Problem Set 2

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Question 2

Prove that $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{N}$.

Response

Proof. Let P_n read "$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all $n \in \mathbb{N}$".

Base case: P_1 reads "$1^3 = 1^2$". Clearly, $1 = 1$ so P_1 holds true.

Inductive Hypothesis: Assume P_n holds true for an arbitrary $n \in \mathbb{N}$. We want to show that P_{n+1} is true.

$$
\begin{aligned}
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= (1^3 + 2^3 + \cdots + n^3) + (n+1)^3 \\
&= (1 + 2 + \cdots + n)^2 + (n+1)^3 && \text{from } P_n \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 && \text{from class, we proved that } \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{1}{4}\Big[n^2(n+1)^2 + 4(n+1)^3\Big] \\
&= \frac{1}{4}\Big[(n+1)^2(n^2 + 4(n+1))\Big] \\
&= \frac{1}{4}\Big[(n+1)^2(n^2 + 4n + 4)\Big] \\
&= \frac{1}{4}\Big[(n+1)^2(n+2)^2\Big] \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \left(\frac{(n+1)(n+2)}{2}\right)^2 \\
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= (1 + 2 + \cdots + n + (n+1))^2 && \text{from class, we proved that } \frac{n(n+1)}{2} = \sum_{i=1}^{n} i
\end{aligned}
$$

By the principle of mathematical induction, since we proved that P_{n+1} holds true for an arbitrary $n \in \mathbb{N}$, P_n holds true for all $n \in \mathbb{N}$. $\square$

Question 6 part (b), (e), (f)

Let $(\mathbb{F}, +, \cdot, \leq)$ be an ordered field (not necessarily $\mathbb{Q}$ or $\mathbb{R}$!) and for any $x \in \mathbb{F}$, define

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1}$$

This is called the *absolute value* function. Notice that $|x| \geq 0$ for every $x \in \mathbb{F}$.

(b) Let $a \in \mathbb{F}$ such that $a \geq 0$. Show that for $x, y \in \mathbb{F}$, $|x - y| \leq a$ if and only if $y - a \leq x \leq y + a$.

(e) Let $x, y \in \mathbb{R}$. Prove that if for any $\varepsilon > 0$, $x \leq y + \varepsilon$, then $x \leq y$. Show that we can also replace $x \leq y + \varepsilon$ with $x < y + \varepsilon$ and obtain $x \leq y$.

(f) Let $x, y \in \mathbb{R}$. Prove that $x = y$ if and only if for any $\varepsilon > 0$, we have $|x - y| < \varepsilon$.

Response

(b) *Proof.* $\implies$ There are two cases:
Case I: $0 \leq x - y$. Then, $|x - y| = x - y$, so

$$\begin{aligned} x - y &\leq a \\ x &\leq y + a \end{aligned}$$

Case II: $x - y < 0$. Then, $|x - y| = -(x - y) = y - x$, so

$$\begin{aligned} y - x &\leq a \\ x &\geq y - a \end{aligned}$$

so, we have that $y - a \leq x \leq y + a$.

$\impliedby$ There are two cases:
Case I: $x \leq y + a$. Note that if $0 \leq x - y \implies |x - y| = x - y$.

$$\begin{aligned} x &\leq y + a \\ a &\geq x - y \\ a &\geq |x - y| \end{aligned}$$

Case II: $y - a \leq x$. Note that if $x - y < 0 \implies |x - y| = -(x - y)$.

$$\begin{aligned} y - a &\leq x \\ a &\geq y - x \\ a &\geq -(x - y) \\ a &\geq |x - y| \end{aligned}$$

In both cases, we have that $|x - y| \leq a$. Therefore, $|x - y| \leq a \iff y - a \leq x \leq y + a$. $\square$

(e) *Proof.* Assume by contradiction that $y < x$. Then, $0 < x - y$. Let $\varepsilon = \frac{1}{2}(x - y)$. Clearly, $0 < \frac{1}{2}(x - y)$ from our assumption. Then,

$$\begin{aligned} \frac{1}{2}(x - y) &< x - y \\ \varepsilon &< x - y \end{aligned}$$

which is a contradiction to the statement that $x - y \leq \varepsilon$. Therefore, if for any $\varepsilon > 0$, $x \leq y + \varepsilon$, then $x \leq y$. $\square$

Proof. Assume by contradiction that $y < x$. Then, $0 < x - y$. Fix $\varepsilon = \frac{1}{2}(x - y)$. Clearly, $0 < \frac{1}{2}(x - y)$ from our assumption. Then,

$$\begin{aligned} \frac{1}{2}(x - y) &< x - y \\ \varepsilon &< x - y \end{aligned}$$

which is a contradiction to the statement that $x - y < \varepsilon$. Therefore, if for any $\varepsilon > 0$, $x < y + \varepsilon$, then $x \leq y$. $\square$

(f) *Proof.* $\implies$ Let $x = y$. We want to prove that $|x - y| < \varepsilon$. $x = y \implies x - y = 0$. Assume $x = y$. Then, $|x - y| = |0| = 0$ by definition of the *absolute value* function. It follows that $|x - y| < \varepsilon \implies 0 < \varepsilon$, using the observation $|x - y| = 0$ from before.

$\impliedby$ Assume by contradiction that $x \neq y$. Then, $0 \leq |x - y|$ by definition of the *absolute value* function. Now take $\varepsilon = \frac{1}{2}|x - y|$. Clearly, $0 < \frac{1}{2}|x - y| < |x - y|$. Then, we have $\frac{1}{2}|x - y| < |x - y| \implies \varepsilon < |x - y|$, which is a contradiction to the statement $|x - y| < \varepsilon$. $\square$

Question 13 part (a)

Assume $\alpha \in \mathbb{R}$. is an upper bound for a set $A \subseteq \mathbb{R}$, which is a non-empty and bounded above. Prove that $\alpha = \sup A$ if and only if for every $\varepsilon > 0$, there exists an $a \in A$ such that $\alpha - \varepsilon \leq a$.

Response

Proof. $\implies$ Let $\alpha = \sup A$ be the supremum for the set $A \subseteq \mathbb{R}$, which is non-empty and bounded above. By definition of the supremum, we have that $a \leq \sup A \leq \alpha$. Since $\sup A = \alpha$ by assumption, we have that $\forall \varepsilon > 0, \alpha - \varepsilon$ is not a supremum, since otherwise it would be an upper bound, but not a least upper bound, which contradicts the definition of supremum. Then, we can write that $\alpha - \varepsilon \leq a$ for some $a \in A$.

$\impliedby$ Fix $\varepsilon > 0$. Take an arbitrary upper bound $\beta \in \mathbb{R}$. Then, we can write that $\alpha - \varepsilon \leq a \leq \beta$ since by assumption, $\forall \varepsilon > 0, \exists a \in A : \alpha - \epsilon \leq a$. Clearly $\alpha - \varepsilon \leq \beta$, so we can rewrite the inequality as $\alpha \leq \beta + \varepsilon \implies \alpha \leq \beta$ from (6(e)). Since β was arbitrary, we have that $\alpha = \sup A$. $\square$

Question 14

Assume that A, B are nonempty subsets of $\mathbb{R}$ that are bounded above and $A \subseteq B$. Show that $\sup A \leq \sup B$.

Response

Proof. Note that $B \subseteq \mathbb{R}$, it is non-empty, and it is bounded above. Therefore, by definition of the supremum, $\sup B$ exists. We now want to show that $\sup A$ exists. Since $A \subseteq B$, by the transitive property of the subset relation, $A \subseteq \mathbb{R}$. By the problem statement, A is also non-empty and bounded above. Therefore, by definition of the supremum, $\sup A$ exists. Note that since $A \subseteq B$, we have $\forall a \in A,\ a \in B \implies \forall a \in A,\ a \leq \sup B$. So, $\sup B$ is an upper bound for A, which implies that $\sup A \leq \sup B$. $\square$