Prove $\inf S \leq \sup S$**:**

Proof. Since $S \neq \emptyset$, $S \subseteq \mathbb{R}$, S is bounded above and below, $\inf S, \sup S$ exist. Since $S \neq \emptyset$, $\exists s \in S$. By definiation, $\inf S \leq s \leq \sup S$ for all $s \in S$. Taking the extremes of the inequality, we get $\inf S \leq \sup S$. $\square$

What if $\inf S = \sup S$**?**
If $\alpha = \inf S = \sup S$, then we know S contains only one element so $\inf S \leq s \leq \sup S \implies \alpha \leq s \leq \alpha \implies s = \alpha$.

Let S **and** T **be nonempty subsets of** $\mathbb{R}$ **with the following property:** $s \leq t$ **for all** $s \in S$ **and** $t \in T$**. Prove** $S \subseteq T \implies \inf T \leq \inf S \leq \inf S \leq \sup T$**:**

Proof. Since both $S, T \neq \emptyset$, $S, T \subseteq \mathbb{R}$, and bounded, $\inf S, \inf T, \sup S, \sup T$ exist. Then, since $S \subseteq T$, $\forall s \in S, s \in T$. Since $\forall t \in T, t \leq \sup T$, $\sup T$ is an upper bound for S. Since $\sup S$ is the *least* upper bound by definition, we have that $\sup S \leq \sup T$. Since $\forall t \in T, \inf T \leq t$, we have that $\inf T$ is a lower bound for S. Since $\inf S$ is the *greatest* lower bound by definition, we have that $\inf T \leq \inf S$. Note that since $S \neq \emptyset$, $\forall s \in S$, $\inf S \leq s \leq \sup S$, so we get the following inequality: $\inf T \leq \inf S \leq s \leq \sup S \leq \sup T$ so $\inf T \leq \inf S \leq \sup S \leq \sup T$ $\square$

Prove that if $a > 0$**, then there exists** $n \in \mathbb{N}$ **such that** $\frac{1}{n} < a < n$**:**

Proof. Multiplying n on both sides of $\frac{1}{n} < a$, we get $1 < na$. By the Archemedian property, since $a, 1 > 0$, there exists an $n \in \mathbb{N}$ s.t. $na > 1$.

Since $a, 1 > 0$ in the inequality $a < 1 \cdot n$, by the Archemedian property, there exists an $n \in \mathbb{N}$ s.t. $n > a$. Therefore, $\frac{1}{n} < a < n$. $\square$

Prove $\lim \frac{(-1)^n}{n} = 0$

Scratch:

$$\begin{aligned}
\left|\frac{(-1)^n}{n} - 0\right| &< \varepsilon \\
\left|\frac{(-1)^n}{n}\right| &< \varepsilon \\
\frac{1}{n} &< \varepsilon \qquad\qquad \text{note: } \left|\frac{(-1)^n}{n} - 0\right| \leq \frac{1}{n} \\
n &> \frac{1}{\varepsilon}
\end{aligned}$$

Proof. Let $\varepsilon > 0$. Let $N \geq \frac{1}{\varepsilon}$. The, $\forall n > N$, we have

$$\begin{aligned}
n &> \frac{1}{\varepsilon} \\
\frac{1}{n} &< \varepsilon \\
\left|\frac{(-1)^n}{n} - 0\right| \leq \frac{1}{n} &< \varepsilon \\
\left|\frac{(-1)^n}{n} - 0\right| &< \varepsilon \qquad \text{taking the extremes of the inequalities}
\end{aligned}$$

Therefore, $\lim \frac{(-1)^n}{n} = 0$. $\square$

Prove $\lim \frac{1}{n^{1/3}} = 0$

Scratch:

$$\left|\frac{1}{n^{1/3}} - 0\right| < \varepsilon$$
$$\frac{1}{n^{1/3}} < \varepsilon$$
$$n > \frac{1}{\varepsilon^3}$$

Proof. Let $\varepsilon > 0$. Let $N \geq \frac{1}{\varepsilon^3}$. Then $\forall n > N$, we have

$$n > \frac{1}{\varepsilon^3} \implies \left|\frac{1}{n^{1/3}} - 0\right| < \varepsilon$$

by the scratch work above. □

Prove $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$

Scratch:

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon$$
$$\left|\frac{6n-3-(6n+4)}{3(3n+2)}\right| < \varepsilon$$
$$\left|\frac{-7}{3(3n+2)}\right| < \varepsilon$$
$$\frac{7}{3(3n+2)} < \varepsilon \qquad \text{note: } \left|\frac{-7}{3(3n+2)}\right| \leq \frac{7}{3(3n+2)}$$
$$\frac{7}{9n+6} < \varepsilon$$
$$n > \frac{7-6\varepsilon}{9\epsilon}$$

Proof. Let $\varepsilon > 0$. Let $N \geq \frac{7-6\varepsilon}{9\epsilon}$. Then $\forall n > N$, we have

$$n > \frac{7-6\varepsilon}{9\epsilon} \implies \left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon$$

by the scratch work above. □

Prove $\lim \frac{n+6}{n^2-6} = 0$

Scratch:

$$\left|\frac{n+6}{n^2-6} - 0\right| < \varepsilon$$
$$\left|\frac{n+6}{n^2-6}\right| < \varepsilon$$

Note that when $n \geq 6$, we have that $|n+6| \leq 2n$, $|n^2-6| \geq \frac{1}{2}n^2$.

$$\left|\frac{n+6}{n^2-6}\right| \leq \frac{2n}{\frac{1}{2}n^2} < \varepsilon$$
$$\frac{4n}{n^2} < \varepsilon$$
$$\frac{4}{n} < \varepsilon$$
$$n > \max\{\frac{4}{\varepsilon}, 6\}$$

Proof. Let $\varepsilon > 0$. Let $N \geq \max\{\frac{4}{\varepsilon}, 6\}$. Then $\forall n > N$, we have

$$n > \frac{4}{\varepsilon} \implies \left|\frac{n+6}{n^2-6}\right| \leq \frac{2n}{\frac{1}{2}n^2} < \varepsilon$$

from the scratch work above. Taking the extremes of both sides of the inequality, we get

$$\left|\frac{n+6}{n^2-6} - 0\right| < \varepsilon$$

$\square$