Exercises and solutions: *Solving systems of equations*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*, and then check your work on a computer.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. Convert the following systems of equations into their matrix form.

a) $$\begin{aligned} 2x + 3y + 75z &= 8 \\ -2y + 2z &= -3 \end{aligned}$$

b) $$\begin{aligned} x - z/2 &= 1/3 \\ 3y + 6z &= 4/3 \end{aligned}$$

c) $$\begin{aligned} s - t &= 6 \\ u + v &= 1 \\ t + u &= 0 \\ 2v + 3t &= 10 \end{aligned}$$

d) $$\begin{aligned} x + y &= 2 \\ x - y &= 0 \end{aligned}$$

2. Convert the following matrix-vector products into their "long-form" equations (i.e., the opposite of the previous exercise).

a) $$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$$

c) $$\begin{bmatrix} 7 & 7 & 8 & 8 & 6 & 7 \\ 1 & 0 & 9 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} q \\ w \\ e \\ r \\ t \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \end{bmatrix}$$

d) $$\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 3 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 6 \\ 2 \end{bmatrix}$$

3. Use Gaussian elimination to compute the echelon form of the following matrices. Compute the rank of the matrix by counting the number of pivots.

a) $$\begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 1 \\ 3 & -2 & 0 \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ -1 & -2 & -4 \\ -4 & 0 & 1 \end{bmatrix}$$

c) $$\begin{bmatrix} 3 & 2 & -4 & 1 \\ 2 & 3 & -2 & 0 \\ 1 & 4 & -1 & -4 \end{bmatrix}$$

4. Use Gauss-Jordan elimination to compute the reduced-row echelon form of the following matrices.

a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$

c) $\begin{bmatrix} -1 & 15 & -25 \\ 7 & -1 & 5 \\ 9 & -31 & 55 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 3 & 5 \\ 5 & 3 & 1 \\ 7 & 9 & 11 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 4 & 3 & 2 \\ 4 & 16 & 12 & 8 \\ 3 & 12 & 9 & 6 \\ 2 & 8 & 6 & 4 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 1 & -3 \\ 6 & -9 & 5 \\ 5 & -2 & -8 \end{bmatrix}$

Answers

1. -

a) $\begin{bmatrix} 2 & 3 & 75 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

c) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \\ v \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \\ 10 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

2. -

a) Not a valid equation!

b) $\begin{aligned} j &= 10 \\ k &= 9 \end{aligned}$

c) $\begin{aligned} 7q + 7w + 8e + 8r + 6t + 7y &= 9 \\ 1q + 9e + r + 2t &= 9 \end{aligned}$

d) $\begin{aligned} s + 3t &= 5 \\ 2s + 4t &= 4 \\ 3s + 4t &= 6 \\ 4s + 2t &= 2 \end{aligned}$

3. Note that the echelon form of a matrix is not unique (although they are all related by row operations). You might get different matrices from what are listed here, but you should get 3 pivots in each case. (*Note*: ps = pivots)

a) $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}$, ps=2,1,-8

b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -10 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, ps=1,-5,-1

c) $\begin{bmatrix} 3 & 2 & -4 & 1 \\ 0 & 5 & 2 & -2 \\ 0 & 0 & 5 & -15 \end{bmatrix}$,ps=3,5,5

4. -

a) $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & 25/52 \\ 0 & 1 & -85/52 \\ 0 & 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$