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1 The Integers

Theorem (Well-Ordering Principle)

Every nonempty set of non-negative integers contain a least element. Mathematically, $\exists a \in S : \forall b \in S, a \leq b$.

Proof. Let S be a set of non-negative integers. Suppose S has no smallest element. Then, $0 \notin S$, because otherwise, 0 would be the smallest element. By induction, suppose $0, 1, \ldots, k \notin S$. Then, $k + 1 \notin S$ since otherwise, it would be the smallest element. Therefore, $S = \emptyset$. $\square$

Definition: Divides

Let $a, b \in \mathbb{Z}$. b **divides** a if $a = bc$ for some $c \in \mathbb{Z}$, written as $b \mid a$.

Proposition: Let $a, b \in \mathbb{Z}, a \neq 0$ such that $b \mid a$. Then $|b| \leq |a|$.

Proof. Let $a, b \in \mathbb{Z}$ such that $b \mid a$ and $a \neq 0$. Then there exists some $c \in \mathbb{Z}$ such that $a = bc$. Since $a \neq 0$, b, c are necessarily nonzero. Applying the absolute value to both sides of the equation, we get $|a| = |bc| = |b||c|$. Since $b, c \neq 0$, we have $|b|, |c| > 0$. Then $|b| \leq |b||c| = |bc| = |a|$, so $|b| \leq |a|$. $\square$

Theorem (Division Algorithm)

Let $a, b \in \mathbb{Z}$ such that $b > 0$. There exists unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ where $0 \leq r < b$.

Proof. **Existence:** Let $a, b \in \mathbb{Z}, b > 0$. Consider the set $S = \{a - bx : x \in \mathbb{Z}\} \cap \mathbb{Z}_{\geq 0}$. Consider $b = -|a|$. Then, $a - (-|a|)x \in S$. By the well-ordering principle, choose the smallest $a - bx \in S$ such that $q := x, r := a - bx$. Then, rearranging r and substituting q for x, we get $a = bq + r \in S$. By construction of S, $0 \leq r$. Suppose $r \geq b$. Then, $0 \leq r - b = (a - bx) - b = a - b(x - 1)$. This implies that $r - b < r$, a contradiction, since $r \in S$ was the least element by choice. Therefore, $0 \leq r < b$.

Uniqueness: Suppose we have $q_1, r_1, q_2, r_2 \in \mathbb{Z}$ such that $a = bq_1 + r_1 = bq_2 + r_2$, where $0 \leq r_1, r_2 < b$. Then, we have

$$\begin{aligned} bq_1 + r_1 &= bq_2 + r_2 \\ bq_1 + r_1 - (bq_2 + r_2) &= 0 \\ b(q_1 - q_2) + (r_1 - r_2) &= 0 \\ b(q_1 - q_2) &= -(r_1 - r_2) \\ b(q_1 - q_2) &= r_2 - r_1 \end{aligned}$$

Since $0 \leq r_1 < b$, we can rewrite the inequality to be $-b < -r_1 \leq 0$. Then, adding $0 \leq r_2 < b$ to the inequality, we get $-b < r_2 - r_1 < b$. Because $b \mid (r_2 - r_1)$, $(r_2 - r_1)$ must be a multiple of b, but since $-b < r_2 - r_1 < b$, we have that $(r_2 - r_1) = 0b = 0$. Then, $b(q_1 - q_2) = r_2 - r_1 = 0$. This implies that $q_1 = q_2$ and $r_1 = r_2$. Therefore, $q_1, r_1 \in \mathbb{Z}$ are unique. $\square$

Definition: Greatest Common Divisor (gcd)

Let $a, b \in \mathbb{Z}$ and either $a \neq 0$ or $b \neq 0$, but not both. The **greatest common divisor** of a and b is the largest integer dividing a and b. We write $\gcd(a, b)$ or (a, b).

$(a, b) \mid a$ and $(a, b) \mid b$. Further, if $c > 0$ divides a and b, then $0 < c \leq (a, b)$.

Theorem (Bezout's Identity)

Let $a, b \in \mathbb{Z}$ with $a \neq 0$ or $b \neq 0$, but not both. Suppose $d = (a, b)$. We can find $x, y \in \mathbb{Z}$ such that $ax + by = d$.

Proof. Let $d = (a, b)$. Consider the set $S = \{ax + by : x, y \in \mathbb{Z}\} \cap \mathbb{Z}_{\geq 0}$. Consider $x = a, y = b$. Then $ax + by = a^2 + b^2 \geq 0 \in S$, so S is not empty. By the well-ordering principle, choose the least element $s = ax + by \in S$ and consider $a = sq + r$ where $0 \leq r < s$. Rearranging the second equation, we get

$$\begin{aligned}
a &= sq + r \\
r &= a - sq \\
&= a - (ax + by)q \\
r &= a(1 - xq) + b(-yq)
\end{aligned}$$

This implies that $r \in S$ since $0 \leq r$ by definition. We also have that $r < s$, but since s was chosen to be the smallest element in S, this forces $r = 0$. Then, $a = sq + r = sq$, so $s \mid a$. Similarly, $b = st$ for some $t \in \mathbb{Z}$, so $s \mid b$. Since $s \mid a$ and $s \mid b$, $s \leq d$. But $d \mid a$ and $d \mid b$ by definition, so $d \mid s$ which implies that $d \leq s$. Therefore, $d = s = ax + by$. □

Theorem

Let $a, b \in \mathbb{Z}$ and suppose $a \mid bc$ and $(a, b) = 1$. Then $a \mid c$.

Proof. Because $(a, b) = 1$, we can write $1 = ax + by$. Also, since $a \mid bc$, there exists some $z \in \mathbb{Z}$ such that $bc = az$. Then

$$\begin{aligned}
c &= cax + cby \\
&= a(cx) + (bc)y \\
&= a(cx) + a(zy) \\
c &= a(cx + zy)
\end{aligned}$$

Therefore, $a \mid c$. □

Corollary

Let $a, b, c \in \mathbb{Z}$ and $(a, b) = 1$. If $a \mid c$ and $b \mid c$, then $ab \mid c$.

Proof. Since $(a, b) = 1$, we have $ax + by = 1$. By definition, since $a \mid c$ and $b \mid c$, there exist $n, m \in \mathbb{Z}$ such that $c = na$ and $c = mb$. Then, we have

$$\begin{aligned} 1 &= ax + by \\ c &= cax + cby \\ &= (bm)ax + (an)by \\ &= (ba)mx + (ab)ny \\ c &= ab(mx + ny) \end{aligned}$$

so $ab \mid c$. $\square$

1.1 Prime Numbers

Definition: Prime

A nonzero non-unit integer p is **prime** if its only divisors are $\pm 1, \pm p$.

Theorem

Let $p \in \mathbb{Z} \setminus \{0, \pm 1\}$. The following statements are equivalent.

(1) p is prime.

(2) If $p \mid bc$, then $p \mid b$ or $p \mid c$ where $b, c \in \mathbb{Z}$.

Proof. **(1)** $\implies$ **(2)** Suppose p is prime and $p \mid bc$. If $p \mid b$, we are done, so suppose $p \nmid b$. Then, $(p, b) = 1$, so we have

$$\begin{aligned} 1 &= px + by \\ c &= cpx + cby \\ &= p(cx) + (bc)y \\ &= p(cx) + (pn)y && p \mid bc \implies bc = pn, n \in \mathbb{Z} \\ &= p(cx) + p(ny) \\ c &= p(cx + ny) \end{aligned}$$

so $p \mid c$.

(1) $\impliedby$ **(2)** To prove the reverse implication, suppose the contrapositive: "If p is not prime, then there exist some $b, c \in \mathbb{Z}$ such that $p \mid bc$ but $p \nmid b$ and $p \nmid c$". Suppose $p \in \mathbb{Z} \setminus \{0, \pm 1\}$ is not prime; i.e. p is composite. Then, p can be written as its unique factorization $q_1 q_2 \cdots q_n$ where $n \geq 2$ and each q_i is prime. Choose $b = q_1$ and $c = q_2 \cdots q_n$. Then $p \mid bc$ because $bc = p$ and $p \mid p$, but $p \nmid b$ and $p \nmid c$ because $|p| > |b|$ and $|p| > |c|$ respectively. $\square$

Theorem

Let $n \in \mathbb{Z} \setminus \{0, \pm 1\}$. n can be written as a product of primes.

Proof. Let $n > 1$. Let S be the set of positive integers greater than 1 that cannot be written as a product of primes. Suppose for the sake of contradiction that S is nonempty. Then by the well-ordering principle, pick a least element $m \in S$. By definition, m is not prime or a product of primes. Because m is not prime, there exists $a \in \mathbb{Z}$ such that $a \neq \pm 1, \pm m$ and $a \mid m$. Then, $m = ab$ for some $b \in \mathbb{Z}$. By definition, $|a| \leq |m|$ and $|b| \leq |m|$. Without loss of generality, assume $a, b > 0$. Note that $b \neq 1$ since otherwise, $a = m$. So, $1 < a, b < m$ and $a, b \notin S$. Because $a, b \notin S$, they are products of primes. But $m = a \cdot b$, so m is a product of primes, a contradiction. Therefore, $S = \emptyset$, so n can be written as a product of primes. $\square$

Theorem (Fundamental Theorem of Arithmetic)

Let $n \in \mathbb{Z} \setminus \{0, \pm 1\}$. Suppose $n = p_1 \cdots p_r$ and $n = q_1 \cdots q_s$ where each p_i, q_j is prime. Then $r = s$ and there is a unique permutation σ on $\{1, \ldots, r\}$ such that $p_i = \pm q_{\sigma(i)}$.

Proof. Let $n \in \mathbb{Z} \setminus \{0, 1\}$. Without loss of generality, suppose n is positive and $n = p_1 \cdots p_r$ and $n = q_1 \cdots q_s$ where each p_i, q_j is prime. Then $p_1 \mid q_1 \cdots q_s$. In particular, $p_1 \mid q_j$ for some $j \leq s$. Because q_j is prime, we necessarily have that $q_j = |p_1|$. Without loss of generality reindex $j = 1$ to get $q_1 = |p_1|$. Then, $p_1 \cdot (p_2 \cdots p_r) = p_1 \cdot (q_2 \cdots q_s) \implies p_2 \cdots p_r = q_2 \cdots q_s$. By induction, we have that $p_r = q_r$. If $r < s$, by the above, we have that $1 = q_{r+1} \cdots q_s$, which implies $q_j = 1$ for each j. A similar argument is said for $s < r$. In either case, we have a contradiction. Therefore, $r = s$ and there is a unique permutation σ on $\{1, \ldots, r\}$ such that $p_i = q_{\sigma(i)}$. $\square$

1.2 Modular Arithmetic

Definition: Well-Defined Functions

A function $f : X \to Y$ is **well-defined** if, for all $a, b \in X$, we have $f(a) = f(b)$ whenever $a = b$.

Definition: Equivalence Relation

A relation R on a set S is any subset of $S \times S$. An **equivalence relation** is a relation with the following properties:

1. Reflexivity: For any $a \in S$, $(a, a) \in R$ (alternatively written as $a \sim a$).
2. Symmetry: For any $(a, b) \in S \times S$, $(a, b) \in R$ implies $(b, a) \in R$ (alternatively written as $a \sim b \implies b \sim a$).
3. Transitivity: For any $a, b, c \in S$, if $(a, b), (b, c) \in R$, then $(a, c) \in R$ (alternatively written as $a \sim b, b \sim c \implies a \sim c$).

Pick $m \in \mathbb{Z}$ to be nonzero. The **Division Algorithm** says that for any $a, b \in \mathbb{Z}$, we can write $a = q_1 m + r_1, b = q_2 m + r_2$ for unique $q_1, q_2, r_1, r_2 \in \mathbb{Z}$ where $0 \leq r_1, r_2 < |m|$.

Definition: Modulo

Define a relation R_m on $\mathbb{Z}$ by saying $(a, b) \in R_m$ if and only if $r_1 = r_2$ (alternatively written as $a \sim b$ if and only if $r_1 = r_2$). We write this as $a \equiv b \pmod{m}$.

Proposition: For any $m \in \mathbb{Z}$ nonzero, R_m is an equivalence relation.

Proof. Let R_m be the relation defined above for $m \in \mathbb{Z}$ nonzero.

(1) For any $a \in \mathbb{Z}$, write $a = bq + r$. Then, since $r = r$, $a \equiv a \pmod{m}$, R_m is reflexive.

(2) Take $a, b \in \mathbb{Z}$ and assume $a \equiv b \pmod{m}$. By the division algorithm, we can write $a = q_1 m + r_1, b = q_2 m + r_2$. By assumption, $a \equiv b \pmod{m}$, so $r_1 = r_2$. Since equality is symmetric, $r_1 = r_2 \iff r_2 = r_1$, so $b \equiv a \pmod{m}$. R_m is symmetric.

(3) Pick $a, b, c \in \mathbb{Z}$ and assume $a \equiv b \pmod{m}, b \equiv c \pmod{m}$. By the division algorithm, we can write $a = q_1 m + r_1, b = q_2 m + r_2, c = q_3 m + r_3$. By assumption, $r_1 = r_2$ and $r_2 = r_3$. Since equality is transitive, $r_1 = r_2, r_2 = r_3 \implies r_1 = r_3$, so $a \equiv c \pmod{m}$. R_m is transitive.

Since R_m satisfies $(1) - (3)$, R_m is an equivalence relation. $\square$

Definition: Equivalence Class

If R is an equivalence relation on a set S, then S can be written as the union of equivalence classes. The **equivalence class** of x is the set $[x] := \{y \in S : (x, y) \in R\}$.

Note: The equivalence classes of R_m are $[0], [1], \ldots, [m-1]$.

Definition: Congruent Modulo n

Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}$ be positive. We say a and b are **congruent modulo** n if $n \mid (a - b)$, written as $a \equiv b \pmod{n}$.

The **integers modulo** n is the set of equivalence classes modulo n, written as $\mathbb{Z}/n, \mathbb{Z}_n, \mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/(n)$.

Definition: Operations on $\mathbb{Z}/n$

Let $n \in \mathbb{Z}$ and $[a], [b] \in \mathbb{Z}/n$. Define

→ $[a] + [b] = [a + b]$

→ $[a][b] = [ab]$

→ For $k \geq 0$, $[a]^k = [a^k]$

Proposition: The operations above are well-defined.

Proof. Let $n \in \mathbb{Z}$ and $[a], [a'], [b], [b'] \in \mathbb{Z}/n$ where $[a] = [a'], [b] = [b']$. Then ($[a] = [a']$ and $[b] = [b']$ implies $n \mid (a - a')$ and $n \mid (b - b')$, so $n \mid (a - a') + (b - b') = (a + b) - (a' + b')$. Therefore, $[a + b] = [a' + b']$. Similarly,

$$\begin{aligned} ab - a'b' &= ab + 0 - a'b' \\ &= ab + (-ab' + ab') - a'b' \\ &= (ab - ab') + (ab' - a'b') \\ ab - a'b' &= a(b - b') + b'(a - a') \end{aligned}$$

Since $n \mid (a - a')$ and $n \mid (b - b')$, $n \mid ab - a'b'$, so $[ab] = [a'b']$. □

Proposition: Let $[a], [b], [c] \in \mathbb{Z}/n$. Then the following properties hold:

(1) $[a] + [b] = [b] + [a]$

(2) $[a] + ([b] + [c]) = ([a] + [b]) + [c]$

(3) $[a] + [0] = [a]$

(4) There exists $x \in \mathbb{Z}$ such that $[a] + x = [0]$

(5) $[a][b] = [b][a]$

(6) $[a]([b][c]) = ([a][b])[c]$

(7) $[a][1] = [a]$

(8) $[a]([b] + [c]) = [a][b] + [a][c]$

Proof. Let $[a], [b], [c] \in \mathbb{Z}/n$. Then

(1) $\underline{[a] + [b]} = [a + b] = [b + a] = \underline{[b] + [a]}$

(2) $\underline{[a] + ([b] + [c])} = [a] + [b + c] = [a + b + c] = [a + b] + [c] = \underline{([a] + [b]) + [c]}$

(3) $\underline{[a] + [0]} = [a + 0] = \underline{[a]}$

(4) Take $x \in \mathbb{Z}$ such that $x = n - a$. Then, $\underline{[a] + x} = [a] + [n - a] = [a - n - a] = [n] = \underline{[0]}$.

(5) $\underline{[a][b]} = [ab] = [ba] = \underline{[b][a]}$

(6) $\underline{[a]([b][c])} = [a][bc] = [abc] = [ab][c] = \underline{([a][b])[c]}$

(7) $\underline{[a][1]} = [a \cdot 1] = [a]$

(8) $\underline{[a]([b] + [c])} = [a][b + c] = [a \cdot (b + c)] = [ab + ac] = [ab] + ac] = \underline{[a][b] + [a][c]}$

□

Definition: Unit and Inverse

Let $n > 1$ be an integer. Consider $[a] \in \mathbb{Z}/n$. If there exists $[b] \in \mathbb{Z}/n$ such that $[a][b] = [1]$, then we say $[a]$ is a **unit** and $[b]$ is the **inverse** of $[a]$, written as $[a]^{-1}$.

Theorem

Let $p > 1$ be an integer. The following statements are equivalent:

(1) p is prime.

(2) Each nonzero $[a] \in \mathbb{Z}/p$ has an inverse.

(3) If $[ab] = [0]$, then either $[a] = [0]$ or $[b] = [0]$

Proof. Let $p > 1$ be an integer.
(1) $\implies$ (2) Take $[a] \in \mathbb{Z}/p$ to be nonzero. Then $p \nmid a$ since p is prime. That is, $(p, a) = 1$. Then $px + ay = 1$, or $[1] = [px + ay] = [px] + [ay]$. But $[px] = [p][x] = [0][x] = [0] \in \mathbb{Z}/p$, so $[1] = [0] + [ay] = [ay] = [a][y]$. Then, $[y]$ is the inverse of $[a]$. Since $[a]$ was arbitrary, this holds for all $[a] \in \mathbb{Z}/p$.

(2) $\implies$ (3) Let $[a], [b] \in \mathbb{Z}/p$ and suppose $[ab] = [0]$. If $[a] = 0$, we are done, so suppose $[a] \neq 0$. Then, $[a]$ has an inverse, so $[a]^{-1}[ab] = [a]^{-1}[a][b] = [1][b] = [b] = [0]$. Therefore, either $[a] = [0]$ or $[b] = [0]$.

(3) $\implies$ (1) Suppose for the sake of contradiction that p is not prime; i.e. p is composite. Then we can find a divisor $a > 0$ such that $a \neq \pm 1, \pm p$. That is, $|1| < a < |p|$. Let $p = ab$. Then $1 < a, b < p$, but $[ab] = [p] = [0]$, a contradiction. $\square$

Theorem

Let $n > 1$ be an integer and $[a] \in \mathbb{Z}/n$. Then $[a]$ has a multiplicative inverse if and only if $(a, n) = 1$.

Proof. **($\implies$)** Suppose $[a]$ has a multiplicative inverse. Then there exists $[x] \in \mathbb{Z}/n$ such that $[a][x] = [1]$. Then

$$\begin{aligned}
[1] &= [a][x] \\
&= [ax] + [0] \\
&= [ax] + [ny] && [ny] = [0] \in \mathbb{Z}/n, y \in \mathbb{Z} \\
[1] &= [ax + ny]
\end{aligned}$$

so $(a, n) = 1$.

($\impliedby$) Suppose $(a, n) = 1$. Then $ax + ny = 1$ for some $x, y \in \mathbb{Z}$, but $[ny] = [0] \in \mathbb{Z}/p$, so $[ax] = [a][x] = [1]$, where $[x]$ is the multiplicative inverse of $[a]$. $\square$

Theorem Chinese Remainder Theorem

Let $m, n \in \mathbb{Z}$ be coprime and positive. Let $a, b \in \mathbb{Z}$. We can find $x \in \mathbb{Z}$ such that

$$\begin{aligned} x &\equiv a \pmod{m} \\ x &\equiv b \pmod{n} \end{aligned}$$

Moreover, if y is another solution, then $y \equiv x \pmod{mn}$.

Proof. Let $m, n \in \mathbb{Z}$ such that $(n, m) = 1$. Then we can write $na + mb = 1$ for some $a, b \in \mathbb{Z}$. Set $x := c(na) + d(mb)$. Then

$$\begin{aligned} [x]_m &= [cna]_m + [dmb]_m \\ &= [n(cn)]_m + [m(db)]_m \\ &= [a(cn)]_m + [0] && [m(db)]_m = [0] \in \mathbb{Z}/m \\ [x]_m &= [a]_m \end{aligned}$$

so $[x]_m = [a]_m$. Similarly, $[x]_n = [b]_n$. So we have

$$\begin{aligned} x &\equiv a \pmod{m} \\ x &\equiv b \pmod{n} \end{aligned}$$

Let y be another solution. Then $[y]_m = [x]_m$ so $m \mid y - x$. Similarly, $n \mid y - x$. But since $(n, m) = 1$, we have that $mn|y - x$, or $[y]_{mn} = [x]_{mn}$. So $y \equiv x \pmod{mn}$. $\square$

Theorem Chinese Remainder Theorem (General)

Let $m_1, \dots, m_n \in \mathbb{Z}$ be positive and pairwise relatively prime (i.e., $(m_i, m_j) = 1$ when $i \neq j$). Let $a_1, \dots, a_n \in \mathbb{Z}$. We can find x such that

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\vdots \\ x &\equiv a_n \pmod{m_n} \end{aligned}$$

Moreover, if y is another solution, then $y \equiv x \mod m_1 m_2 \cdots m_n$

Proof. We will induct on $n \in \mathbb{N}$.
Base case: At $n = 2$, we have $m_1, m_2 \in \mathbb{Z}$ where $(m_1, m_2) = 1$. Then, we can find $p, q \in \mathbb{Z}$ such that $m_1 p + m_2 q = 1$. Then, because $m_2 q \equiv 0 \pmod{m_2}$, we have $m_1 \equiv 1 \pmod{m_2}$. Similarly, $m_2 \equiv 1 \pmod{m_1}$. Consider $x = (m_2 q)r + (m_1 p)s$ for $r, s \in \mathbb{Z}$. Then, since $(m_2 q)r \equiv 0 \pmod{m_2}$, we have $x \equiv (m_1 p)s \equiv s \pmod{m_2}$. Similarly, $x \equiv (m_2 q)r \equiv r \pmod{m_1}$. So, $x \equiv r \pmod{m_1}$ and $x \equiv s \pmod{m_2}$. Now suppose y is another solution. Then, we have $y \equiv x \pmod{m}$, which implies that $m_1 | (y - x)$ and similarly, $m_2 | (y - x)$. Then because $(m_1, m_2) = 1$, we have that $m_1 m_2 | (y - x)$, so $y \equiv x \pmod{m_1 m_2}$.

Inductive step: At $n = n + 1$, we have $m_1, m_2 \in \mathbb{Z}$ where $(m_1, m_2) = 1$. Then by the inductive hypothesis, we have a set of n pairwise coprime integers $m_1, \cdots, m_n$ where $x' \equiv a_i \pmod{m_i}$ for each $i = 1, \cdots, n$. Define $M = \prod_{i=1}^{n} m_i$ and consider $x = x' + sM$ for some $s \in \mathbb{Z}$. Then since $m_i | M$ implies $sM \equiv 0 \pmod{m_i}$ and from the inductive hypothesis, $x' \equiv a_i \pmod{m_i}$, we have $x \equiv x' + sM \equiv x' \equiv a_i \pmod{m_i}$ for $i = 1, \cdots, n$. At m_{n+1}, because $m_{n+1} \nmid M$, we can choose an $s \in \mathbb{Z}$ such that $x \equiv x' + sM \equiv a_{n+1} \pmod{m_{n+1}}$. Now suppose y is another solution. Then $y \equiv x' \pmod{M}$ and $y \equiv a_{n+1} \pmod{m_{n+1}}$. Since $(M, m_{n+1}) = 1$, by the inductive hypothesis, we have that $y \equiv x \pmod{M m_{n+1}}$, so $y \equiv x \pmod{m_1 m_2 \cdots m_{n+1}}$. □

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2 Rings

Definition: Ring

A **ring** R is a nonempty subset with two operations, addition $(+)$ and multiplication $(\cdot)$ such that, for all $a, b, c \in R$, the following properties hold:

(1) $a + b \in R$

(2) $a + (b + c) = (a + b) + c$

(3) $a + b = b + a$

(4) There exists $0 \in R$ such that $0 + a = a + 0 = a$ for all $a \in R$.

(5) For all $a \in R$, there exists $-a$ such that $(-a) + a = a + (-a) = 0$.

(6) $a \cdot b \in R$

(7) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(8) $a \cdot (b + c) = a \cdot b + a \cdot c$

(9)* There exists $1 \in R$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

*A set satisfying (1) - (8) is called a **nonunital ring**. If the set also satisfies (9), it is called a **unital ring**.

→ A ring is **commutative** if, for all $a, b \in R$, $a \cdot b = b \cdot a$.

→ An element $a \in R$ is a **zero divisor** if there exists a nonzero $b \in R$ such that $a \cdot b = 0$ or $b \cdot a = 0$.

→ An element $a \in R$ is a **unit** if there exists $b \in R$ such that $a \cdot b = b \cdot a = 1$, and is called the *inverse* of a, written as a^{-1}.

Proposition: Let $n > 1$, $a \in \mathbb{Z}$. If $(a, n) = 1$, $[a]$ is a unit. Otherwise, it is a zero divisor.

Proof. Let $n > 1$ and $a \in \mathbb{Z}$. There are two cases.

Case (1): $(a, n) = 1$. Then $ax + ny = 1$ so $[ax] = [a][x] = [1]$ where $[x]$ is the inverse of $[a]$, so $[a]$ is a unit.

Case (2): $(a, n) \neq 1$. Then $(a, n) = d$ for $d > 1$. Then, $ax + ny = d$ so $[ax] = [d]$. Since $d|n$, $n = dm$ for some $m \in \mathbb{Z}$. Then since $[d] = [dm] = [0]$, we get $[ax] = [a][x] = [0]$, where $[x]$ is nonzero, so $[a]$ is a zero divisor.

□

Proposition: Let R be a ring and $a, b, c \in R$. The following hold:

(1) The additive identity is unique.

(2) An additive inverse is unique.

(3) If $a + b = a + c$, then $b = c$.

(4) The multiplicative identity is unique.

(5) If a is a unit, then its inverse is unique.

(6) $0 \cdot a = a \cdot 0 = 0$

(7) $(a)(-b) = -ab = (-a)(b)$

(8) $-(-a) = a$

(9) $-(a + b) = -a - b$

(10) $-(a - b) = -a + b$

(11) $(-a)(-b) = ab$

Proof. Let R be a ring. Then

(1) Let $0, 0' \in R$ be two additive identities. Then $\underline{0} = 0 \cdot 0' = 0' \cdot 0 = \underline{0'}$.

(2) Let $a \in R$ have two additive inverses $b, c \in R$. Then
$\underline{b} = 0 + b = (c + a) + b = c + (a + b) = c + 0 = \underline{c}$.

(3) Let $a + b = a + c$. Then $(-a + a) + b = (-a + a) + c \to 0 + b = 0 + c \to b = c$.

(4) $1, 1' \in R$ be two multiplicative identities. Then $\underline{1} = 1 \cdot 1' = 1' \cdot 1 = \underline{1'}$.

(5) Let $a \in R$ be a unit with two multiplicative inverses $b, c \in R$. Then
$\underline{b} = b \cdot 1 = b \cdot (ac) = (ba) \cdot c = 1 \cdot c = \underline{c}$.

(6) Let $a \in R$. Then $0 = (a + a) \cdot 0 = a0 + a0 = a0$. Similarly, $0 = 0a$.

(7) Let $a, b \in R$. Then $a0 = a(b + (-b)) = ab + (a)(-b) \implies (a)(-b) = -ab$. Similarly, $(-a)(b) = -ab$.

(8) Let $a \in R$. Then
$\underline{-(-a)} = 0 - (-a) = (a + (-a)) + (-(-a)) = a + ((-a) - (-a)) = a + 0 = \underline{a}$.

(9) Let $a, b \in R$. Then

$$\begin{aligned}
-(a+b) &= 0-(a+b)) \\
&= 0+0-(a+b)) \\
&= (a-a)+(-b+b)-(a+b) \\
&= a+(-a-b)+b-(a+b) && a-b=a+(-b) \\
&= (-a-b)+(a+b)-(a+b) \\
&= (-a-b)+0 \\
-(a+b) &= -a-b
\end{aligned}$$

(10) Let $a, b \in R$. Then $\underline{-(a-b)} = -(a+(-b)) = -a-(-b) = \underline{-a+b}$.

(11) Let $a, b \in R$. Then $\underline{(-a)(-b)} = a(-(-b)) = \underline{ab}$.

$\square$

2.1 Subrings

Definition: Subring

Let R be a ring. A **subring** $S \subseteq R$ is a subset such that S forms a ring with the same operations and same identities as R. If S forms a nonunital ring with the same operations or forms a ring but $1_s \neq 1_R$, S is a **nonunital subring**.

Let R be a ring. $S \subseteq R$ is a subring of R if and only if it satisfies the following:

(1) $1_R \in S$

(2) S is closed under addition.

(3) S is closed under multiplication.

(4) If $a \in S$, then $-a \in S$.

Definition: Integral Domain

A commutative ring R is an **integral domain** if it has no nonzero zero divisors. That is, if $a, b \in R$ and $ab = 0$, then $a = 0$ or $b = 0$.

Proposition: Let R be an integral domain and $a, b, c \in R$. If $ac = bc$ for $c \neq 0$, then $a = b$.

Proof. Suppose $ac = bc$. Then $ac - bc = 0 \to (a-b)c = 0$. because R is an integral domain, $(a-b) = 0$ or $c = 0$. But since $c \neq 0$ by assumption, $(a-b) = 0$ which implies that $a = b$. $\square$

Definition: Field

Let R be a commutative ring. If all nonzero elements of R are units, R is a field.

Proposition: Every field is an integral domain.

Proof. Let R be a field. Since all nonzero elements of R are units, they cannot be zero divisors. $\square$

Theorem

Every finite integral domain is a field.

Proof. Let R be a finite integral domain $R = \{r_1, \ldots, r_n\}$. Take $r_i \in R$ to be nonzero. Consider $r_iR = \{r_ir_1, \ldots, r_ir_n\} \subseteq R$. Then, $|r_iR| \leq |R|$ since $r_iR \subseteq R$. Take $r_ir_j, r_ir_k \in r_iR$ such that $r_ir_j = r_ir_k$. Then because $r_i \neq 0$, we have $r_ir_j - r_ir_k = 0$, or $(r_j - r_k)r_i = 0$. Since $r_i \neq 0$ by assumption, $(r_j - r_k) = 0 \to r_j = r_k$. So $R \subseteq r_iR$ which implies $|R| \leq |r_iR|$. Because $|r_iR| \leq |R|$ and $|r_iR| \geq |R|$, $|r_iR| = |R|$. $\square$

Definition: Homomorphism

Let R, S be rings. A function $f : R \to S$ is a **ring homomorphism** if

(1) $f(a + b) = f(a) + f(b)$

(2) $f(a \cdot b) = f(a) \cdot f(b)$

(3)* $f(1_R) = 1_S$

*A function satisfying (1), (2), but not (3) is a **nonunital ring homomorphism**.

Proposition: Let R, S be rings and $f : R \to S$ a ring homomorphism. Given $a, b \in R$, the following hold:

(1) $f(0_R) = 0_S$

(2) $f(-a) = -f(a)$

(3) $f(a - b) = f(a) - f(b)$

(4) If $a \in R$ is a unit, then $f(a)$ is a unit and $f(a^{-1}) = [f(a)]^{-1}$.

Proof. Let R, S be rings and $f : R \to S$ a ring homomorphism.

(1) Take any $a \in R$. Then $\underline{f(a) + 0_S} = f(a + 0_R) = \underline{f(a) + f(0_R)}$, so $f(0_R) = 0_S$.

(2) $\underline{0_S} = f(0_R) = f(a+(-a)) = \underline{f(a) + f(-a)}$, so $f(a)+f(-a) = 0_S \implies f(-a) = -f(a)$.

(3) $\underline{f(a - b)} = f(a + (-b)) = f(a) + f(-b) = f(a) + (-f(b)) = \underline{f(a) - f(b)}$.

(4) Let $a \in R$ be a unit. Then there exists $a^{-1} \in R$ such that $aa^{-1} = 1$. Then $\underline{1_S} = f(1_R) = f(aa^{-1}) = \underline{f(a)f(a^{-1})}$ and $\underline{1_S} = f(1_R) = f(a^{-1}a) = \underline{f(a^{-1})f(a)}$, so $f(a)$ is a unit and define $[f(a)]^{-1} := f(a^{-1})$ to get $f(a^{-1}) = [f(a)]^{-1}$.

$\square$

Definition: Isomorphism

Let $f : R \to S$ be a ring homomorphism. f is an isomorphism if f is a bijection. Then R and S are isomorphic, written as $R \simeq S$.

Definition: Kernel and Image

Let $f : R \to S$ be a ring homomorphism.

→ The **kernel** of f is defined as $\ker(f) := \{a \in R : f(a) = 0_S\}$.

→ The **image** of f is defined as $\operatorname{Im}(f) := \{f(a) : a \in R\}$.

Proposition: Given a ring homomorphism $f : R \to S$, the image of f is a subring of S and the kernel of f is a nonunital subring of R.

Proof. Let $f : R \to S$ be a ring homomorphism. Then
$\operatorname{Im}(f)$ is a subring of S: Given $f(a), f(b) \in \operatorname{Im}(f)$, we have the following:

(1) $f(a) + f(b) = f(a + b) \in \operatorname{Im}(f)$.

(2) $f(a)f(b) = f(ab) \in \operatorname{Im}(f)$.

(3) $-f(a) = f(-a) \in \operatorname{Im}(f)$.

(4) $f(1_R) = 1_S \in \operatorname{Im}(f)$.

so $\operatorname{Im}(f)$ is a subring of S.

$\ker(f)$ is a nonunital subring of R: Given $a, b \in \ker(f)$, we have the following:

(1) $f(a + b) = f(a) + f(b) = 0_S + 0_S \in \ker(f)$.

(2) $f(ab) = f(a)f(b) = 0_s \cdot 0_S \in \ker(f)$.

(3) $f(-a) = -f(a) = -0_S = 0_S \in \ker(f)$.

(4) $f(0_R) = 0_S \in \ker(f)$.

so $\ker(f)$ is a nonunital subring of R. □

Proposition: Let $f : R \to S$ be a ring homomorphism. Then, for any $a \in \ker(f)$ and $b \in R$, we have $ab, ba \in \ker(f)$.

Proof. $\underline{f(ab)} = f(a)f(b) = 0_S \cdot f(b) = \underline{0_S} = f(b) \cdot 0_S = f(b)f(a) = \underline{f(ba)} \in \ker(f)$. □

Definition: Initial Object

$\mathbb{Z}$ is the **initial object**. Let R be any ring. Then, there is a unique homomorphism $f : \mathbb{Z} \to R$. At $n = 1$, $1 \mapsto 1_R$. At $n = n + 1$, $n + 1 \mapsto \underbrace{1_R + \cdots + 1_R}_{n \text{ times}} + 1_R$. The same is true for $n < 0$. f as defined above is a well-defined ring homomorphism.

2.2 Ideals

Definition: Ideal

Let R be a ring and $I \subseteq R$ a nonempty subset. I is an **ideal** of R if I is a nonunital subring such that for all $a \in I$ and $x \in R$, $xa, ax \in I$. This is often called the "*absorbing property*".

Remark: The kernel of any ring homomorphism is an ideal. Further, all ideal can be realized as the kernel of a ring homomorphism.

Definition: Principal Ideal

Let R be a commutative ring and $a \in R$. The **principal ideal** (a) is an ideal where $(a) := \{ar : r \in R\}$. We say "*a generates I*". Note that $(a) \iff aR$.

Theorem

Let R be a commutative ring and $a \in R$. Then the principal ideal (a) is an ideal.

Proof. Suppose (a) is the principal ideal. Then, $0 = a \cdot 0 \in (a)$. Given $ar_1, ar_2 \in (a)$, $ar_1 + ar_2 = a(r_1 + r_2) \in (a)$. Take $ar \in (a)$. Then $-ar = a(-r) \in (a)$. Take $ar_1 \in (a), r \in R$. Then $(ar_1)r = a(r_1 r) \in (a)$. Because (a) is a nonunital subring with the absorbing property, it is an ideal. $\square$

Theorem

Let R be a ring and $I_1, \ldots, I_k$ be ideals. Then

(1) $I_1 + \cdots + I_k = \{i_1 + \cdots + i_k : i_j \in I_j\}$ is an ideal.

(2) $I_1 \cap \cdots \cap I_k$ is an ideal.

Proof. Let R be a ring, and $I_1, \cdots, I_k$ be ideals.

$\boldsymbol{I_1 + \cdots + I_k = \{i_1 + \cdots + i_k : i_j \in I_j\}}$ **is an ideal.**

(1) Since I_j is an ideal, $0 \in I_j$ so we get $0 + \cdots 0 = 0 \in I_1 + \cdots + I_k$.

(2) Take two elements $a, b \in I_1 + \cdots + I_k$. We can rewrite a, b as, $a = p_1 + \cdots + p_k$ and $b = q_1 + \cdots + q_k$ for $p_j, q_j \in I_j$. Then $a + b = (p_1 + \cdots + p_k) + (q_1 + \cdots + q_k) = (p_1 + q_1) + \cdots + (p_k + q_k)$, and since $p_j + q_j \in I_j$ for all $j \leq k$, we get $a + b \in I_1 + \cdots + I_k$.

(3) Take any $a \in I_1 + \cdots + I_k$. We can rewrite a as, $a = p_1 + \cdots + p_k$ for $p_j \in I_j$. Consider an element $r \in R$. Then, $ar = (p_1 + \cdots + p_k)r = p_1 r + \cdots + p_k r$. Similarly, $ar = r(p_1 + \cdots + p_k) = rp_1 + \cdots + rp_k$. Since I_j is an ideal, $p_j r, rp_j \in I_j$. Then $ar, ra \in I_1 + \cdots + I_k$.

(4) Let $a := a_1 + \cdots + a_k \in I_1 + \cdots + I_k$. Since I_j is an ideal, there exists $-a \in I_j$, so we get $-a_1 + \cdots + -a_k = -(a_1 + \cdots + a_k) = -a \in I_1 + \cdots + I_k$.

Because $I_1 + \cdots + I_k$ satisfies (1) - (4), $I_1 + \cdots + I_k$ is an ideal.

$I_1 \cap \cdots \cap I_k$ is an ideal.

(1) Since I_j is an ideal, $0 \in I_j$, so $0 \in I_1 \cap \cdots \cap I_k$.

(2) Take two elements $a, b \in I_1 \cap \cdots \cap I_k$. Then since each I_j is an ideal, $a + b \in I_j$. So, $a + b \in I_1 \cap \cdots \cap I_k$.

(3) Take any $a \in I_1 \cap \cdots \cap I_k$. Consider an element $r \in R$. Then, since each I_j is an ideal, $ar, ra \in I_j$. Therefore, $ar, ra \in I_1 \cap \cdots \cap I_k$.

(4) Take any $a \in I_1 \cap \cdots \cap I_k$. Then, since I_j is an ideal, $-a \in I_j$, so $-a \in I_1 \cap \cdots \cap I_k$.

Because $I_1 \cap \cdots \cap I_k$ satisfies (1) - (4), $I_1 \cap \cdots \cap I_k$ is an ideal. □

Definition: Multiple Generators

Let R be a commutative ring and $a_1, \ldots, a_k \in R$. The ideal generated by $a_1, \cdots a_k$ is geiven by $(a_1) + \cdots + (a_k)$ and is written as $(a_1, \ldots, a_k)$.

Proposition: Let F be a field. The only ideal of F are $\{0\}$ and F.

Proof. Let I be a nonzero ideal of F and take $a \in I$. Then, $1 = aa^{-1} \in I$. Because $1 \in I$, $F = (1) = I$. □

2.3 Quotient Rings

Preface: To generalize the construction of $\mathbb{Z}/n$ to general rings, consider the following: given an ideal $I \subseteq R$, define equivalence where $a \sim b$ if $a - b \in I$. We can then inherit $(+, \cdot)$ from R. Given two equivalence classes $[a], [b]$, define $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [ab]$.

Definition: Congruent Modulo I

Let R be a ring, $I \subseteq R$ and ideal, and $a, b \in I$. a and b are **congruent modulo** I if $a - b \in I$. We write $a \equiv b \pmod{I}$, or $a + I = b + I$.

Remark: The notation $a + I := \{a + x : x \in I\}$ is precisely the congruence class modulo I containing a.

Proposition: Let R be a ring and $I \subseteq R$ an ideal. Congruence modulo I is an equivalence relation.

Proof. Let R be a ring and $I \subseteq R$ an ideal.

(1) For any $a \in R$, $a - a = 0 \in I$, so $a \equiv a \pmod{I}$.

(2) Take $a, b \in R$ such that $a \equiv b \pmod{I}$. Then $a - b \in I$. Since I is an ideal, $-(a - b) = b - a \in I$, so $b \equiv a \pmod{I}$.

(3) Let $a, b, c \in R$ such that $a \equiv b \pmod{I}$ and $b \equiv c \pmod{I}$. Then $a - b, b - c \in I$. Then $(a - b) + (b - c) = a + (-b + b) - c = a - c \in I$, so $a \equiv c \pmod{I}$.

Since congruence modulo I satisfies $(1) - (3)$, it is an equivalence relation. □

Theorem

Let R be a ring, $a, b, c, d \in R$, and $I \subseteq R$ and ideal. Suppose $a \equiv c \pmod{I}$, $b \equiv d \pmod{I}$. Then $a + b \equiv c + d \pmod{I}$ and $ab \equiv cd \pmod{I}$.

Proof. Since $a - c, b - d \in I$, we have that $(a - c) + (b - d) = (a + b) - (c + d) \in I$. Then by definition, we have $a + b \equiv c + d \pmod{I}$. Now consider the following:

$$\begin{aligned} ab - cd &= ab + 0 - cd \\ &= ab + (-bc + bc) - cd \\ &= (ab - bc) + (bc - cd) \\ ab - cd &= b(a - c) + c(b - d) \end{aligned}$$

Since $a - c, b - d \in I$, $ab - cd \in I$, so $ab \equiv cd \pmod{I}$. □

Notation: $(a + I) + (b + I) = (a + b) + I$ and $(a + I)(b + I) = ab + I$.

Definition: Quotient Ring

Let R be a ring, $a, b \in$, and $I \subseteq R$ and ideal. The **quotient ring** R/I is the set of congruence classes modulo I with $(+, \cdot)$ defined as $(a + I) + (b + I) = (a + b) + I$ and $(a + I)(b + I) = ab + I$ respectively.

Proposition: R/I is a ring.

Proof. I'm not checking all 9 axioms lol. □

Theorem

Let R be a ring and $I \subseteq R$ and ideal. If R is commutative, then R/I is commutative.

Proof. Take $a + I, b + I \in R/I$. Then $(a + I)(b + I) = ab + I$ and $(a + I)(b + I) = ab + I$, so $ab + I = ba + I \implies (a + I)(b + I) = (b + I)(a + I)$. □

Note: If R/I is commutative, it does ***not*** imply that R is commutative. For example, if $I = R$, then $R/I \simeq \{0\}$.

Definition: Canonical Projection

Let R be a ring, $I \subseteq R$ and ideal. Consider $\pi : R \to R/I$ such that $\pi(a) = a + I$. This map is the **canonical projection**.

Theorem

Let R be a ring, $I \subseteq R$ and ideal. The canonical projection $\pi : R \to R/I$ is a surjective ring homomorphism with $\ker(\pi) = I$.

Proof. Let R be a ring, $I \subseteq R$ and ideal. Let $\pi : R \to R/I$ be the canonical projection from R to R/I. Then

(1) $\pi(a+b) = (a+b) + I = (a+I) + (b+I) = \pi(a) + \pi(b)$.

(2) $\pi(a \cdot b) = (a \cdot b) \cdot I = (a \cdot I) \cdot (b \cdot I) = \pi(a) \cdot \pi(b)$.

(3) $\pi(1_R) = 1 + I = 1_{R/I}$.

so π is a ring homomorphism. Take $a+I \in R/I$. Then $\pi(a) = a+I$. Moreover, if $b \in [a+I]$, then $\pi(b) = a+I$. So π is surjective. Finally, let $a \in I$. Then $\pi(a) = a + I$ but $a \equiv 0 \pmod{I}$, so we have $\pi(a) = a+I = 0_R + I = I$. So, $\ker(\pi) \subseteq I$. Now suppose $\pi(a) = 0_R + I$. Then $[a+I] = [0_R + I]$, or $a \equiv 0_R \pmod{I}$. We can rewrite this to get $a - 0_R = a \in I$, so $I \subseteq \ker(\pi)$. Because $\ker(\pi) \subseteq I$ and $I \subseteq \ker(\pi)$, $\ker(\pi) = I$. $\square$

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Theorem (First Isomorphism Theorem)

Let $f : R \to S$ be a ring homomorphism. The following hold:

$\to$ There exists a unique homomorphism $\overline{f} : R/\ker(f) \to S$ such that $f = \overline{f} \circ \pi$.

$\to$ $R/\ker(f) \simeq \mathrm{Im}(f)$.

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "\pi"'] & S \\
R/\ker(f) \arrow[ur, dashed, "\overline{f}"'] &
\end{tikzcd}$$

Proof. Let $f : R \to S$ be a ring homomorphism. Then

$\overline{f}$ is well-defined: Suppose $a + \ker(f) = a' + \ker(f)$. Then $a - a' \in \ker(f)$, so $f(a - a') = 0 = f(a) - f(a')$. This implies $f(a) = f(a')$, so $\overline{f}$ is well-defined.

$\overline{f}$ is a homomorphism:

(1) $\overline{f}(1_R + \ker(f)) = f(1_R) = 1_S$.

(2) Take $a + \ker(f), b + \ker(f) \in R/\ker(f)$. Then
$$\overline{f}((a+b) + \ker(f)) = f(a+b) = f(a) + f(b) = \overline{f}(a + \ker(f)) + \overline{f}(b + \ker(f))$$

(3) Take $a + \ker(f), b + \ker(f) \in R/\ker(f)$. Then
$$\overline{f}((a \cdot b) + \ker(f)) = f(a \cdot b) = f(a) \cdot f(b) = \overline{f}(a + \ker(f)) \cdot \overline{f}(b + \ker(f))$$

so $\overline{f}$ is a homomorphism.

$f = \overline{f} \circ \pi$: Take $a \in R$. Then, $\underline{\overline{f} \circ \pi(a)} = \overline{f}(\pi(a)) = \overline{f}(a + \ker(f)) = \underline{f(a)}$.

$\overline{f}$ is unique: Suppose we have another function $g : R/\ker(f) \to S$ such that $\overline{f} \neq g$. Then there exists $b \in R/\ker(f)$ such that $g(b + \ker(f) \neq \overline{f}(b + \ker(f))$, so
$$g \circ \pi(a) = g(\pi(a)) = g(a + \ker(f)) \neq \overline{f}(a + \ker(f)) = f(a)$$

Therefore, $\overline{f}$ is unique.

$R/\ker(f) \simeq \mathrm{Im}(f)$: Take $a + \ker(f) \in \ker(\overline{f})$. Then $\overline{f}(a + \ker(f)) = f(a) = 0$. Since $a + \ker(f)$ was arbitrary, this holds for all $a + \ker(f) \in \ker(\overline{f})$, so $\overline{f}$ is **injective**. Now take any $y \in \mathrm{Im}(f)$. Then there is some $z \in R$ such that $f(z) = y$. Set $x := z + \ker(f) \in R/\ker(f)$. Then $\overline{f}(x) = \overline{f}(z + \ker(f)) = f(z) = y$, so $\overline{f}$ is **surjective**. Since $\overline{f}$ is injective and surjective, it is **bijective**, and therefore $R/\ker(f) \simeq \mathrm{Im}(f)$. $\square$

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Theorem (Correspondence Theorem)

Let R be a ring, and $I \subseteq R$ an ideal. Consider the projection $\pi : R \to R/I$ and let $\overline{R} := R/I$. Then

(1) There is a bijective correspondence between ideals in R containing I and ideals of $\overline{R}$ given by $J \mapsto \pi(J) = \{r + I : r \in J\}$ and $\overline{J} \mapsto \pi^{-1}(\overline{J})$ where $J \subseteq R$ and $\overline{J} \subseteq \overline{R}$ are ideals.

(2) If an ideal $J \subseteq R$ corresponds to $\overline{J} \subseteq \overline{R}$, then $R/J \simeq \overline{R}/\overline{J}$.

Proof. **(1)** To show that $\pi(J)$ is an ideal of $\overline{R}$, take $a, b \in \pi(J)$ and $r + I \in \overline{R}$. Then $\pi(a+b) = (a+b)+I = (a+I)+(b+I) = \pi(a)+\pi(b)$ and $(a+I)(r+I) = ar+I \in \pi(J)$. Similarly, $ra + I \in \pi(J)$. To show that $\pi^{-1}(J)$ is an ideal of R, take $a, b \in \pi^{-1}(\overline{J})$. Then note that $\pi(a+b) = (a+b)+I = (a+I)+(b+I) = \pi(a)+\pi(b) \in \overline{J}$, so $a + b \in \pi^{-1}(\overline{J}$. Also, note that $\pi(ar) = ar + I = (a+I)(r+I) \in \overline{J}$, so $ar \in \pi^{-1}(\overline{J})$. Similarly, $rb \in \pi^{-1}(\overline{J})$. So $\pi(J)$ is an ideal of $\overline{R}$ and $\pi^{-1}(\overline{J})$ is an ideal in R.

$\boldsymbol{\pi^{-1}(\pi(J)) = J}$**:** Let $a \in \pi^{-1}(\pi(J))$. Then by definition of the pre-image under π, there exists $x \in J$ such that $\pi(a) = \pi(x) \in \pi(J)$, or $a+I = x+I$, which implies that $a-x \in I \subseteq J$, so $a \in I \subseteq J$. Since a was arbitrary, $\pi^{-1}(\pi(J)) \subseteq J$. Now let $b \in J$. Then by definition, $\pi(b) = b + I$. Then, $\pi^{-1}(\pi(b)) = \pi^{-1}(b+I)$ but by definition of the pre-image, $\pi^{-1}(b+I) = b \in \pi^{-1}(\pi(J))$. Since b was arbitrary, $J \subseteq \pi^{-1}(\pi(J))$. Since we have $\pi^{-1}(\pi(J)) \subseteq J$ and $\pi^{-1}(\pi(J)) \supseteq J$, $\pi^{-1}(\pi(J)) = J$.

$\boldsymbol{\pi(\pi^{-1}(\overline{J})) = \overline{J}}$**:** Let $a + I \in \pi(\pi^{-1}(\overline{J}))$. Then there exists $x \in R$ such that $x \in \pi^{-1}(\overline{J})$ and $\pi(x) = a + I \in \overline{J}$. Since a was arbitrary, $\pi(\pi^{-1}(\overline{J})) \subseteq \overline{J}$. Now let $b + I \in \overline{J}$. Then by definition, $b + I$ is in the image of J under π, so $b \in \pi^{-1}(\overline{J})$. Then $\pi(\pi^{-1}(b+I)) = \pi(b) = b + I \in \pi(\pi^{-1}(\overline{J}))$. Since $b + I$ was arbitrary, $\overline{J} \subseteq \pi(\pi^{-1}(\overline{J}))$. Since $\pi(\pi^{-1}(\overline{J})) \subseteq \overline{J}$ and $\pi(\pi^{-1}(\overline{J})) \supseteq \overline{J}$, $\pi(\pi^{-1}(\overline{J})) = \overline{J}$.

Therefore, there exists a bijective correspondence between the ideals $J \supseteq I$ in R and and the ideals $\overline{J} \subseteq \overline{R}$.

(2) Consider the canonical projection $\phi : \overline{R} \to \overline{R}/\overline{J}$. Since ϕ and π are **surjective**, the composition $\phi \circ \pi : R \to \overline{R}/\overline{J}$ is as well. By the **First Isomorphism Theorem**, we have $\overline{R}/\ker(\phi \circ \pi) \simeq \overline{R}/\overline{J}$.

$\boldsymbol{\ker(\phi \circ \pi) = J}$**:** Let $\overline{J} = \pi(J)$. Take $a \in J$. Then $\phi \circ \pi(a) = \phi(\pi(a)) = \phi(a+I) = (a+I)+\overline{J}$, but since $a + I \in \overline{J}$, we have that $(a+I)+\overline{J} = 0+\overline{J} \in \ker(\phi \circ \pi)$. Since a was arbitrary, $J \subseteq \ker(\phi \circ \pi)$. Now take any $b \in R$ such that $\phi \circ \pi(b) = 0 + \overline{J}$. Then, $(b+I)+\overline{J} = 0+\overline{J}$. By definition, $b+I \in \overline{J} = \pi(J)$. Then $b+I$ is the image of J under π, so $b \in \pi^{-1}(\overline{J}) = \pi^{-1}(\pi(J)) = J$. Since b was arbitrary, $\ker(\phi \circ \pi) \subseteq J$. Since $J \subseteq \ker(\phi \circ \pi)$ and $J \supseteq \ker(\phi \circ \pi)$, $J = \ker(\phi \circ \pi)$.

Therefore, $R/J \simeq \overline{R}/\overline{J}$. $\square$

Theorem (Chinese Remainder Theorem [Rings])

Let R be a commutative ring, $a, b \in R$, and $I, J \subseteq R$ be ideals such that $I + J = R$. We can find $x \in R$ such that

$$\begin{aligned} x &\equiv a \pmod{I} \\ x &\equiv b \pmod{J} \end{aligned}$$

Moreover, if y is another solution, then $y \equiv x \pmod{I \cap J}$.

Proof. Because $I + J = R$, we can find $i \in I$ and $j \in J$ such that $i + j = 1_R$. Then $i \equiv 1 \pmod{J}$ and $j \equiv 1 \pmod{I}$. Consider $x := bi + aj$. Then

$$\begin{aligned} x &= bi + aj \\ &\equiv aj \pmod{I} \\ &\equiv a \cdot 1 \pmod{I} \\ x &\equiv a \pmod{I} \end{aligned}$$

and

$$\begin{aligned} x &= bi + aj \\ &\equiv bi \pmod{J} \\ &\equiv b \cdot 1 \pmod{J} \\ x &\equiv b \pmod{J} \end{aligned}$$

Now suppose that y is another solution. Then $y \equiv x \pmod{I}$ and $y \equiv x \pmod{J}$. By definition, this means that $y - x \in I$ and $y - x \in J$, so $y \equiv x \pmod{I \cap J}$. $\square$

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Theorem (Chinese Remainder Theorem [Isomorphism])

Let R be a ring and $I, J \subseteq R$ be ideals such that $I + J = R$. The quotient rings $(R/I) \times (R/J)$ and $R/(I \cap J)$ are isomorphic.

Proof. Consider $f : (R/I) \times (R/J)$ given by $a \mapsto (a + I, a + J)$. Then

(1) $f(1_R) = (1_R + I, 1_R + J)$

(2) Take $a, b \in R$. Then
$$f(a + b) = ((a + b) + I, (a + b) + J) = (a + I, a + J) + (b + I, b + J) = f(a) + f(b)$$

(3) Take $a, b \in R$.
$$f(a \cdot b) = ((a \cdot b) + I, (a \cdot b) + J) = (a + I, a + J) \cdot (b + I, b + J) = f(a) \cdot f(b)$$

so f is a homomorphism.

Take $(a+I, b+J) \in (R/I) \times (R/J)$. By the **Chinese Remainder Theorem (Rings)**, we can find $x \in R$ such that $x + I = a + I$ and $x + J = a + J$. Then, $f(x) = (a+I, b+J)$, so f is **surjective**. Suppose $f(a) = 0$. Then $a \in I$ and $a \in J$, so $a \in I \cap J$. Now take $a \in I \cap J$. Then $a \in I$ and $a \in J$, so $a + I \in I$ and $a + J \in J$. By the **First Isomorphism Theorem**, we have $\underline{R/(I \cap J)} = R/\ker(f) \simeq \mathrm{Im}(f) = \underline{(R/I) \times (R/J)}$. □

2.4 Prime and Maximal Ideals

Preface: All rings in this subsection are commutative rings.

Definition: Prime Ideal

Let R be a commutative ring and let $I \subsetneq R$ be a proper ideal. I is a **prime ideal** if, whenever $ab \in I$ for $a, b \in R$, we have either $a \in I$ or $b \in I$.

Example: Let R be an integral domain. Then (0) is prime since whenever $ab \in (0)$, we have that either $a \in (0)$ or $b \in (0)$.

Proposition: $(p) \subsetneq \mathbb{Z}$ is a prime ideal if and only if $p \in \mathbb{Z}$ is prime.

Proof. Let $p \in \mathbb{Z}$ be nonzero.

($\Longrightarrow$) Suppose $(p) \subsetneq \mathbb{Z}$ is a prime ideal. Consider $ab \in (p)$. Then either $a \in (p)$ or $b \in (p)$. By definition, we can write $ab = pr$ for some $r \in \mathbb{Z}$, so $p \mid ab$. But we also have that either $a = pq$ or $b = ps$ for some $q, s \in \mathbb{Z}$, so either $p \mid a$ or $p \mid b$. Then, since these two statements:

(1) p is prime.

(2) If $p \mid ab$, then $p \mid a$ or $p \mid b$.

are equivalent, $p \in \mathbb{Z}$ is prime.

($\Longleftarrow$) Suppose $p \in \mathbb{Z}$ is prime and consider the ideal $(p) \subsetneq \mathbb{Z}$. Consider $ab \in \mathbb{Z}$ such that $p \mid ab$. Then either $p \mid a$ or $p \mid b$. Since $p \mid ab$, we have that $ab = pr$ for some $r \in \mathbb{Z}$, so $ab \in (p)$. By a similar argument, either $a \in (p)$ or $b \in (p)$, so $(p) \subsetneq \mathbb{Z}$ is a prime ideal. □

Theorem

Let R be a commutative ring and let $I \subsetneq R$ be a proper ideal. The quotient ring R/I is an integral domain if and only if I is prime.

Proof. Let R be a commutative ring and let $I \subsetneq R$ be a proper ideal.

$(\implies)$ Suppose R/I is an integral domain. Take $ab \in I$. Then $(a+I)(b+I) = ab+I = 0+I$. Since R/I is an integral domain, we have that either $a+I = 0+I$ or $b+I = 0+I$. This implies that either $a \in I$ or $b \in I$, so $I \subsetneq R$ is prime.

$(\impliedby)$ Suppose I is a prime ideal. Take $ab+I \in R/I$. Then $ab+I = (a+I)(b+I) = 0+I$. Since I is a prime ideal, either $a \in I$ or $b \in I$. This implies that either $a+I = I$ or $b+I = I$, so R/I has no zero divisors. This implies that R/I is an integral domain. $\square$

Definition: Maximal Ideal

Let R be a commutative ring and let $I \subsetneq R$ be a proper ideal. I is a **maximal ideal** if, whenever there is an ideal J such that $I \subsetneq J \subseteq R$, we must have $J = R$.

Theorem

Let R be a commutative ring and $I \subsetneq R$ be a maximal ideal. Then I is a prime ideal.

Proof. Let R be a commutative ring and suppose $I \subsetneq R$ is a maximal ideal. Take $ab \in I$. If $a \in I$, then we are done, so suppose not. Then consider $I + (a) \supsetneq I$. Since I is maximal, we have that $I + (a) = R$. Then $1 = x + ar$ for some $x \in I$, $ar \in (a)$. Multiplying both sides by $b \in R$, we get $\underline{b} = b(x + ar) = \underline{bx + abr}$. Since $ab \in I$, we have that $(ab)r \in I$. Further, since $x \in I$, $xb \in I$, so $bx + abr = b \in I$. This implies that I is a prime ideal. $\square$

Note: From now on, I will only state "I is prime/maximal" instead of saying "I is a prime/maximal ideal".

Theorem

Let R be a commutative ring and $I \subsetneq R$ be a proper ideal. I is maximal if and only if R/I is a field.

Proof. Let R be a commutative ring and suppose $I \subsetneq R$ is a proper ideal.

$(\implies)$ Suppose I is maximal. Pick a nonzero $a + I \in R/I$. Since $a + I \neq 0 + I$, $a \notin I$. Consider $I+(a) \supsetneq I$. Since I is maximal, we have that $I+(a) = R$. Then $1 = x+ab$ for some $x \in I$, $ab \in (a)$, so we have $(x+ab)+I = (x+I)+(ab+I) = 1+I$. Since $x \in I$, we have that $x+I = 0+I$. This implies that $(x+I)+(ab+I) = (0+I)+(ab+I) = (a+I)(b+I) = 1+I$. So $a + I \in R/I$ is a unit. Since $a + I \in R/I$ was arbitrary, R/I is a field.

$(\impliedby)$ Suppose R/I is a field. Pick $a \in R \setminus I$. Then $a+I \in R/I$ is nonzero, so there exists $b + I \in R/I$ such that $(a + I)(b + I) = ab + I = 1 + I$. Then $ab - 1 \in I$, so there exists $x \in I$ such that $x = ab - 1$, or $1 = ab - x$. Then since $-x \in I$ and $ab \in (a)$, we have that $ab - x = 1 \in I + (a)$, so $I + (a) = R$. Therefore, I is maximal. $\square$

3 Polynomial Rings over Fields

Preface: Throughout this section, F is a field and $F[x]$ are the polynomials with coefficients in F. Recall that given $f \in F[x]$, we can uniquely express $f(x)$ as $\sum_{i=0}^{n} a_i x^i$, where a_n is nonzero.

Note: The notation $f(x)$ and f are interchangeable.

Definition: Associate

Let $f, g \in F[x]$. f and g are **associates** if there is some nonzero $c \in F$ such that $g = cf$.

Definition: Degree

Let $f \in F[x]$ be expressed as $f(x) = \sum_{i=0}^{n} a_i x^i$, where $a_n \neq 0$. The **degree** of f is written as $\deg(f) = n$.

Let $f, g \in F[x]$. The following hold:

(1) $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$.

(2) $\deg(fg) = \deg(f) + \deg(g)$.

Note: The zero polynomial has a degree of $-\infty$ by convention.

Definition: Monic Polynomial

Let $f \in F[x]$. f is **monic** if its leading term is 1.

Theorem (Division Algorithm [Polynomials])

Let $f, g \in F[x]$ such that $g \neq 0$. Then there are unique polynomials $q, r \in F[x]$ such that $f = gq + r$, where $\deg(r) < \deg(g)$.

Proof. **Existence:** Let $f, g \in F[x]$ such that $g \neq 0$ and consider $S := \{f - sg : s \in F[x]\}$. If s is the zero polynomial, then $f - sg = f - 0g = f \in S$, so S is not empty. Choose $f - sg \in S$ to be of least degree, and define $q := s, r := f - sg$. Then $r = f - sg = f - qg$, or $f = gq + r$. Since $g \neq 0$, we have that $\deg(g) \geq 0$. Suppose for the sake of contradiction that $\deg(r) \geq \deg(g)$. Then $r = \sum_{i=0}^{n} r_i x^i$ and $g = \sum_{i=0}^{m} g_i x^i$ where $n \geq m$. Since $\deg(r) = n, \deg(g) = m$, we have that $r_n \neq 0$ and $g_m \neq 0$; i.e. they are units. Now consider $t := r_n x^n \cdot (g_m x^m)^{-1} = r_n g_m^{-1} x^{n-m}$. Then

$$tg = \left(r_n g_m^{-1} x^{n-m}\right) \cdot \left(\sum_{i=0}^{m} g_i x^i\right) = \left(\sum_{i=0}^{m-1} r_n g_m^{-1} g_i x^{n-m+i}\right) + r_n x^n$$

so

$$\begin{aligned} r - tg &= \left(\sum_{i=0}^{n-1} r_i x^i\right) + r_n x^n - \left(\left(\sum_{i=0}^{m-1} r_n g_m^{-1} g_i x^{n-m+i}\right) + r_n x^n\right) \\ &= \left(\sum_{i=0}^{n-1} r_i x^i\right) - \sum_{i=0}^{m-1} r_n g_m^{-1} g_i x^{n-m+i} \end{aligned}$$

so $\deg(r - tg) \leq n - 1 < n = \deg(r)$. But we have that $r = f - gs$, so we get

$$r - tg = (f - gs) - tg = f - g(s + t)$$

Since $s + t \in F[x]$, we have that $r - tg \in S$, but r was chosen to have the lowest degree and $\deg(r - tg) < \deg(r)$, a contradiction. Therefore, $\deg(r) < \deg(g)$.

Uniqueness: Suppose $f = gq + r = gq' + r'$ for $q, q', r, r' \in F[x]$. Then

$$\begin{aligned} gq + r &= gq' + r' \\ g(q - q') &= r - r' \end{aligned}$$

so $g \mid (r - r')$. But $\deg(r - r') < \deg(g)$, so $r = r'$. Since F is a field and $g \neq 0$, this implies that $q = q'$. Therefore, $q, r \in F[x]$ are unique. $\square$

Definition: Divides (Polynomials)

Let $f, g \in F[x]$. f **divides** g if there is a polynomial $s \in F[x]$ such that $fs = g$. Then f is a **divisor** of g. We write $f \mid g$.

Proposition: Let $f, g \in F[x]$, $g \neq 0$, and suppose f divides g. Then $\deg(f) \leq \deg(g)$.

Proof. Let $f, g \in F[x], g \neq 0$ and suppose $f \mid g$. Then there exists $s \in F[x]$ such that $fs = g$. Since $g \neq 0$, we have that $\deg(g) \geq 0$. Since F is a field, we have that $f \neq 0$ and $s \neq 0$, so $\deg(f) \geq 0$ and $\deg(s) \geq 0$. Then $\deg(g) = \deg(fs) = \deg(f) + \deg(s)$. This implies that $\deg(f) \leq \deg(g)$. □

Definition: Greatest Common Divisor (gcd) (Polynomials)

Let $f, g \in F[x]$ be polynomials such that either $f \neq 0$ or $g \neq 0$. The **greatest common divisor** of f and g is the monic polynomial of largest degree that divides f and g. That is, the greatest common divisor d of f and g is the monic polynomial that satisfies the following:

(1) $d \mid f$ and $d \mid g$.

(2) If $a \mid f$ and $a \mid g$, then $a \mid d$.

If d is the greatest common divisor of f and g, we write $d = \gcd(f, g) = (f, g)$.

Theorem (Bezout's Identity [Polynomials])

Let $f, g \in F[x]$ such that either $f \neq 0$ or $g \neq 0$. There exist $m, n \in F[x]$ such that $fm + gn = d$, where $d = (f, g)$.

Proof. Let $f, g \in F[x]$ such that either $f \neq 0$ or $g \neq 0$. Consider the set $S = \{fm + gn : m, n \in F[x]\}$. If $m = f, n = g$, then since at least one of f, g is nonzero, we have $0 \neq fm + gn = f^2 + g^2 \in S$, so S is not empty. By the well-ordering principle, choose the polynomial $s = fm + gn \in S$ of smallest degree, and consider $f = sq + r$ for $\deg(r) < \deg(g)$. Rearranging the second equation, we get

$$\begin{aligned} f &= sq + r \\ r &= f - sq \\ &= f - (fm + gn)q \\ r &= f(1 - mq) + g(-nq) \end{aligned}$$

This implies that $r \in S$. We also have that $\deg(r) < \deg(g)$, but since s was chosen to be the smallest element in S, this forces $r = 0$. Then $f = sq + r = sq$, so $s \mid f$. Similarly, $s \mid g$. Since $s \mid f$ and $s \mid g$, $s \leq d$. But $d \mid f$ and $d \mid g$ by definition, so $d \mid s$ which implies that $d \leq s$. Therefore, $d = s$, where s is a linear combination of f and g. So, there exist $m, n \in F[x]$ such that $d = fm + gn$, where $d = (f, g)$. □

Theorem

Let $a, b, c \in F[x]$. Suppose $a \mid bc$ such that $(a, b) = 1$. Then $a \mid c$.

Proof. Let $a, b, c \in F[x]$, and suppose $a \mid bc$ such that $(a, b) = 1$. Then we can write 1 as a linear combination of a and b; i.e. $am + bn = 1$ for $m, n \in F[x]$. We also have that $aq = bc$ for some $q \in F[x]$ Then

$$\begin{aligned} 1 &= am + bn \\ c &= c(am + bn) \\ &= acm + (bc)n \\ &= acm + (aq)n \\ c &= a(cm + qn) \end{aligned}$$

which implies that $a \mid c$. $\square$

3.1 Irreducibility

Definition: Irreducible

Let $f \in F[x]$ be nonzero and nonconstant. f is **irreducible** if its only factors are units and associates. Otherwise, f is **reducible**. That is, f is reducible if there exist polynomials $a, b \in F[x]$ of lower degree such that $ab = f$.

Theorem

Let $p \in F[x]$. The following are equivalent statements:

(1) p is irreducible.

(2) If $p \mid ab$, then $p \mid a$ or $p \mid b$.

(3) If $p = ab$, then either a or b is a unit.

Proof. Let $p \in F[x]$.

(1) $\implies$ (2) Suppose p is irreducible and $p \mid ab$. If $p \mid a$, then we are done, so suppose not. Then $p \mid ab$ and $(p, a) = 1$ which implies $p \mid b$.

(2) $\implies$ (3) Suppose that if $p \mid ab$, then $p \mid a$ or $p \mid b$. Let $p = ab$. Then $p \mid p = ab$, so $p \mid a$ or $p \mid b$. Without loss of generality, suppose $p \mid a$. Then $\deg(p) \leq \deg(a)$. But since $p = ab$, we have that $\deg(a), \deg(b) \leq \deg(p)$. So, $\deg(p) = \deg(a)$, which implies that b is a unit.

(3) $\implies$ (1) Suppose that if $p = ab$, then either a or b is a unit. Without loss of generality, suppose a is a unit. Then $\underline{\deg(a)} = 0$, so $\deg(p) = \deg(ab) = \deg(a) + \deg(b) = \underline{\deg(b)}$. This implies that b is an associate of p. Therefore, the only factors of p are units and associates, so p is irreducible. $\square$

Corollary

Let $p \in F[x]$ be irreducible. If $p \mid a_1 \cdots a_n$, then $p \mid a_i$ for some i.

Proof. Let $p \in F[x]$ be irreducible. We will induct on $n \in \mathbb{N}$. At $n = 2$, if $p \mid a_1 a_2$, then $p \mid a_1$ or $p \mid a_2$. Assume the base case holds for some $n \geq 2$. At $n = n + 1$, consider $p \mid a_1 \cdots a_n \cdot a_{n+1}$. Then if $p \mid a_{n+1}$, we are done. Otherwise, by the inductive hypothesis, we have that $p \mid a_i$ for some $i \leq n$. Therefore, if $p \mid a_1 \cdots a_n$, then $p \mid a_i$ for some i. $\square$

Theorem (Unique Factorization [Polynomials])

Let $f \in F[x]$ be nonzero and nonconstant. f can be written a a product of irreducible polynomials. Moreover, if $f = p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_m$ are two irreducible factorizations, then $n = m$ and there is a permutation σ on $\{1, \ldots, n\}$ such that p_i and $q_{\sigma(i)}$ are associates.

Proof. **Existence:** Suppose for the sake of contradiction that there exist polynomials that cannot be written as a product of irreducible polynomials. Let S contain such polynomials. Then since S is not empty, pick f to be the polynomial of least degree. Then if $f = pq$, we have that $\deg(p), \deg(q) \leq \deg(f)$. But f was chosen to be the polynomial with smallest degree, so $p, q \notin S$. Then p, q can be written as a product of irreducible polynomials which implies that f can be written as a product of irreducible polynomials, a contradiction. Therefore, S is empty which implies that all nonzero and nonconstant $f \in F[x]$ can be written as a product of irreducible polynomials.

Uniqueness: Suppose $p_1 \cdots p_n = q_1 \cdots q_m$. Without loss of generality, suppose $n \leq m$. Then $p_1 \mid q_1 \cdots q_m$. Without loss of generality, let $p_1 \mid q_1$. Then p_1 and q_1 are associates since they are both irreducible. Then $q_1 = c_1 p_1$ for some unit $c_1 \in F$, so we have that $p_1 \cdots p_n = c_1 p_1 \cdot q_2 \cdots q_m$. Since F is a field, we can apply the cancellation property to cancel p_1, which yields $p_2 \cdots p_n = c_1 q_2 \cdots q_m$. Continuing this process inductively, we have that $p_{m+1} \cdots p_n = c_1 \cdots c_m$. Suppose for the sake of contradiction that $m < n$. Then $0 < \deg(p_{m+1} \cdots p_n) = \deg(c_1 \cdots c_m) = 0$, a contradiction. Therefore, $m = n$ and there is a unique permuatation σ on $\{1, \ldots, n\}$ such that $p_i = q_{\sigma(i)}$. $\square$

3.2 Roots

Definition: Root

Let $f \in F[x]$. $a \in F$ is a **root** of f if $f(a) = 0$.

Lemma

Let $f \in F[x]$ and let $a \in F[x]$ be a root of f. The remainder of $f(x)$ divided by $x - a$ is $f(a)$.

Proof. Let $f \in F[x]$. We can express f as $f(x) = (x - a)q(x) + r(x)$ for unique $q, r \in F[x]$. Then $\underline{f(a)} = (a - a) + q(a) + r = 0 + r = \underline{r}$. $\square$

Theorem

Let $f \in F[x]$ and $a \in F$. a is a root of f if and only if $x - a$ is a factor of f.

Proof. Let $f \in F[x]$ and $a \in F$.

($\Longrightarrow$) Suppose a is a root of f. We can express f as $f(x) = (x - a)q(x) + r(x)$ for unique $q, r \in F[x]$. Then from the **Lemma** above, we have that $f(a) = r$, but since a is a root, $f(a) = 0$, so $r = 0$ which implies that $f(x) = (x - a)q(x)$, or $(x - a) \mid f$.

($\Longleftarrow$) Suppose $x - a$ is a factor of f. Then $(x - a) \mid f$, or $f(x) = (x - a)q(x)$. Then $f(a) = (a - a)q(a) = 0$. $\square$

Corollary

Let $f \in F[x]$ such that $\deg(f) = n > 0$. f has at most n roots.

Proof. Let $f \in F[x]$. such that $\deg(f) = n > 0$. We will induct on $n \in \mathbb{N}$. At $n = 1$, we have $f(x) = a_0 + a_1 x$. Clearly, f has at most one root. Assume the base case holds for all $1 \leq k < n$. At $k = n$, we can express f as $f(x) = (x - r)q(x)$, where $r \in F$ is a root of f. We have that $\deg(q) = n - 1$, so by the inductive hypothesis, q has at most $n - 1$ roots. Then f has at most $1 + (n - 1) = n$ roots. Since k was arbitrary, this holds for all $n \in \mathbb{N}$. $\square$

3.3 Quotienting by Irreduibles

Theorem

Let $p \in F[x]$ be a nonzero, nonconstant polynomial. The following are equivalent:

(1) p is irreducible.

(2) (p) is maximal.

(3) (p) is prime.

Proof. Let $p \in F[x]$.

(1) $\implies$ (2) Suppose p is irreducible. Consider the ideal $(p) \subseteq F[x]$. Take $a \in F[x] \setminus (p)$. If a is a unit, then $(p) + (a) = F[x]$, so suppose not. Then we have that $(p, a) = 1$, so we can write $pf + ag = 1$ for $f, g \in F[x]$, so $(p) + (a) = (1) = F[x]$. Therefore, (p) is maximal.

(2) $\implies$ (3) Suppose (p) is maximal. Since all maximal ideals are prime, (p) is prime.

(3) $\implies$ (1) Suppose (p) is prime. Consider $ab \in (p)$. Then $ab = pr$ for some $r \in F[x]$, so $p \mid ab$. Then since p is prime, we have that either $a \in (p)$ or $b \in (p)$. Without loss of generality, suppose $a \in (p)$. Then $a = ps$ for some $s \in F[x]$, so $p \mid a$. Since the following statements:

(1) p is irreducible.

(2) If $p \mid ab$, then $p \mid a$ or $p \mid b$.

(3) If $p = ab$, then either a or b is a unit.

are equivalent, p is irreducible. □

Corollary

Let $p \in F[x]$ be a nonzero, nonconstant polynomial. The following are equivalent:

(1) p is irreducible.

(2) $F[x]/(p)$ is a field.

(3) $F[x]/(p)$ is prime.

Note: Let $p \in F[x]$ be an irreducible with $p(x) = \sum_{i=0}^{n} a_i x^i, a_n \neq 0$. The field $F[x]/(p)$ consists of elements that are of the form $(p) + \sum_{i=0}^{n} c_i x^i, c_n, c_i \in F$. Moreover, $\sum_{i=0}^{n} a_i x^i + (p)$ is the zero element. So, $F[x]/(p)$ is $F[x]$ rooted at p.

4 Integral Domains

Preface: Recall that a commutative ring R is an integral domain if, whenever $ab = 0$ for $a, b \in R$, we have either $a = 0$ or $b = 0$.

Definition: Associate (Integral Domains)

Let R be an integral domain, and let $a, b \in R$. a and b are **associates** if there exists a unit c such that $a = bc$.

Proposition: Let the relation that two elements are associates be defined above, and written as $a \sim b$. $\sim$ is an equivalence relation.

Proof. Let R be an integral domain, and let $a, b, c \in R$.

(1) Pick $d = 1$. Then $\underline{a} = a \cdot 1 = \underline{a}$, so a and a are associates. Therefore, $\sim$ is **reflexive**.

(2) Suppose $a \sim b$. Then $a = bd$ for some unit $d \in R$, so there exists $d^{-1} \in R$ such that $dd^{-1} = 1$. Multiplying both sides of the equation by d^{-1}, we get $\underline{ad^{-1}} = bd \cdot d^{-1} = b \cdot 1 = \underline{b}$, so b and a are associates. Therefore, $\sim$ is **symmetric**.

(3) Suppose $a \sim b$ and $b \sim c$. Then $a = bd$, $b = ce$ for units $d, e \in R$. Then $\underline{a} = bd = \underline{(ce)d}$. Since d, e are units, there exist $d^{-1}, e^{-1} \in R$. Consider $d^{-1}e^{-1} \in R$. Multiplying $\overline{d^{-1}e^{-1}}$ to both sides of the equation, we get $\underline{a \cdot d^{-1}e^{-1}} = c(ed) \cdot d^{-1}e^{-1} = ce \cdot 1 \cdot e^{-1} = c \cdot 1 = \underline{c}$, so a and c are associates. Therefore, $\sim$ is **transitive**.

Because $\sim$ satisfies (1) - (3), $\sim$ is an equivalence relation. $\square$

Definition: Divides (Integral Domains)

Let R be an integral domain, and let $a, b \in R$. a **divides** b if we can find $q \in R$ such that $aq = b$. We write $a \mid b$.

Definition: Irreducible (Integral Domains)

Let R be an integral domain, and let $p \in R$ be a nonunit. p is **irreducible** if the only divisors of p are units and associates of p.

Proposition: Let R be an integral domain. $p \in R$ is irreducible if and only if whenever $p = ab$, either a or b is a unit.

Proof. Let R be an integral domain and $p \in R$.
($\Longrightarrow$) Suppose p is irreducible. Then $p \mid p = ab$. If a is a unit, then we are done, so suppose not. Then a is an asociate of p, so b is a unit.

($\Longleftarrow$) Suppose "p = ab implies that either a or b is a unit". Let $a \in R$ such that $a \mid p$. Then $p = ab$ for some $b \in R$. If a is a unit, then b is an associate of p. If b is a unit, then a is an associate of p. In either case, the only factors of p are units and associates, so p is irreducible. $\square$

Definition: Prime (Integral Domains)

Let R be an integral domain and let $p \in R$ be a nonunit. p is prime if, whenever $p \mid ab$, then either $p \mid a$ or $p \mid b$.

Theorem

Let R be an integral domain, and let $p \in R$ be prime. Then p is irreducible.

Proof. Let R be an integral domain. Let $p \in R$ is prime and suppose $p = ab$. Then either $p \mid a$ or $p \mid b$. Without loss of generality, suppose $p \mid a$. Then $a = pc$ for some $c \in R$. Then $p = ab = (pc)b$. Since R is an integral domain, we apply the cancellation property to get $1 = cb$. This implies that b is a unit. $\square$

Note: Irreducibles need not be prime. Take, for example, this bullshit: $R = \mathbb{Z}[\sqrt{-5}]$. Here, 2 and 3 are irreducible but not prime since $2 \cdot 3 = 6 = (1+\sqrt{-5})(1-\sqrt{-5})$, and $2, 3 \mid (1+\sqrt{-5})(1-\sqrt{-5})$ but $2, 3 \nmid (1+\sqrt{-5})$ and $2, 3 \nmid (1-\sqrt{-5})$.

Theorem

Let R be an integral domain, and let $p \in R$. The principal ideal (p) is prime if and only if p is prime.

Proof. Let R be an integral domain and $p \in R$ such that $(p) \subseteq R$ is principal.

$(\implies)$ Suppose (p) is prime. Take $ab \in (p)$. Then $ab = pr$ for some $r \in R$, so $p \mid ab$. Since (p) is prime, either $a \in (p)$ or $b \in (p)$. Then either $p \mid a$ or $p \mid b$, so p is prime.

$(\impliedby)$ Suppose p is prime. Let $a, b \in R$ such that $ab \in (p)$. Then $ab = pr$ for some $r \in R$, so $p \mid ab$. Since p is prime, either $p \mid a$ or $p \mid b$; that is, either $a \in (p)$ or $b \in (p)$. This implies that (p) is prime. $\square$

Notation: Let R be an integral domain. Define R^* to be the nonzero elements of R.

Lemma

Let R be an integral domain. Consider $S(R) := \{(a, b) : a, b \in R; b \neq 0\}]$. The relation $(a, b) \sim (a', b')$ if and only if $ab' = a'b$ forms an equivalence relation.

Proof. Let R be an integral domain, and consider $S(R) := \{(a, b) : a, b \in R; b \neq 0\}]$. Let $(a, b), (c, d), (e, f) \in S(R)$.

(1) $(a, b) \sim (a, b) \iff ab = ba \iff ab = ab \iff (a, b) \sim (a, b)$. Therefore, $\sim$ is **reflexive**.

(2) $(a, b) \sim (c, d) \iff ad = bc \iff ad = bc \iff bc = ad \iff (c, d) \sim (a, b)$. Therefore, $\sim$ is **symmetric**.

(3) Suppose $(a, b) \sim (c, d) \iff ad = bc$ and $(c, d) \sim (e, f) \iff cf = de$. Then

$$\begin{aligned}
ad &= bc \\
(ad)f &= b(cf) \\
(bc)f &= b(de) \\
(af)d &= (be)d \\
af &= be \iff (a, b) \sim (e, f) && d \neq 0, \text{ so apply cancellation property}
\end{aligned}$$

Therefore, $\sim$ is **transitive**.

Because $\sim$ satisfies (1) - (3), $\sim$ is an equivalence relation. $\square$

Definition: Addition and Multiplication in $S(R)$

Define $+$ and $\cdot$ in $S(R)$ by $(a, b) + (c, d) = (ad + bc, bd)$ and $(a, b) \cdot (c, d) = (ab, cd)$.

Lemma

Suppose R is an integral domain. Suppose $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, where $(a, b), (a', b'), (c, d), (c', d') \in S(R)$. Then $(ad, bc) \sim (a'd', b'c')$ and $(ad + bc, bd) \sim (a'd + b'c', b'd')$.

Proof. Suppose R is an integral domain and let $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, where $(a, b), (a', b'), (c, d), (c', d') \in S(R)$. By definition, we have that $ab' = a'b$ and $cd' = c'd$. Then

$$\underline{ad \cdot b'd'} = (ab')(cd') = (a'b)(c'd) = \underline{a'd' \cdot b'c'}$$

and

$$\begin{aligned}
(ad + bc) \cdot b'd' &= adb'd' + bcb'd' \\
&= (ab')dd' + (cd')bb' \\
&= (a'b)dd' + (c'd)bb' && ab' = a'b, cd' = c'd \\
&= (a'd')(bd) + (b'c')(bd) \\
(ad + bc) \cdot b'd' &= (a'd' + b'c') \cdot bd
\end{aligned}$$

So $(ad, bc) \sim (a'd', b'c')$ and $(ad + bc, bd) \sim (a'd + b'c', b'd')$. $\square$

Definition: Field of Fractions

Let R be an integral domain. Define $Frac(R) = S(R)/\sim$ as the **field of fractions** for R, where addition and multiplication are defined by $[(a,b)] + [(c,d)] = [(ad+bc, bd)]$ and $[(a,b)] \cdot [(c,d)] = [(ac, bd)]$, respectively. **Notation:** We will refer to $[(a,b)]$ as $\frac{a}{b}$.

Theorem

Let R be an integral domain. $Frac(R)$ forms a field, and R can be viewed as a subring.

Proof. I'm not checking the ring axioms for $Frac(R)$ lol.

Let R be an integral domain. Take $\frac{a}{b} \in Frac(R)$ to be nonzero. Then since $a, b \neq 0$, the inverse of $\frac{a}{b}$ is $\frac{b}{a}$. Consider the function $f : R \to Frac(R)$ with $r \mapsto \frac{r}{1}$. Then

(1) $f(a+b) = \frac{a+b}{1} = \frac{a}{1} + \frac{b}{1} = f(a) + f(b)$, so f is **closed under addition**.

(2) $f(a \cdot b) = \frac{a \cdot b}{1} = \frac{a}{1} \cdot \frac{b}{1} = f(a) \cdot f(b)$, so f is **closed under multiplication**.

(3) $f(1_R) = \frac{1_R}{1} = 1_{Frac(R)}$, so the **multiplicative identity is preserved**.

so f is a ring homomorphism. Therefore, R is a subring of $Frac(R)$. □

Corollary

Let F be a field. $Frac(F) \simeq F$.

Proof. Let F be a field. Consider the ring homomorphism $f : F \to Frac(F)$ with $r \mapsto \frac{r}{1}$. Take a nonzero $r \in R$. Then $f(r) = \frac{r}{1} \neq 0$, so $r \notin \ker(f)$. This implies that $\ker(f) = \{0\}$, so f is **injective**. Take any $\frac{a}{b} \in Frac(R)$ for $a, b \in R$. Since $b \neq 0$, there exists $b^{-1} \in R$ such that $bb^{-1} = 1$. Consider $x = ab^{-1} \in R$. Then

$$\begin{aligned} a &= a \cdot 1 \\ &= a \cdot bb^{-1} \\ &= ab^{-1} \cdot b \\ a \cdot 1 = x \cdot b &\iff (a,b) \sim (x,1) \end{aligned}$$

so $\underline{f(x)} = \frac{x}{1} = \frac{ab^{-1}}{1} = \underline{\frac{a}{b}}$, which shows that f is **surjective**. Since f is injective and surjective, f is a **bijection**. □

4.1 Euclidean Domains

Definition: Norm

Let R be an integral domain. A **norm** is a non-negative function $N : R \to \mathbb{Z}$ such that

(1) $N(0_R) = 0$.

(2) Given $a, b \in R$ with $b \neq 0$, there exists q such that $a = bq + r$ where $r = 0$ or $N(r) < N(b)$.

Definition: Euclidean Domain

Let R be an integral domain. R is a **Euclidean domain** if there exists a norm function $N : R \to \mathbb{Z}$.

Theorem

Let R be a Euclidean domain, and let $I \subseteq R$ be an ideal. I is principal.

Proof. If $I = \{0\}$, then $I = (0)$ which is principal, so we are done. If $I \neq \{0\}$, Then pick a nonzero $d \in I$ to have the smallest nonzero norm.

($(d) \subseteq I$) Since $d \in I$, we have that $ad, da \in I$ for all $a \in R$ by definition, so $(d) \subseteq I$.

($(d) \supseteq I$) Take $a \in I$. Since $d \neq 0$, we can write $a = dq + r$ for some $q \in R$. Then since $a, dq \in I$, we necessarily have that $r \in I$. Then $N(r) < N(d)$, but d was chosen to have the smallest norm, so r is necessarily 0. Then, $I \ni a = dq \in (d)$, so we have that $I \subseteq (d)$.

Therefore, $(d) = I$, so I is principal. $\square$

Definition: Greatest Common Divisor (Euclidean Domains)

Let R be a commutative ring, and $a, b \in R$ with $b \neq 0$. A **greatest common divisor** of a and b is an element of $d \in R$ such that

(1) $d \mid a$ and $d \mid b$.

(2) Whenever there is another $c \in R$ such that $c \mid a$ and $c \mid b$, then $c \mid d$.

Proposition: Let R be a Euclidean domain and let $a, b \in R$ such that $b \neq 0$, and let d be a greatest common divisor of a and b. Then $d' \in R$ is also a greatest common divisor of a and b if and only if d' is an associate of d.

Proof. Let R be a Euclidean domain and let $a, b \in R$ such that $b \neq 0$, and let d be a greatest common divisor of a and b. Consider $d' \in R$.

$(\implies)$ Suppose d' is also a greatest common divisor of a and b. Then since $d' \mid a$ and $d' \mid b$, by definition, we have $d' \mid d$, so $d = d'p$ for some $p \in R$. But we also have that $d \mid a$ and $d \mid b$, and by definition $d \mid d'$, so $d' = dq$ for some $q \in R$. Then

$$\begin{aligned}
d &= d'p \\
d &= (dq)p \\
1 &= qp && d \neq 0, R \text{ is an integral domain, so apply the cancellation property}
\end{aligned}$$

so d' and d are associates.

$(\impliedby)$ Suppose d' is an associate of d. Then there exists a unit $c \in R$ such that $d = d'c$, so $d' \mid d$ by definition. Since d is a greatest common divisor, we have that $d \mid a$ and $d \mid b$, so $a = dp$, $b = dq$ for $p, q \in R$. This implies that $d' \mid dp = a$ and $d' \mid dq = b$, so $d' \mid a$ and $d' \mid b$, so d' is also a greatest common divisor of a and b.

Therefore, d' is another greatest common divisor for a and b if and only if d' is an associate of d. $\square$

Theorem

Let R be a Euclidean domain, and let $a, b \in R$ such that $b \neq 0$. Suppose d is such that $(d) = (a, b)$. Then d is a greatest common divisor of a and b.

Proof. Let R be a Euclidean domain, and let $a, b \in R$ such that $b \neq 0$. Suppose d is a such that $(d) = (a, b)$. Then $a, b \in (a, b) = (d)$, so we can express them as $a = dp, b = dq$ for $p, q \in R$. This means $d \mid a$ and $d \mid b$. Now suppose that we have $c \in R$ such that $c \mid a$ and $c \mid b$. Then $a = cr, b = cs$ for $r, s \in R$, so we can write $d = ap+bq = (cr)p+(cs)q = c(rp+sq)$, which implies that $c \mid d$. Therefore, d is a greatest common divisor of a and b. $\square$

4.2 Principal Ideal Domains

Definition: Principal Ideal Domain (PID)

Let R be an integral domain. R is a **principal ideal domain (PID)** if every ideal of R is principal. That is, given an ideal $I \subseteq R$, we can find $a \in R$ such that $I = (a)$.

Note: Since all ideals in a Euclidean domain are principal, they are also PID's.

Theorem

Let R be a PID, and let $a, b \in R$ with $b \neq 0$. Let $d \in R$ be such that $(d) = (a, b)$. Then d is a greatest common divisor of R. Moreover, $d' \in R$ is a greatest common divisor of a and b if and only if d' is an associate of d.

Proof. Let R be a principal ideal domain and let $a, b \in R$ such that $b \neq 0$, and let d be a greatest common divisor of a and b. Consider $d' \in R$.

($\implies$) Suppose d' is also a greatest common divisor of a and b. Then since $d' \mid a$ and $d' \mid b$, by definition, we have $d' \mid d$, so $d = d'p$ for some $p \in R$. But we also have that $d \mid a$ and $d \mid b$, and by definition $d \mid d'$, so $d' = dq$ for some $q \in R$. Then

$$\begin{aligned} d &= d'p \\ d &= (dq)p \\ 1 &= qp && d \neq 0, R \text{ is an integral domain, so apply the cancellation property} \end{aligned}$$

so d' and d are associates.

($\impliedby$) Suppose d' is an associate of d. Then there exists a unit $c \in R$ such that $d = d'c$, so $d' \mid d$ by definition. Since d is a greatest common divisor, we have that $d \mid a$ and $d \mid b$, so $a = dp$, $b = dq$ for $p, q \in R$. This implies that $d' \mid dp = a$ and $d' \mid dq = b$, so $d' \mid a$ and $d' \mid b$, so d' is also a greatest common divisor of a and b.

Therefore, d' is another greatest common divisor for a and b if and only if d' is an associate of d. $\square$

Proposition: Let R be a PID and $P \subseteq R$ be a nonzero prime ideal. Then P is maximal.

Proof. Let R be a PID and suppose that $(p) = P \subseteq R$ is a nonzero prime ideal. Suppose $(p) = P \subsetneq M = (m)$. Since $p \in (p) \subsetneq (m)$, $p = mr$ for some $r \in R$. But since (p) is prime, either $m \in P$ or $r \in P$. If $m \in P$, then we are done since $M = (m) \subseteq (p) = P$. If $r \in P$, then $r = ps$ for $s \in R$. Then $p = mr = mps$. Since R is an integral domain and $p \neq 0$, apply the cancellation property to get $1 = ms$, which shows that $(m) = M = R$. Therefore, P is maximal. $\square$

Corollary

Let R be a commutative ring and suppose the polynomial ring $R[x]$ is a PID. Then R is a field.

Proof. Let R be an integral domain and $R[x]$ a principal ideal domain. Consider the principal ideal $(x) \subseteq R[x]$ and a function $f : R[x] \to R$ with $f(p(x)) = p(0)$. Then

- $f(p(x) + q(x)) = p(0) + q(0) = f(p(x)) + f(q(x))$, so f is **closed under addition**.
- $f(p(x) \cdot q(x)) = p(0) \cdot q(0) = f(p(x)) \cdot f(q(x))$, so f is **closed under multiplication**.
- $f(1(x)) = 1$, so f **preserves the multiplicative identity**.

so f is a ring homomorphism. We have that $\ker(f) = \{p(x) : f(p(x)) = 0\} = (x)$, so $\ker(f) = (x)$. To show $\mathrm{Im}(f) = R$, take $a \in R$. Then consider $p \in R$ such that $p(0) = a$. Then $f(p(x)) = p(0) = a \in R$. Therefore, $\mathrm{Im}(f) = R$. Then we have that $R[x]/(x) \simeq R$ by the **First Isomorphism Theorem**.

Note that since $1 \notin (x)$, $(x) \neq R[x]$, so $(x) \subsetneq R[x]$ is a proper ideal. To show that (x) is maximal, consider $(y) \subseteq R[x]$ such that $(y) \supseteq (x)$. If $\deg(y) = 0$, then y is a unit, so $(y) = R[x]$. If $\deg(y) > 0$, then since $x \in (x) \subseteq (y)$, we can write $x = fy$ for some $f \in R[x]$. Then since $\deg(x) = 1$, $\deg(y) \leq \deg(x) = 1$, which means we necessarily have $\deg(y) = 1$. Then x and y are associates, so $(x) = (y)$. Therefore, (x) is maximal, so $R[x]/(x)$ is a field. But since $R[x]/(x) \simeq R$, we have that R is a field. $\square$

Proposition: Let R be a PID and $p \in R$ be irreducible. Then p is prime.

Proof. Suppose p is irreducible and consider $(p) \subseteq I = (a)$. Because $p \in (a)$, we have that $p = ab$ for some $b \in R$. Then a or b is a unit. If a is a unit, then $(a) = I = R$. If b is a unit, then a and b are associates, so $(a) = (p)$. Then either $I = R$ or $I = (p)$, so (p) is maximal and therefore prime. $\square$

Definition: Ascending Chain Condition

Let R be an integral domain. R satisifes the **ascending chain condition** on principal ideals if, whenver we have a chain of inclusions of ideals given by

$$(a_1) \subseteq (a_2) \subseteq \cdots$$

where each $a_i \in R$, there exists a positive integer n such that for all $m \geq n$, we have $(a_m) = (a_n)$.

Lemma

Let R be an integral domain and $I_1 \subseteq I_2 \subseteq \cdots$ be a chain of ideals in R. Their union $\bigcup_j I_j$ is also an ideal.

Proof. Let R be an integral domain and $I_1 \subseteq I_2 \subseteq \cdots$ a chain of ideals in R.

(1) Since I_1 is an ideal, $0 \in I_1 \subseteq \bigcup_j I_j$, so $\bigcup_j I_j$ **preserves the additive identity**.

(2) Take $a \in I_n$ and $b \in I_m$. Without loss of generality, suppose $n \leq m$. Then $a, b \in I_m$, so $a - b \in I_m \subseteq \bigcup_j I_j$, so $\bigcup_j I_j$ is **closed under subtraction**.

(3) Take $a \in I_n$ and $r \in R$. Since I_n is an ideal, $ar, ra \in I_n \subseteq \bigcup_j I_j$, so $\bigcup_j I_j$ is **closed under absorption**.

Since $\bigcup_j I_j$ satisfies (1)-(3), $\bigcup_j I_j$ is an ideal. □

Theorem

A PID satisfies the ascending chain condition on principal ideals.

Proof. Suppose we have an ascending chain of ideals given by

$$(a_1) \subseteq (a_2) \subseteq \cdots$$

Consider their union, $I = \bigcup_j (a_j)$. Because $I \subseteq R$ is principal, we can represent $I = (a)$ for $a \in R$. Then $a \in (a_n)$ for some positive $n \in \mathbb{N}$. This implies that $a \subseteq (a_m)$ for $m \geq n$, so $(a) \subseteq (a_m)$. But we also have that $(a_m) \subseteq I = (a)$, so $(a_m) = (a)$ for every $m \geq n$. In particular, $(a_m) = (a_n)$ for all $m \geq n$. □

Note: This tells us that we do not have ideals that are arbitrarily big but not the entire ring itself. More concretely, the ascending chain condition gives us prime factorizations a PID.

Theorem

Let R be an integral domain that satisfies the ascending chain condition on principal ideals. Let $r \in R$ be nonzero and a nonunit. r can be expressed as a product of irreducible elements.

Proof. Let R be an integral domain that satisfies the ascending chain condition on principal ideals. Let $r \in R$ be nonzero and a nonunit. If r is irreducible, we are done, so suppose not. Suppose for the sake of contradiciton that r cannot be written as a product of irreducibles. Then since r is not irreducible, we can express $r = r_1^1 r_2^1$ such that neither r_1^1 nor r_2^1 are units. Then at least one of r_1^1 or r_2^1 cannot be a product of irreducibles, since otherwise, r would be a product of irreducibles. Without loss of generality, suppose r_1^1 is not a product of irreducibles. Then r_1^1 can be written as $r_1^2 r_2^2$ where neither r_1^2 nor r_2^2 are units. We continue this process inductively to get $r_1^1, \cdots$ where r_1^{i+1} is a proper factor of r_1^i for each i. This gives us a chain of principal ideals given by $(r_1^1) \subsetneq (r_1^2) \subsetneq \cdots$. This is a contradiction to the claim that R satisfies the ascending chain condition. Therefore, r can be expressed as a product of irreducibles. □

Corollary

Because PID's satisfy the ascending chain condition on principal ideals, every nonzero and nonunit decomposes as a product of irreducibles. Further, since irreducibles are prime in a PID, every nonzero and nonunit decomposes as a product of primes.

Theorem

Let R be a PID and $r \in R$. r has a unique prime factorization. That is, if $p_1 \cdots p_n = q_1 \cdots q_m$ are both prime factorizations of r, then $n = m$ and there is a permutation σ on $1, \ldots, n$ such that for every i, we have that p_i and $q_{\sigma(i)}$ are associates.

Proof. Let R be a PID and $r \in R$. Suppose for the sake of contradiction that we have two factorizations $p_1 \cdots p_n, q_1 \cdots q_m$ of r, where p_i, q_j are prime. Then $p_1 \mid q_1 \cdots q_m$. This implies that $p_1 \mid q_i$ for some i. Without loss of generality, suppose $p_1 \mid q_1$. Since p_1, q_1 are irreducible, they are associates, so $q_1 = ap_1$ for $a \in R$ a unit. Then $p_1 \cdots p_n = ap_1 \cdot q_2 \cdots q_m$. Because R is an integral domain, we apply the cancellation property to get $p_2 \cdots p_n = aq_2 \cdots q_m$. Without loss of generality, suppose $n < m$. Continuing this process iteratively, we eventually get $1 = a_1 \cdots a_n \cdot q_{n+1} \cdots q_m$. This implies that $q_{n+1} \cdots q_m$ are units, a contradiction. Therefore, r has a unique factorization and there is a permutation σ on $\{1, \ldots, n\}$ such that $p_i = q_{\sigma(i)}$. $\square$

5 Unique Factorization Domain

Definition: Unique Factorization Domain

Let R be an integral domain. R is a **unique factorization domain** if, given a nonzero and nonunit $r \in R$, the following hold:

(1) r can be factored as a product of irreducibles. That is, we can express $r = p_1 \cdots p_n$ where p_i is irreducible.

(2) The factorization of r is unique. That is, if $p_1 \cdots p_n = q_1 \cdots q_m$ are both factorizations of r, then $n = m$ and there is a permutation σ on $1, \ldots n$ such that for every i, we have that p_i and $q_{\sigma(i)}$ are associates.

Remark: Any PID is a UFD.

Example: Let F be a field. $F[x_1, \ldots, x_n]$ is a UFD, but not a PID since (x_1, x_2) is not principal.

Example: $\mathbb{Z}[x]$ is a UFD, but not a PID since $(2, x)$ is not principal.

Example: If R is a UFD, then $R[x]$ is a UFD.

Theorem

Let R be a UFD and let $r \in R$. r is prime if and only if it is irreducible.

Proof. Let R be a UFD and let $r \in R$.

($\implies$) Since R is an integral domain, primes are irreducible.

($\impliedby$) Suppose $r \in R$ is irreducible and $r \mid ab$ for $a, b \in R$. We can write $ab = rc$ for some $c \in R$. If a is a unit, then there exists $a^{-1} \in R$, so we have $rca^{-1} = b$. This implies that $r \mid b$. If neither a or b is a unit, then consider their unique factorizations $a = p_1 \cdots p_n$ and $b = q_1 \cdots q_m$. Note that c cannot be a unit since otherwise, we have $r = c^{-1}ab$, which implies that r is reducible. Then $c = t_1 \cdots t_s$ for p_i, q_j, t_k all irreducible. We now have that $p_1 \cdots p_n$, $q_1 \cdots q_m$, and $r \cdot t_1 \cdots t_s$ are all factorizations of ab. Therefore, since r is irreducible by assumption, it must be associates with some p_i or q_j. If r and p_i are associates, then $r \mid a$. Similarly, if r and q_j are associates, then $r \mid b$. Therefore, r is prime. $\square$

Theorem

Let R be a UFD and suppose $a, b \in R$. Let $a = up_1^{e_1} \cdots p_n^{e_n}$ and $b = vp_1^{f_1} \cdots p_n^{f_n}$ be prime factorizations where u, v are units and each p_i is a distinct prime. For each n, let $m_i = \min\{e_i, f_i\}$. Then $d = up_1^{m_1} \cdots p_n^{m_n}$ is a greatest common divisor of a and b.

Proof. Let R be a UFD and suppose $a, b \in R$. Let $a = up_1^{e_1} \cdots p_n^{e_n}$ and $b = vp_1^{f_1} \cdots p_n^{f_n}$ be prime factorizations where u, v are units and each p_i is a distinct prime. For each n, let $m_i = \min\{e_i, f_i\}$. Consider $d = up_1^{m_1} \cdots p_n^{m_n}$. Clearly, $d \mid a$ and $d \mid b$ since $m_i \leq e_i, f_i$. Suppose we have $c \in R$ such that $c \mid a$ and $c \mid b$. Then consider the prime factorization $c = wq_1^{g_1} \cdots q_n^{g_n}$, where w is a unit and q_i is a distinct prime. Since $q_i \mid c$, we also have $q_i \mid a$ and $q_i \mid b$, which implies $q_i \mid p_j$ for some p_j. Without loss of generality, suppose $q_i \mid p_i$. Then $g_i \leq \min\{e_i, f_i\} = m_i$, which implies that $c \mid d$. $\square$

Theorem

Let R be an integral domain. R is a UFD if and only if R satisfies the ascending chain condition on principal ideals and irreducible elements of R are prime.

Proof. Let R be an integral domain.

($\implies$) Suppose R is a UFD. Note that since R is a UFD, irreducible elements are prime. Consider the ascending chain of ideals given by

$$(a_1) \subseteq (a_2) \subseteq \cdots$$

for $a_1, a_2, \cdots \in R$. Consider the unique factorization $a_1 = p_1^{r_1} \cdots p_k^{r_k}$, where p_i is a distinct prime. Then $a_n \mid a_1$, so a_n can be written as an associate of $p_1^{s_1} \cdots p_k^{s_k}$ where $0 \leq s_i \leq r_i$. For all $m \geq n$, we have that $(a_n) \subseteq (a_m)$ by construction. Then $a_m \mid a_n$, so we can represent $a_m = p_1^{t_1} \cdots p_k^{t_k}$ where $0 \leq t_i \leq s_i$ for all i. Therefore, R satisfies the ascending chain condition on principal ideals.

($\Longleftarrow$)

$\square$