Partial Proof of Mann's Theorem and Reflections

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1 Introduction and Definitions

Let there be a set A, $A \subseteq \mathbb{N} \cup 0$ and $0 \in A$. Define $A(n)$ to be the number of elements in A from 1 to n inclusive. The Schnirelmann density of A, defined as $\sigma(A)$, is equal to $\inf\limits_{n} \frac{A(n)}{n}, n \in \mathbb{N}$.

In 1942 Edward Mann proved that Schnirelmann density is superadditive, i.e. $\sigma(A \oplus B) \geq \sigma(A) + \sigma(B)$, where $\oplus$ denotes the sumset of A and B. I will borrow some of the notation from his original paper. Let $C = A \oplus B$. Let m denote a natural number not in C, and let m_i denote the i-th such m. Evidently $C(m_i) = m_i - i$ and $m - 1 \geq A(m) + B(m) \, \forall \, m$. Thus let $m_i = A(m_i) + B(m_i) + l_i + 1$. For example, if $A = 0, 1, 6, ...$, $B = 0, 2, 3, 4, ...$, and $7 \notin A \cup B$, then $m_1 = 7$. We can see from below that $C(7) = 6, A(7) + B(7) = 5$, and that $l_1 = 1$, which is represented by a "hole" whereby $n \notin A$ and $n \neq m_1 - b$ for any $b \in B$.

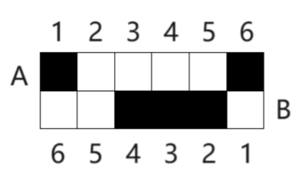

Figure 1: Diagram of $a \in A$ and $7 - b$ for $b \in B$. Clearly for any m, no column will have both squares shaded.

Mann's original proof relied on explicitly constructing the elements of some

l_j by creating a whole new sequence of sets B_1, B_2, and so on. This was rather complicated, and in 1943 Artin and Scherk managed to find a new proof which showed Mann's theorem as a corollary; however despite the claims that their proof was simpler, their proof relied on even more construction than Mann's and was more unintuitive in my opinion. Seeing that there was potentially a proof which did not rely on the construction of a massive number of subsets of C or A and B, I thought that it was a good idea to attempt it. Roughly 2 months later, I have come to the conclusion that this is much harder than it looks.

2 First attempt

For any $n < m_1, 1 = \frac{C(n)}{n} > \frac{C(m_1)}{m_1}$. Similarly for any $n > m_1$ where $n \in C$, let m be the largest $m_i < n$. Then $\frac{C(n)}{n} = \frac{C(m)+k}{m+k} > \frac{C(m)}{m}$. Consider M as the set of all m_i, then $\forall n \in C\, \exists m \in M, \frac{C(m)}{m} < \frac{C(n)}{n}$. Since $M \subseteq \mathbf{N}$, then $\inf\limits_{m \in M} \frac{C(m)}{m} \geq \inf\limits_{n \in N} \frac{C(n)}{n}$. Similarly, $\forall n \in N\, \exists m \in M$ not necessarily distinct from n, such that $\frac{C(n)}{n} \geq \frac{C(m)}{m} \geq \inf\limits_{m \in M} \frac{C(m)}{m}$, meaning that $\inf\limits_{m \in M} \frac{C(m)}{m}$ is a lower bound for $\frac{C(n)}{n}$. Thus $\inf\limits_{m \in M} \frac{C(m)}{m} \leq \inf\limits_{n \in N} \frac{C(n)}{n} \therefore \sigma(M) = \sigma(C)$.

At this point I attempted to apply a similar line of logic towards M by separating it into $M_{\geq}$ and $M_{<}$, where $C(m) \geq A(m) + B(m)\, \forall\, m \in M_{\geq}$, and $C(m) < A(m) + B(m)\, \forall\, m \in M_{<}$. By definition, $l_i = m_i - A(m_i) - B(m_i) - 1 = C(m_i - A(m_i) - B(m_1) + i - 1$, and thus $l_i \geq i - 1\, \forall\, m_i \in M_{\geq}$, whereas $l_j < j - 1\, \forall\, m_j \in M_{<}$. Since we also know that $m - 1 \geq A(m) + B(m)$, therefore $l_1 \geq 0, m_1 \in M_{\geq}$ (this can also be proven using the pigeonhole principle using a similar idea as Fig 1). My hypothesis was that for any $m_j \in M_{<}$, the largest $m_i \in M_{\geq}$ such that $i < j$ would satisfy

$$\frac{C(m_i)}{m_i} < \frac{C(m_j)}{m_j} \tag{2.1}$$

This is a new direction of proof from Mann or Artin and Scherk, and is therefore not guaranteed to be true, though I strongly believe that it is since I have not been able to find a rule by which I could construct a counterexample to 2.1. However this proposition proved to be extremely hard to prove. At the point in time when I was attempting this line of proof, I had not yet found Mann's original paper and was working off of Artin and Scherk's ideas, but those proved to be relatively unrelated. It was around this time that I used the notation $g_j = C(m_j) - A(m_j) - B(m_j)$. As it turns out, this is equivalent to $l_j + 1 - j$ in Mann's notation.

The benefit of thinking in terms of g_j is the observation that for every increment from n to $n+1$, the value of g (if defined similarly for elements of C), will decrease by 1 ($n \in C, A, B$), not change, or increase by 1 ($n \in C, n \notin A, B$). Thus a very weak bound can be obtained:

$$m_{i+1} - m_i - 1 \geq g_i - g_{i+1} \tag{2.2}$$

since the $m_{i+1} - m_i - 1$ increments between the two consecutive m are at most all increments of 1 to the value of g.

It could be the case that a better or more useful bound could be derived. However, the purpose of this bound, or any similar one, is to relate to the following inequality:

$$\begin{gathered}
\text{If } \frac{C(m_j)}{m_j} > \frac{C(m_i)}{m_i} \\
\text{then } \frac{C(m_j)}{m_j} = \frac{C(m_i) + (m_j - m_i) - (j-i)}{m_i + (m_j - m_i)} > \frac{C(m_i)}{m_i} \\
m_i C(m_i) + m_i(m_j - m_i) - m_i(j-i) > m_i C(m_i) + C(m_i)(m_j - m_i) \\
i(m_j - m_i) > m_i(j-i) \\
(m_j - m_i) > (j-i)\frac{m_i}{i}
\end{gathered} \tag{2.3}$$

At this point I guessed that for $m_i \in M_{\geq}, m_{i+1} \in M_{<}$:

$$m_{i+1} - m_i > g_i - g_{i+1} + 1 > g_i + 1 \geq \left\lfloor \frac{m_i}{i} \right\rfloor$$

The first two parts of the inequality are known to be true, and the last part would imply 2.1 for a specific set of cases (namely where $j = i + 1$) due to 2.3 and the fact that $m_{i+1} - m_i \in \mathbb{N}$. Unfortunately the furthest I was able to push this line of logic was that both $g_i + 1$ and $\left\lfloor \frac{m_i}{i} \right\rfloor \geq 1$.

3 Second attempt

Around this point in time I realized that this proof was going nowhere, and by chance I happened to find Mann's original paper which gave me a different idea. It was possible that 2.1 was not true, which would explain why neither Mann nor Artin and Scherk attempted to prove this. Therefore the second attempt uses induction on the following weakened version of 2.1:

$$\forall\, m \in M\, \exists\, k \in M, k \leq m, \frac{A(k) + B(k)}{k} \leq \frac{C(m)}{m} \tag{3.1}$$

If $k = m$, then $m \in M_{\geq}$. 3.1 is also equivalent to

$$\forall\, j\, \exists\, i, i \leq j, jm_i \leq (1 + l_i)m_j \tag{3.1'}$$

Let 3.1' be the inductive hypothesis. It is obviously true for $j = 1$. Thus assume that the hypothesis holds for all $1 \leq i < j$. Now we will assume the following:

$$jm_i > (1 + l_i)m_j\, \forall\, i < j \tag{1}$$

$$l_j < j - 1 \tag{2}$$

From the inductive hypothesis we also get that

$$jm_i > im_j\, \forall\, i < j \tag{3}$$

else (1) would not hold. The idea is to prove that the three conditions cannot all be true at the same time, which is to prove the inductive hypothesis for j by contradiction. Mann managed to prove, without (3), that (2) and (1) cannot both exist because you could construct at least $j-1$ unshaded columns (see Fig 1). As explained, however, it is a somewhat complicated proof.

My approach was to take 3 possible cases:

$$l_i > l_j \,\forall\, i < j \tag{A}$$

$$\exists\, l_i \leq l_j, m_j < m_i + m_{j-l_i-1} \,\forall\, l_i \leq l_j \tag{B}$$

$$\exists\, l_i \leq l_j \text{ such that } m_j \geq m_i + m_{j-l_i-1} \tag{C}$$

We know from the validity of Mann's theorem that in any of the 3 possibilities, there must be some contradiction when we assume that (1), (2), and (3) are all true. However, I have only been able to find a contradiction in case (C).

Suppose under case (C) that $m_j \geq m_i + m_{j-l_i-1}$. Then

$$(j - l_i - 1)m_j + (l_i + 1)m_j \geq jm_i + jm_{j-l_i-1}$$

We know from (3) that

$$(j - l_i - 1)m_j < jm_{j-l_i-1}$$

Hence

$$(l_i + 1)m_j > jm_i$$

which contradicts (1).

For (A), the results only go as far as follows:

$$jm_i > (1 + l_i)m_j > (1 + l_j)m_j$$

$$jm_{j-l_j-1} + jm_i > (1 + l_j)m_j + (j - l_j - 1)m_j$$

$$m_{j-l_j-1} + m_i > m_j$$

In the most extreme case this implies $m_1 > m - j - m_{j-l_j-1}$. This is likely to be incompatible with at least one of the assumptions (1), (2), or (3), though I have not managed to prove so. It is possible that 2.3 would be related in the proof, though the inequality is unfortunately in the opposite direction.

For (B), the idea of proof is mainly to show that by taking the minimum value of l_i, there exists an even lower value l_a, most likely due to the inductive hypothesis. Something of note is that for all $k < j$, we can find some a satisfying $km_a \leq (1+l_a)m_k$ and $l_a \geq a-1$ by repeatedly applying the induction hypothesis until we get $m_a \in M_{\geq}$. This is somewhat reliant on case (A) and I suspect that the method of proof will be similar, if it exists at all.

4 Future direction

Though from the second attempt we know that there must be a contradiction in cases (A) and (B) as well, it is not certain that there will be a nice proof that follows immediately from those assumptions. In fact, it may be the case that Mann's proof, however complicated it is, is the only method through which contradiction can be shown between (1) and (2). It is possible that case (C) is only a convenient subset of cases where the theorem is easily seen to be true, and that a better delineation of cases is needed. Interestingly, the existence of a partial proof of some of the cases does not guarantee that the same approach is workable for the rest of the cases. As with HEP, there will most likely be a follow-up to this attempt.