C&EE 110

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Discussion 5

Question 1

Given:

$$X \leq 31$$
$$\lambda = \frac{1}{\mu} = \frac{1}{44} = 0.0227$$

Then:

(a)

$$\begin{aligned} P(X \leq 31) &= \int_0^{31} \lambda e^{-\lambda x} dx \\ &= \int_0^{31} \frac{1}{44} e^{-\frac{y}{44}} dx \\ &= 1 - e^{\frac{31}{44}} \\ P(X \leq 31) &= 0.5057 \end{aligned}$$

(b) $V(X) = \frac{1}{\lambda^2} = 44^2 = 1936$

Question 2

Given:

$$\begin{aligned} \alpha &= 11.6 \\ \beta &= 2.2 \\ \textit{Range: } &[3.5, 25] \end{aligned}$$

Weibull Distribution:

$$\begin{aligned} E(X) &= \alpha \Gamma \left(1 + \frac{1}{\beta}\right) \\ V(X) &= \alpha^2 \left\{ \Gamma \left(1 + \frac{2}{\beta}\right) - \left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \right\} \\ f(X, \alpha, \beta) &= \left(\frac{\beta}{\alpha}\right) \left(\frac{x}{\alpha}\right)^{\beta - 1} e^{-\left(\frac{x}{\alpha}\right)^\beta} \\ F(X, \alpha, \beta) &= 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta} \end{aligned}$$

(a) $E(X) = 11.6 \cdot \Gamma\left(1 + \frac{1}{2.2}\right) = 10.27$ m/s.

(b) $V(X) = 11.6^2 \cdot \left\{\Gamma\left(1 + \frac{2}{2.2}\right) - \left[\Gamma\left(1 + \frac{1}{2.2}\right)\right]^2\right\} = 23.91 \text{ m/s} \implies \sigma = \sqrt{V(X)} = 4.89$ m/s.

(c) $P(3.5 \leq X \leq 25) = F(X \leq 25) - F(X \leq 3.5) = \left(1 - e^{-\left(\frac{25}{11.6}\right)^{2.22}}\right) - \left(1 - e^{-\left(\frac{3.5}{11.6}\right)^{2.22}}\right) = 0.9284$

(d) $\beta = 1 \implies$ ***Exponential Distribution:***

(i) $E(X) = \frac{1}{\lambda} = \alpha = 11.6$ m/s.

(ii) $V(X) = \frac{1}{\lambda^2} = \alpha^2 = 134.56 \text{ (m/s)}^2$

(iii)

$$\begin{aligned} f(X, \lambda) &= \lambda e^{-\lambda x} \\ F(X, \alpha, \beta) &= 1 - e^{-\lambda x} \\ P(3.5 \leq X \leq 25) &= \left(1 - e^{(11.6)(25)}\right) - \left(1 - e^{(11.6)(3.5)}\right) = 0.624 \end{aligned}$$

Question 3

Given:

$$\mu = 0.35$$
$$\sigma = 0.2$$
$$T = 0.6$$

Log-Normal Distribution: Φ

$$\begin{aligned}
\sigma_T^2 &= \ln\left(\frac{\sigma^2}{\mu^2} + 1\right) \\
&= \ln\left(\frac{0.2^2}{0.35^2} + 1\right) \\
\sigma_T^2 &= 0.283 \\
\mu_T^2 &= \ln\mu - \frac{1}{2}\sigma_T^2 \\
&= \ln 0.35 - \frac{1}{2}(0.283) \\
\mu_T^2 &= -1.19
\end{aligned}$$

(a) $P(T < 0.6) = \Phi\left(\frac{\ln 0.6 - \mu_T}{\sigma_T}\right) = \Phi(1.276) = 0.898.$

(b)

$$\begin{aligned}
P(T > t^*) = 0.95 &\iff 1 - P(T \leq t^*) = 0.95 \implies \Phi\left(\frac{\ln T^* - \mu_T}{\sigma_T}\right) = 0.05 \\
\implies -1.65 &= \frac{\ln T^* - \mu_T}{\sigma_T} \\
&= \exp(-1.65 \cdot \sqrt{0.283} - 1.19) \\
P(T > t^*) &= 0.126 \text{ s}
\end{aligned}$$

Question 4

Given:

$$\mu_X = 100 \ kips$$
$$\sigma_X = 10 \ kips$$
$$\mu_Y = 40 \ kips$$
$$\sigma_Y = 10 \ kips$$

Assumed Normal Distribution:

(a)

$$\mu_Z = \mu_X + \mu_Y = 100 + 40 = 140$$
$$\sigma_Z = \sqrt{\sigma_x^2 + \sigma_Y^2} = \sqrt{10^2 + 10^2} = 14.14 \ kips$$

(b)

$$
\begin{aligned}
P(Z \geq z^*) = 0.05 \iff 1 - P(Z < z^*) &= 0.05 \\
&\implies 1 - P(Z < z^*) = 1 - \Phi\left(\frac{z^* - \mu_Z}{\sigma_Z}\right) \\
&= 0.05 \\
&\implies \Phi\left(\frac{z^* - \mu_Z}{\sigma_Z}\right) = 0.95 \\
&\implies z^* = (1.65)(14.14) + 140 = 163.33 \ \textit{kips}
\end{aligned}
$$

(c) $P(Z < 170) = \Phi\left(\frac{170-140}{14.14}\right) = \Phi(2.12) = 0.983.$

Discussion 6

Question 1

Given:

$$d = \sqrt{r^2 - 40^2}$$

We have that $P(R \leq r) = F(r) = \frac{\textit{length of the fault within distance } r}{\textit{total length of the fault}} = \frac{2\sqrt{r^2-40^2}}{60}$ Then, we have

$$F(r) = \begin{cases} \frac{2\sqrt{r^2-40^2}}{60} & 40 \leq r \leq 50 \\ 1 & r > 50 \\ 0 & r < 40 \end{cases}$$

and

$$f(r) = \begin{cases} \frac{r}{30\sqrt{r^2-40^2}} & 40 \leq r \leq 50 \\ 1 & r > 50 \\ 0 & r < 40 \end{cases}$$

Question 2

Given:

$$f(y_1, y_2) = \begin{cases} k(y_1 + y_2) & 1 \leq y_1 \leq 2,\ 4 \leq y_2 \leq 5 \\ 0 & \textit{else} \end{cases}$$

PDF:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y_1, y_2) dy_1 dy_2 = 1$$

(a)

$$\int_1^2 \int_4^5 f(y_1, y_2) dy_1 dy_2 = \int_1^2 \int_4^5 k(y_1 + y_2) dy_1 dy_2 = 6k$$
$$\implies k = \frac{1}{6}$$

(b)

$$\begin{aligned} P(Y_1 \leq y_1, Y_2 \leq y_2) &= \int_1^{y_1} \int_4^{y_2} \frac{1}{6}(t_1 + t_2) dt_1 dt_2 \\ &= \frac{1}{6} \int_1^{y_1} \left[t_1 t_2 + \frac{1}{2} t_2^2 \right] \Bigg|_4^{y_2} dy_1 \\ &= \frac{1}{6} \int_1^{y_1} \left[t_1(y_2 - 4) + \frac{1}{2} y_2^2 - \frac{16}{2} \right] dy_1 \\ &= \frac{1}{6} \left[\frac{1}{2} t_1^2 (y_2 - 4) + \frac{1}{2} y_2^2 - 8 \right] \Bigg|_1^{y_1} \\ P(Y_1 \leq y_1, Y_2 \leq y_2) &= \frac{1}{12}(y_1 - 1)(y_2 - 4)(y_1 + y_2 + 5) \qquad\qquad 1 \leq y_1 \leq 2,\ 4 \leq y_2 \leq 5 \end{aligned}$$

(c) $P(Y_1 \leq 1.5, Y_2 \leq 4.5) = (1.5 - 1)(4.5 - 4)(1.5 + 4.5 + 5) = 0.229$

(d)

$$
\begin{aligned}
f_1(y_1) &= \int_4^5 \frac{1}{6}(y_1 + y_2)dy_2 \\
&= \frac{1}{6}\int_4^5 (y_1 + y_2)dy_2 \\
&= \frac{1}{6}\left[y_1 + \frac{25}{2} - \frac{16}{2}\right] \\
f_1(y_1) &= \frac{1}{6}\left(y_2 + \frac{3}{2}\right) & 4 \leq y_2 \leq 5
\end{aligned}
$$

(e)

$$
\begin{aligned}
P(Y_1 \leq 1.5 | Y_2 \geq 4.5) &= \frac{P(Y_1 \leq 1.5 \cap Y_2 \geq 4.5)}{P(Y_1 \geq 4.5)} \\
&= \frac{\int_1^{1.5}\int_{4.5}^5 \frac{1}{6}(y_1 + y_2)dy_2 dy_1}{\int_{4.5}^5 \frac{1}{6}\left(y_2 + \frac{3}{2}\right)dy_2} \\
P(Y_1 \leq 1.5 | Y_2 \geq 4.5) &= \frac{0.25}{0.52} = 0.48
\end{aligned}
$$

Discussion 7

Question 1

Given:

$$
\begin{aligned}
n &= 50 \\
\mu &= 12.68 \\
\sigma &= 6.83
\end{aligned}
$$

Assumed Normal Distribution:

$$P(|\overline{X} - \mu| \leq 0) = 0.95$$

$$P\left(\mu - z_{\frac{\alpha}{2}} \cdot \sigma_{\overline{X}} \leq \mu_{\overline{X}} \leq \mu + z_{\frac{\alpha}{2}} \cdot \sigma_{\overline{X}}\right) = 1 - \alpha$$

where the confidence interval is $1 - \alpha = 0.95$ with significance level 0.05.

(a)

$$
\begin{aligned}
z_{\frac{\alpha}{2}} &= z_{\frac{0.05}{2}} = z_{0.025} = 1.96 && \text{from z-table} \\
\sigma_{\overline{X}} &= \frac{\sigma}{\sqrt{n}} = \frac{6.83}{\sqrt{50}} = 0.9695
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&P\left(\mu - z_{\frac{\alpha}{2}} \cdot \sigma_{\overline{X}} \leq \mu_{\overline{X}} \leq \mu + z_{\frac{\alpha}{2}} \cdot \sigma_{\overline{X}}\right) = P\left(12.68 - (1.96)(0.9695) \leq \mu_{\overline{X}} \leq 12.68 - (1.96)(0.9695)\right) = 0.95 \\
&\implies 95\% \textit{ confidence interval is } 12.68 \pm 1.89 \iff (10.79, 14.57)
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(11.09 \leq \mu_{\overline{X}} &\leq 14.27) = 1 - \alpha \\
\implies 14.27 &= \mu + z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}} \\
&= 12.68 + z_{\frac{\alpha}{2}} \cdot 0.9695 \\
z_{\frac{\alpha}{2}} &= 1.646 \\
\implies \frac{\alpha}{2} &= 0.05 \implies \alpha = 0.1, \ \textit{confidence interval is } 100(1-\alpha) = 90\%
\end{aligned}
$$

Question 2-4: See Canvas

Discussion 8/9: See Canvas

MLE

Note: $\ln \iff \log$, $\exp x = e^x$

Given

$$L = \prod_{i=1}^{N} p(x_i) = \Pi_{i=1}^{N} \frac{\lambda^{x_i} \exp(-\lambda)}{x_i!}$$

$$
\begin{aligned}
LL &= \log\left\{\prod_{i=1}^{N} \frac{\lambda^{x_i} \exp(-\lambda)}{x_i!}\right\} \\
&= \sum_{i=1}^{N} \log\left(\lambda^{x_i} \exp(-\lambda)\right) - \log(x_i!) && \log\left(\frac{x}{y}\right) = \log(x) - \log(y) \\
&= \sum_{i=1}^{N} -\lambda + (x_i)\log(\lambda) - \log(x_i!) && \log(xy) = \log(x) + \log(y), \ \log(x^a) = a\log(x), \ \log(e^x) = x \\
LL &= -N\lambda + \sum_{i=1}^{N} [(x_i)\log(\lambda) - \log(x_i!)]
\end{aligned}
$$

Then $\frac{dLL}{d\lambda}$ is

$$\frac{dLL}{d\lambda} = -N\lambda$$