Notes and Solutions for Nielsen and Chuang's *Quantum Computation and Quantum Information*

Warren Alphonso

2019

Contents

Chapter 2

Introduction to quantum mechanics

Notes

Matrix - Linear Operator Congruence

For a matrix to a be a linear operator,

$$A\Big(\sum_i a_i |v_I\rangle\Big) = \sum_i a_i A|v_i\rangle$$

must be true. Note the LHS is the sum of vectors to which A is applied which is certainly equal to the RHS.
Now suppose $A : V \to W$ is a linear operator and that V has basis $|v_i\rangle, \cdots, |v_m\rangle$ and W has basis $|w_i\rangle, \cdots, |w_n\rangle$. Since we know the kth column of a A will be its transformation of $|v_k\rangle$,

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Note this is just saying $A|v_j\rangle$ is equal to the jth column of A, and we can think of $|w_i\rangle$ as the coordinates of the transformed vector. Thus, we can find the matrix representation of any linear operator by finding a matrix A with entries specified by the above equation.

What's so special about the Pauli matrices?

Things to look into:

- they pop up in the Pauli equation
- they form a basis for the vector space of 2×2 Hermitian matrices. Hermitian matrices represent observables? Look at Wikipedia page

Solutions

Exercise 2.1

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0$$

Exercise 2.2

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

because $A|0\rangle$ has coordinate 0 in $|0\rangle$ and coordinate 1 in $|1\rangle$.

If we keep our input bases the same but reorder our output bases as $|1\rangle$ and $|0\rangle$,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Exercise 2.3

We know

$$A|v_i\rangle = \sum_j A_{ji}|w_j\rangle$$

and

$$B|w_j\rangle = \sum_k B_{kj}|x_k\rangle$$

Now we can write

$$\begin{aligned} BA|v_i\rangle = B(A|v_i\rangle) = B\sum_j A_{ji}|w_j\rangle &= \sum_j A_{ji}(B|w_j\rangle) \\ &= \sum_j A_{ji}\sum_k B_{kj}|x_k\rangle \\ &= \sum_k\sum_j B_{kj}A_{ji}|x_k\rangle \\ &= \sum_k (BA)_{ki}|x_k\rangle \end{aligned}$$

We know $\sum_k (BA)_{ki}$ is the matrix representation of operator BA, which the preceding step says is equal to $\sum_k\sum_j B_{kj}A_{ji}$, which is the matrix multiplication BA.

Exercise 2.4

For the same input and output basis, we want some I such that

$$I|v_j\rangle = \sum_i I_{ij}|v_i\rangle = |v_j\rangle$$

which means $I_{ij} = 0$ for all $i \neq j$ and 1 otherwise.

Exercise 2.5

For $|y\rangle, |z_i\rangle \in \mathbb{C}^n$ and $\lambda_i \in C$,

$$\begin{aligned} \Big(|y\rangle, \sum_i \lambda_i|z_i\rangle\Big) &= |y\rangle^* \sum_i \lambda_i|z_i\rangle \\ &= \sum_i \lambda_i |y\rangle^* |z_i\rangle \\ &= \Big(\sum_i \lambda_i^* |z_i\rangle^* |y\rangle\Big)^* \end{aligned}$$

The second and third equalities demonstrate linearity in the second argument and $(|y\rangle, |z\rangle) = (|z\rangle, |w\rangle)^*$. Finally, if $|w\rangle = (w_1, \cdots, w_n)$ where $w_i \in \mathbb{C}^n$, then

$$(|w\rangle, |w\rangle) = \sum_i w_i^* w_i = \sum_i |w_i|^2$$

which proves the non-degeneracy and non-negativity condition.

Exercise 2.6

$$
\begin{aligned}
\left(\sum_i \lambda_i |w_i\rangle, |v\rangle \right) &= \left(|v\rangle^*, \sum_i \lambda_i^* |w_i\rangle^* \right)^* \\
&= \sum_i \lambda_i^* \left(|v\rangle^*, |w_i\rangle^* \right)^* \\
&= \sum_i \lambda_i^* \left(|w_i\rangle, |v\rangle \right)
\end{aligned}
$$