Problem Set 5

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Question 3

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series. Show that:

(a) $\sum_{n=1}^{\infty}(ax_n)$ converges and $\sum_{n=1}^{\infty}(ax_n) = a\sum_{n=1}^{\infty} x_n$ for any $a \in \mathbb{R}$.

(b) Show that $\sum_{n=1}^{\infty}(x_n + y_n)$ converges and $\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$.

(c) Show that the assumption that *both* series converge is necessary for part (b).

(d) Is it true that if $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge then $\sum_{n=1}^{\infty} x_n y_n$ converges?

Response

(a) *Proof.* Given that $\sum_{n=1}^{\infty} x_n$ converges, let $s_n = \sum_{k=1}^{n} x_k$ be the sequence of partial sums of (x_n). Then, since (x_n) converges, we have that $\lim_{n\to\infty} s_n = x$. Let $t_n = \sum_{k=1}^{n} ax_k$. Then, we have:

$$\begin{aligned}
t_n &= \sum_{k=1}^{n} ax_k \\
&= (ax_1 + ax_2 + \cdots + ax_k) && \text{definition of summation} \\
&= a(x_1 + x_2 + \cdots + x_k) && \text{distributivity of } \mathbb{R} \\
&= a\sum_{k=1}^{n} x_k && \text{definition of summation} \\
t_n &= as_n && s_n = \sum_{k=1}^{n} x_k
\end{aligned}$$

So $t_n = \sum_{k=1}^{n} ax_k = a\sum_{k=1}^{n} x_k$. From above, we have that $\lim_{n\to\infty} s_n = x$ and $\lim_{n\to\infty} a = a$ (since a is constant), so both sequences converge. Then by the Algebraic Limit Theorem, we have

$$\lim_{n\to\infty} a \cdot \lim_{n\to\infty} s_n = \lim_{n\to\infty} a \cdot s_n = ax = \lim_{n\to\infty} s_n$$

Since the sequence of partial sums $t_n = \sum_{k=1}^{n} ax_k = a\sum_{k=1}^{n} x_k$ converges, $\sum_{n=1}^{\infty} ax_n = a\sum_{n=1}^{\infty} x_n$ converges. $\square$

(b) *Proof.* Let $s_n = \sum_{k=1}^{n} x_k$, $t_n = \sum_{k=1}^{n} y_k$ be the sequence of partial sums of (x_n), (y_n) respectively. Then, since (x_n), (y_n) converge, we have that $\lim_{n\to\infty} s_n = x$, $\lim_{n\to\infty} t_n = y$. Let $r_n = \sum_{k=1}^{n}(x_k + y_k))$. Then we have:

$$\begin{aligned}
r_n &= \sum_{k=1}^{n}(x_k + y_k) \\
&= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n) && \text{definition of summation} \\
&= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) && \text{associativity of } \mathbb{R} \\
&= \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k && \text{definition of summation} \\
r_n &= s_n + t_n && s_n = \sum_{k=1}^{n} x_k,\ t_n = \sum_{k=1}^{n} y_k
\end{aligned}$$

So $r_n = \sum_{k=1}^{n}(x_k + y_k) = \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k$. From above, we have that $\lim_{n\to\infty} s_n = x$ and $\lim_{n\to\infty} t_n = y$, so both sequences converge. Then, by the Algebraic Limit Theorem, we have

$$\lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n + \lim_{n\to\infty} s_n + t_n = x + y = \lim_{n\to\infty} r_n$$

Since the sequence of partial sums $r_n = \sum_{k=1}^{n}(x_k + y_k) = \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k$ converges, $\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$ converges. $\square$

(c) Assume by contradiction that the requirement that both series converge is not required for (b). Consider $x_n = n$, $y_n = \frac{1}{n^2}$. From lecture, x_n diverges and y_n converges. Then, $\sum_{n=1}^{\infty} x_n + y_n = \sum_{n=1}^{\infty} n + \frac{1}{n^2}$ converges by assumption. However, $\lim_{n\to\infty} \sum_{n=1}^{\infty} n + \frac{1}{n^2}$ does not converge, which is a contradiction. Therefore, both series must converge so that part (b) holds.

(d) No. Consider $x_n = y_n = \frac{(-1)^n}{\sqrt{n}}$. By the Alternating Series Test from lecture, both $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge. However, $\sum_{n=1}^{\infty} x_n y_n$ does not converge since $x_n y_n = \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge from lecture.

Question 7

Study the convergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

(b) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+2}{n^2+1}$

(d) $\sum_{n=1}^{\infty} \frac{n^{\log n}}{(\log n)^n}$

(e) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

(f) $\sum_{n=1}^{\infty} (x_{n+1} - x_n)$ for any sequence (x_n)

Response

(a) Apply the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^2} \right| \left| \frac{n^2}{2^n} \right| &= \lim_{n\to\infty} \left| \frac{2^{n+1}}{2^n} \frac{n^2}{(n+1)^2} \right| \\
&= \lim_{n\to\infty} \left| \frac{2n^2}{(n+1)^2} \right| \\
\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)^2} \right| \left| \frac{n^2}{2^n} \right| &= 2
\end{aligned}$$

Since $2 > 1$, by the ratio test, $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges.

(b)

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{(n+1)^2}{2^{n+1}} \right| \left| \frac{2^n}{n^2} \right| &= \lim_{n\to\infty} \left| \frac{2^n}{2^{n+1}} \frac{(n+1)^2}{n^2} \right| \\
&= \lim_{n\to\infty} \left| \frac{(n+1)^2}{2n^2} \right| \\
&= \frac{1}{2}
\end{aligned}$$

Since $\frac{1}{2} < 1$, by the ratio test, $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges.

(c) Apply the Alternating Series Test: We need to check that

(i) $\left| \frac{n^2+2}{n^2+1} \right|$ is monotonically decreasing: $\frac{\left| \frac{(n+1)^2+2}{(n+1)^2+1} \right|}{\left| \frac{n^2+2}{n^2+1} \right|} = \frac{n^4+4n^3+5n^2+4n+3}{n^4+4n^3+5n^2+5n+2} \leq 1$ so the sequence is monotonically decreasing.

(ii) $\lim_{n\to\infty} \left| \frac{n^2+2}{n^2+1} \right| = 0$:

Note that n^2+1 is never 0 for any n. Therefore, by ALT, we have $\lim_{n\to\infty} \left| \frac{n^2+2}{n^2+1} \right| = 1 \neq 0$.

Since (ii) does not hold, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2+2}{n^2+1}$ diverges.

(d) Apply the root test:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^{\log n}}{(\log n)^n}\right|} &= \lim_{n\to\infty} \frac{n^{\frac{\log n}{n}}}{\log n} \\
&= \lim_{n\to\infty} \frac{e^{\frac{(\log n)^2}{n}}}{\log n} \\
&= \lim_{n\to\infty} e^{\frac{(\log n)^2}{n}} \cdot \lim_{n\to\infty} (\log n)^{-1} \\
&= 1 \cdot 0 \\
\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^{\log n}}{(\log n)^n}\right|} &= 0
\end{aligned}$$

Therefore, by the root test, $\sum_{n=1}^{\infty} \frac{n^{\log n}}{(\log n)^n}$ converges.

(e) Apply the comparison test: Let $x_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$ and $y_n = \frac{1}{n^{\frac{3}{2}}}$. Then, we have that $0 \leq x_n \leq y_n$. Now apply the p-series test to n. Since $p = \frac{3}{2} > 1$, by the p-series test, the series converges. Therefore, x_n also converges. So, $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ converges.

(f) Inconclusive: If the sequence $(x_{n+1} - x_n)$ converges, then so does the series. Otherwise, the series also diverges. Therefore, the convergence of the series is dependent on (x_n).