Problem Set 1

Warren Kim

April 13, 2023

Question 1 part (a)

In Q3(e) in HW1, we proved the De Morgan's laws in propositional logic. Here, we prove the equivalent laws in set theory.

(a) Prove the De Morgan's laws in set theory: Given two sets $A, B \subseteq X$, show that

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

Response

Proof. $(A \cup B)^c \subseteq A^c \cap B^c$
Let $x \in (A \cup B)^c$ Then, by definition, x is neither in A or B; that is, $x \notin A$ and $x \notin B$. But this is equivalent to $x \in A^c$ and $x \in B^c$. Thus, $x \in A^c \cap B^c$, so $(A \cup B)^c \subseteq A^c \cap B^c$.

$A^c \cap B^c \subseteq (A \cup B)^c$
Let $x \in A^c \cap B^c$. Then, $x \notin A$ and $x \notin B$; that is, $x \notin A \cup B$, or $x \in (A \cup B)^c$. Thus, $A^c \cap B^c \subseteq (A \cup B)^c$

Therefore, $(A \cup B)^c = A^c \cap B^c$. □

Question 5 parts (b), (c), (d)

Consider a function $f : X \to Y$ and let $A \subseteq X$ and $B \subseteq X$.

(b) Show that $f(A \cup B) \subseteq f(A) \cup f(B)$.

(c) Let A, B be sets such that $A \cap B \neq \emptyset$. Prove the converse statement $f(A) \cap f(B) \subseteq f(A \cap B)$ is false.
(**Hint:** find a counterexample)

[The converse statement is still false when $A \cap B = \emptyset$ as long as $f(A) \cap f(B) \neq \emptyset$, but imposing $A \cap B \neq \emptyset$ is more interesting. Note that $f(\emptyset) = \emptyset$.]

(d) Give an extra condition on f which makes this statement $f(A) \cap f(B) \subseteq f(A \cap B)$ true and prove this result.

Response

(b) *Proof.* Let $y \in f(A \cap B)$. Then, $\exists x \in A \cap B$ such that $f(x) = y$. Then, $f(x) \in f(A)$ and $f(x) \in f(B)$; that is, $f(x) \in f(A) \cap f(B)$. But $f(x) = y$, so $y \in f(A) \cap f(B)$. Thus, $f(A \cap B) \subseteq f(A) \cap f(B)$. $\square$

(c) *Proof.* Assume by contradiction that $f(A) \cap f(B) \subseteq f(A \cap B)$. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(S) := \{x^2 \in S : \forall x \in S\}$, where $A = \{0, 1\}$, $B = \{-1, 0\}$. Clearly, $f(A \cap B) = \{0\}$, $f(A) = f(B) = \{0, 1\}$. So, $f(A) \cap f(B) = \{0, 1\}$, but $f(A \cap B) = \{0\}$, so $f(A) \cap f(B) \not\subseteq f(A \cap B)$, which is a contradiction to our assumption. $\square$

(d) If f is injective, then $f(A) \cap f(B) \subseteq f(A \cap B)$ is true.

Proof. Consider a function $f : X \to Y$ and let $A \subseteq X$ and $B \subseteq X$. Now, let $y \in f(A) \cap f(B)$. Then, $\exists a \in A$, $\exists b \in B$ such that $f(a) = y = f(b) \implies a = b \implies a \in B,\ b \in A$. Since f is injective. Thus, $x \in A \cap B \implies f(x) \in f(A \cap B)$. But $f(x) = y$, so $y \in f(A) \cap f(B)$. Thus, $f(A) \cap f(B) \subseteq f(A \cap B)$. $\square$

1 Question 7 parts (a), (c), (e)

In class, we saw an axiomatic foundation of N. Making use of the notion of successor, we can make an appropriate definition of + (i.e. addition behaves as we learnt way back). Furthermore, we can make sense of $m - n$ when $m > n$. You may assume these two facts from now on. Now, you will be guided through a foundational construction of $\mathbb{Z}$. Consider the set $\mathbb{N} \times \mathbb{N}$ and the following relation:

$$(m_1, n_1) \sim (m_2, n_2) \text{ if } m_1 + n_2 = n_1 + m_2$$

(Perhaps after the end of this problem, I recommend coming back and trying to understand why the equivalence relation defined as above would work to construct $\mathbb{Z}$. Try to draw a picture.)

(a) Show that $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$ (recall the *cancellative law:* $m + n = m + l$, for $m, n, l \in \mathbb{N}$, then $n = l$)

(c) Use part (b) to show the following: if $[m_1, n_1)] = [m_2, n_2)]$ and $[a_1, b_1)] = [a_2, b_2)]$, then $[(m_1 + a_1, n_1 + b_1)] = [(m_2 + a_2, n_2 + b_2)]$

Part (c) shows us how to define addition + on $\mathbb{Z}$ as follows: we define $[(m, n)] + [(a, b)] = [(m + a, n + b)]$

(e) Show that for every $[(m, n)] \in \mathbb{Z}$, we have $[(m, n)] + [(n, m)] = [(1, 1)]$

Part (e) tells us that the additive inverse of $[(m, n)]$ is $[(n, m)]$ and we write $[(n, m)] = -[(m, n)]$. In particular, we have made sense of what we usually denote by $-n$ for $n \in \mathbb{N}$

Response

(a) *Proof.* Let $(m_1, n_1), (m_2, n_2), (m_3, n_3) \in \mathbb{N} \times \mathbb{N}$. Then,

(Reflexive) $(m_1, n_1) \sim (m_1, n_1) \iff m_1 + n_1 = n_1 + m_1$. Because addition is commutative, $\sim$ is reflexive.

(Symmetric) Because $(m_1, n_1) \sim (m_2, n_2)$, we want to show that $(m_2, n_2) \sim (m_1, n_1)$. $(m_2, n_2) \sim (m_1, n_1) \iff m_2 + n_1 = n_2 + m_1$. Clearly, $\sim$ is symmetric.

(Transitive) If $(m_1, n_1) \sim (m_2, n_2)$ and $(m_2, n_2) \sim (m_3, n_3)$, we want to show that $(m_1, n_1) \sim (m_3, n_3)$. Recall that

$$(1)\ (m_1, n_1) \sim (m_2, n_2) \iff m_1 + n_2 = n_1 + m_2$$

$$(2)\ (m_2, n_2) \sim (m_3, n_3) \iff m_2 + n_3 = n_2 + m_3$$

$$\begin{aligned}
m_2 + n_3 &= n_2 + m_3 && \text{from (1)} \\
m_1 + m_2 + n_3 &= m_1 + n_2 + m_3 && \text{cancellative law} \\
(m_1 + n_3) + m_2 &= (m_1 + n_2) + m_3 && \text{associativity and commutativity of addition} \\
(m_1 + n_3) + m_2 &= (n_1 + m_2) + m_3 && \text{from (2), } m_1 + n_2 = n_1 + m_2 \\
(m_1 + n_3) + m_2 &= (n_1 + m_3) + m_2 && \text{associativity and commutativity of addition} \\
m_1 + n_3 &= n_1 + m_3 && \text{cancellative law}
\end{aligned}$$

So $\sim$ is transitive.

Because $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation on $\mathbb{N} \times \mathbb{N}$. □

(c) *Proof.* Let $[(m_1, n_1)] = [(m_2, n_2)]$ and $[(a_1, b_1)] = [(a_2, b_2)]$. Then, $(m_1, n_1) \sim (m_2, n_2)$ and $(a_1, b_1) \sim (a_2, b_2)$. From (b), $(m_1 + a_1, n_1 + b_1) \sim (m_2 + a_2, n_2 + b_2)$, so $[(m_1 + a_1, n_1 + b_1)] = [(m_2 + a_2, n_2, +b_2)]$ □

(e) *Proof.*

$$\begin{aligned}
[(m,n)] + [(n,m)] &= [(1,1)] \\
[(m+n, n+m)] &= [(1,1)] && \text{from (c)} \\
&\iff (m+n, n+m) \sim (1,1) \\
&\iff (m+n) + 1 = (n+m) + 1 \\
&\iff (m+n) + 1 = (m+n) + 1 && \text{commutativity and associativity of addition}
\end{aligned}$$

□