1. **Linear algebra refresher.**

   (a) Let $\mathbf{Q}$ be a real orthogonal matrix.

      i. To show that $\mathbf{Q^T}$ and $\mathbf{Q^{-1}}$ are also orthogonal, suppose $\mathbf{Q}$ is orthogonal. Then $\mathbf{QQ^T} = \mathbf{Q^TQ} = \mathbf{I}$. Consider $\mathbf{Q^T}$. Recall that $\left(\mathbf{Q^T}\right)^{\mathbf{T}} = \mathbf{Q}$. Then,

$$\mathbf{Q^T}\left(\mathbf{Q^T}\right)^{\mathbf{T}} = \mathbf{Q^TQ} = \mathbf{I} = \mathbf{QQ^T} = \left(\mathbf{Q^T}\right)^{\mathbf{T}}\mathbf{Q^T}$$

      Note that if $\mathbf{Q}$ is orthogonal, then $\mathbf{Q^T} = \mathbf{Q^{-1}}$. Then, since $\mathbf{Q^T}$ is orthogonal, $\mathbf{Q^{-1}}$ is orthogonal.

      ii. To show that $\mathbf{Q}$ has eigenvalues with norm 1, consider

$$\begin{aligned}
\mathbf{Qx} &= \lambda\mathbf{x} && \\
(\mathbf{Qx})^{\mathbf{T}}\mathbf{Qx} &= (\mathbf{Qx})^{\mathbf{T}}\lambda\mathbf{x} && \\
\mathbf{x^TQ^TQx} &= (\lambda\mathbf{x})^{\mathbf{T}}\lambda\mathbf{x} && \mathbf{Qx} = \lambda\mathbf{x} \\
\mathbf{x^TIx} &= \mathbf{x^T}\lambda\lambda\mathbf{x} && \mathbf{Q} \text{ is orthogonal} \\
\mathbf{x^Tx} &= \lambda^2\mathbf{x^Tx} && \mathbf{x^Tx} = \|\mathbf{x}\|^2 \\
\|\mathbf{x}\|^2 &= \lambda^2\|\mathbf{x}\|^2 && \\
\lambda^2 &= 1 &&
\end{aligned}$$

      This implies that $|\lambda| = 1$.

      iii. To show that the determinant of $\mathbf{Q}$ is ± 1, recall that $\mathbb{Q}$ is orthogonal, so we have that $\mathbf{QQ^T} = \mathbf{I}$. Taking the determinant of both sides, we get

$$\det\left(\mathbf{QQ^T}\right) = \det(\mathbf{Q}) \cdot \det\left(\mathbf{Q^T}\right) = \det(\mathbf{I})$$

      Since $\det(\mathbf{I}) = 1$, we have $\det(\mathbf{Q}) \cdot \det\left(\mathbf{Q^T}\right) = 1$. Because $\mathbb{Q}$ is orthogonal, $\det(\mathbf{Q}) = \det\left(\mathbf{Q^T}\right)$, so $[\det(\mathbf{Q})]^2 = 1 \to \det(\mathbf{Q}) = \pm 1$.

      iv. To show that $\mathbf{Q}$ defines a length preserving transformation, consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. By assumption, $\mathbf{Q}$ is an orthogonal matrix, so $\mathbf{QQ^T} = \mathbf{Q^TQ} = \mathbf{I}$. We can represent the linear transformation T by $\mathbf{Q}$, so write $T\mathbf{x} = \mathbf{Qx}$. Then, taking the norm of both sides, we get

$$\begin{aligned}
\|T\mathbf{x}\|^2 &= \|\mathbf{Qx}\|^2 && \\
&= (\mathbf{Qx})^{\mathbf{T}}\mathbf{Qx} && \\
&= \mathbf{x^TQ^TQx} && \\
&= \mathbf{x^TIx} && \mathbf{Q} \text{ is orthogonal} \\
&= \mathbf{x^Tx} && \\
\|T\mathbf{x}\|^2 &= \|\mathbf{x}\|^2 && \mathbf{x^Tx} = \|\mathbf{x}\|^2
\end{aligned}$$

      Taking the square root of both sides, we get $\|T\mathbf{x}\| = \|\mathbf{x}\|$, so $\mathbf{Q}$ is a length preserving transformation.

(b) Let $\mathbf{A}$ be a matrix.

i. Recall that the singular value decomposition of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}}$, where $\mathbf{U} \in \mathbb{R}^{m \times m}, \Sigma \in \mathbb{R}^{m \times n}, \mathbf{V} \in \mathbb{R}^{n \times n}$. Then, we can write $\mathbf{A}\mathbf{A}^{\mathbf{T}} \in \mathbb{R}^{m \times m}$ as

$$\begin{aligned}
\mathbf{A}\mathbf{A}^{\mathbf{T}} &= \left(\mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}}\right)\left(\mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}}\right)^{\mathbf{T}} \\
&= \mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}}\left(\mathbf{V}^{\mathbf{T}}\right)^{\mathbf{T}}\Sigma^{\mathbf{T}}\mathbf{U}^{\mathbf{T}} \\
&= \mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}}\mathbf{V}\Sigma^{\mathbf{T}}\mathbf{U}^{\mathbf{T}} \\
&= \mathbf{U}\Sigma\mathbf{I}\Sigma^{\mathbf{T}}\mathbf{U}^{\mathbf{T}} && \mathbf{V} \text{ is orthogonal} \\
&= \mathbf{U}\Sigma\Sigma^{\mathbf{T}}\mathbf{U}^{\mathbf{T}} \\
&= \mathbf{U}\Sigma^2\mathbf{U}^{\mathbf{T}} && \Sigma \text{ is diagonal}
\end{aligned}$$

Since $\mathbf{U}$ is orthogonal, we have $\mathbf{U}^{\mathbf{T}} = \mathbf{U}^{-1}$. So, $\mathbf{A}\mathbf{A}^{\mathbf{T}} = \mathbf{U}\Sigma^2\mathbf{U}^{\mathbf{T}}$, where $\mathbf{U}$ are the eigenvectors of $\mathbf{A}\mathbf{A}^{\mathbf{T}}$. Then, the left singular vectors of $\mathbf{A}$ are the eigenvectors of $\mathbf{A}\mathbf{A}^{\mathbf{T}}$.

Similarly, we can write $\mathbf{A}^{\mathbf{T}}\mathbf{A} \in \mathbb{R}^{n \times n}$ as

$$\begin{aligned}
\mathbf{A}^{\mathbf{T}}\mathbf{A} &= \left(\mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}}\right)^{\mathbf{T}}\left(\mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}}\right) \\
&= \left(\mathbf{V}^{\mathbf{T}}\right)^{\mathbf{T}}\Sigma^{\mathbf{T}}\mathbf{U}^{\mathbf{T}}\mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}} \\
&= \mathbf{V}\Sigma^{\mathbf{T}}\mathbf{U}^{\mathbf{T}}\mathbf{U}\Sigma\mathbf{V}^{\mathbf{T}} \\
&= \mathbf{V}\Sigma^{\mathbf{T}}\mathbf{I}\Sigma\mathbf{V}^{\mathbf{T}} && \mathbf{U} \text{ is orthogonal} \\
&= \mathbf{V}\Sigma^2\mathbf{V}^{\mathbf{T}} && \Sigma \text{ is diagonal}
\end{aligned}$$

Since $\mathbf{V}$ is orthogonal, we have $\mathbf{V}^{\mathbf{T}} = \mathbf{V}^{-1}$. So, $\mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{V}\Sigma^2\mathbf{V}^{\mathbf{T}}$, where $\mathbf{V}$ are the eigenvectors of $\mathbf{A}^{\mathbf{T}}\mathbf{A}$. Then, the right singular vectors of $\mathbf{A}$ are the eigenvectors of $\mathbf{A}^{\mathbf{T}}\mathbf{A}$.

ii. From the above part, we have that $\mathbf{A}\mathbf{A}^{\mathbf{T}} = \mathbf{U}\Sigma^2\mathbf{U}^{\mathbf{T}}$ and $\mathbf{A}^{\mathbf{T}}\mathbf{A} = \mathbf{V}\Sigma^2\mathbf{V}^{\mathbf{T}}$. Then, the singular values of $\mathbf{A}$ are the square root of the eigenvalues of $\mathbf{A}\mathbf{A}^{\mathbf{T}}$ and $\mathbf{A}^{\mathbf{T}}\mathbf{A}$.

(c) True or False.

i. Every linear operator in an n-dimensional vector space has n distinct eigenvalues.

Response: False. Every linear operator in an n-dimensional vector space has ***at most*** n distinct eigenvalues.

ii. A non-zero sum of two eigenvectors of a matrix $\mathbf{A}$ is an eigenvector.

Response: Consider two eigenvectors $\mathbf{x}, \mathbf{y}$ of a matrix $\mathbf{A} \in \mathbb{R}^2$. There are two cases:

Case 1: If $\mathbf{x}, \mathbf{y}$ correspond to the same eigenvalue λ, the statement is True since $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y} = \lambda\mathbf{x} + \lambda\mathbf{y} = \lambda(\mathbf{x}+\mathbf{y})$

Case 2: If $\mathbf{x}, \mathbf{y}$ correspond to unique eigenvalues $\lambda_{\mathbf{x}}, \lambda_{\mathbf{y}}$, the statement is False since $\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y} = \lambda_{\mathbf{x}}\mathbf{x} + \lambda_{\mathbf{y}}\mathbf{y} \neq \lambda(\mathbf{x}+\mathbf{y})$

iii. If a matrix $\mathbf{A}$ has the positive semidefinite property, i.e., $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$, then its eigenvalues must be non-negative.

Response: True. Suppose a matrix $\mathbf{A}$ has the positive semidefinite property; i.e. $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$. Consider an arbirtrary eigenvalue λ of $\mathbf{A}$. Then, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some eigenvector $\mathbf{x}$. Multiplying both sides by $\mathbf{x}^{\mathbf{T}}$, we get

$$0 \leq \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathbf{T}}\lambda\mathbf{x} = \lambda\mathbf{x}^{\mathbf{T}}\mathbf{x}$$

and since $\mathbf{x}^{\mathbf{T}}x = \|\mathbf{x}\|^2 > 0$ for every $\mathbf{x}$, λ is non-negative.

iv. The rank of a matrix can exceed the number of distinct non-zero eigenvalues.

Response: True. Consider a matrix $\mathbf{A}$ with $\text{rank}(A) = 2$ and an eigenvalue λ with algebraic multiplicity 2. Then, the rank of the matrix exceeds the number of distinct non-zero eigenvalues.

v. A non-zero sum of two eigenvectors of a matrix $\mathbf{A}$ corresponding to the same eigenvalue λ is always an eigenvector.

Response: True. Consider two eigenvectors $\mathbf{x}, \mathbf{y}$ of a matrix $\mathbf{A}$ and suppose $\mathbf{x}, \mathbf{y}$ correspond to the same eigenvalue λ. Then

$$\mathbf{A}(\mathbf{x}+\mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y} = \lambda\mathbf{x} + \lambda\mathbf{y} = \lambda(\mathbf{x}+\mathbf{y})$$