

Problem Set 9

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Question 2

Complete the following truth table:

P	Q	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$

Response

P	Q	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	T

Question 3 part (e)

Given any two statements P and Q, and using C to denote a contradiction (i.e., a statement that is always false), prove that the following statements are tautologies (i.e., they are always true):

$$(e) ((P \wedge \neg Q) \implies C) \implies (P \implies Q)$$

Response

Proof.

P	Q	C	$P \wedge \neg Q$	$(P \wedge \neg Q) \implies C$	$P \implies Q$	$((P \wedge \neg Q) \implies C) \implies (P \implies Q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	F	F	T	T	T
F	F	F	F	T	T	T

Since the statement is true for all cases of P, Q, and C, it is a tautology.

□

Question 8

Convert the following statements into plain English:

$$\forall x \in \mathbb{Z} \exists y \in \mathbb{Z} : x + y = 0$$

$$\exists y \in \mathbb{Z} \forall x \in \mathbb{Z} : x + y = 0$$

Decide on the truth values of each statement and then provide a proof.

Response

$$\forall x \in \mathbb{Z} \exists y \in \mathbb{Z} : x + y = 0$$

"For every integer x , there exists an integer y such that $x + y = 0$ ".

Proof. This statement is true. Let $x \in \mathbb{Z}$. Since \mathbb{Z} is a field, there exists an additive inverse $-x \in \mathbb{Z}$. So, we have

$$x + y = 0$$

$$x + y = x + -x$$

$$y = -x$$

existence of an additive inverse

Substituting $-x$ for y , we get $x + y = x + -x = 0$. □

$$\exists y \in \mathbb{Z} \forall x \in \mathbb{Z} : x + y = 0$$

"There exists an integer y such that $x + y = 0$ for every integer x ".

Proof. We will prove that this statement is false by counter-example. We want to prove that there exists at least one $x \in \mathbb{Z}$ such that $x + y \neq 0$. Let $x = 2$ and $y = -1$. Then, $x + y = 2 + -1 = 1 \neq 0$. Therefore, the statement is false. □

Question 10

- (a) Suppose $p \in \mathbb{N}$. Show that if p^2 is divisible by 5 then p is also divisible by 5.
- (b) Prove $\sqrt{5}$ is irrational.
- (c) Suppose you try the same argument to prove $\sqrt{4}$ is irrational. The proof must fail, but where exactly does it fail?

Response

- (a) *Proof.* To prove the statement, it is equivalent to prove its contrapositive: If p is not divisible by 5, p^2 is not divisible by 5. Note that p can be rewritten as $p = 5s + r$, where $s, r \in \mathbb{N}$. By definition, $r \neq 0$ since $r \in \mathbb{N}$ and $0 \notin \mathbb{N}$. So, p is not divisible by 5. To prove p^2 is not divisible by 5,

$$\begin{aligned} p^2 &= (5s + r)^2 \\ &= 25s^2 + 10sr + r^2 \\ p^2 &= 5(5s^2 + 2sr) + r(r) \end{aligned}$$

Since $r \neq 0$, p^2 is not divisible by 5. □

- (b) *Proof.* Assume by contradiction that $\sqrt{5}$ is rational. By the Rational Zeros Theorem, we can rewrite $\sqrt{5}$ as $x^2 - 5 = 0$, where $x \in \mathbb{Q}$. The only possible solutions are $\pm 1, \pm 5$, neither of which are solutions, which is a contradiction. □
- (c) The proof fails when calculating $x^2 - 4 = 0$. $\pm 1, \pm 2, \pm 4$ are possible solutions, where $x = \pm 2$ is a valid solution and is written as $(x + 2)(x - 2) = 0$.