Notes and Solutions for Nielsen and Chuang's *Quantum Computation and Quantum Information*

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Contents

Chapter 2

Introduction to quantum mechanics

Notes

Notation

For distinct vectors in an orthonormal set, we can write $\langle i|j\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker product and is 1 if $i = j$ and 0 if $i \neq j$.

Matrix - Linear Operator Congruence

For a matrix to a be a linear operator,

$$A\Big(\sum_i a_i |v_I\rangle\Big) = \sum_i a_i A|v_i\rangle$$

must be true. Note the LHS is the sum of vectors to which A is applied which is certainly equal to the RHS.
Now suppose $A : V \to W$ is a linear operator and that V has basis $|v_i\rangle, \cdots, |v_m\rangle$ and W has basis $|w_i\rangle, \cdots, |w_n\rangle$. Since we know the kth column of a A will be its transformation of $|v_k\rangle$,

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Note this is just saying $A|v_j\rangle$ is equal to the jth column of A, and we can think of $|w_i\rangle$ as the coordinates of the transformed vector. Thus, we can find the matrix representation of any linear operator by finding a matrix A with entries specified by the above equation.

What's so special about the Pauli matrices?

Things to look into:

- they pop up in the Pauli equation
- they form a basis for the vector space of 2×2 Hermitian matrices. Hermitian matrices represent observables? Look at Wikipedia page

What does the Completeness Relation say about matrices?

I don't understand why

$$\sum_{ij} \langle w_j|A|v_i\rangle |w_j\rangle \langle v_i|$$

implies A has matrix element $\left\langle w_j \middle| A|v_i\right\rangle$ in the ith column and jth row, with respect to input basis $|v_i\rangle$ and output basis $|w_j\rangle$. page 68.

Solutions

Exercise 2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

because $A|0\rangle$ has coordinate 0 in $|0\rangle$ and coordinate 1 in $|1\rangle$.

If we keep our input bases the same but reorder our output bases as $|1\rangle$ and $|0\rangle$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 2.3

We know

$$A|v_i\rangle = \sum_j A_{ji}|w_j\rangle$$

and

$$B|w_j\rangle = \sum_k B_{kj}|x_k\rangle$$

Now we can write

$$\begin{aligned} BA|v_i\rangle = B(A|v_i\rangle) = B\sum_j A_{ji}|w_j\rangle &= \sum_j A_{ji}(B|w_j\rangle) \\ &= \sum_j A_{ji}\sum_k B_{kj}|x_k\rangle \\ &= \sum_k\sum_j B_{kj}A_{ji}|x_k\rangle \\ &= \sum_k (BA)_{ki}|x_k\rangle \end{aligned}$$

We know $\sum_k (BA)_{ki}$ is the matrix representation of operator BA, which the preceding step says is equal to $\sum_k\sum_j B_{kj}A_{ji}$, which is the matrix multiplication BA.

Exercise 2.4

For the same input and output basis, we want some I such that

$$I|v_j\rangle = \sum_i I_{ij}|v_i\rangle = |v_j\rangle$$

which means $I_{ij} = 0$ for all $i \neq j$ and 1 otherwise.

Exercise 2.5

For $|y\rangle, |z_i\rangle \in \mathbb{C}^n$ and $\lambda_i \in C$,

$$\begin{aligned}\left(|y\rangle, \sum_i \lambda_i |z_i\rangle\right) &= |y\rangle^* \sum_i \lambda_i |z_i\rangle \\ &= \sum_i \lambda_i |y\rangle^* |z_i\rangle \\ &= \left(\sum_i \lambda_i^* |z_i\rangle^* |y\rangle\right)^*\end{aligned}$$

The second and third equalities demonstrate linearity in the second argument and $(|y\rangle, |z\rangle) = (|z\rangle, |w\rangle)^*$. Finally, if $|w\rangle = (w_1, \cdots, w_n)$ where $w_i \in \mathbb{C}^n$, then

$$(|w\rangle, |w\rangle) = \sum_i w_i^* w_i = \sum_i |w_i|^2$$

which proves the non-degeneracy and non-negativity condition.

Exercise 2.6

$$\begin{aligned}\left(\sum_i \lambda_i |w_i\rangle, |v\rangle\right) &= \left(|v\rangle^*, \sum_i \lambda_i^* |w_i\rangle^*\right)^* \\ &= \sum_i \lambda_i^* \left(|v\rangle^*, |w_i\rangle^*\right)^* \\ &= \sum_i \lambda_i^* \left(|w_i\rangle, |v\rangle\right)\end{aligned}$$

Exercise 2.7

$$(|w\rangle, |v\rangle) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0$$

To normalize, divide each vector by $\sqrt{2}$.

Exercise 2.8

Since at each step, we divide by the norm of the vector being added, the set is orthonormal. The set is a basis because at step i, we add the basis vector $|w_i\rangle$ but subtract out the portion that was already in $\mathrm{span}(|v_1\rangle, \cdots, |v_{i-1}\rangle)$, so we still end up spanning the full vector space.

Exercise 2.9

$$\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$\sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Exercise 2.10

$$\begin{aligned}|v_j\rangle\langle v_k| &= I|v_j\rangle\langle v_k|I \\ &= \sum_a |v_a\rangle\langle v_a|v_j\rangle \sum_b \langle v_k|v_b\rangle\langle v_b| \\ &= \sum_{a,b} \delta_{aj}\delta_{kb}|v_a\rangle\langle v_b|\end{aligned}$$

so the element $(|v_j\rangle\langle v_k|)_{ab} = \delta_{aj}\delta_{kb}$.

Exercise 2.11

Each of the Pauli matrices has eigenvalues ±1.
For σ_x,

$$\sigma_{x+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \sigma_{x-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For σ_y,

$$\sigma_{y+} = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } \sigma_{y-} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

For σ_z,

$$\sigma_{z+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \sigma_{z-} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The diagonalization easily follows.

Exercise 2.12

The characteristic equation is $(1-\lambda)^2$, so we have eigenvalue 1. Solving $(A - 1I)|v\rangle = 0$ gives us $|v\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Our diagonal form should be

$$A = |v\rangle \langle v| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Exercise 2.13

$$(|w\rangle \langle v|)^\dagger = \langle v|^\dagger \, |w\rangle^{dagger} = |v\rangle \langle w|$$

Exercise 2.14

Since we know $(a+b)^\dagger = a^\dagger + b^\dagger$, so

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i \left(a_i A_i\right)^\dagger = \sum_i \left(a_i^* A_i^\dagger\right)$$

Exercise 2.15

$$(|v\rangle, A|w\rangle) = (A^\dagger |v\rangle, |w\rangle) = \left(|v\rangle, (A^\dagger)^\dagger |w\rangle\right)$$

since this holds for all $|v\rangle, |w\rangle$, $A = (A^\dagger)^\dagger$.

Exercise 2.16

$$P^2 = \sum_i |i\rangle \langle i| \sum_j |j\rangle \langle j| = \sum_{ij} |i\rangle \langle i|j\rangle \langle j| = \sum_{ij} \delta_{ij} |i\rangle \langle j|$$

Intuitively, projecting some $|v\rangle \in P$ wouldn't change $|v\rangle$ at all.

Exercise 2.17

Since A is normal, we can diagonalize it. If the eigenvalues are all positive, then the adjoint of this diagonal matrix of eigenvalues equals itself. If A is Hermitian, then so is its diagonal form. The adjoint of a diagonal matrix consists of the conjugates of its diagonal entries, so if $A = A^\dagger$, then the diagonal entries (eigenvalues) must all be positive.

Exercise 2.18

For an eigenvector $|v\rangle$, we have $A|v\rangle = \lambda|v\rangle \to \langle v|A^\dagger = \lambda^* \langle v|$. Multiplying these two gives us $\langle v|A^\dagger A|v\rangle = \lambda^*\lambda\langle v|v\rangle$, and because $A^\dagger A = I$,

$$\|v\|^2 = |\lambda|^2 \|v\|^2 \to |\lambda| = 1$$

Exercise 2.19

Omitted because it's just mechanical.

Exercise 2.20

$$\begin{aligned}
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|U^\dagger UAUU^\dagger|v_j\rangle \\
&= \sum_a\sum_b\sum_c\sum_d \langle v_i|w_a\rangle\langle v_a|v_b\rangle\langle w_b|A|w_c\rangle\langle v_c|v_d\rangle\langle w_d|v_j\rangle \\
&= \sum_a\sum_b\sum_c\sum_d \delta_{ab}\delta_{cd}\langle v_i|w_a\rangle A''_{bc}\langle w_d|v_j\rangle
\end{aligned}$$

This tells us $a = b$ and $c = d$, so

$$A'_{ij} = \sum_a\sum_c \langle v_i|w_a\rangle A''_{ac}\langle w_c|v_j\rangle$$

Exercise 2.21

We will prove any Hermitian operator M is diagonal with respect to some orthonormal basis V.
We proceed by induction on dimension d on V.
Base case: $d = 1$
Trivially, M is diagonal.
Inductive hypothesis: Assume $d = n - 1$
Inductive step: Prove $d = n$
Let λ be an eigenvalue of M, P be a projection onto the λ eigenspace, and Q be P's orthogonal complement.
We know $M = (P+Q)M(P+Q)$. First, note that $QMP = \lambda QP = 0$. Now for some $|v\rangle \in P$, $M|v\rangle = M^\dagger|v\rangle = \lambda|v\rangle$ because M is Hermitian, which means $|v\rangle$ is in the eigenspace λ of $M^\dagger$. Now we have $QM^\dagger P|v\rangle = QM^\dagger|v\rangle = \lambda Q|v\rangle = 0$. Taking the adjoint of this gives us $PMQ = 0$. Now we have $M = PMP + QMQ$.
Since $PMP = \lambda P$, QMQ must be nonzero also for $M = PMP + QMQ$ to hold, so they both have dimension less than n. Finally, $PMP = (PMP)^\dagger = PM^\dagger P$ and similarly for QMQ. Since they're both Hermitian, our inductive hypothesis proves the theorem.

Exercise 2.22

For a Hermitian operator A, suppose $A|v\rangle = \lambda|v\rangle$ and $A|w\rangle = \mu|w\rangle$.
Since $\langle v|A = \lambda\langle v|$, we can write

$$\langle v|A^2|w\rangle = \mu^2\langle v|w\rangle = \lambda\mu\langle v|w\rangle$$

where the first equality follows from $A^2|w\rangle = \mu^2|w\rangle$. Since $\lambda \neq \mu$, $\langle v|w\rangle = 0$.

Exercise 2.23

Suppose $|v\rangle$ is an eigenvector of P with eigenvalue λ, $P|v\rangle = \lambda|v\rangle$. Then

$$P|v\rangle = P^2|v\rangle = \lambda^2|v\rangle$$

where the first equality follows from the property $P^2 = P$. Since $\lambda^2 = \lambda$, P's eigenvalues must be 1 or 0.

Exercise 2.24

Let $B = \frac{A+A^\dagger}{2}$ and $C = \frac{A-A^\dagger}{2i}$. This means

$$\langle v|A|v\rangle = \langle v|B|v\rangle + i\langle v|C|v\rangle$$

Since we cannot have an imaginary term in a positive operator, $C = 0$, so $A = A^\dagger$.

Exercise 2.25

We can write $\langle Av|Av\rangle = \langle A^\dagger Av|v\rangle$. But since $\langle Av|Av\rangle$ can also be written as $\|Av\|^2 \geq 0$, $A^\dagger A$ must be positive.

Exercise 2.26

As a tensor product:

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}\Big(|0\rangle + |1\rangle\Big)\otimes\Big(|0\rangle + |1\rangle\Big) = \frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big)$$

As a Kronecker product: Since $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$,

$$\begin{bmatrix} \frac{1}{\sqrt{2}}|\psi\rangle \\ \frac{1}{\sqrt{2}}|\psi\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

As a tensor product:

$$|\psi\rangle^{\otimes 3} = |\psi\rangle\otimes|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}\Big(|0\rangle+|1\rangle\Big)\otimes\frac{1}{2}\Big(|00\rangle+|01\rangle+|10\rangle+|11\rangle\Big) = \frac{1}{2\sqrt{2}}\Big(|000\rangle+|001\rangle+|010\rangle+|011\rangle+|100\rangle+|101\rangle+|110\rangle+|111\rangle\Big)$$

As a Kronecker product:

$$\begin{bmatrix} \frac{1}{\sqrt{2}}|\psi\rangle^{\otimes 2} \\ \frac{1}{\sqrt{2}}|\psi\rangle^{\otimes 2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}^T$$

Exercise 2.27

$$X\otimes Z = \begin{bmatrix} 0\cdot Z & 1\cdot Z \\ 1\cdot Z & 0\cdot Z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$I\otimes X = \begin{bmatrix} 1\cdot X & 0\cdot X \\ 0\cdot X & 1\cdot X \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$X\otimes I = \begin{bmatrix} 0\cdot I & 1\cdot I \\ 1\cdot I & 0\cdot I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Since $I\otimes X \neq X\otimes I$, the tensor product is not commutative.

Exercise 2.28

Writing the Kronecker product,

$$(A \otimes B)^\dagger = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}^\dagger = \begin{bmatrix} A_{11}^* B^\dagger & \cdots & A_{n1}^* B^\dagger \\ \vdots & \ddots & \vdots \\ A_{1n}^* B^\dagger & \cdots & A_{nn}^* B^\dagger \end{bmatrix} = A^\dagger \otimes B^\dagger$$

Proving the transpose is similar and proving the complex conjugate requires only using B^* instead of $B^\dagger$.

Exercise 2.29

Let U_1 and U_2 be unitary.

$$(U_1 \otimes U_2)^\dagger (U_1 \otimes U_2) = (U_1^\dagger \otimes U_2^\dagger)(U_1 \otimes U_2) = U_1^\dagger U_1 \otimes U_2^\dagger U_2 = I \otimes I = I$$

Exercise 2.30

Let $A = A^\dagger$ and $B = B^\dagger$.

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$$

Exercise 2.31

Let A and B be positive operators.

$$\Big((A \otimes B)(|v\rangle \otimes |w\rangle), (|v\rangle \otimes |w\rangle\Big) = \Big(A|v\rangle \otimes B|w\rangle, |v\rangle \otimes |w\rangle\Big) = \langle v|A^\dagger|v\rangle \langle w|B^\dagger|w\rangle = \langle v|A|v\rangle \langle w|B|w\rangle$$

since we know positive operators are Hermitian. Since $\langle v|A|v\rangle$ and $\langle w|B|w\rangle$ are both non-negative, their product is also non-negative.

Exercise 2.32

Let P and Q be projectors. Recall that if $P^2 = P$, P is a projector.

$$(P \otimes Q)(P \otimes Q) = P^2 \otimes Q^2 = P \otimes Q$$

Exercise 2.33

We proceed by induction on n.
Base case: $n = 1$
We know

$$H^{\otimes 1} = H = \frac{1}{\sqrt{2}}\Big[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\Big] = \frac{1}{\sqrt{2}} \sum_{x,y \in \{0,1\}^1} (-1)^{x \cdot y} |x\rangle \langle y|$$

Inductive hypothesis: Assume $n = k - 1$

$$H^{\otimes k-1} = \frac{1}{2^{k-1/2}} \sum_{x,y \in \{0,1\}^{k-1}} (-1)^{x \cdot y} |x\rangle \langle y|$$

Inductive step: Prove $n = k$

$$H^{\otimes k} = H \otimes H^{\otimes k-1} = \frac{1}{2^{k/2}} \sum_{x_1,y_1 \in \{0,1\}} (-1)^{x_1 \cdot y_1} |x_1\rangle \langle y_1| \qquad \otimes \sum_{x_2,y_2 \in \{0,1\}^{k-1}} (-1)^{x_2 \cdot y_2} |x_2\rangle \langle y_2|$$

Since this tensor product only flips the sign of the $H^{\otimes k-1}$ if $x_1 = y_1 = 1$, it is easy to see that concatenating x_1, x_2 and y_1, y_2, would yield the dot product required to flip the signs when we want.

$$H^{\otimes 2} = \frac{1}{\sqrt{2}}\begin{bmatrix} H & H \\ H & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Exercise 2.34

Let $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$. First, we find the eigenvalues and eigenvectors.

$$\det(A - \lambda I) = (\lambda - 7)(\lambda - 1) \rightarrow \lambda = 7, 1$$

with corresponding eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We'll denote the eigenpairs as $(7, |a\rangle)$ and $(1, |b\rangle)$.
So,

$$\sqrt{A} = \sqrt{7}\,|a\rangle\langle a| + 1\,|b\rangle\langle b|$$

and

$$\log(A) = \log(7)\,|a\rangle\langle a|$$

Exercise 2.35

Since $v \cdot \sigma$ is a weighted sum of the Pauli matrices, we know it will have eigenvalues of 1 and -1. Let the eigenpairs of $v \cdot \sigma$ be $(1, |\lambda_1\rangle)$ and $(-1, |\lambda_{-1}\rangle)$.

$$\begin{aligned}
\exp(i\theta v \cdot \sigma) &= e^{i\theta}\,|\lambda_1\rangle\langle\lambda_1| + e^{-i\theta}\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= (\cos\theta + i\sin\theta)|\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta)|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \cos\theta\Big(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|\Big) + i\sin\theta\Big(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|\Big) \\
&= \cos\theta I + i\sin\theta v \cdot \sigma
\end{aligned}$$