Problem Set 4

Warren Kim

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Question 1

(a) Let (x_n), (y_n) and (z_n) be sequences such that $y_n \leq x_n \leq z_n$ for every $n \in \mathbb{N}$ and satisfying $y_n \to x$ and $z_n \to x$ as $n \to \infty$. Show that $x_n \to x$ as $n \to \infty$. (This is known as the *squeeze theorem*. Why?)

(b) Let S be a non-empty subset of $\mathbb{R}$ which is bounded above. Show that there exists a sequence (x_n) of points in S such that $x_n \to \sup S$ as $n \to \infty$.
(Hint: You may find HW3 helpful.)

Once you have an argument for the supremum, do the same for the infimum. That is, if S is a non-empty set in $\mathbb{R}$ which is bounded below, show that there exists a sequnce (y_n) in S such that $y_n \to \inf S$.

Response

(a) *Proof.* Let $\varepsilon > 0$. Then, we have that $|y_n - x| < \varepsilon$ for all $n > N_1$ and $|z_n - x| < \varepsilon$ for all $n > N_2$ since both (y_n) and (z_n) converge. That is, we have that $x - \varepsilon < y_n < x + \varepsilon$ for every $n > N_1$ and $x - \varepsilon < z_n < x + \varepsilon$ for every $n > N_2$. Let $N_3 = \max\{N_1, N_2\}$. Since $y_n \leq x_n \leq z_n \ \forall n \in \mathbb{N}$, we have that for all $n > N_3$, $x - \varepsilon < y_n \leq x_n \leq z_n < x + \varepsilon \implies x - \varepsilon < x_n < x + \varepsilon \implies |x_n - x| < \varepsilon \ \forall n > N_3$. Therefore, $x_n \to x$ as $n \to \infty$. $\square$

This is called the *squeeze theorem* since a function is essentially being "squeezed" between two other functions that converge to the same limit x, forcing the limit of the squeezed function to also be x *assuming the conditions described in (a) are met*.

(b) *Proof.* By LUBP, since S is a non-empty subset of $\mathbb{R}$ that is bounded above, $\sup S$ exists. By definition, $\sup S$ is the *least* upper bound of S, so for all $\varepsilon > 0$, $\exists N \in \mathbb{N} : \forall n > N$, $\sup S - \varepsilon \leq \sup S$. Then, there exists some $(x_n) \in S$ such that $\sup S - \varepsilon \leq x_n$. Let $y_n := \sup S - \frac{1}{n}$. Then, clearly, $y_n \leq x_n \leq \sup S$ for all $n > N$. Let $a_n = \sup S$ and $b_n = \frac{1}{n}$. Clearly, $a_n \to \sup S$ as $n \to \infty$ since it is a constant sequence. From lecture, we have that $b_n \to 0$ as $n \to \infty$. Then, since both a_n and b_n converge, by the Algebraic Limit Theorem, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \sup S - 0 = \sup S$ Then we have that $y_n \leq x_n \leq a_n$ where $y_n \to \sup S$ and $a_n \to \sup S$ as $n \to \infty$. From (a), the squeeze theorem says that $x_n \to \sup S$ as $n \to \infty$. $\square$

Proof. By GLBP, since S is a non-empty subset of $\mathbb{R}$ that is bounded below, $\inf S$ exists. By definition, $\inf S$ is the *greatest* lower bound of S, so for all $\varepsilon > 0$, $\exists N \in \mathbb{N} : \forall n > N$, $\inf S \leq \inf S + \varepsilon$. Then, there exists some $(y_n) \in S$ such that $y_n \leq \inf S + \varepsilon$. Let $x_n := \inf S + \frac{1}{n}$. Then, clearly, $\inf S \leq y_n \leq x_n$ for all $n > N$. Let $a_n = \inf S$ and $b_n = \frac{1}{n}$. Clearly, $a_n \to \inf S$ as $n \to \infty$ since it is a constant sequence. From lecture, we have that $b_n \to 0$ as $n \to \infty$. Then, since both a_n and b_n converge, by the Algebraic Limit Theorem, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \inf S + 0 = \inf S$ Then we have that $a_n \leq y_n \leq x_n$ where $x_n \to \inf S$ and $a_n \to \inf S$ as $n \to \infty$. From (a), the squeeze theorem says that $x_n \to \inf S$ as $n \to \infty$. $\square$

Question 4

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(b) A monotone sequence that diverges but has a convergent subsequence.

(c) A sequence that contains subsequences converging to every point in the infinite set $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$.

(d) An unbounded sequence with a convergent subsequence.

(e) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(f) A Cauchy sequence that is not montone.

(g) A Cauchy sequence with a divergent subsequence.

(h) An unbounded sequence containing a subsequence that is Cauchy.

Response

(a) $x_n = \begin{cases} \frac{1}{n} & n \text{ is even} \\ 1+\frac{1}{n} & n \text{ is odd} \end{cases}$. Clearly, neither $\frac{1}{n}$ nor $1+\frac{1}{n}$ contain 0 or 1 as a term but the subsequence where n is even converges to 0 and the subsequence where n is odd converges to 1.

(b) This is impossible. Assume by contradiction there exists a monotone sequence (x_n) that diverges but has a convergent subsequence. Then, (x_n) must be bounded (either above, below, or both). By the Monotone Convergence Theorem, since (x_n) is monotone and bounded, (x_n) must converge, which is a contradiction to the statement that (x_n) is divergent.

(c) $x_n = \left(1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$

(d) This is impossible since by the Bolzano-Weirstrauss Theorem, every bounded sequence has a convergent subsequence.

(e) This is impossible. Assume by contradiction there exists a sequence that has a bounded subsequence but contains no subsequence that converges. Then, by the Bolzano-Weirstrauss Theorem, every bounded sequence has a convergent subsequence. So, there exists subsequence of the bounded subsequence that converges. Since a subsubsequence is a subsequence of the original sequence, there is at least one subsequence that converges, a contradiction to our assumption.

(f) $x_n = \frac{(-1)^n}{n}$ converges to 0 but is not monotone.

(g) This is impossible since all Cauchy sequences are bounded, and by Bolzano-Weirstrauss, every bounded sequence has a subsequence that converges. Therefore, all subsequences also converge.

(h) $x_n = \begin{cases} 0 & n \text{ is even} \\ n & n \text{ is odd} \end{cases}$. Then, (x_n) is unbounded but the subsequence x_{2n} is Cauchy.

Question 6

Let (x_n) be a Cauchy sequence. Show that the sequence (x_n^{2022}) converges.

Response

Proof. Note that since (x_n) is Cauchy, it converges. Let $x_n \to x$ as $n \to \infty$. Then, for $\varepsilon_0 = 1$, put $N \in \mathbb{N}$ such that for all $n > N$, $|x_n - x| < \varepsilon_0 = 1$. Let $N_1 = \max\{|x_1|, \ldots, |x_{N-1}|, |x| + 1\}$. Then, for all $n \leq N - 1$, $|x_n| \leq N_1$. and for all $n \geq N$, $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| = 1 + |x| \leq N_1$. Taking the extremes of the inequality, we get that $|x_n| \leq N_1$. So, $|x_n| \leq N_1$ for all $n \geq N_1$. Then,

$$
\begin{aligned}
|x_n^{2022} - x^{2022}| &= |(x_n - x)(x_n^{2021} + x^{2020}x + x^{2019}x^2 + \ldots + x^{2021})| \\
&\leq |x_n - x||x_n^{2021} + x^{2020}x + x^{2019}x^2 + \ldots + x^{2021}|
\end{aligned}
$$

Since $|x_n| \leq N_1$ for all $n \in \mathbb{N}$ and $|x| + 1 \leq N_1 \implies |x| \leq N_1 - 1 \leq N_1$, we have that $|x_n|^k \leq N_1^k$ and $|x|^k \leq N_1$ for all $n, k \in \mathbb{N}$. So, $|x_n^{2022} - x^{2022}| \leq |x_n - x|(2022)N_1^{2021}$. Let $\varepsilon > 0$. Then, $|x_n - x|(2022)N_1^{2021} < \varepsilon \iff |x_n - x| < \frac{\varepsilon}{2022N_1^{2021}}$. So, for all $n > N$, we have $|x_n - x| < \frac{\varepsilon}{2022N_1^{2021}} \implies |x_n^{2022} - x^{2022}| \leq |x_n - x|(2022)N_1^{2021}$. Therefore, (x_n^{2022}) converges. $\square$

Question 7(g)

Let (x_n) be a bounded sequence. Show that $\lim_{n\to\infty} x_n$ exists if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$, in which case, $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$ so they all share the same value.

Response

Proof. $\lim_{n\to\infty} x_n$ exists $\implies \limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$
Since (x_n) is bounded, we have that $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$ are finite and exist. From (c), there exists a subsequences (x_{nk}) and (x_{nt}) such that $\lim_{k\to\infty} = \limsup_{n\to\infty} x_n$ and $\lim_{t\to\infty} = \liminf_{n\to\infty} x_n$ respectively. Since $x_n \to x$, we have that $x_{nk} \to x \implies \limsup_{n\to\infty} \leq x$. Since $x_n \to x$, we have that $x_{nt} \to x \implies \liminf_{n\to\infty} \geq x$. Then, we have $\limsup_{n\to\infty} \leq x \leq \liminf_{n\to\infty} \implies \limsup_{n\to\infty} = x = \liminf_{n\to\infty}$ by definition of sup and inf ($\sup A \leq \inf A \iff \sup A = \inf A$).

$\lim_{n\to\infty} x_n$ exists $\impliedby \limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$
Assume $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$. Let $\varepsilon > 0$. Then, since $\limsup_{n\to\infty} = x$, there exists $N_1 \in \mathbb{N} : \forall n > N_1, |x_n - x| < \varepsilon$. Snince $\liminf_{n\to\infty} = x$, there exists $N_2 \in \mathbb{N} : \forall n > N_2, |x_n - x| < \varepsilon$. Take $N = \max\{N_1, N_2\}$. Then we have $|x_n - x| < \varepsilon$ for all $n > N$, which is the definition of a limit. Therefore, $\lim_{n\to\infty} x_n = x$. $\square$