Problem Set 3

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Question 4

Let $x_n = \frac{2n+1}{3n+7}$.

(a) Prove directly, using the definition, that $\lim_{n\to\infty} x_n = \frac{2}{3}$

(b) Prove, using the algebraic limit theorem, that $\lim_{n\to\infty} x_n = \frac{2}{3}$

Response

(a) *Scratch work:*

$$\begin{aligned}
\left|\frac{2n+1}{3n+7} - \frac{2}{3}\right| &< \varepsilon \\
\left|\frac{2n+1}{3n+7} - \frac{2(n+\frac{7}{3})}{3(n+\frac{7}{3})}\right| &< \varepsilon \\
\left|\frac{3(2n+1) - 6n - 7}{3(3n+7)}\right| &< \varepsilon \\
\left|\frac{6n+3-6n-7}{3(3n+7)}\right| &< \varepsilon \\
\left|\frac{-4}{3(3n+7)}\right| &< \varepsilon \\
\frac{4}{9n+49} &< \varepsilon \\
n &> \frac{4-49\varepsilon}{9\varepsilon}
\end{aligned}$$

Proof. Let $\varepsilon > 0$. Let $N > \frac{4-49\varepsilon}{9\varepsilon}$. Then, for all $n > N$, we have

$$\begin{aligned}
n &> \frac{4-49\varepsilon}{9\varepsilon} \\
\frac{4}{9n+49} &< \varepsilon \\
\left|\frac{-4}{3(3n+7)}\right| &< \varepsilon \\
\left|\frac{2n+1}{3n+7} - \frac{2}{3}\right| &< \varepsilon
\end{aligned}$$

so $\lim_{n\to\infty} x_n = \frac{2}{3}$. □

(b) *Proof.*

$$\frac{2n+1}{3n+7} = \frac{2+\frac{1}{n}}{3+\frac{7}{n}}$$

Let $a_n = 2 + \frac{1}{n}$ and $b_n = 3 + \frac{7}{n}$. Then,

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n} && \lim_{n\to\infty}(x_n + y_n) = x + y \text{ by ALT} \\
&= 2 + 0 && \lim_{n\to\infty} c = c,\ \lim_{n\to\infty} \frac{1}{n} = 0 \text{ from lecture and by ALT} \\
\lim_{n\to\infty} a_n &= 2
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} 3 + \lim_{n\to\infty} 7\left(\frac{1}{n}\right) && \lim_{n\to\infty}(x_n + y_n) = x + y \text{ by ALT} \\
&= 3 + 0 && \lim_{n\to\infty} c = c,\ \lim_{n\to\infty} cx_n = cx,\ \lim_{n\to\infty} \frac{1}{n} = 0 \text{ from lecture and by ALT} \\
\lim_{n\to\infty} b_n &= 3
\end{aligned}$$

Since $\lim_{n\to\infty} b_n \neq 0$, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{2}{3}$ ($\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y},\ y \neq 0$ by the algebraic limit theorem). So, $\lim_{n\to\infty} x_n = \frac{2}{3}$ by the algebraic limit theorem. $\square$

Question 10

(a) Let (x_n) be bounded (not necessarily convergent) and assume that $y_n \to 0$ as $n \to \infty$. Show that $x_n y_n \to 0$ as $n \to \infty$. (Why can we not just use the Algebraic limit theorem?)

(b) Let (x_n) be bounded and $y_n \to y$ with $y \neq 0$. Does $(x_n y_n)$ converge? If yes, show it. If not, give a counter-example.

Response

(a) *Proof.* Since (x_n) is bounded, $\exists M \in \mathbb{R}$ such that $|x_n| \leq M$ $\forall n \in \mathbb{N}$. Then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$,

$$\begin{aligned} |y_n - 0| &< \frac{\varepsilon}{M} \\ |y_n| &< \frac{\varepsilon}{M} \end{aligned}$$

Let $N \geq \frac{\varepsilon}{M}$. Then,

$$\begin{aligned} |x_n y_n - 0| &< M \cdot N \\ |x_n y_n| &< M \cdot \frac{\varepsilon}{M} \\ |x_n y_n| &< \varepsilon \end{aligned}$$

Therefore, $x_n y_n \to 0$ as $n \to \infty$. $\square$

We cannot use the algebraic limit theorem since (x_n) is only bounded by the problem statement, so it need not converge.

(b) $(x_n y_n)$ is not necessarily convergent. Consider $x_n = (-1)^n$, $y_n = 1 + \frac{1}{n}$. From lecture, we have that x_n does not converge and that $y_n \to 1$, so $y \neq 0$ by the algebraic limit theorem and lecture. However,

$$\begin{aligned} x_n y_n &= (-1)^n \left(1 + \frac{1}{n}\right) \\ &= (-1)^n + \frac{(-1)^n}{n} \end{aligned}$$

From part (a), we have that $\frac{(-1)^n}{n}$ converges since $\lim_{n\to\infty} \frac{1}{n} = 0$ (set $x_n = (-1)^n$, $y_n = \frac{1}{n}$). However, $(-1)^n$ is not convergent.

Question 12

For the following sequences, provide an example or prove that no souch request is possible. You may appeal to results from lectures.

(a) Sequences (x_n) and (y_n) which both diverge, but whose sum $(x_n + y_n)$ converges.

(b) Sequences (x_n), which converges, and (y_n), which diverges, but whose sum $(x_n + y_n)$ converges.

(c) A convergent sequence (x_n), such that $x_n \neq 0$ for all $n \in \mathbb{N}$ and $(1/x_n)$ diverges.

(d) An unbounded sequence (x_n) and a convergent sequence (y_n) with $(x_n - y_n)$ bounded.

(e) Two sequences (x_n) and (y_n), where $(x_n y_n)$ and (x_n) converge, but (y_n) does not converge.

Response

(a) $x_n = n$, $y_n = -n$. Clearly both $x_n,\ y_n$ diverge, but $x_n + y_n = n + (-n) = (0, 0, \cdots)$ so $(x_n + y_n)$ converges.

(b) This is impossible. By the algebraic limit theorem, if (x_n) and $(x_n + y_n)$ both converge, $(x_n + y_n - x_n) = (y_n)$ also converges, which is a contradiction to the statement that (y_n) diverges.

(c) $x_n = \frac{1}{n}$. x_n converges (from lecture) but $(1/x_n) = 1/\frac{1}{n} = n$ diverges.

(d) This is impossible. Since y_n is bounded (by the theorem that states convergent sequences are bounded, since y_n converges, it is bounded) and $(x_n - y_n)$ is bounded, we have that $|x_n| \leq N_1\ \forall n \in \mathbb{N}$ and $|x_n - y_n| \leq N_2\ \forall n \in \mathbb{N}$. Then, $|x_n - y_n + y_n| \leq N_2 + N_1 \implies |x_n| \leq N_1 + N_2$ means that (x_n) is bounded, which is a contradiction to the statement that (x_n) is unbounded.

(e) $x_n = \frac{1}{n}$, $y_n = n$. x_n converges from lecture and y_n is unbounded and therefore does not converge (by the contrapositive of the theorem that states that convergent sequences are bounded). However, $x_n y_n = \frac{1}{n}(n) = (1, 1, \cdots)$ which converges.