

UNIVERSITY OF CALIFORNIA, BERKELEY

LITERALLY EVERYTHING I KNOW ABOUT

# Linear Algebra

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A very reductionist summary of LINEAR ALGEBRA AND ITS APPLICATIONS by Lay, Lay, and McDonald, as well as LINEAR ALGEBRA DONE WRONG by Treil.

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# Contents

<b>1</b>	<b>Basic Notations</b>	<b>1</b>
1.1	Vector Spaces . . . . .	1
1.2	Linear Combinations . . . . .	1
1.3	Linear Transformations . . . . .	2
1.4	Invertible Transformations and Isomorphisms . . . . .	4
1.5	Subspaces . . . . .	6
<b>2</b>	<b>Systems of Linear Equations</b>	<b>7</b>
2.1	Basic Notation . . . . .	7

# Chapter 1

## Basic Notations

### 1.1 Vector Spaces

**Definition 1.1.1.** A **vector space**  $V$  is a collection of vectors, along with vector addition and scalar multiplication defined such that for vectors  $u, v$ , and  $w$ :

1. Commutative:  $v + w = w + v$
2. Associative:  $(u + v) + w = u + (v + w)$
3. Zero vector:  $v + 0 = v$
4. Additive inverse:  $v + w = 0$
5. Multiplicative identity:  $1v = v$
6. Multiplicative associative:  $(\alpha\beta)v = \alpha(\beta v)$
7. Distribution of scalars:  $\alpha(u + v) = \alpha u + \alpha v$
8. Distribution of vectors:  $(\alpha + \beta)u = \alpha u + \beta u$

These properties are simply to ensure that vector spaces are **abelian groups**.

**Definition 1.1.2.** An  $m \times n$  **matrix** is an array with  $m$  rows and  $n$  columns. Elements of a matrix are called *entries*. Given a matrix  $A$ , its **transpose** is defined as the matrix whose columns are  $A$ 's rows, so  $A^T$  is an  $n \times m$  matrix.

### 1.2 Linear Combinations

**Definition 1.2.1.** A **linear combination** of vectors  $v_1, \dots, v_p \in V$  is a sum of the form

$$\alpha_1 v_1 + \dots + \alpha_p v_p = \sum_{k=1}^p \alpha_k v_k$$

**Definition 1.2.2.** A set of vectors  $v_1, \dots, v_n$  is said to be **linearly independent** if the equation

$$\alpha_1 v_1 + \dots + \alpha_n v_n = 0$$

has only the trivial solution.

**Definition 1.2.3.** A **basis** is a set of vectors  $v_1, \dots, v_n \in V$  such that any vector  $u \in V$  has a *unique* representation as a linear combination

$$u = \alpha_1 v_1 + \dots + \alpha_n v_n$$

The coefficients  $\alpha_1, \dots, \alpha_n$  are called *coordinates* of  $u$ .

Fundamentally, our definition of basis states that a basis must be spanning and unique. In order for a representation to be unique, we know the basis must be linearly independent.

**Theorem 1.2.1.** A set of vectors  $v_1, \dots, v_p \in V$  is a basis if and only if it is linearly independent and complete (spanning).

*Proof.* We already know basis must be linearly independent and spanning, so we just need to prove the other direction.

Suppose the set  $v_1, \dots, v_p$  is linearly independent. Then we know for some vector  $u \in V$ :

$$u = \sum_{k=1}^n \alpha_k v_k$$

All that is remaining is to prove this representation is unique.

Suppose there is another representation,  $u = \sum_{k=1}^n \beta_k v_k$ . Then,

$$\sum_{k=1}^n (\alpha_k - \beta_k) v_k = u - u = 0$$

Since the set is linearly independent, we know  $\alpha_k - \beta_k = 0$ . Thus, the representation is unique. ■

### 1.3 Linear Transformations

**Definition 1.3.1.** A **transformation**  $T$  from set  $X$  to set  $Y$  assigns a value  $y \in Y$  for every value  $x \in X$ :  $y = T(x)$ .  $X$  is called the *domain* of  $T$ ,  $Y$  is called the *codomain* of  $T$ , and the set of all  $T(x)$  is called the *range* of  $T$ .

Let  $V, W$  be vector spaces. A transformation  $T : V \rightarrow W$  is **linear** if:

1.  $T(u + v) = T(u) + T(v)$
2.  $T(\alpha v) = \alpha T(v)$

A mapping  $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$  is *onto*  $\mathbb{F}^m$  if each  $b$  in  $\mathbb{F}^m$  is the image of at least one  $x$  in  $\mathbb{F}^n$ .

A mapping  $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$  is *one-to-one* if each  $b$  in  $\mathbb{F}^m$  is the image of at most one  $x$  in  $\mathbb{F}^n$ .

We can represent linear transformations with matrices. To represent a transformation  $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$ , we need to only how our  $n$  basis vectors are transformed. To see this, note that any vector  $u = \alpha_1 v_1 + \cdots + \alpha_n v_n$ . So  $T(x) = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$ . If we join the vectors  $T(v_1), \dots, T(v_n)$  in a matrix  $A = [T(v_1) \ \cdots \ T(v_n)]$ , we have captured all the information about  $T$ .

**Definition 1.3.2.** There are two ways to approach **matrix-vector multiplication**:

*Column by coordinate rule:* Multiply each column of the matrix by the corresponding coordinate of the vector and add.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

*Row by column rule:* To get entry  $k$  of the result, multiply row  $k$  of the matrix with the column.

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 \\ 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \end{bmatrix}$$

**Definition 1.3.3.** The natural extension to **matrix multiplication** of two matrices  $AB$  is to multiply  $A$  by each column of  $B$ .

$$AB = A \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} Ab_1 & \cdots & Ab_n \end{bmatrix}$$

Using the *row by columns rule*, we can see that the  $(AB)_{j,k} = (\text{row } j \text{ of } A) \cdot (\text{column } k \text{ of } B)$ . This also means  $AB$  is only defined if  $A$  is  $m \times n$  and  $B$  is  $n \times r$ .

Let  $A$  be an  $m \times n$  matrix and let  $B$  and  $C$  have sizes for which the indicated sums and products are defined.

1.  $A(BC) = (AB)C$
2.  $A(B + C) = AB + AC$
3.  $(B + C)A = BA + CA$
4.  $\alpha(AB) = (\alpha A)B = A(\alpha B)$
5.  $I_m A = A = A I_n$

Warnings:

1. In general,  $AB \neq BA$
2. If  $AB = AC$ , then it is **not true** in general that  $B = C$
3. If  $AB = 0$ , then it is **not true** always that  $A = 0$  or  $B = 0$

**Definition 1.3.4.** The **transpose** of  $A$  is the matrix whose columns are formed from the corresponding rows of  $A$ , denoted by  $A^T$ .

1.  $(A^T)^T = A$
2.  $(A + B)^T = A^T + B^T$
3.  $(cA)^T = cA^T$
4.  $(AB)^T = B^T A^T$

To understand the final property, let  $AB$  denote a  $n \times m$  matrix so that

$$AB = \begin{bmatrix} A_{1*} \cdot B_1 & A_{1*} \cdot B_2 & \cdots & A_{1*} \cdot B_m \\ A_{2*} \cdot B_1 & A_{2*} \cdot B_2 & \cdots & A_{2*} \cdot B_m \\ \vdots & \vdots & \ddots & \vdots \\ A_{n*} \cdot B_1 & A_{n*} \cdot B_2 & \cdots & A_{n*} \cdot B_m \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} A_{1*} \cdot B_1 & A_{2*} \cdot B_1 & \cdots & A_{n*} \cdot B_1 \\ A_{1*} \cdot B_2 & A_{2*} \cdot B_2 & \cdots & A_{n*} \cdot B_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_{1*} \cdot B_m & A_{2*} \cdot B_m & \cdots & A_{n*} \cdot B_m \end{bmatrix}$$

where  $A_{\alpha*}$  denotes the  $\alpha$ th row of a  $A$ . Note that  $(AB)_{jk}^T = (\text{row } k \text{ of } A) \cdot (\text{column } j \text{ of } B) = (\text{row } j \text{ of } B^T) \cdot (\text{column } k \text{ of } A^T)$ .

**Definition 1.3.5.** For a *square* matrix  $A$ , its **trace** is the sum of its diagonal entries.

$$\text{trace}(A) = \sum_{k=1}^n a_{k,k}$$

**Theorem 1.3.1.** Let  $A$  and  $B$  be size  $m \times n$  and  $n \times m$ , respectively. Then

$$\text{trace}(AB) = \text{trace}(BA)$$

*Proof.* We need only show that the diagonal entries of  $AB$  are the same as the diagonal entries of  $BA$ . Using the *row by columns rule*, we know  $(AB)_{j,k} = (\text{row } j \text{ of } A) \cdot (\text{column } k \text{ of } B)$ , so when applying to diagonal entries we get  $(AB)_{k,k} = (BA)_{k,k} = (\text{row } k \text{ of } A) \cdot (\text{column } k \text{ of } B)$  ■

## 1.4 Invertible Transformations and Isomorphisms

**Definition 1.4.1.** An  $n \times n$  matrix  $A$  is **invertible** if there is an  $n \times n$  matrix  $A^{-1}$  such that  $A^{-1}A = I$ .

**Theorem 1.4.1.** *If  $A$  and  $B$  are invertible (and such that  $AB$  is defined), then the product  $AB$  is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

*Proof.* Direct computation:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$$

■

**Theorem 1.4.2.** *If  $A$  is invertible, then  $A^T$  is also invertible and*

$$(A^{-1})^T = (A^T)^{-1}$$

*Proof.* Using  $(AB)^T = B^T A^T$ ,

$$(A^{-1})^T(A^T) = (AA^{-1})^T = I$$

and

$$A^T(A^{-1})^T = (A^{-1}A)^T = I$$

■

**Definition 1.4.2.** An invertible linear transformation  $A : V \rightarrow W$  is called an **isomorphism**. The two vector spaces  $V$  and  $W$  for which  $A$  is defined are called **isomorphic**, denoted  $V \cong W$ .

Isomorphic spaces can be considered different representations of the *same* space. The following theorem is an example.

**Theorem 1.4.3.** *Let  $A : V \rightarrow W$  be an isomorphism, and let  $v_1, \dots, v_n$  be a basis in  $V$ . Then  $Av_1, \dots, Av_n$  is a basis in  $W$ .*

*Proof.* Because  $V$  and  $W$  are isomorphic, every  $w \in W$  can be represented as some  $v \in V$  by applying  $A^{-1}$ . For arbitrary  $w \in W$

$$A^{-1}w = v = \sum_{k=1}^n \alpha_k v_k$$

Then we apply  $A$  to get

$$Av = w = \sum_{k=1}^n \alpha_k Av_k$$

■

## 1.5 Subspaces

**Definition 1.5.1.** A **subspace** of vector space  $V$  is a non-empty subset  $V_0 \subset V$  which is also a vector space.

For any linear transformation  $A : V \rightarrow W$ , we can associate the following subspaces:

1. The *null space*, or *kernel*, of  $A$  which consists of all vectors  $v \in V$  such that  $Av = 0$ .
2. The *range* of  $A$  which is the set of all vectors  $w \in W$  which can be represented as  $w = Av$  for  $v \in V$ .

By the *column by coordinate rule*, we know that any vector in  $\text{Range}(A)$  can be represented as a weighted sum of the column vectors of  $A$ , which is why the term Column Space is sometimes used to refer to Range.



## **Chapter 2**

# **Systems of Linear Equations**