

110A HW4

Warren Kim

Winter 2024

Question 1

Let F be a field, and consider the polynomial ring $F[x, y]$ with two variables. Show that $I = (x, y)$ is not a principal ideal (i.e., it cannot be generated by a single element).

Response

Question 2

Let R be a ring, and let I_1, \dots, I_k be ideals. Show that the following sets are ideals:

1. $I_1 + \dots + I_k = \{i_1 + \dots + i_k \mid i_j \in I_j\}$
2. $I_1 \cap I_2 \cap \dots \cap I_k$

Response

1. $I_1 + \dots + I_k = \{i_1 + \dots + i_k \mid i_j \in I_j\}$ is an ideal.

Proof: Let R be a ring, and I_1, \dots, I_k be ideals.

- (a) Closure under addition. Take two elements $a, b \in I_1 + \dots + I_k$. We can rewrite a, b as, $a = p_1 + \dots + p_k$ and $b = q_1 + \dots + q_k$ for $p_j, q_j \in I_j$. Then $a + b = (p_1 + \dots + p_k) + (q_1 + \dots + q_k) = (p_1 + q_1) + \dots + (p_k + q_k)$, and since $p_j + q_j \in I_j$ for all $j \leq k$, we get $a + b \in I_1 + \dots + I_k$, so $I_1 + \dots + I_k$ is closed under addition.
- (b) Absorbing property. Take any $a \in I_1 + \dots + I_k$. We can rewrite a as, $a = p_1 + \dots + p_k$ for $p_j \in I_j$. Consider an element $r \in R$. Then, $ar = (p_1 + \dots + p_k)r = p_1r + \dots + p_kr$. Similarly, $ar = r(p_1 + \dots + p_k) = rp_1 + \dots + rp_k$. Since I_j is an ideal, $p_jr, rp_j \in I_j$, so $ar \in I_1 + \dots + I_k$. Therefore, $I_1 + \dots + I_k$ satisfies the absorbing property.

Because $I_1 + \dots + I_k$ is closed under addition and satisfies the absorbing property, $I_1 + \dots + I_k$ is an ideal. \square

2. $I_1 \cap \dots \cap I_k$ is an ideal.

Proof: Let R be a ring, and I_1, \dots, I_k be ideals.

- (a) Closure under addition. Take two elements $a, b \in I_1 \cap \dots \cap I_k$. Then since each I_j is an ideal, they are closed under addition. So, $a + b \in I_1 \cap \dots \cap I_k$ because $a + b \in I_j$. Therefore, $I_1 \cap \dots \cap I_k$ is closed under addition.
- (b) Absorbing property. Take any $a \in I_1 \cap \dots \cap I_k$. Consider an element $r \in R$. Then, since each I_j is an ideal, they satisfy the absorbing property. Therefore, $ar, ra \in I_1 \cap \dots \cap I_k$ because $ar, ra \in I_j$. Therefore, $I_1 \cap \dots \cap I_k$ satisfies the absorbing property.

Because $I_1 \cap \dots \cap I_k$ is closed under addition and satisfies the absorbing property, $I_1 \cap \dots \cap I_k$ is an ideal. \square

Question 3

Let R be a ring, $a \in R$, and $I \subseteq R$ be an ideal. Show that the set $a + I = \{a + x \mid x \in I\}$ is precisely the congruence class modulo I that contains a . That is, show that $b \equiv a \pmod{I}$ if and only if $b \in a + I$.

Response

Proof: Let R be a ring, $a \in R$, and $I \subseteq R$ be an ideal. Consider the set $a + I = \{a + x \mid x \in I\}$.
(\implies) Suppose $b \equiv a \pmod{I}$ for some $b \in R$. By definition, $b - a \in I$. Then there exists some $i \in I$ such that $b - a = i$. Therefore, $b = a + i$ for some $i \in I$, so $b \in a + I$.
(\impliedby) Suppose $b \in a + I$. By definition, there exists some $i \in I$ such that $b = a + i$. Subtracting a from both sides, we get $b - a = i$, which is in I , so $b \equiv a \pmod{I}$. \square

Question 4

Let $f : R \rightarrow S$ be a ring homomorphism, and suppose $I \subseteq R$ is an ideal such that $I \subseteq \ker(f)$. Show that there is a unique homomorphism $\bar{f} : R/I \rightarrow S$ such that $f = \bar{f} \circ \pi$.

Response

Proof: Let $f : R \rightarrow S$ be a ring homomorphism, and $I \subseteq R$ an ideal such that $I \subseteq \ker(f)$. Consider $\bar{f} : R/I \rightarrow S, a + I \mapsto f(a)$. To show that \bar{f} is a homomorphism:

1. Closed under addition. Let $a + I, b + I \in R/I$. Then

$$\begin{aligned}\bar{f}((a + I) + (b + I)) &= \bar{f}((a + b) + I) \\ &= f(a + b) \\ &= f(a) + f(b) \\ \bar{f}((a + I) + (b + I)) &= \bar{f}(a + I) + \bar{f}(b + I)\end{aligned}$$

so \bar{f} is closed under addition.

2. Closed under multiplication. Let $a + I, b + I \in R/I$. Then

$$\begin{aligned}\bar{f}((a + I) \cdot (b + I)) &= \bar{f}(a \cdot b + I) \\ &= f(a \cdot b) \\ &= f(a) \cdot f(b) \\ \bar{f}((a + I) \cdot (b + I)) &= \bar{f}(a + I) \cdot \bar{f}(b + I)\end{aligned}$$

so \bar{f} is closed under multiplication.

3. Preservation of the additive identity. Let $0_{R/I} := 0 + I \in R/I$. Then

$$\bar{f}(0_{R/I}) = \bar{f}(0 + I) = f(0) = 0_S$$

so \bar{f} preserves the additive identity.

4. Preservation of the multiplicative identity. Let $1_{R/I} := 1 + I \in R/I$. Then

$$\bar{f}(1_{R/I}) = \bar{f}(1 + I) = f(1) = 1_S$$

so \bar{f} preserves the multiplicative identity.

So \bar{f} is a ring homomorphism. To show that $f = \bar{f} \circ \pi$, consider $a \in R$. Then

$$\bar{f} \circ \pi(a) = \bar{f}(\pi(a)) = \bar{f}(a + I) = f(a)$$

so $f = \bar{f} \circ \pi$. To show that \bar{f} is unique, suppose we have another homomorphism $g : R/I \rightarrow S$ such that $f \neq g$. Then

$$g \circ \pi(a) = g(\pi(a)) = g(a + I) \neq f(a) = \bar{f}(a + I) = \bar{f}(\pi(a)) = \bar{f} \circ \pi(a)$$

so \bar{f} is unique. □

Question 5

Let $a \in \mathbb{R}$ be any real number. Show that the quotient ring $\mathbb{R}[x]/(x - a)$ is isomorphic to \mathbb{R} . [hint: you can use, without proof, that a polynomial $p(x)$ has a root a if and only if it can be written $p(x) = (x - a)q(x)$, where $q(x)$ is another polynomial.]

Response

Proof: Let $a \in \mathbb{R}$ and $\mathbb{R}[x]/(x - a)$ be a quotient ring. Consider $f : \mathbb{R}[x] \rightarrow \mathbb{R}$ where $p(x) \mapsto p(a)$. To show that f is a homomorphism:

1. Closed under addition. Let $p(x), q(x) \in \mathbb{R}[x]$. Then

$$f(p(x) + q(x)) = p(a) + q(a) = f(p(x)) + f(q(x))$$

so f is closed under addition.

2. Closed under multiplication. Let $p(x), q(x) \in \mathbb{R}[x]$. Then

$$f(p(x) \cdot q(x)) = p(a) \cdot q(a) = f(p(x)) \cdot f(q(x))$$

so f is closed under multiplication.

3. Preservation of the additive identity. Let $0_{\mathbb{R}[x]} \in \mathbb{R}[x]$. Then

$$f(0_{\mathbb{R}[x]}) = 0(a) = 0_{\mathbb{R}}$$

so f preserves the additive identity.

4. Preservation of the multiplicative identity. Let $1_{\mathbb{R}[x]} \in \mathbb{R}[x]$. Then

$$f(1_{\mathbb{R}[x]}) = 1(a) = 1_{\mathbb{R}}$$

so f preserves the multiplicative identity.

So f is a ring homomorphism. Take an arbitrary $b \in \mathbb{R}$. Then there is some $p(x) \in \mathbb{R}[x]$ such that $f(p(x)) = b$, so f is surjective. Pick $p(x) \in \ker(f)$. Then $f(p(x)) = p(a) = 0$ is only true when $p(x) = (x - a)q(x)$ where $q(x)$ is another polynomial. So, $\ker(f)$ is generated by the principal ideal $(x - a)$. By the First Isomorphism Theorem (proven in class), since $f : \mathbb{R}[x] \rightarrow \mathbb{R}$, we have that $\mathbb{R}[x]/\ker(f) \simeq \text{Im}(f)$. From above, we have that $\ker(f) = (x - a)$ and $\text{Im}(f) = \mathbb{R}$, so $\mathbb{R}[x]/(x - a) \simeq \mathbb{R}$. \square

Question 6

Let R be a commutative ring, and let $I, J \subseteq R$ be ideals. Consider

$$IJ = \{i_1j_1 + \cdots + i_nj_n \mid i_r \in I, j_s \in J, n > 0\}.$$

1. Show that IJ is an ideal.
2. Show that $IJ \subseteq I \cap J$.
3. Show that if $I + J = R$, then $IJ = I \cap J$.

Response

1. IJ is an ideal.

Proof: Let R be a ring, and $I, J \subseteq R$ be ideals. Then, to show that

$$IJ = \{i_1j_1 + \cdots + i_nj_n \mid i_r \in I, j_s \in J, n > 0\}$$

is an ideal:

- (a) Closure under addition. Take two elements $a, b \in IJ$. We can rewrite a, b as, $a = p_1q_1 + \cdots + p_nq_n$ and $b = u_1v_1 + \cdots + u_nv_n$ for $p_r, u_r \in I, q_s, v_s \in J$. Then

$$a + b = (p_1q_1 + \cdots + p_nq_n) + (u_1v_1 + \cdots + u_nv_n)$$

which is a finite sum that can be rewritten in the form $i_1j_1 + \cdots + i_nj_n$ for $i_r \in I, j_s \in J$. Then by definition, $a + b \in IJ$, so IJ is closed under addition.

- (b) Absorbing property. Take any $a \in IJ$. We can rewrite a as, $a = i_1j_1 + \cdots + i_nj_n$ for $i_t \in I, j_s \in J$. Consider an element $t \in R$. Then, $at = (i_1j_1 + \cdots + i_nj_n)t = i_1j_1t + \cdots + i_nj_nt$. Similarly, $at = t(i_1j_1 + \cdots + i_nj_n) = ti_1j_1 + \cdots + ti_nj_n$. Since $I, J \subseteq R$ are ideals, $i_rt, ti_r \in I, j_s \in J$, so $ti_rj_s \in IJ$. Similarly, $i_r \in I, j_st, tj_s \in J$, so $i_rj_st \in IJ$. So IJ satisfies the absorbing property.

Because IJ is closed under addition and satisfies the absorbing property, IJ is an ideal. \square

2. $IJ \subseteq I \cap J$.

Proof: Let R be a ring, and $I, J \subseteq R$ be ideals. Then, from (1), IJ is an ideal in R . Suppose we have an arbitrary element $i_1j_1 + \cdots + i_nj_n \in IJ$. Since $I, J \subseteq R$ are ideals, $i_rj_s \in I$ and $i_rj_s \in J$. That is, $i_rj_s \in I \cap J$. Since $I, J \subseteq R$ are ideals, they are both closed under addition. So, $i_1j_1 + \cdots + i_nj_n \in I \cap J$. Since $i_1j_1 + \cdots + i_nj_n$ was arbitrary, $IJ \subseteq I \cap J$. \square

3. If $I + J = R$, then $IJ = I \cap J$.

Proof: Suppose $I + J = R$. Then $1_I + 1_J = 1_R$. Pick any $a \in I \cap J$. Then, $a = a \cdot 1_R = a \cdot (1_I + 1_J) = a \cdot 1_I + a \cdot 1_J$. Then, $a \cdot 1_I \in I$ because $a, 1_I \in I$. Similarly, $a \cdot 1_J \in J$ because $a, 1_J \in J$. So, $a \cdot 1_I + a \cdot 1_J \in I \cap J$ since $I \cap J$ is an ideal (**Question 2**). Because $a \cdot 1_I + a \cdot 1_J \in I \cap J$ was arbitrary, $I \cap J \subseteq IJ$. From (2), $IJ \subseteq I \cap J$. It follows that $IJ = I \cap J$. \square

Question 7

Let R be a commutative ring. Recall that $r \in R$ is nilpotent if there is some $n > 0$ such that $r^n = 0$.

1. Let $Nil(R)$ be the set of nilpotent elements of R . Show that $Nil(R)$ forms an ideal.
2. Show that $R/Nil(R)$ has no nonzero nilpotent elements.

Response