110A HW4

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Question 1

Let R be a ring and $I \subseteq R$ be an ideal. Let $J \subseteq R$ be an ideal such that $I \subseteq J$, and let $\overline{J} \subseteq \overline{R} = R/I$ be an ideal.

1. Show that $\pi^{-1}(\pi(J)) = J$ and $\pi(\pi^{-1}(\overline{J})) = \overline{J}$. [Recall $\pi : R \to R/I$ is the canonical projection.]

2. Let $\overline{J} = \pi(J)$. Let $\pi : R \to R/I$ and $\phi : \overline{R} \to \overline{R}/\overline{J}$ be canonical projections. Show that $\ker(\phi \circ \pi) = J$.

Response

Proof: Let R be a ring and $I \subseteq R$ be an ideal. Let $J \subseteq R$ be an ideal such that $I \subseteq J$, and let $\overline{J} \subseteq \overline{R} = R/I$ be an ideal.

(1) $\boldsymbol{\pi^{-1}(\pi(J)) = J}$**:** Let $a \in \pi^{-1}(\pi(J))$. Then by definition of the pre-image under π, there exists $x \in J$ such that $\pi(a) = \pi(x) \in \pi(J)$, or $a + I = x + I$, which implies that $a - x \in I \subseteq J$, so $a \in J$. Since a was arbitrary, $\pi^{-1}(\pi(J)) \subseteq J$. Now let $b \in J$. Then by definition, $\pi(b) = b + I$. Then, $\pi^{-1}(\pi(b)) = \pi^{-1}(b + I)$ but by definition of the pre-image, $\pi^{-1}(b + I) = b \in \pi^{-1}(\pi(J))$. Since b was arbitrary, $J \subseteq \pi^{-1}(\pi(J))$. Since we have $\pi^{-1}(\pi(J)) \subseteq J$ and $\pi^{-1}(\pi(J)) \supseteq J$, $\pi^{-1}(\pi(J)) = J$.

$\boldsymbol{\pi(\pi^{-1}(\overline{J})) = \overline{J}}$**:** Let $a + I \in \pi(\pi^{-1}(\overline{J}))$. Then there exists $x \in R$ such that $x \in \pi^{-1}(\overline{J})$ and $\pi(x) = a + I \in \overline{J}$. Since a was arbitrary, $\pi(\pi^{-1}(\overline{J})) \subseteq \overline{J}$. Now let $b + I \in \overline{J}$. Then by definition, $b + I$ is in the image of J under π, so $b \in \pi^{-1}(\overline{J})$. Then $\pi(\pi^{-1}(b + I)) = \pi(b) = b + I \in \pi(\pi^{-1}(\overline{J}))$. Since $b + I$ was arbitrary, $\overline{J} \subseteq \pi(\pi^{-1}(\overline{J}))$. Since $\pi(\pi^{-1}(\overline{J})) \subseteq \overline{J}$ and $\pi(\pi^{-1}(\overline{J})) \supseteq \overline{J}$, $\pi(\pi^{-1}(\overline{J})) = \overline{J}$.

(2) Let $\overline{J} = \pi(J)$. Let $\pi : R \to R/I$ and $\phi : \overline{R} \to \overline{R}/\overline{J}$ be canonical projections. Take $a \in J$. Then $\phi \circ \pi(a) = \phi(\pi(a)) = \phi(a + I) = (a + I) + \overline{J}$, but since $a + I \in \overline{J}$, we have that $(a + I) + \overline{J} = 0 + \overline{J} \in \ker(\phi \circ \pi)$. Since a was arbitrary, $J \subseteq \ker(\phi \circ \pi)$. Now take any $b \in R$ such that $\phi \circ \pi(b) = 0 + \overline{J}$. Then, $(b + I) + \overline{J} = 0 + \overline{J}$. Then by definition, $b + I \in \overline{J} = \pi(J)$ by assumption. Then $b + I$ is the image of J under π, so $b \in \pi^{-1}(\overline{J}) = \pi^{-1}(\pi(J)) = J$. Since b was arbitrary, $\ker(\phi \circ \pi) \subseteq J$. Since $J \subseteq \ker(\phi \circ \pi)$ and $J \supseteq \ker(\phi \circ \pi)$, $J = \ker(\phi \circ \pi)$. $\square$

Question 2

Let $m, n \in \mathbb{Z}$ be nonzero. Show that $(m, n) = 1$ if and only if $\mathbb{Z}/mn \cong \mathbb{Z}/m \times \mathbb{Z}/n$.

Response

($\implies$) Let $m, n \in \mathbb{Z}$ be nonzero such that $\gcd(m, n) = 1$. Let $R = \mathbb{Z}$, $I = (m)$, and $J = (n)$. Then $I + J = R$ since we can represent $(1) := (m)x + (n)y$ for some $x, y \in Z$. Then $R/(I \cap J) \simeq (R/I) \times (R/J)$ but since $I + J = R$, $I \cap J = IJ$, so $R/IJ \simeq (R/I) \times (R/J)$. Substituting I, J, R, we get $\mathbb{Z}/mn \simeq \mathbb{Z}/m \times \mathbb{Z}/n$.

($\impliedby$) Let $\mathbb{Z}/mn \simeq \mathbb{Z}/m \times \mathbb{Z}/n$. Let $d = \gcd(m, n)$.