1 Methods

First, we will prepare the Upwind scheme for the spatial discretization. The conditions for the Upwind scheme are listed below:

$$\phi_{i+1} = \begin{cases} \phi_I & \overline{V} > 0 \\ \phi_{I+1} & \overline{V} < 0 \end{cases} ,$$

and

$$\left(\frac{\partial \phi}{\partial x}\right)_{i-1} = \frac{\phi_I - \phi_{I-1}}{\Delta x} . \tag{1}$$

Discretizing gives:

$$(\rho u_x)_{i+1} \phi_{i+1} - (\rho u_x)_{i-1} \phi_{i-1} = \Gamma \left(\frac{\partial \phi}{\partial x}\right)_{i+1} - \Gamma \left(\frac{\partial \phi}{\partial x}\right)_{i-1}$$

Plugging in our Upwind scheme conditions for velocity greater than zero gives:

$$[-2F - D]\, \phi_{I-1} + [2F + 2D]\, \phi_I + [D]\, \phi_{I+1} = 0 ,$$

where

$$D = \frac{\Gamma}{\Delta x} \qquad F = \frac{\rho u_x}{2} .$$

This equation works for the interior nodes. For the boundary nodes, we will need different equations. For the left boundary:

$$[2F + 2D]\, \phi_I + [D]\, \phi_{I+1} = [2F + D]\, \phi_L .$$

For the right boundary:

$$[-D]\, \phi_{I-1} + [-2F - 2D]\, \phi_I = [-2F + D]\, \phi_R .$$

We have three different time discretizations for this problem: Explicit Euler, Implicit Euler, and Trapezoidal. Explicit Euler:

$$\phi_I^{n+1} = \left[\frac{u_x\, \Delta t}{\Delta x} + \frac{\Gamma\, \Delta t}{\rho\ (\Delta x)^2)}\right] \phi_{I-1}^n + \left[1 - \frac{u_x\, \Delta t}{\Delta x} - \frac{2\Gamma\, \Delta t}{\rho\ (\Delta x)^2}\right] \phi_I^n + \left[\frac{\Gamma\, \Delta t}{\rho\ (\Delta x)^2}\right] \phi_{I+1}^n$$

Implicit Euler:

$$\frac{1}{\rho\, \Delta x} [-\rho u_x\, \Delta t]\, \phi_{I-1}^{n+1} + \frac{1}{\rho\, \Delta x} \left[\rho\, \Delta x + \rho u_x\, \Delta t + 2\frac{\Gamma\, \Delta t}{\Delta x}\right] \phi_I^{n+1} + \frac{1}{\rho\, \Delta x} \left[-\frac{\Gamma\, \Delta t}{\Delta x}\right] \phi_{I+1}^{n+1} = \phi_I^n$$

Trapezoidal:

$$\begin{aligned} &\left[-\frac{\rho u_x\, \Delta t}{2} - \frac{\Gamma\, \Delta t}{2\, \Delta x}\right] \phi_{I-1}^{n+1} + \left[\rho\, \Delta x + \frac{\rho u_x\, \Delta t}{2} + \frac{\Gamma\, \Delta t}{\Delta x}\right] \phi_I^{n+1} + \left[-\frac{\Gamma\, \Delta t}{2\, \Delta x}\right] \phi_{I+1}^{n+1} \\ &\qquad = \left[\frac{\rho u_x\, \Delta t}{2} + \frac{\Gamma\, \Delta t}{2\, \Delta x}\right] \phi_{I-1}^n + \left[\rho\, \Delta x - \frac{\rho u_x\, \Delta t}{2} - \frac{\Gamma\, \Delta t}{\Delta x}\right] \phi_I^n + \left[\frac{\Gamma\, \Delta t}{2\, \Delta x}\right] \phi_{I+1}^n \end{aligned}$$