C&EE 110 Homework 6

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Question 1

(a) Given that $X = 1$, determine the conditional pmf of Y.

(b) Given that two printers are in use on the first floor, what is the conditional pmf of the number of printers in use on the second floor?

(c) Use the result of part (b) to calculate the conditional probability $P(Y \leq 1|X = 2)$.

(d) Given that two printers are in use on the second floor, what is the conditional pmf of the number of printers in use on the first floor?

The answer should be rounded to four decimal places.

Response

(a)

$$P(Y = 0|X = 1) = \frac{P(X = 1, Y = 0)}{P(X = 1)} = \frac{0.07}{0.3} = \mathbf{0.2333}$$

$$P(Y = 1|X = 1) = \frac{P(X = 1, Y = 1)}{P(X = 1)} = \frac{0.18}{0.3} = \mathbf{0.6}$$

$$P(Y = 2|X = 1) = \frac{P(X = 1, Y = 2)}{P(X = 1)} = \frac{0.05}{0.3} = \mathbf{0.1667}$$

(b)

$$P(Y = 0|X = 2) = \frac{P(X = 1, Y = 0)}{P(X = 2)} = \frac{0.05}{0.5} = \mathbf{0.1}$$

$$P(Y = 1|X = 2) = \frac{P(X = 1, Y = 1)}{P(X = 2)} = \frac{0.15}{0.5} = \mathbf{0.3}$$

$$P(Y = 2|X = 2) = \frac{P(X = 1, Y = 2)}{P(X = 2)} = \frac{0.3}{0.5} = \mathbf{0.6}$$

(c)

$$P(Y \leq 1|X = 2) = 1 - P(Y = 2|X = 2) = 1 - 0.6 = 0.4$$

(d)

$$P(Y = 2|X = 0) = \frac{P(X = 1, Y = 0)}{P(Y = 2)} = \frac{0.03}{0.38} = \mathbf{0.0789}$$

$$P(Y = 2|X = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 2)} = \frac{0.05}{0.38} = \mathbf{0.1316}$$

$$P(Y = 2|X = 2) = \frac{P(X = 1, Y = 2)}{P(Y = 2)} = \frac{0.3}{0.38} = \mathbf{0.7895}$$

Question 2

(1) Find the mean of D.

(2) (i) Find the conditional probability density function of Y given $X = 1.2$.

(ii) Find the probability that the whole diameter is less than or equal to 4.8 mm given that the thickness is 1.2 mm.

Response

(1)

$$\begin{aligned}
\mu &= \int_{80.5}^{80.6} \int_{65.1}^{65.2} 100 \cdot \left(\frac{\pi x^2 y}{4} \right) \, dydx \\
&= \int_{80.5}^{80.6} 25\pi x^2 \int_{65.1}^{65.2} y \; dydx \\
&= \int_{80.5}^{80.6} 25\pi x^2 \left(\frac{1}{2} y^2 \Big|_{65.1}^{65.2} \right) dx \\
&= 162.875\pi \int_{80.5}^{80.6} x^2 \; dx \\
&= 162.875\pi \frac{1}{3} x^3 \Big|_{80.5}^{80.6} \\
\mu &= \mathbf{331997.832}
\end{aligned}$$

(2) (i)

$$\begin{aligned}
f_x(x) &= \int_4^5 \frac{4}{15} \left(x + \frac{1}{2} y \right) \; dy \\
&= \frac{4}{15} \int_4^5 \left(x + \frac{1}{2} y \right) \; dy \\
&= \frac{4}{15} \left(xy + \frac{1}{4} y^2 \right) \Big|_4^5 \\
&= \frac{4}{15} \left(x + \frac{9}{4} \right) \Big|_4^5 \\
&= \frac{4}{15} x + \frac{9}{15} \\
f_x(x) &= \frac{1}{15} (4x + 9)
\end{aligned}$$

So,

$$\begin{aligned}
f(y|x) &= \frac{f(x,y)}{f_x(x)} \\
&= \frac{\frac{4}{15} \left(x + \frac{1}{2} y \right)}{\frac{1}{15} (4x + 9)} \\
f(y|x = 1.2) &= \frac{\frac{4}{15} \left(1.2 + \frac{1}{2} y \right)}{\frac{1}{15} (4(1.2) + 9)} \\
f(y|x = 1.2) &= \mathbf{\frac{4.8 + 2y}{13.8}}
\end{aligned}$$

(ii)

$$f(y \leq 4.8 | x = 1.2) = \int_4^{4.8} \frac{4.8 + 2y}{13.8} \; dy = \frac{2.4}{13.8} y \Big|_4^{4.8} + \frac{1}{13.8} y^2 \Big|_4^{4.8} = \mathbf{0.7884}$$

Question 3

(a) If four residential fires are independently being reported on single day, what is the probability that two are in family homes, one is in an apartment, and one is another type of dwelling?

(b) If six residential fires are independently reported on a single day, what is the probability that three or more were reported from apartments?

(c) What is, in expectance, the maximum number of fires to which the fire department will be able to provide assistance without running out of budget?

Response

(a)

$$P(2,1,1) = \frac{4!}{2!1!1!} \cdot 0.73^2 \cdot 0.2 \cdot 0.07 = 0.0895$$

(b)

$$\begin{aligned} P(x \geq 3) &= 1 - P(x < 3) \\ &= 1 - \left(\frac{6!}{0!6!} \cdot 0.2^0 \cdot 0.8^6 + \frac{6!}{1!5!} \cdot 0.2^1 \cdot 0.8^5 + \frac{6!}{2!4!} \cdot 0.2^2 \cdot 0.8^4 \right) \\ P(x \geq 3) &= \mathbf{0.09888} \end{aligned}$$

(c)

$$\begin{aligned} Expectance &= \left\lfloor \frac{budget}{expected\ cost} \right\rfloor \\ expeceted\ cost &= (800 \cdot 0.73) + (500 \cdot 0.2) + (1000 \cdot 0.07) = 754 \\ budget &= 20000 \\ Expectance &= \left\lfloor \frac{20000}{754} \right\rfloor = \mathbf{26\ fires} \end{aligned}$$

Question 4

What is the probability density function of the time duration of your trip?

Response

Given $t = \frac{d}{v}$, we have: $t_1 = \frac{270}{60} = 4.5$, $t_2 = \frac{270}{30} = 9$. Then,

$$F(t) = \frac{60 - v}{30} = \frac{60 - \frac{270}{t}}{30} = 2 - \frac{9}{t}$$

so

$$f(t) = \frac{d}{dt}F(t) = \frac{9}{t^2}$$

Hence,

$$f(t) = \begin{cases} \frac{9}{t^2} & 4.5 \leq t \leq 9 \\ 0 & t < 4.5 \\ 1 & t > 9 \end{cases}$$