110A HW4

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Question 1

Let F be a field, and consider the polynomial ring $F[x, y]$ with two variables. Show that $I = (x, y)$ is not a principal ideal (i.e., it cannot be generated by a single element).

Response

Question 2

Let R be a ring, and let $I_1, \cdots, I_k$ be ideals. Show that the following sets are ideals:

1. $I_1 + \cdots + I_k = \{i_1 + \cdots + i_k | i_j \in I_j\}$
2. $I_1 \cap I_2 \cap \cdots \cap I_k$

Response

Question 3

Let R be a ring, $a \in R$, and $I \subseteq R$ be an ideal. Show that the set $a + I = \{a + x | x \in I\}$ is precisely the congruence class modulo I that contains a. That is, show that $b \equiv a \mod I$ if and only if $b \in a + I$.

Response

Question 4

Let $f : R \to S$ be a ring homomorphism, and suppose $I \subseteq R$ is an ideal such that $I \subseteq \ker(f)$. Show that there is a unique homomorphism $\overline{f} : R/I \to S$ such that $f = \overline{f} \circ \pi$.

Response

Question 5

Let $a \in \mathbb{R}$ be any real number. Show that the quotient ring $\mathbb{R}[x]/(x-a)$ is isomorphic to $\mathbb{R}$. [hint: you can use, without proof, that a polynomial $p(x)$ has a root a if and only if it can be written $p(x) = (x-a)q(x)$, where $q(x)$ is another polynomial.]

Response

Question 6

Let R be a commutative ring, and let $I, J \subseteq R$ be ideals. Consider

$$IJ = \{i_1 j_1 + \cdots + i_n j_n | i_r \in I, j_s \in J, n > 0\}.$$

1. Show that IJ is an ideal.
2. Show that $IJ \subseteq I \cap J$.
3. Show that if $I + J = R$, then $IJ = I \cap J$.

Response

Question 7

Let R be a commutative ring. Recall that $r \in R$ is nilpotent if there is some $n > 0$ such that $r^n = 0$.

1. Let $Nil(R)$ be the set of nilpotent elements of R. Show that $Nil(R)$ forms an ideal.
2. Show that $R/Nil(R)$ has no nonzero nilpotent elements.

Response

Proof: Let R be a commutative ring. First, we will show that $Nil(R)$ is a nonunital subring of R. Define $S := Nil(R)$.

1. **Closure under addition:** Let $a, b \in S$. Then, there exist some $n, m > 0$ such that $a^n = b^m = 0$. Then $(a+b)^{n+m} = (a+b)^n(a+b)^m = 0 \cdot 0 = 0$, so $a + b \in Nil(R)$.

$$\begin{aligned}(a+b)^{n+m} &= \sum_{k=0}^{n+m} \binom{n+m}{k} a^{n+m-k} b^k \\ &= \sum_{k=0}^{n+m} \binom{n+m}{k} a^{n+m-k} b^k\end{aligned}$$

2. **Closure under multiplication:** Let $a, b \in S$. Then, there exist some $n, m > 0$ such that $a^n = b^m = 0$.

3. **Existence of Inverses:**

$\square$