Notes and Solutions for Nielsen and Chuang's *Quantum Computation and Quantum Information*

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Contents

Chapter 2

Introduction to quantum mechanics

Notes

Notation

For distinct vectors in an orthonormal set, we can write $\langle i|j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker product and is 1 if $i = j$ and 0 if $i \neq j$.

Matrix - Linear Operator Congruence

For a matrix to a be a linear operator,

$$A\left(\sum_i a_i |v_I\rangle\right) = \sum_i a_i A |v_i\rangle$$

must be true. Note the LHS is the sum of vectors to which A is applied which is certainly equal to the RHS.
Now suppose $A : V \to W$ is a linear operator and that V has basis $|v_i\rangle, \cdots, |v_m\rangle$ and W has basis $|w_i\rangle, \cdots, |w_n\rangle$. Since we know the kth column of a A will be its transformation of $|v_k\rangle$,

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Note this is just saying $A|v_j\rangle$ is equal to the jth column of A, and we can think of $|w_i\rangle$ as the coordinates of the transformed vector. Thus, we can find the matrix representation of any linear operator by finding a matrix A with entries specified by the above equation.

What's so special about the Pauli matrices?

Things to look into:

- they pop up in the Pauli equation
- they form a basis for the vector space of 2×2 Hermitian matrices. Hermitian matrices represent observables? Look at Wikipedia page

What does the Completeness Relation say about matrices?

I don't understand why

$$\sum_{ij} \langle w_j|A|v_i\rangle |w_j\rangle \langle v_i|$$

implies A has matrix element $\left\langle w_j \middle| A|v_i\right\rangle$ in the ith column and jth row, with respect to input basis $|v_i\rangle$ and output basis $|w_j\rangle$. page 68.

Lucien Hardy Postulates of QM

Solutions

Exercise 2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

because $A|0\rangle$ has coordinate 0 in $|0\rangle$ and coordinate 1 in $|1\rangle$.
If we keep our input bases the same but reorder our output bases as $|1\rangle$ and $|0\rangle$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 2.3

We know

$$A|v_i\rangle = \sum_j A_{ji}|w_j\rangle$$

and

$$B|w_j\rangle = \sum_k B_{kj}|x_k\rangle$$

Now we can write

$$\begin{aligned} BA|v_i\rangle = B(A|v_i\rangle) = B\sum_j A_{ji}|w_j\rangle &= \sum_j A_{ji}(B|w_j\rangle) \\ &= \sum_j A_{ji}\sum_k B_{kj}|x_k\rangle \\ &= \sum_k\sum_j B_{kj}A_{ji}|x_k\rangle \\ &= \sum_k (BA)_{ki}|x_k\rangle \end{aligned}$$

We know $\sum_k (BA)_{ki}$ is the matrix representation of operator BA, which the preceding step says is equal to $\sum_k\sum_j B_{kj}A_{ji}$, which is the matrix multiplication BA.

Exercise 2.4

For the same input and output basis, we want some I such that

$$I|v_j\rangle = \sum_i I_{ij}|v_i\rangle = |v_j\rangle$$

which means $I_{ij} = 0$ for all $i \neq j$ and 1 otherwise.

Exercise 2.5

For $|y\rangle, |z_i\rangle \in \mathbb{C}^n$ and $\lambda_i \in C$,

$$\begin{aligned}\Big(|y\rangle, \sum_i \lambda_i |z_i\rangle\Big) &= |y\rangle^* \sum_i \lambda_i |z_i\rangle \\ &= \sum_i \lambda_i |y\rangle^* |z_i\rangle \\ &= \Big(\sum_i \lambda_i^* |z_i\rangle^* |y\rangle\Big)^*\end{aligned}$$

The second and third equalities demonstrate linearity in the second argument and $(|y\rangle, |z\rangle) = (|z\rangle, |w\rangle)^*$. Finally, if $|w\rangle = (w_1, \cdots, w_n)$ where $w_i \in \mathbb{C}^n$, then

$$(|w\rangle, |w\rangle) = \sum_i w_i^* w_i = \sum_i |w_i|^2$$

which proves the non-degeneracy and non-negativity condition.

Exercise 2.6

$$\begin{aligned}\Big(\sum_i \lambda_i |w_i\rangle, |v\rangle\Big) &= \Big(|v\rangle^*, \sum_i \lambda_i^* |w_i\rangle^*\Big)^* \\ &= \sum_i \lambda_i^* \Big(|v\rangle^*, |w_i\rangle^*\Big)^* \\ &= \sum_i \lambda_i^* \Big(|w_i\rangle, |v\rangle\Big)\end{aligned}$$

Exercise 2.7

$$(|w\rangle, |v\rangle) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0$$

To normalize, divide each vector by $\sqrt{2}$.

Exercise 2.8

Since at each step, we divide by the norm of the vector being added, the set is orthonormal. The set is a basis because at step i, we add the basis vector $|w_i\rangle$ but subtract out the portion that was already in span$(|v_1\rangle, \cdots, |v_{i-1}\rangle)$, so we still end up spanning the full vector space.

Exercise 2.9

$$\begin{aligned}\sigma_x &= |1\rangle\langle 0| + |0\rangle\langle 1| \\ \sigma_y &= i|1\rangle\langle 0| - i|0\rangle\langle 1| \\ \sigma_z &= |0\rangle\langle 0| - |1\rangle\langle 1|\end{aligned}$$

Exercise 2.10

$$\begin{aligned}|v_j\rangle\langle v_k| &= I|v_j\rangle\langle v_k| I \\ &= \sum_a |v_a\rangle\langle v_a|v_j\rangle \sum_b \langle v_k|v_b\rangle\langle v_b| \\ &= \sum_{a,b} \delta_{aj}\delta_{kb} |v_a\rangle\langle v_b|\end{aligned}$$

so the element $(|v_j\rangle\langle v_k|)_{ab} = \delta_{aj}\delta_{kb}$.

Exercise 2.11

Each of the Pauli matrices has eigenvalues ±1.
For σ_x,

$$\sigma_{x+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \sigma_{x-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For σ_y,

$$\sigma_{y+} = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } \sigma_{y-} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

For σ_z,

$$\sigma_{z+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \sigma_{z-} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The diagonalization easily follows.

Exercise 2.12

The characteristic equation is $(1-\lambda)^2$, so we have eigenvalue 1. Solving $(A - 1I)|v\rangle = 0$ gives us $|v\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Our diagonal form should be

$$A = |v\rangle \langle v| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Exercise 2.13

$$(|w\rangle \langle v|)^\dagger = \langle v|^\dagger \, |w\rangle^{dagger} = |v\rangle \langle w|$$

Exercise 2.14

Since we know $(a+b)^\dagger = a^\dagger + b^\dagger$, so

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i \left(a_i A_i\right)^\dagger = \sum_i \left(a_i^* A_i^\dagger\right)$$

Exercise 2.15

$$(|v\rangle, A|w\rangle) = (A^\dagger |v\rangle, |w\rangle) = \left(|v\rangle, (A^\dagger)^\dagger |w\rangle\right)$$

since this holds for all $|v\rangle$, $|w\rangle$, $A = (A^\dagger)^\dagger$.

Exercise 2.16

$$P^2 = \sum_i |i\rangle \langle i| \sum_j |j\rangle \langle j| = \sum_{ij} |i\rangle \langle i|j\rangle \langle j| = \sum_{ij} \delta_{ij} |i\rangle \langle j|$$

Intuitively, projecting some $|v\rangle \in P$ wouldn't change $|v\rangle$ at all.

Exercise 2.17

Since A is normal, we can diagonalize it. If the eigenvalues are all positive, then the adjoint of this diagonal matrix of eigenvalues equals itself. If A is Hermitian, then so is its diagonal form. The adjoint of a diagonal matrix consists of the conjugates of its diagonal entries, so if $A = A^\dagger$, then the diagonal entries (eigenvalues) must all be positive.

Exercise 2.18

For an eigenvector $|v\rangle$, we have $A|v\rangle = \lambda|v\rangle \to \langle v|A^\dagger = \lambda^* \langle v|$. Multiplying these two gives us $\langle v|A^\dagger A|v\rangle = \lambda^*\lambda\langle v|v\rangle$, and because $A^\dagger A = I$,

$$\|v\|^2 = |\lambda|^2 \|v\|^2 \to |\lambda| = 1$$

Exercise 2.19

Omitted because it's just mechanical.

Exercise 2.20

$$\begin{aligned}
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|U^\dagger UAUU^\dagger|v_j\rangle \\
&= \sum_a\sum_b\sum_c\sum_d \langle v_i|w_a\rangle\langle v_a|v_b\rangle\langle w_b|A|w_c\rangle\langle v_c|v_d\rangle\langle w_d|v_j\rangle \\
&= \sum_a\sum_b\sum_c\sum_d \delta_{ab}\delta_{cd}\langle v_i|w_a\rangle A''_{bc}\langle w_d|v_j\rangle
\end{aligned}$$

This tells us $a = b$ and $c = d$, so

$$A'_{ij} = \sum_a\sum_c \langle v_i|w_a\rangle A''_{ac}\langle w_c|v_j\rangle$$

Exercise 2.21

We will prove any Hermitian operator M is diagonal with respect to some orthonormal basis V.
We proceed by induction on dimension d on V.
Base case: $d = 1$
Trivially, M is diagonal.
Inductive hypothesis: Assume $d = n - 1$
Inductive step: Prove $d = n$
Let λ be an eigenvalue of M, P be a projection onto the λ eigenspace, and Q be P's orthogonal complement.
We know $M = (P+Q)M(P+Q)$. First, note that $QMP = \lambda QP = 0$. Now for some $|v\rangle \in P$, $M|v\rangle = M^\dagger|v\rangle = \lambda|v\rangle$ because M is Hermitian, which means $|v\rangle$ is in the eigenspace λ of $M^\dagger$. Now we have $QM^\dagger P|v\rangle = QM^\dagger|v\rangle = \lambda Q|v\rangle = 0$. Taking the adjoint of this gives us $PMQ = 0$. Now we have $M = PMP + QMQ$.
Since $PMP = \lambda P$, QMQ must be nonzero also for $M = PMP + QMQ$ to hold, so they both have dimension less than n. Finally, $PMP = (PMP)^\dagger = PM^\dagger P$ and similarly for QMQ. Since they're both Hermitian, our inductive hypothesis proves the theorem.

Exercise 2.22

For a Hermitian operator A, suppose $A|v\rangle = \lambda|v\rangle$ and $A|w\rangle = \mu|w\rangle$.
Since $\langle v|A = \lambda\langle v|$, we can write

$$\langle v|A^2|w\rangle = \mu^2\langle v|w\rangle = \lambda\mu\langle v|w\rangle$$

where the first equality follows from $A^2|w\rangle = \mu^2|w\rangle$. Since $\lambda \neq \mu$, $\langle v|w\rangle = 0$.

Exercise 2.23

Suppose $|v\rangle$ is an eigenvector of P with eigenvalue λ, $P|v\rangle = \lambda|v\rangle$. Then

$$P|v\rangle = P^2|v\rangle = \lambda^2|v\rangle$$

where the first equality follows from the property $P^2 = P$. Since $\lambda^2 = \lambda$, P's eigenvalues must be 1 or 0.

Exercise 2.24

Let $B = \frac{A+A^\dagger}{2}$ and $C = \frac{A-A^\dagger}{2i}$. This means

$$\langle v|A|v\rangle = \langle v|B|v\rangle + i\langle v|C|v\rangle$$

Since we cannot have an imaginary term in a positive operator, $C = 0$, so $A = A^\dagger$.

Exercise 2.25

We can write $\langle Av|Av\rangle = \langle A^\dagger Av|v\rangle$. But since $\langle Av|Av\rangle$ can also be written as $\|Av\|^2 \geq 0$, $A^\dagger A$ must be positive.

Exercise 2.26

As a tensor product:

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}\Big(|0\rangle + |1\rangle\Big)\otimes\Big(|0\rangle + |1\rangle\Big) = \frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big)$$

As a Kronecker product: Since $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$,

$$\begin{bmatrix}\frac{1}{\sqrt{2}}|\psi\rangle\\ \frac{1}{\sqrt{2}}|\psi\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

As a tensor product:

$$|\psi\rangle^{\otimes 3} = |\psi\rangle\otimes|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big)\otimes\frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big) = \frac{1}{2\sqrt{2}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\Big)$$

As a Kronecker product:

$$\begin{bmatrix}\frac{1}{\sqrt{2}}|\psi\rangle^{\otimes 2}\\ \frac{1}{\sqrt{2}}|\psi\rangle^{\otimes 2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}\end{bmatrix}^T$$

Exercise 2.27

$$X\otimes Z = \begin{bmatrix}0\cdot Z & 1\cdot Z\\ 1\cdot Z & 0\cdot Z\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}$$

$$I\otimes X = \begin{bmatrix}1\cdot X & 0\cdot X\\ 0\cdot X & 1\cdot X\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$X\otimes I = \begin{bmatrix}0\cdot I & 1\cdot I\\ 1\cdot I & 0\cdot I\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}$$

Since $I\otimes X \neq X\otimes I$, the tensor product is not commutative.

Exercise 2.28

Writing the Kronecker product,

$$(A \otimes B)^\dagger = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}^\dagger = \begin{bmatrix} A^*_{11}B^\dagger & \cdots & A^*_{n1}B^\dagger \\ \vdots & \ddots & \vdots \\ A^*_{1n}B^\dagger & \cdots & A^*_{nn}B^\dagger \end{bmatrix} = A^\dagger \otimes B^\dagger$$

Proving the transpose is similar and proving the complex conjugate requires only using B^* instead of $B^\dagger$.

Exercise 2.29

Let U_1 and U_2 be unitary.

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) = U_1^\dagger U_1 \otimes U_2^\dagger U_2 = I \otimes I = I$$

Exercise 2.30

Let $A = A^\dagger$ and $B = B^\dagger$.

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$$

Exercise 2.31

Let A and B be positive operators.

$$\Big((A \otimes B)(|v\rangle \otimes |w\rangle), (|v\rangle \otimes |w\rangle\Big) = \Big(A|v\rangle \otimes B|w\rangle, |v\rangle \otimes |w\rangle\Big) = \langle v|A^\dagger|v\rangle \langle w|B^\dagger|w\rangle = \langle v|A|v\rangle \langle w|B|w\rangle$$

since we know positive operators are Hermitian. Since $\langle v|A|v\rangle$ and $\langle w|B|w\rangle$ are both non-negative, their product is also non-negative.

Exercise 2.32

Let P and Q be projectors. Recall that if $P^2 = P$, P is a projector.

$$(P \otimes Q)(P \otimes Q) = P^2 \otimes Q^2 = P \otimes Q$$

Exercise 2.33

We proceed by induction on n.
Base case: $n = 1$
We know

$$H^{\otimes 1} = H = \frac{1}{\sqrt{2}}\Big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\Big] = \frac{1}{\sqrt{2}} \sum_{x,y \in \{0,1\}^1} (-1)^{x \cdot y} |x\rangle \langle y|$$

Inductive hypothesis: Assume $n = k - 1$

$$H^{\otimes k-1} = \frac{1}{2^{k-1/2}} \sum_{x,y \in \{0,1\}^{k-1}} (-1)^{x \cdot y} |x\rangle \langle y|$$

Inductive step: Prove $n = k$

$$H^{\otimes k} = H \otimes H^{\otimes k-1} = \frac{1}{2^{k/2}} \sum_{x_1,y_1 \in \{0,1\}} (-1)^{x_1 \cdot y_1} |x_1\rangle \langle y_1| \quad \otimes \sum_{x_2,y_2 \in \{0,1\}^{k-1}} (-1)^{x_2 \cdot y_2} |x_2\rangle \langle y_2|$$

Since this tensor product only flips the sign of the $H^{\otimes k-1}$ if $x_1 = y_1 = 1$, it is easy to see that concatenating x_1, x_2 and y_1, y_2, would yield the dot product required to flip the signs when we want.

$$H^{\otimes 2} = \frac{1}{\sqrt{2}}\begin{bmatrix} H & H \\ H & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Exercise 2.34

Let $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$. First, we find the eigenvalues and eigenvectors.

$$\det(A - \lambda I) = (\lambda - 7)(\lambda - 1) \to \lambda = 7, 1$$

with corresponding eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We'll denote the eigenpairs as $(7, |a\rangle)$ and $(1, |b\rangle)$.
So,

$$\sqrt{A} = \sqrt{7}\,|a\rangle\langle a| + 1\,|b\rangle\langle b|$$

and

$$\log(A) = \log(7)\,|a\rangle\langle a|$$

Exercise 2.35

Since $v \cdot \sigma$ is a weighted sum of the Pauli matrices, we know it will have eigenvalues of 1 and -1. Let the eigenpairs of $v \cdot \sigma$ be $(1, |\lambda_1\rangle)$ and $(-1, |\lambda_{-1}\rangle)$.

$$\begin{aligned} \exp(i\theta v \cdot \sigma) &= e^{i\theta}\,|\lambda_1\rangle\langle\lambda_1| + e^{-i\theta}\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\ &= (\cos\theta + i\sin\theta)|\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta)|\lambda_{-1}\rangle\langle\lambda_{-1}| \\ &= \cos\theta\Big(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|\Big) + i\sin\theta\Big(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|\Big) \\ &= \cos\theta I + i\sin\theta v \cdot \sigma \end{aligned}$$

Exercise 2.36

$$\begin{aligned} \mathrm{tr}(X) &= 0 + 0 \\ \mathrm{tr}(Y) &= 0 + 0 \\ \mathrm{tr}(Z) &= 1 - 1 \end{aligned}$$

Exercise 2.37

$$\mathrm{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j A_{ij}B_{ji} = \sum_j \sum_i B_{ji}A_{ij} = \sum_j (BA)_{jj} = \mathrm{tr}(BA)$$

Exercise 2.38

$$\mathrm{tr}(A + B) = \sum_i (A + B)_{ii} = \sum_i A_{ii} + \sum_j B_{jj} = \mathrm{tr}(A) + \mathrm{tr}(B)$$

$$\mathrm{tr}(zA) = \sum_i (zA)_{ii} = z\sum_i A_{ii} = z\,\mathrm{tr}(A)$$

Exercise 2.39

1. (a) Linearity in second argument:

$$(A, \sum_i \lambda_i B) = \text{tr}\left(A^\dagger \sum_i \lambda_i B\right) = \text{tr}\left(\sum_i \lambda_i A^\dagger B\right) = \sum_i \lambda_i \,\text{tr}\left(A^\dagger B\right)$$

(b)

$$\begin{aligned}(B,A)^* = \text{tr}\left(B^\dagger A\right) &= \left(\sum_i \langle i| B^\dagger A |i\rangle\right)^* \\ &= \sum_i \langle i| A^\dagger B |i\rangle \\ &= \text{tr}\left(A^\dagger B\right) \\ &= (A,B)\end{aligned}$$

(c)

$$(A,A) = \text{tr}\left(A^\dagger A\right)$$

Since we know $A^\dagger A$ is a positive operator, the sum of its eigenvalues must be nonnegative. Additionally, the sum of its eigenvalues will only be zero if $A^\dagger A$ is the 0 matrix, which means A was also the 0 matrix.

2. We can fix a basis $|v_1\rangle, \cdots, |v_n\rangle$. In this basis, the columns of any $A \in L_V$ are defined as the vectors A maps the n basis vectors to. Since we need n terms to describe each vector and we map n basis vectors, $\dim L_V = n^2$.

3. Not sure if we need n^2 matrices for a basis, but using the same basis vectors as before, we can define n Hermitian matrices for each i: $A_i = \langle v_i | v_i \rangle$. Since any operator can be written as its action on the basis vectors, linear combinations of these Hermitian matrices span V.

   The set is orthonormal because if the basis vectors are normalized,

$$(A_i, A_i) = \text{tr}\left(A_i^\dagger A_i\right) = \langle v_i | A^\dagger A | v_i \rangle = \langle v_i|v_i\rangle \langle v_i|v_i\rangle \langle v_i|v_i\rangle = 1$$

   and

$$(A_i, A_j) = \text{tr}\left(A_i^\dagger A_j\right) = \text{tr}\left(|v_i\rangle \langle v_i|v_j\rangle \langle v_j|\right) = \text{tr}\left(|v_i\rangle 0 \langle v_j|\right) = 0$$

Exercise 2.40

$$[X,Y] = XY - YX = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}i & 0\\ 0-i & \end{bmatrix} - \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix} = \begin{bmatrix}2i & 0\\ 0 & -2i\end{bmatrix} = 2iZ$$

$$[Y,Z] = YZ - ZY = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = \begin{bmatrix}0 & i\\ i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix} = \begin{bmatrix}0 & 2i\\ 2i & 0\end{bmatrix} = 2iX$$

$$[Z,X] = ZX - XZ = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} - \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix} = 2iY$$

The textbook explains an elegant way to represent this is:

$$[\sigma_j, \sigma_k] = 2i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l$$

where $\epsilon_{jkl} = 0$ except for $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$.

Exercise 2.41

Consider the above exercise and add instead of subtract matrices. It is easy to see each equation would result in 0. This is sufficient for all the examples because $\{A,B\} = \{B,A\}$.

Exercise 2.42

$$\frac{[A,B]+\{A,B\}}{2} = \frac{AB-BA+AB+BA}{2} = \frac{2AB}{2} = AB$$

Exercise 2.43

We know

$$\sigma_j\sigma_k = \frac{[\sigma_j,\sigma_k]+\{\sigma_j,\sigma_k\}}{2}$$

Notice that if $j \neq k$, $\{\sigma_j,\sigma_k\} = 0$, Exercise 2.40 lets us write

$$\sigma_j\sigma_k = \frac{[\sigma_j,\sigma_k]}{2} = i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$$

Notice that if $j = k$, $[\sigma_j,\sigma_k] = 0$, so

$$\sigma_j\sigma_k = \sigma_j\sigma_j = \frac{\{\sigma_j,\sigma_j\}}{2} = \frac{\sigma_j^2+\sigma_j^2}{2} = \sigma_j^2 = I$$

So we arrive at

$$\sigma_j\sigma_k = \delta_{jk}I + i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$$

Exercise 2.44

Combining $AB-BA=0$ and $AB+BA=0$ yields

$$2AB = 0 \rightarrow AB = 0 \rightarrow B = A^{-1}0 = 0$$

Exercise 2.45

$$[A,B]^\dagger = (AB-BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger]$$

Exercise 2.46

$$[A,B] = AB-BA = -(BA-AB) = -[B,A]$$

Exercise 2.47

$$(i[A,B])^\dagger = -i[B^\dagger,A^\dagger] = i[A^\dagger,B^\dagger] = i[A,B]$$

by using the results from the above two exercises.

Exercise 2.48

For a positive matrix P, we have $P = IP$. For a unitary matrix U, we have $U = UI$. For a Hermitian matrix H, we know $J = \sqrt{H^\dagger H} = \sqrt{H^2} = |H|$, so we have $H = U|H|$.

Exercise 2.49

If A is normal, it must have a spectral decomposition $A = \sum_i \lambda_i |i\rangle\langle i|$. We can now write

$$J = \sqrt{A^\dagger A} = \sum_i |\lambda_i||i\rangle\langle i|$$

. Since $U = \sum_i |e_i\rangle\langle i|$,

$$A = UJ = \sum_i |\lambda_i||e_i\rangle\langle i|$$

Exercise 2.50

The calculations aren't turning out pretty so I'm skipping over this one.

Exercise 2.51

$$H^\dagger H = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Since $H = H^\dagger$, $HH^\dagger = I$ follows.

Exercise 2.52

This follows from the above exercise since $H = H^\dagger$.

Exercise 2.53

The characteristic equation $\det(H - \lambda I) = 0$ yields eigenpairs:

$$\left(1, \begin{bmatrix}-1\\ 1-\sqrt{2}\end{bmatrix}\right) \qquad \text{and} \qquad \left(-1, \begin{bmatrix}1-\sqrt{2}\\ 1\end{bmatrix}\right)$$

Exercise 2.54

Since $[A, B] = 0$, we know A and B are simultaneously diagonalizable, so $A = \sum_i a_i |i\rangle\langle i|$ and $B = \sum_i b_i |i\rangle\langle i|$.

$$\begin{aligned}
\exp\left(\sum_i a_i |i\rangle\langle i|\right)\exp\left(\sum_i b_i |i\rangle\langle i|\right) &= |i\rangle\langle i|i\rangle\langle i| \exp\left(\sum_i a_i\right)\exp\left(\sum_i b_i\right)\\
&= |i\rangle\langle i| \exp\left(\sum_i a_i + b_i\right)\\
&= \exp\left(\sum_i a_i + b_i |i\rangle\langle i|\right) = \exp(A+B)
\end{aligned}$$