

Linear Algebra

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A very reductionist summary of key Linear Algebra concepts from *Linear Algebra and its Applications* by Lay, Lay, and McDonald.

1 Systems of Linear Equations

Definition 1.1. A **linear equation** is an equation that can be written in the form

$$a_1x_1 + \dots + a_nx_n = b$$

where b and the coefficients a_k are real or complex numbers.

We can record the important information of a system of linear equations in a matrix. Given the system

$$x_1 - 2x_2 + x_3 = 0$$

$$2x_2 - 8x_3 = 8$$

$$5x_1 - 5x_3 = 10$$

we place the coefficients of each variable aligned in columns

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

This is called the **coefficient matrix** and

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$

is called the **augmented matrix**. The size of a matrix is described as $\mathbf{m} \times \mathbf{n}$ where m denotes the number of rows and n the number of columns.

Definition 1.2. Elementary Row Operations:

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.

2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Definition 1.3. Row Echelon form denotes a matrix with:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in column below a leading entry are zeros.

Reduced Row Echelon form means the leading entry in nonzero rows is 1.

Parallelogram Rule for Addition: If \mathbf{u} and \mathbf{v} in \mathbb{R}^2 are represented as points on the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are 0, \mathbf{u} , and \mathbf{v} .

Definition 1.4. Span $\{v_1, \dots, v_p\}$ denotes the set of all vectors formed by $c_1v_1 + \dots + c_pv_p$.

Definition 1.5. A set of vectors $\{v_1, \dots, v_p\}$ is said to be **linearly independent** if the equation

$$c_1v_1 + \dots + c_pv_p = 0$$

has only the trivial solution.

Theorem 1. *If a set contains more vectors than there are entries in each vector, then the set is linearly independent. That is, the set $\{v_1, \dots, v_p\}$ in \mathbb{R}^n is linearly dependent if $p > n$.*

Proof. Let $A = [v_1 \dots v_p]$. Then A is $n \times p$, and the equation $Ax = 0$ corresponds to a system of n equations in p unknowns. In $Ax = b$, the x vector must have dimension p , so if $p > n$, then there are more variables than equations, so $Ax = 0$ has a nontrivial solution, and the columns of A are linearly dependent. ■

An alternate way to conceptualize matrix multiplication: A **transformation** (or **function** or **mapping**) T from \mathbb{R}^n to \mathbb{R}^m is a rule that assigns to each vector x in \mathbb{R}^n a vector $T(x)$ in \mathbb{R}^m . The set \mathbb{R}^n is called the **domain** of T and \mathbb{R}^m is called the **codomain** of T . The set of all images $T(x)$ is called the **range** of T .

Definition 1.6. A transformation T is **linear** if they preserve vector addition and scalar multiplication. That is:

1. $T(u + v) = T(u) + T(v)$
2. $T(cu) = cT(u)$ for all scalars c

Every matrix transformation is a linear transformation. These two requirements mean that $T(0) = 0$ for linear transformations.

Theorem 2. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a **unique** matrix A such that

$$T(x) = Ax$$

In fact, A is the $m \times n$ matrix whose j th column is the vector $T(e_j)$, where e_j is the j th column of the identity matrix in \mathbb{R}^n :

$$A = [T(e_1) \cdots T(e_n)]$$

Proof. Write $x = I_n x = [e_1 \cdots e_n]x = x_1 e_1 + \cdots + x_n e_n$, and use the linearity of T to compute

$$T(x) = T(x_1 e_1 + \cdots x_n e_n) = x_1 T(e_1) + \cdots + x_n T(e_n)$$

$$[T(e_1) \cdots T(e_n)] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = Ax$$

■

Definition 1.7. A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **onto** \mathbb{R}^m if each b in \mathbb{R}^m is the image of at least one x in \mathbb{R}^n .

Definition 1.8. A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is **one-to-one** if each b in \mathbb{R}^m is the image of at most one x in \mathbb{R}^n .

2 Matrix Algebra

Definition 2.1. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then

$$AB = A \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

Definition 2.2. Let A be an $m \times n$ matrix and let B and C have sizes for which the indicated sums and products are defined.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A = A I_n$

Warnings:

1. In general, $AB \neq BA$
2. If $AB = AC$, then it is **not true** in general that $B = C$

3. If $AB = 0$, then it is **not true** always that $A = 0$ or $B = 0$

Definition 2.3. The **transpose** of A is the matrix whose columns are formed from the corresponding rows of A , denoted by A^T .

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = rA^T$
- $(AB)^T = B^T A^T$

Definition 2.4. An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix A^{-1} such that $A^{-1}A = I$.

- $(A^{-1})^{-1} = A$
- If A and B are $n \times n$ invertible matrices then so is AB . And $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

To compute the inverse, solve the equation $AB = I$, by row-reducing the augmented matrix $[A \ I]$, until you get $[I \ B]$.

3 Determinants

Definition 3.1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is the **determinant** of the matrix. If the determinant is 0, the matrix A is not invertible.

Definition 3.2. To generalize, the determinant of an $n \times n$ matrix A can be computed using a **cofactor expansion** across any row or down any column. The expansion across the i th row is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

where $C_{ij} = (-1)^{i+j}\det(A_{ij})$

Theorem 3. If A is an upper triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal.

Proof. Cofactoring an upper triangular matrix by using the first column ultimately leads to continuously multiplying the upper left item by the determinant of the smaller matrix. For example,

$$A = \begin{bmatrix} 3 & 2 & 9 \\ 0 & 4 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

Then,

$$\det(A) = 3 \cdot \det \begin{bmatrix} 4 & -1 \\ 0 & -8 \end{bmatrix} = 3 \cdot -32 = -96 = 3 \cdot 4 \cdot -8$$

■

Definition 3.3. Determinants after Row Operations

1. If a multiple of a row in matrix A is added to another row to produce matrix B , then $\det(B) = \det(A)$
2. If two rows in A are swapped to produce B , then $\det(B) = -\det(A)$
3. If one row in A is multiplied by k to produce B , then $\det(B) = k \cdot \det(A)$

These identities can be used to easily find determinants of square matrices. Once we reduce a matrix A to upper triangular form B , we know $\det(B) = (-1)^r \det(A)$ if r is the number of row swaps we performed. If we cannot reduce to row echelon form, we know the determinant must be 0 since A must not be invertible.

Theorem 4. If A is an $n \times n$ matrix, then $\det(A^T) = \det(A)$.

Proof. We proceed by induction. The theorem is trivially true for $n = 1$. Assume the theorem is true for $k \times k$ matrices. We will show it holds for $n = k + 1$. The cofactor of a_{1j} in A equals the cofactor of a_{j1} in A^T because it is a $k \times k$ determinant. Thus, the cofactor of $\det(A^T)$ down the first column equals the cofactor of $\det(A)$ across the first row, so A and A^T have equal determinants. Thus, the statement is true for all n . ■

Theorem 5. If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Theorem 6. Cramer's Rule: Let A be an invertible $n \times n$ matrix. For any b in \mathbb{R}^n , the unique solution x of $Ax = b$ has entries given by

$$x_i = \frac{\det(A_i(b))}{\det(A)} \text{ for } i = 1, 2, \dots, n$$

where $A_i(b)$ denotes the matrix obtained by replacing A 's i th column with b .

Proof. Denote the columns of A by a_1, \dots, a_n and the columns of the $n \times n$ identity matrix by e_1, \dots, e_n . If $Ax = b$, the definition of matrix multiplication tells us

$$\begin{aligned} A \cdot I_i(x) &= A \begin{bmatrix} e_1 & \cdots & x & \cdots & e_n \end{bmatrix} = \begin{bmatrix} Ae_1 & \cdots & Ax & \cdots & Ae_n \end{bmatrix} \\ &= \begin{bmatrix} a_1 & \cdots & b & \cdots & a_n \end{bmatrix} = A_i(b) \end{aligned}$$

Using the multiplicative property of determinants,

$$(\det(A))(\det(I_i(x))) = \det(A_i(b))$$

Since $\det(I_i(x))$ is x , we just divide by $\det(A)$. ■

4 Vector Spaces

Some of this is from Chapter 1, but I think it makes more sense to define these concepts here.

Definition 4.1. A **subspace** of \mathbb{R}^n is any set H in \mathbb{R}^n that is closed under addition and scalar multiplication. That is:

1. The zero vector is in H
2. For each u and v in H , the sum $u + v$ is in H
3. For each u in H and each scalar c , the vector cu is in H

Definition 4.2. The **column space** of an $m \times n$ matrix A is the set of all linear combinations of the columns of A , denoted by $\text{Col}(A)$. Since the columns of A are in \mathbb{R}^m , the column space is in \mathbb{R}^m .

Definition 4.3. The **null space** of a matrix A is the set of all solutions for $Ax = 0$, denoted by $\text{Nul}(A)$. When $\text{Nul}(A)$ contains nonzero vectors, the number of vectors in the nullspace equals the number of free variables in $Ax = 0$.

Definition 4.4. A **basis** for a subspace H in \mathbb{R}^n is a linearly independent set in H that spans H .

Using the basis for a subspace H is preferable because any vector in H can only be written in one way as a linear combination of the basis vectors.

Proof. Suppose $\mathbb{B} = \{b_1, \dots, b_p\}$ is a basis for H , and suppose a vector x in H can be generated in two ways:

$$x = c_1b_1 + \cdots + c_pb_p \text{ and } x = d_1b_1 + \cdots + d_pb_p$$

Subtracting gives us:

$$0 = (c_1 - d_1)b_1 + \cdots + (c_p - d_p)b_p$$

Since \mathbb{B} is linearly independent, the weights must all be zero, so $c_j = d_j$ so the two representations are really just the same. ■

Theorem 7. *The pivot columns of a matrix A form a basis for $\text{Col}(A)$.*

Proof. Let B be the reduced echelon form of A . The set of pivot columns of B is linearly independent, since no vector is a linear combination of the vectors that precede it. Since A is *row equivalent* to B , the pivot columns of A are linearly independent as well. Thus, the nonpivot columns of A can be discarded from the spanning set of $\text{Col}(A)$.

Warning: The pivot columns of A are only evident when A has been reduced to *echelon* form. After reducing, make sure to use the **pivot columns of A itself** for the basis of $\text{Col}(A)$. The columns of an echelon form of A are often not in the column space of A . ■

Theorem 8. Unique Representation Theorem: *Let $\mathbb{B} = \{b_1, \dots, b_n\}$ be a basis for a vector space V . Then for each x in V , there exists a unique set of scalars c_1, \dots, c_n such that*

$$x = c_1 b_1 + \dots + c_n b_n$$

Proof. Since \mathbb{B} spans V , we know there exist scalars such that we can form x . Assume x also has the representation

$$x = d_1 b_1 + \dots + d_n b_n$$

Then, after subtracting we have

$$0 = (c_1 - d_1)b_1 + \dots + (c_n - d_n)b_n$$

Since \mathbb{B} is linearly independent, these weights must all be zero so $c_j = d_j$. ■

Because of the unique representation of each vector x in a basis, we can define the coordinates of x relative to the basis \mathbb{B} as the weights c_1, \dots, c_n .

Definition 4.5. Changing coordinates:

$$x = \mathbb{B}[x]_{\mathbb{B}}$$

where \mathbb{B} denotes the matrix whose columns are basis vectors, and $x_{\mathbb{B}}$ denotes the x vector represented by basis coordinates.

To understand this, let $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $\mathbb{B} = \{b_1, b_2\}$. To find $[x]_{\mathbb{B}}$ of x relative to \mathbb{B} ,

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

or

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Since we know the columns of \mathbb{B} are linearly independent, it must be invertible so we can multiply x by \mathbb{B}^{-1} to get $[x]_{\mathbb{B}}$.

Definition 4.6. Change of basis: We can generalize the above further. Let $\mathbb{B} = \{b_1, \dots, b_n\}$ and $\mathbb{C} = \{c_1, \dots, c_n\}$ be bases of a vector space V . Then there is a *unique* $n \times n$ matrix $P_{\mathbb{C} \rightarrow \mathbb{B}}$ such that

$$[x]_{\mathbb{C}} = P_{\mathbb{C} \rightarrow \mathbb{B}} [x]_{\mathbb{B}}$$

The columns of $P_{\mathbb{C} \rightarrow \mathbb{B}}$ are the \mathbb{C} -coordinate vectors of the vectors in the basis \mathbb{B} , that is

$$P_{\mathbb{C} \rightarrow \mathbb{B}} = [[b_1]_{\mathbb{C}} \cdots [b_n]_{\mathbb{C}}]$$

Definition 4.7. An **isomorphism** from V to W is a one-to-one linear transformation.

Definition 4.8. The **dimension** of a nonzero subspace H is the number of vectors in any basis for H . The dimension of the zero subspace $\{0\}$ is defined to be zero.

Definition 4.9. The **rank** of a matrix A is the dimension of the column space of A .

Theorem 9. If a matrix A has n columns, then $\text{Rank}(A) + \text{Dim}(\text{Nul}(A)) = n$.

Proof. An intuitive understanding for this can be achieved by restating the theorem as follows:

$$\left(\text{num of pivot columns}\right) + \left(\text{num of nonpivot columns}\right) = \left(\text{num of columns}\right)$$

■

Definition 4.10. If A is an $m \times n$ matrix, each row has n entries and can be understood as a vector in \mathbb{R}^n . The set of all linear combinations of the row vectors is called the **row space** of A , denoted by $\text{Row}(A)$. Note that $\text{Row}(A) = \text{Col}(A^T)$.

5 Eigenvalues and Eigenvectors

Definition 5.1. An **eigenvector** of an $n \times n$ matrix A is a nonzero vector x such that $Ax = \lambda x$. The scalar λ is called an **eigenvalue** of A if and only if the equation

$$(A - \lambda I)x = 0$$

has a nontrivial solution. The set of all solutions to this equation is the null space of the matrix $A - \lambda I$; this subspace of \mathbb{R}^n is called the **eigenspace** of A .

Theorem 10. The eigenvalues of a triangular matrix are the entries on its main diagonal.

Proof. Consider the 3×3 case. If A is upper triangular, then

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

The scalar λ is an eigenvalue if and only if $(A - \lambda I)x = 0$ has a nontrivial solution, which means the equation must have a free variable, which would only occur if at least one of the values on the diagonal is zero. ■

Theorem 11. *The eigenvectors of A , v_1, \dots, v_r , that correspond to distinct eigenvalues, $\lambda_1, \dots, \lambda_r$, are linearly independent.*

Proof. Suppose $\{v_1, \dots, v_r\}$ is linearly dependent. Let p be the least index such that v_{p+1} is a linear combination of the preceding linearly independent eigenvectors. Then there exist scalars such that

$$c_1 v_1 + \dots + c_p v_p = v_{p+1}$$

Multiply both sides by A , using the fact that $Av_k = \lambda_k v_k$, to get

$$c_1 \lambda_1 v_1 + \dots + c_p \lambda_p v_p = \lambda_{p+1} v_{p+1}$$

We can also multiply both sides of our first equation by λ_{p+1} and then subtract to get

$$c_1 (\lambda_1 - \lambda_{p+1}) v_1 + \dots + c_p (\lambda_p - \lambda_{p+1}) v_p = 0$$

Because we assumed v_{p+1} was the first linearly dependent eigenvector, the set $\{v_1, \dots, v_p\}$ must be linearly independent. That means all $(\lambda_k - \lambda_{p+1})$ should be 0, but because the eigenvalues are distinct they cannot be. Thus, we arrive at a contradiction. ■

Remember that to find eigenvalues, we need to find scalars λ such that

$$(A - \lambda I)x = 0$$

has a nontrivial solution. This is equivalent to the matrix $A - \lambda I$ being not invertible, which is equivalent to $\det(A - \lambda I) = 0$. Writing the determinant as a polynomial involving only λ is called the characteristic equation of a matrix.

Theorem 12. Diagonalization Theorem: *An $n \times n$ matrix A is **diagonalizable** if and only if A has n linearly independent eigenvectors. If this condition is met, we can write*

$$A = PDP^{-1}$$

where P is a matrix whose columns are n linearly independent eigenvectors of A , and D is a diagonal matrix whose diagonal entries are corresponding eigenvalues of A .

Proof. Right multiplying both sides by P gives us $AP = PD$.

$$AP = A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} Av_1 & \cdots & Av_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix}$$

$$PD = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \cdots & \lambda_n v_n \end{bmatrix}$$

Since these are equal and P has an inverse because its columns are linearly independent eigenvectors, $A = PDP^{-1}$. ■

6 Orthogonality and Least Squares