Answering Two Oddly Specific Questions in Financial Modeling

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1 Q1: Portfolio Construction

1.1 Question Statement

We have X individual strategies, each with a Sharpe ratio of 0.50. They all have pairwise correlation ρ. We look at the cases where $\rho = 0.1$, 0, or -0.1.

1.2 Derivation of Portfolio Sharpe

We first define:

$$\begin{aligned}
X &= \text{Number of strategies} \\
\rho &= \text{Pairwise correlation for any pair of strategies} \\
S &= \text{Sharpe of individual strategy} \\
\mu_i &= \text{Mean return of strategy } i \\
\mu &= \text{Vector of mean returns} \\
\sigma_i &= \text{Standard deviation of strategy } i \\
w_i &= \text{Weight of strategy in portfolio (sums to 1)} \\
w &= \text{Vector of weights} \\
S_p, \mu_p, \sigma_p &= \text{Portfolio Sharpe, portfolio mean return, and portfolio standard deviation} \\
D &= diag(\sigma_1, \sigma_2, ..., \sigma_X) \\
K &= \text{Correlation matrix of strategies}
\end{aligned}$$

Given these definitions, we know that:

$$\begin{aligned}
S_p &= \frac{\mu_p}{\sigma_p} \\
&= \frac{w^\top \mu}{\sqrt{w^\top DKDw}}
\end{aligned}$$

where DKD is equivalent to the covariance matrix.

Now we introduce a simplifying assumption: we will first assume that $D = \sigma I$ for some fixed value σ, meaning that all the strategies have the exact same mean return and standard deviation. By symmetry, this implies that the optimal weights for this portfolio are $\frac{1}{X}$ for all strategies.

From the fact that all the off-diagonal entries of K are ρ, we get:

$$\begin{aligned}\frac{w^\top \mu}{\sqrt{w^\top DKDw}} &= \frac{\frac{1}{X} * X * \sigma S}{\sqrt{\sigma^2 \frac{1}{X} * X * \frac{(1+(X-1)\rho)}{X}}} \\ &= \boxed{S\sqrt{\frac{X}{1+(X-1)\rho}}}\end{aligned}$$

This is the optimal Sharpe ratio for a portfolio of X strategies with equal Sharpe S, pairwise correlation ρ, and equal volatility σ. Evidently, its limit as X tends to infinity is $\frac{S}{\sqrt{\rho}}$.

Now let us remove the last assumption of equal volatility. We see that

$$\begin{aligned}S_p &= \frac{\sum_i w_i S \sigma_i}{\sqrt{w^\top DKDw}} \\ &= \frac{S \sum_i w_i \sigma_i}{\sqrt{\sum_i w_i^2 \sigma_i^2 + \rho \sum_{i \neq j} w_i w_j \sigma_i \sigma_j}}\end{aligned}$$

To simplify, let $v_i = w_i \sigma_i$. Now:

$$\begin{aligned}\frac{S \sum_i v_i}{\sqrt{\sum_i v_i^2 + \rho \sum_{i \neq j} v_i v_j}} &= \frac{S \sum_i v_i}{\sqrt{\rho \sum_i v_i^2 + \rho \sum_{i \neq j} v_i v_j + (1-\rho) \sum_i v_i^2}} \\ &= \frac{S \sum_i v_i}{\sqrt{\rho (\sum_i v_i)^2 + (1-\rho) \sum_i v_i^2}} \\ &\leq \frac{S \sum_i v_i}{\sqrt{\rho (\sum_i v_i)^2 + \frac{(1-\rho)}{X} (\sum_i v_i)^2}} \quad \text{by Cauchy-Schwarz} \\ &= \boxed{S\sqrt{\frac{X}{1+(X-1)\rho}}}\end{aligned}$$

Therefore, for a set of strategies satisfying the conditions in the question, their maximum Sharpe ratio occurs when **all volatilities are equal**.

1.3 Discussion of Cases

1.3.1 $\rho = 0.1$

The maximal Sharpe ratio of the strategy increases monotonically as X increases. The upper limit is $\frac{0.5}{\sqrt{0.1}} \approx 1.5811$. At X=84, we are within 95% of the upper limit.

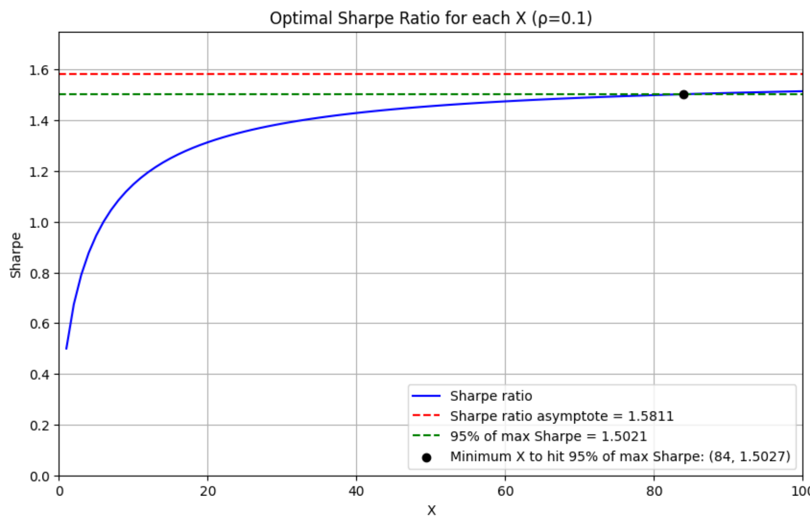

Figure 1: Optimal Sharpe ratio for the portfolio for X from 1 to 100.

1.3.2 $\rho = 0$

The maximal Sharpe ratio simplifies to $S\sqrt{X}$, which is not bounded.

1.3.3 $\rho = -0.1$

We can have at most $-\frac{1}{\rho}$ such strategies each with a negative pairwise correlation ρ. This is because the eigenvalues of K are $1-\rho$ and $1+\rho(X-1)$. These eigenvalues must be nonnegative as the correlation matrix is positive semidefinite, but as we are not considering risk-free assets the eigenvalues of our K should be positive. For negative ρ, this equals $X < 1 - \frac{1}{\rho}$ or $X \leq -\frac{1}{\rho}$. As X increases, the optimal Sharpe ratio increases at an increasing rate, which is different from the other two cases.

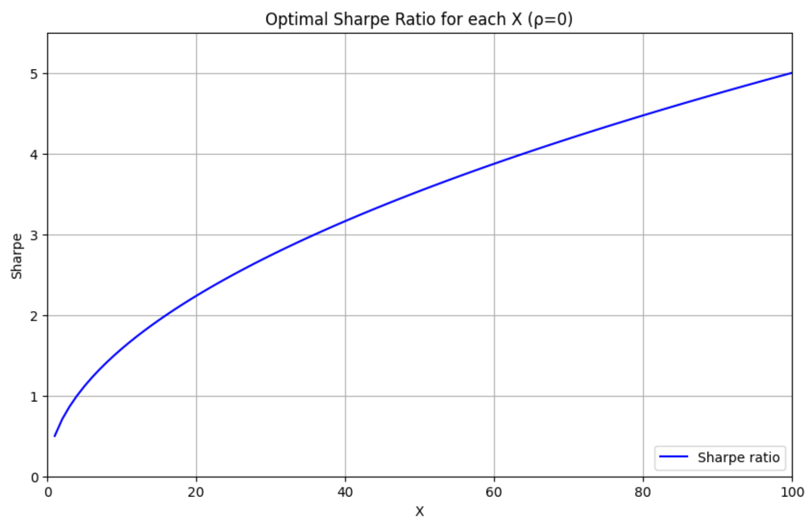

Figure 2: When the strategies are uncorrelated, the optimal Sharpe ratio grows infinitely.

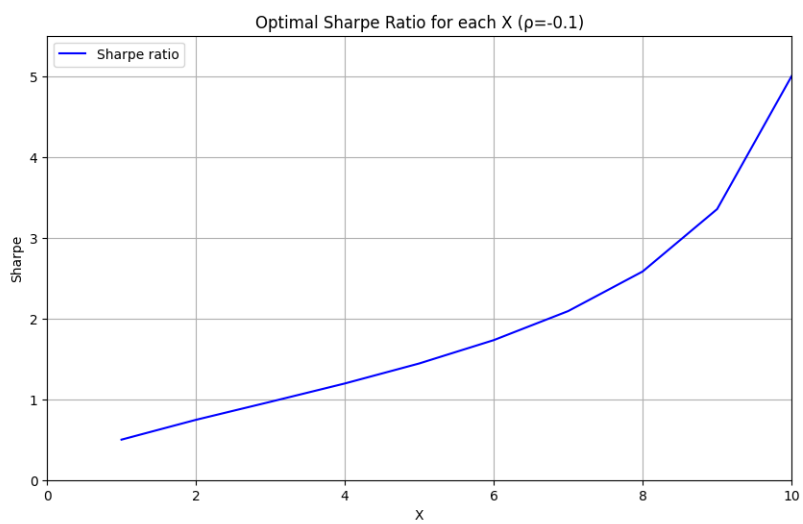

Figure 3: When ρ is negative, there is an upper limit to how many individual strategies can exist to satisfy our requirements.

1.4 Conclusion of Q1

There are two main implications of these findings.

1. Assuming that these strategies can be found, it is optimal to find a family of strategies that are completely uncorrelated.

2. It is better to find strategies that have a low negative pairwise correlation than positive pairwise correlation.

However, this scenario is highly unlikely to exist in the real world. Rather, this provides some insights into the nature of strategies. Firstly, it should intuitively be easier to find an additional strategy in a set of equal Sharpe strategies with positive correlation than it is to find an additional strategy when the correlation is negative, as the marginal payoff for the former is much lower. Secondly, sources of low or uncorrelated returns are highly sought after and thus hard to find. Lastly, if many institutional players are active in the same set of strategies, they are likely to influence liquidity to the extent that these strategies become **more positively correlated**. We saw this in August 2024 when the yen carry trade and the dispersion trade (and probably many other strategies) blew up at the same time, as large investors pulled money out of "unrelated" strategies to meet margin requirements.

2 Q2: Betas and Risk Premia

2.1 Question Statement

We will identify 3 liquid sources of risk premium. We are interested in the following questions:

1. Why does each source of risk premium exist, and who is transferring risk to whom?
2. What are their historical risk and return characteristics? What is its Sharpe ratio minus the risk free rate? How has its Sharpe ratio changed over time?
3. What is the correlation between these strategies?
4. We have a fixed amount of risk to allocate to these strategies. How do we maximize the expected future Sharpe ratio and why is this a good allocation?

2.2 Sources of Risk Premium

Throughout this question, we will assume that the market in question is the US market. Hence, the equity market proxy will be the S&P500, and the proxy risk-free rate will be the 3 month T-bill yield. The liquid instrument which best approximates this is SPDR Bloomberg 1-3 Month T-Bill ETF (BIL). The index that approximates this is IRX, which tracks the 13 week T-Bill yield.

Additionally, I will also only consider instruments that an average retail investor has access to, in terms of both trading and data.

2.2.1 Equity Risk Premium

Equity risk premium (ERP) is the additional return one expects to earn from investing in equities rather than bonds. It exists because the equity market is more volatile than bond markets, therefore if ERP did not exist and the expected return of a bond portfolio was the same as that of an equity portfolio, variance drain would cause the latter to be less attractive in the long term.

In this case, **investors who sell equities** are transferring risk to **investors who buy equities**. Investors who sell equities could be short sellers or equity holders who are selling their shares to buy bonds. When the market is in a bull run, more investors are willing to buy equities than to sell them, thus lowering the risk premium. The opposite happens during a crash, when investors all look to sell and buyers demand a large premium for taking on the risk that equities might tank even more.

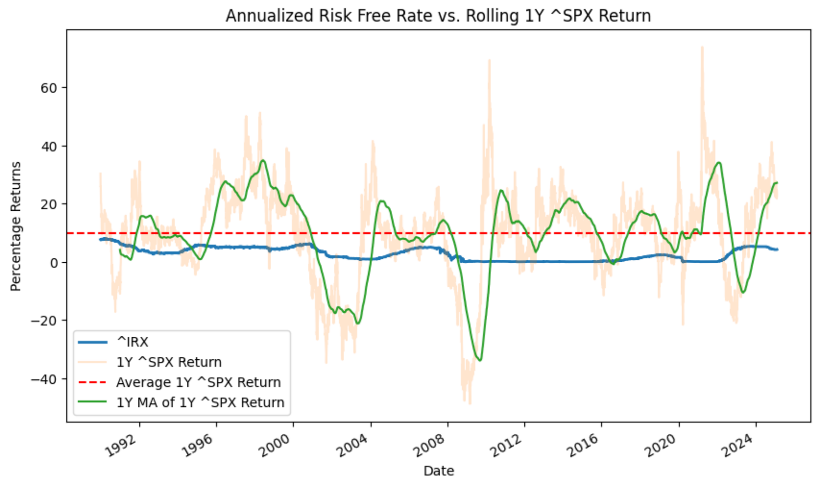

Figure 4: There is clear historical proof of the ERP, though at times it has swung negative in periods of market stress, particularly in bear markets after the dot-com bubble, during the financial crisis, and in the 2022 correction.

One important observation worth pointing out is that when the ERP flips negative, it flips hard and fast. This is true of any risk premium, since the receiver of the premium is being **compensated during normal times to take on outsized risk during abnormal times**.

There is a direct and liquid way to participate in ERP. By being long an ETF tracking SPX and short an ETF tracking the risk free rate, we can get a fairly accurate exposure to ERP, ignoring commissions, ETF tracking error, withholding taxes on dividends, and other miscellaneous costs.

Our portfolio consists of equal nominal positions in long SPY and short BIL. With sufficient starting capital, we can assume that this trade is exact. This portfolio is then rebalanced daily to maintain a 1:1 ratio between the long and short legs.

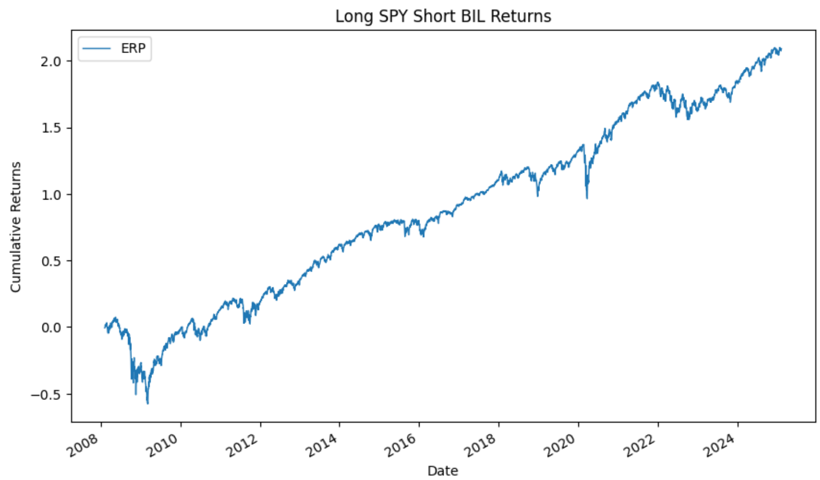

Figure 5: Performance of the long SPY short BIL portfolio. Its annualized Sharpe ratio is 0.38785 after adjusting for the risk-free rate.

From the figure, it can be seen that the graph is relatively linear. This suggests that aside from periods of extreme market stress, such as 2008, the Sharpe ratio of ERP has remained **relatively steady**.

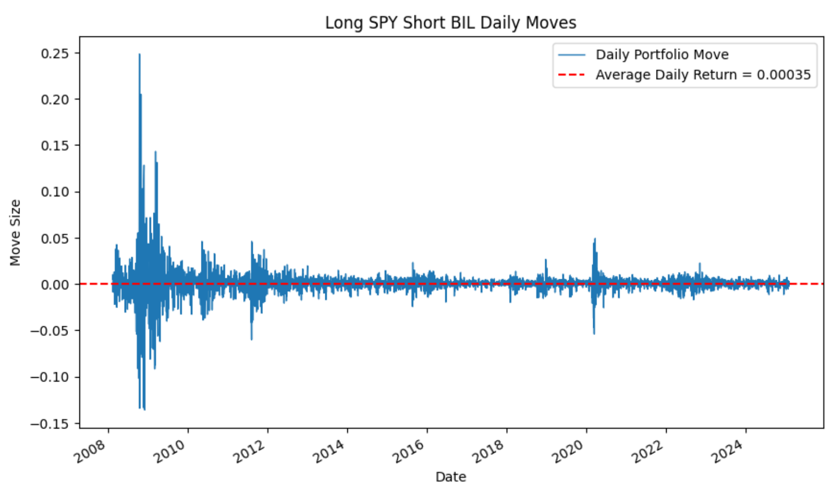

Figure 6: In the earlier days of BIL, lower liquidity sometimes caused drastic moves in the portfolio, especially in Sept-Oct 2008. This became less frequent as time passed.

2.2.2 Volatility Risk Premium

Volatility risk premium (VRP) is the additional return one expects from selling volatility as compared to buying volatility. It exists due to the following characteristics of volatility:

1. There is systematic demand for protection against volatility from investors seeking to improve risk-adjusted returns and institutions with strict risk limits. In other words, **risk-averse investors buy volatility to hedge their portfolios**.

2. The distribution of volatility is heavily right-skewed, and shorts expect additional return for taking on tail risk.

VRP manifests most commonly as **option buyers** paying **option sellers**. This means that the IV priced into options is usually higher than subsequent realized volatility.[1]

Some non-retail friendly manifestations of VRP include variance swaps,[2] log contracts, and other exotic volatility derivatives, which are much more directly exposed to VRP. Alternatively, the (30-day) VIX replicates the square root of the fair value of a 30-day variance swap. However, it can only be traded through futures or ETFs, which are not exact approximations.

The liquid way to trade VRP for retail investors would be to sell an option at the ATM forward price for a fixed weighted maturity of their choice, and delta hedge it with the underlying. In doing so, we are short vega, thus taking on exposure to volatility shocks. In an optimal world, we would hedge this position continuously to adjust for gamma, and the total P/L from this position would be path dependent: if the cost of rebalancing for gamma is lower than the premium earned from theta decay, which would be the case if IV>RV, then we have collected VRP.

Due to limitations on the availability of historical options data,[3] we will use the following proxy: instead of constant delta hedging, we will sell the

[1] There are several ways to measure IV (on SPX). The simplest method is to take the IV of the ATM forward spot price. However at other strikes, options will have different values of IV. Since Black-Scholes assumes that returns on the underlying are normally distributed, whereas in reality most instruments like equities and futures have leptokurtic returns, the Black-Scholes IV increases for strikes further from spot price; this phenomenon is called *skew* and represents the market's adjustment to IV to reflect the fair value of VRP.

[2] Variance swaps can theoretically be approximated by a portfolio of options weighted inversely proportional to the square of their strikes. See: GS volswap notes

[3] The most comprehensive data available comes from tastytrade's options backtester. For our strategy, we roll the 30DTE ATM straddle assuming that puts and calls at the forward ATM strike have 50 delta each. Allowing for some imprecision, this removes the need for delta hedging, which the backtester does not support anyways.

30DTE ATM straddle. This strategy is recalibrated daily to adjust the strike and expiration. Its annualized Sharpe ratio is 0.28248 after adjusting for the risk-free rate.

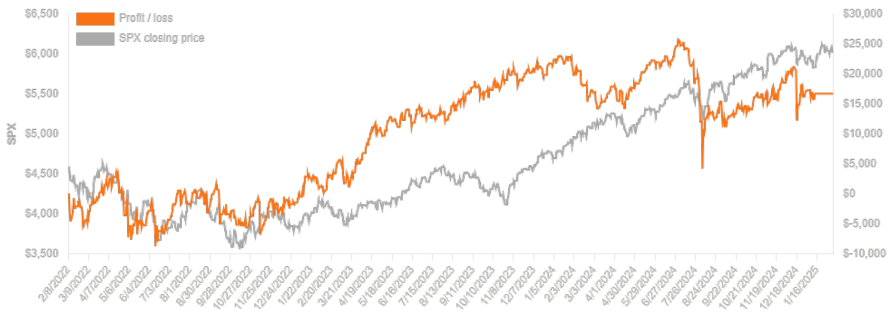

Figure 7: Performance of rolling the 50-delta short straddle on SPX to maintain constant 30DTE maturity. Data provided by tastytrade's backtester. Note that tastytrade's own commissions are factored into the backtest.

There is insufficient data to conclude whether there has been a meaningful change in the Sharpe ratio of VRP over time. However, there is reason to believe that VRP has **increased** since 2022, when 0DTE index options were introduced on every weekday. Increased option buying interest for purely speculative purposes should reasonably contribute to higher VRP. However, this has also led to increased crowding in the short volatility trade, which led to a second "Volmageddon" during the yen carry unwind. In the graph, the precipitous drop in late July 2024 can be attributed to this.

2.2.3 Term Premium

Term premium (TP) is the additional return one expects to earn from investing in long-term bonds rather than rolling short-term bonds over the same period. It exists because there is inherent uncertainty in future rates, thus long-term bond buyers are compensated for holding this source of volatility.

The most common case is the spread between long-term US Treasury bonds and short-term T-bills. The bond issuer, which is the **US Treasury**, is paying **bondholders** TP for holding long-term bonds. The major risk to this strategy is a sudden large increase in rates, which would affect the long-term bonds much more severely because of its duration. The worst case scenario would be where long term yields rise and short term yields fall, as we saw in late 2024 when the bond market believed the Fed's 50bp cut to be excessive.

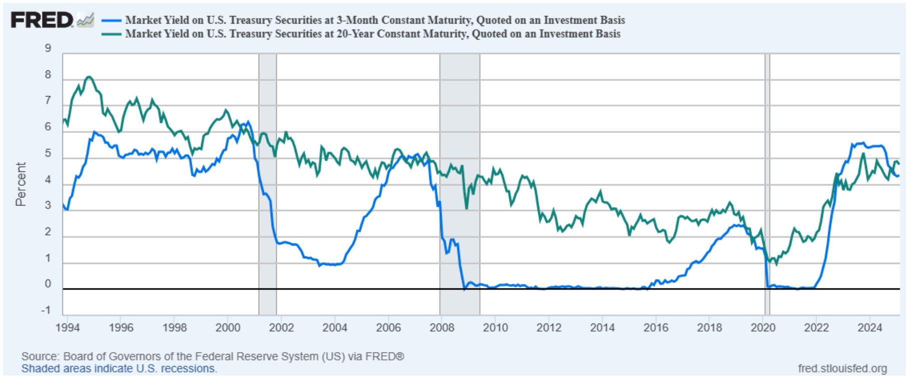

Figure 8: Example of the annualized yields of 20-year bonds and 3-month bills. Notice how the yield curve inverts in 2022, which caused TP strategies to perform very badly during that period.

To trade TP, we use a similar strategy as trading ERP. Our portfolio consists of equal nominal positions in long TLT (average maturity 25.73 years) and short BIL (average maturity 0.12 years), and is rebalanced daily to maintain a 1:1 ratio.

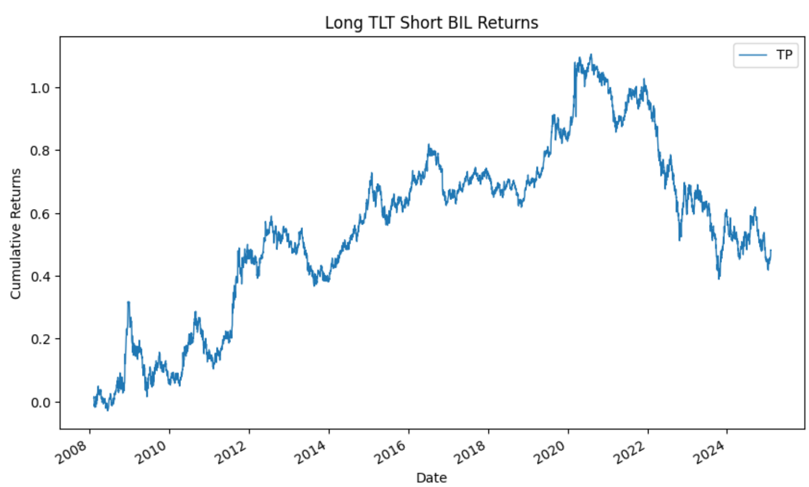

Figure 9: Performance of the long TLT short BIL portfolio. Its annualized Sharpe ratio is 0.18855 after adjusting for the risk-free rate.

Even though the Sharpe ratio is quite low, if we only take the data up until 2020, the Sharpe ratio of this portfolio would have been far higher at 0.47791. Recently, the Sharpe ratio has been negative as TP itself flipped negative during the yield curve inversion.

2.3 Performance and Correlation of Risk Premiums

Due to data limitations, we can only compare the performance of our three risk premiums from 2022 onwards.

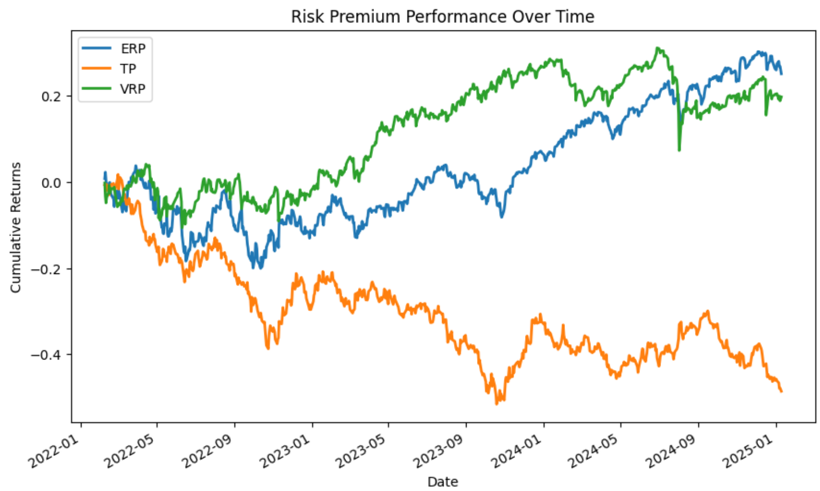

Figure 10: Performance of ERP, TP, and VRP.

Unfortunately, this coincides with the uncharacteristic period of yield curve inversion caused by the Fed's swift rate hikes in 2022. For this period, the correlation matrix between them is:

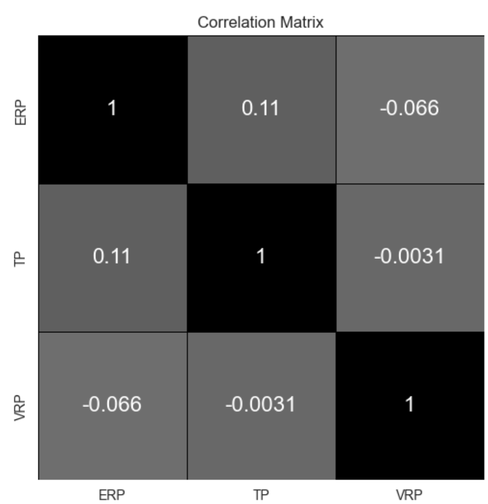

Figure 11: Correlation matrix of ERP, TP, and VRP from 2022-2025.

We can make a few observations:

1. One would have expected VRP and ERP to be positively correlated, since volatility tends to get bid when equities fall. However, on a daily basis during this period, it seems that the two are slightly negatively correlated.

2. TP has positive correlation with the ERP, but arguably this is not the norm. If we look from 2008 to 2020, ERP and TP had a negative correlation of -0.381335, which is expected of a bond strategy vs an equity strategy. (See Figure 12)

3. The correlation coefficients between VRP and ERP are **very low**, despite the final return profiles looking very similar. This is evidence of the **Epps effect**, as our time series are compared on a daily level.

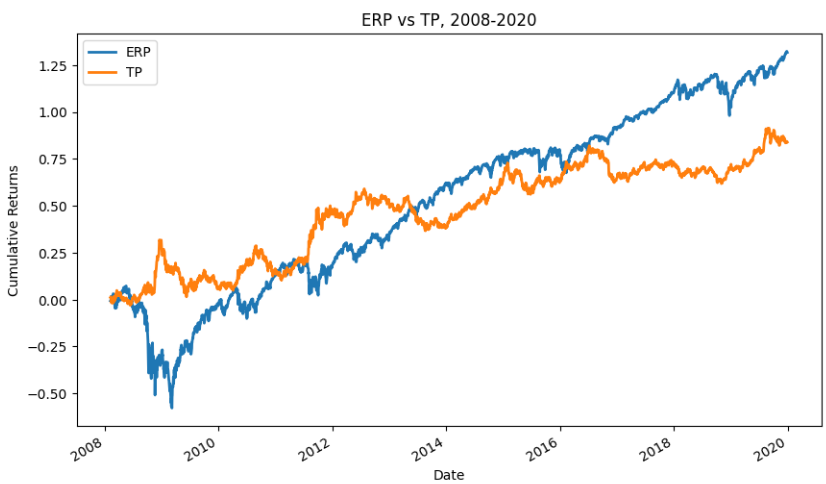

Figure 12: ERP and TP were negatively correlated from 2008-2020.

2.4 Portfolio Allocation and Conclusion of Q2

To maximize the future Sharpe ratio, we will make some assumptions about the future risk and return characteristics of the three strategies. First, we will construct a covariance matrix based on the 2022-2025 data. However, we will replace the cells for ERP and TP with the extended data from 2008-2025 to better reflect their general negative correlation. We will also assume that the future **correlation** between TP and VRP is equal to the correlation between ERP and TP, and derive their covariance from that. Our forward expected covariance matrix (annualized) is thus:

$$\Sigma = \left[\begin{array}{c|ccc} \text{ERP} & 0.042770 & -0.007415 & -0.002220 \\ \text{TP} & -0.007415 & 0.012226 & -0.006416 \\ \text{VRP} & -0.002220 & -0.006416 & 0.032030 \end{array}\right]$$

We also estimate the annualized mean excess returns over the risk-free rate in a similar way:

$$\mu = \begin{bmatrix} 0.080233 \\ 0.020857 \\ 0.050519 \end{bmatrix}$$

By calculating the optimal portfolio as being proportional to $\Sigma^{-1}\mu$, we get:

$$w \propto \left[\begin{array}{c|c} w_{\text{ERP}} & 2.866864 \\ w_{\text{TP}} & 4.890800 \\ w_{\text{VRP}} & 2.755629 \end{array}\right]; \text{ If } \sum_i w_i = 1,\ w = \begin{bmatrix} 0.272689 \\ 0.465201 \\ 0.262109 \end{bmatrix}$$

Given a fixed amount of risk, we can allocate it to these sources of risk premium according to the distribution above, scaled to the required standard deviation. The major assumptions are that as of 2025, the return characteristics of ERP and VRP have not undergone a fundamental shift, and that the returns of TP over the past few years represent a prolonged "drawdown" period instead of a permanent change in the TP mechanism.

Since our goal was to design strategies that gave full exposure to a specific risk premium, these strategies alone are not optimal ways to profit off of the risk premium. For example, we can improve the Sharpe ratio of our volatility strategy by buying OTM puts and calls to make a long butterfly spread. This way, we give up some premium while simultaneously offloading sources of extreme tail risk. In many cases, this is a better strategy than **only holding risk premium**.