Notes and Solutions for Nielsen and Chuang's *Quantum Computation and Quantum Information*

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Contents

Chapter 2

Introduction to quantum mechanics

Notes

Notation

For distinct vectors in an orthonormal set, we can write $\langle i|j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker product and is 1 if $i = j$ and 0 if $i \neq j$.

Matrix - Linear Operator Congruence

For a matrix to a be a linear operator,

$$A\Big(\sum_i a_i |v_I\rangle\Big) = \sum_i a_i A |v_i\rangle$$

must be true. Note the LHS is the sum of vectors to which A is applied which is certainly equal to the RHS.
Now suppose $A : V \to W$ is a linear operator and that V has basis $|v_i\rangle, \cdots, |v_m\rangle$ and W has basis $|w_i\rangle, \cdots, |w_n\rangle$. Since we know the kth column of a A will be its transformation of $|v_k\rangle$,

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Note this is just saying $A|v_j\rangle$ is equal to the jth column of A, and we can think of $|w_i\rangle$ as the coordinates of the transformed vector. Thus, we can find the matrix representation of any linear operator by finding a matrix A with entries specified by the above equation.

What's so special about the Pauli matrices?

Things to look into:

- they pop up in the Pauli equation
- they form a basis for the vector space of 2×2 Hermitian matrices. Hermitian matrices represent observables? Look at Wikipedia page

What does the Completeness Relation say about matrices?

I don't understand why

$$\sum_{ij} \langle w_j|A|v_i\rangle |w_j\rangle \langle v_i|$$

implies A has matrix element $\langle w_j \big| A|v_i\rangle$ in the ith column and jth row, with respect to input basis $|v_i\rangle$ and output basis $|w_j\rangle$. page 68.

Solutions

Exercise 2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

because $A|0\rangle$ has coordinate 0 in $|0\rangle$ and coordinate 1 in $|1\rangle$.

If we keep our input bases the same but reorder our output bases as $|1\rangle$ and $|0\rangle$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 2.3

We know

$$A|v_i\rangle = \sum_j A_{ji}|w_j\rangle$$

and

$$B|w_j\rangle = \sum_k B_{kj}|x_k\rangle$$

Now we can write

$$\begin{aligned} BA|v_i\rangle = B(A|v_i\rangle) = B\sum_j A_{ji}|w_j\rangle &= \sum_j A_{ji}(B|w_j\rangle) \\ &= \sum_j A_{ji}\sum_k B_{kj}|x_k\rangle \\ &= \sum_k \sum_j B_{kj}A_{ji}|x_k\rangle \\ &= \sum_k (BA)_{ki}|x_k\rangle \end{aligned}$$

We know $\sum_k (BA)_{ki}$ is the matrix representation of operator BA, which the preceding step says is equal to $\sum_k \sum_j B_{kj}A_{ji}$, which is the matrix multiplication BA.

Exercise 2.4

For the same input and output basis, we want some I such that

$$I|v_j\rangle = \sum_i I_{ij}|v_i\rangle = |v_j\rangle$$

which means $I_{ij} = 0$ for all $i \neq j$ and 1 otherwise.

Exercise 2.5

For $|y\rangle, |z_i\rangle \in \mathbb{C}^n$ and $\lambda_i \in C$,

$$\begin{aligned}\Big(|y\rangle, \sum_i \lambda_i |z_i\rangle\Big) &= |y\rangle^* \sum_i \lambda_i |z_i\rangle \\ &= \sum_i \lambda_i |y\rangle^* |z_i\rangle \\ &= \Big(\sum_i \lambda_i^* |z_i\rangle^* |y\rangle\Big)^*\end{aligned}$$

The second and third equalities demonstrate linearity in the second argument and $(|y\rangle, |z\rangle) = (|z\rangle, |w\rangle)^*$. Finally, if $|w\rangle = (w_1, \cdots, w_n)$ where $w_i \in \mathbb{C}^n$, then

$$(|w\rangle, |w\rangle) = \sum_i w_i^* w_i = \sum_i |w_i|^2$$

which proves the non-degeneracy and non-negativity condition.

Exercise 2.6

$$\begin{aligned}\Big(\sum_i \lambda_i |w_i\rangle, |v\rangle\Big) &= \Big(|v\rangle^*, \sum_i \lambda_i^* |w_i\rangle^*\Big)^* \\ &= \sum_i \lambda_i^* \Big(|v\rangle^*, |w_i\rangle^*\Big)^* \\ &= \sum_i \lambda_i^* \Big(|w_i\rangle, |v\rangle\Big)\end{aligned}$$

Exercise 2.7

$$(|w\rangle, |v\rangle) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 \\ -1\end{bmatrix} = 1 - 1 = 0$$

To normalize, divide each vector by $\sqrt{2}$.

Exercise 2.8

Since at each step, we divide by the norm of the vector being added, the set is orthonormal. The set is a basis because at step i, we add the basis vector $|w_i\rangle$ but subtract out the portion that was already in $\text{span}(|v_1\rangle, \cdots, |v_{i-1}\rangle)$, so we still end up spanning the full vector space.

Exercise 2.9

$$\begin{aligned}\sigma_x &= |1\rangle\langle 0| + |0\rangle\langle 1| \\ \sigma_y &= i|1\rangle\langle 0| - i|0\rangle\langle 1| \\ \sigma_z &= |0\rangle\langle 0| - |1\rangle\langle 1|\end{aligned}$$

Exercise 2.10

$$\begin{aligned}|v_j\rangle\langle v_k| &= I|v_j\rangle\langle v_k| I \\ &= \sum_a |v_a\rangle\langle v_a|v_j\rangle \sum_b \langle v_k|v_b\rangle\langle v_b| \\ &= \sum_{a,b} \delta_{aj}\delta_{kb} |v_a\rangle\langle v_b|\end{aligned}$$

so the element $(|v_j\rangle\langle v_k|)_{ab} = \delta_{aj}\delta_{kb}$.

Exercise 2.11

Each of the Pauli matrices has eigenvalues ± 1.
For σ_x,

$$\sigma_{x+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \sigma_{x-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For σ_y,

$$\sigma_{y+} = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } \sigma_{y-} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

For σ_z,

$$\sigma_{z+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \sigma_{z-} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The diagonalization easily follows.

Exercise 2.12

The characteristic equation is $(1-\lambda)^2$, so we have eigenvalue 1. Solving $(A - 1I)|v\rangle = 0$ gives us $|v\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Our diagonal form should be

$$A = |v\rangle \langle v| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Exercise 2.13

$$(|w\rangle \langle v|)^\dagger = \langle v|^\dagger \, |w\rangle^{dagger} = |v\rangle \langle w|$$

Exercise 2.14

Since we know $(a+b)^\dagger = a^\dagger + b^\dagger$, so

$$\left(\sum_i a_i A_i \right)^\dagger = \sum_i \left(a_i A_i \right)^\dagger = \sum_i \left(a_i^* A_i^\dagger \right)$$

Exercise 2.15

$$(|v\rangle, A|w\rangle) = (A^\dagger |v\rangle, |w\rangle) = \left(|v\rangle, (A^\dagger)^\dagger |w\rangle \right)$$

since this holds for all $|v\rangle, |w\rangle$, $A = (A^\dagger)^\dagger$.

Exercise 2.16

$$P^2 = \sum_i |i\rangle \langle i| \sum_j |j\rangle \langle j| = \sum_{ij} |i\rangle \langle i|j\rangle \langle j| = \sum_{ij} \delta_{ij} |i\rangle \langle j|$$

Intuitively, projecting some $|v\rangle \in P$ wouldn't change $|v\rangle$ at all.

Exercise 2.17

Since A is normal, we can diagonalize it. If the eigenvalues are all positive, then the adjoint of this diagonal matrix of eigenvalues equals itself. If A is Hermitian, then so is its diagonal form. The adjoint of a diagonal matrix consists of the conjugates of its diagonal entries, so if $A = A^\dagger$, then the diagonal entries (eigenvalues) must all be positive.