Exercises and solutions: *Projections and orthogonalization*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*, and then check your work on a computer.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. The projection formula showed in the lecture was obtained by solving for β in the equation $\mathbf{a}^T(\mathbf{b}-\mathbf{a}\beta)=0$. Solve for β in the equation $\mathbf{a}^T(\mathbf{a}\beta-\mathbf{b})=0$ to see if you get the same result (and to get more practice working with this important equation!).

2. Draw the following lines ($\mathbf{a}$) and points (b). Draw the approximate location of the orthogonal projection of b onto $\mathbf{a}$. Then compute the exact proj$_a(b)$ and compare with your guess.

a) $\mathbf{a}=\begin{bmatrix}-4\\1\end{bmatrix}$, b=$\begin{bmatrix}-3\\2\end{bmatrix}$ **b)** $\mathbf{a}=\begin{bmatrix}-4\\1\end{bmatrix}$, b=$\begin{bmatrix}2\\0\end{bmatrix}$ **c)** $\mathbf{a}=\begin{bmatrix}4\\2\end{bmatrix}$, b=$\begin{bmatrix}2\\1\end{bmatrix}$

3. For the following pairs of vectors, decompose the first into parallel and perpendicular components relative to the second. For $\mathbb{R}^2$ problems, additionally draw all vectors.

a) $\begin{bmatrix}2\\2\end{bmatrix},\begin{bmatrix}0\\-3\end{bmatrix}$ **b)** $\begin{bmatrix}1/2\\0\end{bmatrix},\begin{bmatrix}-1\\2\end{bmatrix}$ **c)** $\begin{bmatrix}0\\0\\3\end{bmatrix},\begin{bmatrix}2\\4\\0\end{bmatrix}$

4. Determine whether the following matrices are orthogonal matrices.

a) $\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}$ **b)** $\frac{1}{5}\begin{bmatrix}3 & 4 & 0\\ -4 & 3 & 0\\ 0 & 0 & 5\end{bmatrix}$ **c)** $\begin{bmatrix}1/\sqrt{2} & 1\sqrt{2} & 0\\ -\sqrt{2}/6 & \sqrt{2}/6 & 2\sqrt{2}/3\\ 2/3 & -2/3 & 2/3\end{bmatrix}$

5. In MATLAB or Python, compute the QR decomposition of a 10x10 matrix of random numbers. You'll notice that matrix $\mathbf{R}$ is upper-triangular. Explain why there are zeros below the diagonal. (Hint: think about building up $\mathbf{R}$ column-wise from the formula $\mathbf{Q^TA}=\mathbf{R}$.)

6. Run the following code in MATLAB:

```
[Q,R]=qr(randn(5,2));
```

Matrix `R` is 5x2 as expected, but matrix `Q` is 5x5, which may seems strange considering that the original matrix is 5x2 – only the first two columns of `Q` correspond to the input matrix. What might be the advantage of having `Q` be square instead of the same size as the input matrix?

Answers

1. Yes, the result is the same!

2. Numbers below are the projection scalar β.

a) 14/17 **b)** -8/17 **c)** 1/2

3. Vectors below are the parallel and perpendicular components

a) $\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ **b)** $\begin{bmatrix} .1 \\ -.2 \end{bmatrix}, \begin{bmatrix} .4 \\ .2 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

4. -

a) No (but try a scaling factor) **b)** Yes **c)** No (but change the final element to 1/3)

5. Remember that in matrix multiplication, the lower-triangular elements are dot products between *later* columns of **Q** and *earlier* columns of **A**. Gram-Schmidt works by setting *later* columns to be orthogonal to *earlier* columns. So, later columns in **Q** are orthogonal to earlier columns in **A**, but later columns in **A** are not necessarily orthogonal to earlier columns in **Q**, which is why the upper-triangular part of the matrix can have nonzero elements.

6. This is done for convenience, because when $\mathbf{Q}$ is square, then $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$. If you consider only the first two columns of $\mathbf{Q}$ (corresponding to an orthogonal basis set for $C(\mathbf{A})$ in this example problem), then $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}_2$ but $\mathbf{Q}\mathbf{Q}^T \neq \mathbf{I}_5$. Try it yourself in code!