Logical Foundations

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1 Introduction

This document presents the logical foundations for a formal system based on first-order logic. The definitions and theorems presented here were designed by the human author. Claude AI assisted with writing proofs and definitions following the author's specifications and design. All AI-generated content in this paper has been carefully reviewed and verified by the human author.

Contents

2 Well Formed Formula

2.1 Syntax

Let $\mathscr{V}$ be an infinite countable set of variable symbols. And $\mathfrak{f}$ be a special predicate symbol.

Definition 2.1. *Let $\mathscr{P}$ be a set of predicate symbols for which $\mathfrak{f} \in \mathscr{P}$ and $\mathscr{F}$ be a set of function symbols. Let $\mathfrak{a} : \mathscr{P} \cup \mathscr{F} \to \mathbb{N}$ be an arity function. then $\mathscr{L} \stackrel{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ is a signature of a language.*

Definition 2.2 (arity)**.** *Let $\mathscr{L} \stackrel{\text{def}}{=} (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

$$\mathscr{P}_k \stackrel{\text{def}}{=} \{p \in \mathscr{P} : \mathfrak{a}(p) = k\}, \tag{1}$$

$$\mathscr{F}_k \stackrel{\text{def}}{=} \{f \in \mathscr{F} : \mathfrak{a}(f) = k\}. \tag{2}$$

Definition 2.3 (recursive: term)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

1. *For any $v \in \mathscr{V}$, v is a **term**.*
2. *For any $c \in \mathscr{F}_0$, c is a **term**.*
3. *For any $f \in \mathscr{F}_k$ and $\tau_1, \ldots, \tau_k$ terms, $f(\tau_1, \ldots, \tau_k)$ is a **term**.*

Definition 2.4 (recursive: atomic formula)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

1. *For any $p \in \mathscr{P}_0$, p is an **atomic formula**.*
2. *For any $p \in \mathscr{P}_k$ and $\tau_1, \ldots, \tau_k$ are terms, $p(\tau_1, \ldots, \tau_k)$ is an **atomic formula**.*

We will denote a set of all atomic formulas as $\mathcal{A}(\mathscr{L})$.

Definition 2.5 (recursive: base formula, well formed formula)**.** *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language.*

1. *For any atomic formula ϕ, ϕ is a **base formula**.*
2. *For any **base formula** ϕ, ϕ is a **well formed formula**.*
3. *For any **well formed formula** ϕ, (ϕ) is a **base formula**.*
4. *For any **base formulas** α, β, $\alpha \to \beta$ is a **well formed formula**.*
5. *For any $v \in \mathscr{V}$ and a **base formula** ϕ, $\forall v.\phi$ is a **well formed formula**.*

We will denote set of all base formulas as $\mathcal{B}(\mathscr{L})$ and set of all well formed formulas as $\mathcal{F}(\mathscr{L})$.

Note 2.6 (syntactic equality)**.** *We use the notation $\stackrel{\text{sx}}{=}$ to denote syntactic equality of terms and formulas. Two terms (or formulas) are syntactically equal if they are the same syntactic construct.*

3 Semantics

Definition 3.1 (**$\mathscr{L}$-structure**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, then a mapping $\mathbf{M}$ such that*

$$\mathbf{M} : \mathscr{P} \cup \mathscr{F} \to \{0,1\} \cup \bigcup_{k=1}^{\infty} \{0,1\}^{(A^k)} \cup A \cup \bigcup_{k=1}^{\infty} A^{(A^k)}, \tag{3}$$

where $\mathbf{M}(\mathfrak{f}) = 0$,

$$\mathbf{M}(\mathscr{P}_0) \subset \{0,1\} \text{ and } \mathbf{M}(\mathscr{P}_k) \subset \{0,1\}^{(A^k)} \text{ for } k = 1, \ldots, \tag{4}$$

and

$$\mathbf{M}(\mathscr{F}_0) \subset A \text{ and } \mathbf{M}(\mathscr{F}_k) \subset A^{(A^k)} \text{ for } k = 1, \ldots \tag{5}$$

is called an $\mathscr{L}$-structure with domain A.

For convenience we will denote values of $\mathbf{M}$ as $p^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(p)$ for $p \in \mathscr{P}$ and $f^{\mathbf{M}} \stackrel{\text{def}}{=} \mathbf{M}(f)$ for $f \in \mathscr{F}$.

Definition 3.2 (**assignment**). *Let A be an arbitrary mathematical domain. Then $j : \mathscr{V} \to A$ is called an variables assignment in A.*

Definition 3.3 (**recursive: term evaluation**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure and $j : \mathscr{V} \to A$. Let τ be an arbitrary term.*

1. *If $\tau \stackrel{\text{sx}}{=} v$ where $v \in \mathscr{V}$, then $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} j(v)$.*

2. *If $\tau \stackrel{\text{sx}}{=} f$ where $f \in \mathcal{F}_0$, then $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} f^{\mathbf{M}}$.*

3. *If $\tau \stackrel{\text{sx}}{=} f(\tau_1, \ldots, \tau_k)$ where $f \in \mathcal{F}_k$ and $\tau_1, \ldots, \tau_k$ are terms, then $\tau^{\mathbf{M},j} \stackrel{\text{def}}{=} f^{\mathbf{M}}(\tau_1^{\mathbf{M},j}, \ldots, \tau_k^{\mathbf{M},j})$.*

Definition 3.4. *Let $\mathscr{L}$ be a signature of a language. Let $\mathbf{M}$ be a $\mathscr{L}$-structure with domain A and let $j : \mathscr{V} \to A$ be an assignment of variables. Then we define*

$$j\frac{a}{x}(v) \stackrel{\text{def}}{=} \begin{cases} a \text{ if } v \text{ is } x, \\ j(v) \text{ otherwise.} \end{cases} \tag{6}$$

For the purpouses of this work, we will define product of infinite number of 0s and 1s.

Definition 3.5. *Let S be an arbitrary set and $a_s \in \{0,1\}$ for any $s \in S$.*

$$\prod_{s \in S} a_s = \begin{cases} 1 \text{ if } a_s = 1 \text{ for all } s \in S, \\ 0 \text{ otherwise .} \end{cases} \tag{7}$$

Definition 3.6 (**recursive: formula evaluation**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$.*

1. *If $\phi \overset{\text{sx}}{=} p$ where $p \in \mathscr{P}$, then $\phi^{\mathbf{M},j} \overset{\text{def}}{=} p^{\mathbf{M}}$.*
2. *If $\phi \overset{\text{sx}}{=} p(\tau_1, \dots, \tau_k)$ where $p \in \mathscr{P}_k$ and $\tau_1, \dots, \tau_k$ are terms, then $\phi^{\mathbf{M},j} \overset{\text{def}}{=} p^{\mathbf{M}}(\tau_1^{\mathbf{M},j}, \dots, \tau_k^{\mathbf{M},j})$.*
3. *If $\phi \overset{\text{sx}}{=} \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then*
$$\phi^{\mathbf{M},j} \overset{\text{def}}{=} 1 - \alpha^{\mathbf{M},j}(1 - \beta^{\mathbf{M},j}). \tag{8}$$
4. *If $\phi \overset{\text{sx}}{=} \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, then*
$$\phi^{\mathbf{M},j} \overset{\text{def}}{=} \prod_{a \in A} \psi^{\mathbf{M},j\frac{a}{x}}. \tag{9}$$

Corollary 3.7. *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*
$$(\phi \to \mathfrak{f})^{\mathbf{M},j} = 1 - \phi^{\mathbf{M},j}. \tag{10}$$

Definition 3.8 (**model**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language and A be an arbitrary mathematical domain, let $\mathbf{M}$ be an $\mathscr{L}$-structure with domain A and $j : \mathscr{V} \to A$. Let $\phi \in \mathcal{F}(\mathscr{L})$, then*
$$\mathbf{M}, j \models \phi \overset{\text{def}}{\iff} \phi^{\mathbf{M},j} = 1. \tag{11}$$

4 Substitutions

4.1 Syntax

Definition 4.1 (**recursive: variable in a term**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $v \in \mathscr{V}$ and τ be a term.*

1. *If $\tau \overset{\text{sx}}{=} v$ then v is a **variable in** term τ.*
2. *If $\tau \overset{\text{sx}}{=} f(\tau_1, \dots \tau_k)$ where $f \in \mathscr{F}_k$ and $\tau_1, \dots \tau_k$ are terms and v is a **variable in** some term τ_i where i is an integer with $1 \le i \le k$, then v is a **variable in** term τ.*
3. *Otherwise v is not a **variable in** a term τ.*

Definition 4.2 (**recursive: variable in a well formed formula**). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $v \in \mathscr{V}$ and $\phi \in \mathcal{F}(\mathscr{L})$.*

1. *If $\phi \stackrel{\text{sx}}{=} p(\tau_1, \ldots, \tau_k)$ where $p \in \mathscr{P}_k$ and $\tau_1, \ldots, \tau_k$ are terms, and v is a **variable in** some term τ_i where i is an integer with $1 \leq i \leq k$, then v is a **variable in** ϕ.*

2. *If $\phi \stackrel{\text{sx}}{=} \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, and v is a **variable in** α or v is a **variable in** β, then v is a **variable in** ϕ.*

3. *If $\phi \stackrel{\text{sx}}{=} \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$, and either $v = x$ or v is a **variable in** ψ, then v is a **variable in** ϕ.*

4. *Otherwise v is not a **variable in** ϕ.*

Definition 4.3 (recursive: free variable in a well formed formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $v \in \mathscr{V}$ and $\phi \in \mathcal{F}(\mathscr{L})$.*

1. *If $\phi \in \mathcal{A}(\mathscr{L})$ and v is a variable in ϕ, then v is a **free variable in** ϕ.*

2. *If $\phi \stackrel{\text{sx}}{=} \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, and v is a **free variable in** α or v is a **free variable in** β, then v is a **free variable in** ϕ.*

3. *If $\phi \stackrel{\text{sx}}{=} \forall x.\psi$ where $x \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$ and v is not x and v is a **free variable in** ψ, then v is a **free variable in** ϕ.*

4. *Otherwise v is not a **free variable in** ϕ.*

Definition 4.4 (recursive: substitution in term). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $x \in \mathscr{V}$ and τ, σ be terms.*

1. *If $\sigma \stackrel{\text{sx}}{=} x$, then $\sigma(x/\tau) \stackrel{\text{def}}{=} \tau$.*

2. *If $\sigma \stackrel{\text{sx}}{=} v$ where $v \in \mathscr{V}$ and v is not x, then $\sigma(x/\tau) \stackrel{\text{def}}{=} v$.*

3. *If $\sigma \stackrel{\text{sx}}{=} c$ where $c \in \mathscr{F}_0$, then $\sigma(x/\tau) \stackrel{\text{def}}{=} c$.*

4. *If $\sigma \stackrel{\text{sx}}{=} f(\sigma_1, \ldots, \sigma_k)$ where $f \in \mathscr{F}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms, then $\sigma(x/\tau) \stackrel{\text{def}}{=} f(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau))$.*

Definition 4.5 (recursive: substitution in well formed formula). *Let $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ be a signature of a language. Let $x \in \mathscr{V}$, τ be a term, and $\phi \in \mathcal{F}(\mathscr{L})$.*

1. *If $\phi \stackrel{\text{sx}}{=} p$ where $p \in \mathscr{P}_0$, then $\phi(x/\tau) \stackrel{\text{def}}{=} p$.*

2. *If $\phi \stackrel{\text{sx}}{=} p(\sigma_1, \ldots, \sigma_k)$ where $p \in \mathscr{P}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms, then $\phi(x/\tau) \stackrel{\text{def}}{=} p(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau))$.*

3. *If $\phi \stackrel{\text{sx}}{=} \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$, then $\phi(x/\tau) \stackrel{\text{def}}{=} \alpha(x/\tau) \to \beta(x/\tau)$.*

4. *If $\phi \stackrel{\text{sx}}{=} \forall y.\psi$ where $y \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$ and $y \stackrel{\text{sx}}{=} x$, then $\phi(x/\tau) \stackrel{\text{def}}{=} \forall y.\psi$.*

5. *If* $\phi \overset{\text{sx}}{=} \forall y.\psi$ *where* $y \in \mathscr{V}$ *and* $\psi \in \mathcal{B}(\mathscr{L})$ *and* y *is not* x *and* y *is not a variable in* τ*, then* $\phi(x/\tau) \overset{\text{def}}{=} \forall y.\psi(x/\tau)$.

6. *Otherwise* $\phi(x/\tau)$ *is* ***not admissible****.*

Lemma 4.6. *Let* $\mathscr{L} = (\mathscr{P}, \mathscr{F}, \mathfrak{a})$ *be a signature of a language. Let* σ *be a term. Let* $x \in \mathscr{V}$ *and* τ *be a term. Let* $y \in \mathscr{V}$ *and* y *is not in term* τ *and* y *is not in term* σ*. Then*

$$\sigma(x/y)(y/\tau) \overset{\text{sx}}{=} \sigma(x/\tau). \tag{12}$$

Proof. We proceed by structural induction on the term σ.
Case 1: $\sigma \overset{\text{sx}}{=} x$.
Then $\sigma(x/y) \overset{\text{sx}}{=} y$, and

$$(\sigma(x/y))(y/\tau) \overset{\text{sx}}{=} y(y/\tau) \overset{\text{sx}}{=} \tau.$$

Also, $\sigma(x/\tau) \overset{\text{sx}}{=} \tau$. Therefore, $\sigma(x/y)(y/\tau) \overset{\text{sx}}{=} \sigma(x/\tau)$.
Case 2: $\sigma \overset{\text{sx}}{=} v$ where $v \in \mathscr{V}$ and $v \neq x$.
Since y is not a **variable in** σ, we have $v \neq y$. Then $\sigma(x/y) \overset{\text{sx}}{=} v$, and

$$(\sigma(x/y))(y/\tau) \overset{\text{sx}}{=} v(y/\tau) \overset{\text{sx}}{=} v$$

(since $v \neq y$). Also, $\sigma(x/\tau) \overset{\text{sx}}{=} v$ (since $v \neq x$). Therefore, $\sigma(x/y)(y/\tau) \overset{\text{sx}}{=} \sigma(x/\tau)$.
Case 3: $\sigma \overset{\text{sx}}{=} c$ where $c \in \mathscr{F}_0$.
Then $\sigma(x/y) \overset{\text{sx}}{=} c$, and

$$(\sigma(x/y))(y/\tau) \overset{\text{sx}}{=} c(y/\tau) \overset{\text{sx}}{=} c.$$

Also, $\sigma(x/\tau) \overset{\text{sx}}{=} c$. Therefore, $\sigma(x/y)(y/\tau) \overset{\text{sx}}{=} \sigma(x/\tau)$.
Case 4: $\sigma \overset{\text{sx}}{=} f(\sigma_1, \ldots, \sigma_k)$ where $f \in \mathscr{F}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms.
Since y is not a **variable in** σ, y is not a **variable in** any σ_i for $i = 1, \ldots, k$. By the definition of substitution in term, $\sigma(x/y) \overset{\text{sx}}{=} f(\sigma_1(x/y), \ldots, \sigma_k(x/y))$. Then

$$(\sigma(x/y))(y/\tau) \overset{\text{sx}}{=} f(\sigma_1(x/y)(y/\tau), \ldots, \sigma_k(x/y)(y/\tau)).$$

By inductive hypothesis, $\sigma_i(x/y)(y/\tau) \overset{\text{sx}}{=} \sigma_i(x/\tau)$ for all $i = 1, \ldots, k$. Therefore,

$$(\sigma(x/y))(y/\tau) \overset{\text{sx}}{=} f(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau)) \overset{\text{sx}}{=} \sigma(x/\tau).$$

This completes the proof by structural induction. $\square$

4.2 Semantics

Lemma 4.7. *Let* $\mathscr{L}$ *be a signature of a language and* A *be an arbitrary mathematical domain. Let* $\mathbf{M}$ *be an* $\mathscr{L}$*-structure with domain* A *and* $j : \mathscr{V} \to A$ *be an assignment. For any term* σ*, any* $x \in \mathscr{V}$*, and any term* τ*, we have*

$$\sigma^{\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}} = (\sigma(x/\tau))^{\mathbf{M},j}. \tag{13}$$

Proof. We proceed by structural induction on the term σ.
Case 1: $\sigma \overset{\text{sx}}{=} x$.
Then $\sigma(x/\tau) \overset{\text{sx}}{=} \tau$ and

$$\sigma^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = (j\frac{\tau^{(\mathbf{M},j)}}{x})(x) = \tau^{\mathbf{M},j} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 2: $\sigma \overset{\text{sx}}{=} v$ where $v \in \mathscr{V}$ and $v \neq x$.
Then $\sigma(x/\tau) \overset{\text{sx}}{=} v$ and

$$\sigma^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = (j\frac{\tau^{(\mathbf{M},j)}}{x})(v) = j(v) = v^{\mathbf{M},j} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 3: $\sigma \overset{\text{sx}}{=} c$ where $c \in \mathscr{F}_0$.
Then $\sigma(x/\tau) \overset{\text{sx}}{=} c$ and

$$\sigma^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = c^{\mathbf{M}} = (\sigma(x/\tau))^{\mathbf{M},j}.$$

Case 4: $\sigma \overset{\text{sx}}{=} f(\sigma_1, \ldots, \sigma_k)$ where $f \in \mathscr{F}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms.

Then $\sigma(x/\tau) \overset{\text{sx}}{=} f(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau))$. By inductive hypothesis, $\sigma_i^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = (\sigma_i(x/\tau))^{\mathbf{M},j}$ for all $i = 1, \ldots, k$. Thus

$$\begin{aligned}
\sigma^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} &= f^{\mathbf{M}}(\sigma_1^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}}, \ldots, \sigma_k^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}}) \\
&= f^{\mathbf{M}}((\sigma_1(x/\tau))^{\mathbf{M},j}, \ldots, (\sigma_k(x/\tau))^{\mathbf{M},j}) \\
&= (f(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau)))^{\mathbf{M},j} \\
&= (\sigma(x/\tau))^{\mathbf{M},j}.
\end{aligned}$$

This completes the proof by structural induction. $\square$

Theorem 4.8. *Let $\mathscr{L}$ be a signature of a language. Let $\phi \in \mathcal{F}(\mathscr{L})$ and let τ be an arbitrary term. If $\phi(x/\tau)$ is admissible, then for any $\mathscr{L}$-structure $\mathbf{M}$ with domain A and for any assignment of variables $j : \mathscr{V} \to A$ we have*

$$(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi \iff (\mathbf{M}, j) \models \phi(x/\tau). \tag{14}$$

Proof. We proceed by structural induction on $\phi \in \mathcal{F}(\mathscr{L})$.
Case 1: $\phi \overset{\text{sx}}{=} p$ where $p \in \mathscr{P}_0$.

By the definition of substitution in well formed formula, $\phi(x/\tau) \stackrel{\text{sx}}{=} p$. By the definition of formula evaluation, $\phi^{\mathbf{M},j} = p^{\mathbf{M}}$ for any assignment j. Thus

$$\begin{aligned}
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi &\iff \phi^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1 \\
&\iff p^{\mathbf{M}} = 1 \\
&\iff \phi^{\mathbf{M},j} = 1 \\
&\iff (\mathbf{M}, j) \models \phi \\
&\iff (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}$$

Case 2: $\phi \stackrel{\text{sx}}{=} p(\sigma_1, \ldots, \sigma_k)$ where $p \in \mathscr{P}_k$ and $\sigma_1, \ldots, \sigma_k$ are terms.
By the definition of substitution in well formed formula, $\phi(x/\tau) \stackrel{\text{sx}}{=} p(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau))$.
By Lemma 4.7, for each term σ_i, we have $\sigma_i^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = (\sigma_i(x/\tau))^{\mathbf{M},j}$. Therefore,

$$\begin{aligned}
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi &\iff \phi^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1 \\
&\iff p^{\mathbf{M}}(\sigma_1^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}}, \ldots, \sigma_k^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}}) = 1 \\
&\iff p^{\mathbf{M}}((\sigma_1(x/\tau))^{\mathbf{M},j}, \ldots, (\sigma_k(x/\tau))^{\mathbf{M},j}) = 1 \\
&\iff (p(\sigma_1(x/\tau), \ldots, \sigma_k(x/\tau)))^{\mathbf{M},j} = 1 \\
&\iff (\phi(x/\tau))^{\mathbf{M},j} = 1 \\
&\iff (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}$$

Case 3: $\phi \stackrel{\text{sx}}{=} \alpha \to \beta$ where $\alpha, \beta \in \mathcal{B}(\mathscr{L})$.
By the definition of substitution in well formed formula, $\phi(x/\tau) \stackrel{\text{sx}}{=} \alpha(x/\tau) \to \beta(x/\tau)$. By inductive hypothesis,

$$\begin{aligned}
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \alpha &\iff (\mathbf{M}, j) \models \alpha(x/\tau), \\
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \beta &\iff (\mathbf{M}, j) \models \beta(x/\tau).
\end{aligned}$$

By the definition of formula evaluation,

$$
\begin{aligned}
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi \iff & \ \phi^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1 \\
\iff & \ 1 - \alpha^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}}(1 - \beta^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}}) = 1 \\
\iff & \ 1 - (\alpha(x/\tau))^{\mathbf{M},j}(1 - (\beta(x/\tau))^{\mathbf{M},j}) = 1 \\
\iff & \ (\alpha(x/\tau) \to \beta(x/\tau))^{\mathbf{M},j} = 1 \\
\iff & \ (\phi(x/\tau))^{\mathbf{M},j} = 1 \\
\iff & \ (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}
$$

Case 4: $\phi \overset{\text{sx}}{\equiv} \forall y.\psi$ where $y \in \mathscr{V}$ and $\psi \in \mathcal{B}(\mathscr{L})$.
Subcase 4a: If $y = x$, then by the definition of substitution, $\phi(x/\tau) \overset{\text{sx}}{\equiv} \forall y.\psi$. Thus

$$
\begin{aligned}
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi \iff & \ \phi^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1 \\
\iff & \ \prod_{a \in A} \psi^{\mathbf{M},(j\frac{\tau^{(\mathbf{M},j)}}{x})\frac{a}{y}} = 1 \\
\iff & \ \prod_{a \in A} \psi^{\mathbf{M},j\frac{a}{y}} = 1 \quad \text{(since } y = x \text{ and } (j\frac{\tau^{(\mathbf{M},j)}}{x})\frac{a}{x} = j\frac{a}{x}\text{)} \\
\iff & \ \phi^{\mathbf{M},j} = 1 \\
\iff & \ (\mathbf{M}, j) \models \phi \\
\iff & \ (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}
$$

Subcase 4b: If $y \neq x$ and y is not a **variable in** τ, then by the definition of substitution, $\phi(x/\tau) \overset{\text{sx}}{\equiv} \forall y.\psi(x/\tau)$. By inductive hypothesis, for any $a \in A$,

$$
(\mathbf{M}, j\frac{a}{y}\frac{\tau^{(\mathbf{M},j\frac{a}{y})}}{x}) \models \psi \iff (\mathbf{M}, j\frac{a}{y}) \models \psi(x/\tau).
$$

Since y is not a **variable in** τ, the evaluation of τ does not depend on the value assigned to y. Therefore, $\tau^{(\mathbf{M},j\frac{a}{y})} = \tau^{(\mathbf{M},j)}$ for all $a \in A$.
Moreover, since $x \neq y$, we have $j\frac{\tau^{(\mathbf{M},j)}}{x}\frac{a}{y} = j\frac{a}{y}\frac{\tau^{(\mathbf{M},j)}}{x}$ for all $a \in A$.

Thus,

$$
\begin{aligned}
(\mathbf{M}, j\frac{\tau^{(\mathbf{M},j)}}{x}) \models \phi &\iff \phi^{\mathbf{M},j\frac{\tau^{(\mathbf{M},j)}}{x}} = 1 \\
&\iff \prod_{a\in A} \psi^{\mathbf{M},(j\frac{\tau^{(\mathbf{M},j)}}{x})\frac{a}{y}} = 1 \\
&\iff \prod_{a\in A} \psi^{\mathbf{M},j\frac{a}{y}\frac{\tau^{(\mathbf{M},j)}}{x}} = 1 \\
&\iff \prod_{a\in A} (\psi(x/\tau))^{\mathbf{M},j\frac{a}{y}} = 1 \quad \text{(by inductive hypothesis)} \\
&\iff (\forall y.\psi(x/\tau))^{\mathbf{M},j} = 1 \\
&\iff (\phi(x/\tau))^{\mathbf{M},j} = 1 \\
&\iff (\mathbf{M}, j) \models \phi(x/\tau).
\end{aligned}
$$

This completes the proof by structural induction. $\square$