

Notes and Solutions for Nielsen and Chuang's *Quantum Computation and Quantum Information*

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## Chapter 2

# Introduction to quantum mechanics

## Notes

### Notation

For distinct vectors in an orthonormal set, we can write  $\langle i|j\rangle = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker product and is 1 if  $i = j$  and 0 if  $i \neq j$ .

### Matrix - Linear Operator Congruence

For a matrix to be a linear operator,

$$A\left(\sum_i a_i |v_i\rangle\right) = \sum_i a_i A|v_i\rangle$$

must be true. Note the LHS is the sum of vectors to which  $A$  is applied which is certainly equal to the RHS.

Now suppose  $A : V \rightarrow W$  is a linear operator and that  $V$  has basis  $|v_i\rangle, \dots, |v_m\rangle$  and  $W$  has basis  $|w_i\rangle, \dots, |w_n\rangle$ . Since we know the  $k$ th column of a  $A$  will be its transformation of  $|v_k\rangle$ ,

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Note this is just saying  $A|v_j\rangle$  is equal to the  $j$ th column of  $A$ , and we can think of  $|w_i\rangle$  as the coordinates of the transformed vector. Thus, we can find the matrix representation of any linear operator by finding a matrix  $A$  with entries specified by the above equation.

### What's so special about the Pauli matrices?

Things to look into:

- they pop up in the Pauli equation
- they form a basis for the vector space of  $2 \times 2$  Hermitian matrices. Hermitian matrices represent observables? Look at Wikipedia page

### What does the Completeness Relation say about matrices?

I don't understand why

$$\sum_{ij} \langle w_j | A | v_i \rangle |w_j\rangle \langle v_i|$$

implies  $A$  has matrix element  $\langle w_j | A | v_i \rangle$  in the  $i$ th column and  $j$ th row, with respect to input basis  $|v_i\rangle$  and output basis  $|w_j\rangle$ .  
page 68.

## Solutions

### Exercise 2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

### Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

because  $A|0\rangle$  has coordinate 0 in  $|0\rangle$  and coordinate 1 in  $|1\rangle$ .

If we keep our input bases the same but reorder our output bases as  $|1\rangle$  and  $|0\rangle$ ,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### Exercise 2.3

We know

$$A|v_i\rangle = \sum_j A_{ji}|w_j\rangle$$

and

$$B|w_j\rangle = \sum_k B_{kj}|x_k\rangle$$

Now we can write

$$\begin{aligned} BA|v_i\rangle &= B(A|v_i\rangle) = B \sum_j A_{ji}|w_j\rangle = \sum_j A_{ji}(B|w_j\rangle) \\ &= \sum_j A_{ji} \sum_k B_{kj}|x_k\rangle \\ &= \sum_k \sum_j B_{kj} A_{ji}|x_k\rangle \\ &= \sum_k (BA)_{ki}|x_k\rangle \end{aligned}$$

We know  $\sum_k (BA)_{ki}$  is the matrix representation of operator  $BA$ , which the preceding step says is equal to  $\sum_k \sum_j B_{kj} A_{ji}$ , which is the matrix multiplication  $BA$ .

### Exercise 2.4

For the same input and output basis, we want some  $I$  such that

$$I|v_j\rangle = \sum_i I_{ij}|v_i\rangle = |v_j\rangle$$

which means  $I_{ij} = 0$  for all  $i \neq j$  and 1 otherwise.

### Exercise 2.5

For  $|y\rangle, |z_i\rangle \in \mathbb{C}^n$  and  $\lambda_i \in \mathbb{C}$ ,

$$\begin{aligned} \left( |y\rangle, \sum_i \lambda_i |z_i\rangle \right) &= |y\rangle^* \sum_i \lambda_i |z_i\rangle \\ &= \sum_i \lambda_i |y\rangle^* |z_i\rangle \\ &= \left( \sum_i \lambda_i^* |z_i\rangle^* |y\rangle \right)^* \end{aligned}$$

The second and third equalities demonstrate linearity in the second argument and  $(|y\rangle, |z\rangle) = (|z\rangle, |w\rangle)^*$ . Finally, if  $|w\rangle = (w_1, \dots, w_n)$  where  $w_i \in \mathbb{C}^n$ , then

$$(|w\rangle, |w\rangle) = \sum_i w_i^* w_i = \sum_i |w_i|^2$$

which proves the non-degeneracy and non-negativity condition.

### Exercise 2.6

$$\begin{aligned} \left( \sum_i \lambda_i |w_i\rangle, |v\rangle \right) &= \left( |v\rangle^*, \sum_i \lambda_i^* |w_i\rangle^* \right)^* \\ &= \sum_i \lambda_i^* \left( |v\rangle^*, |w_i\rangle^* \right)^* \\ &= \sum_i \lambda_i^* \left( |w_i\rangle, |v\rangle \right) \end{aligned}$$

### Exercise 2.7

$$(|w\rangle, |v\rangle) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0$$

To normalize, divide each vector by  $\sqrt{2}$ .

### Exercise 2.8

Since at each step, we divide by the norm of the vector being added, the set is orthonormal. The set is a basis because at step  $i$ , we add the basis vector  $|w_i\rangle$  but subtract out the portion that was already in  $\text{span}(|v_1\rangle, \dots, |v_{i-1}\rangle)$ , so we still end up spanning the full vector space.

### Exercise 2.9

$$\begin{aligned} \sigma_x &= |1\rangle \langle 0| + |0\rangle \langle 1| \\ \sigma_y &= i|1\rangle \langle 0| - i|0\rangle \langle 1| \\ \sigma_z &= |0\rangle \langle 0| - |1\rangle \langle 1| \end{aligned}$$

### Exercise 2.10

$$\begin{aligned} |v_j\rangle \langle v_k| &= I |v_j\rangle \langle v_k| I \\ &= \sum_a |v_a\rangle \langle v_a| v_j \rangle \sum_b \langle v_k| v_b \rangle \langle v_b| \\ &= \sum_{a,b} \delta_{aj} \delta_{kb} |v_a\rangle \langle v_b| \end{aligned}$$

so the element  $(|v_j\rangle \langle v_k|)_{ab} = \delta_{aj} \delta_{kb}$ .

**Exercise 2.11**

Each of the Pauli matrices has eigenvalues  $\pm 1$ .

For  $\sigma_x$ ,

$$\sigma_{x+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \sigma_{x-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For  $\sigma_y$ ,

$$\sigma_{y+} = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } \sigma_{y-} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

For  $\sigma_z$ ,

$$\sigma_{z+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \sigma_{z-} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The diagonalization easily follows.

**Exercise 2.12**

The characteristic equation is  $(1 - \lambda)^2$ , so we have eigenvalue 1. Solving  $(A - 1I)|v\rangle = 0$  gives us  $|v\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Our diagonal form should be

$$A = |v\rangle\langle v| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

**Exercise 2.13**

$$(|w\rangle\langle v|)^\dagger = \langle v|^\dagger |w\rangle^{dagger} = |v\rangle\langle w|$$

**Exercise 2.14**

Since we know  $(a + b)^\dagger = a^\dagger + b^\dagger$ , so

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i (a_i A_i)^\dagger = \sum_i (a_i^* A_i^\dagger)$$

**Exercise 2.15**

$$(|v\rangle, A|w\rangle) = (A^\dagger |v\rangle, |w\rangle) = (|v\rangle, (A^\dagger)^\dagger |w\rangle)$$

since this holds for all  $|v\rangle, |w\rangle$ ,  $A = (A^\dagger)^\dagger$ .

**Exercise 2.16**

$$P^2 = \sum_i |i\rangle\langle i| \sum_j |j\rangle\langle j| = \sum_{ij} |i\rangle\langle i|j\rangle\langle j| = \sum_{ij} \delta_{ij} |i\rangle\langle j|$$

Intuitively, projecting some  $|v\rangle \in P$  wouldn't change  $|v\rangle$  at all.

**Exercise 2.17**

Since  $A$  is normal, we can diagonalize it. If the eigenvalues are all positive, then the adjoint of this diagonal matrix of eigenvalues equals itself. If  $A$  is Hermitian, then so is its diagonal form. The adjoint of a diagonal matrix consists of the conjugates of its diagonal entries, so if  $A = A^\dagger$ , then the diagonal entries (eigenvalues) must all be positive.

**Exercise 2.18**

For an eigenvector  $|v\rangle$ , we have  $A|v\rangle = \lambda|v\rangle \rightarrow \langle v|A^\dagger = \lambda^*\langle v|$ . Multiplying these two gives us  $\langle v|A^\dagger A|v\rangle = \lambda^*\lambda\langle v|v\rangle$ , and because  $A^\dagger A = I$ ,

$$\|v\|^2 = |\lambda|^2 \|v\|^2 \rightarrow |\lambda| = 1$$

**Exercise 2.19**

Omitted because it's just mechanical.

**Exercise 2.20**

$$\begin{aligned} A'_{ij} &= \langle v_i|A|v_j\rangle \\ &= \langle v_i|U^\dagger UAUU^\dagger|v_j\rangle \\ &= \sum_a \sum_b \sum_c \sum_d \langle v_i|w_a\rangle \langle v_a|v_b\rangle \langle w_b|A|w_c\rangle \langle v_c|v_d\rangle \langle w_d|v_j\rangle \\ &= \sum_a \sum_b \sum_c \sum_d \delta_{ab}\delta_{cd} \langle v_i|w_a\rangle A''_{bc} \langle w_d|v_j\rangle \end{aligned}$$

This tells us  $a = b$  and  $c = d$ , so

$$A'_{ij} = \sum_a \sum_c \langle v_i|w_a\rangle A''_{ac} \langle w_c|v_j\rangle$$

**Exercise 2.21**

We will prove any Hermitian operator  $M$  is diagonal with respect to some orthonormal basis  $V$ .

We proceed by induction on dimension  $d$  on  $V$ .

**Base case:**  $d = 1$

Trivially,  $M$  is diagonal.

**Inductive hypothesis:** Assume  $d = n - 1$

**Inductive step:** Prove  $d = n$

Let  $\lambda$  be an eigenvalue of  $M$ ,  $P$  be a projection onto the  $\lambda$  eigenspace, and  $Q$  be  $P$ 's orthogonal complement.

We know  $M = (P + Q)M(P + Q)$ . First, note that  $QMP = \lambda QP = 0$ . Now for some  $|v\rangle \in P$ ,  $M|v\rangle = M^\dagger|v\rangle = \lambda|v\rangle$  because  $M$  is Hermitian, which means  $|v\rangle$  is in the eigenspace  $\lambda$  of  $M^\dagger$ . Now we have  $QM^\dagger P|v\rangle = QM^\dagger|v\rangle = \lambda Q|v\rangle = 0$ . Taking the adjoint of this gives us  $PMQ = 0$ . Now we have  $M = PMP + QMQ$ .

Since  $PMP = \lambda P$ ,  $QMQ$  must be nonzero also for  $M = PMP + QMQ$  to hold, so they both have dimension less than  $n$ . Finally,  $PMP = (PMP)^\dagger = PM^\dagger P$  and similarly for  $QMQ$ . Since they're both Hermitian, our inductive hypothesis proves the theorem.

**Exercise 2.22**

For a Hermitian operator  $A$ , suppose  $A|v\rangle = \lambda|v\rangle$  and  $A|w\rangle = \mu|w\rangle$ .

Since  $\langle v|A = \lambda\langle v|$ , we can write

$$\langle v|A^2|w\rangle = \mu^2 \langle v|w\rangle = \lambda\mu \langle v|w\rangle$$

where the first equality follows from  $A^2|w\rangle = \mu^2|w\rangle$ . Since  $\lambda \neq \mu$ ,  $\langle v|w\rangle = 0$ .

**Exercise 2.23**

Suppose  $|v\rangle$  is an eigenvector of  $P$  with eigenvalue  $\lambda$ ,  $P|v\rangle = \lambda|v\rangle$ . Then

$$P|v\rangle = P^2|v\rangle = \lambda^2|v\rangle$$

where the first equality follows from the property  $P^2 = P$ . Since  $\lambda^2 = \lambda$ ,  $P$ 's eigenvalues must be 1 or 0.

### Exercise 2.24

Let  $B = \frac{A+A^\dagger}{2}$  and  $C = \frac{A-A^\dagger}{2i}$ . This means

$$\langle v|A|v\rangle = \langle v|B|v\rangle + i\langle v|C|v\rangle$$

Since we cannot have an imaginary term in a positive operator,  $C = 0$ , so  $A = A^\dagger$ .

### Exercise 2.25

We can write  $\langle Av|Av\rangle = \langle A^\dagger Av|v\rangle$ . But since  $\langle Av|Av\rangle$  can also be written as  $\|Av\|^2 \geq 0$ ,  $A^\dagger A$  must be positive.

### Exercise 2.26

As a tensor product:

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

As a Kronecker product: Since  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ ,

$$\begin{bmatrix} \frac{1}{\sqrt{2}}|\psi\rangle \\ \frac{1}{\sqrt{2}}|\psi\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

As a tensor product:

$$|\psi\rangle^{\otimes 3} = |\psi\rangle \otimes |\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)$$

As a Kronecker product:

$$\begin{bmatrix} \frac{1}{\sqrt{2}}|\psi\rangle^{\otimes 2} \\ \frac{1}{\sqrt{2}}|\psi\rangle^{\otimes 2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}^T$$

### Exercise 2.27

$$X \otimes Z = \begin{bmatrix} 0 \cdot Z & 1 \cdot Z \\ 1 \cdot Z & 0 \cdot Z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$I \otimes X = \begin{bmatrix} 1 \cdot X & 0 \cdot X \\ 0 \cdot X & 1 \cdot X \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$X \otimes I = \begin{bmatrix} 0 \cdot I & 1 \cdot I \\ 1 \cdot I & 0 \cdot I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Since  $I \otimes X \neq X \otimes I$ , the tensor product is not commutative.



### Exercise 2.28

Writing the Kronecker product,

$$(A \otimes B)^\dagger = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}^\dagger = \begin{bmatrix} A_{11}^*B^\dagger & \cdots & A_{n1}^*B^\dagger \\ \vdots & \ddots & \vdots \\ A_{1n}^*B^\dagger & \cdots & A_{nn}^*B^\dagger \end{bmatrix} = A^\dagger \otimes B^\dagger$$

Proving the transpose is similar and proving the complex conjugate requires only using  $B^*$  instead of  $B^\dagger$ .

### Exercise 2.29

Let  $U_1$  and  $U_2$  be unitary.

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) = U_1^\dagger U_1 \otimes U_2^\dagger U_2 = I \otimes I = I$$

### Exercise 2.30

Let  $A = A^\dagger$  and  $B = B^\dagger$ .

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$$

### Exercise 2.31

Let  $A$  and  $B$  be positive operators.

$$\left( (A \otimes B)(|v\rangle \otimes |w\rangle), (|v\rangle \otimes |w\rangle) \right) = \left( A|v\rangle \otimes B|w\rangle, |v\rangle \otimes |w\rangle \right) = \langle v|A^\dagger|v\rangle \langle w|B^\dagger|w\rangle = \langle v|A|v\rangle \langle w|B|w\rangle$$

since we know positive operators are Hermitian. Since  $\langle v|A|v\rangle$  and  $\langle w|B|w\rangle$  are both non-negative, their product is also non-negative.

### Exercise 2.32

Let  $P$  and  $Q$  be projectors. Recall that if  $P^2 = P$ ,  $P$  is a projector.

$$(P \otimes Q)(P \otimes Q) = P^2 \otimes Q^2 = P \otimes Q$$

### Exercise 2.33

We proceed by induction on  $n$ .

**Base case:**  $n = 1$

We know

$$H^{\otimes 1} = H = \frac{1}{\sqrt{2}} \left[ (|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1| \right] = \frac{1}{\sqrt{2}} \sum_{x,y \in \{0,1\}^1} (-1)^{x \cdot y} |x\rangle \langle y|$$

**Inductive hypothesis:** Assume  $n = k - 1$

$$H^{\otimes k-1} = \frac{1}{2^{k-1/2}} \sum_{x,y \in \{0,1\}^{k-1}} (-1)^{x \cdot y} |x\rangle \langle y|$$

**Inductive step:** Prove  $n = k$

$$H^{\otimes k} = H \otimes H^{\otimes k-1} = \frac{1}{2^{k/2}} \sum_{x_1, y_1 \in \{0,1\}} (-1)^{x_1 \cdot y_1} |x_1\rangle \langle y_1| \otimes \sum_{x_2, y_2 \in \{0,1\}^{k-1}} (-1)^{x_2 \cdot y_2} |x_2\rangle \langle y_2|$$

Since this tensor product only flips the sign of the  $H^{\otimes k-1}$  if  $x_1 = y_1 = 1$ , it is easy to see that concatenating  $x_1, x_2$  and  $y_1, y_2$ , would yield the dot product required to flip the signs when we want.

$$H^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{bmatrix} H & H \\ H & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$