C&EE 110 Homework 8

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Question 1

Let Y have probability density function:

$$f_Y(y) = \begin{cases} \frac{3(\theta^2 - y^2)}{2\theta^3}, & 0 < y < \theta \\ 0, & elsewhere \end{cases}$$

(a) Show that $\frac{Y}{\theta}$ can be used as a pivitol quantity.

(b) Use the pivotal quantity from part (a) to find a 90% confidence limit for θ.

Response

(a) We are given:

$$U = \frac{Y}{\theta} \implies Y = U\theta = h^{-1}(u)$$
$$\frac{d}{du}h^{-1}(u) = \frac{d}{du}U\theta = \theta$$

Then, we have:

$$\begin{aligned} f_u(U) &= f_y\left(h^{-1}(u)\right)\left|\frac{d}{du}h^{-1}\right| \\ &= \frac{3(\theta^2 - (u\theta)^2)}{2\theta^3}\,|\theta| \\ &= \frac{3\theta^3(1-u^2)}{2\theta^3} \\ f_u(U) &= \frac{3(1-u^2)}{2} \end{aligned}$$

$f_u(U)$ does not depend on $\theta \implies U$ can be used as a pivotal quantity.

(b)

$$\begin{aligned} 0.10 &= \int_0^a f_u(U)du = \int_0^a \frac{3(1-u^2)}{2}du \\ &= \frac{3}{2}\int_0^a (1-u^2)du \\ &= \frac{1}{2}\left(3u - u^3\right)\Big|_0^a \\ a &= 0.067 \\ \implies \theta &= \frac{Y}{0.067} \end{aligned}$$

Question 2

The breaking strength of hockey stick shafts made of two different graphite-Kevlar composites yield the following results (in newtons):

- Composite A: 487.3 444.5 467.7 456.3 449.7 459.2 478.9 461.5 477.2
- Composite B: 488.5 501.2 475.3 467.2 462.5 499.7 470.0 469.5 481.5 485.2 509.3 479.3 478.3 491.5

Find a 98% confidence interval for the difference between the mean breaking strengths of hockey stick shafts made of the two materials for two cases:

(a) The population variances are not necessarily the same.

(b) The population variances can be assumed to be similar.

Response

(a) We are given for A:

$$\begin{aligned} s &= 14.238 \\ s^2 &= 202.718 \\ n &= 9 \\ \overline{x} &= 464.7 \end{aligned}$$

We are given for B:

$$\begin{aligned} s &= 13.942 \\ s^2 &= 194.383 \\ n &= 14 \\ \overline{x} &= 482.786 \end{aligned}$$

Then we have:

$$\begin{aligned} v &= \frac{\left(\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}\right)^2}{\frac{\left(\frac{s_A^2}{n_A}\right)^2}{n_A - 1} + \frac{\left(\frac{s_B^2}{n_B}\right)^2}{n_B - 1}} \\ &= \frac{\left(\frac{202.718}{9} + \frac{194.38}{14}\right)^2}{\frac{\left(\frac{202.718}{9}\right)^2}{9-1} + \frac{\left(\frac{194.38}{14}\right)^2}{14-1}} \\ &= 16.941 \\ \implies df &= 17 \end{aligned}$$

$\alpha = 0.02 \implies t_{17,\ 0.02} = 2.567$. Then,

$$\begin{aligned} (\mu_A - \mu_B) \pm t_{17,\ 0.02}\sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}} \\ (464.7 - 482.786) \pm 2.567\sqrt{\frac{202.718}{9} + \frac{194.383}{14}} \\ [-33.575, -2.597] \end{aligned}$$

(b) We have:

$$\begin{aligned}
S_p^2 &= \sqrt{\frac{(n_A - 1)S_1^2 + (n_B - 1)S_2^2}{n_1 + n_2 - 2}} \\
&= \frac{(9-1)(202.718) + (14-1)(194.383)}{9 + 14 - 2} \\
&= \frac{4148.717}{21} \\
S_p^2 &= 197.558 \\
S_p &= 14.056
\end{aligned}$$

Then,

$$\begin{aligned}
\left(\overline{Y}_A - \overline{Y}_B\right) \pm t_{df,\frac{\alpha}{2}} S_P \sqrt{\frac{1}{n_A} + \frac{1}{n_B}} \\
(464.7 - 482.786) \pm t_{21,0.02}(14.056)\sqrt{\frac{1}{9} + \frac{1}{14}} \\
[-33.206, -2.965]
\end{aligned}$$

Question 3

The following table shows the results of a uniaxial strength testing for a new steel alloy that UCLA is developing. The underlying distribution for the strength of this new material is estimated to be normal.

Table 1:Results for the uniaxial strength in MPa.

415	418	341	427
394	377	429	366
406	379	407	398
384	384	385	395

(a) Obtain the Maximum Likelihood Estimators for the mean and variance of the underlying distribution (derive them analytically, showing the complete process).

(b) Using the Maximum Likelihood Estimators obtained in part (a), compute the point esimates for the mean and variance based on the data shown in Table 1.

(c) An alternative version of this material is being developed in parallel, and the objective is to test whether a different percentage in carbon can increase the strength of the steel alloy with respect to the original formula (part (a), (b)). For this, 45 samples with this alternative recipe are forged and tested, obtaining measurements with mean strength of $412.6[MPa]$ and a standard deviation of $15.5MPa$.

  (i) Set up the null and alternative hypothesis. Is this a two-tailed test or one-tailed test?

  (ii) Can you conclude that the new recipe effectively increases the strength of the material? Use a confidence interval of $\alpha = 0.05$.

Response

(a) We have that the underlying distribution is estimated to be normal. Then, we have $X_1, \ldots, X_n$.

$$\begin{aligned} L(\mu, \sigma^2) &= f(x_1, \ldots, x_n|\mu, \sigma^2) \\ &= \prod_{i=1}^{n} f(x_i|\mu, \sigma^2) \\ &= \prod_{i=1}^{n} \left\{ \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x_i-\mu)^2}{2\sigma^2}\right]\right\} \\ L(\mu, \sigma^2) &= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] \end{aligned}$$

Then,

$$\ln\left[L(\mu, \sigma^2\right] = -\frac{n}{2}\ln\left(\sigma^2\right) - \frac{n}{2}\ln\left(2\pi\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

The MLE of μ and σ^2 are values such that $\ln\left[L(\mu, \sigma^2\right]$ is a max. So, we have:

$$\frac{\partial\left\{\ln\left[L(\mu, \sigma^2)\right]\right\}}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

and

$$\frac{\partial\left\{\ln\left[L(\mu, \sigma^2)\right]\right\}}{\partial\sigma^2} = -\left(\frac{n}{2}\right)\left(\frac{1}{\sigma^2}\right) + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2$$

solving for when the derivatives are 0, we get:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\hat{\mu}) = 0 \implies \sum_{i=1}^{n}x_i - n\hat{\mu} = 0 \implies \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}x_i = \overline{x}$$

and

$$-\left(\frac{n}{\hat{\sigma}^2}\right) + \frac{1}{\hat{\sigma}^4}\sum_{i=1}^{n}(x_i - \overline{x})^2 = 0 \implies \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

Then, $\overline{x}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum\limits_{i=1}^{n}(x_i - \overline{x})^2$ are MLE's of μ and σ^2.

(b)

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} x_i = 394.063$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - 394.063)^2 = 536.596$$

(c) We are given: $\overline{x} = 412.6$ and $s = 15.5$. Then,

(i)

Null Hypothesis: $H_0 : \hat{\mu}_n \leq 394.063$
Alternative Hypothesis: $H_a : \hat{\mu}_n > 394.063$

This is a one tailed test because we are testing for an increase in the parameter.

(ii) We reject the H_0 if $z > z_\alpha$:

$$\begin{aligned} z &= \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}} \\ &= \frac{412.6 - 394.063}{\frac{15.5}{\sqrt{45}}} \\ z &= 8.02 \end{aligned}$$

A cumulative area of $0.95 \implies z_\alpha = 1.64$. Then,

$$z = 8.02 > 1.64 = z_\alpha \implies \textit{reject } H_0$$

Thus, we reject H_0.