110A HW9

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Question 1

Let R be a Euclidean domain, and let $a, b \in R$, such that $b \neq 0$, and let d be a greatest common divisor of a and b. Show that $d' \in R$ is also a greatest common divisor of a and b if and only if d' is an associate of d.
[Hint: Your proof should also work for PIDs.]

Response

Proof: Let R be a Euclidean domain, $a, b \in R$ such that $b \neq 0$, and d be a greatest common divisor of a and b.
$(\implies)$ Suppose d' is another greatest common divisor of a and b. Then $d' \mid a$ and $d' \mid b$, so $d' \mid d$. Then $d = d'x$ for some $x \in R$. But since $d \mid a$ and $d \mid b$, we have that $d \mid d'$, so $d' = dy$ for some $y \in R$. Then $d = d'x = (dy)x$. Since $d \neq 0$, apply the cancellation property to get $1 = yx$, which shows that x is a unit. This means that d' is an associate of d.

$(\impliedby)$ Suppose d' is an associate of d'. Then $d = d'x$ for some unit $x \in R$. Since d is a greatest common divisor of a and b, we have that $d \mid a$ and $d \mid b$, which can be written as $a = dp$, $b = dq$ for some $p, q \in R$. Then $a = dp = (d'x)p = d'(xp)$ and $b = dq = (d'x)q = d'(xq)$. This shows that $d' \mid a$ and $d' \mid b$. Now suppose that $c \mid a$ and $c \mid b$. Then $c \mid a = d'(xp)$ and $c \mid d'(xq)$, so $c \mid d'$. Therefore, d' is another greatest common divisor of a and b.

Therefore, $d' \in R$ is also a greatest common divisor of a and b if and only if d' is an associate of d.
$\square$

Question 2

Let R be a Euclidean domain, and let N be a norm. Show that $N' : R \to \mathbb{Z}$ given by $N'(a) = \min_{r \neq 0} N(ar)$ forms a norm. Moreover, show that $N'(a) \leq N'(ab)$ for nonzero $a, b \in R$

Response

Proof: $\square$

Question 3

Let F be a field. Show that the function $N : F \to \mathbb{Z}$ given by $N(a) = 0$ for all $a \in F$ gives a norm on F. Conclude that every field is a Euclidean domain.
[we briefly discussed this in class.]

Response

Question 4

Let R be an integral domain. Suppose $R[x]$ is a principal ideal domain. Show that R must be a field.
[Hint: Think about (x).]

Response

Proof: Let R be an integral domain and $R[x]$ a principal ideal domain. Consider the principal ideal $(x) \subseteq R[x]$ and a function $f : R[x] \to R$ with $f(p(x)) = p(0)$. Then

- $f(p(x) + q(x)) = p(0) + q(0) = f(p(x)) + f(q(x))$, so f is **closed under addition**.
- $f(p(x) \cdot q(x)) = p(0) \cdot q(0) = f(p(x)) \cdot f(q(x))$, so f is **closed under multiplication**.
- $f(1(x)) = 1$, so f **preserves the multiplicative identity**.

so f is a ring homomorphism. We have that $\ker(f) = \{p(x) : f(p(x)) = 0\} = (x)$, so $\ker(f) = (x)$. To show $\mathrm{Im}(f) = R$, take $a \in R$. Then consider $p \in R$ such that $p(0) = a$. Then $f(p(x)) = p(0) = a \in R$. Therefore, $\mathrm{Im}(f) = R$. Then by the **First Isomorphism Theorem**, we have that $R[x]/(x) \simeq R$.

Note that since $1 \notin (x)$, $(x) \neq R[x]$, so $(x) \subsetneq R[x]$ is a proper ideal. To show that (x) is maximal, consider $(y) \subseteq R[x]$ such that $(y) \supseteq (x)$. If $\deg(y) = 0$, then y is a unit, so $(y) = R[x]$. If $\deg(y) > 0$, then since $x \in (x) \subseteq (y)$, we can write $x = fy$ for some $f \in R[x]$. Then since $\deg(x) = 1$, $\deg(y) \leq \deg(x) = 1$, which means we necessarily have $\deg(y) = 1$. Then x and y are associates, so $(x) = (y)$. Therefore, (x) is maximal, so $R[x]/(x)$ is a field. But since $R[x]/(x) \simeq R$, we have that R is a field. $\square$

Question 5

Let R be a PID, and let $I \subseteq R$ be a prime ideal. Show that R/I is a PID.

Response

Question 6

Let R be an integral domain. Prove that R is a PID if and only if (i) every ideal of R is finitely generated (i.e., every ideal $I \subseteq R$ can be written $I = (x_1, \cdots x_n)$ for $x_i \in R$) and (ii) whenever $a, b \in R$, the ideal (a, b) is principal.

Response

Question 7

Let R be an integral domain, and let $I_1 \subseteq I_2 \subseteq \cdots$ be a chain of ideal in R. Show their union $\bigcup_j I_j$ is also an ideal.

Response

Question 8

Let R be a UFD, and let $a, b, c \in R$. Suppose $a|c$ and $b|c$, and that 1 is a greatest common divisor of a and b. Show that $ab|c$.

Response

Question 9

Let R be an integral domain. Show that R is a UFD if and only if R satisfies the ascending chain condition on principal ideals and irreducible elements of R are prime.

Response