Problem Set 9

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Section 5.1 Question 2 part (a), (d)

For each of the following linear operators T on a vector space V, compute the determinant of T and the characteristic polynomial of T.

(a) $V = \mathbb{R}^2,\ T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a - b \\ 5a + 3b \end{pmatrix}$

(d) $V = \mathcal{M}_{2\times 2}(\mathbb{R}),\ T(A) = 2A^t - A$

Response

(a) Let $\beta = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be a basis for $\mathbb{R}^2$. Then, we have

$$\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ [T]_\beta &= \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} \end{aligned}$$

Taking the determinant of $[T]_\beta$, we get

$$\det([T]_\beta) = 6 - -5 = 11$$

To compute the characteristic polynomial,

$$\begin{aligned} \det([T]_\beta - \lambda I) &= \begin{vmatrix} 2-\lambda & -1 \\ 5 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)(3-\lambda) - -5 \\ &= \lambda^2 - 5\lambda + 6 + 5 \\ \det([T]_\beta - \lambda I) &= \lambda^2 - 5\lambda + 11 \end{aligned}$$

(d) Let $\beta = \{e_1, e_2, e_3, e_4\}\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\}$ be a basis for $\mathcal{M}_{2\times 2}$. Then, we have

$$\begin{aligned}
T\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} &= 2\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\\
&= \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix}\\
&= 2e_1 + 0e_2 + 0e_3 + 0e_4\\
T\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} &= 2\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\\
&= \begin{pmatrix}0 & -1\\2 & 0\end{pmatrix}\\
&= 0e_1 + -1e_2 + 2e_3 + 0e_4\\
T\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} &= 2\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\\
&= \begin{pmatrix}0 & 2\\-1 & 0\end{pmatrix}\\
&= 0e_1 + 2e_2 + -1e_3 + 0e_4\\
T\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} &= 2\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\\
&= \begin{pmatrix}0 & -1\\2 & 0\end{pmatrix}\\
&= 0e_1 + 0e_2 + 0e_3 + 1e_4\\
[T]_\beta &= \begin{pmatrix}1 & 0 & 0 & 0\\0 & -1 & 2 & 0\\0 & 2 & -1 & 0\\0 & 0 & 0 & 1\end{pmatrix}
\end{aligned}$$

Taking the determinant of $[T]_\beta$, we get

$$\begin{aligned}
\det([T]_\beta) &= \begin{vmatrix}1 & 0 & 0 & 0\\0 & -1 & 2 & 0\\0 & 2 & -1 & 0\\0 & 0 & 0 & 1\end{vmatrix}\\
&= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} - a_{14}C_{14}\\
&= 1\begin{vmatrix}-1 & 2 & 0\\2 & -1 & 0\\0 & 0 & 1\end{vmatrix} + 0 + 0 + 0\\
&= a_{11}(a_{31}C_{31} - a_{32}C_{32} + a_{33}C_{33})\\
&= 1\left(0 + 0 + 1\begin{vmatrix}-1 & 2\\2 & -1\end{vmatrix}\right)\\
&= 1(1(1-4))\\
\det([T]_\beta) &= -3
\end{aligned}$$

To compute the characteristic polynomial,

$$
\begin{aligned}
\det([T]_\beta - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -1-\lambda & 2 & 0 \\ 0 & 2 & -1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} \\
&= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} - a_{14}C_{14} \\
&= 1-\lambda \begin{vmatrix} -1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} + 0 + 0 + 0 \\
&= a_{11}(a_{31}C_{31} - a_{32}C_{32} + a_{33}C_{33}) \\
&= (1-\lambda)\left(0 + 0 + 1 - \lambda \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix}\right) \\
&= (1-\lambda)((1-\lambda)((-1-\lambda)^2 - 4)) \\
&= (1-\lambda)^2((-1-\lambda)^2 - 4) \\
&= (1-\lambda)^2(\lambda^2 + 2\lambda + 1 - 4) \\
&= (1-\lambda)^2(\lambda^2 + 2\lambda - 3) \\
&= (\lambda^2 - 2\lambda + 1)(\lambda^2 + 2\lambda - 3) \\
&= (\lambda^4 + 2\lambda^3 - 3\lambda^2) - (2\lambda^3 + 4\lambda^2 - 6\lambda) + (\lambda^2 + 2\lambda - 3) \\
&= \lambda^4 - 6\lambda^2 + 8\lambda - 3
\end{aligned}
$$

Section 5.1 Question 9

(a) Prove that the linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.

(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(c) State and prove results analogous to (a) and (b) for matrices.

Response

(a) *Proof.* $\Longrightarrow$

Let T be invertible. We want to prove that 0 is not an eigenvalue of T. Recall that T is invertible if and only if $\det(T) \neq 0$. This implies that $N(T - \lambda I) = \{0\}$. Thus, 0 cannot be an eigenvalue of T.

$\Longleftarrow$

Assume 0 is not an eigenvalue of T by the previous statement. Then, we know that there exists no eigenvector x such that $T(x) = 0$. Therefore, we have that $N(T) = \{0\}$. From the rank-nullity theorem, we have

$$\begin{aligned}
nullity(T) + rank(T) &= dim(V) \\
0 + rank(T) &= dim(V) \\
dim(R(T)) &= dim(V) \\
dim(W) &= dim(V)
\end{aligned}$$

Thus, T is invertible. □

(b) *Proof.* $\Longrightarrow$

Assume a scalar λ is an eigenvalue with eigenvector $x \in V$ of T. We are given that T is invertible, so we have

$$\begin{aligned}
T(x) &= \lambda x \\
T^{-1}T(x) &= T^{-1}(\lambda x) \\
x &= \lambda T^{-1}(x) \\
\lambda^{-1}x &= \lambda^{-1}\lambda T^{-1}(x) && \text{from (a), we know that } \lambda \neq 0 \\
\lambda^{-1}x &= T^{-1}(x) && \lambda^{-1}\lambda = 1
\end{aligned}$$

Thus, λ^{-1} is an eigenvalue of T^{-1}.

$\Longleftarrow$

Assume a scalar λ^{-1} is an eigenvalue with eigenvector $y \in V$ of T^{-1}. We are given that T^{-1} is invertible, so we have

$$\begin{aligned}
T^{-1}(y) &= \lambda^{-1}y \\
TT^{-1}(y) &= T(\lambda^{-1}y) \\
y &= \lambda^{-1}T(y) \\
\lambda y &= \lambda\lambda^{-1}T(y) && \text{from (a), we know that } \lambda \neq 0 \\
\lambda y &= T(y) && \lambda\lambda^{-1} = 1
\end{aligned}$$

Thus, λ is an eigenvalue of T. □

(c) Analogous proof for (a): An $n \times n$ matrix A is invertible if and only if zero is not an eigenvalue of A.

Proof. $\Longrightarrow$
Let A be invertible. We want to prove that 0 is not an eigenvalue of A. If A is invertible, this means that it is one-to-one and onto, meaning there is no non-zero vector such that $Ax = 0$. So, 0 cannot be an eigenvalue of A.

$\Longleftarrow$
Assume 0 is not an eigenvalue by the previous statement. Then, we know that the only vector that satisfies $Ax = 0$ is the zero vector. This implies that A is one-to-one, which also implies that A is invertible. $\square$

Analogous proof for (b): Given that A is invertible, prove that a scalar λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

Proof. $\Longrightarrow$
Assume a scalar λ is an eigenvalue with eigenvector $x \in V$ of T. We are given that T is invertible, so we have

$$\begin{aligned}
A(x) &= \lambda x \\
A^{-1}A(x) &= A^{-1}(\lambda x) \\
x &= \lambda A^{-1}(x) \\
\lambda^{-1}x &= \lambda^{-1}\lambda A^{-1}(x) && \text{from (a), we know that } \lambda \neq 0 \\
\lambda^{-1}x &= A^{-1}(x) && \lambda^{-1}\lambda = 1
\end{aligned}$$

Thus, λ^{-1} is an eigenvalue of A^{-1}.

$\Longleftarrow$
Assume a scalar λ^{-1} is an eigenvalue with eigenvector $y \in V$ of A^{-1}. We are given that A^{-1} is invertible, so we have

$$\begin{aligned}
A^{-1}(y) &= \lambda^{-1}y \\
AA^{-1}(y) &= A(\lambda^{-1}y) \\
y &= \lambda^{-1}A(y) \\
\lambda y &= \lambda\lambda^{-1}A(y) && \text{from (a), we know that } \lambda \neq 0 \\
\lambda y &= A(y) && \lambda\lambda^{-1} = 1
\end{aligned}$$

Thus, λ is an eigenvalue of A. $\square$

Section 5.1 Question 12

A scalar matrix is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Response

(a) *Proof.* Let A and λI be similar square matrices. We want to prove that $A = \lambda I$. Let $A = B\lambda IB^{-1}$, where B is invertible. Then we have

$$\begin{aligned} A &= B\lambda IB^{-1} \\ &= \lambda(BIB^{-1}) \\ &= \lambda(BB^{-1}) && BI = B \\ &= \lambda(I) \\ A &= \lambda I \end{aligned}$$

□

(b) *Proof.* Let A be a diagonalizable matrix having only one eigenvalue λ. Let $A = BDB^{-1}$, where B is invertible and D is diagonal. Since A only has one eigenvalue, D must be the scalar matrix λI. From (a), we have $A = \lambda I$, so A is a scalar matrix. □

(c) *Proof.* Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Taking the characteristic polynomial of the matrix, we get

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)^2 - 0 \\ \det(A - \lambda I) &= (1-\lambda)^2 \end{aligned}$$

Because we only have one eigenvalue $\lambda = 1$, there is no ordered basis with 2 linearly independent vectors. By definition, A is diagonalizable if and only if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix. Because there is no such ordered basis for A, A is not diagnoalizable. □

Section 5.1 Question 15

For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

Response

Proof. We want to prove that A and A^t have the same characteristic polynomial. Recall that the characteristic polynomial is defined as $f(x) = \det(A - \lambda I)$. Let $f(x)$ and $g(x)$ be the characteristic polynomials for A and A^t respectively. Then we have

$$\begin{aligned} f(x) &= \det(A - \lambda I) \\ &= \det((A - \lambda I)^t) && \det(A) = \det(A^t) \\ &= \det(A^t - \lambda I^t) \\ &= \det(A^t - \lambda I) \\ f(x) &= g(x) \end{aligned}$$

$\square$

Section 5.1 Question 18

Let T be the linear operator on $\mathcal{M}_{n\times n}(\mathbb{R})$ defined by $T(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find an ordered basis β for $\mathcal{M}_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is diagonal.

(d) Find an ordered basis β for $\mathcal{M}_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is diagonal for $n > 2$.

Response

(a) We want to show that the only eigenvalues for T are ± 1. Let λ be an eigenvalue of T and A be its corresponding eigenvector. Then we have

$$\begin{aligned}
T(A) &= \lambda A \\
A^t &= \lambda A \\
T(A^t) &= T(\lambda A) \\
&= \lambda T(A) \\
A &= \lambda T(A) && (A^t)^t = A \\
&= \lambda(\lambda A) && T(A) = \lambda A \\
&= \lambda^2 A \\
0 &= \lambda^2 A - A \\
&= (\lambda^2 - 1)A
\end{aligned}$$

Since we defined A to be an eigenvector of T, by definition it cannot be 0. So, we have

$$\begin{aligned}
(\lambda^2 - 1)A &= 0 \\
(\lambda + 1)(\lambda - 1) &= 0 \\
\lambda &= \pm 1
\end{aligned}$$

So, the only eigenvalues of T are ± 1.

(b) For $\lambda = -1$

$$\begin{aligned}
T(A) &= \lambda A \\
&= -1A \\
A^t &= -A && T(A) = A^t
\end{aligned}$$

When $\lambda = -1$, the matrix A is skew-symmetric, so the set of skew-symmetric matrices are eigenvectors that correspond to $\lambda = -1$.

For $\lambda = 1$

$$\begin{aligned}
T(A) &= \lambda A \\
&= 1A \\
A^t &= A && T(A) = A^t
\end{aligned}$$

When $\lambda = 1$, the matrix A is symmetric, so the set of symmetric matrices are eigenvectors that correspond to $\lambda = 1$.

(c) $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

(d) Let B_{ii} be the $n \times n$ matrix with the ii^{th} element 1, and all others 0. Then, we have $\beta = (B_{ii})_{i=1,2,\ldots,n} \cup (B_{ij} + B_{ji})_{i>j} \cup (B_{ij} - B_{ji})_{i>j}$

Section 5.2 Question 2

For each of the following matrices $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

Response

(a)

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)^2 - 0 \end{aligned}$$

(a) Clearly, $(1-\lambda)^2 = (1-\lambda)(1-\lambda)$ splits.

(b) Solving for λ, we get $\lambda = 1$ with multiplicity 2. We test for $multiplicity = n - rank(A - \lambda I)$

$$\begin{aligned} multiplicity &= n - rank(A - 1I) \\ 2 &= 2 - rank\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ &= 2 - 1 \\ 2 &\neq 1 \end{aligned}$$

This matrix is not diagonalizable.

(b)

$$\begin{aligned} \det(A - \lambda I) &= \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)^2 - 9 \\ &= \lambda^2 - 2\lambda + 1 - 9 \\ &= \lambda^2 - 2\lambda - 8 \end{aligned}$$

(a) Clearly, $\lambda^2 - \lambda - 8 = (\lambda + 2)(\lambda - 4)$ splits.

(b) Solving for λ, we get $\lambda = -2, 4$. We test for $multiplicity = n - rank(A - \lambda I)$
When $\lambda = -2$

$$\begin{aligned} multiplicity &= n - rank(A - -2I) \\ 1 &= rank(A + 4I) \\ &= 2 - rank\begin{pmatrix} 3+3 \\ 3+3 \end{pmatrix} \\ &= 2 - 1 \\ 1 &= 1 \end{aligned}$$

When $\lambda = 4$

$$\begin{aligned} multiplicity &= n - rank(A - 4I) \\ 1 &= rank(A + 4I) \\ &= 2 - rank\begin{pmatrix} -3+3 \\ 3+-3 \end{pmatrix} \\ &= 2 - 1 \\ 1 &= 1 \end{aligned}$$

Therefore, this matrix is diagonalizable.

To find the eigenvectors, we find $(A - \lambda I)x = 0$
For $\lambda = -2$

$$\begin{aligned} 0 &= (A - \lambda I)x \\ &= (A - -2I)x \\ &= \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

For $\lambda = 4$

$$\begin{aligned} 0 &= (A - \lambda I)x \\ &= (A - 4I)x \\ &= \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

So we have $Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. To find the diagonal matrix D, recall $D = Q^{-1}AQ$

$$\begin{aligned} D &= Q^{-1}AQ \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 4 \\ 2 & 4 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} -4 & 0 \\ 0 & 8 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \end{aligned}$$

$Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$

(c)

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} \\
&= (1-\lambda)(2-\lambda) - 12 \\
&= \lambda^2 - 3\lambda + 2 - 12 \\
&= \lambda^2 - 3\lambda - 10
\end{aligned}$$

(a) Clearly, $\lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$ splits

(b) Solving for λ, we get $\lambda = -2, 5$. We test for $multiplicity = n - rank(A - \lambda I)$
When $\lambda = -2$

$$\begin{aligned}
multiplicity &= n - rank(A - \lambda I) \\
1 &= 2 - rank(A - -2I) \\
&= 2 - rank \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \\
&= 2 - 1 \\
1 &= 1
\end{aligned}$$

When $\lambda = 5$

$$\begin{aligned}
multiplicity &= n - rank(A - \lambda I) \\
1 &= 2 - rank(A - 5I) \\
&= 2 - rank \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix} \\
&= 2 - 1 \\
1 &= 1
\end{aligned}$$

Therefore, this matrix is diagonalizable.

To find the eigenvectors, we find $(A - \lambda I)x = 0$
For $\lambda = -2$

$$\begin{aligned}
0 &= (A - \lambda I)x \\
&= (A - -2I)x \\
&= \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 4 \\ -3 \end{pmatrix}
\end{aligned}$$

For $\lambda = 5$

$$\begin{aligned}
0 &= (A - \lambda I)x \\
&= (A - 5I)x \\
&= \begin{pmatrix} -4 & 4 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{aligned}$$

So we have $Q = \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix}$. To find the diagonal matrix D, recall $D = Q^{-1}AQ$

$$\begin{aligned} D &= Q^{-1}AQ \\ &= \frac{1}{7}\begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -8 & 5 \\ 6 & 5 \end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix} -14 & 0 \\ 0 & 35 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix} \end{aligned}$$

$Q = \begin{pmatrix} 4 & 1 \\ -3 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$

(d)

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & -4 & 0 \\ 8 & -5-\lambda & 0 \\ 6 & -6 & 3-\lambda \end{vmatrix} \\ &= a_{13}C_{13} - a_{23}C_{23} + a_{33}C_{33} \\ &= 0 - 0 + (3-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{vmatrix} \\ &= (3-\lambda)((7-\lambda)(-5-\lambda) - -32) \\ &= (3-\lambda)(\lambda^2 - 2\lambda - 35 + 32) \\ &= (3-\lambda)(\lambda^2 - 2\lambda - 3) \\ &= (3-\lambda)(\lambda - 3)(\lambda + 1) \\ &= (\lambda - 3)^2(\lambda + 1) \end{aligned}$$

(a) Clearly, $(\lambda - 3)^2(\lambda + 1) = (\lambda - 3)(\lambda - 3)(\lambda + 1)$ splits

(b) Solving for λ, we get $\lambda = -1, 3$. We test for $multiplicity = n - rank(A - \lambda I)$
When $\lambda = -1$

$$\begin{aligned} multiplicity &= n - rank(A - \lambda I) \\ 1 &= 3 - rank(A - -1I) \\ &= 3 - rank\begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{pmatrix} \\ &= 3 - 2 \\ 1 &= 1 \end{aligned}$$

When $\lambda = 3$

$$\begin{aligned} multiplicity &= n - rank(A - \lambda I) \\ 2 &= 3 - rank(A - 3I) \\ &= 3 - rank\begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \\ &= 3 - 1 \\ 2 &= 2 \end{aligned}$$

Therefore, this matrix is diagonalizable.

To find the eigenvectors, we find $(A - \lambda I)x = 0)$
For $\lambda = -1$

$$\begin{aligned} 0 &= (A - \lambda I)x \\ &= (A - -1I)x \\ &= \begin{pmatrix} 8 & -4 & 0 \\ 8 & -4 & 0 \\ 6 & -6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} \end{aligned}$$

For $\lambda = 3$

$$\begin{aligned} 0 &= (A - \lambda I)x \\ &= (A - 3I)x \\ &= \begin{pmatrix} 4 & -4 & 0 \\ 8 & -8 & 0 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

So, we have $Q = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$. To find the diagonal matrix D, recall $D = Q^{-1}AQ$

$$\begin{aligned} D &= Q^{-1}AQ \\ &= \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 \\ 4 & -2 & 0 \\ 3 & -3 & 2 \end{pmatrix} \begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -1 & 1 & 0 \\ 4 & -2 & 0 \\ 3 & -3 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 & 0 \\ -4 & 3 & 0 \\ -3 & 0 & 3 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \end{aligned}$$

$$Q = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \ D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(e)

$$
\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 0-\lambda & 0 & 1 \\ 1 & 0-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} \\
&= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} \\
&= -\lambda \begin{vmatrix} -\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} - 0 + 1 \begin{vmatrix} 1 & -\lambda \\ 0 & 1 \end{vmatrix} \\
&= -\lambda(\lambda^2 - \lambda - -1) + (1-0) \\
&= -\lambda(\lambda^2 + \lambda + 1) + 1 \\
&= \lambda^3 - \lambda^2 + \lambda - 1
\end{aligned}
$$

(a) $\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda^2+1)(1-\lambda)$ cannot split

This matrix is not diagonalizable.

(f)

$$
\begin{aligned}
\det(A-\lambda I) &= \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} \\
&= a_{11}C_{11} - a_{21}C_{21} + a_{31}C_{31} \\
&= (1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} - 0 + 0 \\
&= (1-\lambda)((1-\lambda)(3-\lambda) - 0) \\
&= (1-\lambda)(1-\lambda)(3-\lambda)
\end{aligned}
$$

(a) Clearly, the characteristic polynomial $(1-\lambda)(1-\lambda)(3-\lambda)$ splits

(b) Solving for λ, we get $\lambda = 1, 3$. We test for $multiplicity = n - rank(A - \lambda I)$
When $\lambda = 1$

$$
\begin{aligned}
multiplicity &= n - rank(A - \lambda I) \\
1 &= 3 - rank(A - 1I) \\
&= 3 - rank \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \\
&= 3 - 2 \\
1 &= 1
\end{aligned}
$$

When $\lambda = 3$

$$
\begin{aligned}
multiplicity &= n - rank(A - \lambda I) \\
2 &= 3 - rank(A - 3I) \\
&= 3 - rank \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\
&= 3 - 2 \\
2 &\neq 1
\end{aligned}
$$

This matrix is not diagonalizable.

(g)

$$\begin{aligned}
\det(A-\lambda I) &= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{vmatrix} \\
&= a_{11}C_{11} - a_{12}C_{12} + a_{13}C_{13} \\
&= (3-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ -1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 2 & 4-\lambda \\ -1 & -1 \end{vmatrix} \\
&= (3-\lambda)((4-\lambda)(1-\lambda) - -2) - (2-2\lambda - -2) + (-2-(-4+\lambda)) \\
&= (3-\lambda)((4-\lambda)(1-\lambda)+2) - (4-2\lambda) + (2-\lambda) \\
&= (3-\lambda)((4-\lambda)(1-\lambda)+2) + (-2+\lambda) \\
&= \lambda^3 - 8\lambda^2 + 20\lambda - 16
\end{aligned}$$

(a) Clearly, $\lambda^3 - 8\lambda^2 + 20\lambda - 16 = (\lambda-2)(\lambda-2)(\lambda-4)$ splits

(b) Solving for λ, we get $\lambda = 2, 4$. We test for $multiplicity = n - rank(A-\lambda I)$
When $\lambda = 2$

$$\begin{aligned}
multiplicity &= n - rank(A-\lambda I) \\
2 &= 3 - rank(A-1I) \\
&= 3 - rank\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} \\
&= 3-1 \\
2 &= 2
\end{aligned}$$

When $\lambda = 4$

$$\begin{aligned}
multiplicity &= n - rank(A-\lambda I) \\
1 &= 3 - rank(A-3I) \\
&= 3 - rank\begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} \\
&= 3-2 \\
1 &= 1
\end{aligned}$$

Therefore, this matrix is diagonalizable.

To find the eigenvectors, we find $(A-\lambda I)x = 0)$
For $\lambda = 2$

$$\begin{aligned}
0 &= (A-\lambda I)x \\
&= (A-2I)x \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}
\end{aligned}$$

For $\lambda = 4$

$$\begin{aligned}
0 &= (A - \lambda I)x \\
&= (A - 4I)x \\
&= \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},
\end{aligned}$$

So, we have $Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$. To find the diagonal matrix D, recall $D = Q^{-1}AQ$

$$\begin{aligned}
D &= Q^{-1}AQ \\
&= \frac{1}{2} \begin{pmatrix} -1 & -1 & -3 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} -1 & -1 & -3 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 4 \\ 0 & -2 & 8 \\ -2 & 0 & -4 \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}$$

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix}, \; D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Section 5.2 Question 3 part (a), (c)

For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(a) $V = P_3(\mathbb{R})$ and T is defined by $T(f(x)) = f'(x) + f''(x)$.

(c) $V = \mathbb{R}^3$ and T is defined by $T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}$

Response

(a) Let $f(x) = a_1 + a_2x + a_3x^2 + a_4x^3$. Then, $f'(x) = a_2 + 2a_3x + 3a_4x^2$ and $f''(x) = 2a_3 + 6a_4x$. So,

$$\begin{aligned} T(a_1 + a_2x + a_3x^2 + a_4x^3) &= a_2 + 2a_3x + 3a_4x^2 + 2a_3 + 6a_4x \\ &= a_2 + 2a_3 + (2a_3 + 6a_4)x + 3a_4x^2 \end{aligned}$$

Let γ be the standard basis for V. Then, we have

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To get the characteristic polynomial, we do

$$\begin{aligned} \det([T]_\gamma - \lambda I) &= \begin{pmatrix} 0-\lambda & 1 & 2 & 0 \\ 0 & 0-\lambda & 2 & 6 \\ 0 & 0 & 0-\lambda & 3 \\ 0 & 0 & 0 & 0-\lambda \end{pmatrix} \\ &= \begin{pmatrix} -\lambda & 1 & 2 & 0 \\ 0 & -\lambda & 2 & 6 \\ 0 & 0 & -\lambda & 3 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} \\ &= \lambda^4 \end{aligned}$$

(a) Clearly, $\lambda^4 = (-\lambda)(-\lambda)(-\lambda)(-\lambda)$ splits

(b) Solving for λ, we get that $\lambda = 0$ with multiplicity 4. We test for $multiplicity = n - rank(A - \lambda I)$.

$$\begin{aligned} multiplicity &= n - rank([T]_\gamma - 0I) \\ 4 &= 4 - rank\begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= 4 - 3 \\ 4 &\neq 1 \end{aligned}$$

This linear operator is not diagonalizable.

(c) Let γ be the standard basis for V. Then, we have

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

To get the chraracteristic polynomial, we do

$$
\begin{aligned}
\det([T]_\gamma - \lambda I) &= \begin{pmatrix} 0-\lambda & 1 & 0 \\ -1 & 0-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\
&= \begin{pmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} \\
&= a_{13}C_{13} - a_{23}C_{23} + a_{33}C_{33} \\
&= 0 - 0 + (2-\lambda)\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} \\
&= (2-\lambda)(\lambda^2 - -1) \\
&= (2-\lambda)(\lambda^2 + 1) \\
&= (2\lambda^2 + 2 - \lambda^3 - \lambda) \\
&= -\lambda^3 + 2\lambda^2 - \lambda + 2 \\
&= \lambda^3 - 2\lambda^2 + \lambda - 2
\end{aligned}
$$

(a) Clearly, $\lambda^3 - 2\lambda^2 + \lambda - 2$ cannot be split

This matrix is not diagonalizable

Section 5.2 Question 9 part (a)

Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.

(a) Prove that the characteristic polynomial for T splits.

Response

Proof. The chraracteristic polynomial is defined as $f(\lambda) = \det([T]_\beta - \lambda I)$. Since $[T]_\beta$ is upper triangular, we can rewrite this as $f(\lambda) = \prod_{i=1}^{n}(([T]_\beta)_{ii} - \lambda)$, which splits. $\square$

Section 5.2 Question 10

Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding multiplicities $m_1, m_2, \ldots, m_k$. Suppose that β is a basis for V such that $[T]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[T]_\beta$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and that each λ, occurs m_i times $(1 \leq i \leq k)$

Response

The characteristic polynomial is defined by $f(\lambda) = \det([T]_\beta - \lambda I)$. Since $[T]_\beta$ is upper triangular, we know that $\det([T]_\beta - \lambda I)$ is also upper triangular, so we can rewrite the characteristic polynomial as $f(\lambda) = \prod_{i=1}^{k}(([T]_\beta)_{ii}) - \lambda I)$. This shows that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the diagonal entries of $[T]_\beta$ and also that each λ_i occurs m_i times $(1 \leq i \leq k)$.

Section 5.2 Question 13

Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T , λ^{-1} is an eigenvalue of T^{-1} (Exercise 9 of Section 5.1). Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Response

(a) *Proof.* Let $E_\lambda(T)$ be the eigenspace of T corresponding to λ. From (5.1.9), we have that for any eigenvalue of T, λ^{-1} is an eigenvalue of T^{-1}. We must prove that x is an eigenvector of T corresponding to λ, if and only if x is an eigenvector of T^{-1} corresponding to λ^{-1}.

$$\begin{aligned}
T(x) &= T(\lambda\lambda^{-1}x) \\
&= T(\lambda(\lambda^{-1}x)) \\
&= T(\lambda T^{-1}(x)) && \lambda^{-1}x = T^{-1}(x) \\
&= \lambda T T^{-1}(x) \\
&= \lambda x
\end{aligned}$$

$$\begin{aligned}
T^{-1}(x) &= T^{-1}(\lambda^{-1}\lambda x) \\
&= T^{-1}(\lambda^{-1}(\lambda x)) \\
&= T^{-1}(\lambda^{-1}T(x)) && \lambda x = T(x) \\
&= \lambda^{-1}T^{-1}(T(x)) \\
T^{-1}(x) &= \lambda^{-1}x
\end{aligned}$$

So, x is an eigenvector of T corresponding to λ if and only if x is also an eigenvector of T corresponding to λ^{-1}. So, we can write that $E_\lambda(T) = E_\lambda^{-1}(T^{-1})$. Therefore, the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}. □

(b) *Proof.* Given that T is diagonalizable, we know that it has n linearly independent eigenvectors. From (a), we have that any eigenvector of T is also an eigenvector of T^{-1}. So, T^{-1} also has n linearly independent eigenvectors. Thus, T^{-1} is also diagonalizable. □