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Research Report

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Title of Research Report

On Standard Auctions with Entry Costs

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Abstract

It is often profitable for sellers in auctions to exclude prospective buyers through means such as positive reserve prices and bidding fees. In certain cases, these costs may arise naturally and not through any intentional design on the seller's part. In this paper, we examine auctions where participation by buyers entails certain costs which are not passed on to the seller. Examples include administrative fees in sales by tender and due diligence expenses in mergers and acquisitions. We consider both the case where buyers are risk-neutral and where they are risk-averse, through two standard auction formats: the first-price auction and the second-price auction. We hypothesise that the existence of these costs will affect the seller's revenue and explore the implications that the auction format has on auction revenue under different assumptions.

Keywords: Auctions, Auctions with Entry, Risk Aversion, Mechanism Design, First-price Auctions, Second-price Auctions

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1 Introduction

The study of the distribution of resources in economics is inextricably linked to mechanism design for efficient allocation (i.e. mechanisms allowing resources to accumulate in individuals with the highest appraisement). Indeed, such studies of efficient mechanisms have real-world significance, particularly in the context of privatisation of state-owned enterprises and assets, where the pursuit of social efficiency cannot be achieved with the market.

William Vickrey (1961) explored the prospect of a second-price sealed-bid auction being an efficient mechanism – the auctioned commodity is awarded to the bidder with the highest valuation – as an alternative to the competitive model. Vickrey's observations were generalised to formulate the revenue-equivalence principle, (Riley and Samuelson, 1981; Myerson, 1981) which states that the projected revenue and bidder profits will be uniform regardless of the distribution of values for an extensive class of auction modes, on the understanding that bidders adopt equilibrium strategies. The theorem appears to, however, defy the empirical observation that objects are hardly auctioned off in multiple bidding rules concurrently. As such, artworks, wines and antiques are traded under English rules, while government tenders are awarded through sealed-bid auctions. Instances of variations in auction formats (e.g. the United States Treasury Bills) are confronted with opposition and contradict the literature, given that a consequence of the principle is that the bidder's expected surplus is invariant across auction institutions.

From the aforementioned inconsistencies, it can be concluded that the revenue-equivalence principle holds with the primary premises of risk neutrality, symmetrical bidders, the independence of bidders' information of valuations and absence of collaboration between bidders. Recent auction literature has examined the consequences of modifying the premises on revenue or efficiency. Holt (1980), Riley and Samuelson (1981) and Maskin and Riley (1984) established that, in auctions with risk-averse bidders, a seller should favour sealed-bid auctions, even in the situation where he/she is risk-averse. When the precondition of independence of private information is abolished and reserve prices are correlated, Milgrom and Weber (1982) prove that the English rules will produce higher revenue. Finally, projected revenue will be higher in sealed-bid auctions when bidder collusion in English auctions are considered (Graham and Marshall, 1989; McAfee and McMillan, 1992).

In models of adverse selection, it is often profitable for sellers to institute multiple screening mechanisms (e.g. reserve prices and entry costs) that exclude a fraction of prospective bidders with valuations below the screening threshold from participating in the auction process (Riley and Samuelson, 1981). The proposition with reserve prices can be illustrated through a comparison of revenue in a second-price auction where the highest and second-highest bid is less than the reserve price respectively. In the former instance, the seller bears the probability of the object being unsold; in the latter, however, he/she collects the reserve price as opposed to the second-highest bid. An entry cost is a levy bidders pay prior to the auction to submit bids. To induce a similar screening effect as a reserve price, the seller could peg the bidding fee to the expected payoff of a bidder with an appraisal equivalent to the reserve price, such that bidders with valuation under the screening threshold are effectively excluded (Riley and Samuelson, 1981). A substantial collection of studies in auction literature have scrutinised single-object auctions with risk-neutral bidders to calculate the efficiency properties of auctions (Harris and Raviv, 1981; Riley and Samuelson, 1981). Myerson (1981) and Riley and Samuelson's (1981) groundbreaking research have determined that revenue maximisation for such auctions involve a positive degree of screening. It is commonly acknowledged that the seller would be indifferent between establishing an entry cost and a reserve price as screening mechanisms (Krishna, 2010).

Nonetheless, prior auction literature on screening mechanisms has largely, if not exclusively, studied the implications of an individual screening mechanism on revenue. In this paper, we explore the correlation between a combination of reserve prices and hybrid bidding fees, which encompass fixed costs collected by the seller and external costs incurred by the bidder during the auction that is irretrievably lost, and revenue equivalence in the single-object first-price and second-price auction respectively. The aforenamed screening mechanisms may exist concurrently in real-world auctions, as evident in mergers and acquisitions where the market value of the target company's shareholder equities typically resembles a reserve price and bidders are expected to offer a control premium to shareholders, in addition to due diligence expenses incurred regardless of the outcome of the merger or acquisition. In the following chapters, we present the basic auction model and derive the optimal bidding strategies in sealed-bid first-price (Proposition 2.1 and 2.2) and second-price auction rules (Proposition 2.3) to show the revenue equivalence principle established by Myerson (1981) and Riley and Samuelson (1981) (Proposition 2.4). Thereafter, in chapter 3, we calculate the ex post payment in both auction institutions respectively (Proposition 3.1 – 3.2) where partial entry costs are incurred by bidders to establish that revenue equivalence holds in any combination of a reserve price and partial entry costs (Proposition 3.3 – 3.4).

We ultimately extend our findings on the seller's screening decisions in both first-price and second-price auctions to the situation where bidders are risk-averse, calculating the ex ante expected revenue in first-price and second-price auctions with non-transferable entry costs and risk-averse bidders (Proposition 4.1 – 4.2). We subsequently model the divergence in revenue (Proposition 4.3) to conclude that the first-price auction under risk aversion is revenue-maximising at higher values of non-transferable costs, while, at low values, auctions with risk neutrality yield the highest revenue. The line of literature on screening mechanisms in risk aversion includes Li, Lu and Zhao (2015) and Chakraborty (2017). Li, Lu and Zhao (2015) illustrate the bidding and entry equilibrium in first-price and ascending auction institutions with selective entry, elucidating risk-averse bidders' endogenous participation decisions. The authors provide that bidders' entry behaviour diverge under decreasing, constant and increasing absolute risk aversion between the auction modes. Their paper, however, analyses the first-price and ascending auctions (the latter is strategically different from a second-price sealed-bid auction in the sense that bidders in a Vickrey auction are only apprised of the distribution of values) and proves the deviation in entry behaviour without calculating revenue. Given that revenue equivalence does not hold in risk aversion (Matthews, 1983; Maskin and Riley, 1984) and the equilibrium entry condition differs between the first-price and ascending auctions with entry costs (Li, Lu and Zhao, 2015), our paper complements their findings and contributes to auction literature by proving the base case of revenue with partial entry costs and difference in revenue for first-price and second-price sealed-bid auctions with entry costs and risk aversion. In his paper investigating the ramifications of optimal screening by reserve prices and entry costs in risk aversion in terms of revenue and efficiency, Chakraborty (2017) concludes that entry costs expand bidders' payoff uncertainty such that it is productive for rent extraction in the first-price auction, but not in the second-price and English ascending auctions. Furthermore, he deduced that aggressive screening in a first-price auction may potentially reverse the auction's ex post efficiency ranking which indicates its relative efficiency, noting that sellers are incentivised to establish a positive level of screening through bidding fees and reserve prices in auctions with risk-averse bidders. Differentiating from prior literature, we extend our observations of auctions with cost evaporation where entry costs are not necessarily transferred to the seller to the case of risk aversion, deriving the revenue of auctions with combinations of partial entry costs and evaluating the implications of cost evaporation on revenue equivalence without considering reserve prices. Our conclusions for screening mechanisms in first-price and second-price auctions with risk-neutral and risk-averse bidders have important implications for revenue maximisation.

2 Equilibrium in First- and Second-Price Auctions

Assume an auction with N risk-neutral potential bidders, each with independent and identically distributed valuations v drawn from the continuous and fully-differentiable function $F(v)$ with support over $[0, \omega]$ and density $f(v)$. In order to understand the revenue outcomes of different mechanisms, we first illustrate how bidders decide what amount to bid by deriving the optimal bidding strategy $\beta(v)$ for first- and second-price auctions. Unless otherwise stated, $F_{N:N}$ is the distribution of the N-th order statistic in ascending order, i.e. highest-order statistic out of N bids , i.e. the distribution of the highest bid out of N bids. Since the scope of this paper is limited to single-object auctions, F_N will be used to denote $F_{N:N}$. F^N will not be used to refer to any order statistic in this paper.

The general timeline of all auctions analysed in this paper is as follows:

At $t = 0$, bidders are informed of their individual realisations of $F(v)$ and $f(v)$. N and bidders' identical utility functions $u(v)$ are common knowledge to all bidders as well as the seller. If there is a cost, all bidders are informed of the total cost incurred. If there is a reserve price, all bidders are informed of the reserve price. The seller chooses between a first or second price auction, and the rules of the chosen mechanism are made known to all bidders: if a first-price auction is chosen, the highest bidder wins and pays the highest bid; if a second-price auction is chosen, the highest bidder wins and pays the second-highest bid.

At $t = 1$, bidders choose whether or not they want to enter.

At $t = 2$, the auction proceeds according to the rules of the mechanism chosen.

Proposition 2.1

In a first-price auction with no entry costs, the optimal bidding strategy for bidder i with valuation v_i is

$$\beta(v_i) = \frac{1}{F_{(N-1)}(v_i)} \int_0^{v_i} y f_{(N-1)}(y)\ dy$$

where $F_{(N-1)}(v_i)$ denotes the distribution of the highest-order statistic in $(N-1)$ bidders, with density $f_{(N-1)}(v_i) = F'_{(N-1)}(v_i) = (N-1)F_{(N-2)}(v_i)f(v_i)$.

Proof

For a bidder i with valuation v_i, the pay-off function is given by

$$\Pi_i = v_i - b_i \text{ if } b_i > \max_{j \neq i} \ b_j$$

and

$$\Pi_i = 0 \text{ if } b_i < \max_{j \neq i} \; b_j$$

If $b_i = \max_{j \neq i} \; b_j$, every bidder bidding b_i has an equal chance of winning the object.

Suppose that each bidder $j \neq i$ with valuation v_j follows a symmetric, increasing and differentiable strategy $\beta(v_j)$. We wish to identify the optimal bid value b_i, for bidder i.

Bidder i wins the auction whenever he submits the highest bid, i.e.

$$\max_{j \neq i} \beta(v_j) < b_i$$

Since β is an increasing function,

$$\max_{j \neq i} \beta(v_j) = \beta(\max_{j \neq i} v_j) = \beta(Y_{(N-1)})$$

where $Y_{(N-1)}$ denotes the highest valuation among the remaining $(N-1)$ bidders, i.e. all bidders except bidder i.

Therefore, bidder i wins when $\beta(Y_{(N-1)}) < b_i \Rightarrow Y_{(N-1)} < \beta^{-1}(b_i)$, and hence

$$\Pi(b_i, v_i) = \text{Probability that bidder } i \text{ wins} \times \text{Payoff if bidder } i \text{ wins}$$

or,

$$\Pi(b_i, v_i) = F_{(N-1)}(\beta^{-1}(b_i)) \times (v_i - b_i)$$

Maximising this with respect to b_i yields:

$$f_{(N-1)}(\beta^{-1}(b_i))(\tfrac{d}{db_i}(\beta^{-1}(b_i)))(v_i - b_i) - F_{(N-1)}(\beta^{-1}(b_i)) = 0$$

or,

$$\tfrac{f_{(N-1)}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}(v_i - b_i) - F_{(N-1)}(\beta^{-1}(b_i)) = 0$$

As will be proven in proposition 2.2, $b_i = \beta(v_i)$ at symmetric equilibrium, such that the expression for maximised payoff with respect to b_i can be manipulated into the differential equation

$$f_{(N-1)}(v_i)(v_i - \beta(v_i)) = F_{(N-1)}(v_i)\beta'(v_i)$$

or,

$$F_{(N-1)}(v_i)\beta'(v_i) + f_{(N-1)}(v_i)\beta(v_i) = v_i f_{(N-1)}(v_i)$$

Using the product rule, we have:

$$\tfrac{d}{dv_i} F_{(N-1)}(v_i)\beta(v_i) = v_i f_{(N-1)}(v_i)$$

Integrating on both sides yields:

$$F_{(N-1)}(v_i)\beta(v_i) = \int_0^{v_i} y f_{(N-1)}(y)\ dy + C,\ C \in \mathbb{R}$$

Since a bidder with valuation 0 would not submit a positive bid, lest he make a loss if he wins the auction, $\beta(0) = 0$, such that

$$F_{(N-1)}(0)\beta(0) = \int_0^{0} y f_{(N-1)}(y)\ dy + C \Rightarrow C = 0$$

$$\beta(v_i) = \frac{1}{F_{(N-1)}(v_i)} \int_0^{v_i} y f_{(N-1)}(y)\ dy$$

or, by the definition of conditional expectations,

$$\beta(v_i) = E[Y_{(N-1)} | Y_{(N-1)} < v_i]\ \blacksquare$$

The above proof requires that $b_i = \beta(v_i)$ is a symmetric equilibrium for all bidders, which is indeed true, as will be shown in the next proposition.

Proposition 2.2

For bidder i with valuation v_i, bidding $b_i = \beta(v_i)$ gives a payoff at least as high as any other value of b_i and thus there is a symmetric equilibrium where all bidders bid $\beta(v_i) = \frac{1}{F_{(N-1)}(v_i)} \int_0^{v_i} y f_{(N-1)}(y)\ dy$.

Proof

Assume that all bidders except bidder i follow the bidding strategy β. We argue that it is optimal for bidder i to also follow β, because bidding any other value x_i will not increase his expected payoff.

Recalling that β is an increasing and continuous function, it is evident that the bidder with the highest valuation places the highest bid and wins the auction. Therefore, it is not optimal for bidder i to bid $x_i > \beta(\omega)$. Where $u_i = \beta^{-1}(x_i)$ denotes the valuation for which x_i is the bid given by β, the expected payoff from bidding $x_i < \beta(\omega)$ is:

$$\Pi(x_i, v_i) = F_{(N-1)}(u_i) \times (v_i - \beta(u_i))$$

or, using the result of proposition 2.1,

$$\Pi(x_i, v_i) = F_{(N-1)}(u_i) v_i - \int_0^{u_i} y f_{(N-1)}(y)\ dy$$

or, using integration by parts

$$\Pi(x_i, v_i) = F_{(N-1)}(u_i) v_i - F_{(N-1)}(u_i) u_i + \int_0^{u_i} F_{(N-1)}(y)\ dy$$

or,

$$\Pi(x_i, v_i) = F_{(N-1)}(u_i)(v_i - u_i) + \int\limits_0^{u_i} F_{(N-1)}(y)\ dy$$

This implies that $\Pi(\beta(v_i), v_i) - \Pi(x_i, v_i)$ can be expressed as:

$$F_{(N-1)}(v_i)(v_i - v_i) + \int\limits_0^{v_i} F_{(N-1)}(y)\ dy\ - F_{(N-1)}(u_i)(v_i - u_i) - \int\limits_0^{u_i} F_{(N-1)}(y)\ dy$$

or,

$$\int\limits_0^{v_i} F_{(N-1)}(y)\ dy - F_{(N-1)}(u_i)(v_i - u_i) - \int\limits_0^{u_i} F_{(N-1)}(y)\ dy$$

or,

$$\int\limits_{u_i}^{v_i} F_{(N-1)}(y)\ dy - F_{(N-1)}(u_i)(v_i - u_i)$$

Given that

$$\int\limits_{u_i}^{v_i} F_{(N-1)}(y)\ dy = \lim_{n\to\infty} \sum_{r=1}^{n}\ F_{(N-1)}(u_i + \tfrac{r(v_i-u_i)}{n})(\tfrac{v_i-u_i}{n})$$

which, since $F_{(N-1)}$ is continuous and increasing, means that

$$F_{(N-1)}(u_i) \leq F_{(N-1)}(u_i + \tfrac{r(v_i-u_i)}{n})$$

Hence,

$$\lim_{n\to\infty} \sum_{r=1}^{n} F_{(N-1)}(u_i)(\tfrac{v_i-u_i}{n}) \leq \lim_{n\to\infty} \sum_{r=1}^{n} F_{(N-1)}(u_i + \tfrac{r(v_i-u_i)}{n})(\tfrac{v_i-u_i}{n})$$

or,

$$\lim_{n\to\infty} nF_{(N-1)}(u_i)(\tfrac{v_i-u_i}{n}) \leq \lim_{n\to\infty} \sum_{r=1}^{n}\ F_{(N-1)}(u_i + \tfrac{r(v_i-u_i)}{n})(\tfrac{v_i-u_i}{n})$$

or,

$$F_{(N-1)}(u_i)(v_i - u_i) \leq \int\limits_{u_i}^{v_i} F_{(N-1)}(y)\ dy$$

Hence,

$$\int\limits_{u_i}^{v_i} F_{(N-1)}(y)\ dy - F_{(N-1)}(u_i)(v_i - u_i) \geq 0$$

as shown in the figure below.

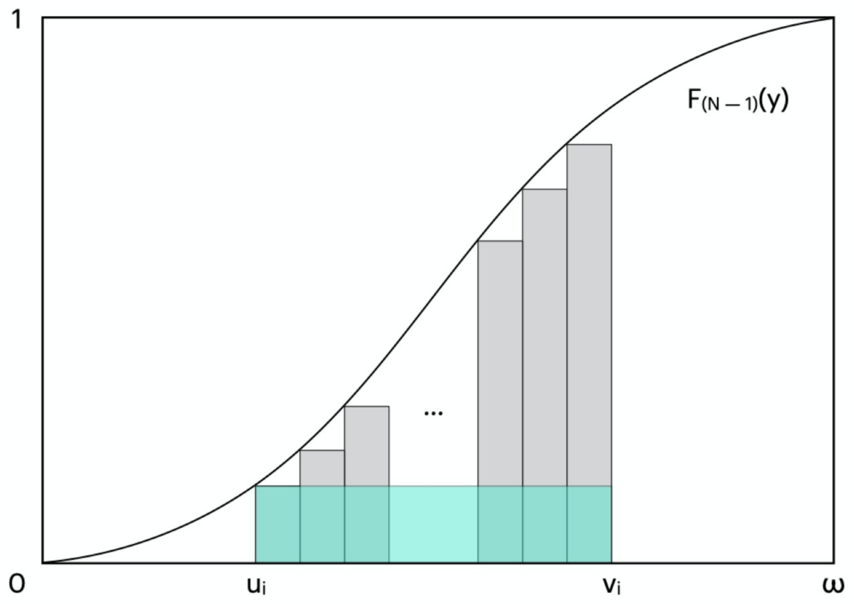

Figure 2.2.1 Graphical depiction that $\int_{u_i}^{v_i} F_{(N-1)}(y)\ dy - F_{(N-1)}(u_i)(v_i - u_i) \geq 0$

This is shown in the graph above, where the area in grey is $\sum_{r=1}^{n} F_{(N-1)}(u_i + \frac{r(v_i-u_i)}{n})(\frac{v_i-u_i}{n})$ as $n \to +\infty$, and the area in green is $F_{(N-1)}(u_i)(v_i - u_i)$.

This means that, for bidder i, bidding $b_i = \beta(v_i)$ gives a payoff at least as high as any alternative bid. β is thus a symmetric equilibrium strategy, i.e. $b_i = \beta(v_i)$ at symmetric equilibrium. ■

Proposition 2.3

In a second-price auction, it is a weakly dominant strategy for each bidder i to place a bid, b_i, that is equal to his valuation v_i.

Proof

For a bidder i with valuation v_i, the pay-off function is:

$$\Pi_i = v_i - \max_{j \neq i}\ b_j \text{ if } b_i > \max_{j \neq i}\ b_j$$

and

$$\Pi_i = 0 \text{ if } b_i < \max_{j \neq i}\ b_j$$

If $b_i = \max_{j \neq i}\ b_j$, every bidder bidding b_i has an equal chance of winning the object.

We compare the payoffs where bidder i bids v_i and where he bids $b_i > v_i$:

When $b_i > v_i > \max_{j \neq i} \ b_j$, bidder i wins the object and obtains a payoff of $v_i - \max_{j \neq i} \ b_j \geq 0$ when he bids $b_i > v_i$, and also when he bids v_i.

When $\max_{j \neq i} \ b_j > b_i > v_i$, bidder i does not win the object i.e. obtains a payoff of zero when he bids $b_i > v_i$, and also when he bids v_i ($\max_{j \neq i} \ b_j \geq b_i > v_i$ occurs with probability 0).

When $b_i > \max_{j \neq i} \ b_j > v_i$, bidder i wins the object when he bids $b_i > v_i$, but his payoff is $\max_{j \neq i} b_j - v_i < 0$, which is negative, and in fact less than if he bid v_i, lost the auction, and obtained a payoff of zero.

Therefore, bidding v_i gives a payoff at least as high as bidding $b_i > v_i$.

Similarly, we can compare the payoffs where bidder i bids v_i and where he bids $b_i < v_i$:

When $b_i < v_i < \max_{j \neq i} \ b_j$, bidder i does not win the object, i.e. he obtains a payoff of zero when he bids $b_i < v_i$, and also when he bids v_i.

When $\max_{j \neq i} \ b_j < b_i < v_i$, bidder i wins the object and obtains a payoff of $v_i - \max_{j \neq i} \ b_j \geq 0$ when he bids $b_i < v_i$, and also when he bids v_i.

When $b_i < \max_{j \neq i} \ b_j < v_i$, bidder i does not win the object when he bids $b_i < v_i$, such that he obtains a payoff of zero, which is less than if he bids v_i, in which case his payoff would be $v_i - \max_{j \neq i} \ b_j \geq 0$.

Therefore, bidding v_i gives a payoff at least as high as any other strategy. Hence, bidding v_i is a weakly dominant strategy at equilibrium. ■

Proposition 2.4

The expected revenue in a first-price auction is the same as that in a second-price auction, *ceteris paribus.*

Proof

From propositions 2.1 and 2.3, the expected payment by a bidder i with valuation v_i is, in a first-price auction:

$$\text{Probability that bidder } i \text{ wins} \times \text{Expected payment received when bidder } i \text{ wins}$$

or, in a second-price auction:

Probability that bidder i wins
$\times$ Expected value of second-highest bid when v_i is the highest bid

Using the expression for the expected bid from a bidder i with valuation v_i from propositions 2.1 and 2.3, both of these expressions are equal to:

$$\int_0^{v_i} y f_{(N-1)}(y)\ dy$$

Where $[0, \omega]$ is the range of all possible values a bidder's valuation can take, the *ex ante* expected payment of a particular bidder in either auction format is:

Sum of (Expected *ex post* payment from bidder given that he has valuation v_i
$\times$ Probability that the bidder has valuation v_i) for all values of v_i between 0 and ω inclusive

or,

$$\int_0^{\omega} \int_0^{v_i} y f_{(N-1)}(y)\ dy f(v_i)\ dv_i$$

or, by exchanging the order of integration,

$$\int_0^{\omega} \int_y^{\omega} f(v_i)\ dv_i y f_{(N-1)}(y)\ dy$$

or,

$$\int_0^{\omega} [F(y)]_y^{\omega} y f_{(N-1)}(y)\ dy$$

or, because $F(\omega) = 0$, ω being the upper bound of the support,

$$\int_0^{\omega} y(1 - F(y)) f_{(N-1)}(y)\ dy$$

Where both auctions have N potential bidders, the expected revenue in either auction format is:

Number of bidders $\times$ *Ex ante* expected payment of a particular bidder

or,

$$N \int_0^{\omega} y(1 - F(y)) f_{(N-1)}(y)\ dy$$

Recalling that $F_N(y) = F(y)^N$, $f_N(y) = NF(y)^{(N-1)}f(y)$, where $F_N(y)$ is the distribution of the highest-order statistic in N bids, $F_{(N-1):N}(y)$ can be expressed as:

Probability that y is higher than the $(N-1)$-th order statistic

or,

Probability that y is higher than the N-th order statistic
+ Binomial probability that y is higher than the $(N-1)$-th order statistic
but lower than the N-th order statistic

or,

Probability that y is higher than the N-th order statistic
+ ($\binom{N}{1}$ ways of choosing the N-th order statistic
$\times$ Probability that y is higher than exactly $(N-1)$ bids but lower than 1 bid)

or,

$$F_N(y) + \binom{N}{1}F(y)^{(N-1)}(1 - F(y))$$

or,

$$F_N(y) + NF(y)^{(N-1)}(1 - F(y))$$

or,

$$NF(y)^{(N-1)} - (N-1)F(y)^N$$

Hence, $f_{(N-1):N}(y)$ can be expressed as:

$$N(N-1)F_{(N-2):N}(y)f(y) - N(N-1)F_{(N-1):N}(y)f(y)$$

or,

$$Nf_{(N-1)}(y) - NF_N(y)f_{(N-1)}(y)$$

or,

$$N(1 - F_N(y))f_{(N-1)}(y)$$

Substituting the expression for into the expression for expected revenue yields:

$$\text{Expected revenue} = N\int_0^{\omega} y(1 - F(y))f_{(N-1)}(y)\ dy$$

or,

$$\text{Expected revenue} = \int_0^{\omega} y f_{(N-1):N}(y)\ dy$$

or, by definition,

$$\text{Expected revenue} = E[Y_{(N-1):N}]$$

Where $Y_{(N-1):N}$ is the second-highest-order statistic in N bids, i.e. the second-highest bid. Since $Y_{(N-1):N}$ is identical for both auction formats, the expected revenues from both auction formats are identical. ■

3 Equilibrium with Costs and Reserve Prices

In this section, we consider how costs incurred by bidders will affect the payment and revenue received by sellers. Let c^* be a fixed cost incurred by every bidder, and let $c^* = c_1 + c_2$, where c_1 denotes the portion of the fixed cost that is passed on to the seller, and c_2 denotes the portion that is irretrievably lost i.e. external costs incurred by the bidder in the process of bidding. For propositions 3.1 and 3.2, we first consider the case where c^* consists wholly of c_2, i.e. $c_1 = 0$. Building on that, propositions 3.3 and 3.4 generalise to the case where c_1 takes on a non-zero value i.e. the case of partial entry fees, as well as the case of reserve prices, which, as Krishna (2010) notes, are an equivalent form of screening.

v^* denotes the valuation at which a bidder is indifferent to entering the auction, meaning that his expected payoff from entering is the same as his expected payoff from not entering. In this section, $F_{(N-1)}(v)$ and $f_{(N-1)}(v)$ denote the same distributions as in Section 2.

Proposition 3.1

When $c_1 = 0$, the expected *ex post* payment in a first-price auction from bidder i with valuation v_i is

$$\int_{v^*}^{v_i} y f_{(N-1)}(y)\ dy$$

Proof

In a first-price auction with non-refundable third-party entry costs applying to all bidders, we seek an expression for the expected *ex post* payment from a bidder i with valuation v_i.

Assuming that bidders are risk-neutral, the utility function is linear, i.e. $u(v_i) = v_i$.

Since $u(0) = 0$, the indifference condition is:

Payoff if bidder with valuation v^* wins $\times$ Probability that bidder with valuation v^* wins
+Payoff if bidder with valuation v^* loses $\times$ Probability that bidder with valuation v^* loses

or,

$$(v^* - \beta(v^*) - c^*)F_{(N-1)}(v^*) - c^*(1 - F_{(N-1)}(v^*)) = 0$$

or,

$$(v^* - \beta(v^*))F_{(N-1)}(v^*) - c^* = 0$$

and, hence,

$$(v^* - \beta(v^*))F_{(N-1)}(v^*) = c^*$$

Additionally, it is intuitively obvious that a bidder with valuation $v < c^*$ will not enter the auction. Therefore, all bidders entering the auction will have valuation $v > c^*$.

A bidder with valuation v^* will not bid $b > v^*$. Neither will he bid $b = v^*$, because if he were to win with such a bid, he could have captured a greater profit by bidding $b < v^*$, since there would be no bidders below him. Also, a bidder who would have lost by bidding $b = v^*$ would lose as much by bidding $b < v^*$. Seeking to maximise profits, he will therefore bid the lowest possible value of b where $b < v^*$, i.e. $\beta(v^*) = 0$. Therefore,

$$v^* \ F_{(N-1)}(v^*) = c^*$$

Assuming that β is a symmetric equilibrium strategy as in our previous proof, we can use the expression for expected payoff from our previous proof, the only change being to deduct the fixed cost.

$$\Pi(b, v_i) = F_{(N-1)}(\beta^{-1}(b)) \times (v_i - b) - c^*$$

Where c^* is a constant, we can directly apply the procedure from our previous proof, maximising the above expression with respect to b to obtain:

$$\tfrac{f_{(N-1)}(\beta^{-1}(b))}{\beta'(\beta^{-1}(b))}(v_i - b) - F_{(N-1)}(\beta^{-1}(b)) = 0$$

or, using our previous proof,

$$\tfrac{d}{dv_i}F_{(N-1)}(v_i)\beta(v_i) = v_i f_{(N-1)}(v_i)$$

Note that the existence of v^* truncates $F_{(N-1)}$. Therefore, whereas expected payoff was zero for a bidder with valuation zero in our second section, expected payoff is zero for a bidder with valuation v^* here. Hence, integrating both sides, the lower bound of the definite integral is not zero but v^* here, such that

$$F_{(N-1)}(v_i)\beta(v_i) = \int\limits_{v^*}^{v_i} y f_{(N-1)}(y) \ dy + C$$

When $v_i = v^*$,

$$F_{(N-1)}(v^*)\beta(v^*) = C = 0, \text{ since } \beta(v^*) = 0$$

Hence,

$$\beta(v_i) = \tfrac{1}{F_{(N-1)}(v_i)} \int\limits_{v^*}^{v_i} y f_{(N-1)}(y) \ dy$$

Hence, the expected *ex post* payment received by seller from bidder with valuation v_i is:

$$F_{(N-1)}(v_i)\beta(v_i)$$

or,

$$\int_{v^*}^{v_i} y f_{(N-1)}(y)\ dy \ \blacksquare$$

Proposition 3.2

Where $c_1 = 0$, the expected *ex post* payment in a second-price auction from bidder i with valuation v_i is

$$\int_{v^*}^{v_i} y f_{(N-1)}(y)\ dy$$

Equivalence between both auction formats accordingly holds for expected *ex post* payment, as shown in proposition 2.4.

Proof

We assume that the fixed cost c^* is the same as in proposition 3.1.

Bidder i, who will bid his valuation v_i as proven in proposition 2.2, wins the auction if $v_i > \max\limits_{j \neq i} b_j$. If there are two more more bidders participating, i.e. $N \geq 2$,he pays $Y_{(N-1)}$. If he is the only bidder participating, he will get the object at the reserve price, which is assumed to be zero in the current section, as that is the 'second highest bid'.

When the buyer is indifferent to participating,

$$(v^* - E[Y_{(N-1)}|Y_{(N-1)} < v^*] - c^*)F_{(N-1)}(v^*) - c^*(1 - F_{(N-1)}(v^*)) = 0$$

Since no bidder with a valuation less than v^* will enter,

$$E[Y_{(N-1)}|Y_{(N-1)} < v^*] = 0$$

This results in the same indifference condition as in proposition 3.1, given that c^* remains the same. Therefore, the valuation at which a bidder is indifferent to entering the auction is the same as in proposition 3.1.

Therefore, the expected *ex post* payment received by seller from bidder with valuation v_i is:

$$\begin{gathered}\text{Expected payment received when bidder } i \text{ wins where } N \geq 2 \\ \times \text{ Probability that bidder } i \text{ wins where } N \geq 2 \\ + \text{ Expected payment received when bidder } i \text{ wins where } N = 1 \\ \times \text{ Probability that bidder } i \text{ wins where } N = 1\end{gathered}$$

or,

$$(F_{(N-1)}(v_i) - F_{(N-1)}(v^*))E[Y_{(N-1)}|v^* < Y_{(N-1)} < v_i] + (0)F_{(N-1)}(v^*)$$

or,

$$(F_{(N-1)}(v_i) - F_{(N-1)}(v^*))(\tfrac{1}{F_{(N-1)}(v_i)-F_{(N-1)}(v^*)})\int_{v^*}^{v_i} yf_{(N-1)}(y)\ dy$$

or,

$$\int_{v^*}^{v_i} yf_{(N-1)}(y)\ dy$$

This expression, being the same as the one found in proposition 3.1, shows that expected *ex post* payment for a bidder with valuation v_i is the same in first- and second-price auctions. By proposition 2.4, the revenue generated in both cases must also be the same, thus revenue equivalence holds. ■

Proposition 3.3

The revenue equivalence proven in proposition 2.4 holds with any combination of partial costs and reserve prices r.

Proof

Where there is a reserve price r in addition to entry costs c_1 and c_2, bidders with valuation v^*, who are indifferent between entering and not entering the auction, will bid r as that is the lowest price that the seller could possibly accept. Hence,

$$\beta(v^*) = r$$

Assuming $c^* = c_1 + c_2$, in a first-price auction,

$$(v^* - \beta(v^*))F_{(N-1)}(v^*) = c^*$$

$$rF_{(N-1)}(v^*) = v^*F_{(N-1)}(v^*) - c^*$$

Substituting the result from proposition 3.1,

$$F_{(N-1)}(v_i)\beta(v_i) = \int_{v^*}^{v_i} yf_{(N-1)}(y)\ dy + C$$

When $v_i = v^*$,

$$rF_{(N-1)}(v^*) = C = v^*F_{(N-1)}(v^*) - c^*, \text{ since } \beta(v^*\ ) = r$$

Note that the seller's expected *ex post* payment received from bidder i should also include c^*, which is paid regardless whether bidder i wins or loses. Hence, the expected *ex post* payment received from bidder i in a first-price auction is:

$$\int_{v^*}^{v_i} yf_{(N-1)}(y)\ dy + v^* F_{(N-1)}(v^*) - c^* + c_1$$

or,

$$\int_{v^*}^{v_i} yf_{(N-1)}(y)\ dy + v^* F_{(N-1)}(v^*) - c_2$$

In a second-price auction,

$$(v^* - r)F_{(N-1)}(v^*) = c^*$$

this being the same as the indifference condition in proposition 3.1, v^* being identical.

By the same method of proof in proposition 3.2, the expected *ex post* payment received from bidder i in a second-price auction is:

$$\int_{v^*}^{v_i} yf_{(N-1)}(y)\ dy + rF_{(N-1)}(v^*) + c_1$$

or,

$$\int_{v^*}^{v_i} yf_{(N-1)}(y)\ dy + v^* F_{(N-1)}(v^*) - c^* + c_1$$

or,

$$\int_{v^*}^{v_i} yf_{(N-1)}(y)\ dy + v^* F_{(N-1)}(v^*) - c_2$$

As the expected *ex post* payment received from bidder i is the same in both first-price auctions and second-price auctions, revenue equivalence holds by the result of proposition 2.4 for any combination of c_1, c_2 and r, thus proving that any combination of partial costs and reserve prices will result in revenue equivalence. ■

Proposition 3.4

In two auctions with the same non-refundable entry cost c_2, the seller's revenue from placing an entry fee $k = c_1$ is the same as the revenue from placing a reserve price $\frac{k}{F_{(N-1)}(v^*)}$.

Proof

Where there is an entry fee k,

$$(v^* - \beta(v^*))F_{(N-1)}(v^*) = k$$

As shown in proposition 3.1, $\beta(v^*) = 0$. Therefore,

$$v^* F_{(N-1)}(v^*) = k$$

Where there is a reserve price r instead of an entry fee, we wish to find the value of r for which $v^*_{\text{reserve}} = v^*_{\text{entry}}$, thus screening out the same potential bidders.

With a reserve price,

$$\beta(v^*) = r$$

$$(v^* - r)F_{(N-1)}(v^*) = 0$$

$$v^* F_{(N-1)}(v^*) = rF_{(N-1)}(v^*)$$

When $r = \frac{k}{F_{(N-1)}(v^*)}$,

$$v^* F_{(N-1)}(v^*) = (\tfrac{k}{F_{(N-1)}(v^*)})F_{(N-1)}(v^*)$$

or,

$$v^* F_{(N-1)}(v^*) = k$$

Hence $v^*_{\text{reserve}} = v^*_{\text{entry}}$.

Using the procedure of proposition 2.4, the expected revenue with entry fees is:

$$N[\int_{v^*}^{\omega}(\int_{v^*}^{v_i} y f_{(N-1)}(y)\ dy + k) f(v_i)\ dv_i + \int_0^{v^*} 0\ dv_i]$$

or,

$$N[\int_{v^*}^{\omega} \int_{y}^{\omega} f(v_i)\ dv_i\ y f_{(N-1)}(y)\ dy + \int_{v^*}^{\omega} k f(v_i)\ dv_i]$$

or,

$$N[\int_{v^*}^{\omega} y(1 - F(y)) f_{(N-1)}(y)\ dy] + Nk(1 - F(v^*))$$

or,

$$E[Y_{(N-1):N}] + N[k(1 - F(v^*)) - \int_0^{v^*} y(1 - F(y))\ f_{(N-1)}(y)\ dy]$$

The expected revenue with reserve prices is:

$$N\{\int_{v^*}^{\omega}[\int_{v^*}^{v_i} y f_{(N-1)}(y)\ dy + (\tfrac{k}{F_{(N-1)}(v^*)})F_{(N-1)}(v^*)] f(v_i)\ dv_i + \int_0^{v^*} 0\ dv_i\}$$

or,

$$N[\int_{v^*}^{\omega}(\int_{v^*}^{v_i} y f_{(N-1)}(y)\ dy + k) f(v_i)\ dv_i + \int_0^{v^*} 0\ dv_i]$$

which is therefore equal to the expected revenue with entry fees, and therefore equal to

$$E[Y_{(N-1):N}] + N[k(1 - F(v^*)) - \int_0^{v^*} y(1 - F(y))\ f_{(N-1)}(y)\ dy]\ \blacksquare$$

4 Equilibrium under Risk Aversion

It is generally known that revenue equivalence does not hold under risk aversion (Matthews, 1983; Maskin and Riley, 1984) in the basic case with no entry costs or reserve prices. In this section, we consider how revenue in first- and second-price auctions under risk aversion is impacted by the existence of non-transferrable costs.

In this section, u denotes constant absolute risk aversion. For a bidder i with valuation v_i, it is taken in this section that $u(v_i) = 1 - e^{-\rho v_i}$ and $u'(v_i) = \rho e^{-\rho v_i}$.

Proposition 4.1

In a first-price auction with risk-averse bidders and non-transferrable entry costs, the expected *ex ante* revenue for bidder i with valuation v_i is

$$N\int_{v^*}^{\omega}\{\frac{F_{(N-1)}(v_i)}{\rho}\ln[\frac{1}{F_{(N-1)}(v_i)}(\int_{v^*}^{v_i} f_{(N-1)}(y)e^{\rho y}\ dy + F_{(N-1)}(v^*))]\}f_{(N-1)}(v_i)\ dv_i + N(1-F(v^*))c_1$$

Proof

For a bidder i with valuation v_i, the payoff function, $\Pi(\beta(v_i), v_i)$ is given by:

Probability that bidder i wins $\times$ Payoff if bidder i wins
+ Probability that bidder i loses $\times$ Payoff if bidder i loses

or,

$$F_{(N-1)}(\beta^{-1}(b_i))u(v_i - \beta(v_i) - c^*) - (1 - F_{(N-1)}(\beta^{-1}(b_i)))u(-c^*)$$

The indifference condition, where $\Pi(\beta(v_i), v_i) = 0$, is therefore:

$$F_{(N-1)}(v^*)u(v^* - \beta(v^*) - c^*) - (1 - F_{(N-1)}(v^*))u(-c^*) = 0$$

as $\beta(v^*) = 0$ by definition.

By product rule, $\frac{d\Pi(\beta(v_i),v_i)}{db_i}$ can be expressed as:

$$f_{(N-1)}(\beta^{-1}(b_i))(\frac{d}{db_i}(\beta^{-1}(b_i)))u(v_i - \beta(v_i) - c^*) - F_{(N-1)}(\beta^{-1}(b_i))u'(v_i - \beta(v_i) - c^*)$$
$$-f_{(N-1)}(\beta^{-1}(b_i))(\frac{d}{db_i}(\beta^{-1}(b_i)))u(-c^*)$$

or,

$$\frac{f_{(N-1)}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i)))}u(v_i - \beta(v_i) - c^*) - F_{(N-1)}(\beta^{-1}(b_i))u'(v_i - \beta(v_i) - c^*) - \frac{f_{(N-1)}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}u(-c^*)$$

At the maximum value of $\Pi(\beta(v_i), v_i)$,

$$\frac{d\Pi(\beta(v_i),v_i)}{d\beta(v_i)} = 0$$

or,

$$\frac{f_{(N-1)}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i)))}u(v_i - \beta(v_i) - c^*) - F_{(N-1)}(\beta^{-1}(b_i))u'(v_i - \beta(v_i) - c^*) - \frac{f_{(N-1)}(\beta^{-1}(b_i))}{\beta'(\beta^{-1}(b_i))}u(-c^*) = 0$$

or, where $\beta^{-1}(b_i) = v_i$ at symmetric equilibrium,

$$\frac{f_{(N-1)}(v_i)}{\beta'(v_i)}(u(v_i - \beta(v_i) - c^*) - u(-c^*)) = F_{(N-1)}(v_i)u'(v_i - \beta(v_i) - c^*)$$

or,

$$\frac{f_{(N-1)}(v_i)(u(v_i-\beta(v_i)-c^*)-u(-c^*))}{F_{(N-1)}(v_i)u'(v_i-\beta(v_i)-c^*)} = \beta'(v_i)$$

or, recalling that $u(v_i) = 1 - e^{-\rho v_i}$ and $u'(v_i) = \rho e^{-\rho v_i}$,

$$\frac{f_{(N-1)}(v_i)[1-e^{-\rho(v_i-\beta(v_i)-c^*)}-1+e^{\rho c^*}]}{\rho F_{(N-1)}(v_i)e^{-\rho(v_i-\beta(v_i)-c^*)}} = \beta'(v_i)$$

or,

$$\frac{f_{(N-1)}(v_i)[e^{\rho c^*}-e^{-\rho v_i}e^{\rho\beta(v_i)}e^{\rho c^*}]}{\rho F_{(N-1)}(v_i)e^{-\rho v_i}e^{\rho\beta(v_i)}e^{\rho c^*}} = \beta'(v_i)$$

or,

$$\frac{f_{(N-1)}(v_i)[1-e^{-\rho v_i}e^{\rho\beta(v_i)}]}{F_{(N-1)}(v_i)e^{-\rho v_i}} = \rho\beta'(v_i)e^{\rho\beta(v_i)}$$

Letting $h(v_i) = e^{\rho\beta(v_i)}$ and $h'(v_i) = \rho\beta'(v_i)e^{\rho\beta(v_i)}$, we obtain

$$\frac{f_{(N-1)}(v_i)(1-e^{-\rho v_i}h(v_i))}{F_{(N-1)}(v_i)e^{-\rho v_i}} = h'(v_i)$$

or,

$$\frac{f_{(N-1)}(v_i)(e^{\rho v_i}-h(v_i))}{F_{(N-1)}(v_i)} = h'(v_i)$$

or,

$$\frac{f_{(N-1)}(v_i)e^{\rho v_i}}{F_{(N-1)}(v_i)} = h'(v_i) + \frac{f_{(N-1)}(v_i)}{F_{(N-1)}(v_i)}h(v_i)$$

Using integrating factor $\mu(v_i) = e^{\int \frac{f_{(N-1)}(v_i)}{F_{(N-1)}(v_i)}\,dv_i} = F_{(N-1)}(v_i)$,

$$h'(v_i)F_{(N-1)}(v_i) + h(v_i)f_{(N-1)}(v_i) = f_{(N-1)}(v_i)e^{\rho v_i}$$

or,

$$\frac{d}{dv_i}(h(v_i)F_{(N-1)}(v_i)) = f_{(N-1)}(v_i)e^{\rho v_i}$$

or,

$$h(v_i)F_{(N-1)}(v_i) = \int_{v^*}^{v_i} f_{(N-1)}(y)e^{\rho y}\ dy + k,\ k \in \mathbb{R}$$

where k is the constant of integration.
Since $\beta(v^*) = 0$, $h(v^*) = e^{\rho\beta(v^*)} = e^{\rho(0)} = 1$, and hence

$$F_{(N-1)}(v^*) = \int_{v^*}^{v^*} f_{(N-1)}(y)e^{\rho y}\ dy + k$$

or,

$$k = F_{(N-1)}(v^*)$$

and hence

$$h(v_i)F_{(N-1)}(v_i) = \int_{v^*}^{v_i} f_{(N-1)}(y)e^{\rho y}\ dy + F_{(N-1)}(v^*)$$

Recalling that $h(v_i) = e^{\rho\beta(v_i)}$, we obtain

$$e^{\rho\beta(v_i)} = \frac{1}{F_{(N-1)}(v_i)}\left(\int_{v^*}^{v_i} f_{(N-1)}(y)e^{\rho y}\ dy + F_{(N-1)}(v^*)\right)$$

$$\beta(v_i) = \frac{1}{\rho}\ln\left[\frac{1}{F_{(N-1)}(v_i)}\left(\int_{v^*}^{v_i} f_{(N-1)}(y)e^{\rho y}\ dy + F_{(N-1)}(v^*)\right)\right]$$

As shown in previous sections, expected *ex ante* revenue is:

Sum of (Expected *ex post* payment from bidder given that he has valuation v_i $\times$ Probability that the bidder has valuation v_i) for all values of v_i between 0 and ω inclusive

or,

$$N\int_{v^*}^{\omega} (F_{(N-1)}(v_i)\beta(v_i) + c_1) f_{(N-1)}(v_i) dv_i$$

$$N\int_{v^*}^{\omega} \left\{\frac{F_{(N-1)}(v_i)}{\rho}\ln\left[\frac{1}{F_{(N-1)}(v_i)}\left(\int_{v^*}^{v_i} f_{(N-1)}(y)e^{\rho y}\ dy + F_{(N-1)}(v^*)\right)\right] + c_1\right\} f_{(N-1)}(v_i)\ dv_i$$

or,

$$N\int_{v^*}^{\omega} \left\{\frac{F_{(N-1)}(v_i)}{\rho}\ln\left[\frac{1}{F_{(N-1)}(v_i)}\left(\int_{v^*}^{v_i} f_{(N-1)}(y)e^{\rho y}\ dy + F_{(N-1)}(v^*)\right)\right]\right\} f_{(N-1)}(v_i)\ dv_i + N\int_{v^*}^{\omega} c_1 f_{(N-1)}(v_i)\ dv_i$$

or,

$$N\int_{v^*}^{\omega} \left\{\frac{F_{(N-1)}(v_i)}{\rho}\ln\left[\frac{1}{F_{(N-1)}(v_i)}\left(\int_{v^*}^{v_i} f_{(N-1)}(y)e^{\rho y}\ dy + F_{(N-1)}(v^*)\right)\right]\right\} f_{(N-1)}(v_i)\ dv_i + N(1 - F(v^*))c_1 \blacksquare$$

Proposition 4.2

In a second-price auction with risk-averse bidders and non-transferrable entry costs, the expected revenue is

$$N[\int_{v^*}^{\omega} y(1-F(y))f_{(N-1)}(y)\ dy] + N(1-F(v^*))c_1$$

Proof

Using the same procedure as in proposition 2.3, it can be found that each bidder i will place a bid equal to his valuation v_i, since placing a bid more or less than v_i will not increase his payoff.

In such an auction, the indifference condition, as in proposition 4.1, is:

$$F_{(N-1)}(v^*)u(v^* - E[Y_{(N-1)}|Y_{(N-1)} < v^*] - c^*) - (1 - F_{(N-1)}(v^*))u(-c^*) = 0$$

where $E[Y_{(N-1)}|Y_{(N-1)} < v^*] = 0$, as shown in previous sections.

The value of v^* is hence identical in first- and second-price auctions with risk aversion, the indifference condition being the same.

We observe that this auction is identical to the second-price auction described in proposition 3.2, the only difference being that bidders are risk-averse here, whereas they are risk-neutral in Section 3. The revenue expression here is therefore identical to that in proposition 3.4, with the only exception being that the indifference condition and value of v^* are changed as a result of risk aversion. The revenue expression is hence

$$N[\int_{v^*}^{\omega} y(1-F(y))f_{(N-1)}(y)\ dy] + N(1-F(v^*))c_1$$

albeit with a different value of v^*.

Proposition 4.3

In cases where bidders are risk-averse, the first-price auction generates more revenue than the second-price auction regardless of changes in costs.

Proof

We take the case where c^* is fixed and c_2 is varied, as well as the case where c_1 is fixed and c_2 is varied, recalling that $c^* = c_1 + c_2$. Throughout the examples, we assume $F(v_i) = v_i$ over support $[0, 1]$, ie. a uniform distribution from 0 to 1. First, assuming that $N = 5$, $\rho = 10$ and $c^* = 0.05$, the respective revenue curves of the first-price auction (blue) and second-price auction (green) in risk aversion are shown in Fig. 4.3.1.

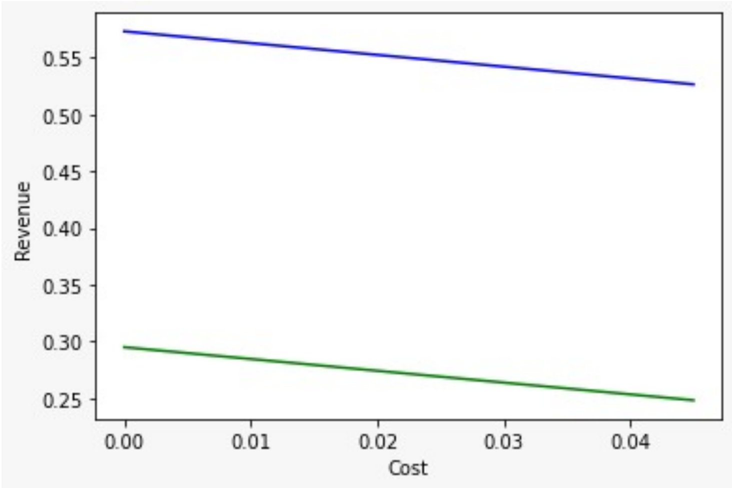

Figure 4.3.1 Change in expected *ex ante* revenue as non-transferable cost increases

Next, we take the case where c_1 is fixed, i.e. the seller charges a fixed entry fee. In this case, $c_1 = 0.05$ and $\rho = 2$. However, N is varied from 2, which is the minimum number of potential bidders for an effective second price auction, to 11. Here, we find that revenue decreases in both the first- and second-price auction when non-transferable cost increases, but that revenue for the first-price auction (blue) continues to be higher than that for the second-price auction (green) in risk aversion, as shown in Fig. 4.3.2.

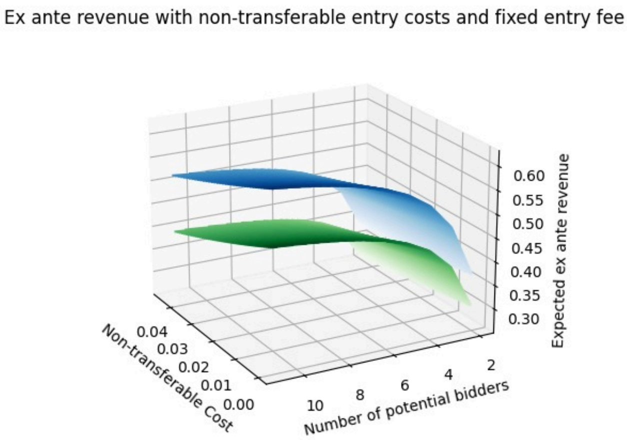

Figure 4.3.2 Change in expected ex ante revenue as a function of N and c_2

Lastly, and most interestingly, when we take the case where c^* is fixed, ie. the total cost to the bidders is the same, we find that the first-price auction still fares better than the second-price auction in terms of seller revenue under risk aversion. However, as shown in Fig. 4.3.3 and 4.3.4 on the next page (two angles of the same graph for clarity), the first- and second-price auctions under risk neutrality, which have been shown to be equivalent in terms of revenue, actually result in higher revenue for the seller when non-transferable cost. However, when c_2 increases, we see that sellers are actually negatively affected to a greater extent under risk neutrality, to the point that their expected revenue drops below that of the risk-averse first-price auction. We also notice that the revenues under risk aversion taper off more quickly when N increases as compared to the risk-neutral case. As in previous diagrams, blue and green represent the risk-averse first-price auction and second-price auction respectively, whereas red represents the risk-neutral first- or second-price auction.

An intuitive reason for this change can be found in the fact that the indifference condition in the risk neutral case is different from the one in the risk averse case, which changes the rate of change of the bidding functions to a change in the cost and the number of bidders.

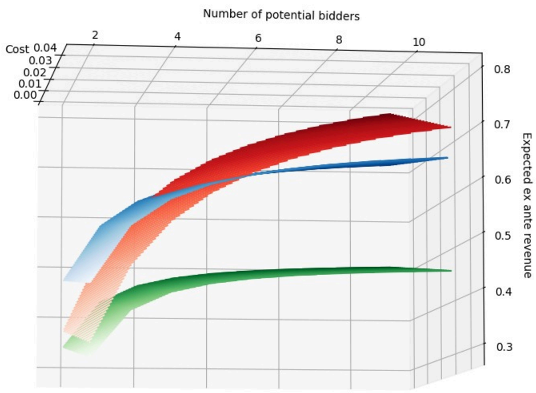

Figure 4.3.3 Angle 1 of the fixed-c^* graph. Notice how the risk averse first-price auction performs better with fewer potential bidders.

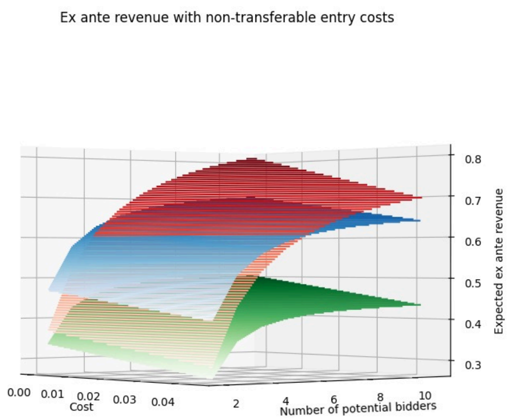

Figure 4.3.4 Angle 2 of the fixed-c^* graph. As shown clearly in the graph, the intersection between the blue and red surfaces is diagonal, which shows that the revenue curve under risk neutrality decreases more quickly than the revenue curve under risk aversion when c_2 is increased.

5 Conclusion

Prior auction literature has concluded that the revenue equivalence principle does not hold in the base case of risk aversion without reserve prices and entry fees. Our paper extends the exploration of revenue maximisation when there are non-transferable fees incurred during the auction, which affects the optimal bid of the bidders and hence the expected revenue of the seller. Firstly, we establish that, under risk neutrality, revenue equivalence holds in any combination of reserve prices and partial entry costs. Secondly, we see that the first-price auction continues to generate higher expected revenue than the second-price auction with non-transferable costs. Thirdly, we see that the number of bidders as well as the non-transferable costs have a significant impact on whether or not an auction in the risk-neutral setting generates more revenue than the first price auction in the risk-averse setting. Differentiating from prior theoretical work on auctions with risk-averse bidders that study the bidding equilibrium and divergence in entry behaviour, our approach explicitly derives the *ex ante* revenue from auctions with cost evaporation through consideration of the equilibrium bidding strategies.

However, our paper does not account for the combination of hybrid bidding fees and reserve prices for auctions with risk-averse bidders in terms of revenue maximisation. It is an area for future research to overcome the hurdle of proving optimality with reserve prices.

6 References

Chakraborty, Indranil. "Optimal Screening in Auctions with Risk Averse Bidders." (2017)

Graham, Daniel A., and Robert C. Marshall. "Collusive Bidder Behavior at Single-Object Second-Price and English Auctions." *Journal of Political Economy* 95, no. 6 (1987): 1217-239. doi:10.1086/261512.

Harris, Milton, and Artur Raviv. "Allocation Mechanisms and the Design of Auctions." *Econometrica* 49, no. 6 (1981): 1477-499. doi:10.2307/1911413.

Holt, Charles A. "Competitive Bidding for Contracts under Alternative Auction Procedures." *Journal of Political Economy* 88, no. 3 (1980): 433-45. doi:10.1086/260878.

Krishna, Vijay. Auction Theory. Amsterdam: Elsevier, Academic Press, 2010.

Li, Tong, Jingfeng Lu, and Li Zhao. "Auctions with Selective Entry and Risk Averse Bidders: Theory and Evidence." *The RAND Journal of Economics* 46, no. 3 (2015): 524-45. doi10.1111/1756-2171.12096.

Maskin, Eric, and John Riley. "Monopoly with Incomplete Information." *The RAND Journal of Economics* 15, no. 2 (1984): 171-96. doi:10.2307/2555674.

Matthews, Steven A., "Selling to Risk Averse Buyers with Unobservable Tastes." *Journal of Economic Theory*, (1983), 370-400. doi:10.1016/0022-0531(83)90113-8.

McAfee, R. Preston, and John McMillan. "Bidding Rings." *The American Economic Review* 82, no. 3 (1992): 579-99.

Milgrom, Paul R., and Robert J. Weber. "A Theory of Auctions and Competitive Bidding." *Econometrica* 50, no. 5 (1982): 1089-122. doi:10.2307/1911865.

Myerson, Roger B. "Optimal Auction Design." *Mathematics of Operations Research* 6, no. 1 (1981): 58-73. doi:10.1287/moor.6.1.58.

Riley, John G., and William F. Samuelson. "Optimal Auctions." *The American Economic Review* 71, no. 3 (1981): 381-92.

Vickrey, William. "Counterspeculation, Auctions and Competitive Sealed Tenders." *The Journal of Finance* 16, no. 1 (1961): 8-37. doi:10.1111/j.1540-6261.1961.tb02789.x.

7 Appendix

The programs used to create the graphs in Proposition 4.3 can be viewed here.

In order to view some of the 3D graphs, it is recommended to run the program offline, as Google Colab does not support interactive Matlab plots in its Jupyter Notebook environment.