

110A HW5

Warren Kim

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Question 1

Let R be a ring and $I \subseteq R$ be an ideal. Let $J \subseteq R$ be an ideal such that $I \subseteq J$, and let $\bar{J} \subseteq \bar{R} = R/I$ be an ideal.

1. Show that $\pi^{-1}(\pi(J)) = J$ and $\pi(\pi^{-1}(\bar{J})) = \bar{J}$. [Recall $\pi : R \rightarrow R/I$ is the canonical projection.]
2. Let $\bar{J} = \pi(J)$. Let $\pi : R \rightarrow R/I$ and $\phi : \bar{R} \rightarrow \bar{R}/\bar{J}$ be canonical projections. Show that $\ker(\phi \circ \pi) = J$.

Response

Proof: Let R be a ring and $I \subseteq R$ be an ideal. Let $J \subseteq R$ be an ideal such that $I \subseteq J$, and let $\bar{J} \subseteq \bar{R} = R/I$ be an ideal.

(1) $\pi^{-1}(\pi(J)) = J$: Let $a \in \pi^{-1}(\pi(J))$. Then by definition of the pre-image under π , there exists $x \in J$ such that $\pi(a) = \pi(x) \in \pi(J)$, or $a + I = x + I$, which implies that $a - x \in I \subseteq J$, so $a \in J$. Since a was arbitrary, $\pi^{-1}(\pi(J)) \subseteq J$. Now let $b \in J$. Then by definition, $\pi(b) = b + I$. Then, $\pi^{-1}(\pi(b)) = \pi^{-1}(b + I)$ but by definition of the pre-image, $\pi^{-1}(b + I) = b \in \pi^{-1}(\pi(J))$. Since b was arbitrary, $J \subseteq \pi^{-1}(\pi(J))$. Since we have $\pi^{-1}(\pi(J)) \subseteq J$ and $\pi^{-1}(\pi(J)) \supseteq J$, $\pi^{-1}(\pi(J)) = J$.

$\pi(\pi^{-1}(\bar{J})) = \bar{J}$: Let $a + I \in \pi(\pi^{-1}(\bar{J}))$. Then there exists $x \in R$ such that $x \in \pi^{-1}(\bar{J})$ and $\pi(x) = a + I \in \bar{J}$. Since a was arbitrary, $\pi(\pi^{-1}(\bar{J})) \subseteq \bar{J}$. Now let $b + I \in \bar{J}$. Then by definition, $b + I$ is in the image of J under π , so $b \in \pi^{-1}(\bar{J})$. Then $\pi(\pi^{-1}(b + I)) = \pi(b) = b + I \in \pi(\pi^{-1}(\bar{J}))$. Since $b + I$ was arbitrary, $\bar{J} \subseteq \pi(\pi^{-1}(\bar{J}))$. Since $\pi(\pi^{-1}(\bar{J})) \subseteq \bar{J}$ and $\pi(\pi^{-1}(\bar{J})) \supseteq \bar{J}$, $\pi(\pi^{-1}(\bar{J})) = \bar{J}$.

(2) Let $\bar{J} = \pi(J)$. Let $\pi : R \rightarrow R/I$ and $\phi : \bar{R} \rightarrow \bar{R}/\bar{J}$ be canonical projections. Take $a \in J$. Then $\phi \circ \pi(a) = \phi(\pi(a)) = \phi(a + I) = (a + I) + \bar{J}$, but since $a + I \in \bar{J}$, we have that $(a + I) + \bar{J} = 0 + \bar{J} \in \ker(\phi \circ \pi)$. Since a was arbitrary, $J \subseteq \ker(\phi \circ \pi)$. Now take any $b \in R$ such that $\phi \circ \pi(b) = 0 + \bar{J}$. Then, $(b + I) + \bar{J} = 0 + \bar{J}$. Then by definition, $b + I \in \bar{J} = \pi(J)$ by assumption. Then $b + I$ is the image of J under π , so $b \in \pi^{-1}(\bar{J}) = \pi^{-1}(\pi(J)) = J$. Since b was arbitrary, $\ker(\phi \circ \pi) \subseteq J$. Since $J \subseteq \ker(\phi \circ \pi)$ and $J \supseteq \ker(\phi \circ \pi)$, $J = \ker(\phi \circ \pi)$. \square

Question 2

Let $m, n \in \mathbb{Z}$ be nonzero. Show that $(m, n) = 1$ if and only if $\mathbb{Z}/mn \cong \mathbb{Z}/m \times \mathbb{Z}/n$.

Response

(\implies) Let $m, n \in \mathbb{Z}$ be nonzero such that $\gcd(m, n) = 1$. Let $R = \mathbb{Z}$, $I = (m)$, and $J = (n)$. Then $I + J = R$ since we can represent $(1) := (m)x + (n)y$ for some $x, y \in \mathbb{Z}$. Then $R/(I \cap J) \simeq (R/I) \times (R/J)$ but since $I + J = R$, $I \cap J = IJ$, so $R/IJ \simeq (R/I) \times (R/J)$. Substituting I, J, R , we get $\mathbb{Z}/mn \simeq \mathbb{Z}/m \times \mathbb{Z}/n$.

(\impliedby) Let $\mathbb{Z}/mn \simeq \mathbb{Z}/m \times \mathbb{Z}/n$. Suppose for the sake of contradiction that $d = \gcd(m, n) > 1$. Since $\mathbb{Z}/mn \simeq \mathbb{Z}/m \times \mathbb{Z}/n$, there exists a bijection $f : \mathbb{Z}/mn \rightarrow \mathbb{Z}/m \times \mathbb{Z}/n$. Consider $([m]_m, [n]_n) = ([0]_m, [0]_n) \in \mathbb{Z}/m \times \mathbb{Z}/n$. Then since f is bijective, there exists $x \in \mathbb{Z}/mn$ such that $f([x]_{mn}) = ([0]_m, [0]_n)$. Put $x := d \cdot \min\{m, n\}$. Without loss of generality, assume $n < m$. Then $f([x]_{mn}) = f([dn]_{mn}) = ([dn]_m, [dn]_n) = ([0]_m, [0]_n)$ since $d \mid m$ and $d \mid n$ by definition. Because $d < m$, $[dn]_{mn} = [x]_{mn} \neq [0]_{mn}$. Since $\ker(f) \neq \{0\}$, f is not injective and therefore not bijective, a contradiction.

Question 3

Let R be a (commutative) ring and $I_1, I_2, I_3 \subseteq R$ be ideals such that $I_1 + I_3 = R$ and $I_2 + I_3 = R$. Show that $(I_1 \cap I_2) + I_3 = R$.

Response

Let R be a commutative ring and $I_1, I_2, I_3 \subseteq R$ be ideals such that $I_1 + I_3 = R$ and $I_2 + I_3 = R$. **$(I_1 \cap I_2) + I_3 \subseteq R$:** Take $a \in (I_1 \cap I_2) + I_3$. Then since $I_1 + I_3 = R$ and $I_2 + I_3 = R$, $a \in R$ since $a \in I_1 + I_3 = R$ and $a \in I_2 + I_3 = R$.

$R \subseteq (I_1 \cap I_2) + I_3$: Pick any $x \in R$. Since $I_1 + I_3 = R$ and $I_2 + I_3 = R$, there exist $a \in I_1$, $b \in I_2$, $c, d \in I_3$ such that $a + c = 1$ and $b + d = 1$. Then

$$\begin{aligned} 1 &= (a + c)(b + d) \\ &= ab + ad + cb + cd \\ 1 &= ab + ((ad + cb) + cd) \end{aligned}$$

Then $ab \in I_1 \cap I_2$ because $a \in I_1$, we have $ab \in I_1$, and similarly, $b \in I_2$. Also, $(ad + cb) + cd \in I_3$ since $cd \in I_3$, so $ab + ((ad + cb) + cd) \in (I_1 \cap I_2) + I_3$. Then multiplying by x on both sides, we get $x(ab) + x((ad + cb) + cd) = x \in (I_1 \cap I_2) + I_3$.

Since $(I_1 \cap I_2) + I_3 \subseteq R$ and $(I_1 \cap I_2) + I_3 \supseteq R$, $(I_1 \cap I_2) + I_3 = R$.

Question 4

Let R be a (commutative) ring and let $I_1, I_2, I_3 \subseteq R$ be ideals. Suppose that $I_i + I_j = R$ for $i \neq j$. Let a_1, a_2, a_3 be any ideals. Show that there is some $x \in R$ such that

$$\begin{aligned}x &\equiv a_1 \pmod{I_1} \\x &\equiv a_2 \pmod{I_2} \\x &\equiv a_3 \pmod{I_3}.\end{aligned}$$

Response

Let R be a commutative ring and let $I_1, I_2, I_3 \subseteq R$ be ideals where $I_i + I_j = R$ for $i \neq j$. Let $a_1, a_2, a_3 \in R$. Then $I_1 + I_2 = R$, $I_1 + I_3 = R$, and $I_2 + I_3 = R$, so

$$\begin{aligned}(I_2 \cap I_3) + I_1 &= R \\(I_1 \cap I_3) + I_2 &= R \\(I_1 \cap I_2) + I_3 &= R\end{aligned}$$

from (Question 3). Then there exist

$$\begin{aligned}p &\in I_1, q \in I_2 \cap I_3 \text{ such that } p + q = 1_R \\r &\in I_2, s \in I_1 \cap I_3 \text{ such that } r + s = 1_R \\u &\in I_3, v \in I_1 \cap I_2 \text{ such that } u + v = 1_R\end{aligned}$$

Define $x := a_1(qu) + a_2(ps) + a_3(rv)$. Then

$$\begin{aligned}x &= a_1(qu) + a_2(ps) + a_3(rv) \equiv a_1(qu) \pmod{I_1} & ps &\in I_1, rv \in I_1 \cap I_3 \subseteq I_1 \\x &= a_1(qu) + a_2(ps) + a_3(rv) \equiv a_2(ps) \pmod{I_2} & r &\in I_2, qu \in I_2 \cap I_3 \subseteq I_2 \\x &= a_1(qu) + a_2(ps) + a_3(rv) \equiv a_3(rv) \pmod{I_3} & u &\in I_3, ps \in I_1 \cap I_3 \subseteq I_3\end{aligned}$$

so

$$\begin{aligned}x &\equiv a_1 \pmod{I_1} \\x &\equiv a_2 \pmod{I_2} \\x &\equiv a_3 \pmod{I_3}\end{aligned}$$

1 Midterm Review — Do NOT turn this stuff in.

Note: Just like all other homework problems, the problems above are fair game for the midterm!

1. Let R be a commutative ring. We say that $r \in R$ is nilpotent if there is some $n > 0$ such that $r^n = 0$.
 - (a) Let $P \subseteq R$ be a prime ideal. Show that if $r \in R$ is nilpotent, then $r \in P$.
 - (b) Let $N \subseteq R$ be the set of all nilpotent elements of R . Show that N forms an ideal.
2. Let R be any nonzero ring. Show that R has a subring that is isomorphic to \mathbb{Z} or \mathbb{Z}/n (for some positive integer $n > 0$).
3. Let R be a ring and let I_1, I_2, I_3, \dots be ideals such that the ideals are nested in an ascending manner: $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$.
Show that $\bigcup_{i=1}^{\infty} I_i$ is an ideal.
4. Let $m, n, d \in \mathbb{Z}$. Show the following are equivalent:
 - (a) There is a homomorphism $f : \mathbb{Z}/d \rightarrow \mathbb{Z}/m \times \mathbb{Z}/n$;
 - (b) We have $m|d$ and $n|d$.