Number-Theoretic Significance of li (x)

(Assuming Riemann's hypothesis that all non-real zeros of $\zeta(z)$ have a real part of $\frac{1}{2}$)

5.1.50 $\quad \text{li}(x)-\pi(x)=O(\sqrt{x}\ln x) \qquad (x\to\infty)$

$\pi(x)$ is the number of primes less than or equal to x.

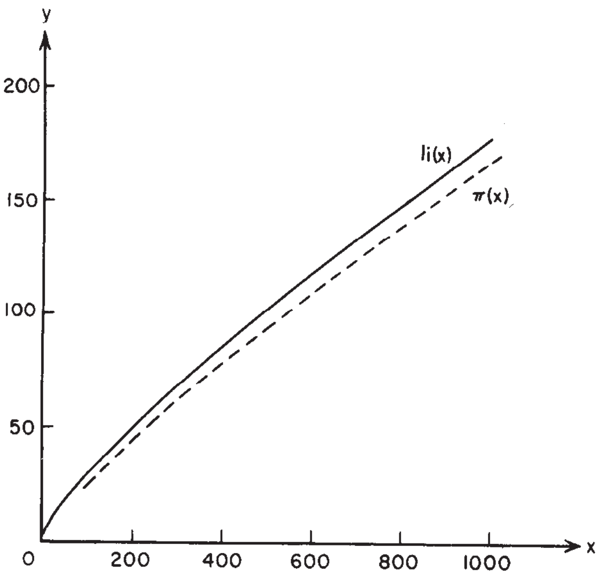

FIGURE 5.5. $y=\text{li}(x)$ *and* $y=\pi(x)$

Asymptotic Expansion

5.1.51

$$E_n(z)\sim\frac{e^{-z}}{z}\left\{1-\frac{n}{z}+\frac{n(n+1)}{z^2}-\frac{n(n+1)(n+2)}{z^3}+\ldots\right\} \qquad (|\arg z|<\tfrac{3}{2}\pi)$$

Representation of $E_n(x)$ for Large n

5.1.52

$$E_n(x)=\frac{e^{-x}}{x+n}\left\{1+\frac{n}{(x+n)^2}+\frac{n(n-2x)}{(x+n)^4}+\frac{n(6x^2-8nx+n^2)}{(x+n)^6}+R(n,x)\right\}$$

$$-.36n^{-4}\leq R(n,x)\leq\left(1+\frac{1}{x+n-1}\right)n^{-4} \qquad (x>0)$$

Polynomial and Rational Approximations [5]

5.1.53 $\qquad 0\leq x\leq 1$

$$E_1(x)+\ln x=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\epsilon(x)$$

$$|\epsilon(x)|<2\times10^{-7}$$

$a_0=-.57721\ 566$	$a_3=\ \ .05519\ 968$
$a_1=\ \ .99999\ 193$	$a_4=-.00976\ 004$
$a_2=-.24991\ 055$	$a_5=\ \ .00107\ 857$

5.1.54 $\qquad 1\leq x<\infty$

$$xe^xE_1(x)=\frac{x^2+a_1x+a_2}{x^2+b_1x+b_2}+\epsilon(x)$$

$$|\epsilon(x)|<5\times10^{-5}$$

$a_1=2.334733$	$b_1=3.330657$
$a_2=\ .250621$	$b_2=1.681534$

5.1.55 $\qquad 10\leq x<\infty$

$$xe^xE_1(x)=\frac{x^2+a_1x+a_2}{x^2+b_1x+b_2}+\epsilon(x)$$

$$|\epsilon(x)|<10^{-7}$$

$a_1=4.03640$	$b_1=5.03637$
$a_2=1.15198$	$b_2=4.19160$

5.1.56 $\qquad 1\leq x<\infty$

$$xe^xE_1(x)=\frac{x^4+a_1x^3+a_2x^2+a_3x+a_4}{x^4+b_1x^3+b_2x^2+b_3x+b_4}+\epsilon(x)$$

$$|\epsilon(x)|<2\times10^{-8}$$

$a_1=\ 8.57332\ 87401$	$b_1=\ 9.57332\ 23454$
$a_2=18.05901\ 69730$	$b_2=25.63295\ 61486$
$a_3=\ 8.63476\ 08925$	$b_3=21.09965\ 30827$
$a_4=\ \ .26777\ 37343$	$b_4=\ 3.95849\ 69228$

5.2. Sine and Cosine Integrals

Definitions

5.2.1 $\qquad \text{Si}(z)=\int_0^z\frac{\sin t}{t}\,dt$

5.2.2 [6]

$$\text{Ci}(z)=\gamma+\ln z+\int_0^z\frac{\cos t-1}{t}\,dt \qquad (|\arg z|<\pi)$$

5.2.3 [7] $\qquad \text{Shi}(z)=\int_0^z\frac{\sinh t}{t}\,dt$

5.2.4 [7]

$$\text{Chi}(z)=\gamma+\ln z+\int_0^z\frac{\cosh t-1}{t}\,dt \qquad (|\arg z|<\pi)$$

[5] The approximation **5.1.53** is from E. E. Allen, Note 169, MTAC 8, 240 (1954); approximations **5.1.54** and **5.1.56** are from C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955; approximation **5.1.55** is from C. Hastings, Jr., Note 143, MTAC 7, 68 (1953) (with permission).

[6] Some authors [5.14], [5.16] use the entire function $\int_0^z(1-\cos t)dt/t$ as the basic function and denote it by $\text{Cin}(z)$. We have

$$\text{Cin}(z)=-\text{Ci}(z)+\ln z+\gamma.$$

[7] The notations $\text{Sih}(z)=\int_0^z\sinh t\,dt/t$, $\text{Cinh}(z)=\int_0^z(\cosh t-1)dt/t$ have also been proposed [5.14.]

5.2.5 $$\mathrm{si}(z)=\mathrm{Si}(z)-\frac{\pi}{2}$$

Auxiliary Functions

5.2.6 $$f(z)=\mathrm{Ci}(z)\sin z-\mathrm{si}(z)\cos z$$

5.2.7 $$g(z)=-\mathrm{Ci}(z)\cos z-\mathrm{si}(z)\sin z$$

Sine and Cosine Integrals in Terms of Auxiliary Functions

5.2.8 $$\mathrm{Si}(z)=\frac{\pi}{2}-f(z)\cos z-g(z)\sin z$$

5.2.9 $$\mathrm{Ci}(z)=f(z)\sin z-g(z)\cos z$$

Integral Representations

5.2.10 $$\mathrm{si}(z)=-\int_0^{\frac{\pi}{2}} e^{-z\cos t}\cos(z\sin t)dt$$

5.2.11 $$\mathrm{Ci}(z)+E_1(z)=\int_0^{\frac{\pi}{2}} e^{-z\cos t}\sin(z\sin t)dt$$

5.2.12 $$f(z)=\int_0^\infty \frac{\sin t}{t+z}dt=\int_0^\infty \frac{e^{-zt}}{t^2+1}dt \qquad (\mathscr{R}z>0)$$

5.2.13 $$g(z)=\int_0^\infty \frac{\cos t}{t+z}dt=\int_0^\infty \frac{te^{-zt}}{t^2+1}dt \qquad (\mathscr{R}z>0)$$

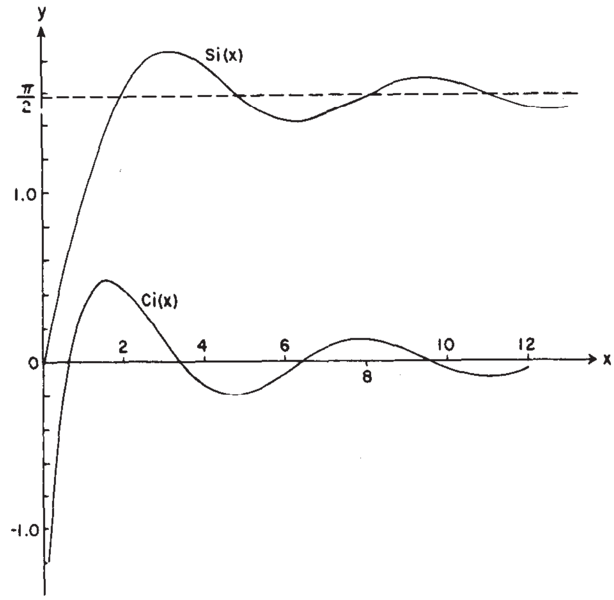

FIGURE 5.6. $y=\mathrm{Si}(x)$ *and* $y=\mathrm{Ci}(x)$

Series Expansions

5.2.14 $$\mathrm{Si}(z)=\sum_{n=0}^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}$$

5.2.15 $$\mathrm{Si}(z)=\pi\sum_{n=0}^\infty J^2_{n+\frac{1}{2}}\left(\frac{z}{2}\right)$$

5.2.16 $$\mathrm{Ci}(z)=\gamma+\ln z+\sum_{n=1}^\infty \frac{(-1)^n z^{2n}}{2n(2n)!}$$

5.2.17 $$\mathrm{Shi}(z)=\sum_{n=0}^\infty \frac{z^{2n+1}}{(2n+1)(2n+1)!}$$

5.2.18 $$\mathrm{Chi}(z)=\gamma+\ln z+\sum_{n=1}^\infty \frac{z^{2n}}{2n(2n)!}$$

Symmetry Relations

5.2.19 $$\mathrm{Si}(-z)=-\mathrm{Si}(z),\ \mathrm{Si}(\bar{z})=\overline{\mathrm{Si}(z)}$$

5.2.20
$$\mathrm{Ci}(-z)=\mathrm{Ci}(z)-i\pi \qquad (0<\arg z<\pi)$$
$$\mathrm{Ci}(\bar{z})=\overline{\mathrm{Ci}(z)}$$

Relation to Exponential Integral

5.2.21
$$\mathrm{Si}(z)=\frac{1}{2i}[E_1(iz)-E_1(-iz)]+\frac{\pi}{2} \qquad (|\arg z|<\frac{\pi}{2})$$

5.2.22 $$\mathrm{Si}(ix)=\frac{i}{2}[\mathrm{Ei}(x)+E_1(x)] \qquad (x>0)$$

5.2.23
$$\mathrm{Ci}(z)=-\frac{1}{2}[E_1(iz)+E_1(-iz)] \qquad (|\arg z|<\frac{\pi}{2})$$

5.2.24 $$\mathrm{Ci}(ix)=\frac{1}{2}[\mathrm{Ei}(x)-E_1(x)]+i\frac{\pi}{2} \qquad (x>0)$$

Value at Infinity

5.2.25 $$\lim_{x\to\infty}\mathrm{Si}(x)=\frac{\pi}{2}$$

Integrals

(For more extensive tables of integrals see [5.3], [5.6], [5.11], [5.12], [5.13].)

5.2.26 $$\int_z^\infty \frac{\sin t}{t}dt=-\mathrm{si}(z) \qquad (|\arg z|<\pi)$$

5.2.27 $$\int_z^\infty \frac{\cos t}{t}dt=-\mathrm{Ci}(z) \qquad (|\arg z|<\pi)$$

5.2.28 $$\int_0^\infty e^{-at}\mathrm{Ci}(t)dt=-\frac{1}{2a}\ln(1+a^2) \qquad (\mathscr{R}a>0)^*$$

5.2.29 $$\int_0^\infty e^{-at}\mathrm{si}(t)dt=-\frac{1}{a}\arctan a \qquad (\mathscr{R}a>0)$$

5.2.30 $$\int_0^\infty \cos t\,\mathrm{Ci}(t)dt=\int_0^\infty \sin t\,\mathrm{si}(t)dt=-\frac{\pi}{4}$$

*See page II.

5.2.31 $$\int_0^\infty \mathrm{Ci}^2(t)\,dt=\int_0^\infty \mathrm{si}^2(t)\,dt=\frac{\pi}{2}$$

5.2.32* $$\int_0^\infty \mathrm{Ci}(t)\,\mathrm{si}(t)\,dt=\ln 2$$

5.2.33

$$\int_0^1 \frac{(1-e^{-at})\cos bt}{t}\,dt=\frac{1}{2}\ln\left(1+\frac{a^2}{b^2}\right)+\mathrm{Ci}(b)$$
$$+\mathscr{R}E_1(a+ib)\quad(a \text{ real}, b>0)$$

Asymptotic Expansions

5.2.34

$$f(z)\sim\frac{1}{z}\left(1-\frac{2!}{z^2}+\frac{4!}{z^4}-\frac{6!}{z^6}+\ldots\right)\qquad(|\arg z|<\pi)$$

5.2.35

$$g(z)\sim\frac{1}{z^2}\left(1-\frac{3!}{z^2}+\frac{5!}{z^4}-\frac{7!}{z^6}+\ldots\right)\qquad(|\arg z|<\pi)$$

Rational Approximations [8]

5.2.36 $1\le x<\infty$

$$f(x)=\frac{1}{x}\left(\frac{x^4+a_1x^2+a_2}{x^4+b_1x^2+b_2}\right)+\epsilon(x)$$

$$|\epsilon(x)|<2\times10^{-4}$$

$a_1=7.241163$ $\quad b_1=9.068580$

$a_2=2.463936$ $\quad b_2=7.157433$

5.2.37 $1\le x<\infty$

$$g(x)=\frac{1}{x^2}\left(\frac{x^4+a_1x^2+a_2}{x^4+b_1x^2+b_2}\right)+\epsilon(x)$$

$$|\epsilon(x)|<10^{-4}$$

$a_1=7.547478$ $\quad b_1=12.723684$ *

$a_2=1.564072$ $\quad b_2=15.723606$ *

5.2.38 $1\le x<\infty$

$$f(x)=\frac{1}{x}\left(\frac{x^8+a_1x^6+a_2x^4+a_3x^2+a_4}{x^8+b_1x^6+b_2x^4+b_3x^2+b_4}\right)+\epsilon(x)$$

$$|\epsilon(x)|<5\times10^{-7}$$

$a_1=38.027264$ $\quad b_1=40.021433$

$a_2=265.187033$ $\quad b_2=322.624911$

$a_3=335.677320$ $\quad b_3=570.236280$

$a_4=38.102495$ $\quad b_4=157.105423$

5.2.39 $1\le x<\infty$

$$g(x)=\frac{1}{x^2}\left(\frac{x^8+a_1x^6+a_2x^4+a_3x^2+a_4}{x^8+b_1x^6+b_2x^4+b_3x^2+b_4}\right)+\epsilon(x)$$

$$|\epsilon(x)|<3\times10^{-7}$$

$a_1=42.242855$ $\quad b_1=48.196927$

$a_2=302.757865$ $\quad b_2=482.485984$

$a_3=352.018498$ $\quad b_3=1114.978885$

$a_4=21.821899$ $\quad b_4=449.690326$

Numerical Methods

5.3. Use and Extension of the Tables

Example 1. Compute Ci (.25) to 5D.

From **Tables 5.1** and **4.2** we have

$$\frac{\mathrm{Ci}(.25)-\ln(.25)-\gamma}{(.25)^2}=-.249350,$$

$$\mathrm{Ci}(.25)=(.25)^2(-.249350)+(-1.38629)$$
$$+.577216=-.82466.$$

Example 2. Compute Ei (8) to 5S.

From **Table 5.1** we have $xe^{-x}\mathrm{Ei}(x)=1.18185$ for $x=8$. From **Table 4.4**, $e^8=2.98096\times10^3$. Thus Ei (8) $=440.38$.

*See page II.

[8] From C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

Example 3. Compute Si (20) to 5D.

Since $1/20=.05$ from **Table 5.2** we find $f(20)=.049757$, $g(20)=.002464$. From **Table 4.8**, $\sin 20=.912945$, $\cos 20=.408082$. Using **5.2.8**

$$\mathrm{Si}(20)=\frac{\pi}{2}-f(20)\cos 20-g(20)\sin 20$$
$$=1.570796-.022555=1.54824.$$

Example 4. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=1.275$, $N=10$.

If x is less than about five, the recurrence relation **5.1.14** can be used in increasing order of n without serious loss of accuracy.

By quadratic interpolation in **Table 5.1** we get $E_1(1.275)=.1408099$, and from **Table 4.4**, $e^{-1.275}=.2794310$. The recurrence formula **5.1.14** then yields