

Exercises and solutions: *Matrix spaces*

The only way to learn mathematics is *to solve math problems*. Watching and re-watching video lectures is important and helpful, but it's not enough. If you really want to learn linear algebra, you need to solve problems *by hand*, and then check your work on a computer.

Below are some practice problems to solve. You can find many more by searching the Internet.

Exercises

1. For each matrix-vector pair, determine whether the vector is in the column space of the matrix, and if so, the coefficients that map the vector into that column space.

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

f) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

g) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$

h) $\begin{bmatrix} -1 & 5 & 2 \\ -7 & 9 & 8 \\ -1 & 4 & \pi \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. Same as the previous exercise but for the row space.

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}^T$

b) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}^T$

c) $\begin{bmatrix} 1 & 6 \\ 2 & 12 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \end{bmatrix}^T$

d) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}^T$

3. For each matrix-set pair, determine whether the vector set can form a basis for the column space of the matrix.

a) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

b) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\}$

c) $\begin{bmatrix} 3 & 6 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \left\{ \begin{bmatrix} 1 \\ 0 \\ 1/3 \end{bmatrix} \right\}$

d) $\begin{bmatrix} 0 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

e) $\begin{bmatrix} e^\pi & 3^e \\ \sqrt[3]{3.7} & e^{e^2} \end{bmatrix}, \left\{ \begin{bmatrix} -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

4. Determine whether the following matrices have a null space. If so, provide basis vector(s) for that null space. Recall that a basis vector for the null space is a vector \mathbf{v} such that $\mathbf{A}\mathbf{v} = \mathbf{0}$ (excluding $\mathbf{v} = \mathbf{0}$).

a) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 5 \end{bmatrix}$

d) $\begin{bmatrix} 3 & 1 & 5 \\ 4 & 1 & 0 \end{bmatrix}$

5. Fill in the blanks (*dim*=dimensionality) for matrix $\mathbf{A} \in \mathbb{R}^{2 \times 3}$

a) $\dim(C(\mathbf{A})) = 0$, $\dim(N(\mathbf{A}^T)) = \text{---}$

c) $\dim(C(\mathbf{A})) = 2$, $\dim(N(\mathbf{A}^T)) = \text{---}$

e) $\dim(N(\mathbf{A})) = 0$, $\dim(R(\mathbf{A})) = \text{---}$

g) $\dim(N(\mathbf{A})) = 2$, $\dim(R(\mathbf{A})) = \text{---}$

b) $\dim(C(\mathbf{A})) = 1$, $\dim(N(\mathbf{A}^T)) = \text{---}$

d) $\dim(C(\mathbf{A})) = 3$, $\dim(N(\mathbf{A}^T)) = \text{---}$

f) $\dim(N(\mathbf{A})) = 1$, $\dim(R(\mathbf{A})) = \text{---}$

h) $\dim(N(\mathbf{A})) = 3$, $\dim(R(\mathbf{A})) = \text{---}$

Answers

1. -

- a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ b) not in column space c) not in column space d) not in column space
 e) sizes don't match f) $\begin{bmatrix} 3 \\ -3 \end{bmatrix}$ g) $\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$ h) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2. -

- a) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ b) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ c) Not in the row space d) $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

3. -

- a) No. Any column space basis must be a single vector that is a multiple of $[1 \ 2]^T$.
 b) Yes: $C(\mathbf{M}) = \mathbb{R}^2$, so any independent set of two vectors can be a basis.
 c) Yes
 d) Yes
 e) Yes for the same reason as (b).

4. -

- a) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ b) No null space c) No null space d) $\begin{bmatrix} 1 \\ -4 \\ 1/5 \end{bmatrix}$

5. -

- a) 2 b) 1
 c) 0 d) Trick question; $\dim(C(\mathbf{A}))$ cannot be greater than 2.
 e) Trick question; $\dim(N(\mathbf{A}))$ must be >0 . f) 2
 g) 1 h) 0