Partial Order: $\forall x,y,z \in A$: Reflexive: $x\mathcal{R}x$, Anti-symmetric: $x\mathcal{R}y, y\mathcal{R}x \implies x=y$, Transitive: $x\mathcal{R}y, y\mathcal{R}z \implies x\mathcal{R}z$. **Total:** $\forall x,y \in A, x\mathcal{R}y \vee y\mathcal{R}x$
Equivalence Relation: $\forall x,y,z \in A$: Reflexive: $x\mathcal{R}x$, Symmetric: $x\mathcal{R}y = y\mathcal{R}x$, Transitive: $x\mathcal{R}y, y\mathcal{R}z \implies x\mathcal{R}z$. **Eq. Class:** $[x] := \{y \in A : x \sim y\}$
Induction: Base step: ***(i)*** *P_1 is true.* **Inductive Hypothesis:** ***(ii)*** *Assume P_n is true for some $n \in \mathbb{N}$. Prove P_{n+1} is true.* Then, P_n is true $\forall n \in \mathbb{N}$. **Ordered Fields:** A field with a partial order ($\leq$) s.t.: ***(i)*** If $x,y,z \in \mathbb{F},\ x<y \implies x+z<x+y$, ***(ii)*** $x,y \in \mathbb{F},\ x,y>0 \implies xy>0$
Algebraic Number: a is algebraic if it solves $c_nx^n + \cdots + c_1x + c_0 = 0$ for some $n \in \mathbb{N}, c_0, c_n \in \mathbb{Z}, c_n \neq 0$ (e.g. $\sqrt[n]{2}$. ***Note:*** $\mathbb{Q} \subset \{algebraic\ numbers\}$)
RZT: *Suppose $c_0, \cdots, c_n \in \mathbb{Z},\ r \in \mathbb{Q}$ satisfies $c_nr^n + \cdots + c_1r + c_0 = 0$ for some $n \in \mathbb{N},\ c_n \neq 0$. Let $r = \frac{c}{d}, c, d \in \mathbb{Z}, d \neq 0$, be coprime. Then $c,\ d$ divides c_0, c_n.*
LUBP: Given $A \subseteq \mathbb{E}$ where $\mathbb{E}$ is an ordered set, $\exists \sup A \in \mathbb{E} \iff A \neq \emptyset,\ A \subseteq \mathbb{E}$, A is bounded above. $\sup A := \alpha,\ \exists \alpha, \beta \in \mathbb{E}$ s.t. $\forall a \in A,\ a \leq \alpha \leq \beta$.
GLBP: Given $A \subseteq \mathbb{E}$ where $\mathbb{E}$ is an ordered set, $\exists \inf A \in \mathbb{E} \iff A \neq \emptyset,\ A \subseteq \mathbb{E}$, A is bounded below. $\inf A := \alpha,\ \exists \alpha, \beta \in \mathbb{E}$ s.t. $\forall a \in A,\ \beta \leq \alpha \leq a$.
Archemedian Property: If $y \in \mathbb{R},\ x>0$, then $\exists n \in \mathbb{N}$ s.t. $n \cdot x > y$. Put $x=1 : \exists n \in \mathbb{N}$ s.t. $n>y$. Put $y=1 : \exists n \in \mathbb{N}$ s.t. $n \cdot x > 1 \rightsquigarrow x > \frac{1}{n} > 0$.
Density of $\mathbb{Q}$ in $\mathbb{R}$: $\forall x,y \in \mathbb{R} : x<y, \exists p \in \mathbb{Q} : x<p<y$
Sequence: A function $f : \mathbb{N} \to \mathbb{R} \iff n \mapsto f(n) \iff n \mapsto f_n$ e.g. $(1, \frac{1}{2}, \frac{1}{3}, \cdots),\ x_n = \frac{1}{n}\ \forall n \in \mathbb{N},\ \{x_n : n \in \mathbb{N}\},\ (x_n)_{n=1}^{\infty},\ (x_n)_{n \in \mathbb{N}}$
Convergent: A sequence (x_n) converges to $x \in \mathbb{R}$ if: $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n > N,\ |x_n - x| < \varepsilon$. We write $(x_n) \to x$ as $n \to \infty$ or $\lim_{n \to \infty} x_n := x$, where x is the **limit** of (x_n).
Divergent: A sequence that does not **converge**.
Absolute Value: $|x| = \{x \text{ if } x \geq 0\},\ \{-x \text{ if } x<0\} \implies |x| \geq 0$. ***(i)*** $|xy| = |x||y|$, ***(ii)*** $|x-y| \leq z \iff z \leq x-y \leq z \iff y-z \leq x \leq y+z$
Triangle Inequality: $|x+y| \leq |x|+|y| \implies |x-y| = |x+(-z+z)-y| \leq |x-z|+|z-y|\ \forall x,y,z \in \mathbb{R}$.
Unique Limits: $x_n \to x,\ x_n \to y \implies x=y$. $|x-y| = |x+(-x+x)-y| \leq |x_n-x|+|x_n-y| = \varepsilon$ if $|x_n-x|, |x_n-y| \leq \frac{\varepsilon}{2}$.
Algebraic Limit Theorem: $x_n \to x, y_n \to y$: ***(i)*** $ax_n \to ax$, ***(ii)*** $x_n \pm y_n \to x \pm y$, ***(iii)*** $x_n \cdot y_n \to x \cdot y$ ***(iv)*** $\frac{x_n}{y_n} \to \frac{x}{y}, y \neq 0$

Prove $\inf S \leq \sup S$:

Proof. Since $S \neq \emptyset$, $S \subseteq \mathbb{R}$, S is bounded above and below, $\inf S, \sup S$ exist. Since $S \neq \emptyset$, $\exists s \in S$. By definiation, $\inf S \leq s \leq \sup S$ for all $s \in S$. Taking the extremes of the inequality, we get $\inf S \leq \sup S$. □

What if $\inf S = \sup S$? If $\alpha = \inf S = \sup S$, then we know S contains only one element so $\inf S \leq s \leq \sup S \implies \alpha \leq s \leq \alpha \implies s = \alpha$.

Let S and T be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$. Prove $S \subseteq T \implies \inf T \leq \inf S \leq \inf S \leq \sup T$:

Proof. Since both $S, T \neq \emptyset$, $S, T \subseteq \mathbb{R}$, and bounded, $\inf S, \inf T, \sup S, \sup T$ exist. Then, since $S \subseteq T$, $\forall s \in S, s \in T$. Since $\forall t \in T, t \leq \sup T$, $\sup T$ is an upper bound for S. Since $\sup S$ is the *least* upper bound by definition, we have that $\sup S \leq \sup T$. Since $\forall t \in T, \inf T \leq t$, we have that $\inf T$ is a lower bound for S. Since $\inf S$ is the *greatest* lower bound by definition, we have that $\inf T \leq \inf S$. Note that since $S \neq \emptyset$, $\forall s \in S$, $\inf S \leq s \leq \sup S$, so we get the following inequality: $\inf T \leq \inf S \leq \sup S \leq \sup T$. □