Random Walks on Infinite Restricted Planar Graphs

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1 Introduction

Define a random walk W_t over the integer number line as follows: at time t, $W_t = W_{t-1} - 1$ or $W_{t-1} + 1$ with equal probability. More rigorously, let Z be a random variable that takes the values of 1 or -1 with equal probability, and let $Z_1, Z_2 \ldots Z_t$ be i.i.d. variables distributed according to Z, then $W_t = \sum_{i=1}^{t} Z_i$. We would like to consider whether a random walk with the initial state $Z_0 = 0$ would return to 0, i.e. $\exists\, k > 0$ such that $Z_k = 0$. This is called a *recurrent* walk, and walks that are not guaranteed to return to their origin are *transient*. We take this as a base case before considering lattices in $\mathbb{R}^2$, Pólya's recurrence theorem, and general infinite restricted partitions of $\mathbb{R}^2$.

2 Simple walks in 1D and 2D

In the 1D case, W_t can clearly only return to the origin for even t. For even t, let $P(t) = P(W_t = 0) = \binom{t}{t/2}\frac{1}{2^t}$. Suppose $t = 2k$, then

$$\begin{aligned} P(2k) &= \binom{2k}{k}\frac{1}{2^{2k}} \\ &= \frac{(2k)!}{(k!)^2 2^{2k}} \\ &= \frac{(2k)!}{(2k!!)^2} \\ &= \frac{(2k-1)!!}{(2k)!!} \end{aligned}$$

For a walk to be recurrent, the sum of all $P(t)$ needs to be infinite. If it is finite, the Borel-Cantelli lemma implies that only a finite set of t gives $W_t = 0$, which means that the walk is not recurrent.[1] Hence, the series $P(2k)$ diverges.

To find a closed form for the series, consider

$$\begin{aligned} \frac{P(2(k+1))}{P(2k)} &= \frac{2k+1}{2k+2} \\ 2(k+1)P(2(k+1)) - 2kP(2k) &= P(2k) \\ \therefore \sum_{k=1}^{\infty} P(2k) &= 2\sum_{k=1}^{\infty}(k+1)P(2(k+1)) - kP(2k) \\ &= 2\left(\lim_{k\to\infty}(k+1)P(2(k+1)) - \frac{1}{2}\right) \\ &= \lim_{k\to\infty}\frac{(2k+1)!!}{(2k)!!} - 1 \\ &= \lim_{k\to\infty}(2k+1)\frac{(2k)!}{(k!)^2 2^{2k}} - 1 \end{aligned}$$

[1] We defined a recurrent walk as one which is guaranteed to return to its origin, but if it returns to its origin once it is guaranteed to return to its origin again due to its recurrence property, and therefore it must return to its origin or any other point an infinite number of times.

Using Stirling's approximation:

$$\begin{aligned}
\lim_{k\to\infty}(2k+1)\frac{(2k)!}{(k!)^2 2^{2k}} &= (2k+1)\frac{\sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k}}{2\pi k\left(\frac{k}{e}\right)^{2k} 2^{2k}} \\
&= \lim_{k\to\infty}\frac{2k+1}{\sqrt{\pi k}} \\
&= \infty
\end{aligned}$$

If we only consider the sequence $P(2k)$, we see that it asymptotically converges to $\frac{1}{\sqrt{\pi k}}$ as k tends to infinity, and by considering its integral we can roughly conclude that the series diverges. Heuristically, this suggests that the 2-dimensional case, which is essentially two 1D walks combined, asymptotically approaches $\frac{1}{\pi k}$, for which the integral also approaches infinity as k becomes infinitely large.[2] Adding any more dimensions, however, will result in a convergent sequence by this heuristic argument. In fact, Pólya's recurrence theorem states that random walks are recurrent in 1D and 2D but transient in 3D and above. To briefly show why the 2D case holds, let $P_2(t)$ be the 2D equivalent of $P(t)$, and consider the Wallis product for π:

$$\begin{aligned}
\frac{\pi}{2} &= \left(\frac{2\cdot 2\cdot 4\cdot 4\cdot\ \dots}{1\cdot 3\cdot 3\cdot 5\cdot\ \dots}\right) \\
&= \lim_{N\to\infty}\prod_{k=1}^{N}\frac{1}{2k+1}\left(\frac{2k}{2k-1}\right)^2 \\
&= \lim_{k\to\infty}\frac{1}{(2k+1)P(2k)^2} \\
&= \lim_{k\to\infty}\frac{1}{(2k+1)P_2(2k)}
\end{aligned}$$

Therefore as $n\to\infty, P_2(2k)\to\frac{1}{\pi(k+1/2)}$. Since $P_2(t)$ is $O(t^{-1})$, the series is divergent and hence the walk is recurrent.

[2] Consider two 1D walks placed orthogonally and combine the random moves at every time step, giving 4 possible diagonal moves that are orthogonal to each other. Rotate and resize the plane accordingly to fit the resultant walk onto the 2D lattice.

3 Doyle's electrical circuit formulation

Since between $\mathbb{R}^2$ and $\mathbb{R}^3$ a clear shift in the behavior of random walks on the lattice appears, the natural question to ask is to what extent similar structures in the plane will support recurrent walks. Consider other possible ways to tessellate $\mathbb{R}^2$, such as by triangles or hexagons. There does not appear to be an easy to way to find an asymptotic form of a similarly defined $P(t)$, since in these cases the set of possible moves returning to the origin at time t cannot be found by only considering moves in the inverse direction.[3] The additional structures on these tilings require a different type of tool to analyze.

There happens to be an alternative method formulated by Peter Doyle, who proved that for any graph upon which a random walk is defined, a corresponding electrical circuit with 1-Ω resistors on every edge and a 1-V battery between the start and end points of the desired walk is recurrent if the effective resistance between the two points is infinite, and transient otherwise.[4] For an infinite graph, we simply need to ground all points at infinity and consider the effective resistance to infinity of the circuit.

To illustrate this approach, consider the set of vertices of distance 2 from the origin on a 2D lattice, $S(2)$. We place 1-Ω resistors on all edges bounded by $S(2)$, then ground all the vertices in $S(2)$ and apply a voltage between the origin and $S(2)$. We are interested in finding the effective resistance of such a configuration at infinity, or $\lim_{n\to\infty} S(n)$.

Doyle's primary insight is to use Rayleigh's shorting rule to simplify the calculation of effective resistance. The rule states that shorting two points on a circuit can only decrease its effective resistance, which means that a lower bound can be obtained. On the 2D lattice, we can short every edge on the $2n$-square around the origin, which essentially leaves $8n+4$ parallel resistors between each tier. Summing the resistances of each level to infinity gives $\sum_{n=1}^{\infty} \frac{1}{8n+4} = \infty$.

[3] For example, there is a natural isomorphism between the triangular lattice and the additive group generated by the 6th roots of unity on the complex plane.

[4] `https://math.dartmouth.edu/~doyle/docs/walks/walks.pdf` and `https://math.dartmouth.edu/~pw/math100w13/mare.pdf`

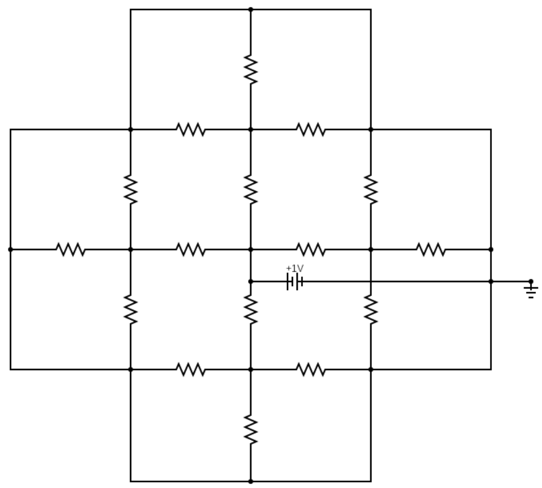

Figure 1: $S(2)$ circuit.

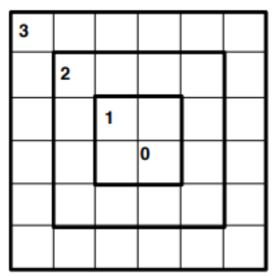

Figure 2: Illustration of the shorting rule by Shrirang Mare.

Using this method, it becomes much easier to find a lower bound for the effective resistance of a circuit, and thus to determine whether the original walk is recurrent or not. For a triangular lattice in $\mathbb{R}^2$, we simply observe that its effective resistance is bounded below by $\sum_{n=1}^{\infty} \frac{1}{12n+6}$, which is divergent. In general, if we can construct sets of points to short, and the number of edges (parallel resistors in the worst case, since we want the resistance to be as high as possible) from the n-th set to the next is $O(n)$, then a random walk on this graph is recurrent.

We have the following general descriptions of a random partition of $\mathbb{R}^2$ into infinite faces of finite area by a graph G, which generalizes the idea of lattices and tessellations:

- G is planar and simple.
- $|G| = \infty$
- The supremum of $\deg(v)$ is finite.
- There are no faces of infinite size when $\mathbb{R}^2$ is partitioned by G.
- The infimum of the edge lengths is greater than 0.

The first three requirements are straightforward. The fourth and fifth requirements are needed to exclude trees, for which the number of edges between each level grows exponentially. A binary tree of height n, for example, has effective resistance $1 - \left(\frac{1}{2}\right)^n$ given by $2r_{n+1} - 1 = r_n$, which converges to 1 as $n \to \infty$.

Since the plane can be divided into infinitely many faces of finite size in this manner, let A be the set of areas of every face and $a = \inf A$. Next divide the plane into squares of area a and let the origin be at one of the intersections of the square lattice. We can now imagine squares centered at the origin of size $4n^2a$, and since each small square is at least as small as the smallest face, the number of faces within the n-th square is bounded by $f(n) < 4n^2a = O(n^2)$. Since there is no infinite face and every edge has length greater than some e, every face is bounded by a finite cycle C. Let $s = \sup C$. Then the number of vertices within the n-th square is bounded by $v(n) < 4n^2as = O(n^2)$. By the fact that $e \leq 3v - 6 = O(v)$ for any planar graph, $e(n)$ is also $O(n^2)$. If $d(n) = e(n) - e(n-1)$ grows faster than a linear function, then its series grows faster than $O(n^2)$. Thus, $d(n)$ represents the increment in the number of edges for each increment in the size of the square and is linear in n. By shorting all edges that do not intersect the bold squares (see Fig. 2), we see that the number of edges joining each shorted level is $O(n)$, and hence the walk is recurrent.

4 Conclusion

We have proven that any graph in this restricted family of infinite planar graphs generate recurrent random walks. However, the converse is not true, since recurrent walks on nonplanar graphs do exist. Doyle's original monograph contained a purely probabilistic proof that a random walk is recurrent if and only if its graph could be embedded in a k-fuzz of a 2D lattice. Using his result, planarity is not necessarily a relevant criterion for recurrence in random walks, though for the purposes of considering tessellations and tilings of $\mathbb{R}^2$ our method is perhaps more pertinent.