Complete Nesting in Randomly Generated Interior Circles

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1 Introduction

Any two points P_1 and P_2 chosen from $\mathbb{R}^2$ uniquely define a circle, which is the set of all points X for which $\angle P_1 X P_2 = \frac{\pi}{2}$. We are interested in randomly choosing two such points from within a disk and calculating the probability that the circle generated from these points nests completely inside the disk, so that the union of the two circular regions is the original disk. We denote a completely nesting circle as a *valid* circle, and those which partially fall outside the disk as *invalid*.

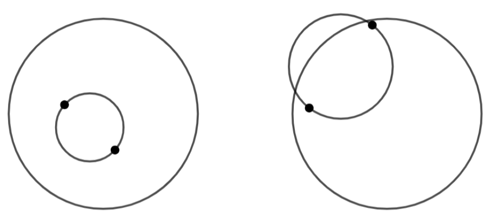

Figure 1: On the left is a valid circle, whereas on the right is an invalid circle.

For simplicity, we assume that the disk from which we choose our points is a unit disk centered at the origin. Let $f(r, \theta) = \frac{1}{\pi}$ be the probability density of a point being chosen at any point (r, θ). Later, as we explore the generalizations of this problem, we let $f(P) = \frac{1}{\lambda(A)}$, where $\lambda(A)$ denotes the Lebesgue measure of the set from which we take our two random points.

2 Circles in circles

On the surface, the probability could be thought of as the expected value of an indicator function (1 if circle is valid, 0 if invalid) over all possible circles generated from two points within the unit disk. Due to the rotational symmetry of a disk, the expected value of this indicator over circles centered at a particular *point* ought to be θ-invariant.

We can express the two points as vectors p_1 and p_2. The midpoint of these points is $m = \frac{p_1+p_2}{2}$. Let $r_m = \|m\|$ and $\theta_m = \cos^{-1}(\frac{m \cdot \mathbf{i}}{r_m})$, where $\mathbf{i}$ is the basis vector parallel to the x-axis. We now define a new probability density function over the disk, $g(r, \theta)$, as the probability distribution of (r_m, θ_m) being at (r, θ).

The end goal of this is to find the expected value of

$$P(\text{circle is valid} \mid \text{center of circle is } (r_m, \theta_m)) \times g(r_m, \theta_m), \tag{*}$$

since this would encompass all possible combinations of p_1 and p_2.

Finding $g(r_m, \theta_m)$ involves the area of possible coordinates for any m in which p_1 and p_2 can lie. Consider a disk reflected about a point M. If we choose any other point P_1 within the disk, it will have a mirror image P_2 in the disk reflected about M. If P_1 lies within the intersection of the disk and its reflection, then P_2 will also lie within the intersection. Hence, P_2 lies within the original disk if and only if P_1 lies within the intersection of the disk and its reflection about M.

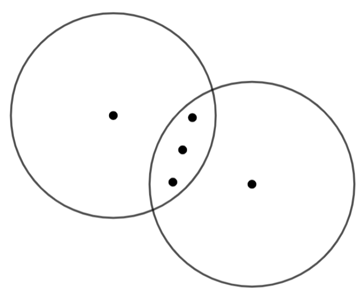

Figure 2

From this, we conclude that for any m, the set of coordinates from which p_1 and p_2 could have come from form a leaf-shaped overlapping area. This area is made up of two identical minor segments formed by the chord passing through (r_m, θ_m) and perpendicular to m. (From now on, (r_m, θ_m) will be denoted as (r, θ) and will refer to the specific values at m for simplicity.) We calculate this area as

$$2(\cos^{-1} r - r\sqrt{1-r^2}) \tag{2.1}$$

We know that $g(r, \theta)$ will be proportional to the ratio of the intersection to the area of the unit disk, since the pair (m, p_1) also uniquely generates p_2, implying that only one point needs to be selected from the intersection to uniquely define any pair (p_1, p_2). Hence, $g(r, \theta) = \frac{1}{k} \frac{2(\cos^{-1} r - r\sqrt{1-r^2})}{\pi}$, where k is a normalizing constant that scales the function to become a probability distribution. To find k, we simply integrate the rest of the function over $r \in [0, 1]$ and $\theta \in [0, 2\pi)$:

$$k = \int_0^{2\pi} \int_0^1 \frac{2(\cos^{-1} r - r\sqrt{1-r^2})}{\pi} r \, dr d\theta$$

Since g is θ-invariant:

$$= 4\int_0^1 (\cos^{-1} r - r\sqrt{1-r^2})r\,dr$$

$$= 4\int_{\frac{\pi}{2}}^0 -u\sin u\cos u + \sin^2 u\cos^2 u\,du$$

$$= (-\frac{\pi}{2}\cos\pi + 0) - \int_0^{\frac{\pi}{2}} \cos 2u\,du - \int_0^{\frac{\pi}{2}} \frac{1-\cos 4u}{2}\,du$$

$$= \frac{\pi}{2} - 0 - \frac{\pi}{4} + 0$$

$$= \frac{\pi}{4}$$

hence $g(r,\theta) = \frac{8(\cos^{-1} r - r\sqrt{1-r^2})}{\pi^2}$.

Returning to (*), we can now calculate the probability that (p_1, p_2) define a valid circle given that they fall within the leaf-shaped intersection. The radius of the circle formed by choosing p_1 given a certain m is $\|m - p_1\|$. If $\|m - p_1\| > (1-r)$, then m can be extended to intersect the circle outside of the unit disk. Thus, the maximum value of $\|m - p_1\| = (1-r)$, and choosing p_1 from within this radius will generate a valid circle. The probability of randomly choosing a p_1 from the leaf-shaped intersection which falls within this circle is $\frac{\pi(1-r)^2}{2(\cos^{-1} r - r\sqrt{1-r^2})}$.

To solve the original problem, we solve

$$\int_0^{2\pi}\int_0^1 \frac{\pi(1-r)^2}{2(\cos^{-1} r - r\sqrt{1-r^2})}\frac{8(\cos^{-1} r - r\sqrt{1-r^2})}{\pi^2} r\,dr d\theta$$

$$= 8\int_0^1 r(1-r)^2\,dr$$

$$= 8(\frac{1}{2} - \frac{2}{3} + \frac{1}{4})$$

$$= \frac{2}{3} \quad \square$$

3 Circles in regular polygons

We consider the first generalization of this problem, which asks for the probability $P(n)$ that a random circle generated from a similar method as above fits inside an n-sided polygon. The framework of (*) still applies, though it is significantly more difficult to find the area of intersection between an n-sided polygon and its reflection. For simplicity, let the polygon be centered at the origin and let the apothem of the polygon be 1. Denote the area of the polygon as A_n and the area of overlap between the n-sided polygon and its reflection about M as $R_n(M)$. Then

$$P(n) = \int\limits_M \frac{\text{Area in which } p_1 \text{ generates a valid circle}}{R_n(M)} \frac{R_n(M)}{kA_n}\, dM$$

$$= \frac{1}{kA_n} \int\limits_M (\text{Area in which } p_1 \text{ generates a valid circle})\, dM, \tag{3.1}$$

which indicates that we do not need to find a closed-form expression for $R_n(M)$, as long as we can find $k = \int\limits_M \frac{R_n(M)}{A_n}\, dM$ without explicitly solving for $R_n(M)$.

First notice that the n-sided polygon is made up of $2n$ congruent right triangles meeting at the origin. The adjacent edge to the angle formed at the origin is the apothem, hence the opposite edge has length $(1)\tan\frac{2\pi}{2n} = \tan\frac{\pi}{n}$. For any M, the edge closest to it is clearly the edge belonging to the triangle that it falls within. Hence, it suffices to limit the area over which M is chosen to one of the triangular slices, and then solve for $P(n)$ using this value.

We first consider the triangle in the first quadrant which borders the x-axis. For any point $M = (x, y)$ in this triangle, the shortest distance from it to the edge of the polygon is also its perpendicular distance from the edge belonging to this triangle. The perpendicular distance is $(1 - x)$, and since the distance from M to any other point on the edge is greater than this, the circle of area $\pi(1-x)^2$ encompasses all the possible points P_1 for which there exists a P_2 such that their midpoint is M and they form a valid circle within the polygon.

Integrating this over the triangle gives

$$\int_0^1 \int_0^{x \tan\frac{\pi}{n}} \pi(1-x)^2\, dydx$$

$$= \tan\frac{\pi}{n} \int_0^1 \pi x(1-x)^2\, dx$$

$$= \frac{\pi \tan\frac{\pi}{n}}{12}$$

Since this only considers one of the $2n$ triangles, the value of this integral over all possible M is

$$\frac{2n\pi\tan\frac{\pi}{n}}{12} = \frac{\pi A_n}{6} \tag{3.2}$$

Now we need to find $\int\limits_M \frac{R_n(M)}{A_n}\, dM$ to treat the distribution of midpoints in the polygon as a probability distribution. Consider the indicator function i_A defined over a set A:

$$i_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

We can define as A as the set of points on the interior of the polygon, and A' as the set of points on the interior of the mirror image of A about M. However, using the vector representation of M as m, consider:

$$A' = \{-x + 2m \mid x \in A\}$$

Thus for any vector $x \in \mathbb{R}^2$, we can tell if it is in $A \cap A'$ by checking if $-x+2m \in A$. If this is fulfilled, then $i_A(-x+2m) = 1$, and trivially $i_A(x) = 1$ too. Thus

$$\begin{aligned} R_n(M) &= \int\limits_x i_A(-x+2m) i_A(x)\, dx \\ &= (i_A * i_A)(2m) \end{aligned}$$

which is the convolution of i_A to itself. To find the expected value of $R_n(M)$, we integrate this over all possible $m \in A$:

$$\mathbb{E}[R_n(M)] = \frac{\int (i_A * i_A)(2m)\, dm}{A_n}$$

Using the substitution $s = 2m$, we have

$$\det \mathbf{J}_s = \frac{1}{2} \times \frac{1}{2} - 0 = \frac{1}{4}$$

$$\begin{aligned} \int (i_A * i_A)(2m)\, dm &= \int (i_A * i_A)(s) \frac{1}{4}\, ds \\ &= \frac{1}{4}\Big(\int i_A(s) ds\Big)^2 = \frac{1}{4} A_n^2 \end{aligned}$$

Thus:

$$\mathbb{E}[R_n(M)] = k = \frac{A_n}{4} \tag{3.3}$$

Solving again for (3.1), we get

$$\begin{aligned} P(n) &= \frac{1}{kA_n} \times \frac{\pi A_n}{6} \\ &= \frac{2\pi}{3A_n} \quad \square \end{aligned}$$

To verify this, we take $\lim_{n\to\infty} P(n) = \frac{2\pi}{3\pi} = \frac{2}{3}$.

4 (n-1)-spheres in n-balls

Now we consider the second generalization, which asks for the probability $Q(n)$ that a random $(n-1)$-sphere defined by two points in a unit n-ball lies completely in the interior of the n-ball. By a similar argument as in the case for the 1-sphere in the 2-ball, the overlapping volume between two n-balls reflected about some point M constitutes the measure of the set of points which could form $(n-1)$-spheres centered at M. In similar form to section 2, we let

$$Q(n) = \frac{1}{k}\int_0^{2\pi}\underbrace{\int_0^{\pi}...\int_0^{\pi}}_{n-2}\int_0^1 \frac{V_n(1-r)}{V_n(1)}\left|\det\frac{\delta(x_1,...,x_n)}{\delta(r,\theta_1,...,\theta_{n-1})}\right| drd\theta_1...d\theta_{n-1}$$

$$k = \int_0^{2\pi}\underbrace{\int_0^{\pi}...\int_0^{\pi}}_{n-2}\int_0^1 \frac{2C_n(r)}{V_n(1)}\left|\det\frac{\delta(x_1,...,x_n)}{\delta(r,\theta_1,...,\theta_{n-1})}\right| drd\theta_1...d\theta_{n-1}$$

where $V_n(r)$ is the volume of a n-ball with radius r, $C_n(r)$ is the volume of an n-dimensional hyperspherical cap from an n-ball of radius 1 with an orthogonal distance of r away from the origin, and $\theta_1,...,\theta_{n-1}$ are the angular measurements of M taken from each of the first $n-1$ axes. Note that the integrand is separable in both cases, which allows us to change the order of integration later.

Using existing results,[1] we have

$$C_n(r) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})}I_n(r), \text{where } I_n(r) = \int_0^{\cos^{-1} r}\sin^n\theta\, d\theta$$

$$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}r^n$$

$$\left|\det\frac{\delta(x_1,...,x_n)}{\delta(r,\theta_1,...,\theta_{n-1})}\right| = r^{n-1}\sin^{n-2}\theta_1\sin^{n-3}\theta_2...\sin\theta_{n-2}$$

Since we can separate the integrand while calculating $Q(n)$ and k, we first calculate

$$F(n-1) = \int_0^{2\pi}\underbrace{\int_0^{\pi}...\int_0^{\pi}}_{n-2}\sin^{n-2}\theta_1\sin^{n-3}\theta_2...\sin\theta_{n-2}\, d\theta_1...d\theta_{n-1}$$

This integrand is also separable, so we define

$$\begin{aligned} G(n) &= \int_0^{\pi}\sin^n x\, dx \\ &= \int_0^{\pi}(1-\cos^2 x)\sin^{n-2}x\, dx \\ &= G(n-2) - \left(\frac{\cos x}{n-1}\sin^{n-1}x\Big|_0^{\pi} + \int_0^{\pi}\frac{\sin^n x}{n-1}\, dx\right) \\ &= G(n-2) - \frac{1}{n-1}G(n) \end{aligned}$$

[1] https://docsdrive.com/pdfs/ansinet/ajms/2011/66-70.pdf

Rearranging, we get

$$G(n) = \frac{n-1}{n} G(n-2) \tag{4.1}$$

To find the closed form expression of $G(n)$, we can express $G(n)$ using the double factorial as

$$G(n) = \begin{cases} \frac{(n-1)!!}{n!!}(\pi) & \text{if } n \equiv 0 \mod 2 \\ \\ \frac{(n-1)!!}{n!!}(2) & \text{if } x \equiv 1 \mod 2 \end{cases}$$

We use the property[2] that

$$x!! = \begin{cases} \frac{\Gamma(\frac{x}{2}+1)2^{\frac{x+1}{2}}}{\sqrt{\pi}} & \text{if } x \equiv 0 \mod 2 \\ \\ \Gamma(\frac{x}{2}+1)2^{\frac{x}{2}} & \text{if } x \equiv 1 \mod 2 \end{cases}$$

to conclude that $G(n) = \frac{\sqrt{\pi}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)}$. By separating the functions in the integrand of $F(n-1)$, we get

$$\begin{aligned} F(n-1) &= 2\pi \prod_{k=1}^{n-2} \frac{\sqrt{\pi}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)} \\ &= 2\pi \times \pi^{\frac{n-2}{2}} \times \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \times \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n-1}{2})} \times ... \times \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \end{aligned} \tag{4.2}$$

We can thus rewrite $Q(n)$ and k as follows:

$$Q(n) = \frac{F(n-1)}{k} \int_0^1 \frac{V_n(1-r)}{V_n(1)} r^{n-1} \, dr$$

$$k = F(n-1) \int_0^1 \frac{2C_n(r)}{V_n(1)} r^{n-1} \, dr$$

Using the formulas for C_n and V_n, this simplifies to

$$\begin{aligned} Q(n) &= \frac{\int_0^1 \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} (1-r)^n r^{n-1} \, dr}{2\int_0^1 \frac{\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n+1}{2})} I_n(r) r^{n-1} \, dr} \\ &= \frac{\sqrt{\pi}\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{n}{2}+1)} \times \frac{\int_0^1 (1-r)^n r^{n-1} \, dr}{\int_0^1 I_n(r) r^{n-1} \, dr} \end{aligned} \tag{4.3}$$

[2] https://mathworld.wolfram.com/DoubleFactorial.html

Due to the recursive nature of I_n, finding a closed-form expression for it is unfeasible. Thus, we leave the denominator of 4.3 in integral form and solve for the numerator:

$$\int_0^1 (1-r)^n r^{n-1}\,dr = 2\int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta \sin^{2n-1}\theta\,d\theta,\ r=\sin^2\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}}\sin^{2n-1}2\theta\frac{\cos 2\theta+1}{2}\,d\theta$$

$$= \frac{1}{2^{2n-1}}\left(\frac{\sin^{2n}2\theta}{4n}\Bigg|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\sin^{2n-1}2\theta\,d\theta\right)$$

$$= \frac{1}{2^{2n}}\int_0^{\pi}\sin^{2n-1}\phi\,d\phi,\ \phi=2\theta$$

Using the notation of $G(n)$ above, this equals

$$\frac{G(2n-1)}{2^{2n}} = \frac{\sqrt{\pi}\Gamma(n)}{2^{2n}\Gamma(n+\frac{1}{2})}$$

Using the Legendre duplication formula, we can simplify this as

$$\frac{\sqrt{\pi}\Gamma(n)^2}{2^{2n}2^{1-2n}\sqrt{\pi}\Gamma(2n)}$$

$$= \frac{\Gamma(n)^2}{2\Gamma(2n)}$$

$$= \frac{n!(n-1)!}{(2n)!}$$

With a similar principle, we can now simplify 4.3 as

$$Q(n) = \frac{\sqrt{\pi}\Gamma(\frac{n+1}{2})^2 n!(n-1)!}{2\sqrt{\pi}2^{-n}n!(2n)!\int_0^1 I_n(r)r^{n-1}\,dr}$$

$$= \frac{2^{n-1}\Gamma(\frac{n+1}{2})^2(n-1)!}{(2n)!\int_0^1 I_n(r)r^{n-1}\,dr} \quad \blacksquare$$

5 Conclusion

Using computational methods, the theoretical results above can be approximated to a relatively high degree of certainty.[3] As the results show, even though the 2-ball is already the optimal 2-dimensional shape to fit completely nested circles in, additional dimensions greatly reduce the likelihood of complete nesting for $(n-1)$-spheres in n-balls.

[3]https://github.com/warrenjch/area-test/blob/main/areatest.ipynb