Problem Set 6

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Question 2 part (b)

Show that L_A is closed.

Response

Proof. Let $x \in L_{L_A}$. Then, x is a limit point of L_A, so for all $\varepsilon > 0$, there exists some $y \in L_A \cap (x - \varepsilon, x + \varepsilon)$ such that $y \neq x$. Since $x \neq y$ by definition of a neighborhood, y is also a limit point of L_A, but $y \in L_A$ since for any two sets A, B, if $x \in A \cap B$ then $x \in A$. Since y was arbitrary, this holds for every limit point in L_A. So, $x \in L_A$. Since x was arbitrary, this holds for any point in L_{L_A}, so L_A is closed. $\square$

Question 3

Show that $c \in A$ is an isolated point if and only if it is not a limit point of A.

Response

Proof. ($c \in A$ is an isolated point $\implies$ $c \in A$ is not a limit point)
Let $c \in A$ be an isolated point. Then, by defintion, there exists some $\varepsilon > 0$ such that $A \cap (c-\varepsilon, c+\varepsilon) = \{c\}$ which implies that for any $y \neq c,\ y \notin \{c\}$. Then by the neighborhood definition of a limit point, $c \in A$ cannot be a limit point since $\exists \varepsilon > 0 : A \cap (c-\varepsilon, c+\varepsilon) : y \neq c = \emptyset \iff \neg[\forall \varepsilon > 0,\ \exists y \in A \cap (c-\varepsilon, c+\varepsilon) : y \neq c]$, or the negation of the neighborhood definition of a limit point. Therefore, if $c \in A$ is an isolated point, then it is not a limit point of A.

($c \in A$ is an isolated point $\impliedby$ $c \in A$ is not a limit point)
Let $c \in A$ not be a limit point. Then, by the neighborhood definition of a limit, $\neg[\forall \varepsilon > 0,\ \exists y \in A \cap (c-\varepsilon, c+\varepsilon) : y \neq c] \iff \exists \varepsilon > 0 : A \cap (c-\varepsilon, c+\varepsilon) : y \neq c = \emptyset$. That is, the only point in the intersection of A and $(c-\varepsilon, c+\varepsilon)$ is c itself, since $c \in A$ by assumption. This is precisely the definition of an isolated point. Therefore, if $c \in A$ is not a limit point, then it is an isolated point.

Since we proved both directions, the proof is complete. $\square$

Question 4 part (a)

(a) Let $a, b \in \mathbb{R}$. Prove that the interval $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ is a closed set.

Response

Proof. Assume by contradiction that $[a, b]$ is not closed. Then there exists an $x \in L_{[a,b]}$ but $x \notin [a, b]$. By the neighborhood definition of a limit, for all $\varepsilon > 0$, there exists some $y \in [a, b] \cap (x - \varepsilon, x + \varepsilon)$. There are two cases:

Case I: $x < a$

Choose $\varepsilon = a - x > 0$ (by assumption, $x < a$). Then, $[a, b] \cap (x - (a - x), x + (a - x)) = [a, b] \cap (x - \varepsilon, a) = \emptyset$, a contradiction that x is a limit point of $[a, b]$.

Case II: $x > b$

Choose $\varepsilon = x - b > 0$ (by assumption, $x > b$). Then, $[a, b] \cap (x - (x - b), x + (x - b)) = [a, b] \cap (b, x + \varepsilon) = \emptyset$, a contradiction that x is a limit point of $[a, b]$.

Since x was arbitrary, this holds for any $x \in \mathbb{R} \setminus [a, b]$. In either case, we reach a contradiction. Therefore, $[a, b]$ must be closed. $\square$

Question 5 part (c)

Prove the following by using the (ε, δ)-definition of the functional limit:
(c) $\lim\limits_{x \to 1} \frac{x^2-x+1}{x+1} = \frac{1}{2}$

Response

Scratch

$$\begin{aligned}
\left|\frac{x^2-x+1}{x+1} - \frac{1}{2}\right| &= \left|\frac{2(x^2-x+1)-(x+1)}{2(x+1)}\right| \\
&= \left|\frac{2x^2-3x+1}{2(x+1)}\right| \\
&= \left|\frac{(2x-1)(x-1)}{2(x+1)}\right| \\
&\leq \left|\frac{2x-1}{x+1}\delta\right| < \varepsilon
\end{aligned}$$

Let $\delta = 1$

$$\begin{aligned}
|x-1| < 1 &\implies 0 < x < 2 \\
|2x-1| < 1 &\implies -\frac{1}{2} < 2x-1 < \frac{3}{2} \\
|x+1| < 1 &\implies 1 < x+1 < 3
\end{aligned}$$

So $1 < x < \frac{3}{2}$ to ensure that all three conditions are met. Then,

$$\begin{aligned}
\left|f(x) - \frac{1}{2}\right| &\leq \frac{3}{2}\delta < \varepsilon \\
\delta &< \frac{2}{3}\varepsilon
\end{aligned}$$

Proof. Let $\varepsilon > 0$. Choose $\delta = \min\left\{1, \frac{2}{3}\varepsilon\right\}$. If $|x-1| < \delta$, then $x+1 \neq 0$ and $\frac{x^2-x+1}{x+1} \leq \left|\frac{2x-1}{x+1}\delta\right| < \varepsilon$. Therefore, $\lim\limits_{x \to 1} \frac{x^2-x+1}{x+1} = \frac{1}{2}$. □

Question 7 part (a)

(a) $f(x) = \frac{x}{|x|}$

Response

(i) $\lim_{x\to 0^+} f(x)$:
$\exists (x_n) : x_n > 0, x_n \to 1$. Then, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(\lim_{n\to\infty} \frac{x_n}{|x_n|}) = 1$, so $\lim_{x\to 0^+} \frac{x}{|x|} = 1$.

(ii) $\lim_{x\to 0^-} f(x)$:
$\exists (x_n) : x_n < 0, x_n \to -1$. Then, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(\lim_{n\to\infty} \frac{x_n}{|x_n|}) = -1$, so $\lim_{x\to 0^-} \frac{x}{|x|} = -1$.

(iii) $\lim_{x\to 0} f(x)$:
Since $\lim_{x\to 0^-} \frac{x}{|x|} \neq \lim_{x\to 0^+} \frac{x}{|x|}$, $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

(iv) $\lim_{x\to +\infty} f(x)$:
$\exists (x_n) : x_n \to 1$. Then, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(\lim_{n\to\infty} \frac{x_n}{|x_n|}) = 1$, so $\lim_{x\to +\infty} \frac{x}{|x|} = 1$.

(v) $\lim_{x\to -\infty} f(x)$:
$\exists (x_n) : x_n \to -1$. Then, $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(\lim_{n\to\infty} \frac{x_n}{|x_n|}) = -1$, so $\lim_{x\to -\infty} \frac{x}{|x|} = -1$.