C&EE 110 Homework 4

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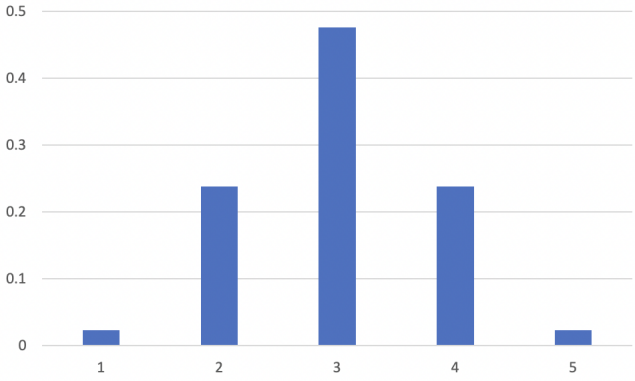
May 10, 2023

Problem 1

(a) Y has a **hypergeometric probability distribution** since we remove elements from the sample size without replacing them.

(b)

$$P(Y = y) = \frac{\binom{5}{y}\binom{5}{4-y}}{\binom{10}{4}}, \; y = 0, 1, 2, 3, 4$$



y	P(y)
0	0.02380952
1	0.23809524
2	0.47619048
3	0.23809524
4	0.02380952

(c)

$$P(Y \geq 2) = \sum_{y=2}^{4} P(y) = 0.738$$

Problem 2

(a)

$$\begin{aligned} p(y) &= \frac{\lambda^y}{y!}e^{-\lambda} && \lambda = 5 \cdot 2 = 10,\ y \leq 6 \\ &= \sum_{y=0}^{6} \frac{10^y}{y!}e^{-10} \\ p(y) &= 0.130 \end{aligned}$$

(b)

$$\begin{aligned} p(y) &= \frac{\lambda^y}{y!}e^{-\lambda} && \lambda = 5,\ y < 3 \\ &= \sum_{y=0}^{2} \frac{10^y}{y!}e^{-10} \\ p(y) &= 0.125 \end{aligned}$$

(c)

$$m(t) = \left(\frac{1}{6}\right)e^t + \left(\frac{2}{6}\right)e^{2t} + \left(\frac{3}{6}\right)e^{3t}$$

(*i*)

$$E(Y) = m'(0) = \left(\frac{1}{6}\right)e^t + \left(\frac{4}{6}\right)e^{2t} + \left(\frac{9}{6}\right)e^{3t} = \frac{7}{3}$$

(*ii*)

$$\begin{aligned} V(Y) &= E'(Y^2) - [E(Y)]^2 \\ &= m''(0) - [m'(0)]^2 \\ &= \left[\left(\frac{1}{6}\right)e^t + \left(\frac{8}{6}\right)e^{2t} + \left(\frac{27}{6}\right)e^{3t}\right] - \left[\frac{7}{3}\right]^2 \\ V(Y) &= \frac{5}{9} \end{aligned}$$

(*iii*)

$$\begin{aligned} m(t) &= E(e^{tY}) \\ &= \sum_y e^{ty}p(y) \\ &= \frac{1}{6}e^t + \frac{2}{6}e^{2t} + \frac{3}{6}e^{3t} \\ &= \sum_y [1 + 2e^t + 3e^{2t} + \ldots]\frac{1}{6}e^t \end{aligned}$$

y	P(y)
1	1/6
2	1/3
3	1/2

Problem 3

(a)

$$\begin{aligned} 1 &= \int_0^\infty ce^{-4x}dx \\ &= \frac{c}{4} \\ c &= 4 \end{aligned}$$

(b)

$$\begin{aligned} \int_0^x ce^{-4x}dx &= \int_0^x 4e^{-4x}dx \\ &= -e^{-4x} - (-1) \\ F(x) &= \begin{cases} 0 & x < 0 \\ 1 - e^{-4x} & x \geq 0 \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} P(2 < X < 5) &= F(2 < x < 5) \\ &= F(5) - F(2) \\ &= [1 - e^{-4(5)}] - [1 - e^{-4(2)}] \\ &= e^{-8} - e^{-20} \\ P(2 < X < 5) &= 3.35 \times 10^{-4} \end{aligned}$$

(d)

$$\begin{aligned} F(X \leq 0) &= F(0) - F(-2) \\ &= \left[\frac{1}{2} + \frac{3}{32}\left(4(0) - \frac{(0)^3}{3}\right)\right] - \left[\frac{1}{2} + \frac{3}{32}\left(4(-2) - \frac{(-2)^3}{3}\right)\right] \\ &= \frac{1}{2} \end{aligned}$$

(e)

$$F(\phi_{0.5}) = 0.5 \implies \frac{1}{2} + \frac{3}{32}\left(4\phi_{0.5} - \frac{\phi_{0.5}^3}{3}\right) = 0.5 \implies \phi_{0.5} = 0$$

(f)

$$\begin{aligned} f(x) = F'(x) &= \frac{3}{32}(4 - x^2) \\ f(1) &= \frac{1}{2} + \frac{3}{32}(4 - 1) = \frac{9}{32} \end{aligned}$$