1. **Linear algebra refresher.**

   (a) Let $\mathbf{Q}$ be a real orthogonal matrix.

      i. To show that $\mathbf{Q^T}$ and $\mathbf{Q^{-1}}$ are also orthogonal, suppose $\mathbf{Q}$ is orthogonal. Then $\mathbf{QQ^T} = \mathbf{Q^TQ} = \mathbf{I}$. Consider $\mathbf{Q^T}$. We want to show

$$\mathbf{Q^T}\left(\mathbf{Q^T}\right)^{\mathbf{T}} = \mathbf{I}$$

      Recall that $\left(\mathbf{Q^T}\right)^{\mathbf{T}} = \mathbf{Q}$. Then substituting $\left(\mathbf{Q^T}\right)^{\mathbf{T}}$ with $\mathbf{Q}$, we get

$$\mathbf{Q^TQ} = \mathbf{QQ^T} = \mathbf{I}$$

      Note that if $\mathbf{Q}$ is orthogonal, then $\mathbf{Q^T} = \mathbf{Q^{-1}}$. Then, since $\mathbf{Q^T}$ is orthogonal, $\mathbf{Q^{-1}}$ is orthogonal.

      ii. To show that $\mathbf{Q}$ has eigenvalues with norm 1, consider

$$\begin{aligned}
\mathbf{Qx} &= \lambda\mathbf{x} \\
(\mathbf{Qx})^{\mathbf{T}}\mathbf{Qx} &= (\mathbf{Qx})^{\mathbf{T}}\lambda\mathbf{x} \\
\mathbf{x^TQ^TQx} &= (\lambda\mathbf{x})^{\mathbf{T}}\lambda\mathbf{x} && \mathbf{Qx} = \lambda\mathbf{x} \\
\mathbf{x^TIx} &= \lambda^2\mathbf{x^Tx} && \mathbf{Q} \text{ is orthogonal} \\
\mathbf{x^Tx} &= \lambda^2\mathbf{x^Tx} && \mathbf{x^Tx} = \|\mathbf{x}\|^2 \\
\|\mathbf{x}\|^2 &= \lambda^2\|\mathbf{x}\|^2 \\
\lambda^2 &= 1
\end{aligned}$$

      This implies that $|\lambda| = 1$.

      iii. To show that the determinant of $\mathbf{Q}$ is ± 1, we have that $\mathbf{QQ^T} = \mathbf{I}$. Taking the determinant of both sides, we get

$$\det\left(\mathbf{QQ^T}\right) = \det(\mathbf{Q}) \cdot \det\left(\mathbf{Q^T}\right) = \det(\mathbf{I})$$

      Since $\det(\mathbf{I}) = 1$, we have $\det(\mathbf{Q}) \cdot \det\left(\mathbf{Q^T}\right) = 1$. Recall that $\det(\mathbf{Q}) = \det\left(\mathbf{Q^T}\right)$, so $[\det(\mathbf{Q})]^2 = 1 \rightarrow \det(\mathbf{Q}) = \pm 1$.

      iv. To show that $\mathbf{Q}$ defines a length preserving transformation, consider a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$. By assumption, $\mathbf{Q}$ is an orthogonal matrix, so $\mathbf{QQ^T} = \mathbf{Q^TQ} = \mathbf{I}$. We can represent the linear transformation T by $\mathbf{Q}$, so $T\mathbf{x} = \mathbf{Qx}$. Then, taking the norm of both sides, we get

$$\begin{aligned}
\|T\mathbf{x}\|^2 &= \|\mathbf{Qx}\|^2 \\
&= (\mathbf{Qx})^{\mathbf{T}}\mathbf{Qx} \\
&= \mathbf{x^TQ^TQx} \\
&= \mathbf{x^TIx} && \mathbf{Q} \text{ is orthogonal} \\
&= \mathbf{x^Tx} \\
\|T\mathbf{x}\|^2 &= \|\mathbf{x}\|^2 && \mathbf{x^Tx} = \|\mathbf{x}\|^2
\end{aligned}$$

Taking the square root of both sides, we get $\|T\mathbf{x}\| = \|\mathbf{x}\|$, so $\mathbf{Q}$ is a length preserving transformation.

(b) Let $\mathbf{A}$ be a matrix.

i.