1 Methods

$$\int\limits_S \rho\overline{u}\phi \cdot \overline{n} dS = \int\limits_S \Gamma\frac{\partial\phi}{\partial x} \cdot \overline{n} dS + \int\limits_V q_\phi dV \tag{1}$$

We can approximate the two surface integrals as follows:

$$\int\limits_S \rho\overline{u}\phi \cdot \overline{n} dS = \Big(\rho\underline{(u)}S\phi\Big)_{i+1} - \Big(\rho\underline{(u)}S\phi\Big)_{i-1} \ ,$$

and

$$\int\limits_S \Gamma\frac{\partial\phi}{\partial x} \cdot \overline{n} dS = \left(\Gamma S\frac{\partial\phi}{\partial x}\right)_{i+1} - \left(\Gamma S\frac{\partial\phi}{\partial x}\right)_{i-1} \ .$$

We can approximate the volume integral as follows:

$$\int\limits_V q_\phi dV = \overline{q}V \ .$$

If we assume S is constant our transport equation becomes:

$$\Big(\rho\bar{(u)}\phi\Big)_{i+1} - \Big(\rho\bar{(u)}\phi\Big)_{i-1} = \left(\Gamma\frac{\partial\phi}{\partial x}\right)_{i+1} - \left(\Gamma\frac{\partial\phi}{\partial x}\right)_{i-1} + \overline{q}V \ .$$

We can apply the Central Differencing Scheme approximations:

$$\phi_{i+1} = \frac{\phi_{I+1} + \phi_I}{2} \qquad \phi_{i-1} = \frac{\phi_{I-1} + \phi_I}{2}$$

$$\left(\frac{\partial\phi}{\partial x}\right)_{i+1} = \frac{\phi_{I+1} - \phi_I}{\delta x} \qquad \left(\frac{\partial\phi}{\partial x}\right)_{i-1} = \frac{\phi_I - \phi_{I-1}}{\delta x}$$

The transport equation now looks like:

$$\left(\frac{\rho\overline{u}}{2}\right)_{i+1}(\phi_{I+1} + \phi_I) - \left(\frac{\rho\overline{u}}{2}\right)_{i-1}(\phi_{I-1} + \phi_I) = \frac{\Gamma}{\Delta x}(\phi_{I+1} - \phi_I) - \frac{\Gamma}{\Delta x}(\phi_I - \phi_{I-1}) + \overline{q}_I \, \Delta x \ .$$

For this problem, the density and velocity are constant throughout the geometry. We can group terms and get the transport equation in a useful form to sovle.

$$\left[-\frac{\rho\overline{u}}{2} - \frac{\Gamma}{\Delta x}\right]\phi_{I-1} + \left[\frac{2\Gamma}{\Delta x}\right]\phi_I + \left[\frac{\rho\overline{u}}{2} - \frac{\Gamma}{\Delta x}\right]\phi_{I+1} = \overline{q}_I \, \Delta x \tag{2}$$

We can rewrite this equation as:

$$a_I\phi_{I-1} + b_I\phi_I + c_I\phi_{I+1} = Q_I \ . \tag{3}$$

This equation works for middle nodes, but the edge nodes do not have enough neighbor nodes.

2 Results

Figure 1 shows the two solutions for Case 1: 5 control volumes and $\overline{u} = 0.1\,\mathrm{m\,s^{-1}}$. Figure 2 shows the two solutions for Case 2: 5 control volumes and $\overline{u} = 2.5\,\mathrm{m\,s^{-1}}$. Figure 3 shows the two solutions for Case 3: 20 control volumes and $\overline{u} = 2.5\,\mathrm{m\,s^{-1}}$.

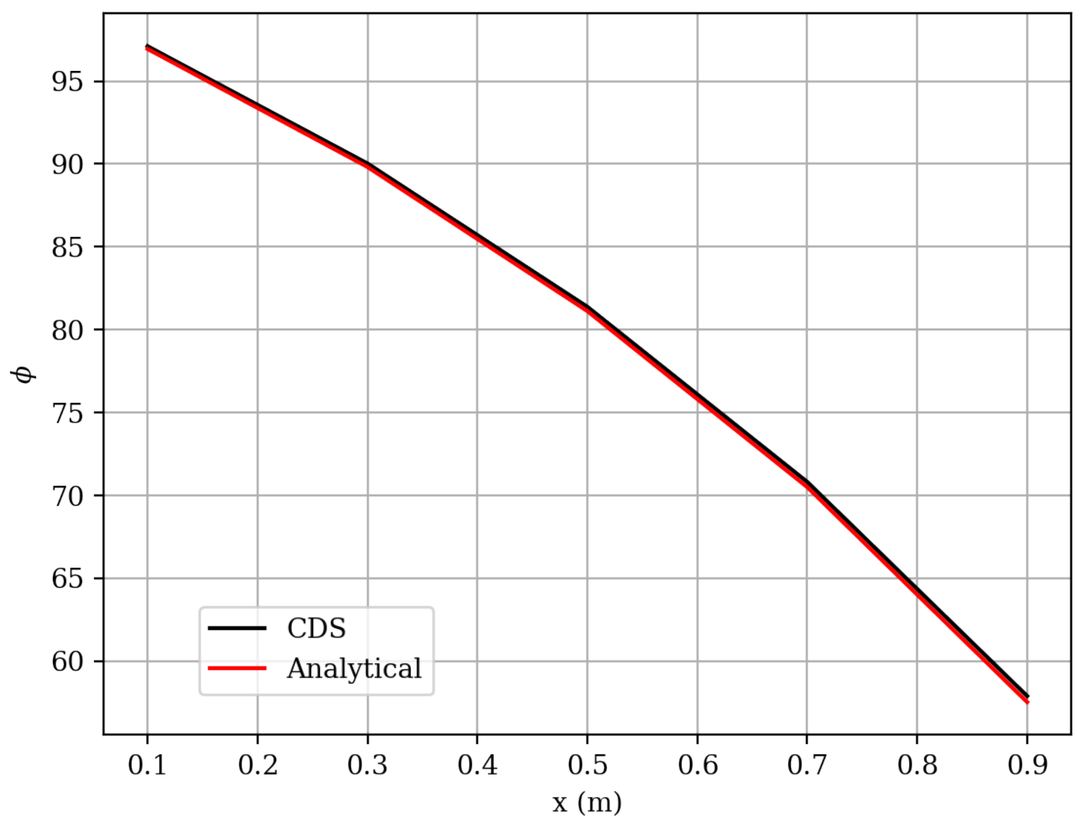

Figure 1: Central Difference Scheme and Analytical Solutions for Case 1.

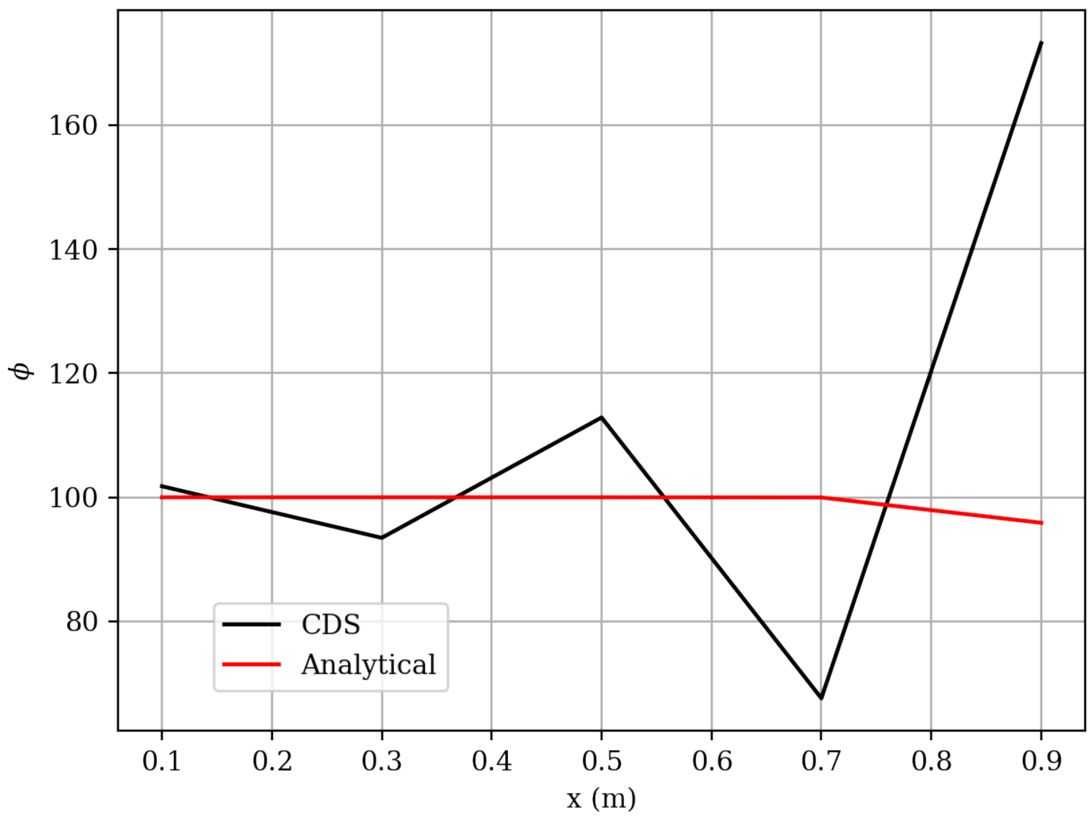

Figure 2: Central Difference Scheme and Analytical Solutions for Case 2.

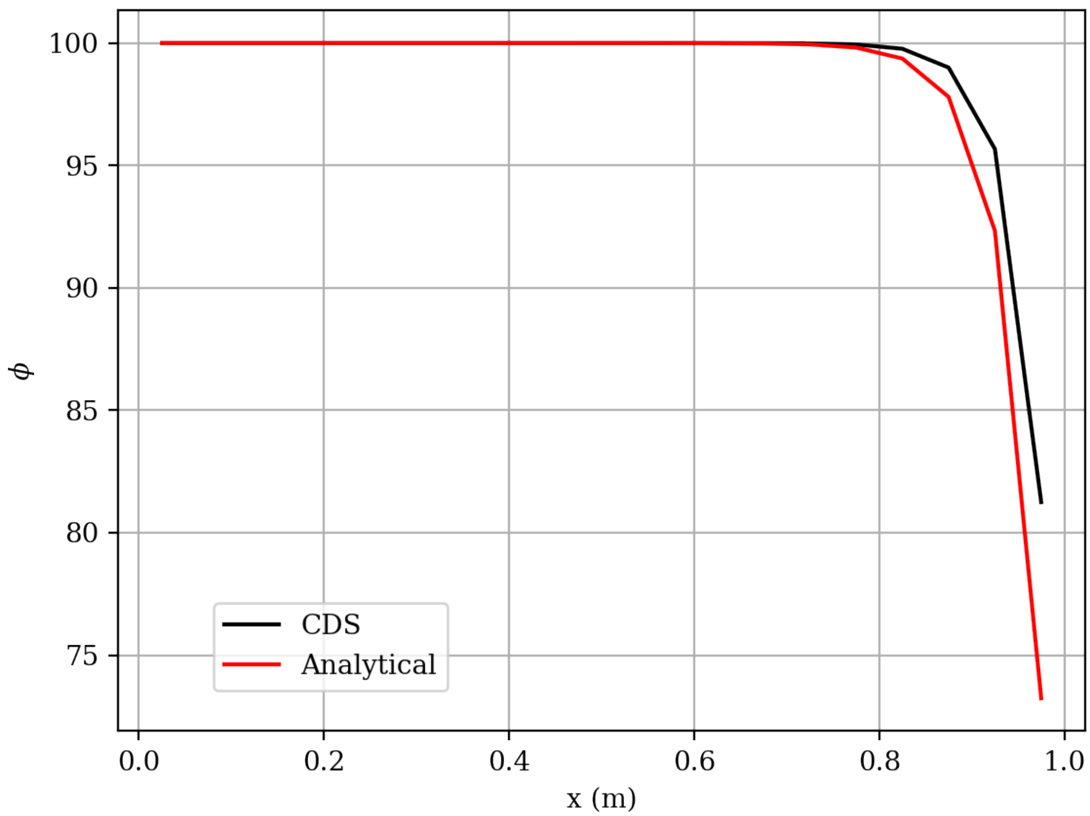

Figure 3: Central Difference Scheme and Analytical Solutions for Case 3.

Table 1 lists the error values calculated for each case.

Table 1: Error values for each case.

Case	Error
1	0.26294912888831474
2	26.174593696190538
3	0.6567406338821179

3 Discussion