# Notes and Solutions for Nielsen and Chuang's *Quantum Computation and Quantum Information*

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# Chapter 1

# Introduction and overview

## Notes

"Instead of looking at quantum systems purely as phenomena to be explained..., they looked at them as systems that can be *designed*...No longer is the quantum world taken merely as presented, but instead it can be *created*."

### Bloch Spheres

For a complex number $z = x + iy$, where $x$ and $y$ are real, we can write

$$|z|^2 = z^* z = (x - iy)(x + iy) = x^2 + y^2$$

Using the polar representation gives us $z = r(\cos\theta + i\sin\theta)$. Substituting Euler's identity ($e^{i\theta} = \cos\theta + i\sin\theta$) results in

$$z = re^{i\theta}$$

We know qubits have complex amplitudes, so we can write

$$|\psi\rangle = r_\alpha e^{i\theta_\alpha}|0\rangle + r_\beta e^{i\theta_\beta}|1\rangle$$

where all four variables are real.
It turns out the term $e^{i\gamma}$ doesn't affect the probabilities $|\alpha|^2$ or $\left|\beta\right|^2$:

$$\left|e^{i\gamma}\alpha\right|^2 = (e^{i\gamma}\alpha)^*(e^{i\gamma}\alpha) = (e^{-i\gamma}\alpha^*)(e^{i\gamma}\alpha) = \alpha^*\alpha = |\alpha|^2$$

so we can multiply by $e^{-i\theta_\alpha}$ to get

$$|\psi\rangle = r_\alpha|0\rangle + r_\beta e^{i\theta}|1\rangle$$

with three real parameters: $r_\alpha$, $r\beta$, and $\theta = \theta_\beta - \theta_\alpha$.
Switching back to Cartesian coordinates and recalling the normalization condition, we have

$$\begin{aligned}|r_\alpha|^2 + \left|x + iy\right|^2 &= r_\alpha^2 + (x - iy)(x + iy)\\ &= r_\alpha^2 + x^2 + y^2 = 1\end{aligned}$$

which is the equation for a sphere. Renaming $r_\alpha$ as $z$, we can use the identities

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

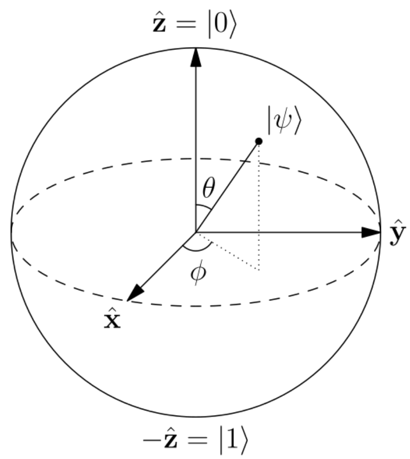
and after subbing in $r = 1$, we can write

$$\begin{aligned}|\psi\rangle &= z|0\rangle + (x + iy)|1\rangle\\ &= \cos\theta|0\rangle + \sin\theta(\cos\phi + i\sin\phi)|1\rangle\\ &= \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle\end{aligned}$$

Notice that $\theta = 0 \to |\psi\rangle = |0\rangle$ and $\theta = \frac{\pi}{2} \to |\psi\rangle = e^{i\phi}|1\rangle$, which suggests that $0 \leq \theta \leq \frac{\pi}{2}$ generates all superpositions. In fact, one can easily check that plugging in $\theta' = \pi - \theta$ and $\phi' = \phi + \pi$ (the opposite point on the sphere) results in $-|\psi\rangle$. Since the lower hemisphere of the sphere differs only by a phase factor of -1, we can choose to only consider the upper hemisphere.
To map points in the upper hemisphere onto a sphere, we can write

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.




The poles represent classical states. When a qubit is measured, it has higher probability of collapsing to the pole it's closer to. This representation makes it clear that the Pauli Z gate results in only a *phase change* because it does not affect the state the qubit will collapse to.[1] Now that we've derived Bloch spheres, there are two key properties we must understand:

1. *Orthogonality of opposite points:* Let $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$, and let $|\varphi\rangle$ the opposite point on the Bloch sphere,

$$|\varphi\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|0\rangle - e^{i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|1\rangle$$

Using the identity $\cos(a+b) = \cos a\cos b - \sin a\sin b$, we get

$$(\varphi, \psi) = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\pi-\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\pi-\theta}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

2. *Rotations:* The Pauli X, Y, and Z gates are so-called because when exponentiated they yield rotation operators, which rotate the Bloch vector $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ about the $x$, $y$, and $z$ axes. For example, the exponentiated Pauli X gate is

$$R_X(\theta) = e^{-i\theta X/2}$$

The exponentiated Pauli Y and Z gates are similarly defined.

To understand this, note that in the special case where $A^2 = I$ (which holds for all the Pauli matrices),

$$\begin{aligned} e^{i\theta A} &= I + i\theta A - \frac{\theta^2 I}{2!} - i\frac{\theta^3 A}{3!} + \frac{\theta^4 I}{4!} + i\frac{\theta^5 A}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right)I + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)A \\ &= \cos(\theta)I + i\sin(\theta)A \end{aligned}$$

Now we can exponentiate the Pauli X gate as

$$R_X(\theta) = e^{-i\theta X/2} = \cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}X = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

[1] Image from Anastasios Kyrillidis's notes.

Let's consider $R_X(\pi)$,

$$R_X(\pi) = \begin{bmatrix} \cos\frac{\pi}{2} & -i\sin\frac{\pi}{2} \\ -i\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -iX$$

This is a 180 degree rotation across the x-axis since we swap amplitudes and the phase changes.[2] **Todo:** Need an actual reason for why.

### Why must quantum gates be unitary?

For a quantum state $\left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$, we know $|\alpha|^2 + \left|\beta\right|^2 = 1$ must be true. Another way of writing this is $(\left|\psi\right\rangle, \left|\psi\right\rangle) = 1$. This must also be true after the application of a quantum gate $U$. Thus, we have

$$\begin{aligned} 1 &= (U\left|\psi\right\rangle, U\left|\psi\right\rangle) = (\left|\psi\right\rangle, \left|\psi\right\rangle) \\ &= (U^\dagger U\left|\psi\right\rangle, \left|\psi\right\rangle) = (\left|\psi\right\rangle, \left|\psi\right\rangle) \end{aligned}$$

which means $U^\dagger U = I$ is a fundamental constraint on quantum gates.

### Decomposing single qubit operations

Move to section 4.2 according to Box 1.1 on p20.

### What does the matrix representation of a quantum gate mean?

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This $U_{CN}$ gate acts on two qubits, so it is a transformation on basis vectors $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, and $|1\rangle \otimes |1\rangle$. The columns tell us that an input of the third basis vector, $|1\rangle \otimes |0\rangle$, will output the fourth basis vector, $|1\rangle \otimes |1\rangle$.
**Todo:** What does this say about superpositions?

### Why must quantum gates be reversible?

Since quantum gates are unitary, we know $U^\dagger U = I$, which means quantum gates are invertible, and the inverse is also a quantum gate because it is unitary.
One implication of this is that classical gates like XOR and NAND have no quantum cousins because these gates are irreversible since they result in a loss of information.
Well, technically, the Toffoli gate can be used to simulate NAND gates (and therefore all classical gates), but it can only do so because it explicitly preserves all input bits.

## Solutions

### Exercise 1.1

2 evaluations. If we get the same value and guess the constant function, there's a 25% chance it was the balanced function.

### Exercise 1.2

If we could fully identify the state, we can just bitwise add our result to $|0\rangle$ to get a clone of our original state.
For the converse, if we had a device that could clone quantum states, we could continue cloning and measuring the clones to get an arbitrary level of accuracy of the original quantum state.

[2] Most of this section is from Ian Glendinning's talk.

# Chapter 2

# Introduction to quantum mechanics

## Notes

### Notation

For distinct vectors in an orthonormal set, we can write $\langle i|j\rangle = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker product and is 1 if $i = j$ and 0 if $i \neq j$.

### Matrix - Linear Operator Congruence

For a matrix to a be a linear operator,

$$A\left(\sum_i a_i |v_I\rangle\right) = \sum_i a_i A |v_i\rangle$$

must be true. Note the LHS is the sum of vectors to which $A$ is applied which is certainly equal to the RHS.
Now suppose $A : V \to W$ is a linear operator and that $V$ has basis $|v_i\rangle, \cdots, |v_m\rangle$ and $W$ has basis $|w_i\rangle, \cdots, |w_n\rangle$. Since we know the $k$th column of a $A$ will be its transformation of $|v_k\rangle$,

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Note this is just saying $A|v_j\rangle$ is equal to the $j$th column of $A$, and we can think of $|w_i\rangle$ as the coordinates of the transformed vector. Thus, we can find the matrix representation of any linear operator by finding a matrix $A$ with entries specified by the above equation.

### What's so special about the Pauli matrices?

Things to look into:

- they pop up in the Pauli equation
- they form a basis for the vector space of $2 \times 2$ Hermitian matrices. Hermitian matrices represent observables? Look at Wikipedia page

### What does the Completeness Relation say about matrices?

I don't understand why

$$\sum_{ij} \langle w_j|A|v_i\rangle |w_j\rangle \langle v_i|$$

implies $A$ has matrix element $\left\langle w_j \middle| A|v_i\right\rangle$ in the $i$th column and $j$th row, with respect to input basis $|v_i\rangle$ and output basis $|w_j\rangle$. page 68.

### Lucien Hardy Postulates of QM

## Solutions

### Exercise 2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

### Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

because $A|0\rangle$ has coordinate 0 in $|0\rangle$ and coordinate 1 in $|1\rangle$.
If we keep our input bases the same but reorder our output bases as $|1\rangle$ and $|0\rangle$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

### Exercise 2.3

We know

$$A|v_i\rangle = \sum_j A_{ji}|w_j\rangle$$

and

$$B|w_j\rangle = \sum_k B_{kj}|x_k\rangle$$

Now we can write

$$\begin{aligned} BA|v_i\rangle = B(A|v_i\rangle) = B\sum_j A_{ji}|w_j\rangle &= \sum_j A_{ji}(B|w_j\rangle) \\ &= \sum_j A_{ji}\sum_k B_{kj}|x_k\rangle \\ &= \sum_k \sum_j B_{kj}A_{ji}|x_k\rangle \\ &= \sum_k (BA)_{ki}|x_k\rangle \end{aligned}$$

We know $\sum_k (BA)_{ki}$ is the matrix representation of operator $BA$, which the preceding step says is equal to $\sum_k \sum_j B_{kj}A_{ji}$, which is the matrix multiplication $BA$.

### Exercise 2.4

For the same input and output basis, we want some $I$ such that

$$I|v_j\rangle = \sum_i I_{ij}|v_i\rangle = |v_j\rangle$$

which means $I_{ij} = 0$ for all $i \neq j$ and 1 otherwise.

## Exercise 2.5

For $|y\rangle, |z_i\rangle \in \mathbb{C}^n$ and $\lambda_i \in C$,

$$\begin{aligned}\Big(|y\rangle, \sum_i \lambda_i |z_i\rangle\Big) &= |y\rangle^* \sum_i \lambda_i |z_i\rangle \\ &= \sum_i \lambda_i |y\rangle^* |z_i\rangle \\ &= \Big(\sum_i \lambda_i^* |z_i\rangle^* |y\rangle\Big)^*\end{aligned}$$

The second and third equalities demonstrate linearity in the second argument and $(|y\rangle, |z\rangle) = (|z\rangle, |w\rangle)^*$. Finally, if $|w\rangle = (w_1, \cdots, w_n)$ where $w_i \in \mathbb{C}^n$, then

$$(|w\rangle, |w\rangle) = \sum_i w_i^* w_i = \sum_i |w_i|^2$$

which proves the non-degeneracy and non-negativity condition.

## Exercise 2.6

$$\begin{aligned}\Big(\sum_i \lambda_i |w_i\rangle, |v\rangle\Big) &= \Big(|v\rangle^*, \sum_i \lambda_i^* |w_i\rangle^*\Big)^* \\ &= \sum_i \lambda_i^* \Big(|v\rangle^*, |w_i\rangle^*\Big)^* \\ &= \sum_i \lambda_i^* \Big(|w_i\rangle, |v\rangle\Big)\end{aligned}$$

## Exercise 2.7

$$(|w\rangle, |v\rangle) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 - 1 = 0$$

To normalize, divide each vector by $\sqrt{2}$.

## Exercise 2.8

Since at each step, we divide by the norm of the vector being added, the set is orthonormal. The set is a basis because at step $i$, we add the basis vector $|w_i\rangle$ but subtract out the portion that was already in span$(|v_1\rangle, \cdots, |v_{i-1}\rangle)$, so we still end up spanning the full vector space.

## Exercise 2.9

$$\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$\sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

## Exercise 2.10

$$\begin{aligned}|v_j\rangle\langle v_k| &= I|v_j\rangle\langle v_k| I \\ &= \sum_a |v_a\rangle\langle v_a|v_j\rangle \sum_b \langle v_k|v_b\rangle\langle v_b| \\ &= \sum_{a,b} \delta_{aj}\delta_{kb} |v_a\rangle\langle v_b|\end{aligned}$$

so the element $(|v_j\rangle\langle v_k|)_{ab} = \delta_{aj}\delta_{kb}$.

## Exercise 2.11

Each of the Pauli matrices has eigenvalues ±1.
For $\sigma_x$,

$$\sigma_{x+} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \sigma_{x-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For $\sigma_y$,

$$\sigma_{y+} = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ and } \sigma_{y-} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

For $\sigma_z$,

$$\sigma_{z+} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \sigma_{z-} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The diagonalization easily follows.

## Exercise 2.12

The characteristic equation is $(1-\lambda)^2$, so we have eigenvalue 1. Solving $(A - 1I)|v\rangle = 0$ gives us $|v\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Our diagonal form should be

$$A = |v\rangle\langle v| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

## Exercise 2.13

$$(|w\rangle\langle v|)^\dagger = \langle v|^\dagger |w\rangle^{dagger} = |v\rangle\langle w|$$

## Exercise 2.14

Since we know $(a+b)^\dagger = a^\dagger + b^\dagger$, so

$$\left(\sum_i a_i A_i\right)^\dagger = \sum_i \left(a_i A_i\right)^\dagger = \sum_i \left(a_i^* A_i^\dagger\right)$$

## Exercise 2.15

$$(|v\rangle, A|w\rangle) = (A^\dagger |v\rangle, |w\rangle) = \left(|v\rangle, (A^\dagger)^\dagger |w\rangle\right)$$

since this holds for all $|v\rangle$, $|w\rangle$, $A = (A^\dagger)^\dagger$.

## Exercise 2.16

$$P^2 = \sum_i |i\rangle\langle i| \sum_j |j\rangle\langle j| = \sum_{ij} |i\rangle\langle i|j\rangle\langle j| = \sum_{ij} \delta_{ij} |i\rangle\langle j|$$

Intuitively, projecting some $|v\rangle \in P$ wouldn't change $|v\rangle$ at all.

## Exercise 2.17

Since $A$ is normal, we can diagonalize it. If the eigenvalues are all positive, then the adjoint of this diagonal matrix of eigenvalues equals itself. If $A$ is Hermitian, then so is its diagonal form. The adjoint of a diagonal matrix consists of the conjugates of its diagonal entries, so if $A = A^\dagger$, then the diagonal entries (eigenvalues) must all be positive.

## Exercise 2.18

For an eigenvector $|v\rangle$, we have $A|v\rangle = \lambda|v\rangle \to \langle v|A^\dagger = \lambda^* \langle v|$. Multiplying these two gives us $\langle v|A^\dagger A|v\rangle = \lambda^*\lambda\langle v|v\rangle$, and because $A^\dagger A = I$,

$$\|v\|^2 = |\lambda|^2 \|v\|^2 \to |\lambda| = 1$$

## Exercise 2.19

Omitted because it's just mechanical.

## Exercise 2.20

$$\begin{aligned}
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|U^\dagger UAUU^\dagger|v_j\rangle \\
&= \sum_a\sum_b\sum_c\sum_d \langle v_i|w_a\rangle \langle v_a|v_b\rangle \langle w_b|A|w_c\rangle \langle v_c|v_d\rangle \langle w_d|v_j\rangle \\
&= \sum_a\sum_b\sum_c\sum_d \delta_{ab}\delta_{cd} \langle v_i|w_a\rangle A''_{bc} \langle w_d|v_j\rangle
\end{aligned}$$

This tells us $a = b$ and $c = d$, so

$$A'_{ij} = \sum_a\sum_c \langle v_i|w_a\rangle A''_{ac} \langle w_c|v_j\rangle$$

## Exercise 2.21

We will prove any Hermitian operator $M$ is diagonal with respect to some orthonormal basis $V$.
We proceed by induction on dimension $d$ on $V$.
**Base case:** $d = 1$
Trivially, $M$ is diagonal.
**Inductive hypothesis:** Assume $d = n - 1$
**Inductive step:** Prove $d = n$
Let $\lambda$ be an eigenvalue of $M$, $P$ be a projection onto the $\lambda$ eigenspace, and $Q$ be $P$'s orthogonal complement.
We know $M = (P+Q)M(P+Q)$. First, note that $QMP = \lambda QP = 0$. Now for some $|v\rangle \in P$, $M|v\rangle = M^\dagger|v\rangle = \lambda|v\rangle$ because $M$ is Hermitian, which means $|v\rangle$ is in the eigenspace $\lambda$ of $M^\dagger$. Now we have $QM^\dagger P|v\rangle = QM^\dagger|v\rangle = \lambda Q|v\rangle = 0$. Taking the adjoint of this gives us $PMQ = 0$. Now we have $M = PMP + QMQ$.
Since $PMP = \lambda P$, $QMQ$ must be nonzero also for $M = PMP + QMQ$ to hold, so they both have dimension less than $n$. Finally, $PMP = (PMP)^\dagger = PM^\dagger P$ and similarly for $QMQ$. Since they're both Hermitian, our inductive hypothesis proves the theorem.

## Exercise 2.22

For a Hermitian operator $A$, suppose $A|v\rangle = \lambda|v\rangle$ and $A|w\rangle = \mu|w\rangle$.
Since $\langle v|A = \lambda\langle v|$, we can write

$$\langle v|A^2|w\rangle = \mu^2\langle v|w\rangle = \lambda\mu\langle v|w\rangle$$

where the first equality follows from $A^2|w\rangle = \mu^2|w\rangle$. Since $\lambda \neq \mu$, $\langle v|w\rangle = 0$.

## Exercise 2.23

Suppose $|v\rangle$ is an eigenvector of $P$ with eigenvalue $\lambda$, $P|v\rangle = \lambda|v\rangle$. Then

$$P|v\rangle = P^2|v\rangle = \lambda^2|v\rangle$$

where the first equality follows from the property $P^2 = P$. Since $\lambda^2 = \lambda$, $P$'s eigenvalues must be 1 or 0.

## Exercise 2.24

Let $B = \frac{A+A^\dagger}{2}$ and $C = \frac{A-A^\dagger}{2i}$. This means

$$\langle v|A|v\rangle = \langle v|B|v\rangle + i\langle v|C|v\rangle$$

Since we cannot have an imaginary term in a positive operator, $C = 0$, so $A = A^\dagger$.

## Exercise 2.25

We can write $\langle Av|Av\rangle = \langle A^\dagger Av|v\rangle$. But since $\langle Av|Av\rangle$ can also be written as $\|Av\|^2 \geq 0$, $A^\dagger A$ must be positive.

## Exercise 2.26

As a tensor product:

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}\Big(|0\rangle + |1\rangle\Big)\otimes\Big(|0\rangle + |1\rangle\Big) = \frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big)$$

As a Kronecker product: Since $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$,

$$\begin{bmatrix}\frac{1}{\sqrt{2}}|\psi\rangle\\ \frac{1}{\sqrt{2}}|\psi\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

As a tensor product:

$$|\psi\rangle^{\otimes 3} = |\psi\rangle\otimes|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}\Big(|0\rangle + |1\rangle\Big)\otimes\frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big) = \frac{1}{2\sqrt{2}}\Big(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\Big)$$

As a Kronecker product:

$$\begin{bmatrix}\frac{1}{\sqrt{2}}|\psi\rangle^{\otimes 2}\\ \frac{1}{\sqrt{2}}|\psi\rangle^{\otimes 2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}}\end{bmatrix}^T$$

## Exercise 2.27

$$X\otimes Z = \begin{bmatrix}0\cdot Z & 1\cdot Z\\ 1\cdot Z & 0\cdot Z\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}$$

$$I\otimes X = \begin{bmatrix}1\cdot X & 0\cdot X\\ 0\cdot X & 1\cdot X\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}$$

$$X\otimes I = \begin{bmatrix}0\cdot I & 1\cdot I\\ 1\cdot I & 0\cdot I\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}$$

Since $I\otimes X \neq X\otimes I$, the tensor product is not commutative.

## Exercise 2.28

Writing the Kronecker product,

$$(A\otimes B)^\dagger = \begin{bmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{bmatrix}^\dagger = \begin{bmatrix} A^*_{11}B^\dagger & \cdots & A^*_{n1}B^\dagger \\ \vdots & \ddots & \vdots \\ A^*_{1n}B^\dagger & \cdots & A^*_{nn}B^\dagger \end{bmatrix} = A^\dagger \otimes B^\dagger$$

Proving the transpose is similar and proving the complex conjugate requires only using $B^*$ instead of $B^\dagger$.

## Exercise 2.29

Let $U_1$ and $U_2$ be unitary.

$$(U_1\otimes U_2)^\dagger(U_1\otimes U_2) = (U_1^\dagger\otimes U_2^\dagger)(U_1\otimes U_2) = U_1^\dagger U_1 \otimes U_2^\dagger U_2 = I\otimes I = I$$

## Exercise 2.30

Let $A = A^\dagger$ and $B = B^\dagger$.

$$(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger = A\otimes B$$

## Exercise 2.31

Let $A$ and $B$ be positive operators.

$$\Big((A\otimes B)(|v\rangle\otimes|w\rangle), (|v\rangle\otimes|w\rangle\Big) = \Big(A|v\rangle\otimes B|w\rangle, |v\rangle\otimes|w\rangle\Big) = \langle v|A^\dagger|v\rangle\langle w|B^\dagger|w\rangle = \langle v|A|v\rangle\langle w|B|w\rangle$$

since we know positive operators are Hermitian. Since $\langle v|A|v\rangle$ and $\langle w|B|w\rangle$ are both non-negative, their product is also non-negative.

## Exercise 2.32

Let $P$ and $Q$ be projectors. Recall that if $P^2 = P$, $P$ is a projector.

$$(P\otimes Q)(P\otimes Q) = P^2\otimes Q^2 = P\otimes Q$$

## Exercise 2.33

We proceed by induction on $n$.
**Base case:** $n = 1$
We know

$$H^{\otimes 1} = H = \frac{1}{\sqrt{2}}\Big[(|0\rangle+|1\rangle)\langle 0| + (|0\rangle-|1\rangle)\langle 1|\Big] = \frac{1}{\sqrt{2}}\sum_{x,y\in\{0,1\}^1}(-1)^{x\cdot y}|x\rangle\langle y|$$

**Inductive hypothesis:** Assume $n = k-1$

$$H^{\otimes k-1} = \frac{1}{2^{k-1/2}}\sum_{x,y\in\{0,1\}^{k-1}}(-1)^{x\cdot y}|x\rangle\langle y|$$

**Inductive step:** Prove $n = k$

$$H^{\otimes k} = H\otimes H^{\otimes k-1} = \frac{1}{2^{k/2}}\sum_{x_1,y_1\in\{0,1\}}(-1)^{x_1\cdot y_1}|x_1\rangle\langle y_1| \qquad \otimes \sum_{x_2,y_2\in\{0,1\}^{k-1}}(-1)^{x_2\cdot y_2}|x_2\rangle\langle y_2|$$

Since this tensor product only flips the sign of the $H^{\otimes k-1}$ if $x_1 = y_1 = 1$, it is easy to see that concatenating $x_1, x_2$ and $y_1, y_2$, would yield the dot product required to flip the signs when we want.

$$H^{\otimes 2} = \frac{1}{\sqrt{2}}\begin{bmatrix} H & H \\ H & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

## Exercise 2.34

Let $A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$. First, we find the eigenvalues and eigenvectors.

$$\det(A - \lambda I) = (\lambda - 7)(\lambda - 1) \rightarrow \lambda = 7, 1$$

with corresponding eigenvectors

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We'll denote the eigenpairs as $(7, |a\rangle)$ and $(1, |b\rangle)$.
So,

$$\sqrt{A} = \sqrt{7}\,|a\rangle\langle a| + 1\,|b\rangle\langle b|$$

and

$$\log(A) = \log(7)\,|a\rangle\langle a|$$

## Exercise 2.35

Since $v \cdot \sigma$ is a weighted sum of the Pauli matrices, we know it will have eigenvalues of 1 and -1. Let the eigenpairs of $v \cdot \sigma$ be $(1, |\lambda_1\rangle)$ and $(-1, |\lambda_{-1}\rangle)$.

$$\begin{aligned}
\exp(i\theta v \cdot \sigma) &= e^{i\theta}\,|\lambda_1\rangle\langle\lambda_1| + e^{-i\theta}\,|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= (\cos\theta + i\sin\theta)|\lambda_1\rangle\langle\lambda_1| + (\cos\theta - i\sin\theta)|\lambda_{-1}\rangle\langle\lambda_{-1}| \\
&= \cos\theta\Big(|\lambda_1\rangle\langle\lambda_1| + |\lambda_{-1}\rangle\langle\lambda_{-1}|\Big) + i\sin\theta\Big(|\lambda_1\rangle\langle\lambda_1| - |\lambda_{-1}\rangle\langle\lambda_{-1}|\Big) \\
&= \cos\theta I + i\sin\theta v \cdot \sigma
\end{aligned}$$

## Exercise 2.36

$$\begin{aligned}
\operatorname{tr}(X) &= 0 + 0 \\
\operatorname{tr}(Y) &= 0 + 0 \\
\operatorname{tr}(Z) &= 1 - 1
\end{aligned}$$

## Exercise 2.37

$$\operatorname{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j A_{ij}B_{ji} = \sum_j \sum_i B_{ji}A_{ij} = \sum_j (BA)_{jj} = \operatorname{tr}(BA)$$

## Exercise 2.38

$$\operatorname{tr}(A + B) = \sum_i (A + B)_{ii} = \sum_i A_{ii} + \sum_j B_{jj} = \operatorname{tr}(A) + \operatorname{tr}(B)$$

$$\operatorname{tr}(zA) = \sum_i (zA)_{ii} = z\sum_i A_{ii} = z\operatorname{tr}(A)$$

## Exercise 2.39

1. (a) Linearity in second argument:

$$(A, \sum_i \lambda_i B) = \mathrm{tr}\left(A^\dagger \sum_i \lambda_i B\right) = \mathrm{tr}\left(\sum_i \lambda_i A^\dagger B\right) = \sum_i \lambda_i \,\mathrm{tr}\left(A^\dagger B\right)$$

(b)

$$\begin{aligned}(B,A)^* = \mathrm{tr}\left(B^\dagger A\right) &= \left(\sum_i \langle i|B^\dagger A|i\rangle\right)^* \\ &= \sum_i \langle i|A^\dagger B|i\rangle \\ &= \mathrm{tr}\left(A^\dagger B\right) \\ &= (A,B)\end{aligned}$$

(c)

$$(A,A) = \mathrm{tr}\left(A^\dagger A\right)$$

Since we know $A^\dagger A$ is a positive operator, the sum of its eigenvalues must be nonnegative. Additionally, the sum of its eigenvalues will only be zero if $A^\dagger A$ is the 0 matrix, which means $A$ was also the 0 matrix.

2. We can fix a basis $|v_1\rangle, \cdots, |v_n\rangle$. In this basis, the columns of any $A \in L_V$ are defined as the vectors $A$ maps the $n$ basis vectors to. Since we need $n$ terms to describe each vector and we map $n$ basis vectors, $\dim L_V = n^2$.

3. Not sure if we need $n^2$ matrices for a basis, but using the same basis vectors as before, we can define $n$ Hermitian matrices for each $i$: $A_i = \langle v_i|v_i\rangle$. Since any operator can be written as its action on the basis vectors, linear combinations of these Hermitian matrices span $V$.

   The set is orthonormal because if the basis vectors are normalized,

$$(A_i, A_i) = \mathrm{tr}\left(A_i^\dagger A_i\right) = \langle v_i|A^\dagger A|v_i\rangle = \langle v_i|v_i\rangle \langle v_i|v_i\rangle \langle v_i|v_i\rangle = 1$$

   and

$$(A_i, A_j) = \mathrm{tr}\left(A_i^\dagger A_j\right) = \mathrm{tr}\left(|v_i\rangle \langle v_i|v_j\rangle \langle v_j|\right) = \mathrm{tr}\left(|v_i\rangle\, 0 \,\langle v_j|\right) = 0$$

## Exercise 2.40

$$[X,Y] = XY - YX = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}i & 0\\0-i & \end{bmatrix} - \begin{bmatrix}-i & 0\\0 & i\end{bmatrix} = \begin{bmatrix}2i & 0\\0 & -2i\end{bmatrix} = 2iZ$$

$$[Y,Z] = YZ - ZY = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = \begin{bmatrix}0 & i\\i & 0\end{bmatrix} - \begin{bmatrix}0 & -i\\-i & 0\end{bmatrix} = \begin{bmatrix}0 & 2i\\2i & 0\end{bmatrix} = 2iX$$

$$[Z,X] = ZX - XZ = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} - \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix} = 2iY$$

The textbook explains an elegant way to represent this is:

$$[\sigma_j, \sigma_k] = 2i \sum_{l=1}^{3} \epsilon_{jkl}\sigma_l$$

where $\epsilon_{jkl} = 0$ except for $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$.

## Exercise 2.41

Consider the above exercise and add instead of subtract matrices. It is easy to see each equation would result in 0. This is sufficient for all the examples because $\{A,B\} = \{B,A\}$.

## Exercise 2.42

$$\frac{[A,B]+\{A,B\}}{2} = \frac{AB-BA+AB+BA}{2} = \frac{2AB}{2} = AB$$

## Exercise 2.43

We know

$$\sigma_j\sigma_k = \frac{[\sigma_j,\sigma_k]+\{\sigma_j,\sigma_k\}}{2}$$

Notice that if $j \neq k$, $\{\sigma_j,\sigma_k\} = 0$, Exercise 2.40 lets us write

$$\sigma_j\sigma_k = \frac{[\sigma_j,\sigma_k]}{2} = i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$$

Notice that if $j = k$, $[\sigma_j,\sigma_k] = 0$, so

$$\sigma_j\sigma_k = \sigma_j\sigma_j = \frac{\{\sigma_j,\sigma_j\}}{2} = \frac{\sigma_j^2+\sigma_j^2}{2} = \sigma_j^2 = I$$

So we arrive at

$$\sigma_j\sigma_k = \delta_{jk}I + i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$$

## Exercise 2.44

Combining $AB-BA=0$ and $AB+BA=0$ yields

$$2AB = 0 \to AB = 0 \to B = A^{-1}0 = 0$$

## Exercise 2.45

$$[A,B]^\dagger = (AB-BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger,A^\dagger]$$

## Exercise 2.46

$$[A,B] = AB-BA = -(BA-AB) = -[B,A]$$

## Exercise 2.47

$$(i[A,B])^\dagger = -i[B^\dagger,A^\dagger] = i[A^\dagger,B^\dagger] = i[A,B]$$

by using the results from the above two exercises.

## Exercise 2.48

For a positive matrix $P$, we have $P = IP$. For a unitary matrix $U$, we have $U = UI$. For a Hermitian matrix $H$, we know $J = \sqrt{H^\dagger H} = \sqrt{H^2} = |H|$, so we have $H = U|H|$.

## Exercise 2.49

If $A$ is normal, it must have a spectral decomposition $A = \sum_i \lambda_i |i\rangle\langle i|$. We can now write

$$J = \sqrt{A^\dagger A} = \sum_i |\lambda_i||i\rangle\langle i|$$

. Since $U = \sum_i |e_i\rangle\langle i|$,

$$A = UJ = \sum_i |\lambda_i||e_i\rangle\langle i|$$

## Exercise 2.50

The calculations aren't turning out pretty so I'm skipping over this one.