P}_k$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $\phi \in \mathcal{B}(\mathcal{L})$ such that q does not occur in ϕ . Then $q(\mathbf{x}) \leftrightarrow \phi$ is a weak tautology with respect to $\{q\}$.

Proof. Take any \mathcal{L} -structure \mathbf{M} with domain A and any assignment $j : \mathcal{V} \rightarrow A$.

We construct a new structure $\mathbf{M}' \in \mathcal{M}_q(\mathbf{M})$ as follows: for the predicate q , we define

$$q^{\mathbf{M}'}(\mathbf{a}) = \phi^{\mathbf{M},j} \frac{\mathbf{a}}{\mathbf{x}} \quad (2.68)$$

for any $\mathbf{a} \in A^k$.

For all other symbols $s \in (\mathcal{P} \cup \mathcal{F}) \setminus \{q\}$, we have $s^{\mathbf{M}'} = s^{\mathbf{M}}$.

Note that $\mathbf{M}' \in \mathcal{M}_q(\mathbf{M})$ by construction.

We need to show that $\mathbf{M}', j \models q(\mathbf{x}) \leftrightarrow \phi$.

We have

$$q(\mathbf{x})^{\mathbf{M}',j} = q^{\mathbf{M}'}(\mathbf{x}^{\mathbf{M}',j}) = \phi^{\mathbf{M},j} \frac{\mathbf{x}^{\mathbf{M}',j}}{\mathbf{x}}. \quad (2.69)$$

Since \mathbf{x} are distinct variables, we have $j \frac{\mathbf{x}^{\mathbf{M}',j}}{\mathbf{x}} = j$, thus

$$q(\mathbf{x})^{\mathbf{M}',j} = \phi^{\mathbf{M},j}. \quad (2.70)$$

Since q does not occur in ϕ , and \mathbf{M}' agrees with \mathbf{M} on all symbols except q , we have $\phi^{\mathbf{M},j} = \phi^{\mathbf{M}',j}$.

Therefore $q(\mathbf{x})^{\mathbf{M}',j} = \phi^{\mathbf{M}',j}$.

By Theorem 2.3.5, this means $\mathbf{M}', j \models q(\mathbf{x}) \leftrightarrow \phi$.

Therefore, $q(\mathbf{x}) \leftrightarrow \phi$ is a weak tautology with respect to $\{q\}$. \square

Lemma 2.6.8. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathcal{P}_k$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Let $\phi, \psi \in \mathcal{B}(\mathcal{L})$ such that q does not occur in both ϕ and ψ .

Let \mathbf{M} be \mathcal{L} -structure. Then

$$\mathbf{M}' \models (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi \text{ for any } \mathbf{M}' \in \mathcal{M}_q(\mathbf{M}) \implies \mathbf{M} \models \psi. \quad (2.71)$$

Proof. By Lemma 2.6.7, $q(\mathbf{x}) \leftrightarrow \phi$ is a weak tautology with respect to $\{q\}$.

Since q does not occur in ψ , by Lemma 2.6.6 with $S = \{q\}$ and ϕ replaced by $q(\mathbf{x}) \leftrightarrow \phi$, we have: if $\mathbf{M}' \models (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi$ for any $\mathbf{M}' \in \mathcal{M}_q(\mathbf{M})$, then $\mathbf{M} \models \psi$. \square

Lemma 2.6.9. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a non-negative integer, let $f \in \mathcal{F}_k$, $x \in \mathcal{V}$ and $\mathbf{u} \in \mathcal{V}^{\nabla k}$. Let $\phi \in \mathcal{B}(\mathcal{L})$ such that f does not occur in ϕ and $\phi(x/f(\mathbf{u}))$ is admissible. Then $(\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))$ is a weak tautology with respect to $\{f\}$.

Proof. Take any \mathcal{L} -structure \mathbf{M} with domain A and any assignment $j : \mathcal{V} \rightarrow A$. Let's fix some $a^* \in A$. We construct a new structure $\mathbf{M}' \in \mathcal{M}_f(\mathbf{M})$ as follows: for the function f , we define

$$f^{\mathbf{M}'}(\mathbf{a}) = \begin{cases} \text{chosen } a_0 & \text{if } \phi^{\mathbf{M},j} \frac{\mathbf{a} a_0}{\mathbf{u} x} = 1 \text{ for some } a_0 \in A, \\ a^* & \text{otherwise.} \end{cases} \quad (2.72)$$

for any $\mathbf{a} \in A^k$.

For all other symbols $s \in (\mathcal{P} \cup \mathcal{F}) \setminus \{f\}$, we have $s^{\mathbf{M}'} = s^{\mathbf{M}}$.

Note that $\mathbf{M}' \in \mathcal{M}_f(\mathbf{M})$ by construction.

We need to show that $\mathbf{M}', j \models (\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))$.

Case 1: $(\exists x.\phi)^{\mathbf{M}',j} = 0$.

Then trivially $\mathbf{M}', j \models (\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))$.

Case 2: $(\exists x.\phi)^{\mathbf{M}',j} = 1$.

Since f does not occur in ϕ , and \mathbf{M}' agrees with \mathbf{M} on all symbols except f , we have

$$\phi^{\mathbf{M}',j} \frac{a}{x} = \phi^{\mathbf{M},j} \frac{a}{x} \text{ for any } a \in A.$$

By Theorem 2.3.6, there exists some $a_0 \in A$ such that $\phi^{\mathbf{M}',j} \frac{a_0}{x} = 1$, which means

$$\phi^{\mathbf{M},j} \frac{\mathbf{u}^{\mathbf{M},j} a_0}{x} = \phi^{\mathbf{M},j} \frac{a_0}{x} = 1.$$

Now, we have $f^{\mathbf{M}'}(\mathbf{u}^{\mathbf{M}',j}) = a_0$ by construction (choosing the witness a_0).

$$\text{We have } \phi^{\mathbf{M},j} \frac{\mathbf{u}^{\mathbf{M},j} a_0}{x} = \phi^{\mathbf{M},j} \frac{a_0}{x}.$$

By Theorem 2.4.5, since $\phi(x/f(\mathbf{u}))$ is admissible,

$$\phi(x/f(\mathbf{u}))^{\mathbf{M}',j} = \phi^{\mathbf{M}',j} \frac{f(\mathbf{u})^{\mathbf{M}',j}}{x}. \quad (2.73)$$

Since $f(\mathbf{u})^{\mathbf{M}',j} = f^{\mathbf{M}'}(\mathbf{u}^{\mathbf{M}',j}) = a_0$, we have

$$\phi(x/f(\mathbf{u}))^{\mathbf{M}',j} = \phi^{\mathbf{M}',j} \frac{a_0}{x} = \phi^{\mathbf{M},j} \frac{a_0}{x} = 1. \quad (2.74)$$

Therefore $\mathbf{M}', j \models (\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))$.

Thus $(\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))$ is a weak tautology with respect to $\{f\}$. \square

Lemma 2.6.10. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $f \in \mathcal{F}_k$, $x \in \mathcal{V}$ and $\mathbf{u} \in \mathcal{V}^{\nabla k}$. Let $\phi, \psi \in \mathcal{B}(\mathcal{L})$ such that f does not occur in both ϕ and ψ , and $\phi(x/f(\mathbf{u}))$ is admissible. Let \mathbf{M} be an \mathcal{L} -structure. Then

$$\mathbf{M}' \models ((\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))) \rightarrow \psi \text{ for any } \mathbf{M}' \in \mathcal{M}_f(\mathbf{M}) \implies \mathbf{M} \models \psi. \quad (2.75)$$

Proof. By Lemma 2.6.9, $(\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))$ is a weak tautology with respect to $\{f\}$. Since f does not occur in ψ , by Lemma 2.6.6 with $S = \{f\}$ and ϕ replaced by $(\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))$, we have: if $\mathbf{M}' \models ((\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))) \rightarrow \psi$ for any $\mathbf{M}' \in \mathcal{M}_f(\mathbf{M})$, then $\mathbf{M} \models \psi$. \square

Next, we will define a relation $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$. The intuition behind is that $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ holds when ψ_2 is created from ψ_1 by replacing some or none ϕ_1 which occurs in ψ_1 by ϕ_2 .

Definition 2.6.11 (replace relation). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $\phi_1, \phi_2 \in \mathcal{F}(\mathcal{L})$.

Let $\psi_1, \psi_2 \in \mathcal{F}(\mathcal{L})$. We will define recursively $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$.

1. If $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$, then $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ holds.
2. If $\psi_1 \doteq p(\boldsymbol{\eta})$ and $\psi_2 \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$, then $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ holds.
3. If $\psi_1 \doteq \alpha_1 \rightarrow \beta_1$ and $\psi_2 \doteq \alpha_2 \rightarrow \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathcal{L})$, then $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ iff $\text{rep}_{\phi_1, \phi_2}(\alpha_1, \alpha_2)$ and $\text{rep}_{\phi_1, \phi_2}(\beta_1, \beta_2)$.

4. If $\psi_1 \doteq \forall u. \gamma_1$ and $\psi_2 \doteq \forall u. \gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathcal{L})$, then $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ iff $\text{rep}_{\phi_1, \phi_2}(\gamma_1, \gamma_2)$.
5. Otherwise $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ does not hold.

For a given $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$ we will say that an occurrence of ϕ_1 in ψ_1 is **active** if it's indeed replaced by ϕ_2 in ψ_2 .

Theorem 2.6.12. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{B}(\mathcal{L})$ such that $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$. For any \mathcal{L} -structure \mathbf{M} ,

$$\mathbf{M} \models \phi_1 \leftrightarrow \phi_2 \implies \mathbf{M} \models \psi_1 \leftrightarrow \psi_2. \quad (2.76)$$

Proof. We proceed by structural induction on the relation $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$.

Assume that $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathcal{V} \rightarrow A$. By Theorem 2.3.5, this means $\phi_1^{\mathbf{M}, j} = \phi_2^{\mathbf{M}, j}$ for any $j : \mathcal{V} \rightarrow A$.

Take an arbitrary $j : \mathcal{V} \rightarrow A$. We will show that $\psi_1^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$.

Case 1: $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$.

Then $\psi_1^{\mathbf{M}, j} = \phi_1^{\mathbf{M}, j} = \phi_2^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$.

By Theorem 2.3.5, $\psi_1^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$ implies $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 2: $\psi_1 \doteq p(\eta)$ and $\psi_2 \doteq p(\eta)$ where $\eta \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\eta|}$.

Then $\psi_1 \doteq \psi_2$, so trivially $\psi_1^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$ and $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 3: $\psi_1 \doteq \alpha_1 \rightarrow \beta_1$ and $\psi_2 \doteq \alpha_2 \rightarrow \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathcal{L})$, and $\text{rep}_{\phi_1, \phi_2}(\alpha_1, \alpha_2)$ and $\text{rep}_{\phi_1, \phi_2}(\beta_1, \beta_2)$.

By the inductive hypothesis, since $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathcal{V} \rightarrow A$, we have:

- $M, j \models \alpha_1 \leftrightarrow \alpha_2$ for any $j : \mathcal{V} \rightarrow A$, which means $\alpha_1^{\mathbf{M}, j} = \alpha_2^{\mathbf{M}, j}$ for any $j : \mathcal{V} \rightarrow A$.
- $M, j \models \beta_1 \leftrightarrow \beta_2$ for any $j : \mathcal{V} \rightarrow A$, which means $\beta_1^{\mathbf{M}, j} = \beta_2^{\mathbf{M}, j}$ for any $j : \mathcal{V} \rightarrow A$.

By the definition of formula evaluation for implication:

$$\begin{aligned} \psi_1^{\mathbf{M}, j} &= (\alpha_1 \rightarrow \beta_1)^{\mathbf{M}, j} = 1 - \alpha_1^{\mathbf{M}, j}(1 - \beta_1^{\mathbf{M}, j}) \\ &= 1 - \alpha_2^{\mathbf{M}, j}(1 - \beta_2^{\mathbf{M}, j}) = (\alpha_2 \rightarrow \beta_2)^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}. \end{aligned}$$

Therefore, $M, j \models \psi_1 \leftrightarrow \psi_2$.

Case 4: $\psi_1 \doteq \forall u. \gamma_1$ and $\psi_2 \doteq \forall u. \gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathcal{L})$, and $\text{rep}_{\phi_1, \phi_2}(\gamma_1, \gamma_2)$.

By the inductive hypothesis, since $M, j' \models \phi_1 \leftrightarrow \phi_2$ for any $j' : \mathcal{V} \rightarrow A$, we have $M, j' \models \gamma_1 \leftrightarrow \gamma_2$ for any $j' : \mathcal{V} \rightarrow A$.

In particular, for any $a \in A$, taking $j' = j \frac{a}{u}$, we have $M, j \frac{a}{u} \models \gamma_1 \leftrightarrow \gamma_2$, which means

$$\gamma_1 \frac{a}{u} = \gamma_2 \frac{a}{u}.$$

By the definition of formula evaluation for universal quantification:

$$\begin{aligned} \psi_1^{\mathbf{M}, j} &= (\forall u. \gamma_1)^{\mathbf{M}, j} = \prod_{a \in A} \gamma_1 \frac{a}{u} \\ &= \prod_{a \in A} \gamma_2 \frac{a}{u} = (\forall u. \gamma_2)^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}. \end{aligned}$$

Therefore, $M, j \models \psi_1 \leftrightarrow \psi_2$.

Since this holds for arbitrary $j : \mathcal{V} \rightarrow A$, we conclude that $M, j \models \psi_1 \leftrightarrow \psi_2$ for any $j : \mathcal{V} \rightarrow A$.

This completes the proof by structural induction. \square

Theorem 2.6.13. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{B}(\mathcal{L})$ such that $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$. Let k be a non-negative integer and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. No free variable of $\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)$ is under a quantifier which binds any active occurrence of ϕ_1 in ψ_1 . Let \mathbf{M} be an arbitrary \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$. Then

$$\mathbf{M}, j \models \forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2) \implies \mathbf{M}, j \models \psi_1 \leftrightarrow \psi_2. \quad (2.77)$$

Proof. Assume $\mathbf{M}, j \models \forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)$.

By Theorem 2.3.8, we have

$$(\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2))^{\mathbf{M}, j} = \prod_{\mathbf{a} \in A^k} (\phi_1 \leftrightarrow \phi_2)^{\mathbf{M}, j} \frac{\mathbf{a}}{\mathbf{x}} = 1. \quad (2.78)$$

Since the product equals 1, each factor must equal 1. Therefore, $(\phi_1 \leftrightarrow \phi_2)^{\mathbf{M}, j} \frac{\mathbf{a}}{\mathbf{x}} = 1$ for all $\mathbf{a} \in A^k$.

By Theorem 2.3.5, this means $\phi_1^{\mathbf{M}, j} \frac{\mathbf{a}}{\mathbf{x}} = \phi_2^{\mathbf{M}, j} \frac{\mathbf{a}}{\mathbf{x}}$ for all $\mathbf{a} \in A^k$.

We will now show $\psi_1^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$ by structural induction on the relation $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$.

Case 1: $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$.

Choosing $\mathbf{a} = \mathbf{x}^{\mathbf{M}, j}$, we have $j \frac{\mathbf{a}}{\mathbf{x}} = j$, thus:

$$\psi_1^{\mathbf{M}, j} = \phi_1^{\mathbf{M}, j} = \phi_1^{\mathbf{M}, j} \frac{\mathbf{x}^{\mathbf{M}, j}}{\mathbf{x}} = \phi_2^{\mathbf{M}, j} \frac{\mathbf{x}^{\mathbf{M}, j}}{\mathbf{x}} = \phi_2^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}. \quad (2.79)$$

Case 2: $\psi_1 \doteq p(\boldsymbol{\eta})$ and $\psi_2 \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathcal{L})^*$ and $p \in \mathcal{P}_{|\boldsymbol{\eta}|}$.

Then $\psi_1 \doteq \psi_2$, so trivially $\psi_1^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}$.

Case 3: $\psi_1 \doteq \alpha_1 \rightarrow \beta_1$ and $\psi_2 \doteq \alpha_2 \rightarrow \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathcal{L})$, and $\text{rep}_{\phi_1, \phi_2}(\alpha_1, \alpha_2)$ and $\text{rep}_{\phi_1, \phi_2}(\beta_1, \beta_2)$.

By the inductive hypothesis on the replace relation, we have $\alpha_1^{\mathbf{M}, j} = \alpha_2^{\mathbf{M}, j}$ and $\beta_1^{\mathbf{M}, j} = \beta_2^{\mathbf{M}, j}$.

By the definition of formula evaluation for implication:

$$\psi_1^{\mathbf{M}, j} = (\alpha_1 \rightarrow \beta_1)^{\mathbf{M}, j} = 1 - \alpha_1^{\mathbf{M}, j} (1 - \beta_1^{\mathbf{M}, j}) = 1 - \alpha_2^{\mathbf{M}, j} (1 - \beta_2^{\mathbf{M}, j}) = (\alpha_2 \rightarrow \beta_2)^{\mathbf{M}, j} = \psi_2^{\mathbf{M}, j}. \quad (2.80)$$

Case 4: $\psi_1 \doteq \forall u. \gamma_1$ and $\psi_2 \doteq \forall u. \gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathcal{L})$, and $\text{rep}_{\phi_1, \phi_2}(\gamma_1, \gamma_2)$.

Subcase 4a: There is no active occurrence of ϕ_1 in γ_1 . Then $\gamma_1 \doteq \gamma_2$ and $\mathbf{M}, j \models \psi_1 \leftrightarrow \psi_2$ trivially.

Subcase 4b: There is an active occurrence of ϕ_1 in γ_1 . Then by assumption, u is not a free variable of $\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)$.

Take any $a \in A$. Since $j \frac{a}{u}$ agrees on all free variables of $\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)$, by Theorem 2.2.24, we have $\mathbf{M}, j \frac{a}{u} \models \forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)$.

Now, by the inductive hypothesis applied to assignment $j \frac{a}{u}$, we have $\gamma_1^{\mathbf{M},j} \frac{a}{u} = \gamma_2^{\mathbf{M},j} \frac{a}{u}$. Since $a \in A$ was arbitrary chosen, we have

$$\gamma_1^{\mathbf{M},j} \frac{a}{u} = \gamma_2^{\mathbf{M},j} \frac{a}{u} \quad (2.81)$$

for all $a \in A$.

By the definition of formula evaluation for universal quantification:

$$\psi_1^{\mathbf{M},j} = (\forall u. \gamma_1)^{\mathbf{M},j} = \prod_{a \in A} \gamma_1^{\mathbf{M},j} \frac{a}{u} = \prod_{a \in A} \gamma_2^{\mathbf{M},j} \frac{a}{u} = (\forall u. \gamma_2)^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}. \quad (2.82)$$

Therefore, $\psi_1^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$, which by Theorem 2.3.5 means $\mathbf{M}, j \models \psi_1 \leftrightarrow \psi_2$. \square

Corollary 2.6.14. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{B}(\mathcal{L})$ such that $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$. Let k be a non-negative integer and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. No free variable of $\forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2)$ is under a quantifier which binds any active occurrence of ϕ_1 in ψ_1 . Let \mathbf{M} be an arbitrary \mathcal{L} -structure. Then

$$\mathbf{M} \models \forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2) \rightarrow \psi_1 \implies \mathbf{M} \models \forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2) \rightarrow \psi_2, \quad (2.83)$$

and

$$\mathbf{M} \models \forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2) \rightarrow \psi_2 \implies \mathbf{M} \models \forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2) \rightarrow \psi_1. \quad (2.84)$$

Proof. Take an arbitrary assignment $j : \mathcal{V} \rightarrow A$.

Case 1: $(\forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2))^{\mathbf{M},j} = 0$.

Then trivially

$$(\forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2) \rightarrow \psi_1)^{\mathbf{M},j} = (\forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2) \rightarrow \psi_2)^{\mathbf{M},j}. \quad (2.85)$$

Case 2: $(\forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2))^{\mathbf{M},j} = 1$.

Then $\mathbf{M}, j \models \forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2)$.

By Theorem 2.6.13, since $\mathbf{M}, j \models \forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2)$, we have $\mathbf{M}, j \models \psi_1 \leftrightarrow \psi_2$.

By Theorem 2.3.5, this means $\psi_1^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$. And thus

$$(\forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2) \rightarrow \psi_1)^{\mathbf{M},j} = (\forall \mathbf{x}. (\phi_1 \leftrightarrow \phi_2) \rightarrow \psi_2)^{\mathbf{M},j}. \quad (2.86)$$

This is enough to complete the proof. \square

Lemma 2.6.15. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let $c \in \mathcal{F}_0$, and $x \in \mathcal{V}$. Let $\psi \in \mathcal{F}(\mathcal{L})$ such that c does not occur in ψ . Let \mathbf{M} be \mathcal{L} -structure. Then

$$\mathbf{M}' \models \psi(x/c) \text{ for any } \mathbf{M}' \in \mathcal{M}_c(\mathbf{M}) \implies \mathbf{M} \models \psi. \quad (2.87)$$

Proof. Since $c \in \mathcal{F}_0$, $\psi(x/c)$ is always an admissible substitution.

Take any \mathcal{L} -structure \mathbf{M} with domain A . Assume that assumption of implication (2.87) holds. We need to show that $\mathbf{M} \models \psi$. Take any assignment $j : \mathcal{V} \rightarrow A$. We need to show that $\mathbf{M}, j \models \psi$.

We construct \mathbf{M}' such that:

$$c^{\mathbf{M}'} = j(x), \quad (2.88)$$

and for all other predicates and functions, \mathbf{M}' agrees with \mathbf{M} . Note that $\mathbf{M}' \in \mathcal{M}_c(\mathbf{M})$. By assumption of implication (2.87), we have $\mathbf{M}' \models \psi(x/c)$. In particular, $\mathbf{M}', j \models \psi(x/c)$.

By Theorem 2.4.5, since $\psi(x/c)$ is an admissible substitution, we have

$$\mathbf{M}', j \models \psi(x/c) \iff \mathbf{M}', j \frac{c^{\mathbf{M}',j}}{x} \models \psi. \quad (2.89)$$

Since $c^{\mathbf{M}',j} = c^{\mathbf{M}'} = j(x)$, we have

$$j \frac{j(x)}{x} = j. \quad (2.90)$$

Therefore, $\mathbf{M}', j \models \psi$. Since c does not occur in ψ , and \mathbf{M}' differs from \mathbf{M} only in the interpretation of c , we have $\psi^{\mathbf{M}',j} = \psi^{\mathbf{M},j}$. Therefore, $\mathbf{M}, j \models \psi$. Since this holds for any assignment $j : \mathcal{V} \rightarrow A$, we have $\mathbf{M} \models \psi$. \square

2.7 Context Rules

Theorem 2.7.1 (modus ponens in context). *Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let \mathbf{M} be an \mathcal{L} -structure with domain A and $j : \mathcal{V} \rightarrow A$ be an assignment. Let $\alpha, \beta, \gamma \in \mathcal{F}(\mathcal{L})$. If $\mathbf{M}, j \models \gamma \rightarrow \alpha$ and $\mathbf{M}, j \models \gamma \rightarrow (\alpha \rightarrow \beta)$, then $\mathbf{M}, j \models \gamma \rightarrow \beta$.*

Proof. We need to show $(\gamma \rightarrow \beta)^{\mathbf{M},j} = 1$, i.e., $1 - \gamma^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) = 1$.

We consider two cases.

Case 1: $\gamma^{\mathbf{M},j} = 0$. Then

$$(\gamma \rightarrow \beta)^{\mathbf{M},j} = 1 - 0 \cdot (1 - \beta^{\mathbf{M},j}) = 1. \quad (2.91)$$

Case 2: $\gamma^{\mathbf{M},j} = 1$, i.e., $\mathbf{M}, j \models \gamma$.

Since $\mathbf{M}, j \models \gamma \rightarrow \alpha$ and $\mathbf{M}, j \models \gamma$, by Theorem 2.6.1, $\mathbf{M}, j \models \alpha$.

Since $\mathbf{M}, j \models \gamma \rightarrow (\alpha \rightarrow \beta)$ and $\mathbf{M}, j \models \gamma$, by Theorem 2.6.1, $\mathbf{M}, j \models \alpha \rightarrow \beta$.

Since $\mathbf{M}, j \models \alpha$ and $\mathbf{M}, j \models \alpha \rightarrow \beta$, by Theorem 2.6.1, $\mathbf{M}, j \models \beta$, so $\beta^{\mathbf{M},j} = 1$.

Therefore,

$$(\gamma \rightarrow \beta)^{\mathbf{M},j} = 1 - 1 \cdot (1 - 1) = 1. \quad (2.92)$$

In both cases, $(\gamma \rightarrow \beta)^{\mathbf{M},j} = 1$, hence $\mathbf{M}, j \models \gamma \rightarrow \beta$. \square

Chapter 3

Inference Rules

Let \mathcal{F}_k^g be an infinite and countable set of special function symbols of arity k for $k = 0, \dots$

Let \mathcal{P}_k^g be an infinite and countable set of special predicate symbols of arity k for $k = 0, \dots$

$$\mathcal{F}^g \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \mathcal{F}_k^g \quad (3.1)$$

$$\mathcal{P}^g \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \mathcal{P}_k^g \quad (3.2)$$

Definition 3.0.1.

$$\mathcal{L}^g \stackrel{\text{def}}{=} (\mathcal{F}^g, \mathcal{P}^g, \mathbf{a}^g), \quad (3.3)$$

where $\mathbf{a}^g(p) = k$ for any $p \in \mathcal{P}_k^g$ and $\mathbf{a}^g(f) = k$ for any $f \in \mathcal{F}_k^g$ for any integer $k \geq 0$.

It can be constructed in a way that Definition 3.0.1 holds and \mathcal{L}^g is a signature of a language.

Definition 3.0.2. Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathbf{a})$ be signature of a language and $\mathcal{F}^g \cap \mathcal{F} = \emptyset$ and $\mathcal{P}^g \cap \mathcal{P} = \emptyset$. We will consider extention of the language with signature

$$\mathcal{L}^G \stackrel{\text{def}}{=} (\mathcal{F}^g \cup \mathcal{F}, \mathcal{P}^g \cup \mathcal{P}, \mathbf{a} \cup \mathbf{a}^g). \quad (3.4)$$

Note that \mathcal{L}^G is also a signature of a language and $\mathcal{L} \subset \mathcal{L}^G$.

When we consider \mathcal{L}^G -structures $\mathbf{M}_1, \mathbf{M}_2$, we say that they agree on \mathcal{L} iff $\mathbf{M}_1(\mathcal{M} \cup \mathcal{F}) = \mathbf{M}_2(\mathcal{M} \cup \mathcal{F})$ (i.e. $M'|_{\mathcal{L}} = M|_{\mathcal{L}}$).

Definition 3.0.3 (modus ponens). Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathbf{a})$ be a signature of a language. We will define a function $Mod_{\mathcal{L}} : \mathcal{F}(\mathcal{L}) \times \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{F}(\mathcal{L})$.

$$Mod_{\mathcal{L}}(\psi, \alpha) \stackrel{\text{def}}{=} \begin{cases} \beta & \text{for } \psi \doteq \alpha \rightarrow \beta \text{ and } \alpha, \beta \in \mathcal{B}(\mathcal{L}), \\ \mathbf{t} & \text{otherwise.} \end{cases} \quad (3.5)$$

Definition 3.0.4 (generalisation). Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathbf{a})$ be a signature of a language. We will define a family of functions $Gen_{\mathcal{L}}^x : \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{F}(\mathcal{L})$ for $x \in \mathcal{V}$.

$$Gen_{\mathcal{L}}^x(\psi) \stackrel{\text{def}}{=} \begin{cases} \forall x. \psi & \text{for } \psi \in \mathcal{B}(\mathcal{L}), \\ \mathbf{t} & \text{otherwise.} \end{cases} \quad (3.6)$$

Definition 3.0.5 (replacement). Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathbf{a})$ be a signature of a language. We will define a family of functions $\text{Rep}_{\mathcal{L}}^{\psi_1, \psi_2} : \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{F}(\mathcal{L})$ for $\psi_1, \psi_2 \in \mathcal{B}(\mathcal{L})$.

$$\text{Rep}_{\mathcal{L}}^{\psi_1, \psi_2}(\phi) \stackrel{\text{def}}{=} \begin{cases} \psi_1 \leftrightarrow \psi_2 \text{ for } \phi \doteq \phi_1 \leftrightarrow \phi_2 \text{ where } \phi_1, \phi_2 \in \mathcal{B}(\mathcal{F}) \text{ and } \text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2), \\ \text{t otherwise.} \end{cases} \quad (3.7)$$

Definition 3.0.6 (replacement with quantification). Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathbf{a})$ be a signature of a language. Let m be a non-negative integer. We will define a family of functions $\text{RepQ}_{\mathcal{L}}^{\psi_2, \mathbf{x}} : \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{F}(\mathcal{L})$ for $\psi_2 \in \mathcal{B}(\mathcal{L})$ and $\mathbf{x} \in \mathcal{V}^{\nabla m}$.

$$\text{RepQ}_{\mathcal{L}}^{\psi_2, \mathbf{x}}(\phi) \stackrel{\text{def}}{=} \begin{cases} (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_2 \text{ for } \phi \doteq (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_1 \text{ where } \phi_1, \phi_2, \psi_1 \in \mathcal{B}(\mathcal{L}), \\ \text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2), \\ \text{and no free variable of } \forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2) \text{ is under a quantifier} \\ \text{which binds any active occurrence of } \phi_1 \text{ in } \psi_1, \\ \text{t otherwise.} \end{cases} \quad (3.8)$$

Definition 3.0.7 (reverse replacement with quantification). Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathbf{a})$ be a signature of a language. Let m be a non-negative integer. We will define a family of functions $\text{RepQRev}_{\mathcal{L}}^{\psi_1, \mathbf{x}} : \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{F}(\mathcal{L})$ for $\psi_1 \in \mathcal{B}(\mathcal{L})$ and $\mathbf{x} \in \mathcal{V}^{\nabla m}$.

$$\text{RepQRev}_{\mathcal{L}}^{\psi_1, \mathbf{x}}(\phi) \stackrel{\text{def}}{=} \begin{cases} (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_1 \text{ for } \phi \doteq (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_2 \text{ where } \phi_1, \phi_2, \psi_2 \in \mathcal{B}(\mathcal{L}), \\ \text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2), \\ \text{and no free variable of } \forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2) \text{ is under a quantifier} \\ \text{which binds any active occurrence of } \phi_1 \text{ in } \psi_1, \\ \text{t otherwise.} \end{cases} \quad (3.9)$$

Definition 3.0.8 (definition reduction). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a non-negative integer.

We will define a family of functions $\text{Red}_{\mathcal{L}}^{q(\mathbf{x})} : \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{F}(\mathcal{L})$ for $q \in \mathcal{P}_k$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$.

$$\text{Red}_{\mathcal{L}}^{q(\mathbf{x})}(\gamma) \stackrel{\text{def}}{=} \begin{cases} \psi \text{ for } \gamma \doteq (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi \text{ where } \psi, \phi \in \mathcal{B}(\mathcal{L}) \\ \text{and } q \text{ does not occur in } \psi \text{ and } \phi, \\ \text{t otherwise.} \end{cases} \quad (3.10)$$

Definition 3.0.9 (Skolemization schema). Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a non-negative integer.

We will define a family of functions $\text{Sko}_{\mathcal{L}}^{f(\mathbf{u}), x} : \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{F}(\mathcal{L})$ for $f \in \mathcal{F}_k$, $\mathbf{u} \in \mathcal{V}^{\nabla k}$, and $x \in \mathcal{V}$.

$$\text{Sko}_{\mathcal{L}}^{f(\mathbf{u}), x}(\gamma) \stackrel{\text{def}}{=} \begin{cases} \psi \text{ for } \gamma \doteq ((\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))) \rightarrow \psi \text{ where } \psi, \phi \in \mathcal{B}(\mathcal{L}), \\ f \text{ does not occur in } \psi \text{ and } \phi, \\ \text{and } \phi(x/f(\mathbf{u})) \text{ is admissible,} \\ \text{t otherwise.} \end{cases} \quad (3.11)$$

Chapter 4

Abstract Rules Theory

Definition 4.0.1. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k_1, k_2 be non-negative integers. We will say that r is an inference rule on $\mathcal{F}(\mathcal{L})$ iff $r : \mathcal{F}(\mathcal{L})^{k_1} \rightarrow \mathcal{F}(\mathcal{L})^{k_2}$. Moreover, we will define $\dim r \stackrel{\text{def}}{=} (k_1, k_2)$.

Definition 4.0.2. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathcal{L})^{k_1} \rightarrow \mathcal{F}(\mathcal{L})^{k_2}$, we will say that r is **strictly truth-preserving** iff for any \mathcal{L} -structure \mathbf{M} and any $\phi \in \mathcal{F}(\mathcal{L})^{k_1}$

$$\mathbf{M} \models \phi \implies \mathbf{M} \models r(\phi). \quad (4.1)$$

Theorem 4.0.3. Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathfrak{a})$ be a signature of a language. Then $\text{Mod}_{\mathcal{L}}$ is strictly truth preserving.

Proof. Take any \mathcal{L} -structure \mathbf{M} and any $(\psi, \alpha) \in \mathcal{F}(\mathcal{L})^2$ such that $\mathbf{M} \models (\psi, \alpha)$. By definition, $\mathbf{M} \models (\psi, \alpha)$ means $\mathbf{M} \models \psi$ and $\mathbf{M} \models \alpha$.

We need to show that $\mathbf{M} \models \text{Mod}_{\mathcal{L}}(\psi, \alpha)$.

Case 1: $\psi \doteq \alpha \rightarrow \beta$ where $\alpha, \beta \in \mathcal{B}(\mathcal{L})$.

Then $\text{Mod}_{\mathcal{L}}(\psi, \alpha) = \beta$.

Take any assignment $j : \mathcal{V} \rightarrow A$. We need to show that $\mathbf{M}, j \models \beta$.

Since $\mathbf{M} \models \alpha$, we have $\mathbf{M}, j \models \alpha$.

Since $\mathbf{M} \models \psi$, we have $\mathbf{M}, j \models \alpha \rightarrow \beta$.

By Theorem 2.6.1, since $\mathbf{M}, j \models \alpha$ and $\mathbf{M}, j \models \alpha \rightarrow \beta$, we have $\mathbf{M}, j \models \beta$.

Since this holds for any assignment $j : \mathcal{V} \rightarrow A$, we have $\mathbf{M} \models \beta = \text{Mod}_{\mathcal{L}}(\psi, \alpha)$.

Case 2: Otherwise.

Then $\text{Mod}_{\mathcal{L}}(\psi, \alpha) = \top$, which is trivially satisfied by any structure. \square

Theorem 4.0.4. Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathfrak{a})$ be a signature of a language and let $x \in \mathcal{V}$. Then $\text{Gen}_{\mathcal{L}}^x$ is strictly truth preserving.

Proof. Take any \mathcal{L} -structure \mathbf{M} and any $\psi \in \mathcal{F}(\mathcal{L})$ such that $\mathbf{M} \models \psi$.

We need to show that $\mathbf{M} \models \text{Gen}_{\mathcal{L}}^x(\psi)$.

Case 1: $\psi \in \mathcal{B}(\mathcal{L})$.

Then $\text{Gen}_{\mathcal{L}}^x(\psi) = \forall x. \psi$.

By Theorem 2.6.2, since $\mathbf{M} \models \psi$, we have $\mathbf{M} \models \forall x. \psi = \text{Gen}_{\mathcal{L}}^x(\psi)$.

Case 2: $\psi \notin \mathcal{B}(\mathcal{L})$.

Then $\text{Gen}_{\mathcal{L}}^x(\psi) = \top$, which is trivially satisfied by any structure. \square

Theorem 4.0.5. Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathfrak{a})$ be a signature of a language and let $\psi_1, \psi_2 \in \mathcal{B}(\mathcal{L})$. Then $\text{Rep}_{\mathcal{L}}^{\psi_1, \psi_2}$ is strictly truth preserving.

Proof. Take any \mathcal{L} -structure \mathbf{M} and any $\phi \in \mathcal{F}(\mathcal{L})$ such that $\mathbf{M} \models \phi$.

We need to show that $\mathbf{M} \models \text{Rep}_{\mathcal{L}}^{\psi_1, \psi_2}(\phi)$.

Case 1: $\phi \doteq \phi_1 \leftrightarrow \phi_2$ where $\phi_1, \phi_2 \in \mathcal{B}(\mathcal{L})$ and $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$.

Then $\text{Rep}_{\mathcal{L}}^{\psi_1, \psi_2}(\phi) = \psi_1 \leftrightarrow \psi_2$.

Since $\mathbf{M} \models \phi_1 \leftrightarrow \phi_2$, by Theorem 2.6.12, we have $\mathbf{M} \models \psi_1 \leftrightarrow \psi_2 = \text{Rep}_{\mathcal{L}}^{\psi_1, \psi_2}(\phi)$.

Case 2: Otherwise.

Then $\text{Rep}_{\mathcal{L}}^{\psi_1, \psi_2}(\phi) = \mathbf{t}$, which is trivially satisfied by any structure. \square

Theorem 4.0.6. Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathbf{a})$ be a signature of a language, let $\psi_2 \in \mathcal{B}(\mathcal{L})$ and let m be a non-negative integer, $\mathbf{x} \in \mathcal{V}^{\nabla m}$. Then $\text{RepQ}_{\mathcal{L}}^{\psi_2, \mathbf{x}}$ is strictly truth preserving.

Proof. Take any \mathcal{L} -structure \mathbf{M} and any $\phi \in \mathcal{F}(\mathcal{L})$ such that $\mathbf{M} \models \phi$.

We need to show that $\mathbf{M} \models \text{RepQ}_{\mathcal{L}}^{\psi_2, \mathbf{x}}(\phi)$.

Case 1: $\phi \doteq (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_1$ where $\phi_1, \phi_2, \psi_1 \in \mathcal{B}(\mathcal{L})$, $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$, and no free variable of $\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)$ is under a quantifier which binds any active occurrence of ϕ_1 in ψ_1 .

Then $\text{RepQ}_{\mathcal{L}}^{\psi_2, \mathbf{x}}(\phi) = (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_2$.

Since $\mathbf{M} \models (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_1$, by Corollary 2.6.14 (with the replace relation $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$), we have $\mathbf{M} \models (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_2 = \text{RepQ}_{\mathcal{L}}^{\psi_2, \mathbf{x}}(\phi)$.

Case 2: Otherwise.

Then $\text{RepQ}_{\mathcal{L}}^{\psi_2, \mathbf{x}}(\phi) = \mathbf{t}$, which is trivially satisfied by any structure. \square

Theorem 4.0.7. Let $\mathcal{L} = (\mathcal{F}, \mathcal{P}, \mathbf{a})$ be a signature of a language, let $\psi_1 \in \mathcal{B}(\mathcal{L})$ and let m be a non-negative integer, $\mathbf{x} \in \mathcal{V}^{\nabla m}$. Then $\text{RepQRev}_{\mathcal{L}}^{\psi_1, \mathbf{x}}$ is strictly truth preserving.

Proof. Take any \mathcal{L} -structure \mathbf{M} and any $\phi \in \mathcal{F}(\mathcal{L})$ such that $\mathbf{M} \models \phi$.

We need to show that $\mathbf{M} \models \text{RepQRev}_{\mathcal{L}}^{\psi_1, \mathbf{x}}(\phi)$.

Case 1: $\phi \doteq (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_2$ where $\phi_1, \phi_2, \psi_2 \in \mathcal{B}(\mathcal{L})$, $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$, and no free variable of $\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)$ is under a quantifier which binds any active occurrence of ϕ_1 in ψ_1 .

Then $\text{RepQRev}_{\mathcal{L}}^{\psi_1, \mathbf{x}}(\phi) = (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_1$.

Since $\mathbf{M} \models (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_2$, by Corollary 2.6.14 (using the second implication with the replace relation $\text{rep}_{\phi_1, \phi_2}(\psi_1, \psi_2)$), we have $\mathbf{M} \models (\forall \mathbf{x}.(\phi_1 \leftrightarrow \phi_2)) \rightarrow \psi_1 = \text{RepQRev}_{\mathcal{L}}^{\psi_1, \mathbf{x}}(\phi)$.

Case 2: Otherwise.

Then $\text{RepQRev}_{\mathcal{L}}^{\psi_1, \mathbf{x}}(\phi) = \mathbf{t}$, which is trivially satisfied by any structure. \square

Definition 4.0.8. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathcal{L}^G)^{k_1} \rightarrow \mathcal{F}(\mathcal{L}^G)^{k_2}$, we will say that r is **weakly truth-preserving** iff for any \mathcal{L}^G -structure \mathbf{M} and any $\phi \in \mathcal{F}(\mathcal{L}^G)^{k_1}$

$$\mathbf{M}' \models \phi \text{ for any } \mathcal{L}^G\text{-structure } \mathbf{M}' \text{ for which } \mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}} \implies \mathbf{M} \models r(\phi). \quad (4.2)$$

Corollary 4.0.9. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathcal{L}^G)^{k_1} \rightarrow \mathcal{F}(\mathcal{L}^G)^{k_2}$. If r is strictly truth-preserving, then r is weakly truth-preserving.

Proof. Assume r is strictly truth-preserving.

Take any \mathcal{L}^G -structure \mathbf{M} and any $\phi \in \mathcal{F}(\mathcal{L}^G)^{k_1}$ such that $\mathbf{M}' \models \phi$ for all \mathcal{L}^G -structure \mathbf{M}' for which $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}$.

In particular, taking $\mathbf{M}' = \mathbf{M}$, we have $\mathbf{M} \models \phi$.

Since r is strictly truth-preserving and $\mathbf{M} \models \phi$, we have $\mathbf{M} \models r(\phi)$.

Therefore, r is weakly truth-preserving. \square

Theorem 4.0.10. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathcal{P}_k^g$ and $\mathbf{x} \in \mathcal{V}^{\nabla k}$. Then $\text{Red}_{\mathcal{L}^G}^{q(\mathbf{x})}$ is weakly truth-preserving.

Proof. Take any \mathcal{L}^G -structure \mathbf{M} and any $\gamma \in \mathcal{F}(\mathcal{L}^G)$ such that $\mathbf{M}' \models \gamma$ for all \mathcal{L}^G -structure \mathbf{M}' for which $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}$.

We need to show that $\mathbf{M} \models \text{Red}_{\mathcal{L}^G}^{q(\mathbf{x})}(\gamma)$.

Case 1: $\gamma \doteq (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi$ where $\psi, \phi \in \mathcal{B}(\mathcal{L})$ and q does not occur in ψ and ϕ .

Then $\text{Red}_{\mathcal{L}}^{q(\mathbf{x})}(\gamma) = \psi$.

We need to show that $\mathbf{M} \models \psi$.

Since $q \in \mathcal{P}_k^g$ and $q \notin \mathcal{P}$, for any $\mathbf{M}' \in \mathcal{M}_q(\mathbf{M})$ (where \mathbf{M}' is an \mathcal{L}^G -structure), we have $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}$, so by assumption $\mathbf{M}' \models \gamma$, which means $\mathbf{M}' \models (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi$.

By Lemma 2.6.8, since $\mathbf{M}' \models (q(\mathbf{x}) \leftrightarrow \phi) \rightarrow \psi$ for any $\mathbf{M}' \in \mathcal{M}_q(\mathbf{M})$, we have $\mathbf{M} \models \psi$.

Therefore, $\mathbf{M} \models \text{Red}_{\mathcal{L}^G}^{q(\mathbf{x})}(\gamma)$.

Case 2: Otherwise.

Then $\text{Red}_{\mathcal{L}}^{q(\mathbf{x})}(\gamma) = t$, which is trivially satisfied by any structure.

Therefore, $\text{Red}_{\mathcal{L}^G}^{q(\mathbf{x})}$ is weakly truth-preserving. \square

Theorem 4.0.11. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let k be a non-negative integer, let $f \in \mathcal{F}_k^g$, $\mathbf{u} \in \mathcal{V}^{\nabla k}$, and $x \in \mathcal{V}$. Then $\text{Sko}_{\mathcal{L}^G}^{f(\mathbf{u}), x}$ is weakly truth-preserving.

Proof. Take any \mathcal{L}^G -structure \mathbf{M} and any $\gamma \in \mathcal{F}(\mathcal{L}^G)$ such that $\mathbf{M}' \models \gamma$ for all \mathcal{L}^G -structure \mathbf{M}' for which $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}$.

We need to show that $\mathbf{M} \models \text{Sko}_{\mathcal{L}^G}^{f(\mathbf{u}), x}(\gamma)$.

Case 1: $\gamma \doteq ((\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))) \rightarrow \psi$ where $\psi, \phi \in \mathcal{B}(\mathcal{L}^G)$, f does not occur in ψ and ϕ , and $\phi(x/f(\mathbf{u}))$ is admissible.

Then $\text{Sko}_{\mathcal{L}^G}^{f(\mathbf{u}), x}(\gamma) = \psi$.

We need to show that $\mathbf{M} \models \psi$.

Since $f \in \mathcal{F}_k^g$ and $f \notin \mathcal{F}$, for any $\mathbf{M}' \in \mathcal{M}_f(\mathbf{M})$ (where \mathbf{M}' is an \mathcal{L}^G -structure), we have $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}$, so by assumption $\mathbf{M}' \models \gamma$, which means $\mathbf{M}' \models ((\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))) \rightarrow \psi$.

By Lemma 2.6.10, since $\mathbf{M}' \models ((\exists x.\phi) \rightarrow \phi(x/f(\mathbf{u}))) \rightarrow \psi$ for any $\mathbf{M}' \in \mathcal{M}_f(\mathbf{M})$, we have $\mathbf{M} \models \psi$.

Therefore, $\mathbf{M} \models \text{Sko}_{\mathcal{L}^G}^{f(\mathbf{u}), x}(\gamma)$.

Case 2: Otherwise.

Then $\text{Sko}_{\mathcal{L}^G}^{f(\mathbf{u}), x}(\gamma) = t$, which is trivially satisfied by any structure.

Therefore, $\text{Sko}_{\mathcal{L}^G}^{f(\mathbf{u}), x}$ is weakly truth-preserving. \square

Definition 4.0.12. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let \mathcal{R} be a family of inference rules on $\mathcal{F}(\mathcal{L})$. Let $\Gamma \subset \mathcal{F}(\mathcal{L})$. We will say that Γ is closed with respect to \mathcal{R} iff for any $r \in \mathcal{R}$ with $\dim r = (k_1, k_2)$, we have $r(\Gamma^{k_1}) \subset \Gamma^{k_2}$.

Definition 4.0.13. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let \mathcal{R} be a family of inference rules on $\mathcal{F}(\mathcal{L})$. Let $T \subset \mathcal{F}(\mathcal{L})$.

$$Cn(T, \mathcal{R}) \stackrel{\text{def}}{=} \bigcap \{\Gamma : T \subset \Gamma \text{ and } \Gamma \text{ is closed with respect to } \mathcal{R}\}. \quad (4.3)$$

Definition 4.0.14. Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathbf{a})$ be a signature of a language. Let \mathbf{M} be an \mathcal{L}^G -structure.

$$\mathcal{G}(\mathcal{L}^G, \mathbf{M}) \stackrel{\text{def}}{=} \{\phi \in \mathcal{F}(\mathcal{L}^G) : \mathbf{M} \models \phi, \text{ and } (\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}) \implies \mathbf{M}' \models \phi\} \quad (4.4)$$

for any \mathcal{L}^G -structures \mathbf{M}' .

Theorem 4.0.15. *Let $\mathcal{L} = (\mathcal{P}, \mathcal{F}, \mathfrak{a})$ be a signature of a language and \mathbf{M} be an \mathcal{L}^G -structure. Let \mathcal{R} be a family of weakly truth-preserving inference rules on $\mathcal{F}(\mathcal{L}^G)$. If $T \subset \mathcal{G}(\mathcal{L}^G, \mathbf{M})$, then $Cn(T, \mathcal{R}) \subset \mathcal{G}(\mathcal{L}^G, \mathbf{M})$.*

Proof. We will show that $\mathcal{G}(\mathcal{L}^G, \mathbf{M})$ is closed with respect to \mathcal{R} and $T \subset \mathcal{G}(\mathcal{L}^G, \mathbf{M})$. Then by definition of $Cn(T, \mathcal{R})$, we have $Cn(T, \mathcal{R}) \subset \mathcal{G}(\mathcal{L}^G, \mathbf{M})$.

Since $T \subset \mathcal{G}(\mathcal{L}^G, \mathbf{M})$ by assumption, it remains to show that $\mathcal{G}(\mathcal{L}^G, \mathbf{M})$ is closed with respect to \mathcal{R} .

Take any $r \in \mathcal{R}$. Let $\dim r = (k_1, k_2)$. We need to show that $r(\mathcal{G}(\mathcal{L}^G, \mathbf{M})^{k_1}) \subset \mathcal{G}(\mathcal{L}^G, \mathbf{M})^{k_2}$.

Take an arbitrary $\phi \in \mathcal{G}(\mathcal{L}^G, \mathbf{M})^{k_1}$. By Definition 4.0.14, we have $\mathbf{M} \models \phi$.

Take any \mathcal{L}^G -structure \mathbf{M}' with $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}$. Again by Definition 4.0.14, we have $\mathbf{M}' \models \phi$.

And since r is weakly truth preserving, we have $\mathbf{M} \models r(\phi)$.

Take an arbitrary \mathcal{L}^G -structure \mathbf{M}'' with $\mathbf{M}'|_{\mathcal{L}} = \mathbf{M}''|_{\mathcal{L}}$.

Since $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}$, $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}''|_{\mathcal{L}}$ and by Definition 4.0.14 we have $\mathbf{M}'' \models \phi$.

Since r is weakly truth-preserving and $\mathbf{M}'' \models \phi$ for all \mathbf{M}'' with $\mathbf{M}'|_{\mathcal{L}} = \mathbf{M}''|_{\mathcal{L}}$, we have $\mathbf{M}' \models r(\phi)$. We have shown that $\mathbf{M} \models r(\phi)$ and that for any \mathcal{L}^G -structure \mathbf{M}' with $\mathbf{M}|_{\mathcal{L}} = \mathbf{M}'|_{\mathcal{L}}$, we have $\mathbf{M}' \models r(\phi)$. Thus $r(\phi) \in \mathcal{G}(\mathcal{L}^G, \mathbf{M})^{k_2}$, which completes the proof. \square