# Problem Set 8

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## Section 2.5 Question 6

For each matrix $A$ and ordered basis $\beta$, find $[L_A]_\beta$. Also, find an invertible matrix $Q$ such that $[L_A]_\beta = Q^{-1}AQ$.

(a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

(c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

(d) $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

### Response

(a) $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $Q^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

$$\begin{aligned} [L_A]_\beta &= Q^{-1}AQ \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1+3 & 1+6 \\ 1+1 & 1+2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 7 \\ 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 8+-2 & 14+-3 \\ -4+2 & -7+3 \end{pmatrix} \\ [L_A]_\beta &= \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix} \end{aligned}$$

(b) $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $Q^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$$\begin{aligned} [L_A]_\beta &= Q^{-1}AQ \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}+1 & \frac{1}{2}+-1 \\ 1+\frac{1}{2} & 1+-\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2}+\frac{3}{2} & -\frac{1}{2}+\frac{1}{2} \\ \frac{3}{2}+-\frac{3}{2} & -\frac{1}{2}+-\frac{1}{2} \end{pmatrix} \\ [L_A]_\beta &= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

(c) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$,

$$\begin{aligned}
Q^{-1} &= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \\
Q^{-1} &= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
[L_A]_\beta &= Q^{-1}AQ \\
&= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1+1+-1 & 1+0+-1 & 1+1+-2 \\ 2+0+1 & 2+0+1 & 2+0+2 \\ 1+1+0 & 1+0+0 & 1+1+0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 3 & 3 & 4 \\ 2 & 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1+3+-2 & 0+3+-1 & 0+4+-2 \\ 1+-3+0 & 0+-3+0 & 0+-4+0 \\ -1+0+2 & 0+0+1 & 0+0+2 \end{pmatrix} \\
[L_A]_\beta &= \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}
\end{aligned}$$

(d) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$

$$\begin{aligned}
Q^{-1} &= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & -1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 2 & 0 & 1 \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right) \\
&= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & -\frac{2}{6} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right) \\
Q^{-1} &= \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
[L_A]_\beta &= Q^{-1}AQ \\
&= \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 13+1+-8 & 13+-1+0 & 13+1+4 \\ 1+13+-8 & 1+-13+0 & 1+13+4 \\ 4+4+-20 & 4+-4+0 & 4+4+10 \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 6 & 12 & 18 \\ 6 & -12 & 18 \\ -12 & 0 & 18 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & -2 \\ 3 & -3 & 0 \\ 2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \\ -2 & 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1+1+4 & 2+-2+0 & 3+3+-6 \\ 3+-3+0 & 6+6+0 & 9+-9+0 \\ 2+2+-4 & 4+-4+0 & 6+6+6 \end{pmatrix} \\
[L_A]_\beta &= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}
\end{aligned}$$

## Section 2.5 Question 10 part (a)

(a) Prove that if $A$ and $B$ are similar $n \times n$ matrices, then $tr(A) = tr(B)$. *Hint:* Use Exercise 13 of Section 2.3.

### Response

*Proof.* Let $A$ and $B$ be similar matrices. Then, there exists an invertible matrix $C$ such that $CBC^{-1} = A$. So we have

$$
\begin{aligned}
tr(A) &= tr(CBC^{-1}) & \\
&= tr((BC^{-1})C) & tr(AB) = tr(BA) \\
&= tr(B(C^{-1}C)) & \\
&= tr(BI) & C^{-1}C = I \\
tr(A) &= tr(B) & BI = B
\end{aligned}
$$

$\square$

## Section 4.4 Question 2

Evaluate the determinant of the following $2 \times 2$ matrices.

(a) $\begin{pmatrix} 4 & -5 \\ 2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 7 \\ 3 & 8 \end{pmatrix}$

(c) $\begin{pmatrix} 2+i & -1+3i \\ 1-2i & 3-i \end{pmatrix}$

(d) $\begin{pmatrix} 3 & 4i \\ -6i & 2i \end{pmatrix}$

### Response

Note that the determinant of a $2 \times 2$ matrix can be calculated by

$$\det(A) = ad - bc, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(a) $\det(A) = (4 \cdot 3) - (-5 \cdot 2) = 12 - -10 = 22$

(b) $\det(A) = (-1 \cdot 8) - (7 \cdot 3) = -8 - 21 = -29$

(c)

$$\begin{aligned} \det(A) &= ((2+i)(3-i)) - ((-1+3i)(1-2i)) \\ &= (6 + -2i + 3i + 1) - (-1 + 2i + 3i + 6) \\ &= (7+i) - (5+5i) \\ \det(A) &= 2 - 4i \end{aligned}$$

(d)

$$\begin{aligned} \det(A) &= ((3+0i)(0+2i)) - ((0+4i)(0-6i)) \\ &= 6i - 24 \\ \det(A) &= -24 + 6i \end{aligned}$$

## Section 4.4 Question 3 parts (a) - (d)

Evaluate the determinant of the following matrices in the manner indicated

(a) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the first row

(b) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the first column

(c) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the second column

(d) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the third row

### Response

(a) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the first row

$$\begin{aligned}
\det(A) &= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} \\
&= 0 - 1\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ 2 & 3 \end{vmatrix} \\
&= (0 - -6) + 2(-3 - 0) \\
&= -6 + -6 \\
\det(A) &= -12
\end{aligned}$$

(b) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the first column

$$\begin{aligned}
\det(A) &= a_{11}C_{11} - a_{21}C_{21} + a_{31}C_{31} \\
&= 1\begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} - 0 + -1\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} \\
&= 1(0 - 15) + -1(0 - 2) \\
&= -15 + 2 \\
\det(A) &= -13
\end{aligned}$$

(c) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the second column

$$\begin{aligned}
\det(A) &= -a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\
&= -1\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} + 0 - 3\begin{vmatrix} 0 & 2 \\ -1 & -3 \end{vmatrix} \\
&= -1(0 - -6) - 3(0 - -2) \\
&= -6 - 6 \\
\det(A) &= -12
\end{aligned}$$

(d) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the third row

$$\begin{aligned}
\det(A) &= a_{31}C_{31} - a_{32}C_{32} + a_{33}C_{33} \\
&= -1\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} + 0 \\
&= -1(0-2) - 3(5-0) \\
&= 2 - 15 \\
\det(A) &= -13
\end{aligned}$$

## Section 4.4 Question 4 parts (a) - (d)

Evaluate the determinant of the following matrices by any legitimate method.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{pmatrix}$

### Response

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$\begin{aligned} \det(A) &= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} \\ &= 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(45-48) - 2(36-42) + 3(32-35) \\ &= -3 - 2(-6) + 3(-3) \\ &= -3 + 12 - 9 \\ \det(A) &= 0 \end{aligned}$$

(b) $\begin{pmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

$$\begin{aligned} \det(A) &= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} \\ &= -1\begin{vmatrix} -8 & 1 \\ 2 & 5 \end{vmatrix} - 3\begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} + 2\begin{vmatrix} 4 & -8 \\ 2 & 2 \end{vmatrix} \\ &= -1(-40-2) - 3(20-2) + 2(8 - -16) \\ &= 42 - 54 + 48 \\ \det(A) &= 36 \end{aligned}$$

(c) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$

$$\begin{aligned} \det(A) &= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} \\ &= 0 - 1\begin{vmatrix} 1 & -5 \\ 6 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 6 & -4 \end{vmatrix} \\ &= -1(3-30) + 1(-4-12) \\ &= -33 - 16 \\ \det(A) &= -49 \end{aligned}$$

(d) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{pmatrix} \equiv \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & -2 \\ 3 & -1 & 2 \end{pmatrix}$

$$\begin{aligned} \det(A) &= -a_{21}C_{21} + a_{22}C_{22} - a_{23}C_{23} \\ &= -0 + 0 - -2\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} \\ &= 2(-1 - -6) \\ \det(A) &= 10 \end{aligned}$$

# Section 5.1 Question 3 part (a)

For each of the followin linear operators $T$ on a vector space $V$ and ordered bases $\beta$, compute $[T]_\beta$, and determine whether $\beta$ is a basis consisting of eigenvectors of $T$.

(a) $V = \mathbb{R}^2, T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

## Response

$$\begin{aligned} T\begin{pmatrix} 1 \\ 2 \end{pmatrix} &= \begin{pmatrix} 10(1) - 6(2) \\ 17(1) - 10(2) \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -3 \end{pmatrix} \\ T\begin{pmatrix} 1 \\ 2 \end{pmatrix} &= 0\begin{pmatrix} 1 \\ 2 \end{pmatrix} + -1\begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 10(2) - 6(3) \\ 17(2) - 10(3) \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 4 \end{pmatrix} \\ T\begin{pmatrix} 2 \\ 3 \end{pmatrix} &= 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}$$

$$[T]_\beta = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$$

Because the matrix is not a diagnoal matrix, $\beta$ does not contain the eigenvectors of $T$.

# Section 5.1 Question 4 part (a)

For each of the following matrices $A \in \mathcal{M}_{n\times n}(F)$,

(i) Determine all the eigenvalues of $A$.

(ii) For each eigenvalue $\lambda$ of $A$, find the set of eigenvectors corresponding to $\lambda$.

(iii) If possible, find a basis for $F^n$ consisting of eigenvectors of $A$.

(iv) If successful in finding such a basis, determine an invertible matrix $Q$ and diagonal matrix $D$ such that $Q^{-1}AQ = D$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = \mathbb{R}$

## Response

(i) Determine all the eigenvalues of $A$.

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)(2-\lambda) - 6 \\
&= 2 - \lambda - 2\lambda + \lambda^2 - 6 \\
\det(A - \lambda I) &= \lambda^2 - 3\lambda - 4
\end{aligned}$$

$$\begin{aligned}
\lambda^2 - 3\lambda - 4 &= (\lambda - 4)(\lambda + 1) \\
\lambda_1 &= 4 \\
\lambda_2 &= -1
\end{aligned}$$

(ii) For each eigenvalue $\lambda$ of $A$, find the set of eigenvectors corresponding to $\lambda$.
If $\lambda = 4$

$$\begin{aligned}
B_1 &= A - \lambda_1 I \\
&= \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} -3x_1 & 2x_2 \\ 3x_1 & -2x_2 \end{pmatrix} \\
x_1 &= 2 \\
x_2 &= 3
\end{aligned}$$

If $\lambda = -1$

$$\begin{aligned} B_1 &= A - \lambda_1 I \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 & 2x_2 \\ 3x_1 & 3x_2 \end{pmatrix} \\ x_1 &= 1 \\ x_2 &= -1 \end{aligned}$$

So, the eigenvectors for $\lambda = 4, \lambda = -1$ are $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

(iii) If possible, find a basis for $F^n$ consisting of eigenvectors of $A$.

One possible basis for $F^n$ consisting of the eigenvectors is $\beta = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

(iv) If successful in finding such a basis, determine an invertible matrix $Q$ and diagonal matrix $D$ such that $Q^{-1}AQ = D$.

$$\begin{aligned} Q &= \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \\ Q^{-1} &= \frac{1}{(-2-3)} \begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \\ D &= Q^{-1}AQ \\ &= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2+6 & 1+-2 \\ 6+6 & 3+-2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 8 & -1 \\ 12 & 1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 8+12 & -1+1 \\ 24+-24 & -3+-2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 20 & 0 \\ 0 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$Q = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}, D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

# Section 5.1 Question 10

Prove that the eigenvalues of an upper triangular matrix $M$ are the diagonal entries of $M$.

## Response

*Proof.* Let $M$ be an $n \times n$ upper triangular matrix. Then, the characteristic polynomial is

$$\begin{aligned} \det(M - \lambda I) &= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{12} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} - \lambda \end{vmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) \end{aligned}$$

By the fact that the determinant of an upper triangular matrix is the product of its diagonal entries. So, the eigenvalues of an upper triangular matrix $M$ are $\lambda_i = a_{ii}, 1 \leq i \leq n$, or the diagonal entries of $M$. □

## Section 5.1 Question 11

Let $V$ be a finite-dimensional vector space, and let $\lambda$ be any scalar.

(a) For any ordered basis $\beta$ for $V$, prove that $[\lambda I_V]_\beta = \lambda I$.

(b) Compute the characteristic polynomial of $\lambda I_V$.

(c) Show that $\lambda I_V$ is diagonalizable and has only one eigenvalue.

### Response

(a) *Proof.* Let $T$ be a linear transformation such that $T(v) = \lambda v$. Then, we can rewrite this as

$$\begin{aligned} T(v) &= \lambda v \\ &= \lambda I v \\ T &= \lambda I_V \end{aligned}$$

Note that since this is true, we can write

$$\begin{aligned} [T]_\beta v &= [\lambda]_\beta v \\ &= \lambda I v \\ [\lambda I_V]_\beta v &= \lambda I v && \text{from the previous part, } T = \lambda I_V \end{aligned}$$

So, for any ordered basis $\beta$ for $V$, we have that $[\lambda I_V]_\beta v = \lambda I v$. □

(b)

$$\begin{aligned} \det(T - \lambda' I) &= \det(\lambda I_V - \lambda' I) \\ &= \begin{vmatrix} \lambda - \lambda' & 0 & 0 & \cdots & 0 \\ 0 & \lambda - \lambda' & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda - \lambda' \end{vmatrix} \\ &= (\lambda - \lambda')^{dim(V)} \\ &= (\lambda - \lambda')^n && \text{let } n = dim(V) \end{aligned}$$

So, the characteristic polynomial of $\lambda I_V$ is $(\lambda - \lambda')^n$.

(c) Note that $\lambda I$ is a diagonal matrix, so $\lambda I_V$ is diagonalizable. From part (b), we have $\lambda I_V$ is $(\lambda - \lambda')^n$, so solving for lambda, we get $\lambda' = \lambda$, so $\lambda I_V$ only has one eigenvalue.

## Section 5.1 Question 13 part (a)

(a) Prove that similar matrices have the same characteristic polynomial.

### Response

*Proof.* Let $A$ and $B$ be similar matrices. First, we must prove that there exists a matrix $C$ such that $CBC^{-1} = A$. So we have

$$\begin{aligned}
\det(A) &= \det(CBC^{-1}) \\
&= \det(C)\det(B)\det(C^{-1}) \\
&= \det(B)\det(C^{-1})\det(C) && \det(A)\det(B) = \det(B)\det(A) \\
&= \det(B)\det(C^{-1}C) \\
&= \det(B)\det(I) && C^{-1}C = I \\
\det(A) &= \det(B) && \det(I) = 1
\end{aligned}$$

Now, define the characteristic polynomial to be $\det(A - \lambda I)$. So we have

$$\begin{aligned}
\det(A - \lambda I) &= \det(CBC^{-1} - \lambda I) \\
&= \det(CBC^{-1} - \lambda CIC^{-1}) \\
&= \det(C(B - \lambda I)C^{-1}) \\
\det(A - \lambda I) &= \det(C)\det(B - \lambda I)\det(C^{-1})
\end{aligned}$$

Therefore, similar matrices have the same characteristic polynomial. $\square$