110A HW3

Warren Kim

Winter 2024

Question 1

Let R be a ring. Show that $1 = 0$ if and only if $R = \{0\}$.

Response

Proof: ($\implies$) Let R be a ring and suppose $1 = 0$. Then, for any $a \in R$, we can write $a = 1 \cdot a = a \cdot 1$. But since $1 = 0$, we have $a = 0 \cdot a = a \cdot 0 = 0$, so $a = 0$. Because a was arbitrary, $a = 0$ is the only element in R.
($\impliedby$) Let R be a ring and let it be defined by $R = \{0\}$. Then, because it's a ring, there exists an element $1_R \in R$ such that $1_R \cdot a = a \cdot 1_R = a$ for any $a \in R$. Because 0 is the only element in R, set $1_R = 0$. Then, since 0 is the only element in R, we have that $a = 0$, so $a \cdot 1_R = 1_R \cdot a = 0 = a = 0 \cdot a = a \cdot 0$. $\square$

Question 2

Let R be a ring, and consider the associated polynomial ring $R[x]$.

1. Show that R is commutative if and only if $R[x]$ is commutative.
2. Suppose R is commutative. Show that R is an integral domain if and only if $R[x]$ is an integral domain.

Response

Proof:

1. ($\implies$) Suppose R is a commutative ring. Then, consider the associated polynomial ring $R[x]$. Note that x is commutative with all $a \in R$; i.e. $ax = xa$. Then, suppose we have two elements $\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{m} b_j x^j \in R$ for some $n, m \in \mathbb{Z}_{>0}$. Then

$$\begin{aligned}\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) &= \sum_{i=0}^{n}\sum_{j=0}^{m} a_i b_j x^{i+j} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m} b_j a_i x^{j+i} && \text{addition in } \mathbb{Z} \text{ and } R \text{ are commutative} \\ &= \left(\sum_{j=0}^{m} b_j x^j\right)\left(\sum_{i=0}^{n} a_i x^i\right)\end{aligned}$$

so $R[x]$ is commutative.
($\impliedby$) Suppose $R[x]$ is a commutative ring. Then given two elements $\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{m} b_j x^j \in R$ for some $n, m \in \mathbb{Z}_{>0}$, we have that for any $i < n$ and $j < m$, $(a_i x^i)(b_j x^j) = (b_j x^j)(a_i x^i)$. Then $(a_i x^i)(b_j x^j) = a_i x^i b_j x^j = b_j a_i x^{i+j} = b_j x^j a_i x^i = (b_j x^j)(a_i x^i)$. So, $a_i b_j = b_j a_i$, and since $a_i, b_j \in R$, R must be commutative.

$\square$

Question 3

Prove the parts of Proposition 2.1 (in the notes) that were not proved in class.

Response

Question 4

Let R and S be rings, and let $f : R \to S$ be a ring homomorphism. Let $a, b \in R$. Prove the following:

1. $f(a - b) = f(a) - f(b)$.
2. If $a \in R$ is a unit, then $f(a)$ is a unit as well, with $f(a^{-1}) = f(a)^{-1}$.

Response

Question 5

Consider the Gaussian integers, given by $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$, where $i^2 = -1$. Consider the map $f : \mathbb{Z}[i] \to \mathbb{Z}[i]$ where $a + bi \mapsto a - bi$. Show f is an isomorphism.

Response

Question 6

Let R be a ring. We say that $a \in R$ is nilpotent if there is some integer n such that $a^n = 0$. Show that $1 + a$ is a unit.

Response

Question 7

We say that a ring R is a Boolean ring if, for every $a \in R$, we have $a^2 = a$.

1. Show that a Boolean ring R is commutative.
2. Suppose R is a Boolean ring and an integral domain. Show that $|R| = 2$. [Hint: show that any nonzero element must be 1.]

Response

Question 8

Let R and S be rings. Show that if R and S are isomorphic, then $R[x]$ and $S[x]$ are isomorphic.

Response