Notes and Solutions for Nielsen and Chuang's *Quantum Computation and Quantum Information*

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Contents

Chapter 1

Introduction and overview

Notes

"Instead of looking at quantum systems purely as phenomena to be explained..., they looked at them as systems that can be *designed*...No longer is the quantum world taken merely as presented, but instead it can be *created*."

Bloch Spheres

For a complex number $z = x + iy$, where x and y are real, we can write

$$|z|^2 = z^* z = (x - iy)(x + iy) = x^2 + y^2$$

Using the polar representation gives us $z = r(\cos\theta + i\sin\theta)$. Substituting Euler's identity ($e^{i\theta} = \cos\theta + i\sin\theta$) results in

$$z = re^{i\theta}$$

We know qubits have complex amplitudes, so we can write

$$|\psi\rangle = r_\alpha e^{i\theta_\alpha} |0\rangle + r_\beta e^{i\theta_\beta} |1\rangle$$

where all four variables are real.
It turns out the term $e^{i\gamma}$ doesn't affect the probabilities $|\alpha|^2$ or $\left|\beta\right|^2$:

$$\left|e^{i\gamma}\alpha\right|^2 = (e^{i\gamma}\alpha)^*(e^{i\gamma}\alpha) = (e^{-i\gamma}\alpha^*)(e^{i\gamma}\alpha) = \alpha^*\alpha = |\alpha|^2$$

so we can multiply by $e^{-i\theta_\alpha}$ to get

$$|\psi\rangle = r_\alpha |0\rangle + r_\beta e^{i\theta} |1\rangle$$

with three real parameters: r_α, $r\beta$, and $\theta = \theta_\beta - \theta_\alpha$.
Switching back to Cartesian coordinates and recalling the normalization condition, we have

$$\begin{aligned} |r_\alpha|^2 + \left|x + iy\right|^2 &= r_\alpha^2 + (x - iy)(x + iy) \\ &= r_\alpha^2 + x^2 + y^2 = 1 \end{aligned}$$

which is the equation for a sphere. Renaming r_α as z, we can use the identities

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

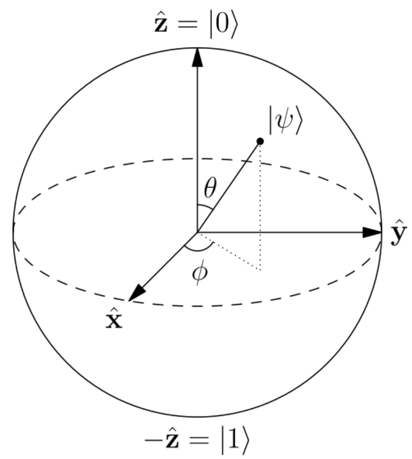
and after subbing in $r = 1$, we can write

$$\begin{aligned} |\psi\rangle &= z|0\rangle + (x + iy)|1\rangle \\ &= \cos\theta|0\rangle + \sin\theta(\cos\phi + i\sin\phi)|1\rangle \\ &= \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle \end{aligned}$$

Notice that $\theta = 0 \to |\psi\rangle = |0\rangle$ and $\theta = \frac{\pi}{2} \to |\psi\rangle = e^{i\phi}|1\rangle$, which suggests that $0 \le \theta \le \frac{\pi}{2}$ generates all superpositions. In fact, one can easily check that plugging in $\theta' = \pi - \theta$ and $\phi' = \phi + \pi$ (the opposite point on the sphere) results in $-|\psi\rangle$. Since the lower hemisphere of the sphere differs only by a phase factor of -1, we can choose to only consider the upper hemisphere.
To map points in the upper hemisphere onto a sphere, we can write

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

where $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.



The poles represent classical states. When a qubit is measured, it has higher probability of collapsing to the pole it's closer to. This representation makes it clear that the Pauli Z gate results in only a *phase change* because it does not affect the state the qubit will collapse to.[1] Now that we've derived Bloch spheres, there are two key properties we must understand:

1. *Orthogonality of opposite points:* Let $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, and let $|\varphi\rangle$ the opposite point on the Bloch sphere,

$$|\varphi\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle - e^{i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

   Using the identity $\cos(a+b) = \cos a\cos b - \sin a \sin b$, we get

$$(\varphi,\psi) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

2. *Rotations:* The Pauli X, Y, and Z gates are so-called because when exponentiated they yield rotation operators, which rotate the Bloch vector $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ about the x, y, and z axes. For example, the exponentiated Pauli X gate is

$$R_X(\theta) = e^{-i\theta X/2}$$

   The exponentiated Pauli Y and Z gates are similarly defined.

   To understand this, note that in the special case where $A^2 = I$ (which holds for all the Pauli matrices),

$$\begin{aligned} e^{i\theta A} &= I + i\theta A - \frac{\theta^2 I}{2!} - i\frac{\theta^3 A}{3!} + \frac{\theta^4 I}{4!} + i\frac{\theta^5 A}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right)I + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)A \\ &= \cos(\theta)I + i\sin(\theta)A \end{aligned}$$

   Now we can exponentiate the Pauli X gate as

$$R_X(\theta) = e^{-i\theta X/2} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

[1] Image from Anastasios Kyrillidis's notes.

Let's consider $R_X(\pi)$,

$$R_X(\pi) = \begin{bmatrix} \cos\frac{\pi}{2} & -i\sin\frac{\pi}{2} \\ -i\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -iX$$

This is a 180 degree rotation across the x-axis since we swap amplitudes and the phase changes.[2] **Todo:** Need an actual reason for why.

Why must quantum gates be unitary?

For a quantum state $\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$, we know $|\alpha|^2 + \left|\beta\right|^2 = 1$ must be true. Another way of writing this is $(\left|\psi\right\rangle, \left|\psi\right\rangle) = 1$. This must also be true after the application of a quantum gate U. Thus, we have

$$\begin{aligned} 1 &= (U\left|\psi\right\rangle, U\left|\psi\right\rangle) = (\left|\psi\right\rangle, \left|\psi\right\rangle) \\ &= (U^\dagger U\left|\psi\right\rangle, \left|\psi\right\rangle) = (\left|\psi\right\rangle, \left|\psi\right\rangle) \end{aligned}$$

which means $U^\dagger U = I$ is a fundamental constraint on quantum gates.

Decomposing single qubit operations

Move to section 4.2 according to Box 1.1 on p20.

What does the matrix representation of a quantum gate mean?

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This U_{CN} gate acts on two qubits, so it is a transformation on basis vectors $|0\rangle\otimes|0\rangle$, $|0\rangle\otimes|1\rangle$, $|1\rangle\otimes|0\rangle$, and $|1\rangle\otimes|1\rangle$. The columns tell us that an input of the third basis vector, $|1\rangle\otimes|0\rangle$, will output the fourth basis vector, $|1\rangle\otimes|1\rangle$.
Todo: What does this say about superpositions?

Why must quantum gates be reversible?

Since quantum gates are unitary, we know $U^\dagger U = I$, which means quantum gates are invertible, and the inverse is also a quantum gate because it is unitary.
One implication of this is that classical gates like XOR and NAND have no quantum cousins because these gates are irreversible since they result in a loss of information.
Well, technically, the Toffoli gate can be used to simulate NAND gates (and therefore all classical gates), but it can only do so because it explicitly preserves all input bits.

Solutions

Exercise 1.1

2 evaluations. If we get the same value and guess the constant function, there's a 25% chance it was the balanced function.

Exercise 1.2

If we could fully identify the state, we can just bitwise add our result to $|0\rangle$ to get a clone of our original state.
For the converse, if we had a device that could clone quantum states, we could continue cloning and measuring the clones to get an arbitrary level of accuracy of the original quantum state.

[2]Most of this section is from Ian Glendinning's talk.

Chapter 2

Introduction to quantum mechanics

Exercise 2.1: (Linear dependence: example)

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

Exercise 2.2: (Matrix representations: example)
A is the NOT gate,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Exercise 2.3: (Matrix representation for operator products)
Figure 2.12 implies that we can write any linear transformation as a matrix whose columns are the transformed basis vectors. So

$$A = \begin{bmatrix} A\left|v_1\right\rangle & \cdots & A\left|v_n\right\rangle \end{bmatrix}$$

so

$$BA = \begin{bmatrix} BA\left|v_1\right\rangle & \cdots & BA\left|v_n\right\rangle \end{bmatrix}$$

I think this is a sufficient answer, though I'm not completely sure what the exercise is asking us to demonstrate.
Exercise 2.4: (Matrix representation for identity)
By definition of the identity operator,

$$I\left|v_i\right\rangle = \left|v_i\right\rangle$$

Using Figure 2.12,

$$I\left|v_i\right\rangle = \sum_i I_{ii}\left|v_i\right\rangle$$

which means $I_{ii} = 1$.
Exercise 2.5:
1.

$$(\left|v\right\rangle, \lambda\left|w\right\rangle) = v_1^* \lambda w_1 + \cdots + v_n^* \lambda w_n = \lambda(\left|v\right\rangle, \left|w\right\rangle)$$

2.

$$(\left|v\right\rangle, \left|w\right\rangle) =$$

Exercise 2.53: What are the eigenvalues and eigenvectors of H?
Solving $det(H - \lambda I) = 0$,

$$det\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = det\left(\begin{bmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0$$

So $\lambda^2 = 1$.
For $\lambda = 1$, the eigenvector is $\frac{1}{-1+\sqrt{2}}$.