1 Methods

First, we will prepare the Upwind scheme for the spatial discretization. The conditions for the Upwind scheme are listed below:

$$\phi_{i+1} = \begin{cases} \phi_I & \overline{V} > 0 \\ \phi_{I+1} & \overline{V} < 0 \end{cases},$$

and

$$\left(\frac{\partial \phi}{\partial x}\right)_{i-1} = \frac{\phi_I - \phi_{I-1}}{\Delta x}. \tag{1}$$

Discretizing gives:

$$(\rho u_x)_{i+1}\,\phi_{i+1} - (\rho u_x)_{i-1}\,\phi_{i-1} = \Gamma\left(\frac{\partial \phi}{\partial x}\right)_{i+1} - \Gamma\left(\frac{\partial \phi}{\partial x}\right)_{i-1}$$

Plugging in our Upwind scheme conditions for velocity greater than zero gives:

$$[-2F - D]\,\phi_{I-1} + [2F + 2D]\,\phi_I + [D]\,\phi_{I+1} = 0\,,$$

where

$$D = \frac{\Gamma}{\Delta x} \qquad F = \frac{\rho u_x}{2}\,.$$

This equation works for the interior nodes. For the boundary nodes, we will need different equations. For the left boundary:

$$[2F + 2D]\,\phi_I + [D]\,\phi_{I+1} = [2F + D]\,\phi_L\,.$$

For the right boundary:

$$[-D]\,\phi_{I-1} + [-2F - 2D]\,\phi_I = [-2F + D]\,\phi_R\,.$$

We have three different time discretizations for this problem: Explicit Euler, Implicit Euler, and Trapezoidal.

1.1 Explicit Euler

Inner Nodes:

$$\phi_I^{n+1} = \left[\frac{u_x\,\Delta t}{\Delta x} + \frac{\Gamma\,\Delta t}{\rho\,(\Delta x)^2)}\right]\phi_{I-1}^n + \left[1 - \frac{u_x\,\Delta t}{\Delta x} - \frac{2\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_I^n + \left[\frac{\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_{I+1}^n \tag{2}$$

Left Node (Node 1):

$$\phi_I^{n+1} = \left[\frac{u_x\,\Delta t}{\Delta x}\right]\phi_L + \left[1 - \frac{u_x\,\Delta t}{\Delta x} - \frac{3\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_I^n + \left[\frac{\Gamma\,\Delta t}{\Delta x}\right]\phi_{I+1}^n \tag{3}$$

Right Node (Node N):

$$\phi_I^{n+1} = \left[\frac{u_x\,\Delta t}{\Delta x} + \frac{\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_{I-1}^n + \left[1 - \frac{u_x\,\Delta t}{\Delta x} - \frac{3\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_I^n + \left[\frac{2\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_R \tag{4}$$

1.2 Implicit Euler

Inner Nodes:

$$\left[\frac{-u_x\,\Delta t}{\Delta x} - \frac{\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_{I-1}^{n+1} + \left[1 + \frac{u_x\,\Delta t}{\Delta x} + \frac{2\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_I^{n+1} + \left[-\frac{\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_{I+1}^{n+1} = \phi_I^n \quad (5)$$

Left Node (Node 1):

$$\left[1 + \frac{u_x\,\Delta t}{\Delta x} + \frac{3\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_I^{n+1} + \left[-\frac{\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_{I+1}^{n+1} = \phi_I^n + \left[\frac{u_x\,\Delta t}{\Delta x} + \frac{2\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_L \quad (6)$$

Right Node (Node N):

$$\left[-\frac{u_x\,\Delta t}{\Delta x} - \frac{\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_{I-1}^{n+1} + \left[1 + \frac{u_x\,\Delta t}{\Delta x} + \frac{3\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_I^{n+1} = \phi_I^n + \left[\frac{2\Gamma\,\Delta t}{\rho\,(\Delta x)^2}\right]\phi_R \quad (7)$$

1.3 Trapezoidal

$$\left[-\frac{\rho u_x\,\Delta t}{2} - \frac{\Gamma\,\Delta t}{2\,\Delta x}\right]\phi_{I-1}^{n+1} + \left[\rho\,\Delta x + \frac{\rho u_x\,\Delta t}{2} + \frac{\Gamma\,\Delta t}{\Delta x}\right]\phi_I^{n+1} + \left[-\frac{\Gamma\,\Delta t}{2\,\Delta x}\right]\phi_{I+1}^{n+1}$$
$$= \left[\frac{\rho u_x\,\Delta t}{2} + \frac{\Gamma\,\Delta t}{2\,\Delta x}\right]\phi_{I-1}^{n} + \left[\rho\,\Delta x - \frac{\rho u_x\,\Delta t}{2} - \frac{\Gamma\,\Delta t}{\Delta x}\right]\phi_I^{n} + \left[\frac{\Gamma\,\Delta t}{2\,\Delta x}\right]\phi_{I+1}^{n}$$

2 Results

2.1 Explict Euler

Norm:

Table 1: Norm values for Explicit Euler.

K	Norm
0.2	1.55418029575927
2.0	8.3196861106867e+245
20.0	inf

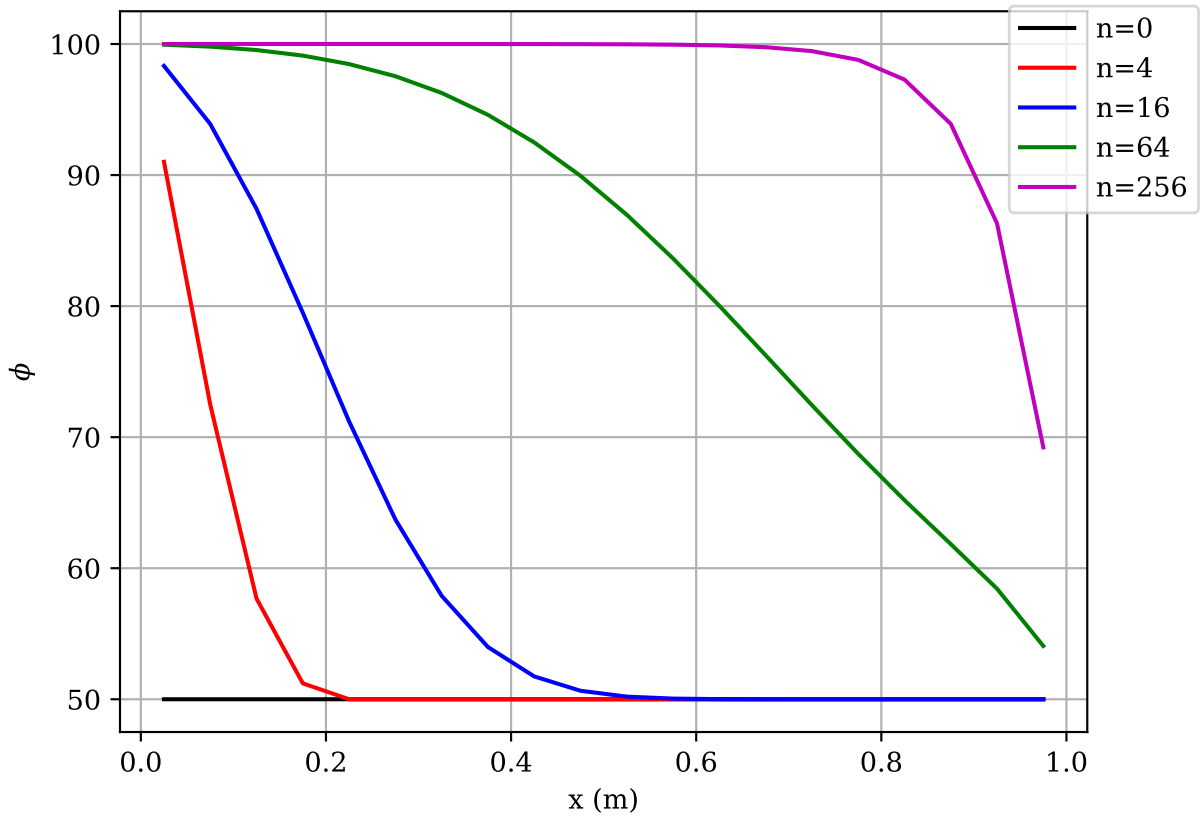

Figure 1: Explicit Euler solution for case 1: $K = 0.2$.

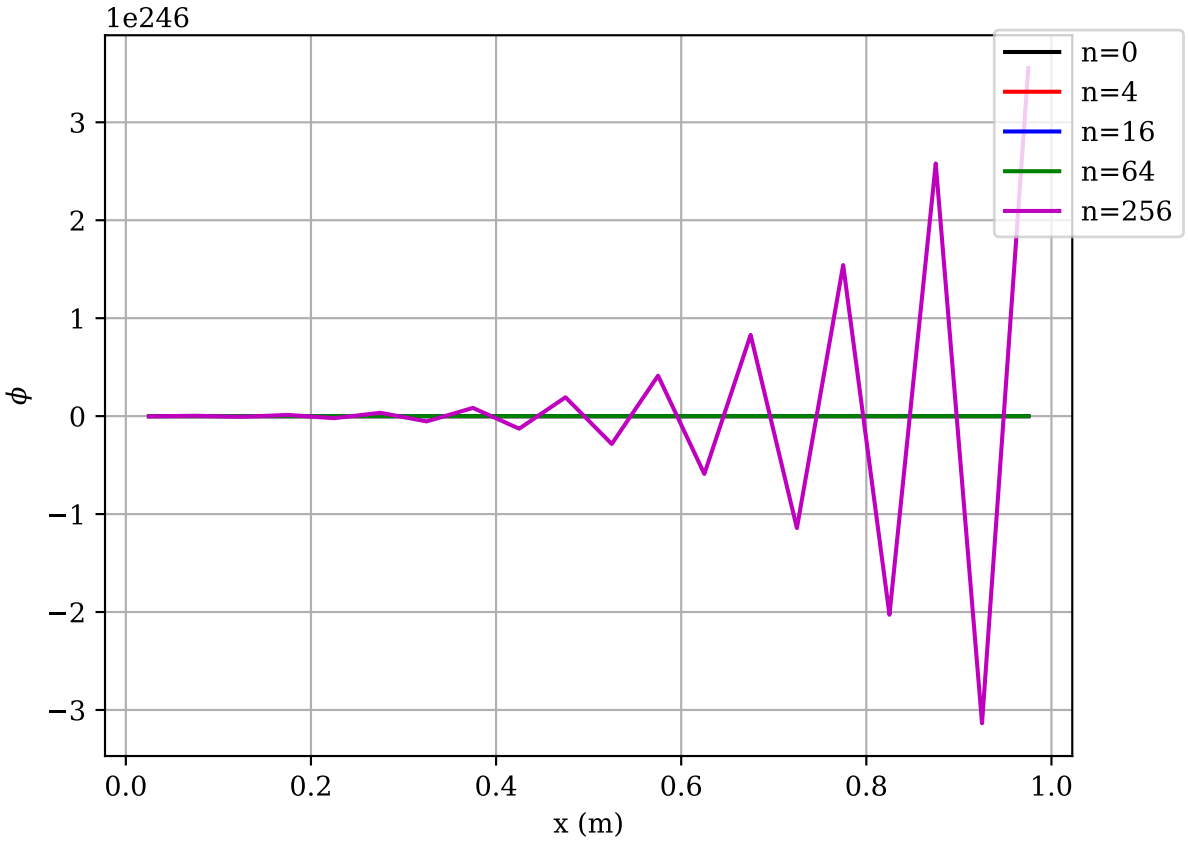

Figure 2: Explicit Euler solution for case 2: $K = 2.0$.

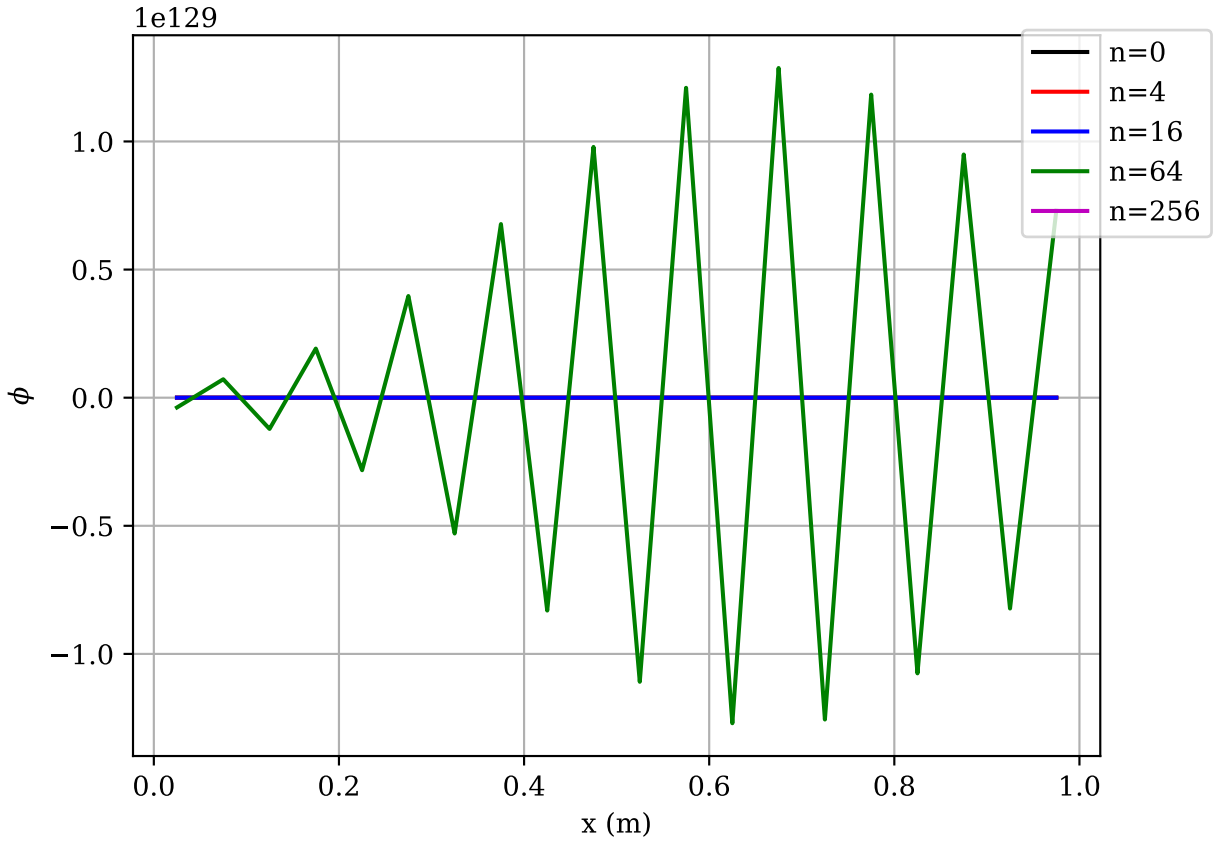

Figure 3: Explicit Euler solution for case 3: $K = 20.0$.

2.2 Implicit Euler

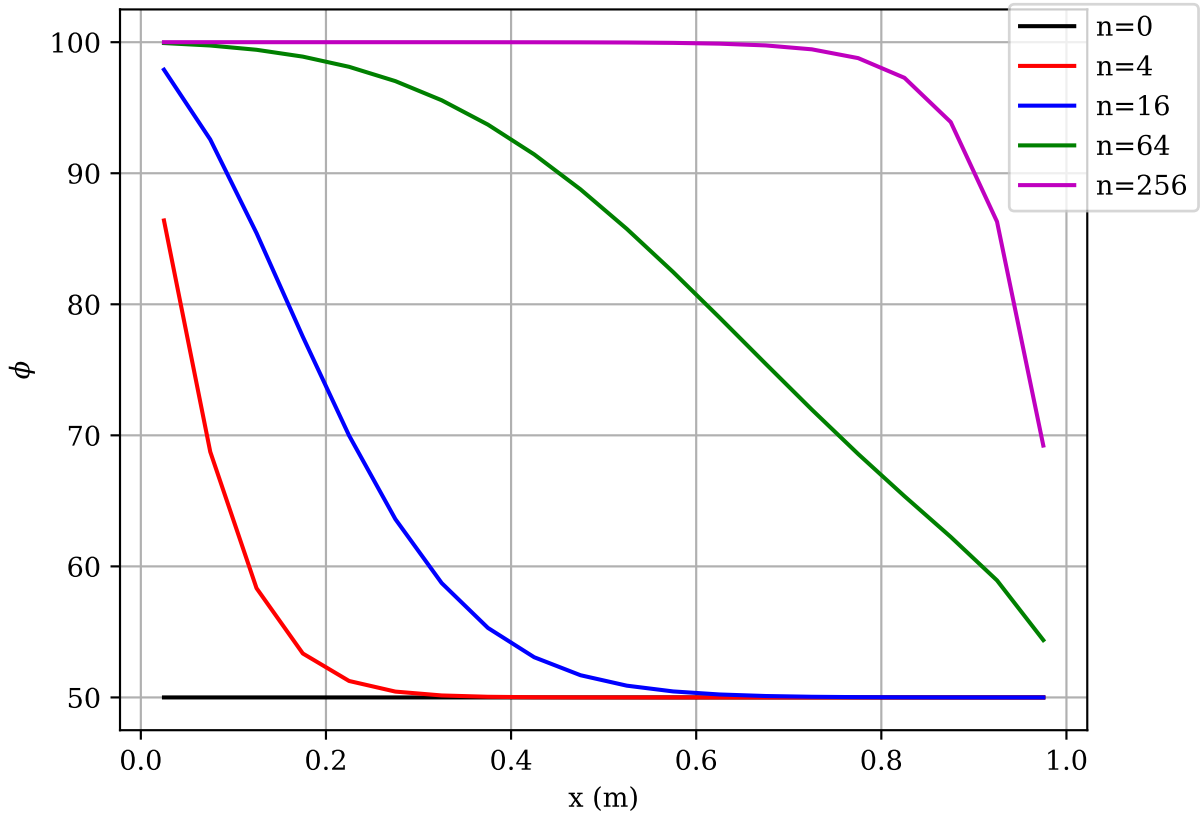

Figure 4: Implicit Euler solution for case 1: $K = 0.2$.

Norm:

Table 2: Norm values for Implicit Euler.

K	Norm
0.2	1.5567368462357045
2.0	1.5504768792236276
20.0	1.5504768792236157

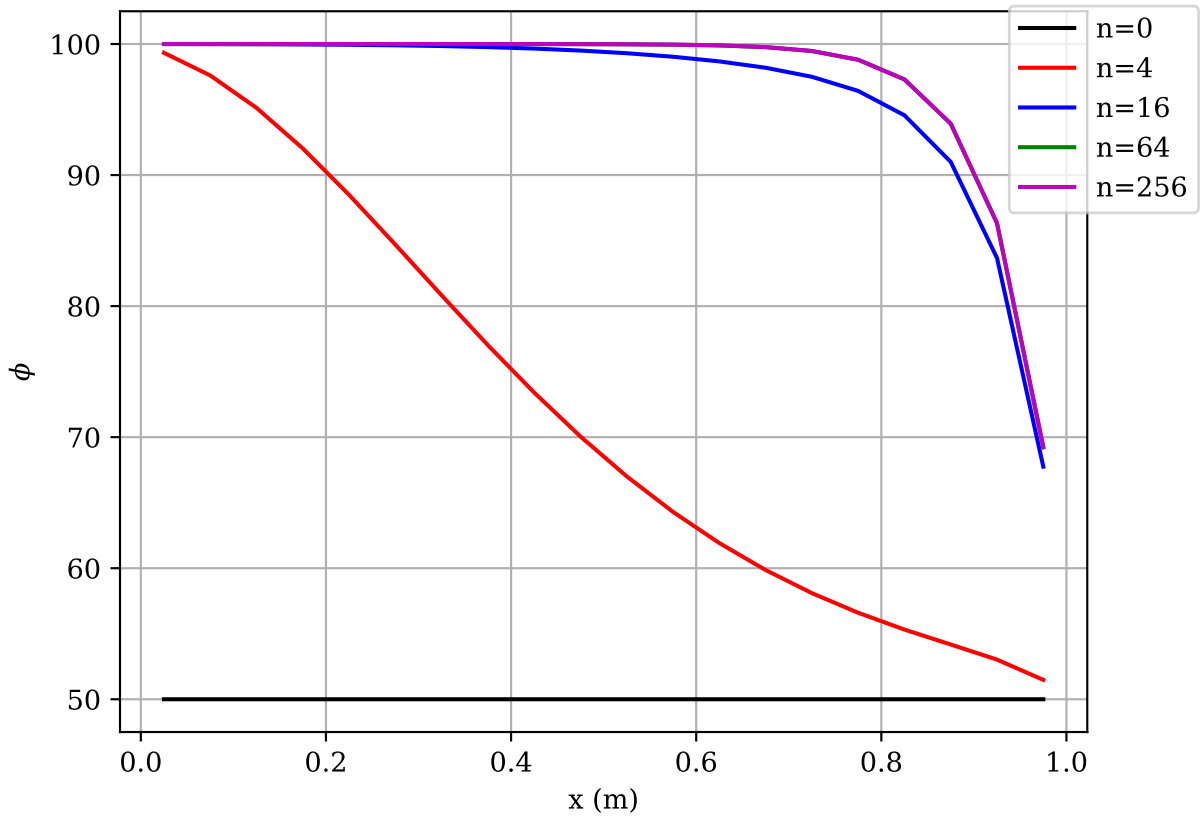

Figure 5: Implicit Euler solution for case 2: $K = 2.0$.

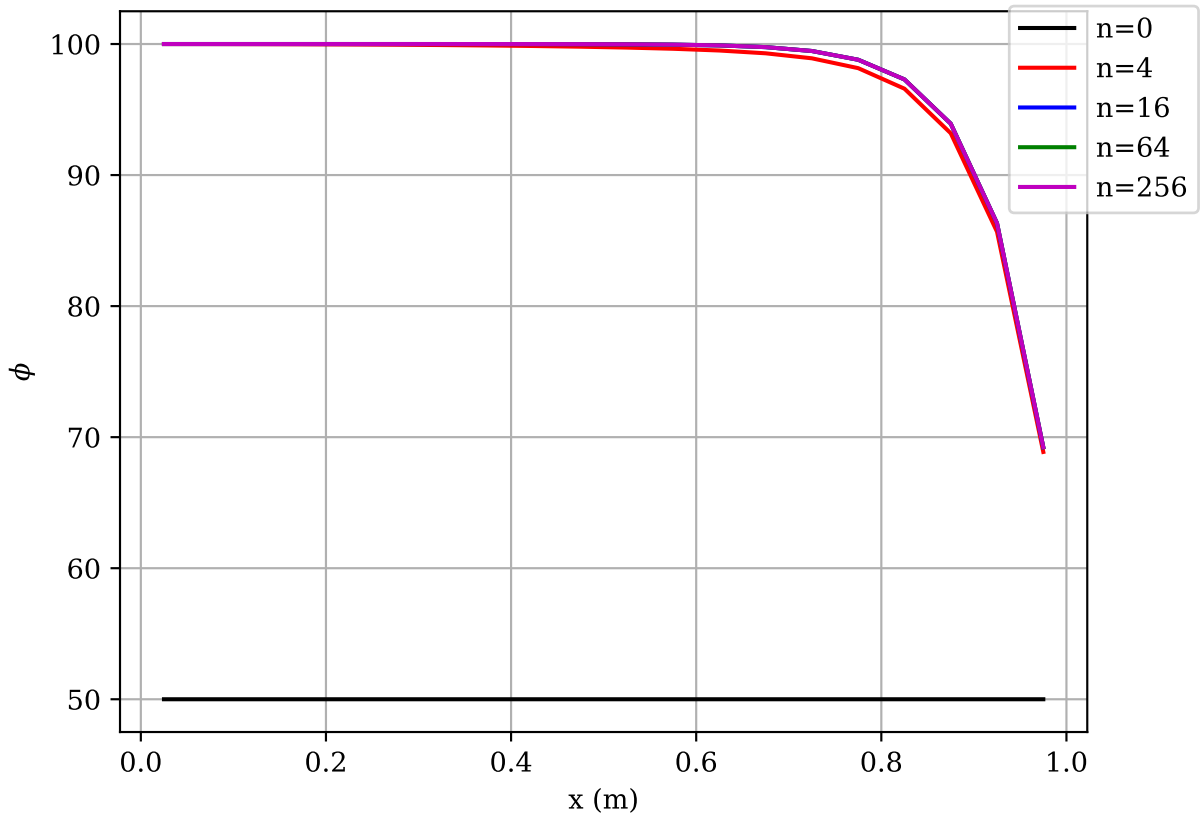

Figure 6: Implicit Euler solution for case 3: $K = 20.0$.