

# Problem Set 2

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## Question 2

Prove that  $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$  for all  $n \in \mathbb{N}$ .

### Response

*Proof.* Let  $P_n$  read " $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$  for all  $n \in \mathbb{N}$ ".

**Base case:**  $P_1$  reads " $1^3 = 1^2$ ". Clearly,  $1 = 1$  so  $P_1$  holds true.

**Inductive Hypothesis:** Assume  $P_n$  holds true for an arbitrary  $n \in \mathbb{N}$ . We want to show that  $P_{n+1}$  is true.

$$\begin{aligned}
 1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= (1^3 + 2^3 + \cdots + n^3) + (n+1)^3 \\
 &= (1 + 2 + \cdots + n)^2 + (n+1)^3 && \text{from } P_n \\
 &= \left( \frac{n(n+1)}{2} \right)^2 + (n+1)^3 && \text{from class, we proved that } \sum_{i=1}^n i = \frac{n(n+1)}{2} \\
 &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
 &= \frac{1}{4} \left[ n^2(n+1)^2 + 4(n+1)^3 \right] \\
 &= \frac{1}{4} \left[ (n+1)^2(n^2 + 4(n+1)) \right] \\
 &= \frac{1}{4} \left[ (n+1)^2(n^2 + 4n + 4) \right] \\
 &= \frac{1}{4} \left[ (n+1)^2(n+2)^2 \right] \\
 &= \frac{(n+1)^2(n+2)^2}{4} \\
 &= \left( \frac{(n+1)(n+2)}{2} \right)^2
 \end{aligned}$$

$$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = (1 + 2 + \cdots + n + (n+1))^2 \quad \text{from class, we proved that } \frac{n(n+1)}{2} = \sum_{i=1}^n i$$

By the principle of mathematical induction, since we proved that  $P_{n+1}$  holds true for an arbitrary  $n \in \mathbb{N}$ ,  $P_n$  holds true for all  $n \in \mathbb{N}$ .  $\square$

## Question 6 part (b), (e), (f)

Let  $(\mathbb{F}, +, \cdot, \leq)$  be an ordered field (not necessarily  $\mathbb{Q}$  or  $\mathbb{R}$ !) and for any  $x \in \mathbb{F}$ , define

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad (1)$$

This is called the *absolute value* function. Notice that  $|x| \geq 0$  for every  $x \in \mathbb{F}$ .

- (b) Let  $a \in \mathbb{F}$  such that  $a \geq 0$ . Show that for  $x, y \in \mathbb{F}$ ,  $|x - y| \leq a$  if and only if  $y - a \leq x \leq y + a$ .
- (e) Let  $x, y \in \mathbb{R}$ . Prove that if for any  $\varepsilon > 0$ ,  $x \leq y + \varepsilon$ , then  $x \leq y$ . Show that we can also replace  $x \leq y + \varepsilon$  with  $x < y + \varepsilon$  and obtain  $x \leq y$ .
- (f) Let  $x, y \in \mathbb{R}$ . Prove that  $x = y$  if and only if for any  $\varepsilon > 0$ , we have  $|x - y| < \varepsilon$ .

## Response

(b) *Proof.*  $\implies$  There are two cases:

**Case I:**  $0 \leq x - y$ . Then,  $|x - y| = x - y$ , so

$$\begin{aligned} x - y &\leq a \\ x &\leq y + a \end{aligned}$$

**Case II:**  $x - y < 0$ . Then,  $|x - y| = -(x - y) = y - x$ , so

$$\begin{aligned} y - x &\leq a \\ x &\geq y - a \end{aligned}$$

so, we have that  $y - a \leq x \leq y + a$ .

$\Leftarrow$  There are two cases:

**Case I:**  $x \leq y + a$ . Note that if  $0 \leq x - y \implies |x - y| = x - y$ .

$$\begin{aligned} x &\leq y + a \\ a &\geq x - y \\ a &\geq |x - y| \end{aligned}$$

**Case II:**  $y - a \leq x$ . Note that if  $x - y < 0 \implies |x - y| = -(x - y)$ .

$$\begin{aligned} y - a &\leq x \\ a &\geq y - x \\ a &\geq -(x - y) \\ a &\geq |x - y| \end{aligned}$$

So,  $|x - y| \leq a$ .

In both cases, we have that  $|x - y| \leq a$ . Therefore,  $|x - y| \leq a \iff y - a \leq x \leq y + a$ .  $\square$

- (e) *Proof.* Assume by contradiction that  $y < x$ . Then,  $0 < x - y$ . Fix  $\varepsilon = \frac{1}{2}(x - y)$ . Clearly,  $0 < \frac{1}{2}(x - y)$  from our assumption. Then,

$$\begin{aligned} \frac{1}{2}(x - y) &< x - y \\ \varepsilon &< x - y \end{aligned}$$

which is a contradiction to the statement that  $x - y \leq \varepsilon$ . Therefore, if for any  $\varepsilon > 0$ ,  $x \leq y + \varepsilon$ , then  $x \leq y$ .  $\square$

*Proof.* Assume by contradiction that  $y < x$ . Then,  $0 < x - y$ . Fix  $\varepsilon = \frac{1}{2}(x - y)$ . Clearly,  $0 < \frac{1}{2}(x - y)$  from our assumption. Then,

$$\begin{aligned}\frac{1}{2}(x - y) &< x - y \\ \varepsilon &< x - y\end{aligned}$$

which is a contradiction to the statement that  $x - y < \varepsilon$ . Therefore, if for any  $\varepsilon > 0$ ,  $x < y + \varepsilon$ , then  $x \leq y$ .  $\square$

(f) *Proof.*  $\implies$  Let  $x = y$ . We want to prove that  $|x - y| < \varepsilon$ .  $x = y \implies x - y = 0$ . Then,  $|x - y| = |0| = 0$  by definition of the *absolute value* function. Substituting  $|x - y| = 0$ , we get  $|x - y| < \varepsilon = 0 < \varepsilon$ . Clearly, for any  $\varepsilon > 0$ ,  $0 < \varepsilon$  holds true.

$\Leftarrow$  Assume by contradiction that  $x \neq y$ . Then,  $0 \leq |x - y|$  by definition of the *absolute value* function. Now take  $\varepsilon = \frac{1}{2}|x - y|$ . Clearly,  $0 < \frac{1}{2}|x - y| < |x - y|$ . Then, we have  $\frac{1}{2}|x - y| < |x - y| \implies \varepsilon < |x - y|$ , which is a contradiction to the statement for any  $\varepsilon > 0$ ,  $|x - y| < \varepsilon$ .  $\square$

### Question 13 part (a)

Assume  $\alpha \in \mathbb{R}$  is an upper bound for a set  $A \subseteq \mathbb{R}$ , which is a non-empty and bounded above. Prove that  $\alpha = \sup A$  if and only if for every  $\varepsilon > 0$ , there exists an  $a \in A$  such that  $\alpha - \varepsilon \leq a$ .

#### Response

*Proof.*  $\implies$  Let  $\alpha = \sup A$ . Assume by contradiction that  $\exists \varepsilon > 0$  such that  $\forall a \in A$ , we have  $a < \alpha - \varepsilon$ . Then  $\alpha - \varepsilon$  is an upper bound for  $A$ . But  $\alpha - \varepsilon < \alpha$ , which is a contradiction to the statement that  $\alpha$  is the *least* upper bound for  $A$ . Therefore,  $\forall \varepsilon > 0, \exists a \in A$  such that  $\alpha - \varepsilon \leq a$ .

$\Leftarrow$  Assume  $\forall \varepsilon > 0$ , there exists some  $a \in A$  such that  $\alpha - \varepsilon \leq a$ . Assume by contradiction that  $\alpha \neq \sup A$ . Since  $\sup A$  is the least upper bound for  $A$ , we have that  $\sup A < \alpha$  since  $\alpha$  is an upper bound by the problem statement. Let  $\varepsilon = \alpha - \sup A$ . Clearly, since  $\sup A < \alpha$ ,  $0 < \varepsilon$ . Then, there exists some  $a \in A$  such that  $\alpha - \varepsilon \leq a$ . Substituting  $\varepsilon = \alpha - \sup A$ , we have  $\alpha - (\alpha - \sup A) \leq a \implies \sup A \leq a$  which is a contradiction to the statement that  $\sup A$  is a supremum for  $A$ .  $\square$

## Question 14

Assume that  $A, B$  are nonempty subsets of  $\mathbb{R}$  that are bounded above and  $A \subseteq B$ . Show that  $\sup A \leq \sup B$ .

### Response

*Proof.* Note that  $B \subseteq \mathbb{R}$ , it is non-empty, and it is bounded above. Therefore, by definition of the supremum,  $\sup B$  exists. We now want to show that  $\sup A$  exists. Since  $A \subseteq B$ , by the transitive property of the subset relation,  $A \subseteq \mathbb{R}$ . By the problem statement,  $A$  is also non-empty and bounded above. Therefore, by definition of the supremum,  $\sup A$  exists. Note that since  $A \subseteq B$ , we have  $\forall a \in A, a \in B \implies \forall a \in A, a \leq \sup B$ . So,  $\sup B$  is an upper bound for  $A$ . Since  $\sup A$  is the *least* upper bound for  $A$ , by the definition of the supremum,  $\sup A \leq \sup B$ .  $\square$