Foundations of Computational Logic

Michał Stanisław Wójcik

January 9, 2026

Chapter 1

Introduction

1.1 Acknowledgements

This document presents the logical foundations for a formal system based on first-order logic. The net of definitions and theorems presented here were designed by a human author - Michał Stanisław Wójcik. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed, verified and corrected by Michał Stanisław Wójcik who takes the full responsibility for the content of this work.

Contents

Chapter 2

First Order Logic

2.1 Preliminaries

We will use $\mathbb{N}$ to denote a set of all nonnegative integers.

Definition 2.1.1. *Let S be an arbitrary set.*

1. $S^0 \stackrel{\text{def}}{=} \{\emptyset\}$.
2. $S^k \stackrel{\text{def}}{=} \underbrace{S \times \cdots \times S}_{k}$.
3. $S^* \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^k$.

Note that $\emptyset$ is treated as empty sequence.

Remark 2.1.2. *For a sequence $\boldsymbol{s} \in S^k$, we denote the i-th element of the sequence by s_i where $i \in \{1, \dots, k\}$. let also $|\boldsymbol{s}| \stackrel{\text{def}}{=} k$ denote a length of $\boldsymbol{s}$.*

Note that according to the above convention $|\emptyset| = 0$.

Definition 2.1.3. *Let S be an arbitrary set.*

1. $S^{\nabla 0} \stackrel{\text{def}}{=} \{\emptyset\}$.
2. $S^{\nabla k} \stackrel{\text{def}}{=} \{\boldsymbol{s} \in S^k : \boldsymbol{s}_i \neq \boldsymbol{s}_j \text{ for } i \neq j\}$.
3. $S^{\nabla} \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} S^{\nabla k}$.

Sometimes if not disambiguous, we will treat $s \in S^{\nabla}$ notationally as set, so we will write $s_i \in s$, or for a set a, $a \setminus s$, $a \subset s$ etc.

2.2 First Order Logic Formula

2.2.1 Syntax

Let $\mathfrak{L}$ denote a set of all logical symbols “ (”, “) ”, “ $\rightarrow$ ”, “ . ”, “, ”, “ $\forall$ ”. Let $\mathscr{V}$ be an infinite countable set of variable symbols. And $\mathfrak{f}$ be a special predicate symbol.

Definition 2.2.1. *Let $\mathscr{P}$ be a set of predicate symbols for which $\mathfrak{f} \in \mathscr{P}$ and $\mathscr{F}$ be a set of function symbols. Let $\mathfrak{a} : \mathscr{P} \cup \mathscr{F} \rightarrow \mathbb{N}$ be an arity function. then $\mathscr{L} \stackrel{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ is a signature of a language.*

If $\mathscr{L}' = (\mathscr{P}', \mathscr{F}', \mathfrak{a}')$ is a language signature and $\mathscr{P} \subset \mathscr{P}'$ and $\mathscr{F} \subset \mathscr{F}'$ and $\mathfrak{a} \subset \mathfrak{a}'$, we will slightly abuse notation by writing $\mathscr{L} \subset \mathscr{L}'$ meaning that $\mathscr{L}'$ is an extention of $\mathscr{L}$ or $\mathscr{L}$ is a restriction of $\mathscr{L}'$.
To write first order logic formulas for a given language signature $\mathscr{L}$, we will use symbols from

$$\mathcal{S}(\mathscr{L}) \stackrel{\text{def}}{=} \mathfrak{L} \cup \mathscr{V} \cup \mathscr{P} \cup \mathscr{F}. \tag{2.1}$$

All formulas will be then a subset of $\mathcal{S}(\mathscr{L})^*$. Elements of $\mathcal{S}(\mathscr{L})^*$ will be called strings.

Note 2.2.2 (syntactic equality). *We use the notation $\doteq$ to denote equality of strings. We will sometimes overload it when cast of some object to string is obvious.*

When we stack symbols together from left to right, we mean that we concatenate them in the same order. E.g let p be a predicate symbol and x, y be variable symbols, then $p(x, y)$ is just a sequence of 6 symbols: " p ", "(", " x ", " , ", " y ", ") ".

For convenience we will use an abreviated style of writing, in which for any symbol $s \in \mathscr{P} \cup \mathscr{F}$ and for any seqence of strings $\boldsymbol{x} \in \mathcal{S}(\mathscr{L})^*$

$$s(\boldsymbol{x}) \doteq \begin{cases} s(x_1, \ldots, x_k) \text{ for } |\boldsymbol{x}| > 0, \\ s \text{ for } |\boldsymbol{x}| = 0. \end{cases} \tag{2.2}$$

Definition 2.2.3 (arity). *Let $\mathscr{L} \stackrel{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

$$\mathscr{P}_k \stackrel{\text{def}}{=} \{p \in \mathscr{P} : \mathfrak{a}(p) = k\}, \tag{2.3}$$

$$\mathscr{F}_k \stackrel{\text{def}}{=} \{f \in \mathscr{F} : \mathfrak{a}(f) = k\}. \tag{2.4}$$

Definition 2.2.4 (atomic term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

1. *For any $v \in \mathscr{V}$, v is an **atomic term**.*
2. *For any $c \in \mathscr{F}_0$, c is an **atomic term**.*

Definition 2.2.5 (recursive: term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all terms $\mathcal{T}(\mathscr{L})$.*

Let $s \in \mathcal{S}(\mathscr{L})^*$.

1. *If $s \doteq v$ where $v \in \mathscr{V}$, then $s \in \mathcal{T}(\mathscr{L})$.*
2. *If $s \doteq f(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\tau}|}$, then $s \in \mathcal{T}(\mathscr{L})$.*
3. *Otherwise $s \notin \mathcal{T}(\mathscr{L})$.*

Definition 2.2.6 (atomic formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define a set of all atomic formulas $\mathcal{A}(\mathscr{L})$.*

$$\mathcal{A}(\mathscr{L}) \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \{p(\boldsymbol{\tau}) : p \in \mathscr{P}_k, \boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k\}. \tag{2.5}$$

Definition 2.2.7 (recursive: base formula, well-formed formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. We will define recursively a set of all base formulas $\mathcal{B}(\mathscr{L})$ and a set of all well-formed formulas $\mathcal{F}(\mathscr{L})$.*

Let $s \in \mathcal{S}(\mathscr{L})^*$.

1. *If* $s \in \mathcal{A}(\mathscr{L})$*, then* $s \in \mathcal{B}(\mathscr{L})$*.*

2. *If* $s \in \mathcal{B}(\mathscr{L})$*, then* $s \in \mathcal{F}(\mathscr{L})$*.*

3. *If* $s \doteq (\phi)$ *where* $\phi \in \mathcal{F}(\mathscr{L})$*, then* $s \in \mathcal{B}(\mathscr{L})$*.*

4. *If* $s \doteq \alpha \to \beta$ *where* $\alpha, \beta \in \mathcal{B}(\mathscr{L})$*, then* $s \in \mathcal{F}(\mathscr{L})$*.*

5. *If* $s \doteq \forall v.\phi$ *where* $\phi \in \mathcal{B}(\mathscr{L})$*, then* $s \in \mathcal{F}(\mathscr{L})$*.*

6. *Otherwise* $s \notin \mathcal{F}(\mathscr{L})$ *and* $s \notin \mathcal{B}(\mathscr{L})$*.*

Definition 2.2.8 (**recursive: variable in term**)**.** *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. We will define recursively a set* $var(\tau)$ *of all variables in a term* τ*.*

Let $v \in \mathscr{V}$ *and* $\tau \in \mathcal{T}(\mathscr{L})$*.*

1. *If* $\tau \doteq v$ *then* $v \in var(\tau)$*.*

2. *If* $\tau \doteq f(\boldsymbol{\sigma})$ *where* $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ *and* $f \in \mathscr{F}_{|\boldsymbol{\sigma}|}$ *and* $v \in var(\boldsymbol{\sigma}) \overset{\text{def}}{=} \bigcup_{k=1}^{|\boldsymbol{\sigma}|} var(\sigma_k)$*, then* $v \in var(\tau)$*.*

3. *Otherwise* $v \notin var(\tau)$*.*

Definition 2.2.9 (**recursive: variable in formula**)**.** *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. We will define recursively a set* $var(\phi)$ *of all variables in a formula* ϕ*.*

Let $v \in \mathscr{V}$ *and* $\phi \in \mathcal{F}(\mathscr{L})$*.*

1. *If* $\phi \doteq p(\boldsymbol{\tau})$ *where* $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ *and* $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$ *and* $v \in var(\boldsymbol{\tau})$*, then* $v \in var(\phi)$*.*

2. *If* $\phi \doteq \alpha \to \beta$ *where* $\alpha, \beta \in \mathcal{B}(\mathscr{L})$*, and* $v \in var(\alpha)$ *or* $v \in var(\beta)$*, then* $v \in var(\phi)$*.*

3. *If* $\phi \doteq \forall x.\psi$ *where* $x \in \mathscr{V}$ *and* $\psi \in \mathcal{B}(\mathscr{L})$*, and either* $v = x$ *or* $v \in var(\psi)$*, then* $v \in var(\phi)$*.*

4. *Otherwise* $v \notin var(\phi)$*.*

Definition 2.2.10 (**recursive: free variable in formula**)**.** *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. We will define recursively a set* $free(\phi)$ *of all free variables in a formula* ϕ*.*

Let $v \in \mathscr{V}$ *and* $\phi \in \mathcal{F}(\mathscr{L})$*.*

1. *If* $\phi \in \mathcal{A}(\mathscr{L})$ *and* $v \in var(\phi)$*, then* $v \in free(\phi)$*.*

2. *If* $\phi \doteq \alpha \to \beta$ *where* $\alpha, \beta \in \mathcal{B}(\mathscr{L})$*, and* $v \in free(\alpha)$ *or* $v \in free(\beta)$*, then* $v \in free(\phi)$*.*

3. *If* $\phi \doteq \forall x.\psi$ *where* $\psi \in \mathcal{B}(\mathscr{L})$*,* $x \in \mathscr{V}$ *and* $x \neq v$*, and* $v \in free(\psi)$*, then* $v \in free(\phi)$*.*

4. *Otherwise* $v \notin free(\phi)$*.*

2.2.2 Semantics

Definition 2.2.11 (**$\mathscr{L}$-structure**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and $A \neq \emptyset$ be an arbitrary mathematical domain, then a mapping $\mathbf{M}$ such that*

$$\mathbf{M} : \mathscr{P} \cup \mathscr{F} \to \bigcup_{k=0}^{\infty} \{0,1\}^{(A^k)} \cup \bigcup_{k=0}^{\infty} A^{(A^k)}, \tag{2.6}$$

where

$$\begin{aligned} &\mathbf{M}(\mathfrak{f}) = (\emptyset \mapsto 0), \\ &\mathbf{M}(\mathscr{P}_k) \subset \{0,1\}^{(A^k)} \text{ for } k \in \mathbb{N}, \\ &\mathbf{M}(\mathscr{F}_k) \subset A^{(A^k)} \text{ for } k \in \mathbb{N}. \end{aligned}$$

is called an $\mathscr{L}$-structure with domain A.

For convenience we will denote values of $\mathbf{M}$ as $p^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(p)$ for $p \in \mathscr{P}$ and $f^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(f)$ for $f \in \mathscr{F}$. Note that $p^{\mathbf{M}} : A^k \to \{0,1\}$ and $f^{\mathbf{M}} : A^k \to A$.

For a $\mathscr{L}$-structure $\mathbf{M}$, we will use a standard notation of restriction of a function $\mathbf{M}|_S$, where $S \subset \mathscr{P} \cup \mathscr{F}$. Also if we have a signatures of language $\mathscr{L}, \mathscr{L}'$ for which $\mathscr{L} \subset \mathscr{L}'$ (i.e. $\mathscr{L}'$ is an extenion of $\mathscr{L}$), for a $\mathscr{L}'$-signature $\mathbf{M}'$, $\mathbf{M}'|_{\mathscr{L}} \stackrel{\text{def}}{=} \mathbf{M}'|_{\mathscr{P} \cup \mathscr{F}}$. Note that $\mathbf{M}'|_{\mathscr{L}}$ is an $\mathscr{L}$-structure.

Definition 2.2.12 (**assignment**). *Let A be an arbitrary mathematical domain. Then $j : \mathscr{V} \to A$ is called a variables assignment in A.*

Definition 2.2.13 (**recursive: term evaluation**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure and $j : \mathscr{V} \to A$.*

Let $\tau \in \mathcal{T}(\mathscr{L})$.

1. *If $\tau \doteq v$ where $v \in \mathscr{V}$, then $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} j(v)$.*
2. *If $\tau \doteq f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathcal{F}_{|\boldsymbol{\sigma}|}$, then $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} f^{\mathbf{M}}(\boldsymbol{\sigma}^{\mathbf{M},j})$ where*

$$\boldsymbol{\sigma}^{\mathbf{M},j} \stackrel{\text{def}}{=} \begin{cases} (\sigma_1^{\mathbf{M},j}, \ldots, \sigma_k^{\mathbf{M},j}) \text{ for } k = |\boldsymbol{\sigma}| > 0, \\ \emptyset \text{ for } |\boldsymbol{\sigma}| = 0. \end{cases} \tag{2.7}$$

Definition 2.2.14. *Let $\mathscr{L}$ be a signature of a language. Let $\mathbf{M}$ be a $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$ be an assignment of variables. Then we define*

$$j\frac{a}{x}(v) \stackrel{\text{def}}{=} \begin{cases} a \text{ if } v = x, \\ j(v) \text{ otherwise.} \end{cases} \tag{2.8}$$

Definition 2.2.15. *Let $\mathscr{L}$ be a signature of a language. Let $\mathbf{M}$ be a $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$ be an assignment of variables. Let k be a positive integer and let $\boldsymbol{x} \in V^{\nabla k}$, and $\boldsymbol{a} \in A^k$.*
Then we define

$$j\frac{\boldsymbol{a}}{\boldsymbol{x}}(v) \stackrel{\text{def}}{=} \begin{cases} a_i \text{ if } v = x_i \text{ for some } i \in \{1, \ldots, k\}, \\ j(v) \text{ otherwise.} \end{cases} \tag{2.9}$$

For the purpouses of this work, we will define multiplication product of infinite number of 0s and/or 1s.

Definition 2.2.16. *Let S be an arbitrary set and $a_s \in \{0,1\}$ for any $s \in S$.*

$$\prod_{s\in S} a_s = \begin{cases} 1 \ \textit{if } a_s = 1 \textit{ for all } s \in S, \\ 0 \ \textit{otherwise} . \end{cases} \tag{2.10}$$

Definition 2.2.17 (recursive: formula evaluation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$.*

1. *If $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$, then $\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} p^{\mathbf{M}}(\boldsymbol{\tau}^{\mathbf{M},j})$.*
2. *If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then*

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}). \tag{2.11}$$

3. *If $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, then*

$$\phi^{\mathbf{M},j} \stackrel{\text{def}}{=} \prod_{a\in A} \psi^{\mathbf{M},j\frac{a}{x}}. \tag{2.12}$$

Corollary 2.2.18. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*

$$(\phi \to \mathfrak{f})^{\mathbf{M},j} = 1 - \phi^{\mathbf{M},j}. \tag{2.13}$$

Definition 2.2.19 (interpretation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*

$$\mathbf{M}, j \models \phi \stackrel{\text{def}}{\iff} \phi^{\mathbf{M},j} = 1. \tag{2.14}$$

Definition 2.2.20 (model). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ and let $\mathbf{M}$ be an $\mathscr{L}$-structure. Let $\phi \in \mathcal{F}(\mathscr{L})$. We define*

$$\mathbf{M} \models \phi \tag{2.15}$$

iff for any assignment $j : \mathscr{V} \to A$, we have $M, j \models \phi$.

Definition 2.2.21 (semantic consequence). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\Gamma \subset \mathcal{F}(\mathscr{L})$ and $\phi \in \mathcal{F}(\mathscr{L})$. We define that $\Gamma \models \phi$ iff for any $\mathscr{L}$-structure $\mathbf{M}$, if $\mathbf{M} \models \psi$ for all $\psi \in \Gamma$, then $\mathbf{M} \models \phi$.*

Definition 2.2.22 (tautology). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\phi \in \mathcal{F}(\mathscr{L})$. We define that $\models \phi$ iff $\emptyset \models \phi$.*

Lemma 2.2.23. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j_1, j_2 : \mathscr{V} \to A$. Let $\tau \in \mathcal{T}(\mathscr{L})$. If $j_1\big(var(\tau)\big) = j_2\big(var(\tau)\big)$, then*

$$\tau^{\mathbf{M},j_1} = \tau^{\mathbf{M},j_2}. \tag{2.16}$$

Proof. We proceed by structural induction on the term τ.
Case 1: $\tau \doteq v$ where $v \in \mathscr{V}$.
Then $\mathrm{var}(\tau) = \{v\}$. Since $j_1(\mathrm{var}(\tau)) = j_2(\mathrm{var}(\tau))$, we have $j_1(v) = j_2(v)$. By the definition of term evaluation,

$$\tau^{\mathbf{M},j_1} = j_1(v) = j_2(v) = \tau^{\mathbf{M},j_2}.$$

Case 2: $\tau \doteq f(\boldsymbol{\sigma})$ where $\boldsymbol{\sigma} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\sigma}|}$.
Then $\mathrm{var}(\tau) = \mathrm{var}(\boldsymbol{\sigma}) = \bigcup_{i=1}^{|\boldsymbol{\sigma}|} \mathrm{var}(\sigma_i)$. Since $j_1(\mathrm{var}(\tau)) = j_2(\mathrm{var}(\tau))$, we have $j_1(\mathrm{var}(\sigma_i)) = j_2(\mathrm{var}(\sigma_i))$ for all $i \in \{1, \ldots, |\boldsymbol{\sigma}|\}$. By inductive hypothesis, $\sigma_i^{\mathbf{M},j_1} = \sigma_i^{\mathbf{M},j_2}$ for all i. Therefore, $\boldsymbol{\sigma}^{\mathbf{M},j_1} = \boldsymbol{\sigma}^{\mathbf{M},j_2}$. By the definition of term evaluation,

$$\tau^{\mathbf{M},j_1} = f^{\mathbf{M}}(\boldsymbol{\sigma}^{\mathbf{M},j_1}) = f^{\mathbf{M}}(\boldsymbol{\sigma}^{\mathbf{M},j_2}) = \tau^{\mathbf{M},j_2}.$$

This completes the proof by structural induction. □

Theorem 2.2.24. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j_1, j_2 : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$. If $j_1\big(\mathit{free}(\phi)\big) = j_2\big(\mathit{free}(\phi)\big)$, then*

$$\phi^{\mathbf{M},j_1} = \phi^{\mathbf{M},j_2}. \tag{2.17}$$

Proof. We proceed by structural induction on the formula ϕ.
Case 1: $\phi \doteq p(\boldsymbol{\tau})$ where $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\tau}|}$.
Then $\mathrm{free}(\phi) = \mathrm{var}(\phi) = \mathrm{var}(\boldsymbol{\tau}) = \bigcup_{i=1}^{|\boldsymbol{\tau}|} \mathrm{var}(\tau_i)$. Since $j_1(\mathrm{free}(\phi)) = j_2(\mathrm{free}(\phi))$, we have $j_1(\mathrm{var}(\tau_i)) = j_2(\mathrm{var}(\tau_i))$ for all i. By Lemma 2.2.23, $\tau_i^{\mathbf{M},j_1} = \tau_i^{\mathbf{M},j_2}$ for all i. Therefore, $\boldsymbol{\tau}^{\mathbf{M},j_1} = \boldsymbol{\tau}^{\mathbf{M},j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M},j_1} = p^{\mathbf{M}}(\boldsymbol{\tau}^{\mathbf{M},j_1}) = p^{\mathbf{M}}(\boldsymbol{\tau}^{\mathbf{M},j_2}) = \phi^{\mathbf{M},j_2}.$$

Case 2: $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.
Then $\mathrm{free}(\phi) = \mathrm{free}(\alpha) \cup \mathrm{free}(\beta)$. Since $j_1(\mathrm{free}(\phi)) = j_2(\mathrm{free}(\phi))$, we have $j_1(\mathrm{free}(\alpha)) = j_2(\mathrm{free}(\alpha))$ and $j_1(\mathrm{free}(\beta)) = j_2(\mathrm{free}(\beta))$. By inductive hypothesis, $\alpha^{\mathbf{M},j_1} = \alpha^{\mathbf{M},j_2}$ and $\beta^{\mathbf{M},j_1} = \beta^{\mathbf{M},j_2}$. By the definition of formula evaluation,

$$\phi^{\mathbf{M},j_1} = 1 - \alpha^{\mathbf{M},j_1}(1 - \beta^{\mathbf{M},j_1}) = 1 - \alpha^{\mathbf{M},j_2}(1 - \beta^{\mathbf{M},j_2}) = \phi^{\mathbf{M},j_2}.$$

Case 3: $\phi \doteq \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$.

Then $\mathrm{free}(\phi) = \mathrm{free}(\psi) \setminus \{x\}$. For any $a \in A$, the assignments $j_1\frac{a}{x}$ and $j_2\frac{a}{x}$ agree on $\mathrm{free}(\psi)$ because:

- For $v \in \mathrm{free}(\psi) \setminus \{x\}$, we have $v \in \mathrm{free}(\phi)$, so $(j_1\frac{a}{x})(v) = j_1(v) = j_2(v) = (j_2\frac{a}{x})(v)$.
- For $v = x$, we have $(j_1\frac{a}{x})(x) = a = (j_2\frac{a}{x})(x)$.

Thus $j_1\frac{a}{x}(\mathrm{free}(\psi)) = j_2\frac{a}{x}(\mathrm{free}(\psi))$. By inductive hypothesis, $\psi^{\mathbf{M},j_1\frac{a}{x}} = \psi^{\mathbf{M},j_2\frac{a}{x}}$ for all $a \in A$. By the definition of formula evaluation,

$$\phi^{\mathbf{M},j_1} = \prod_{a \in A} \psi^{\mathbf{M},j_1\frac{a}{x}} = \prod_{a \in A} \psi^{\mathbf{M},j_2\frac{a}{x}} = \phi^{\mathbf{M},j_2}.$$

This completes the proof by structural induction. □

2.3 Syntactic Sugar

In this section, we introduce common logical connectives as syntactic abbreviations of formulas using implication and the false predicate $\mathfrak{f}$.

Definition 2.3.1 (**logical connectives as syntactic sugar**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\alpha, \beta \in \mathcal{B}(\mathscr{L})$. We define the following syntactic abbreviations:*

$$\mathfrak{t} \stackrel{\text{def}}{=} \mathfrak{f} \to \mathfrak{f} \tag{2.18}$$

$$\sim \alpha \stackrel{\text{def}}{=} \alpha \to \mathfrak{f}, \tag{2.19}$$

$$\alpha \wedge \beta \stackrel{\text{def}}{=} (\alpha \to (\beta \to \mathfrak{f})) \to \mathfrak{f}, \tag{2.20}$$

$$\alpha \vee \beta \stackrel{\text{def}}{=} (\alpha \to \mathfrak{f}) \to \beta, \tag{2.21}$$

$$\alpha \leftrightarrow \beta \stackrel{\text{def}}{=} (\alpha \to \beta) \wedge (\beta \to \alpha). \tag{2.22}$$

For any $v \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, we also define:

$$\exists v.\psi \stackrel{\text{def}}{=} \sim (\forall v. \sim \psi). \tag{2.23}$$

The following theorems characterize the semantic behavior of these connectives.

Theorem 2.3.2 (**negation evaluation**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\alpha \in \mathcal{B}(\mathscr{L})$, then*

$$(\sim \alpha)^{\mathbf{M},j} = 1 - \alpha^{\mathbf{M},j}. \tag{2.24}$$

Proof. By definition, $\sim \alpha \doteq \alpha \to \mathfrak{f}$. By the definition of formula evaluation for implication and the fact that $\mathfrak{f}^{\mathbf{M},j} = 0$,

$$(\sim \alpha)^{\mathbf{M},j} = (\alpha \to \mathfrak{f})^{\mathbf{M},j} = 1 - \alpha^{\mathbf{M},j}(1 - \mathfrak{f}^{\mathbf{M},j}) = 1 - \alpha^{\mathbf{M},j}(1 - 0) = 1 - \alpha^{\mathbf{M},j}. \tag{2.25}$$

□

Theorem 2.3.3 (**conjunction evaluation**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then*

$$(\alpha \wedge \beta)^{\mathbf{M},j} = \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}. \tag{2.26}$$

Proof. By definition, $\alpha \wedge \beta \doteq (\alpha \to (\beta \to \mathfrak{f})) \to \mathfrak{f}$. By the definition of formula evaluation for implication and the fact that $\mathfrak{f}^{\mathbf{M},j} = 0$,

$$\begin{aligned}(\alpha \wedge \beta)^{\mathbf{M},j} &= ((\alpha \to (\beta \to \mathfrak{f})) \to \mathfrak{f})^{\mathbf{M},j} \\ &= 1 - (\alpha \to (\beta \to \mathfrak{f}))^{\mathbf{M},j}(1 - 0) \\ &= 1 - (\alpha \to (\beta \to \mathfrak{f}))^{\mathbf{M},j}.\end{aligned}$$

Now, expanding the inner implication:

$$\begin{aligned}(\alpha \to (\beta \to \mathfrak{f}))^{\mathbf{M},j} &= 1 - \alpha^{\mathbf{M},j}(1 - (\beta \to \mathfrak{f})^{\mathbf{M},j}) \\ &= 1 - \alpha^{\mathbf{M},j}(1 - (1 - \beta^{\mathbf{M},j}(1 - 0))) \\ &= 1 - \alpha^{\mathbf{M},j}(1 - (1 - \beta^{\mathbf{M},j})) \\ &= 1 - \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}.\end{aligned}$$

Therefore,

$$(\alpha \wedge \beta)^{\mathbf{M},j} = 1 - (1 - \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}) = \alpha^{\mathbf{M},j} \cdot \beta^{\mathbf{M},j}. \tag{2.27}$$

$\square$

Theorem 2.3.4 (disjunction evaluation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then*

$$(\alpha \vee \beta)^{\mathbf{M},j} = 1 - (1 - \alpha^{\mathbf{M},j})(1 - \beta^{\mathbf{M},j}). \tag{2.28}$$

Proof. By definition, $\alpha \vee \beta \doteq (\alpha \to \mathfrak{f}) \to \beta$. By the definition of formula evaluation for implication:

$$\begin{aligned}(\alpha \vee \beta)^{\mathbf{M},j} &= ((\alpha \to \mathfrak{f}) \to \beta)^{\mathbf{M},j} \\ &= 1 - (\alpha \to \mathfrak{f})^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) \\ &= 1 - (1 - \alpha^{\mathbf{M},j})(1 - \beta^{\mathbf{M},j}).\end{aligned}$$

$\square$

Theorem 2.3.5 (biconditional evaluation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then*

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = 1 \textit{ if and only if } \alpha^{\mathbf{M},j} = \beta^{\mathbf{M},j}. \tag{2.29}$$

Proof. By definition, $\alpha \leftrightarrow \beta \doteq (\alpha \to \beta) \wedge (\beta \to \alpha)$. By Theorem 2.3.3,

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = (\alpha \to \beta)^{\mathbf{M},j} \cdot (\beta \to \alpha)^{\mathbf{M},j}. \tag{2.30}$$

Expanding using the definition of implication:

$$(\alpha \leftrightarrow \beta)^{\mathbf{M},j} = (1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}))(1 - \beta^{\mathbf{M},j}(1 - \alpha^{\mathbf{M},j})).$$

This product equals 1 if and only if both factors equal 1, which happens if and only if:

- $\alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) = 0$, and
- $\beta^{\mathbf{M},j}(1 - \alpha^{\mathbf{M},j}) = 0$.

These conditions hold if and only if $\alpha^{\mathbf{M},j} = \beta^{\mathbf{M},j}$. $\square$

Theorem 2.3.6 (existential quantifier evaluation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $v \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, then*

$$(\exists v.\psi)^{\mathbf{M},j} = 1 - \prod_{a \in A}(1 - \psi^{\mathbf{M},j\frac{a}{v}}). \tag{2.31}$$

Proof. By definition, $\exists v.\psi \doteq\sim (\forall v. \sim \psi)$. By Theorem 2.3.2,

$$(\exists v.\psi)^{\mathbf{M},j} = 1 - (\forall v. \sim \psi)^{\mathbf{M},j}. \tag{2.32}$$

By the definition of formula evaluation for universal quantification,

$$(\forall v. \sim \psi)^{\mathbf{M},j} = \prod_{a \in A}(\sim \psi)^{\mathbf{M},j\frac{a}{v}}. \tag{2.33}$$

By Theorem 2.3.2, for each $a \in A$,

$$(\sim \psi)^{\mathbf{M},j\frac{a}{v}} = 1 - \psi^{\mathbf{M},j\frac{a}{v}}. \tag{2.34}$$

Therefore,

$$(\exists v.\psi)^{\mathbf{M},j} = 1 - \prod_{a \in A}(1 - \psi^{\mathbf{M},j\frac{a}{v}}). \tag{2.35}$$

□

2.4 Substitutions

2.4.1 Syntax

Definition 2.4.1 (recursive: substitution in a term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$. Let $\sigma \in \mathcal{T}(\mathscr{L})$. We will define recursively $\sigma(\boldsymbol{x}/\boldsymbol{\tau})$. Let's establish that $\sigma(\emptyset/\emptyset) \stackrel{\text{def}}{=} \sigma$.*

1. *If $\sigma \doteq x_i$ for some $i \in \{1, \ldots, k\}$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \tau_i$.*
2. *If $\sigma \doteq v$ where $v \in \mathscr{V}$ and $v \neq x$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} v$.*
3. *If $\tau \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\eta}|}$, then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$ where*

$$\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \begin{cases} (\eta_1(\boldsymbol{x}/\boldsymbol{\tau}), \ldots, \eta_k(\boldsymbol{x}/\boldsymbol{\tau})) \text{ for } k = |\boldsymbol{\eta}| > 0, \\ \emptyset \text{ for } |\boldsymbol{\eta}| = 0. \end{cases} \tag{2.36}$$

Definition 2.4.2 (recursive: admissible substitution in a formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$.*

*Let $\phi \in \mathcal{F}(\mathscr{L})$. We will define recursively $\sigma(\boldsymbol{x}/\boldsymbol{\tau})$. Let's establish that $\phi(\emptyset/\emptyset) \stackrel{\text{def}}{=} \phi$ and is **admissible**.*

1. *If $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$ and is **admissible**.*
2. *If $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$ and $\alpha(\boldsymbol{x}/\boldsymbol{\tau}), \beta(\boldsymbol{x}/\boldsymbol{\tau})$ are **admissible**, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau})$ and is **admissible**.*
3. *If $\phi \doteq \forall u.\psi$ where $u \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$:*
   Let $\boldsymbol{v}$ be $\boldsymbol{x}$ with x_i removed when $x_i = u$ or $x_i \notin free(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from $\boldsymbol{x}$.

   (a) *If $\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin var(\boldsymbol{\sigma})$, then $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \stackrel{\text{def}}{=} \forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ and is **admissable**.*
   (b) *Otherwise $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is **not admissible**.*

Definition 2.4.3 (recursive: admissible predicate substitution). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Let $q \in \mathscr{P}_k$ and $\phi \in \mathcal{F}(\mathscr{L})$.*

Let $\psi \in \mathcal{F}(\mathscr{L})$. We will define recursively $\psi(q(\boldsymbol{x})/\phi)$.

1. *If* $\psi \doteq p(\boldsymbol{\eta})$ *where* $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ *and* $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.
   - (a) *If* $p = q$ *and* $\phi(\boldsymbol{x}/\boldsymbol{\eta})$ *is admissible, then* $\psi(q(\boldsymbol{x})/\phi) \stackrel{\text{def}}{=} \phi(\boldsymbol{x}/\boldsymbol{\eta})$ *and is* ***admissible***.
   - (b) *If* $p \neq q$, *then* $\psi(q(\boldsymbol{x})/\phi) \stackrel{\text{def}}{=} \psi$ *and is* ***admissible***.
2. *If* $\psi \doteq \alpha \to \beta$ *where* $\alpha, \beta \in \mathcal{B}(\mathscr{L})$ *and* $\alpha(q(\boldsymbol{x})/\phi)$, $\beta(q(\boldsymbol{x})/\phi)$ *are* ***admissible***, *then* $\psi(q(\boldsymbol{x})/\phi) \stackrel{\text{def}}{=} \alpha(q(\boldsymbol{x})/\phi) \to \beta(q(\boldsymbol{x})/\phi)$ *and is* ***admissible***.
3. *If* $\psi \doteq \forall u.\gamma$ *where* $u \in \mathscr{V}$ *and* $\gamma \in \mathcal{B}(\mathscr{L})$:
   - (a) *If* q *occurs in* γ, $u \notin \mathit{free}(\phi) \setminus \boldsymbol{x}$, $\gamma(q(\boldsymbol{x})/\phi)$ *is* ***admissible***, *then* $\psi(q(\boldsymbol{x})/\phi) \stackrel{\text{def}}{=} \forall u.\gamma(q(\boldsymbol{x})/\phi)$ *and is* ***admissible***.
   - (b) *If* q *does not occur in* γ, *then* $\psi(q(\boldsymbol{x})/\phi) \stackrel{\text{def}}{=} \forall u.\gamma$ *and is* ***admissible***.
   - (c) *Otherwise* $\psi(q(\boldsymbol{x})/\phi)$ *is* ***not admissible***.

2.4.2 Semantics

Lemma 2.4.4. *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language and* A *be an arbitrary mathematical domain. Let* $\mathbf{M}$ *be an* $\mathscr{L}$*-structure with domain* A *and* $j : \mathscr{V} \to A$ *be an assignment.*
Let k *be a positive integer and let* $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ *and* $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$.

Then for any term σ, *we have*

$$\sigma^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}. \tag{2.37}$$

Proof. We proceed by structural induction on the term σ.
Case 1: $\sigma \doteq x_i$ for some $i \in \{1, \ldots, k\}$.
Then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \tau_i$ and

$$\sigma^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = \Big(j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\Big)(x_i) = \tau_i^{\mathbf{M},j} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 2: $\sigma \doteq v$ where $v \in \mathscr{V}$ and $v \neq x$.
Then $\sigma(x/\tau) \doteq v$ and

$$\sigma^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = \Big(j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\Big)(v) = j(v) = v^{\mathbf{M},j} = (\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 3: $\sigma \doteq f(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $f \in \mathscr{F}_{|\boldsymbol{\eta}|}$.

Then $\sigma(\boldsymbol{x}/\boldsymbol{\tau}) \doteq f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$. By inductive hypothesis, $\boldsymbol{\eta}^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{(\mathbf{M},j)}}{\boldsymbol{x}}} = \boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}$. Thus

$$\begin{aligned}\sigma^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} &= f^{\mathbf{M}}\left(\boldsymbol{\eta}^{\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}}\right) = f^{\mathbf{M}}\left(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}\right) \\ &= f(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = \sigma(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.\end{aligned}$$

This completes the proof by structural induction. □

Theorem 2.4.5. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $\phi \in \mathcal{F}(\mathscr{L})$. Let k be a positive integer and let $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$ and $\boldsymbol{\tau} \in \mathcal{T}(\mathscr{L})^k$. If $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is admissible, then for any $\mathscr{L}$-structure $\mathbf{M}$ with domain A and for any assignment of variables $j : \mathscr{V} \to A$ we have*

$$(\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}) \models \phi \iff \mathbf{M}, j \models \phi(\boldsymbol{x}/\boldsymbol{\tau}). \tag{2.38}$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathscr{L})$.

Case 1: $\phi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.

By the definition of substitution in well formed formula, $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))$. By Lemma 2.4.4, we have $\boldsymbol{\eta}^{\mathbf{M},j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = \boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}$. Therefore,

$$\phi^{\mathbf{M},j\frac{\tau^{\mathbf{M},j}}{x}} = p^{\mathbf{M}}\left(\boldsymbol{\eta}^{\mathbf{M},j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}}\right) = p^{\mathbf{M}}(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}) = p(\boldsymbol{\eta}(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = \phi(\boldsymbol{x}/\boldsymbol{\tau})^{\mathbf{M},j}.$$

Case 2: $\phi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.

By the definition of substitution in well formed formula, $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau})$. By inductive hypothesis,

$$\begin{aligned}(\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}) \models \alpha &\iff \mathbf{M}, j \models \alpha(\boldsymbol{x}/\boldsymbol{\tau}),\\ (\mathbf{M}, j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}) \models \beta &\iff \mathbf{M}, j \models \beta(\boldsymbol{x}/\boldsymbol{\tau}).\end{aligned}$$

By the definition of formula evaluation,

$$\begin{aligned}\phi^{\mathbf{M},j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} &= 1 - \alpha^{\mathbf{M},j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}}\left(1 - \beta^{\mathbf{M},j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}}\right) = 1 - (\alpha(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}(1 - (\beta(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}) = 1\\ &= (\alpha(\boldsymbol{x}/\boldsymbol{\tau}) \to \beta(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j} = (\phi(\boldsymbol{x}/\boldsymbol{\tau}))^{\mathbf{M},j}\end{aligned}$$

Case 3: $\phi \doteq \forall u.\psi$ where $u \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$:

Let $\boldsymbol{v}$ be $\boldsymbol{x}$ with x_i removed when $x_i = u$ or $x_i \notin \text{free}(\psi)$. And let $\boldsymbol{\sigma}$ be $\boldsymbol{\tau}$ with τ_i removed for which x_i was removed from $\boldsymbol{x}$.

Since $\phi(\boldsymbol{x}/\boldsymbol{\tau})$ is admissible, $\psi(\boldsymbol{v}/\boldsymbol{\sigma})$ is **admissible** and $u \notin \text{var}(\boldsymbol{\sigma})$, and we have $\phi(\boldsymbol{x}/\boldsymbol{\tau}) \doteq \forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma})$. Thus

$$\phi^{\mathbf{M},j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}} = \prod_{a \in A} \psi^{\mathbf{M},j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\frac{a}{u}} \tag{2.39}$$

By choice of $\boldsymbol{v}$, $j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\frac{a}{u}$ and $j\frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}}\frac{a}{u}$ agrees on free(ψ) therefore by Theorem 2.2.24 we have

$$
\begin{aligned}
\prod_{a\in A}\psi^{\mathbf{M},j\frac{\boldsymbol{\tau}^{\mathbf{M},j}}{\boldsymbol{x}}\frac{a}{u}} &= \prod_{a\in A}\psi^{\mathbf{M},j\frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}}\frac{a}{u}} \\
&= \prod_{a\in A}\psi^{\mathbf{M},j\frac{a}{u}\frac{\boldsymbol{\sigma}^{\mathbf{M},j}}{\boldsymbol{v}}} \quad \big(\text{since } u \text{ not in } \boldsymbol{v}\big) \\
&= \prod_{a\in A}\psi(\boldsymbol{v}/\boldsymbol{\sigma})^{\mathbf{M},j\frac{a}{u}} \quad \big(\text{by inductive hypothesis}\big) \\
&= (\forall u.\psi(\boldsymbol{v}/\boldsymbol{\sigma}))^{\mathbf{M},j} = \phi(\boldsymbol{x}/\boldsymbol{\tau})^{M,j}.
\end{aligned}
$$

This completes the proof by structural induction. □

Lemma 2.4.6. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $q \in \mathscr{P}$. Let $\phi \in \mathcal{F}(\mathscr{L})$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Let $j_0 : free(\phi) \setminus \boldsymbol{x} \to A$, where A is a domain of an $\mathscr{L}$-structure M for which*

$$M, j \models q(\boldsymbol{x}) \text{ iff } M, j \models \phi \tag{2.40}$$

for any j that is an extention of j_0. If $\psi \in \mathcal{F}(\mathscr{L})$ and $\psi(q(\boldsymbol{x})/\phi)$ is admissible, then

$$M, j \models \psi \text{ iff } M, j \models \psi(q(\boldsymbol{x})/\phi) \tag{2.41}$$

for any j that is an extention of j_0.

Proof. We proceed by structural induction on $\psi \in \mathcal{F}(\mathscr{L})$ following Definition 2.4.3.
Take any $j : \mathscr{V} \to A$ which is an extension of j_0.
Case 1: $\psi \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.
Subcase 1a: $p = q$ and $\phi(\boldsymbol{x}/\boldsymbol{\eta})$ is admissible.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \phi(\boldsymbol{x}/\boldsymbol{\eta})$. By the definition of formula evaluation,

$$\psi^{M,j} = q^M(\boldsymbol{\eta}^{M,j}) = q(\boldsymbol{x})^{M,j\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}}}.$$

Since $j\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}}$ is an extension of j_0, by (2.40), we have

$$q(\boldsymbol{x})^{M,j\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}}} = \phi^{M,j\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}}}.$$

By Theorem 2.4.5, since $\phi(\boldsymbol{x}/\boldsymbol{\eta})$ is admissible,

$$\phi^{M,j\frac{\boldsymbol{\eta}^{M,j}}{\boldsymbol{x}}} = \phi(\boldsymbol{x}/\boldsymbol{\eta})^{M,j}.$$

Therefore, $\psi^{M,j} = \phi(\boldsymbol{x}/\boldsymbol{\eta})^{M,j} = \psi(q(\boldsymbol{x})/\phi)^{M,j}$.
Subcase 1b: $p \neq q$.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \psi$, so $\psi^{M,j} = \psi(q(\boldsymbol{x})/\phi)^{M,j}$ holds trivially.
Case 2: $\psi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$ and $\alpha(q(\boldsymbol{x})/\phi)$, $\beta(q(\boldsymbol{x})/\phi)$ are admissible.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \alpha(q(\boldsymbol{x})/\phi) \to \beta(q(\boldsymbol{x})/\phi)$. By inductive hypothesis, $\alpha^{M,j} = \alpha(q(\boldsymbol{x})/\phi)^{M,j}$ and $\beta^{M,j} = \beta(q(\boldsymbol{x})/\phi)^{M,j}$. By the definition of formula

evaluation,

$$\begin{aligned}\psi^{M,j} &= 1-\alpha^{M,j}(1-\beta^{M,j})\\ &= 1-\alpha(q(\boldsymbol{x})/\phi)^{M,j}(1-\beta(q(\boldsymbol{x})/\phi)^{M,j})\\ &= (\alpha(q(\boldsymbol{x})/\phi)\to\beta(q(\boldsymbol{x})/\phi))^{M,j}\\ &= \psi(q(\boldsymbol{x})/\phi)^{M,j}.\end{aligned}$$

Case 3: $\psi \doteq \forall u.\gamma$ where $u \in \mathscr{V}$ and $\gamma \in \mathcal{B}(\mathscr{L})$.
Subcase 3a: q does not occur in γ.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \forall u.\gamma = \psi$, so $\psi^{M,j} = \psi(q(\boldsymbol{x})/\phi)^{M,j}$ holds trivially.
Subcase 3b: q occurs in γ. Since $\psi(q(\boldsymbol{x})/\phi)$ is admissible, $u \notin \operatorname{free}(\phi)\setminus\boldsymbol{x}$ and $\gamma(q(\boldsymbol{x})/\phi)$ is admissible.
Then by Definition 2.4.3, $\psi(q(\boldsymbol{x})/\phi) \doteq \forall u.\gamma(q(\boldsymbol{x})/\phi)$.

For any $a \in A$, let $j' = j\frac{a}{u}$. Since $u \notin \operatorname{free}(\phi)\setminus\boldsymbol{x}$, the assignment j' is also an extension of j_0. By inductive hypothesis, $\gamma^{M,j'} = \gamma(q(\boldsymbol{x})/\phi)^{M,j'}$.
This holds for all $a \in A$, therefore by the definition of formula evaluation,

$$\begin{aligned}\psi^{M,j} &= \prod_{a\in A}\gamma^{M,j\frac{a}{u}}\\ &= \prod_{a\in A}\gamma(q(\boldsymbol{x})/\phi)^{M,j\frac{a}{u}}\\ &= (\forall u.\gamma(q(\boldsymbol{x})/\phi))^{M,j}\\ &= \psi(q(\boldsymbol{x})/\phi)^{M,j}.\end{aligned}$$

This completes the proof by structural induction. □

Theorem 2.4.7. *Let $\mathscr{L} = (\mathscr{P},\mathscr{F},\mathfrak{a})$ be a signature of a language and let $q \in \mathscr{P}$. Let $\phi \in \mathcal{F}(\mathscr{L})$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. If $\models \psi$ and $\psi(q(\boldsymbol{x})/\phi)$ is admissible, then $\models \psi(q(\boldsymbol{x})/\phi)$.*

Proof. Let's introduce a new predicate symbol q' such that $q' \notin \mathscr{P}$. Let $\mathscr{L}' = (\mathscr{P}\cup\{q'\},\mathscr{F},\mathfrak{a}')$ such that $\mathfrak{a}'$ is an extension of $\mathfrak{a}$ such that $\mathfrak{a}'(q') = \mathfrak{a}(q)$. Let $k = \mathfrak{a}(q)$. Let ψ' be ψ with each q symbol replaced by q'. Since $\models \psi$, also $\models \psi'$. Take any non-empty domain A. Take any $\mathscr{L}$-structure $\mathbf{M}$ with domain A. Take any $j_0 : \operatorname{free}(\phi)\setminus\boldsymbol{x} \to A$. We will define an $\mathscr{L}'$ structure.

$$\mathbf{M}'_{j_0}(s) = \begin{cases}\mathbf{M}'(s) \text{ for } s \in \mathscr{P}\cup\mathscr{F},\\ \left(A^k \ni \boldsymbol{a} \mapsto \phi^{M,l\frac{\boldsymbol{a}}{\boldsymbol{x}}}\right) \text{ for } s = q'.\end{cases} \tag{2.42}$$

Where $l : \mathscr{V} \to A$ is some fixed extension of j_0 (note that choice of l does not matter for the above definition). Take any $j : \mathscr{V} \to A$ which is an extension of j_0. Since $\models \psi'$, we have $M'_{j_0}, j \models \psi'$. Note that

$$q'(\boldsymbol{x})^{M'_{j_0},j} = (q')^{M'_{j_0}}(\boldsymbol{x}^{M'_{j_0},j}) = \phi^{M'_{j_0},l\frac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}}}. \tag{2.43}$$

Since $l\frac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}}$ and $j\frac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}}$ agrees on $\operatorname{free}(\phi)$, by Theorem 2.2.24 we have

$$\phi^{M'_{j_0},l\frac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}}} = \phi^{M'_{j_0},j\frac{\boldsymbol{x}^{M'_{j_0},j}}{\boldsymbol{x}}} = \phi^{M'_{j_0},j}. \tag{2.44}$$

We have shown that

$$q'(\boldsymbol{x})^{M'_{j_0},j} = \phi^{M'_{j_0},j}. \tag{2.45}$$

Thus by Theorem 2.4.7, we have $M'_{j_0}, j \models \psi'(q(\boldsymbol{x})/\phi)$. Note that $\psi'(q'(\boldsymbol{x})/\phi) \doteq \psi(q(\boldsymbol{x})/\phi)$ and therefore $\psi'(q'(\boldsymbol{x})/\phi) \in \mathcal{F}(\mathscr{L})$. Then by construction of M'_{j_0}, we have $M, j \models \psi'(q'(\boldsymbol{x})/\phi)$, and consequently

$$M, j \models \psi(q(\boldsymbol{x})/\phi). \tag{2.46}$$

Since $j_0 : \text{free}(\phi) \setminus \boldsymbol{x} \to A$ was arbitrarily chosen, (2.46) holds for arbirary $j : \mathscr{V} \to A$. This completes the proof. □

2.5 Preliminaries for Rules

Theorem 2.5.1 (modus ponens). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$ be an assignment Let $\alpha, \beta \in \mathcal{F}(\mathscr{L})$. If $\mathbf{M}, j \models \alpha$ and $\mathbf{M}, j \models \alpha \to \beta$, then $\mathbf{M}, j \models \beta$.*

Proof. We have $\mathbf{M}, j \models \alpha$, which means $\alpha^{\mathbf{M},j} = 1$.
Since $\Gamma \models \alpha \to \beta$, we have $\mathbf{M}, j \models \alpha \to \beta$, which means $(\alpha \to \beta)^{\mathbf{M},j} = 1$.
By the definition of formula evaluation for implication,

$$(\alpha \to \beta)^{\mathbf{M},j} = 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}) = 1. \tag{2.47}$$

Since $\alpha^{\mathbf{M},j} = 1$, we have

$$1 - 1 \cdot (1 - \beta^{\mathbf{M},j}) = 1, \tag{2.48}$$

which simplifies to $1 - (1 - \beta^{\mathbf{M},j}) = 1$, hence $\beta^{\mathbf{M},j} = 1$.
Therefore, $\mathbf{M}, j \models \beta$. □

Theorem 2.5.2 (generalization). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\mathbf{M}$ be an $\mathscr{L}$-structure. Let $v \in \mathscr{V}$ and $\phi \in \mathcal{B}(\mathscr{L})$. If $\mathbf{M} \models \phi$, then $\mathbf{M} \models \forall v.\phi$.*

Proof. Take any non-empty domain A and any $\mathscr{L}$-structure $\mathbf{M}$ with domain A, such that $\mathbf{M} \models \phi$. Take any $j : \mathscr{V} \to A$. We need to show that $\mathbf{M}, j \models \forall v.\phi$.
By the definition of formula evaluation for universal quantification,

$$(\forall v.\phi)^{\mathbf{M},j} = \prod_{a \in A} \phi^{\mathbf{M},j\frac{a}{v}}. \tag{2.49}$$

Since $\mathbf{M} \models \phi$, we have $\mathbf{M}, j\frac{a}{v} \models \phi$, which means $\phi^{\mathbf{M},j\frac{a}{v}} = 1$.
This holds for all $a \in A$, therefore

$$(\forall v.\phi)^{\mathbf{M},j} = \prod_{a \in A} \phi^{\mathbf{M},j\frac{a}{v}} = \prod_{a \in A} 1 = 1. \tag{2.50}$$

Hence $\mathbf{M}, j \models \forall v.\phi$. And since $j : \mathscr{V} \to A$ was arbitrary, then $\mathbf{M} \models \forall v.\phi$. □

Definition 2.5.3. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A. For any $S \subset \mathscr{P} \cup \mathscr{F}$,*

$$\begin{aligned} \mathscr{M}_S(\mathbf{M}) \stackrel{\text{def}}{=} \{\mathbf{M}' : \mathbf{M}' & \textit{is } \mathscr{L}\textit{-structure with domain } A \\ & \textit{and } \mathbf{M}'(s) = \mathbf{M}(s) \textit{ for all } s \in (\mathscr{P} \cup \mathscr{F}) \setminus S \ \}. \end{aligned} \tag{2.51}$$

For $S = \{s\}$, we will write $\mathscr{M}_s$ instead of $\mathscr{M}_{\{s\}}$ as long as not ambiguous.

Definition 2.5.4. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ and $S \subset \mathscr{P} \cup \mathscr{F}$. Let $\phi \in \mathcal{F}(\mathscr{L})$. We define that ϕ is* ***assignment-satisfable with respect to*** *S iff for any $\mathscr{L}$-structure $\mathbf{M}$ with domain A and any assignment $j : \mathscr{V} \to \mathcal{A}$, there exists $\mathbf{M}' \in \mathscr{M}_S(\mathbf{M})$ such that $\mathbf{M}', j \models \phi$.*

Lemma 2.5.5. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $S \subset \mathscr{P} \cup \mathscr{F}$. Let $\phi, \psi \in \mathcal{B}(\mathscr{L})$ be such that ϕ is assignment-satisfable with respect to S and no symbol from S occurs in ψ. Let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$. Then*

$$\mathbf{M}', j \models \phi \to \psi \textit{ for any } \mathbf{M}' \in \mathscr{M}_S(\mathbf{M}) \implies \mathbf{M}, j \models \psi. \tag{2.52}$$

Proof. Assume $\mathbf{M}', j \models \phi \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_S(\mathbf{M})$.
Since ϕ is assignment-satisfable with respect to S, for the given structure $\mathbf{M}$ with domain A and assignment $j : \mathscr{V} \to A$, there exists $\mathbf{M}'' \in \mathscr{M}_S(\mathbf{M})$ such that $\mathbf{M}'', j \models \phi$.
By assumption, since $\mathbf{M}'' \in \mathscr{M}_S(\mathbf{M})$, we have $\mathbf{M}'', j \models \phi \to \psi$.
Since $\mathbf{M}'', j \models \phi$ and $\mathbf{M}'', j \models \phi \to \psi$, by Theorem 2.5.1, we have $\mathbf{M}'', j \models \psi$.
Since no symbol from S occurs in ψ, and $\mathbf{M}'' \in \mathscr{M}_S(\mathbf{M})$ means $\mathbf{M}''$ agrees with $\mathbf{M}$ on all symbols outside S, we have $\psi^{\mathbf{M}'', j} = \psi^{\mathbf{M}, j}$.
Therefore $\mathbf{M}, j \models \psi$. □

Lemma 2.5.6. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and let $S \subset \mathscr{P} \cup \mathscr{F}$. Let $\phi, \psi \in \mathcal{B}(\mathscr{L})$ be such that ϕ is assignment-satisfable and no symbol from S occurs in ψ. Let $\mathbf{M}$ be an $\mathscr{L}$-structure. Then*

$$\mathbf{M}' \models \phi \to \psi \textit{ for any } \mathbf{M}' \in \mathscr{M}_S(\mathbf{M}) \implies \mathbf{M} \models \psi. \tag{2.53}$$

Proof. Assume $\mathbf{M}' \models \phi \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_S(\mathbf{M})$.
Let A be the domain of $\mathbf{M}$. Take an arbitrary assignment $j : \mathscr{V} \to A$.
By assumption, for any $\mathbf{M}' \in \mathscr{M}_S(\mathbf{M})$, we have $\mathbf{M}' \models \phi \to \psi$. In particular, $\mathbf{M}', j \models \phi \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_S(\mathbf{M})$.
By Lemma 2.5.5, since ϕ is assignment-satisfable, no symbol from S occurs in ψ, and $\mathbf{M}', j \models \phi \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_S(\mathbf{M})$, we have $\mathbf{M}, j \models \psi$.
Since j was arbitrary, we conclude $\mathbf{M} \models \psi$. □

Lemma 2.5.7. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathscr{P}_k$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Let $\phi \in \mathcal{B}(\mathscr{L})$ such that q does not occur in ϕ. Then $q(\boldsymbol{x}) \leftrightarrow \phi$ is assignment-satisfable with respect to $\{q\}$.*

Proof. Take any $\mathscr{L}$-structure $\mathbf{M}$ with domain A and any assignment $j : \mathscr{V} \to A$.
We construct a new structure $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$ as follows: for the predicate q, we define

$$q^{\mathbf{M}'}(\boldsymbol{a}) = \phi^{\mathbf{M}, j \frac{\boldsymbol{a}}{\boldsymbol{x}}} \tag{2.54}$$

for any $\boldsymbol{a} \in A^k$.
For all other symbols $s \in (\mathscr{P} \cup \mathscr{F}) \setminus \{q\}$, we have $s^{\mathbf{M}'} = s^{\mathbf{M}}$.
Note that $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$ by construction.
We need to show that $\mathbf{M}', j \models q(\boldsymbol{x}) \leftrightarrow \phi$.

We have

$$q(\boldsymbol{x})^{\mathbf{M}',j} = q^{\mathbf{M}'}(\boldsymbol{x}^{\mathbf{M}',j}) = \phi^{\mathbf{M},j\frac{\boldsymbol{x}^{\mathbf{M}',j}}{\boldsymbol{x}}}. \tag{2.55}$$

Since $\boldsymbol{x}$ are distinct variables, we have $j\frac{\boldsymbol{x}^{\mathbf{M}',j}}{\boldsymbol{x}} = j$, thus

$$q(\boldsymbol{x})^{\mathbf{M}',j} = \phi^{\mathbf{M},j}. \tag{2.56}$$

Since q does not occur in ϕ, and $\mathbf{M}'$ agrees with $\mathbf{M}$ on all symbols except q, we have $\phi^{\mathbf{M},j} = \phi^{\mathbf{M}',j}$.
Therefore $q(\boldsymbol{x})^{\mathbf{M}',j} = \phi^{\mathbf{M}',j}$.
By Theorem 2.3.5, this means $\mathbf{M}', j \models q(\boldsymbol{x}) \leftrightarrow \phi$.
Therefore, $q(\boldsymbol{x}) \leftrightarrow \phi$ is assignment-satisfable with respect to $\{q\}$. $\square$

Lemma 2.5.8. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathscr{P}_k$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Let $\phi, \psi \in \mathcal{B}(\mathscr{L})$ such that q does not occur in both ϕ and ψ.*
Let $\mathbf{M}$ be $\mathscr{L}$-structure. Then

$$\mathbf{M}' \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi \text{ for any } \mathbf{M}' \in \mathscr{M}_q(\mathbf{M}) \implies \mathbf{M} \models \psi. \tag{2.57}$$

Lemma 2.5.9. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $f \in \mathscr{F}_k$, $x \in \mathscr{V}$ and $\boldsymbol{u} \in \mathscr{V}^{\nabla k}$. Let $\phi \in \mathcal{B}(\mathscr{L})$ such that f does not occur in ϕ and $\phi(x/f(\boldsymbol{u}))$ is admissible. Then $(\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))$ is assignment-satisfable with respect to $\{f\}$.*

Proof. Take any $\mathscr{L}$-structure $\mathbf{M}$ with domain A and any assignment $j : \mathscr{V} \to A$. Let's fix some $a^* \in A$. We construct a new structure $\mathbf{M}' \in \mathscr{M}_f(\mathbf{M})$ as follows: for the function f, we define

$$f^{\mathbf{M}'}(\boldsymbol{a}) = \begin{cases} \text{chosen } a_0 & \text{if } \phi^{\mathbf{M},j\frac{\boldsymbol{a}}{\boldsymbol{u}}\frac{a_0}{x}} = 1 \text{ for some } a_0 \in A, \\ a^* & \text{otherwise.} \end{cases} \tag{2.58}$$

for any $\boldsymbol{a} \in A^k$.
For all other symbols $s \in (\mathscr{P} \cup \mathscr{F}) \setminus \{f\}$, we have $s^{\mathbf{M}'} = s^{\mathbf{M}}$.
Note that $\mathbf{M}' \in \mathscr{M}_f(\mathbf{M})$ by construction.
We need to show that $\mathbf{M}', j \models (\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))$.
Case 1: $(\exists x.\phi)^{\mathbf{M}',j} = 0$.
Then trivially $\mathbf{M}', j \models (\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))$.
Case 2: $(\exists x.\phi)^{\mathbf{M}',j} = 1$.
Since f does not occur in ϕ, and $\mathbf{M}'$ agrees with $\mathbf{M}$ on all symbols except f, we have $\phi^{\mathbf{M}',j\frac{a}{x}} = \phi^{\mathbf{M},j\frac{a}{x}}$ for any $a \in A$.

By Theorem 2.3.6, there exists some $a_0 \in A$ such that $\phi^{\mathbf{M}',j\frac{a_0}{x}} = 1$, which means $\phi^{\mathbf{M},j\frac{\boldsymbol{u}^{\mathbf{M},j}}{\boldsymbol{u}}\frac{a_0}{x}} = \phi^{\mathbf{M},j\frac{a_0}{x}} = 1$.
Now, we have $f^{\mathbf{M}'}(\boldsymbol{u}^{\mathbf{M}',j}) = a_0$ by construction (choosing the witness a_0).
We have $\phi^{\mathbf{M},j\frac{\boldsymbol{u}^{\mathbf{M},j}}{\boldsymbol{u}}\frac{a_0}{x}} = \phi^{\mathbf{M},j\frac{a_0}{x}}$.

By Theorem 2.4.5, since $\phi(x/f(\boldsymbol{u}))$ is admissible,

$$\phi(x/f(\boldsymbol{u}))^{\mathbf{M}',j} = \phi^{\mathbf{M}',j\frac{f(\boldsymbol{u})^{\mathbf{M}',j}}{x}}. \tag{2.59}$$

Since $f(\boldsymbol{u})^{\mathbf{M}',j} = f^{\mathbf{M}'}(\boldsymbol{u}^{\mathbf{M}',j}) = a_0$, we have

$$\phi(x/f(\boldsymbol{u}))^{\mathbf{M}',j} = \phi^{\mathbf{M}',j\frac{a_0}{x}} = \phi^{\mathbf{M},j\frac{a_0}{x}} = 1. \tag{2.60}$$

Therefore $\mathbf{M}', j \models (\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))$.
Thus $(\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))$ is assignment-satisfable with respect to $\{f\}$. □

Lemma 2.5.10. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $f \in \mathscr{F}_k$, $x \in \mathscr{V}$ and $\boldsymbol{u} \in \mathscr{V}^{\nabla k}$. Let $\phi, \psi \in \mathcal{B}(\mathscr{L})$ such that f does not occur in both ϕ and ψ, and $\phi(x/f(\boldsymbol{u}))$ is admissible. Let $\mathbf{M}$ be an $\mathscr{L}$-structure. Then*

$$\mathbf{M}' \models ((\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))) \to \psi \textit{ for any } \mathbf{M}' \in \mathscr{M}_f(\mathbf{M}) \implies \mathbf{M} \models \psi. \tag{2.61}$$

Proof. By Lemma 2.5.9, $(\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))$ is assignment-satisfable with respect to $\{f\}$.
Since f does not occur in ψ, by Lemma 2.5.6 with $S = \{f\}$ and ϕ replaced by $(\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))$, we have: if $\mathbf{M}' \models ((\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))) \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_f(\mathbf{M})$, then $\mathbf{M} \models \psi$. □

Proof. By Lemma 2.5.7, $q(\boldsymbol{x}) \leftrightarrow \phi$ is assignment-satisfable.
Since q does not occur in ψ, by Lemma 2.5.6 with $S = \{q\}$ and ϕ replaced by $q(\boldsymbol{x}) \leftrightarrow \phi$, we have: if $\mathbf{M}' \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$, then $\mathbf{M} \models \psi$. □

Next, we will define a relation $\mathrm{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$. The intuition behind is that $\mathrm{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$ holds when ψ_2 is created from ψ_1 by replacing some or none ϕ_1 which occurs in ψ_1 by ϕ_2.

Definition 2.5.11 (replace relation). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $\phi_1, \phi_2 \in \mathcal{F}(\mathscr{L})$.*

Let $\psi_1, \psi_2 \in \mathcal{F}(\mathscr{L})$. We will define recursively $rep_{\phi_1,\phi_2}(\psi_1, \psi_2)$.

1. *If $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$, then $rep_{\phi_1,\phi_2}(\psi_1, \psi_2)$ holds.*
2. *If $\psi_1 \doteq p(\boldsymbol{\eta})$ and $\psi_2 \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$, then $rep_{\phi_1,\phi_2}(\psi_1, \psi_2)$ holds.*
3. *If $\psi_1 \doteq \alpha_1 \to \beta_1$ and $\psi_2 \doteq \alpha_2 \to \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathscr{L})$, then $rep_{\phi_1,\phi_2}(\psi_1, \psi_2)$ iff $rep_{\phi_1,\phi_2}(\alpha_1, \alpha_2)$ and $rep_{\phi_1,\phi_2}(\beta_1, \beta_2)$.*
4. *If $\psi_1 \doteq \forall u.\gamma_1$ and $\psi_2 \doteq \forall u.\gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathscr{L})$, then $rep_{\phi_1,\phi_2}(\psi_1, \psi_2)$ iff $rep_{\phi_1,\phi_2}(\gamma_1, \gamma_2)$.*
5. *Otherwise $rep_{\phi_1,\phi_2}(\psi_1, \psi_2)$ does not hold.*

Theorem 2.5.12. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary non-empty mathematical domain. Let $\phi_1, \phi_2, \psi_1, \psi_2 \in \mathcal{B}(\mathscr{L})$ such that $rep_{\phi_1,\phi_2}(\psi_1, \psi_2)$. For any $\mathscr{L}$-structure $\mathbf{M}$,*

$$\mathbf{M} \models \phi_1 \leftrightarrow \phi_2 \implies \mathbf{M} \models \psi_1 \leftrightarrow \psi_2. \tag{2.62}$$

Proof. We proceed by structural induction on the relation $\mathrm{rep}_{\phi_1,\phi_2}(\psi_1,\psi_2)$.
Assume that $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathscr{V} \to A$. By Theorem 2.3.5, this means $\phi_1^{\mathbf{M},j} = \phi_2^{\mathbf{M},j}$ for any $j : \mathscr{V} \to A$.
Take an arbitrary $j : \mathscr{V} \to A$. We will show that $\psi_1^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$.
Case 1: $\psi_1 \doteq \phi_1$ and $\psi_2 \doteq \phi_2$.
Then $\psi_1^{\mathbf{M},j} = \phi_1^{\mathbf{M},j} = \phi_2^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$.
By Theorem 2.3.5, $\psi_1^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$ implies $M, j \models \psi_1 \leftrightarrow \psi_2$.
Case 2: $\psi_1 \doteq p(\boldsymbol{\eta})$ and $\psi_2 \doteq p(\boldsymbol{\eta})$ where $\boldsymbol{\eta} \in \mathcal{T}(\mathscr{L})^*$ and $p \in \mathscr{P}_{|\boldsymbol{\eta}|}$.
Then $\psi_1 \doteq \psi_2$, so trivially $\psi_1^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}$ and $M, j \models \psi_1 \leftrightarrow \psi_2$.
Case 3: $\psi_1 \doteq \alpha_1 \to \beta_1$ and $\psi_2 \doteq \alpha_2 \to \beta_2$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{B}(\mathscr{L})$, and $\mathrm{rep}_{\phi_1,\phi_2}(\alpha_1,\alpha_2)$ and $\mathrm{rep}_{\phi_1,\phi_2}(\beta_1,\beta_2)$.
By the inductive hypothesis, since $M, j \models \phi_1 \leftrightarrow \phi_2$ for any $j : \mathscr{V} \to A$, we have:

- $M, j \models \alpha_1 \leftrightarrow \alpha_2$ for any $j : \mathscr{V} \to A$, which means $\alpha_1^{\mathbf{M},j} = \alpha_2^{\mathbf{M},j}$ for any $j : \mathscr{V} \to A$.
- $M, j \models \beta_1 \leftrightarrow \beta_2$ for any $j : \mathscr{V} \to A$, which means $\beta_1^{\mathbf{M},j} = \beta_2^{\mathbf{M},j}$ for any $j : \mathscr{V} \to A$.

By the definition of formula evaluation for implication:

$$\begin{aligned}\psi_1^{\mathbf{M},j} &= (\alpha_1 \to \beta_1)^{\mathbf{M},j} = 1 - \alpha_1^{\mathbf{M},j}(1 - \beta_1^{\mathbf{M},j})\\ &= 1 - \alpha_2^{\mathbf{M},j}(1 - \beta_2^{\mathbf{M},j}) = (\alpha_2 \to \beta_2)^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}.\end{aligned}$$

Therefore, $M, j \models \psi_1 \leftrightarrow \psi_2$.
Case 4: $\psi_1 \doteq \forall u.\gamma_1$ and $\psi_2 \doteq \forall u.\gamma_2$ where $\gamma_1, \gamma_2 \in \mathcal{B}(\mathscr{L})$, and $\mathrm{rep}_{\phi_1,\phi_2}(\gamma_1,\gamma_2)$.
By the inductive hypothesis, since $M, j' \models \phi_1 \leftrightarrow \phi_2$ for any $j' : \mathscr{V} \to A$, we have $M, j' \models \gamma_1 \leftrightarrow \gamma_2$ for any $j' : \mathscr{V} \to A$.
In particular, for any $a \in A$, taking $j' = j\frac{a}{u}$, we have $M, j\frac{a}{u} \models \gamma_1 \leftrightarrow \gamma_2$, which means $\gamma_1^{\mathbf{M},j\frac{a}{u}} = \gamma_2^{\mathbf{M},j\frac{a}{u}}$.
By the definition of formula evaluation for universal quantification:

$$\begin{aligned}\psi_1^{\mathbf{M},j} &= (\forall u.\gamma_1)^{\mathbf{M},j} = \prod_{a\in A} \gamma_1^{\mathbf{M},j\frac{a}{u}}\\ &= \prod_{a\in A} \gamma_2^{\mathbf{M},j\frac{a}{u}} = (\forall u.\gamma_2)^{\mathbf{M},j} = \psi_2^{\mathbf{M},j}.\end{aligned}$$

Therefore, $M, j \models \psi_1 \leftrightarrow \psi_2$.
Since this holds for arbitrary $j : \mathscr{V} \to A$, we conclude that $M, j \models \psi_1 \leftrightarrow \psi_2$ for any $j : \mathscr{V} \to A$.
This completes the proof by structural induction. □

Lemma 2.5.13. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $c \in \mathscr{F}_0$, and $x \in \mathscr{V}$. Let $\psi \in \mathcal{F}(\mathscr{L})$ such that c does not occur in ψ. Let $\mathbf{M}$ be $\mathscr{L}$-structure. Then*

$$\mathbf{M}' \models \psi(x/c) \text{ for any } \mathbf{M}' \in \mathscr{M}_c(\mathbf{M}) \implies \mathbf{M} \models \psi. \tag{2.63}$$

Proof. Since $c \in \mathcal{F}_0$, $\psi(x/c)$ is always an admissible substitution.
Take any $\mathscr{L}$-structure $\mathbf{M}$ with domain A. Assume that assumption of implication (2.63) holds. We need to show that $\mathbf{M} \models \psi$. Take any assignment $j : \mathscr{V} \to A$. We need to show that $\mathbf{M}, j \models \psi$.

We construct $\mathbf{M'}$ such that:

$$c^{\mathbf{M'}} = j(x), \tag{2.64}$$

and for all other predicates and functions, $\mathbf{M'}$ agrees with $\mathbf{M}$. Note that $\mathbf{M'} \in \mathscr{M}_c(\mathbf{M})$. By assumption of implication (2.63), we have $\mathbf{M'} \models \psi(x/c)$. In particular, $\mathbf{M'}, j \models \psi(x/c)$.

By Theorem 2.4.5, since $\psi(x/c)$ is an admissible substitution, we have

$$\mathbf{M'}, j \models \psi(x/c) \iff \mathbf{M'}, j\frac{c^{\mathbf{M'},j}}{x} \models \psi. \tag{2.65}$$

Since $c^{\mathbf{M'},j} = c^{\mathbf{M'}} = j(x)$, we have

$$j\frac{j(x)}{x} = j. \tag{2.66}$$

Therefore, $\mathbf{M'}, j \models \psi$. Since c does not occur in ψ, and $\mathbf{M'}$ differs from $\mathbf{M}$ only in the interpretation of c, we have $\psi^{\mathbf{M'},j} = \psi^{\mathbf{M},j}$. Therefore, $\mathbf{M}, j \models \psi$. Since this holds for any assignment $j : \mathscr{V} \to A$, we have $\mathbf{M} \models \psi$. $\square$

Chapter 3

Inference Rules

Let $\mathscr{F}^g_k$ be an ininite and countable set of special function symbols of arity k for $k = 0, \ldots$.
Let $\mathscr{P}^g_k$ be an ininite and countable set of special predicate symbols of arity k for $k = 0, \ldots$.

$$\mathscr{F}^g \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \mathscr{F}^g_k \tag{3.1}$$

$$\mathscr{P}^g \stackrel{\text{def}}{=} \bigcup_{k=0}^{\infty} \mathscr{P}^g_k \tag{3.2}$$

Definition 3.0.1.

$$\mathscr{L}^g \stackrel{\text{def}}{=} (\mathscr{F}^g, \mathscr{P}^g, \mathfrak{a}^g), \tag{3.3}$$

where $\mathfrak{a}^g(p) = k$ *for any* $p \in \mathscr{P}^g_k$ *and* $\mathfrak{a}^g(f) = k$ *for any* $f \in \mathscr{F}^g_k$ *for any integer* $k \geq 0$.

Definition 3.0.2. *Let* $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ *be signature of a language and* $\mathscr{F}^g \cap \mathscr{F} = \emptyset$ *and* $\mathscr{P}^g \cap \mathscr{P} = \emptyset$. *We will consider extention of the language with signature*

$$\mathscr{L}^G \stackrel{\text{def}}{=} (\mathscr{F}^g \cup \mathscr{F}, \mathscr{P}^g \cup \mathscr{P}, \mathfrak{a} \cup \mathfrak{a}^g). \tag{3.4}$$

Note that $\mathscr{L}^G$ is also a signature of a language and $\mathscr{L} \subset \mathscr{L}^G$.
When we consider $\mathscr{L}^G$-structures $\mathbf{M}_1, \mathbf{M}_2$, we say that they agree on $\mathscr{L}$ iff $\mathbf{M}_1(\mathscr{M} \cup \mathscr{F}) = \mathbf{M}_2(\mathscr{M} \cup \mathscr{F})$ (i.e. $M'|_{\mathscr{L}} = M|_{\mathscr{L}}$).

Definition 3.0.3 (**modus ponens**). *Let* $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ *be a signature of a language. We will define a function* $Mod_{\mathscr{L}} : \mathcal{F}(\mathscr{L}) \times \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$.

$$Mod_{\mathscr{L}}(\psi, \alpha) \stackrel{\text{def}}{=} \begin{cases} \beta \text{ for } \psi \doteq \alpha \to \beta \text{ and } \alpha, \beta \in \mathcal{B}(\mathscr{L}), \\ \mathfrak{t} \text{ otherwise.} \end{cases} \tag{3.5}$$

Definition 3.0.4 (**generalisation**). *Let* $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ *be a signature of a language. We will define a family of functions* $Gen^x_{\mathscr{L}} : \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$ *for* $x \in \mathscr{V}$.

$$Gen^x_{\mathscr{L}}(\psi) \stackrel{\text{def}}{=} \begin{cases} \forall x.\psi \text{ for } \psi \in \mathcal{B}(\mathscr{L}), \\ \mathfrak{t} \text{ otherwise.} \end{cases} \tag{3.6}$$

Definition 3.0.5 (**replacement**). *Let* $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ *be a signature of a language. We will define a family of functions* $Rep_{\mathscr{L}}^{\psi_1,\psi_2} : \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$ *for* $\psi_1, \psi_2 \in \mathcal{B}(\mathscr{L})$.

$$Rep_{\mathscr{L}}^{\psi_1,\psi_2}(\phi) \stackrel{\text{def}}{=} \begin{cases} \psi_1 \leftrightarrow \psi_2 \text{ for } \phi \doteq \phi_1 \leftrightarrow \phi_2 \text{ where } \phi_1, \phi_2 \in \mathcal{B}(\mathcal{F}) \text{ and } rep_{\phi_1,\phi_2}(\psi_1,\psi_2), \\ \mathfrak{t} \text{ otherwise.} \end{cases} \tag{3.7}$$

Definition 3.0.6 (**definition reduction**). *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. Let* k *be a non-negative integer.*
We will define a family of functions $Red_{\mathscr{L}}^{q(\boldsymbol{x})} : \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$ *for* $q \in \mathscr{P}_k$ *and* $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$.

$$Red_{\mathscr{L}}^{q(\boldsymbol{x})}(\gamma) \stackrel{\text{def}}{=} \begin{cases} \psi \text{ for } \gamma \doteq (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi \text{ where } \psi, \phi \in \mathcal{B}(\mathscr{L}) \\ \qquad\qquad\qquad\qquad \text{and } q \text{ does not occur in } \psi \text{ and } \phi, \\ \mathfrak{t} \text{ otherwise.} \end{cases} \tag{3.8}$$

Definition 3.0.7 (**Skolemization schema**). *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. Let* k *be a non-negative integer.*
We will define a family of functions $Sko_{\mathscr{L}}^{f(\boldsymbol{u}),x} : \mathcal{F}(\mathscr{L}) \to \mathcal{F}(\mathscr{L})$ *for* $f \in \mathscr{F}_k$, $\boldsymbol{u} \in \mathscr{V}^{\nabla k}$, *and* $x \in \mathscr{V}$.

$$Sko_{\mathscr{L}}^{f(\boldsymbol{u}),x}(\gamma) \stackrel{\text{def}}{=} \begin{cases} \psi \text{ for } \gamma \doteq ((\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))) \to \psi \text{ where } \psi, \phi \in \mathcal{B}(\mathscr{L}), \\ \qquad\qquad\qquad\qquad f \text{ does not occur in } \psi \text{ and } \phi, \\ \qquad\qquad\qquad\qquad \text{and } \phi(x/f(\boldsymbol{u})) \text{ is admissible,} \\ \mathfrak{t} \text{ otherwise.} \end{cases} \tag{3.9}$$

Chapter 4

Abstract Rules Theory

Definition 4.0.1. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k_1, k_2 be non-negative integer. We will say that r is an inference rule on $\mathcal{F}(\mathscr{L})$ iff $r : \mathcal{F}(\mathscr{L})^{k_1} \to \mathcal{F}(\mathscr{L})^{k_2}$. Moreover, we will define $\dim r \overset{\text{def}}{=} (k_1, k_2)$.*

Definition 4.0.2. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathscr{L})^{k_1} \to \mathcal{F}(\mathscr{L})^{k_2}$, we will say that r is **strictly truth-preserving** iff for any $\mathscr{L}$-structure $\mathbf{M}$ and any $\boldsymbol{\phi} \in \mathcal{F}(\mathscr{L})^{k_1}$*

$$\mathbf{M} \models \boldsymbol{\phi} \implies \mathbf{M} \models r(\boldsymbol{\phi}). \tag{4.1}$$

Theorem 4.0.3. *Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language. Then $Mod_{\mathscr{L}}$ is strictly truth preserving.*

Proof. Take any $\mathscr{L}$-structure $\mathbf{M}$ and any $(\psi, \alpha) \in \mathcal{F}(\mathscr{L})^2$ such that $\mathbf{M} \models (\psi, \alpha)$.
By definition, $\mathbf{M} \models (\psi, \alpha)$ means $\mathbf{M} \models \psi$ and $\mathbf{M} \models \alpha$.
We need to show that $\mathbf{M} \models \mathrm{Mod}_{\mathscr{L}}(\psi, \alpha)$.
Case 1: $\psi \doteq \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.
Then $\mathrm{Mod}_{\mathscr{L}}(\psi, \alpha) = \beta$.
Take any assignment $j : \mathscr{V} \to A$. We need to show that $\mathbf{M}, j \models \beta$.
Since $\mathbf{M} \models \alpha$, we have $\mathbf{M}, j \models \alpha$.
Since $\mathbf{M} \models \psi$, we have $\mathbf{M}, j \models \alpha \to \beta$.
By Theorem 2.5.1, since $\mathbf{M}, j \models \alpha$ and $\mathbf{M}, j \models \alpha \to \beta$, we have $\mathbf{M}, j \models \beta$.
Since this holds for any assignment $j : \mathscr{V} \to A$, we have $\mathbf{M} \models \beta = \mathrm{Mod}_{\mathscr{L}}(\psi, \alpha)$.
Case 2: Otherwise.
Then $\mathrm{Mod}_{\mathscr{L}}(\psi, \alpha) = \mathfrak{t}$, which is trivially satisfied by any structure. □

Theorem 4.0.4. *Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language and let $x \in \mathscr{V}$. Then $Gen^x_{\mathscr{L}}$ is strictly truth preserving.*

Proof. Take any $\mathscr{L}$-structure $\mathbf{M}$ and any $\psi \in \mathcal{F}(\mathscr{L})$ such that $\mathbf{M} \models \psi$.
We need to show that $\mathbf{M} \models \mathrm{Gen}^x_{\mathscr{L}}(\psi)$.
Case 1: $\psi \in \mathcal{B}(\mathscr{L})$.
Then $\mathrm{Gen}^x_{\mathscr{L}}(\psi) = \forall x.\psi$.
By Theorem 2.5.2, since $\mathbf{M} \models \psi$, we have $\mathbf{M} \models \forall x.\psi = \mathrm{Gen}^x_{\mathscr{L}}(\psi)$.
Case 2: $\psi \notin \mathcal{B}(\mathscr{L})$.
Then $\mathrm{Gen}^x_{\mathscr{L}}(\psi) = \mathfrak{t}$, which is trivially satisfied by any structure. □

Theorem 4.0.5. *Let $\mathscr{L} = (\mathscr{F}, \mathscr{P}, \mathfrak{a})$ be a signature of a language and let $\psi_1, \psi_2 \in \mathcal{B}(\mathscr{L})$. Then $Rep^{\psi_1,\psi_2}_{\mathscr{L}}$ is strictly truth preserving.*

Proof. Take any $\mathscr{L}$-structure $\mathbf{M}$ and any $\phi \in \mathcal{F}(\mathscr{L})$ such that $\mathbf{M} \models \phi$.
We need to show that $\mathbf{M} \models \mathrm{Rep}_{\mathscr{L}}^{\psi_1,\psi_2}(\phi)$.
Case 1: $\phi \doteq \phi_1 \leftrightarrow \phi_2$ where $\phi_1, \phi_2 \in \mathcal{B}(\mathscr{L})$ and $\mathrm{rep}_{\phi_1,\phi_2}(\psi_1, \psi_2)$.
Then $\mathrm{Rep}_{\mathscr{L}}^{\psi_1,\psi_2}(\phi) = \psi_1 \leftrightarrow \psi_2$.
Since $\mathbf{M} \models \phi_1 \leftrightarrow \phi_2$, by Theorem 2.5.12, we have $\mathbf{M} \models \psi_1 \leftrightarrow \psi_2 = \mathrm{Rep}_{\mathscr{L}}^{\psi_1,\psi_2}(\phi)$.
Case 2: Otherwise.
Then $\mathrm{Rep}_{\mathscr{L}}^{\psi_1,\psi_2}(\phi) = \mathfrak{t}$, which is trivially satisfied by any structure. □

Definition 4.0.6. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathscr{L}^G)^{k_1} \to \mathcal{F}(\mathscr{L}^G)^{k_2}$, we will say that r is **weakly truth-preserving** iff for any $\mathscr{L}^G$-structure $\mathbf{M}$ and any $\boldsymbol{\phi} \in \mathcal{F}(\mathscr{L}^G)^{k_1}$*

$$\mathbf{M}' \models \boldsymbol{\phi} \text{ for any } \mathscr{L}^G\text{-structure } \mathbf{M}' \text{ for which } \mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}} \implies \mathbf{M} \models r(\boldsymbol{\phi}). \tag{4.2}$$

Corollary 4.0.7. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k_1, k_2 be non-negative integers and let $r : \mathcal{F}(\mathscr{L})^{k_1} \to \mathcal{F}(\mathscr{L})^{k_2}$. If r is strictly truth-preserving, then r is weakly truth-preserving.*

Proof. Assume r is strictly truth-preserving.
Take any $\mathscr{L}^G$-structure $\mathbf{M}$ and any $\boldsymbol{\phi} \in \mathcal{F}(\mathscr{L})^{k_1}$ such that $\mathbf{M}' \models \boldsymbol{\phi}$ for all $\mathscr{L}^G$-structure $\mathbf{M}'$ for which $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$.
In particular, taking $\mathbf{M}' = \mathbf{M}$, we have $\mathbf{M} \models \boldsymbol{\phi}$.
Since r is strictly truth-preserving and $\mathbf{M} \models \boldsymbol{\phi}$, we have $\mathbf{M} \models r(\boldsymbol{\phi})$.
Therefore, r is weakly truth-preserving. □

Theorem 4.0.8. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $q \in \mathscr{P}_k^g$ and $\boldsymbol{x} \in \mathscr{V}^{\nabla k}$. Then $\mathrm{Red}_{\mathscr{L}^G}^{q(\boldsymbol{x})}$ is weakly truth-preserving.*

Proof. Take any $\mathscr{L}^G$-structure $\mathbf{M}$ and any $\gamma \in \mathcal{F}(\mathscr{L}^G)$ such that $\mathbf{M}' \models \gamma$ for all $\mathscr{L}^G$-structure $\mathbf{M}'$ for which $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$.
We need to show that $\mathbf{M} \models \mathrm{Red}_{\mathscr{L}^G}^{q(\boldsymbol{x})}(\gamma)$.
Case 1: $\gamma \doteq (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$ where $\psi, \phi \in \mathcal{B}(\mathscr{L})$ and q does not occur in ψ and ϕ.
Then $\mathrm{Red}_{\mathscr{L}}^{q(\boldsymbol{x})}(\gamma) = \psi$.
We need to show that $\mathbf{M} \models \psi$.
Since $q \in \mathscr{P}_k^g$ and $q \notin \mathscr{P}$, for any $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$ (where $\mathbf{M}'$ is an $\mathscr{L}^G$-structure), we have $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$, so by assumption $\mathbf{M}' \models \gamma$, which means $\mathbf{M}' \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$.
By Lemma 2.5.8, since $\mathbf{M}' \models (q(\boldsymbol{x}) \leftrightarrow \phi) \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_q(\mathbf{M})$, we have $\mathbf{M} \models \psi$.
Therefore, $\mathbf{M} \models \mathrm{Red}_{\mathscr{L}^G}^{q(\boldsymbol{x})}(\gamma)$.
Case 2: Otherwise.
Then $\mathrm{Red}_{\mathscr{L}}^{q(\boldsymbol{x})}(\gamma) = \mathfrak{t}$, which is trivially satisfied by any structure.
Therefore, $\mathrm{Red}_{\mathscr{L}^G}^{q(\boldsymbol{x})}$ is weakly truth-preserving. □

Theorem 4.0.9. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let k be a non-negative integer, let $f \in \mathscr{F}_k^g$, $\boldsymbol{u} \in \mathscr{V}^{\nabla k}$, and $x \in \mathscr{V}$. Then $\mathrm{Sko}_{\mathscr{L}^G}^{f(\boldsymbol{u}),x}$ is weakly truth-preserving.*

Proof. Take any $\mathscr{L}^G$-structure $\mathbf{M}$ and any $\gamma \in \mathcal{F}(\mathscr{L}^G)$ such that $\mathbf{M}' \models \gamma$ for all $\mathscr{L}^G$-structure $\mathbf{M}'$ for which $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$.
We need to show that $\mathbf{M} \models \mathrm{Sko}_{\mathscr{L}^G}^{f(\boldsymbol{u}),x}(\gamma)$.
Case 1: $\gamma \doteq ((\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))) \to \psi$ where $\psi, \phi \in \mathcal{B}(\mathscr{L}^G)$, f does not occur in ψ and ϕ, and $\phi(x/f(\boldsymbol{u}))$ is admissible.
Then $\mathrm{Sko}_{\mathscr{L}^G}^{f(\boldsymbol{u}),x}(\gamma) = \psi$.
We need to show that $\mathbf{M} \models \psi$.

Since $f \in \mathscr{F}_k^g$ and $f \notin \mathscr{F}$, for any $\mathbf{M}' \in \mathscr{M}_f(\mathbf{M})$ (where $\mathbf{M}'$ is an $\mathscr{L}^G$-structure), we have $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$, so by assumption $\mathbf{M}' \models \gamma$, which means $\mathbf{M}' \models ((\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))) \to \psi$.
By Lemma 2.5.10, since $\mathbf{M}' \models ((\exists x.\phi) \to \phi(x/f(\boldsymbol{u}))) \to \psi$ for any $\mathbf{M}' \in \mathscr{M}_f(\mathbf{M})$, we have $\mathbf{M} \models \psi$.
Therefore, $\mathbf{M} \models \mathrm{Sko}_{\mathscr{L}^G}^{f(\boldsymbol{u}),x}(\gamma)$.
Case 2: Otherwise.
Then $\mathrm{Sko}_{\mathscr{L}^G}^{f(\boldsymbol{u}),x}(\gamma) = \mathfrak{t}$, which is trivially satisfied by any structure.
Therefore, $\mathrm{Sko}_{\mathscr{L}^G}^{f(\boldsymbol{u}),x}$ is weakly truth-preserving. □

Definition 4.0.10. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $\mathcal{R}$ be a family of inference rules on $\mathcal{F}(\mathscr{L})$. Let $\Gamma \subset \mathcal{F}(\mathscr{L})$. We will say that Γ is closed with respect to $\mathcal{R}$ iff for any $r \in \mathcal{R}$ with $\dim r = (k_1, k_2)$, we have $r(\Gamma^{k_1}) \subset \Gamma^{k_2}$.*

Definition 4.0.11. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $\mathcal{R}$ be a family of inference rules on $\mathcal{F}(\mathscr{L})$. Let $T \subset \mathcal{F}(\mathscr{L})$.*

$$Cn(T, \mathcal{R}) \stackrel{\text{def}}{=} \bigcap \{\Gamma : T \subset \Gamma \textit{ and } \Gamma \textit{ is closed with respect to } \mathcal{R}\}. \tag{4.3}$$

Definition 4.0.12. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $\mathbf{M}$ be an $\mathscr{L}^G$-structure.*

$$\mathcal{G}(\mathscr{L}^G, \mathbf{M}) \stackrel{\text{def}}{=} \{\phi \in \mathcal{F}(\mathscr{L}^G) : \begin{array}{l} \mathbf{M} \models \phi, \text{ and } (\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}) \implies \mathbf{M}' \models \phi \\ \text{for any } \mathscr{L}^G\text{-structures } \mathbf{M}'. \end{array}\} \tag{4.4}$$

Theorem 4.0.13. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and $\mathbf{M}$ be an $\mathscr{L}^G$-structure. Let $\mathcal{R}$ be a family of weakly truth-preserving inference rules on $\mathcal{F}(\mathscr{L}^G)$. If $T \subset \mathcal{G}(\mathscr{L}^G, \mathbf{M})$, then $Cn(T, \mathcal{R}) \subset \mathcal{G}(\mathscr{L}^G, \mathbf{M})$.*

Proof. We will show that $\mathcal{G}(\mathscr{L}^G, \mathbf{M})$ is closed with respect to $\mathcal{R}$ and $T \subset \mathcal{G}(\mathscr{L}^G, \mathbf{M})$. Then by definition of $\mathrm{Cn}(T, \mathcal{R})$, we have $\mathrm{Cn}(T, \mathcal{R}) \subset \mathcal{G}(\mathscr{L}^G, \mathbf{M})$.
Since $T \subset \mathcal{G}(\mathscr{L}^G, \mathbf{M})$ by assumption, it remains to show that $\mathcal{G}(\mathscr{L}^G, \mathbf{M})$ is closed with respect to $\mathcal{R}$.
Take any $r \in \mathcal{R}$. Let $\dim r = (k_1, k_2)$. We need to show that $r(\mathcal{G}(\mathscr{L}^G, \mathbf{M})^{k_1}) \subset \mathcal{G}(\mathscr{L}^G, \mathbf{M})^{k_2}$.
Take an arbitrary $\boldsymbol{\phi} \in \mathcal{G}(\mathscr{L}^G, \mathbf{M})^{k_1}$. By Definition 4.0.12, we have $\mathbf{M} \models \boldsymbol{\phi}$.
Take any $\mathscr{L}^G$-structure $\mathbf{M}'$ with $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$. Again by Definition 4.0.12, we have $\mathbf{M}' \models \boldsymbol{\phi}$.
And since r is weakly truth preserving, we have $\mathbf{M} \models r(\boldsymbol{\phi})$.
Take an arbitrary $\mathscr{L}^G$-structure $\mathbf{M}''$ with $\mathbf{M}'|_{\mathscr{L}} = \mathbf{M}''|_{\mathscr{L}}$.
Since $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$, $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}''|_{\mathscr{L}}$ and by Definition 4.0.12 we have $\mathbf{M}'' \models \boldsymbol{\phi}$.
Since r is weakly truth-preserving and $\mathbf{M}'' \models \boldsymbol{\phi}$ for all $\mathbf{M}''$ with $\mathbf{M}'|_{\mathscr{L}} = \mathbf{M}''|_{\mathscr{L}}$, we have $\mathbf{M}' \models r(\boldsymbol{\phi})$. We have shown that $\mathbf{M} \models r(\boldsymbol{\phi})$ and that for any $\mathscr{L}^G$-structure $\mathbf{M}'$ with $\mathbf{M}|_{\mathscr{L}} = \mathbf{M}'|_{\mathscr{L}}$, we have $\mathbf{M}' \models r(\boldsymbol{\phi})$. Thus $r(\boldsymbol{\phi}) \in \mathcal{G}(\mathscr{L}^G, \mathbf{M})^{k_2}$, which completes the proof. □