Problem Set 0

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Question 2

Complete the following truth table:

P	Q	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$

Response

P	Q	$\neg Q$	$P \wedge Q$	$P \vee Q$	$P \implies Q$	$P \iff Q$
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	T

Question 3 part (e)

Given any two statements P and Q, and using C to denote a contradiction (i.e., a statement that is always false), prove that the following statements are tautologies (i.e., they are always true):

(e) $((P \land \neg Q) \implies C) \implies (P \implies Q)$

Response

Proof.

P	Q	C	$P \land \neg Q$	$(P \land \neg Q) \implies C$	$P \implies Q$	$((P \land \neg Q) \implies C) \implies (P \implies Q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	F	F	T	T	T
F	F	F	F	T	T	T

Since the statement is true regardless of the truth values of P, Q, and C, it is a tautology. □

Question 8

Convert the following statements into plain English:

$$\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} : x + y = 0$$

$$\exists y \in \mathbb{Z}\ \forall x \in \mathbb{Z} : x + y = 0$$

Decide on the truth values of each statement and then provide a proof.

Response

$\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z} : x + y = 0$
"For every integer x, there exists an integer y such that $x + y = 0$".

Proof. This statement is true. Let $x \in \mathbb{Z}$. Since $\mathbb{Z}$ is a field, there exists an additive inverse $-x \in \mathbb{Z}$. So, we have

$$\begin{aligned} x + y &= 0 \\ x + y &= x + -x && \text{existence of an additive inverse} \\ y &= -x \end{aligned}$$

Substituting $-x$ for y, we get $x + y = x + -x = 0$. □

$\exists y \in \mathbb{Z}\ \forall x \in \mathbb{Z} : x + y = 0$
"There exists an integer y such that $x + y = 0$ for every integer x".

Proof. We will prove that this statement is false by counter-example. We want to prove that there exists at least one $x \in \mathbb{Z}$ such that $x + y \neq 0$. Let $x = 2$ and $y = -1$. Then, $x + y = 2 + -1 = 1 \neq 0$. Therefore, the statement is false. □

Question 10

(a) Suppose $p \in \mathbb{N}$. Show that if p^2 is divisible by 5 then p is also divisible by 5.

(b) Prove $\sqrt{5}$ is irrational.

(c) Suppose you try the same argument to prove $\sqrt{4}$ is irrational. The proof must fail, but where exactly does it fail?

Response

(a) *Proof.* To prove the statement, it is equivalent to prove its contrapositive: If p is not divisible by 5, p^2 is not divisible by 5. Note that p can be rewritten as $p = 5s + r$, where $s, r \in \mathbb{N}$. By definition, $r \neq 0$ since $r \in \mathbb{N}$ and $0 \notin \mathbb{N}$. So, p is not divisible by 5. To prove p^2 is not divisible by 5,

$$\begin{aligned} p^2 &= (5s + r)^2 \\ &= 25s^2 + 10sr + r^2 \\ p^2 &= 5(5s^2 + 2sr) + r(r) \end{aligned}$$

Since $r \neq 0$, p^2 is not divisible by 5. □

(b) *Proof.* Assume by contradiction that $\sqrt{5} \in \mathbb{Q}$. By definition, we can rewrite $\sqrt{5}$ as the fraction $\frac{p}{q}$, where p and q are coprime. Then we have

$$\begin{aligned} \sqrt{5} &= \frac{p}{q} && && (1) \\ 5 &= \frac{p^2}{q^2} && && (2) \\ 5q^2 &= p^2 && && (3) \\ q^2 &= \frac{1}{5}p^2 && \text{from (a), since } p^2 \text{ is divisible by 5, } p \text{ is also divisible by 5} && (4) \\ q^2 &= \frac{1}{5}(5r)^2 && \text{since } \mathbb{Q} \text{ is a field, } \exists r \in \mathbb{Q} : p = 5r && (5) \\ &= \frac{1}{5}(25r^2) && && (6) \\ q^2 &= 5r^2 && && (7) \\ \frac{1}{5}q^2 &= r^2 && \text{from (a), since } q^2 \text{ is divisible by 5, } q \text{ is also divisible by 5} && (8) \end{aligned}$$

which means that both p and q are divisible by 5, which is a contradiction. Therefore, $\sqrt{5}$ is irrational. □

(c) The proof fails because the statement ”if p^2 is divisible by 4, p is divisible by 4” does not always hold (e.g. $p = 2$). Therefore, we cannot assume that $p = 4r$ in (4) and (8).