

# Problem Set 3

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## Question 4

Let  $x_n = \frac{2n+1}{3n+7}$ .

- (a) Prove directly, using the definition, that  $\lim_{n \rightarrow \infty} x_n = \frac{2}{3}$   
 (b) Prove, using the algebraic limit theorem, that  $\lim_{n \rightarrow \infty} x_n = \frac{2}{3}$

## Response

(a) *Scratch work:*

$$\begin{aligned} \left| \frac{2n+1}{3n+7} - \frac{2}{3} \right| &< \varepsilon \\ \left| \frac{2n+1}{3n+7} - \frac{2(n+\frac{7}{3})}{3(n+\frac{7}{3})} \right| &< \varepsilon \\ \left| \frac{3(2n+1) - 6n - 7}{3(3n+7)} \right| &< \varepsilon \\ \left| \frac{6n+3-6n-7}{3(3n+7)} \right| &< \varepsilon \\ \left| \frac{-4}{3(3n+7)} \right| &< \varepsilon \\ \frac{4}{9n+49} &< \varepsilon \\ n &> \frac{4-49\varepsilon}{9\varepsilon} \end{aligned}$$

*Proof.* Let  $\varepsilon > 0$ . Let  $N > \frac{4-49\varepsilon}{9\varepsilon}$ . Then, for all  $n > N$ , we have

$$\begin{aligned} n &> \frac{4-49\varepsilon}{9\varepsilon} \\ \frac{4}{9n+49} &< \varepsilon \\ \left| \frac{-4}{3(3n+7)} \right| &< \varepsilon \\ \left| \frac{2n+1}{3n+7} - \frac{2}{3} \right| &< \varepsilon \end{aligned}$$

so  $\lim_{n \rightarrow \infty} x_n = \frac{2}{3}$ . □

(b) *Proof.*

$$\frac{2n+1}{3n+7} = \frac{2 + \frac{1}{n}}{3 + \frac{7}{n}}$$

Let  $a_n = 2 + \frac{1}{n}$  and  $b_n = 3 + \frac{7}{n}$ . Then,

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{1}{n} & \lim_{n \rightarrow \infty} (x_n + y_n) &= x + y \\ &= 2 + 0 & \lim_{n \rightarrow \infty} c &= c, \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \text{ from lecture} \\ \lim_{n \rightarrow \infty} a_n &= 2 \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} b_n &= \lim_{n \rightarrow \infty} 3 + \lim_{n \rightarrow \infty} 7 \left( \frac{1}{n} \right) & \lim_{n \rightarrow \infty} (x_n + y_n) &= x + y \\ &= 3 + 0 & \lim_{n \rightarrow \infty} c &= c, \quad \lim_{n \rightarrow \infty} cx_n = cx, \quad \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \text{ from lecture} \\ \lim_{n \rightarrow \infty} b_n &= 3 \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} b_n \neq 0$ , we have  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{2}{3}$  ( $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$ ,  $y \neq 0$  by the algebraic limit theorem). So,  $\lim_{n \rightarrow \infty} x_n = \frac{2}{3}$  by the algebraic limit theorem.  $\square$

## Question 10

- (a) Let  $(x_n)$  be bounded (not necessarily convergent) and assume that  $y_n \rightarrow 0$  as  $n \rightarrow \infty$ . Show that  $x_n y_n \rightarrow 0$  as  $n \rightarrow \infty$ . (Why can we not just use the Algebraic limit theorem?)
- (b) Let  $(x_n)$  be bounded and  $y_n \rightarrow y$  with  $y \neq 0$ . Does  $(x_n y_n)$  converge? If yes, show it. If not, give a counter-example.

## Response

- (a) *Proof.* Since  $(x_n)$  is bounded,  $\exists M \in \mathbb{R}$  such that  $|x_n| \leq M \forall n \in \mathbb{N}$ . Then,  $\forall \varepsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that  $\forall n > N$ ,

$$\begin{aligned} |y_n - 0| &< \frac{\varepsilon}{M} \\ |y_n| &< \frac{\varepsilon}{M} \end{aligned}$$

so

$$\begin{aligned} |x_n y_n - 0| &< M \cdot \frac{\varepsilon}{M} \\ |x_n y_n| &< \varepsilon \end{aligned}$$

for all  $n > N$ . Therefore,  $x_n y_n \rightarrow 0$  as  $n \rightarrow \infty$ . □

We cannot use the algebraic limit theorem since  $(x_n)$  is only bounded by the problem statement, so it need not converge.

- (b)  $(x_n y_n)$  is not necessarily convergent. Consider  $x_n = (-1)^n$ ,  $y_n = 1 + \frac{1}{n}$ . From lecture, we have that  $x_n$  does not converge and that  $y_n \rightarrow 1$ , so  $y \neq 0$  by the algebraic limit theorem and lecture. However,

$$\begin{aligned} x_n y_n &= (-1)^n \left(1 + \frac{1}{n}\right) \\ &= (-1)^n + \frac{(-1)^n}{n} \end{aligned}$$

From part (a), we have that  $\frac{(-1)^n}{n}$  converges to 0 since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . However,  $(-1)^n$  is not convergent.

## Question 12

For the following sequences, provide an example or prove that no such request is possible. You may appeal to results from lectures.

- (a) Sequences  $(x_n)$  and  $(y_n)$  which both diverge, but whose sum  $(x_n + y_n)$  converges.
- (b) Sequences  $(x_n)$ , which converges, and  $(y_n)$ , which diverges, but whose sum  $(x_n + y_n)$  converges.
- (c) A convergent sequence  $(x_n)$ , such that  $x_n \neq 0$  for all  $n \in \mathbb{N}$  and  $(1/x_n)$  diverges.
- (d) An unbounded sequence  $(x_n)$  and a convergent sequence  $(y_n)$  with  $(x_n - y_n)$  bounded.
- (e) Two sequences  $(x_n)$  and  $(y_n)$ , where  $(x_n y_n)$  and  $(x_n)$  converge, but  $(y_n)$  does not converge.

## Response

- (a)  $x_n = n$ ,  $y_n = -n$ . Clearly both  $x_n$ ,  $y_n$  diverge, but  $x_n + y_n = n + (-n) = (0, 0, \dots)$  so  $(x_n + y_n)$  converges.
- (b) This is impossible. By the algebraic limit theorem, if  $(x_n + y_n)$  converges, then  $(x_n)$  and  $(y_n)$  must both converge. Therefore,  $(y_n)$  cannot diverge if  $(x_n + y_n)$  converges.
- (c)  $x_n = \frac{1}{n}$ .  $x_n$  converges (from lecture) but  $(1/x_n) = 1/\frac{1}{n} = n$  diverges.
- (d) This is impossible. Since  $y_n$  is bounded (by the theorem that states convergent sequences are bounded, since  $y_n$  converges, it is bounded) and  $(x_n - y_n)$  is bounded, we have that  $|x_n| \leq N_1 \forall n \in \mathbb{N}$  and  $|x_n - y_n| \leq N_2 \forall n \in \mathbb{N}$ . Then,  $|x_n - y_n + y_n| \leq N_2 + N_1 \implies |x_n| \leq N_1 + N_2$  means that  $(x_n)$  is bounded, which is a contradiction to the statement that  $(x_n)$  is unbounded.
- (e)  $x_n = \frac{1}{n}$ ,  $y_n = n$ .  $x_n$  converges from lecture and  $y_n$  is unbounded and therefore does not converge (by the contrapositive of the theorem that states that convergent sequences are bounded). However,  $x_n y_n = \frac{1}{n}(n) = (1, 1, \dots)$  which converges.