110A HW6

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Question 1

Consider $\mathbb{Z}$, and let $p \in \mathbb{Z}$ be nonzero. Show that (p) is a prime ideal if and only if p is prime.

Response

Proof: Let $p \in \mathbb{Z}$ be nonzero.
$(\implies)$ Suppose that (p) is a prime ideal. Consider $ab \in (p)$. This means that $p \mid ab$ since we can represent $ab = pr$ for some $r \in \mathbb{Z}$. If $ab \in (p)$, then by definition either $a \in (p)$ or $b \in (p)$. If $b \in (p)$, then we are done, so suppose not. Then $a \in (p)$; that is, $p \mid a$. Since the following two statements

1. p is prime.
2. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

are equivalent and $a, b \in \mathbb{Z}$ were arbitrary, p is prime.

$(\impliedby)$ Suppose that p is prime. Suppose $p \mid ab$. Then by definition, either $p \mid a$ or $p \mid b$. Without loss of generality, suppose $p \mid a$ and consider $(p) \subseteq \mathbb{Z}$. Then $ab \in (p)$ since $p \mid ab$. But since $p \mid a$, $a \in (p)$. Since $a, b \in \mathbb{Z}$ were arbitrary, (p) is a prime ideal.

Since we proved both directions, (p) is a prime ideal if and only if p is prime. $\square$

Question 2

Let $R = \mathbb{Z}/1024$, and consider the principal ideal $I = ([2]) \subseteq R$. Show that I is maximal.

Response

Proof: Let $R = \mathbb{Z}/1024$ and consider the principal ideal $I = ([2]) \subseteq R$. Then $1 \notin I$, so $I \subsetneq R$ is a proper ideal. Note that $([2])$ contains all even elements of $\mathbb{Z}/1024$. Suppose we have some ideal $J \subseteq R$ such that $J \supsetneq I$. Then there exists $[a] \in J$ such that a is odd. Since J contains I, $[2] \in J$. Then $[a] - [2] \in J$ is also odd. Then we have that $[a] - [2q] \in J$ for $q \in \mathbb{Z}$. Since a is odd, we can represent $a := 2k + 1$ for some $k \in \mathbb{Z}$. Put $k := q$. Then $[a] - [2q] = [2q + 1 - 2q] = [1] \in J$. Since $[1] \in J$, this implies that $J = R$. Thus, I is maximal. $\square$

Question 3

Let $f : R \to S$ be surjective, and let $P \subseteq S$ be a prime ideal. Show that $f^{-1}(P) \subseteq R$ is a prime ideal.

Response

Question 4

Let $f : R \to S$ be surjective, and let $M \subseteq S$ be a maximal ideal. Shw that $f^{-1}(M) \subseteq R$ is maximal.

Response