Problem Set 5

Warren Kim

May 11, 2023

Question 3

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ be convergent series. Show that:

(a) $\sum_{n=1}^{\infty}(ax_n)$ converges and $\sum_{n=1}^{\infty}(ax_n) = a\sum_{n=1}^{\infty} x_n$ for any $a \in \mathbb{R}$.

(b) Show that $\sum_{n=1}^{\infty}(x_n + y_n)$ converges and $\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$.

(c) Show that the assumption that *both* series converge is necessary for part (b).

(d) Is it true that if $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge then $\sum_{n=1}^{\infty} x_n y_n$ converges?

Response

(a) *Proof.* Given that $\sum_{n=1}^{\infty} x_n$ converges, let $s_n = \sum_{k=1}^{n} x_k$ be the sequence of partial sums of (x_n). Then, since (x_n) converges, we have that $\lim_{n\to\infty} s_n = x$. Let $t_n = \sum_{k=1}^{n} ax_k$. Then, we have:

$$\begin{aligned}
t_n &= \sum_{k=1}^{n} ax_k \\
&= (ax_1 + ax_2 + \cdots + ax_k) && \text{definition of summation} \\
&= a(x_1 + x_2 + \cdots + x_k) && \text{distributivity of } \mathbb{R} \\
&= a\sum_{k=1}^{n} x_k && \text{definition of summation} \\
t_n &= as_n && s_n = \sum_{k=1}^{n} x_k
\end{aligned}$$

So $t_n = \sum_{k=1}^{n} ax_k = a\sum_{k=1}^{n} x_k$. From above, we have that $\lim_{n\to\infty} s_n = x$ and $\lim_{n\to\infty} a = a$ (since a is constant), so both sequences converge. Then by the Algebraic Limit Theorem, we have

$$\lim_{n\to\infty} a \cdot \lim_{n\to\infty} s_n = \lim_{n\to\infty} a \cdot s_n = ax = \lim_{n\to\infty} s_n$$

Since the sequence of partial sums $t_n = \sum_{k=1}^{n} ax_k = a\sum_{k=1}^{n} x_k$ converges, $\sum_{n=1}^{\infty} ax_n = a\sum_{n=1}^{\infty} x_n$ converges. □

(b) *Proof.* Let $s_n = \sum_{k=1}^{n} x_k$, $t_n = \sum_{k=1}^{n} y_k$ be the sequence of partial sums of (x_n), (y_n) respectively. Then, since (x_n), (y_n) converge, we have that $\lim_{n\to\infty} s_n = x$, $\lim_{n\to\infty} t_n = y$. Let $r_n = \sum_{k=1}^{n}(x_k + y_k))$. Then we have:

$$\begin{aligned}
r_n &= \sum_{k=1}^{n}(x_k + y_k) \\
&= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n) && \text{definition of summation} \\
&= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) && \text{associativity of } \mathbb{R} \\
&= \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k && \text{definition of summation} \\
r_n &= s_n + t_n && s_n = \sum_{k=1}^{n} x_k,\ t_n = \sum_{k=1}^{n} y_k
\end{aligned}$$

So $r_n = \sum_{k=1}^{n}(x_k + y_k) = \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k$. From above, we have that $\lim_{n\to\infty} s_n = x$ and $\lim_{n\to\infty} t_n = y$, so both sequences converge. Then, by the Algebraic Limit Theorem, we have

$$\lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n + \lim_{n\to\infty} s_n + t_n = x + y = \lim_{n\to\infty} r_n$$

Since the sequence of partial sums $r_n = \sum_{k=1}^{n}(x_k + y_k) = \sum_{k=1}^{n} x_k + \sum_{k=1}^{n} y_k$ converges, $\sum_{n=1}^{\infty}(x_n + y_n) = \sum_{n=1}^{\infty} x_n + \sum_{n=1}^{\infty} y_n$ converges. □

(c) Show that the assumption that *both* series converge is necessary for part (b).

(d) Is it true that if $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge then $\sum_{n=1}^{\infty} x_n y_n$ converges?