Problem Set 7

Warren Kim

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Section 2.3 Question 16

Let V be a finite-dimensional vector space, and let $T : V \to V$ be linear.

(a) If $rank(T) = rank(T^2)$, prove that $R(T) \cap N(T) = \{0\}$. Deduce that $V = R(T) \oplus N(T)$ (see the exercises of Section 1.3).

(b) Prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k.

Response

(a) *Proof.* We want to prove that if $rank(T) = rank(T^2)$, then $R(T) \cap N(T) = \{0\}$. Let some $x \in N(T)$. Then, it follows that $T(x) = 0$ by the definition of the null space. Applying the T again, we get that $T(T(x)) = T(0) = 0$. Thus, $N(T) \in N(T^2)$. Now, using the dimension theorem, we have that

$$nullity(T) + rank(T) = dim(V)$$

Note that since $rank(T) = rank(T^2)$, we can rewrite this as

$$nullity(T^2) + rank(T^2) = dim(V)$$

Setting the two equations equal, we get

$$\begin{aligned} nullity(T) + rank(T) &= nullity(T^2) + rank(T^2) \\ nullity(T) &= nullity(T^2) + (rank(T^2) - rank(T)) \\ nullity(T) &= nullity(T^2) \end{aligned}$$

Therefore, we have that $N(T) = N(T^2)$.

Now, let some $y \in R(T) \cap N(T)$; that is, $y \in R(T)$ and $y \in N(T)$. By the definition of $R(T)$, if $y \in R(T)$, then there exists an $x \in V$ such that $T(x) = y$. By the definition of $N(T)$, if $y \in N(T)$, we have

$$\begin{aligned} T(y) &= 0 \\ T(T(x)) &= 0 & T(x) = 0 \\ y &= 0 \end{aligned}$$

Therefore, since $y = 0$ for an arbitrary y, we have that $R(T) \cap N(T) = \{0\}$. To show that $V = R(T) \oplus N(T)$, recall that the rank-nullity theorem can be rewritten as

$$\begin{aligned} nullity(T) + rank(T) &= dim(V) \\ dim(N(T) + R(T)) &= dim(V) \\ &= dim(R(T) + N(T) - (R(T) \cap N(T))) \\ &= dim(R(T) + N(T) - 0) \\ dim(N(T) + R(T)) &= dim(R(T) \oplus N(T)) \end{aligned}$$

Since we proved that $R(T) \cap NT(T) = \{0\}$, we deduced that $V = R(T) \oplus N(T)$. $\square$

(b) *Proof.* We want to prove that $V = R(T^k) \oplus N(T^k)$ for some positive integer k. From part (a), we have that $R(T) = R(T^2)$ and $V = R(T) \oplus N(T)$ where $k = 1$. Let this be true for some arbitrary k. We need to prove that this holds true when $k = k + 1$. Note that $\forall y \in R(T^{k+1})$, $\exists x \in V$ such that

$$\begin{aligned} T^{k+1}(x) &= y \\ T^k(T(x)) &= y \\ R(T^{k+1}) &\subset R(T^k) & y \in R(T^k) \\ dim(R(T^{k+1})) &\leq dim(R(T^k)) \\ rank(T^{k+1}) &\leq rank(T^k) \end{aligned}$$

So we have that $rank(T^{k+a}) \leq rank(T^k)$, where $a \geq 0$. Now, we want to prove that $rank(T^{k+1}) = rank(T^k)$. Note that at most, $rank(T) = n$. So, the samllest possible rank, denoted by $rank(T^j)$, is $rank(T^j) \geq 0$. Now, consider when $k = j$

$$\begin{aligned} rank(T^{k+a}) &= rank(T^k) \\ &\leq rank(T^k) && \text{but } rank(T^k) \text{ is the smallest possible rank} \\ rank(T^{k+a}) &\geq rank(T^k) \end{aligned}$$

Therefore, since we have that both $rank(T^{k+a}) \leq rank(T^k)$ and $rank(T^{k+a}) \geq rank(T^k)$, it must be true that $rank(T^k) = rank(T^{k+a})$. Now, let $a = k$. We have that $rank(T^k) = rank(T^{2k})$, and from part (a), we have that $V = R(T) \oplus N(T)$. $\square$

Section 2.3 Question 18

Let β be an ordered basis for a finite-dimensional vector space V, and let $T : V \to V$ be linear. Prove that, for any nonnegative integer k, $[T^k]_\beta = ([T]_\beta)^k$.

Response

Proof. Let $\beta = \{b_1, b_2, \ldots, b_n\}$ be the ordered basis for V. By definition of a linear transformation, for some arbitrary vector $b_j \in \beta$, we have

$$T(b_j) = \sum_{i=1}^{n} a_{j,i} b_j, \ 1 \leq j \leq n$$

Now, consider $T^2(b_j)$

$$\begin{aligned} T^2(b_j) &= T(T(b_j)) \\ &= T\left(\sum_{i=1}^{n} a_{j,i} b_j\right. \\ &= \sum_{i=1}^{n} a_{j,i} T(b_j) \\ &= \sum_{i=1}^{n} a_{j,i} \sum_{k=1}^{n} a_{i,k} b_j && 1 \leq k \leq n \\ &= \sum_{k=1}^{n} \sum_{i=1}^{n} a_{j,i} a_{i,k} b_j \end{aligned}$$

So,

$$[T]_\beta = \begin{pmatrix} \sum_{i=1}^{n} a_{j,i} a_{i,1} \\ \sum_{i=1}^{n} a_{j,i} a_{i,2} \\ \vdots \\ \sum_{i=1}^{n} a_{j,i} a_{i,n} \end{pmatrix}$$

Let $k = 1$. Then we have

$$\begin{aligned} [T]_\beta^2 &= \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{pmatrix} \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{n} a_{1,i} a_{i,1} & \ldots & a_{1,i} a_{i,1} \\ \vdots & ddots & \vdots \\ \sum_{i=1}^{n} a_{n,i} a_{i,n} & \ldots & a_{n,i} a_{i,n} \end{pmatrix} \\ &= ([T]_\beta)^2 \end{aligned}$$

Now, we want to prove this holds true when $k = k + 1$. Consider

$$\begin{aligned} ([T]_\beta)^{k+1} &= ([T]_\beta)^k ([T]_\beta) \\ &= [T^k]_\beta [T]_\beta \\ &= [T^k T]_\beta \\ &= [T^{k+1}]_\beta \end{aligned}$$

Since we have shown the general case holds when $k = k + 1$, this concludes the induction. Thus, we have proved tha $[T^k]_\beta = ([T]_\beta)^k$ for any nonnegative integer k. $\square$

Section 2.4 Question 1 part (a) - (e)

Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β respectively, $T : V \to W$ is linear, and A and B are matrices.

(a) $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$.

(b) T is invertible if and only if T is one-to-one and onto.

(c) $T = L_A$, where $A = [T]_\alpha^\beta$.

(d) $\mathcal{M}_{2\times 3}(F)$ is isomorphic to F^5.

(e) $P_n(F)$ is isomorphic to $P_m(F)$ if and only if $n = m$.

Response

(a) False

(b) True

(c) False

(d) False

(e) True

Section 2.4 Question 3

Which of the following pairs of vector spaces are isomorphic? Justify your answers.

(a) F^3 and $P_3(F)$.

(b) F^4 and $P_3(F)$.

(c) $\mathcal{M}_{2\times2}(\mathbb{R})$ and $P_3(\mathbb{R})$.

(d) $V = \{A \in \mathcal{M}_{2\times2}(\mathbb{R}) : tr(A) = 0\}$ and $\mathbb{R}^4$.

Response

(a) This pair of vector spaces is **not** isomorphic, since $dim(F^3) \neq dim(P_3(F))$, or $3 \neq 4$.

(b) This pair of vector spaces **is** isomorphic, since $dim(F^4) = dim(P_3(F))$, or $4 = 4$.

(c) This pair of vector spaces **is** isomorphic, since $dim(\mathcal{M}_{2\times2}(\mathbb{R})) = dim(P_3(\mathbb{R}))$, or $4 = 4$.

(d) This pair of vector spaces is **not** isomorphic, since $dim(V) \neq dim(\mathbb{R}^4)$, or $3 \neq 4$.

Section 2.4 Question 6

Prove that if A is invertile and $AB = O$, then $B = O$.

Response

Proof. Let A be invertible defined by the problem statement. Then we have

$$\begin{aligned} AB &= O \\ A^{-1}AB &= AO \\ IB &= O && A^{-1}A = I \text{ and } AO = O \\ B &= O && IB = B \end{aligned}$$

□

Section 2.4 Question 7

Let A be an $n \times n$ matrix.

(a) Suppose that $A^2 = O$. Prove that A is not invertible.

(b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

Response

(a) *Proof.* We want to prove that if $A^2 = O$, A is not invertible. Assume by contradiction that A is invertible. Then, we have

$$\begin{aligned}
A^2 &= O & & \\
A^{-1}AA &= A^{-1}O & & A^2 = AA \\
IAA^{-1} &= OA^{-1} & & A^{-1}A = I \text{ and } A^{-1}O = O \\
II &= O & & AA^{-1} = I \text{ and } OA^{-1} = O \\
I &= O & & II = I
\end{aligned}$$

which is a contradiction, since I cannot be the zero matrix O. Therefore, A is not invertible. □

(b) *Proof.* Let $AB = O$ for some nonzero $n \times n$ matrix B defined by the problem statement. A cannot be invertible. Assume by contradiction that A is invertible. Then, we have

$$\begin{aligned}
AB &= O & & \\
A^{-1}AB &= AO & & \\
IB &= O & & A^{-1}A = I \text{ and } AO = O \\
B &= O & & IB = B
\end{aligned}$$

but B must be nonzero by our earlier definition, which is a contradiction. Therefore, A cannot be invertible if B is nonzero. □

Section 2.4 Question 14

Let

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}.$$

Construct an isomorphism from V to F^3.

Response

Let $A \in V$ and $T : V \to F^3$ defined by

$$T \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} = (a, b, c)$$

To prove that T is linear, let $A, B \in V$ and $d \in F$. Then, we have

$$T(dA + B) = (da_1 + a_2, db_1 + b_2, dc_1 + c_2)$$

Therefore, T is linear. Now, its null space is

$$\begin{aligned} T(A) &= 0 \\ T \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} &= (0, 0, 0) \end{aligned}$$

$$\begin{aligned} a &= 0 \\ a + b &= 0 \\ b &= 0 && \text{substitute } a = 0 \\ c &= 0 \end{aligned}$$

Therefore, $N(T) = \{0\}$. By observation, we have that β is a basis for V, where $\beta = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, so we have that $dim(V) = 3$. So by the rank-nullity theorem, we have that

$$\begin{aligned} nullity(T) + rank(T) &= dim(V) \\ 0 + rank(T) &= 3 \\ rank(T) &= 3 \end{aligned}$$

Clearly, we have that $N(T) = \{0\}$ and $rank(T) = dim(V)$; that is, $R(T) = V$. So T is both one-to-one and onto, and by definition, this means that T is invertible and is an isomorphism.

Section 2.4 Question 16

Let B be an $n \times n$ matrix. Define $\Phi : \mathcal{M}_{n\times n}(F) \to \mathcal{M}_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

Response

Proof. To prove that Φ is an isomorphism, we must show that it is linear and invertible. Let $A, C \in \mathcal{M}_{2\times 2}$ and $d \in F$. Then, we have

$$\begin{aligned} \Phi(dA + C) &= B^{-1}(dA + C)B \\ &= dB^{-1}AB + B^{-1}CB \\ \Phi(dA + C) &= d\Phi(A) + \Phi(B) \end{aligned}$$

Therefore, Φ is linear. To show that Φ is invertible, let $\Phi^{-1} : \mathcal{M}_{n\times n}(F) \to \mathcal{M}_{n\times n}(F)$ be defined by

$$\Phi^{-1} = BAB^{-1}$$

Then, we have that

$$\begin{aligned} \Phi^{-1}(\Phi(A)) &= B(B^{-1}AB)B^{-1} \\ &= BB^{-1}ABB^{-1} \\ &= IAI && B^{-1}B = I = BB^{-1} \\ &= A && IA = A \text{ and } AI = A \end{aligned}$$

Note that $\Phi^{-1}(\Phi(A)) = A$; therefore, Φ^{-1} is the inverse of Φ. Since we have shown that Φ is both linear and invertible, we can say that it is an isomorphism. □

Section 2.5 Question 1

Label the following statements as true or false.

(a) Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x_1', x_2', \ldots, x_n'\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$.

(b) Every change of coordinate matrix is invertible.

(c) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$.

(d) The matrices $A, B \in \mathcal{M}_{n\times n}(F)$ are called similar if $B = Q^t AQ$ for some $Q \in \mathcal{M}_{n\times n}(F)$.

(e) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[T]_\beta$ is similar to $[T]_\gamma$.

Response

(a) False

(b) True

(c) True

(d) False

(e) True

Section 2.5 Question 2 part (a) and (c)

For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$

(c) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$

Response

(a)

$$\begin{aligned}(a_1, a_2) &= a_1(1, 0) + a_2(0, 1)\\ (b_1, b_2) &= b_1(1, 0) + b_2(0, 1)\end{aligned}$$

So $[x_1']_\beta = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $[x_2']_\beta = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, where $x_1' = (a_1, a_2), x_2' = (b_1, b_2)$ Then, we have

$$Q = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

(b)

$$\begin{aligned}(1, 0) &= 3(2, 5) + 5(-1, -3) && \text{LCM of 5 and 3 is 15}\\ &= (6, 15) + (-5, 15)\\ &= (1, 0)\\ (0, 1) &= -1(2, 5) + -2(-1, -3) && \text{LCM of 1 and 2 is 2}\\ &= (-2, -5) + (2, 6)\\ &= (0, 1)\end{aligned}$$

So $[x_1']_\beta = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $[x_2']_\beta = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$, where $x_1' = (1, 0), x_2' = (0, 1)$ Then, we have

$$Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$$

Section 2.5 Question 4

Let T be the linear operator on $\mathbb{R}^2$, and let

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix},$$

let β be the standard ordered basis for $\mathbb{R}^2$, and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

to find $[T]_{\beta'}$.

Response

Note that $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. To find $[T]_\beta$, we do

$$\begin{aligned} 2a+b &= 2(1,0) + 1(0,1) \\ a-3b &= 1(1,0) + -3(0,1) \end{aligned}$$

So $[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$. Now, we calculate $[T]_{\beta'}$ by applying the equation defined earlier.

$$\begin{aligned} [T]_{\beta'} &= Q^{-1}T]_\beta Q^{-1} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 4+-1 & 2+3 \\ -2+1 & -1+-3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 5 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3+5 & 3+10 \\ -1+-4 & -3+-6 \end{pmatrix} \\ [T]_{\beta'} &= \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix} \end{aligned}$$

Section 2.5 Question 5

Let T be the linear operator on $P_1(\mathbb{R})$ defined by $T(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1 + x, 1 - x\}$. Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

to find $[T]_{\beta'}$.

Response

Note that $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. To find $[T]_\beta$, we do

$$\begin{aligned} p(1) &= 0 = 0(1) + 0(x) \\ p(x) &= 1 = 1(1) + 0(x) \end{aligned}$$

So $[T]_\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Now, we calculate $[T]_{\beta'}$ by applying the equation defined earlier.

$$\begin{aligned} [T]_{\beta'} &= Q^{-1}T]_\beta Q^{-1} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0+0 & \frac{1}{2}+0 \\ 0+0 & \frac{1}{2}+0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0+\frac{1}{2} & 0+-\frac{1}{2} \\ 0+\frac{1}{2} & 0+-\frac{1}{2} \end{pmatrix} \\ [T]_{\beta'} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \end{aligned}$$