Assumption we use for p21 – p22

1. $[a_p, a^\dagger_{p'}] = (2\pi)^3 \delta^{(3)}(p-p')$
2. $[a_p, a_{p'}] = 0 \quad [a^\dagger_p, a^\dagger_{p'}] = 0$
3. $a_p|0\rangle = 0$

P.21 $[a_p, a^+_{p'}] = (2\pi)^3 \delta^{(3)}(p-p')$, $[a_p, a_{p'}] = 0$ $[a^+_p, a^+_{p'}] = 0$

$$[a_p e^{ipx} + a^+_p e^{-ipx}, a_{p'} e^{ip'x'} - a^+_{p'} e^{-ip'x'}] = e^{-i(px-p'x')}[a^+_p, a_{p'}] - e^{i(px-p'x')}[a_p, a^+_{p'}]$$

$$\iint -e^{-i(px-p'x)}(2\pi)^3 \delta^{(3)}(p-p') - e^{i(px-p'x')}(2\pi)^3 \delta^{(3)}(p-p')\, dpp'$$

$$(2\pi)^3 \iint f(p,p')(-e^{-i(px-p'x')} - e^{i(px-p'x')})\delta^{(3)}(p-p')\, dpp' =$$

$$(2\pi)^3 \int f(p,p)(-e^{-ip(x-x')} - e^{ip(x-x')})\, dp = \frac{i}{\cancel{2}(2\pi)^3}\int \cancel{2}e^{-p(x-x')}\, dp =$$

$$f(p,p) = \frac{1}{(2\pi)^6}\frac{(-i)}{2}$$

$$= \frac{i}{(2\pi)^3}(2\pi)^3 \delta^{(3)}(x-x') = i\,\delta^{(3)}(x-x')$$

recall: $\boxed{(2\pi)^{-n}\int e^{-ixp}\, dp = \delta^{(n)}(x)}$

p21-22

$$A = \int \frac{d^3p}{(2\pi)^3} f(p)\, a_p^\dagger a_p$$

Let $|\Omega\rangle = a^\dagger_{p_1}\dots a^\dagger_{p_n}|0\rangle$ $\quad p_i \neq p_j$ for $i \neq j$ $\quad \mathfrak{Z} = \mathbb{R}^3 \setminus \cup O_{p_i}(\varepsilon)$

$a_p|0\rangle = 0$

$$a_p a^\dagger_{p_0}|0\rangle = [a_p, a^\dagger_{p_0}]|0\rangle = (2\pi)^3 \delta^{(3)}(p-p_0)|0\rangle$$

$$A|\Omega\rangle = \int d^3p\, (2\pi)^{-3} f(p)\, a_p^\dagger a_p|\Omega\rangle = \sum_i \int_{O_{p_i}(\varepsilon)} d^3p\, (2\pi)^{-3} f(p)\, a_p^\dagger a_p|\Omega\rangle + \int_{\mathfrak{Z}} \underline{\qquad} =$$

$$= \sum_i \int_{O_{p_i}(\varepsilon)} d^3p\, (2\pi)^{-3} f(p)\, a_p^\dagger a_p\, a^\dagger_{p_1}\dots a^\dagger_{p_{i-1}} a^\dagger_{p_{i+1}} \dots a^\dagger_{p_i}|0\rangle =$$

$$= \sum_i \int_{O_{p_i}(\varepsilon)} d^3p\, \cancel{(2\pi)^{-3}} f(p)\, a_p^\dagger a^\dagger_{p_1}\dots a^\dagger_{p_{i-1}} a^\dagger_{p_{i+1}}\, \delta^{(3)}(p-p_i) \cancel{(2\pi)^3}|0\rangle = \Big(\sum_i f(p_i)\Big)|\Omega\rangle$$

$$A|\Omega\rangle = \Big(\sum_i f(p_i)\Big)|\Omega\rangle$$ spectrum

Hamiltonian

For $H = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p$, $\omega_p = \sqrt{p^2+m^2}$ we have spectrum $\sum_i \omega_{p_i}$

for $P^k = \int \frac{d^3p}{(2\pi)^3} p^k a_p^\dagger a_p$ we have spectrum $\sum_i p_i^k$

momentum operator

p21-22

$$N = \int \frac{d^3p}{(2\pi)^3} a_p^\dagger a_p$$

$$a_p a_{p_0}^\dagger a_{p_0}^\dagger |0\rangle = \left(a_{p_0}^\dagger a_p + [a_p, a_{p_0}^\dagger]\right) a_{p_0}^\dagger |0\rangle = a_{p_0}^\dagger a_p a_{p_0}^\dagger |0\rangle + (2\pi)^3 \delta^{(3)}(p-p_0) a_{p_0}^\dagger |0\rangle =$$

$$= a_{p_0}^\dagger (2\pi)^3 \delta^{(3)}(p-p_0) + (2\pi)^3 \delta^{(3)}(p-p_0) a_{p_0}^\dagger |0\rangle = 2(2\pi)^3 \delta^{(3)}(p-p_0) a_{p_0}^\dagger |0\rangle$$

$$N a_{p_0}^\dagger a_{p_0}^\dagger |0\rangle = \int \frac{d^3p}{\cancel{(2\pi)^3}} 2\cancel{(2\pi)^3} \delta^{(3)}(p-p_0) a_p^\dagger a_{p_0}^\dagger |0\rangle = 2 a_{p_0}^\dagger a_{p_0}^\dagger |0\rangle$$

Analogously, we can show:

$$|\Omega\rangle = \underbrace{a_{p_1}^\dagger \cdots a_{p_1}^\dagger}_{n_1} \cdots \underbrace{a_{p_k}^\dagger \cdots a_{p_k}^\dagger}_{n_k} |0\rangle$$

$$A = \int \frac{d^3p}{(2\pi)^3} f(p) a_p^\dagger a_p \qquad A|\Omega\rangle = \left(\sum_i n_i f(p_i)\right)|\Omega\rangle$$

II

$$a_p \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n-1} |0\rangle = (n-1)(2\pi)^3 \delta^{(3)}(p-p_0) \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n-2} |0\rangle$$

$$a_p \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n} |0\rangle = \left(a^+_{p_0} a_p + [a_p, a^+_{p_0}]\right) \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n-1} |0\rangle =$$

$$= a^+_{p_0} (n-1)(2\pi)^3 \delta^{(3)}(p-p_0) \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n-2} |0\rangle + (2\pi)^3 \delta^{(3)}(p-p_0) \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n-1} |0\rangle =$$

$$= n (2\pi)^3 \delta^{(3)}(p-p_0) \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n-1} |0\rangle$$

$$a_p \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n} |0\rangle = n (2\pi)^3 \delta^{(3)}(p-p_0) \underbrace{a^+_{p_0} \dots a^+_{p_0}}_{n-1} |0\rangle$$

$$\boxed{a_p \left|{n \atop p_0}\right\rangle = (2\pi)^3 \delta^{(3)}(p-p_0)\, n \left|{n-1 \atop p_0}\right\rangle}$$

p22 $$H = \int \frac{d^3p}{(2\pi)^3} E_p a_p^\dagger a_p \qquad [A,B]^\dagger = [B^\dagger, A^\dagger]$$

$$[H, a_{p_0}^\dagger] = \int \frac{d^3p}{(2\pi)^3} E_p [a_p^\dagger a_p, a_{p_0}^\dagger] = \int \frac{d^3p}{(2\pi)^3} E_p \left(a_p^\dagger a_p a_{p_0}^\dagger - a_{p_0}^\dagger a_p^\dagger a_p\right) =$$

$$= \int \frac{d^3p}{(2\pi)^3} E_p \left(a_p^\dagger a_p a_{p_0}^\dagger - a_p^\dagger a_{p_0}^\dagger a_p\right) = \int \frac{d^3p}{(2\pi)^3} E_p a_p^\dagger [a_p, a_{p_0}^\dagger] = \int \frac{d^3p}{\cancel{(2\pi)^3}} E_p a_p^\dagger \cancel{(2\pi)^3} \delta^{(3)}(p - p_0) =$$

$$= E_{p_0} a_{p_0}^\dagger$$

$$\boxed{[H, a_{p_0}^\dagger] = E_{p_0} a_{p_0}^\dagger}$$

$$[a_{p_0}, H] = E_{p_0} a_{p_0}$$

$$\boxed{[H, a_{p_0}] = -E_{p_0} a_{p_0}}$$

p25

From commutators we have: we will show $H^n a_p = a_p(H-E_p)^n$. Assume

$$H a_p = a_p(H-E_p)$$

$$H^{n-1} a_p = a_p(H-E_p)^{n-1} \text{ then } H^n a_p = H H^{n-1} a_p = H a_p (H-E_p)^{n-1} = a_p (H-E_p)^n.$$

Analogusly: $H^n \hat{a}_p = \hat{a}_p (H+E_p)^n$.

time evolution of a_p is given by $t \mapsto e^{iHt} a_p e^{-iHt}$, then

$$e^{iHt} a_p e^{-iHt} = a_p e^{i(H-E_p)t} e^{-iHt} = a_p e^{-iE_p t}$$

analogously

$$e^{iHt} \hat{a}_p e^{-iHt} = \hat{a}_p e^{i(H+E_p)t} e^{-iHt} = \hat{a}_p e^{iE_p t}$$

$$\boxed{\begin{aligned} e^{iHt} a_p e^{-iHt} &= a_p e^{-iE_p t} \\ e^{iHt} \hat{a}_p e^{-iHt} &= \hat{a}_p e^{iE_p t} \end{aligned}}$$

p23

$d^4p \; \delta(p^2 - m^2)\,\Theta(p^0)$ is invariant for orthochronous Λ

$$E_p^2 = \mathbf{p}^2 + m^2$$

$$E_p^2 - \mathbf{p}^2 = m^2$$

$$d^4p \; \delta(p^2 - m^2)\,\Theta(p^0) = d^4p \; \delta(p^2 - E_p^2 + \mathbf{p}^2)\,\Theta(p^0) = d^4p \; \delta(p_0^2 - E_p^2)\,\Theta(p^0) =$$

$$= d^4p \, \frac{1}{2E_p} \, \delta(p_0 - E_p)\,\Theta(p^0) = d^3p \, dp_0 \, \frac{1}{2E_p} \, \delta(p_0 - E_p)\,\Theta(p_0) = d^3p \, \frac{1}{2E_p}$$

$\frac{d^3p}{E_p}$ is invariant under orthochronous Λ

p27

$$D(z) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot z} = \int \frac{d^4p}{(2\pi)^3} \Theta(p^0)\,\delta(p^2-m^2)\, e^{-ip\cdot z} =$$

$$= \int \frac{d^4\Lambda p}{(2\pi)^3} \theta((\ p)^0)\,\delta((\ p)^2-m^2)\, e^{-i(\Lambda p)\cdot(\Lambda z)} =$$

$$= \int \frac{d^4\Lambda p}{(2\pi)^3} \Theta((\cancel{\Lambda^{-1}\Lambda}\, p)^0)\,\delta((\cancel{\Lambda^{-1}\Lambda}\, p)^2-m^2)\, e^{-i(\Lambda p)\cdot(\Lambda z)} =$$

$$= \int \frac{d^4\Lambda p}{(2\pi)^3} \Theta((\Lambda p)^0)\,\delta((\Lambda p)^2-m^2)\, e^{-i(\Lambda p)\cdot(\Lambda z)} =$$

$$= \int \frac{d^4p}{(2\pi)^3} \Theta(p^0)\,\delta(p^2-m^2)\, e^{-ip\cdot(\Lambda z)} = D(\Lambda z)$$

Λ must be orthochronous!

Lorentz Transformation

$$g(\Lambda(x+y),\Lambda(x+y)) = g(x+y,x+y)$$

$$\cancel{g(\Lambda x,\Lambda x)} + 2g(\Lambda x,\Lambda y) + \cancel{g(\Lambda y,\Lambda y)} = \cancel{g(x,x)} + 2g(x,y) + \cancel{g(y,y)}$$

$$\underline{\cancel{2}g(\Lambda x,\Lambda y) = g(x,y)}$$

Λ is a General Lorenz transformation iff $\Lambda^{\mu}{}_{\partial}\Lambda^{\nu}{}_{\rho}\eta_{\mu\nu} = \eta_{\partial\rho}$

This is equivalent of $g(\Lambda x,\Lambda y) = g(x,y)$ [g - Minkovsky tensor]

x is time-like iff $g(x,x) > 0$

x in null if $g(x,x) = 0$

x is space-like if $g(x,x) < 0$

$(x^0)^2 - (\vec{x})^2 \geqslant 0$, $(x^0)^2 \geqslant (\vec{x})^2 \Rightarrow$

If x is time-like and $x \neq 0$, then $x^0 \neq 0$.

If x is time-like $(g(x,x) > 0)$ any y $y \neq 0$ is time-like or null $(g(y,y) \geqslant 0)$,
then $\mathrm{sgn}(g(x,y)) = \mathrm{sgn}(x^0 y^0) \neq 0$

Let $\mathcal{J}$ be set of all time-like vectors. Let $x \sim y \iff g(x,y) > 0$

$\sim$ is equivalence relation on $\mathcal{J}$

($g(x,y) > 0$ and $g(y,z) > 0$ $\quad x^0 y^0 > 0$ and $y^0 z^0 > 0$ then $y^0 z^0 > 0$.)

$\mathcal{J}^+ = \{x \in \mathcal{J} : x^0 > 0\}$, $\mathcal{J}^- = \{x \in \mathcal{J} : x^0 < 0\}$

Obviously $\quad$ If $x \in \mathcal{J}^+$ and $y \sim x \Rightarrow y \in \mathcal{J}^+$

If $x \in \mathcal{J}^-$ and $y \sim x \Rightarrow y \in \mathcal{J}^-$

$x, y \in \mathcal{J}^+ \Rightarrow x \sim y$ $\quad \mathcal{J}^+ \cup \mathcal{J}^- = \mathcal{J}$ thus there are only two class of

$x, y \in \mathcal{J}^- \Rightarrow x \sim y$ $\quad$ time orientations.

abstraction. We call them time orientations.

$x, y \in \mathcal{J}^+ \Rightarrow \lambda_1 x + \lambda_2 y \in \mathcal{J}^+$ for $\lambda_1, \lambda_2 > 0$ $\quad$ $\mathcal{J}$ there are cones.

$x, y \in \mathcal{J}^- \Rightarrow \lambda_1 x + \lambda_2 y \in \mathcal{J}^-$ for $\lambda_1, \lambda_2 > 0$

Note that for any General Lorentz transformation it is either

$$\Lambda \mathcal{J}^+ = \mathcal{J}^+$$

or

$$\Lambda \mathcal{J}^+ = \mathcal{J}^-$$

proof take $x \in \mathcal{J}^+$ then Λx is still time-like thus $\Lambda x \in \mathcal{J}^+$ or $\Lambda x \in \mathcal{J}^-$. Assume $\Lambda x \in \mathcal{J}^+$, take $y \in \mathcal{J}^+$ $g(x,y) > 0$ thus $y \in \mathcal{J}^+$ but $g(\Lambda x, \Lambda y) = g(x,y) > 0$ thus $\Lambda y \in \mathcal{J}^+$. The same for $\Lambda x \in \mathcal{J}^-$

Λ is orthochronous if $\Lambda^0_0 \geq 1$

Λ is orthochronous

$$\Updownarrow$$

Λ preserve time orientation of all nonzero null vectors

$$\Updownarrow$$

Λ preserve time orientation of all time-like vectors

Λ is proper iff $\det \Lambda = 1$

$$\frac{d}{d\varepsilon} U(\varepsilon)^{\nu}_{\mu} x^{\mu}\Big|_{\varepsilon=0} = A^{\nu}_{\mu} x^{\mu} \qquad \text{Generators}$$

$$\frac{d}{d\varepsilon} \hat{U}(\varepsilon)(f)\Big|_{\varepsilon=0}(x) = \frac{d}{d\varepsilon} f(U(\varepsilon)x)\Big|_{\varepsilon=0} = \frac{\partial f}{\partial x^{\mu}} (U(\varepsilon)x) \frac{d U(\varepsilon)}{d\varepsilon} x\Big|_{\varepsilon=0} =$$

$$= \frac{\partial f}{\partial x^{\mu}}(x) A^{\mu}_{\nu} x^{\nu} = \hat{A}(f)(x)$$

$$(\hat{A}(\hat{B}(f)))(x) = \left(\hat{A}\left(x \mapsto \frac{\partial f}{\partial x^{\mu}} B^{\mu}_{\nu} x^{\nu}\right)\right)(x) = \frac{\partial}{\partial x^{\eta}}\left(\frac{\partial f}{\partial x^{\mu}} B^{\mu}_{\nu} x^{\nu}\right) A^{\eta}_{\sigma} x^{\sigma} =$$

$$= \left(\frac{\partial^2 f}{\partial x^{\eta}\partial x^{\mu}} B^{\mu}_{\nu} x^{\nu} + \frac{\partial f}{\partial x^{\mu}} B^{\mu}_{\nu} \frac{\partial x^{\nu}}{\partial x^{\eta}}\right) A^{\eta}_{\sigma} x^{\sigma} = \frac{\partial^2 f}{\partial x^{\eta}\partial x^{\mu}} B^{\mu}_{\nu} A^{\eta}_{\sigma} x^{\nu} x^{\sigma} + \frac{\partial f}{\partial x^{\mu}} B^{\mu}_{\eta} A^{\eta}_{\sigma} x^{\sigma}$$

$$(\hat{B}\hat{A}(f))(x) = \frac{\partial^2 f}{\partial x^{\eta}\partial x^{\mu}} A^{\mu}_{\nu} B^{\eta}_{\sigma} x^{\nu} x^{\sigma} + \frac{\partial f}{\partial x^{\mu}} A^{\mu}_{\eta} B^{\eta}_{\sigma} x^{\sigma}$$

$$[\hat{A}, \hat{B}](f)(x) = -\nabla f \cdot ([A, B]x)$$

If $\tilde{A} f = i \hat{A} f$

$$[\tilde{A}, \tilde{B}] f = -[\hat{A}, \hat{B}] f = \nabla f \cdot [A, B]$$

Thus

$$(\hat{A} f)(x) = i \frac{\partial f}{\partial x^\mu} A^\mu_\nu x^\nu$$ has the same commutation relations as A.