Exercises and solutions: *Matrix inverse*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*, and then check your work on a computer.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. Compute the inverse (if it exists) of the following matrices.

a) $\begin{bmatrix} 1 & 4 \\ 4 & 4 \end{bmatrix}$ **b)** $\begin{bmatrix} 2 & 1/2 \\ 3 & 1/3 \end{bmatrix}$ **c)** $\begin{bmatrix} 2 & 1/2 \\ 3 & 3/4 \end{bmatrix}$ **d)** $\begin{bmatrix} -9 & 3 \\ -9 & -8 \end{bmatrix}$

2. Use the row-reduction method to compute the inverse of the following matrices, or to discover that the matrix is singular.

a) $\begin{bmatrix} -4 & 7 \\ 3 & -8 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 3 & 0 & 0 & 1 \\ 0 & 4 & 5 & 0 \\ 5 & 8 & 2 & -17 \end{bmatrix}$ **d)** $\begin{bmatrix} 1 & 0 & 2 & 5 \\ 3 & 0 & 0 & 1 \\ 0 & 4 & 5 & 0 \\ 0 & 3 & 4 & 0 \end{bmatrix}$

3. Compute the inverse of the following matrices, or determine that the inverse does not exist. Confirm that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ **d)** $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

e) $\begin{bmatrix} 4 & -4 \\ 1 & 6 \end{bmatrix}$ **f)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$ **g)** $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **h)** $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 7 & 6 & 0 \end{bmatrix}$

i) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **j)** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **k)** $\begin{bmatrix} 3 & 1 \\ 0 & b \end{bmatrix}$ **l)** $\begin{bmatrix} 2 & 1 & 0 & 6 \\ -1 & 0 & 4 & 0 \\ 2 & 0 & 3 & -4 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

4. The inverse of the inverse is the original matrix. Is that $\mathbf{A}^{-1}\mathbf{A}^{-1}$ or $(\mathbf{A}^{-1})^{-1}$? Think of an answer and then confirm it empirically using the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$.

5. For the following matrices and vectors, compute $\mathbf{A}_n^{-1}$ and use it to solve for $\mathbf{x}$ in $\mathbf{A}_n\mathbf{x} = \mathbf{b}_n$.

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & -3 \\ 2 & 2 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 4 \\ 2 & 2 & 1 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} 9 & 0 & 1 \\ 3 & 1 & 0 \\ 0 & 6 & 1 \end{bmatrix}$$

$$\mathbf{b}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 6 \\ -3 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix}$$

a) $\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$ **b)** $\mathbf{A}_1\mathbf{x} = \mathbf{b}_2$ **c)** $\mathbf{A}_2\mathbf{x} = \mathbf{b}_1$ **d)** $\mathbf{A}_2\mathbf{x} = \mathbf{b}_2$
e) $\mathbf{A}_3\mathbf{x} = \mathbf{b}_3$ **f)** $\mathbf{A}_3\mathbf{x} = \mathbf{b}_4$ **g)** $\mathbf{A}_4\mathbf{x} = \mathbf{b}_3$ **h)** $\mathbf{A}_4\mathbf{x} = \mathbf{b}_4$

6. Compute the left inverse of the following matrices. Then multiply that one-sided inverse by the original matrix.

a) $\begin{bmatrix} 0 & 1 \\ 4 & 1 \\ 7 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 6 & 2 \\ 7 & 3 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 3 \\ 4 & 6 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$

7. Apply the MCA algorithm to compute the inverse of the following 2x2 matrix. Notice how the full algorithm produces the shortcut you learned in the lecture.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Answers

1. -

a) $\frac{1}{3}\begin{bmatrix}-1 & 1\\ 1 & -1/4\end{bmatrix}$ **b)** $\frac{-1}{5/6}\begin{bmatrix}1/3 & -1/2\\ -3 & 2\end{bmatrix}$ **c)** No inverse. **d)** $\frac{1}{99}\begin{bmatrix}-8 & -3\\ 9 & -9\end{bmatrix}$

2. -

a) $\frac{1}{11}\begin{bmatrix}-8 & -7\\ -3 & -4\end{bmatrix}$ **b)** $\begin{bmatrix}-1 & 0 & 2\\ 0 & 1/2 & -1/2\\ 1 & 0 & -1\end{bmatrix}$ **c)** No inverse.

d) $\begin{bmatrix}-1/14 & 5/14 & -3/7 & 4/7\\ 0 & 0 & 4 & -5\\ 0 & 0 & -3 & 4\\ 3/14 & -1/14 & 9/7 & -12/7\end{bmatrix}$

3. -

a) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ **b)** $\begin{bmatrix}.25 & 0\\ 0 & -.5\end{bmatrix}$ **c)** $\frac{1}{3}\begin{bmatrix}-1 & 2\\ 2 & -1\end{bmatrix}$ **d)** no inverse

e) $\frac{1}{28}\begin{bmatrix}6 & 4\\ -1 & 4\end{bmatrix}$ **f)** $\frac{1}{3}\begin{bmatrix}3 & 0 & 0\\ 0 & -1 & 2\\ 0 & 2 & -1\end{bmatrix}$ **g)** $\frac{1}{3}\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$ **h)** no inverse

i) $\begin{bmatrix}1/2 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1/3\end{bmatrix}$ **j)** $\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$ **k)** $\frac{1}{3b}\begin{bmatrix}b & -1\\ 0 & 3\end{bmatrix}$ **l)** $\frac{1}{27}\begin{bmatrix}8 & -3 & 4 & -8\\ -22 & -12 & 16 & 49\\ 2 & 6 & 1 & -2\\ 11/2 & 3 & -4 & -11/2\end{bmatrix}$

4. $(\mathbf{A}^{-1})^{-1}$ is the right answer, which you can confirm in the example matrix. $\mathbf{A}^{-1}\mathbf{A}^{-1}$ is the same thing as $(\mathbf{A}^{-1})^2$

5. -

a) $\frac{1}{18}\begin{bmatrix}-4\\ 19\end{bmatrix}$ **b)** $\frac{1}{3}\begin{bmatrix}-4\\ 11/2\end{bmatrix}$ **c)** $\frac{1}{12}\begin{bmatrix}11\\ -5\end{bmatrix}$ **d)** $\frac{1}{4}\begin{bmatrix}1\\ -7\end{bmatrix}$

e) $\frac{1}{16}\begin{bmatrix}103\\ -45\\ 12\end{bmatrix}$ **f)** $\frac{1}{16}\begin{bmatrix}-37\\ 39\\ 12\end{bmatrix}$ **g)** $\frac{1}{9}\begin{bmatrix}11/3\\ 16\\ -24\end{bmatrix}$ **h)** $\frac{1}{9}\begin{bmatrix}25/3\\ 2\\ -3\end{bmatrix}$

6. Matrices below are $(\mathbf{A}^{\mathsf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathsf{T}}$.

a) $\frac{1}{74}\begin{bmatrix}-11 & 1 & 10\\ 65 & 21 & -12\end{bmatrix}$ **b)** $\frac{1}{16}\begin{bmatrix}12 & -8\\ -28 & 24\end{bmatrix}$ **c)** $\frac{1}{314}\begin{bmatrix}-32 & 46 & 81 & -87\\ 34 & 10 & -37 & 63\end{bmatrix}$