

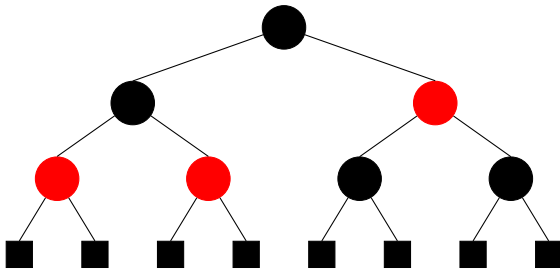
# Balanced Trees (*Red-Black* Trees)

Warren Kim

### Quick Definition

### Definition

A **red-black** tree is a type of **self-balancing** binary search tree that guarantees  $O(\log n)$  performance.



# Applications

***Red-Black*** Trees have a variety of applications. Some include:

- Implementing 2-4 trees<sup>1</sup>. 2-4 Trees  $\simeq$  ***Red-Black*** Trees.
- Linux CPU scheduler (Completely Fair Scheduler)
- Linux Virtual Memory Areas (VMA)
- STL Data Structures (e.g. C++'s `std::map`, Java's `HashMap`)
- Graph algorithm optimizations (for AI/ML)[e.g. K-mean clustering]
- Priority Queues (e.g. Bounded Queries)

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<sup>1</sup>2-4 (sometimes called 2-3-4) trees are a subset of  $B^+$ -trees.

# Motivation

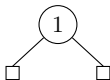
*Why do we want balanced binary trees?*

- Raw binary search tree performance is highly dependant on input order.
- We want to ensure  $\mathcal{O}(\log n)$  performance.

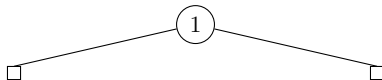
# Example

Suppose we have the input set  $\{1, 2, 3, 4, 5, 6, 7\}$  and consider two input orders:

$\{1, 2, 3, 4, 5, 6, 7\}$



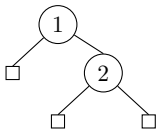
$\{1, 7, 2, 6, 3, 5, 4\}$



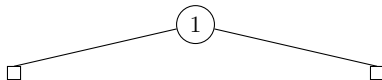
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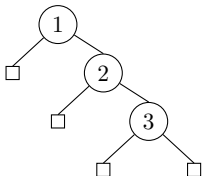
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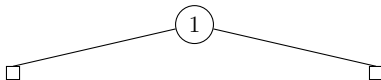
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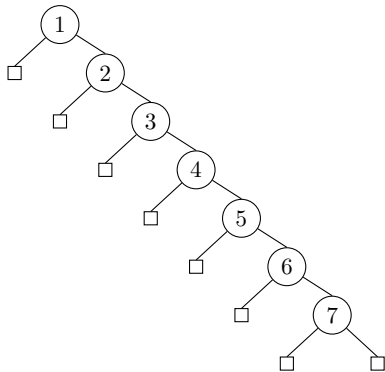
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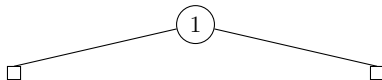
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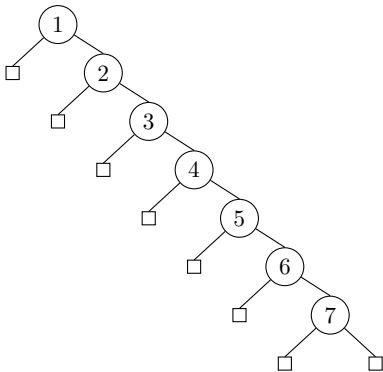




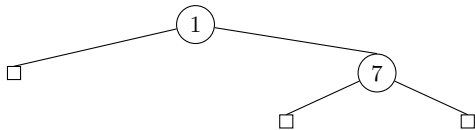
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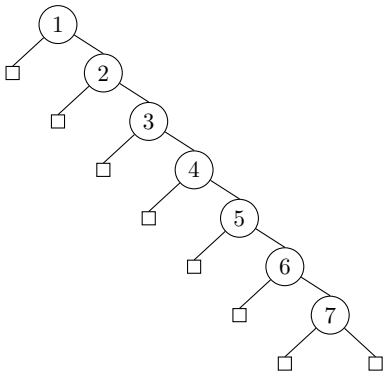
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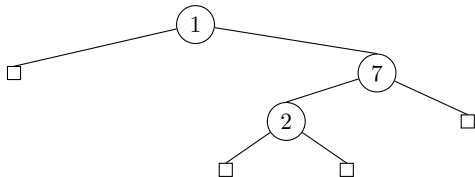
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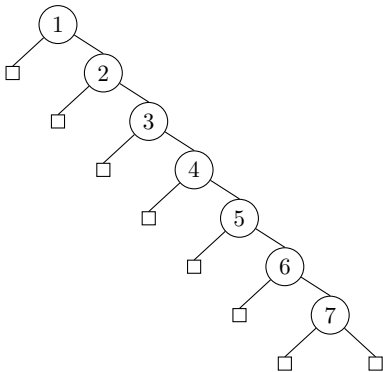
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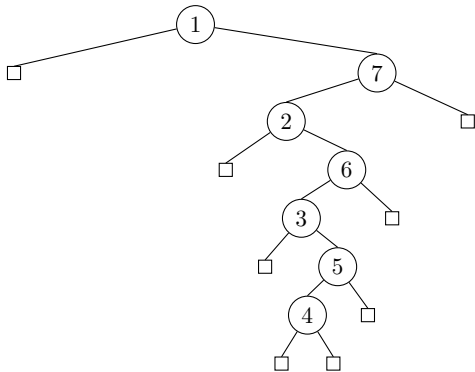
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# Intuition

*How do we balance binary trees?*

We can *dynamically* balance the tree!

→ We can add metadata<sup>2</sup> to our `Node` struct.

→ We can define a set of conditions that enforce balance.

---

<sup>2</sup>Metadata: Additional member variables

# Definition and Properties

## Definition

A **red-black** tree is a type of **self-balancing** binary search tree that guarantees  $\mathcal{O}(\log n)$  search, insertion, and deletion operations with the following properties:

- (i) *Color*: Every node is either **red** or **black**

```
enum Color { RED, BLACK };

struct Node {
    Color color;
    Node *left;
    Node *right;
    Node *parent;

    int data;
};
```

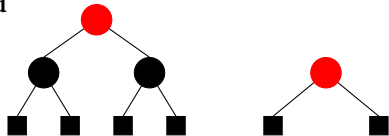
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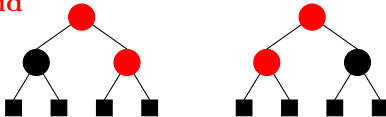
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- (ii) *Internal*: A **red** node does not have a **red** child

Valid



Invalid

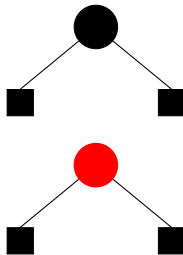


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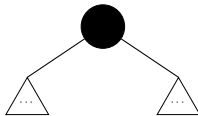


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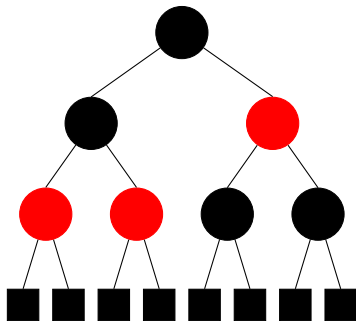


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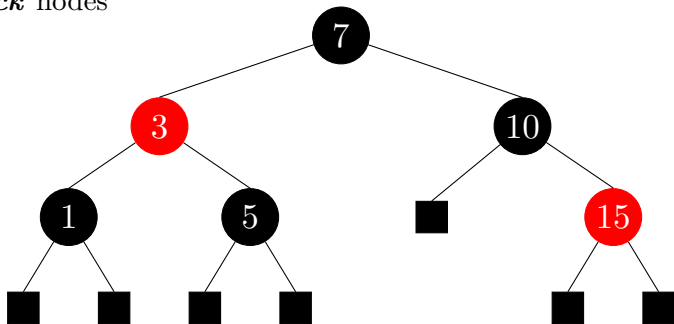
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- (v) *Depth:* Every path from the root to *any* null pointer passes through the same number of **black** nodes



# Depth Property

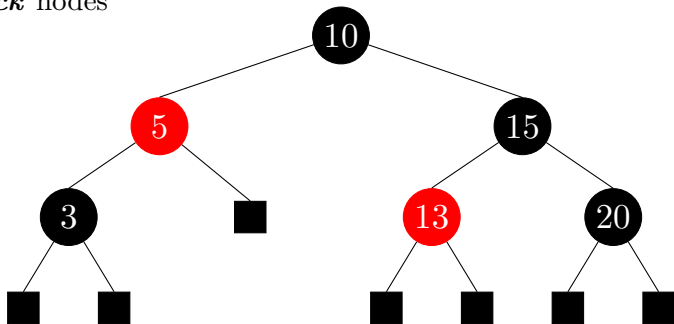
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# Depth Property

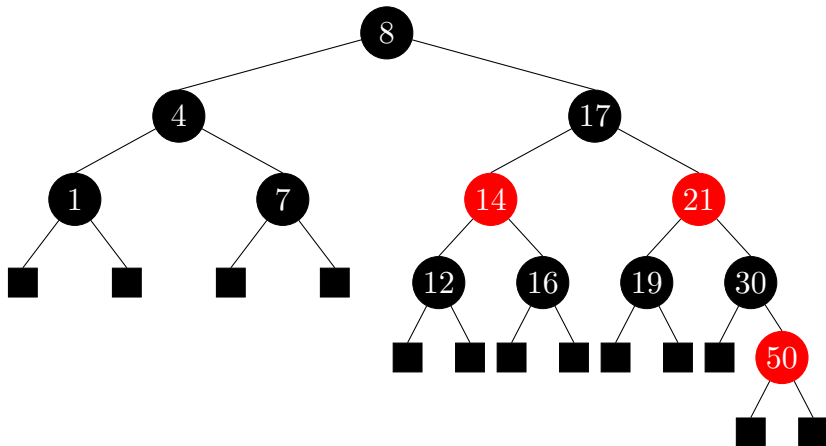
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- (iii) Fix double *red* violations, if any.

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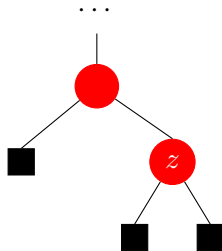
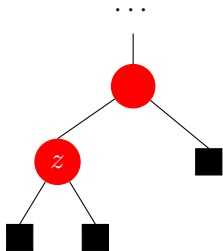
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- (ii) Color  $z$  *red*.
- (iii) Fix double *red* violations, if any.
- (iv) Recursively fix violations upward.

# Double Red Violations

Recall *Property (ii)*: A **red** node does not have a **red** child.

When we insert our node  $z$  (**red** by definition), its parent may be **red**. Below are examples of such cases.



# Fixing Double Red Violations

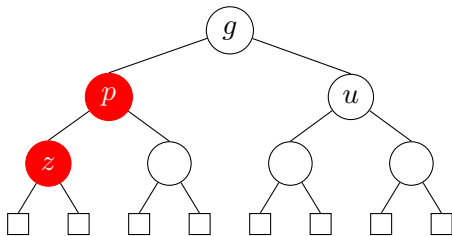
## Terminology

With respect to inserted node  $z$ ,

→ *Parent* ( $p$ ):  $z$ 's direct parent

→ *Uncle* ( $u$ ):  $p$ 's sibling

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# Fixing Double Red Violations

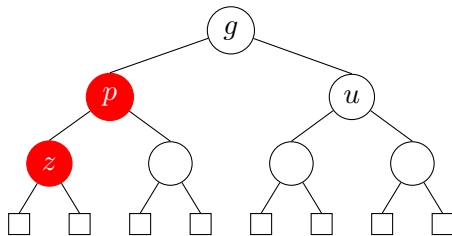
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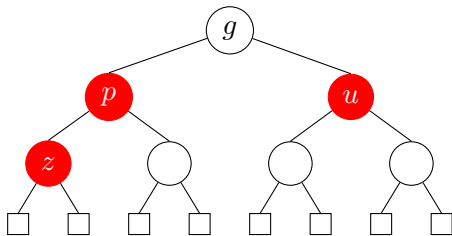
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(i) **Recolor:** If both the *parent* and *uncle* are **red**, perform a *recolor*.

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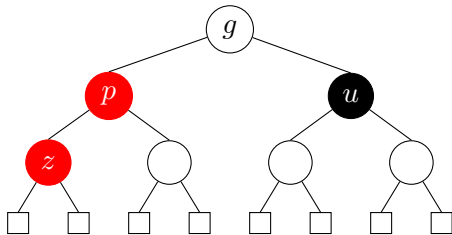
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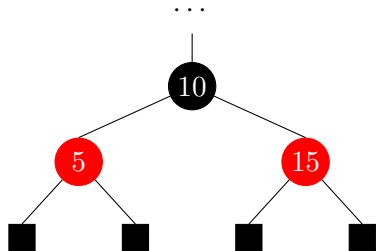
→ *Grandparent* ( $g$ ):  $p$ 's parent



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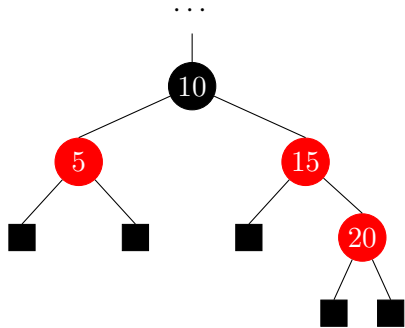
- (i) **Recolor:** If both the *parent* and *uncle* are **red**, perform a *recolor*.
- (ii) **Restructure:** If the *parent* is **red** but the *uncle* is **black**, perform a *tri-node restructure*.

# Recolor

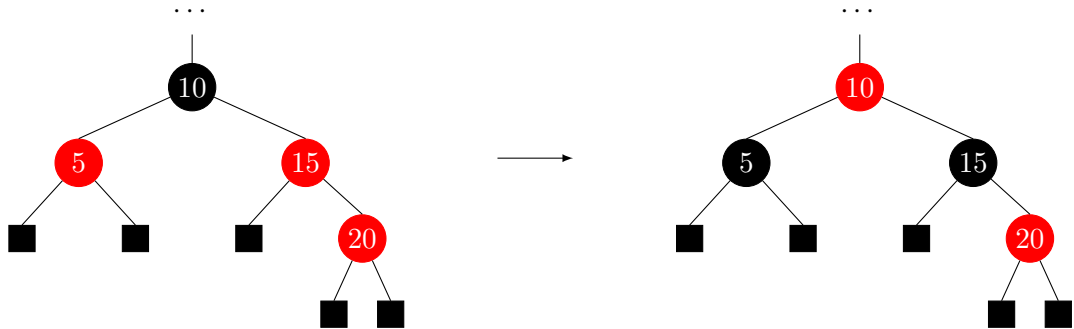




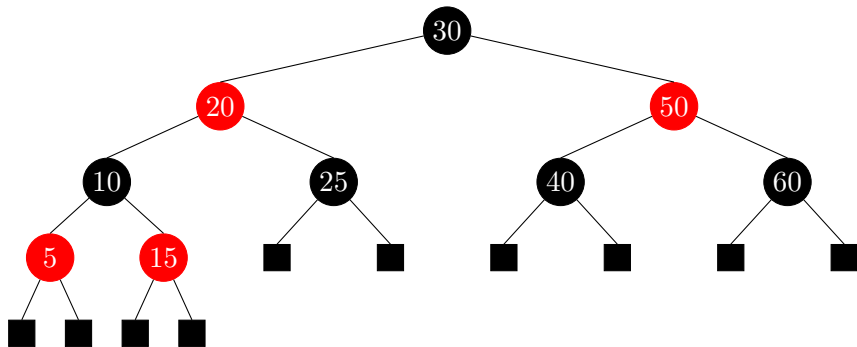
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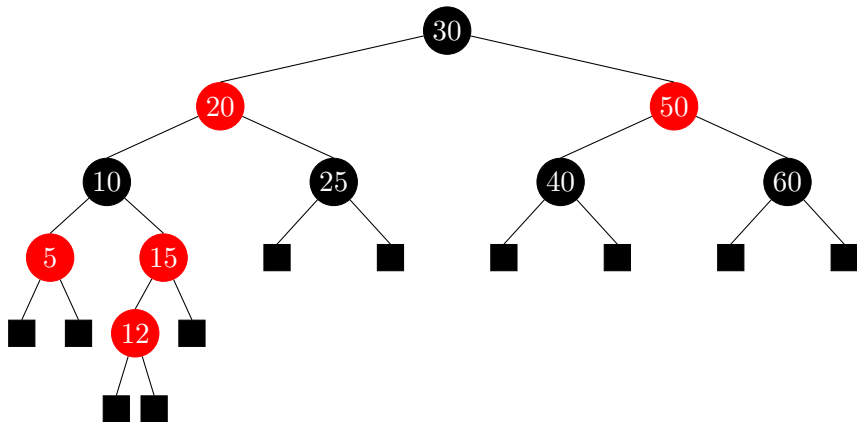
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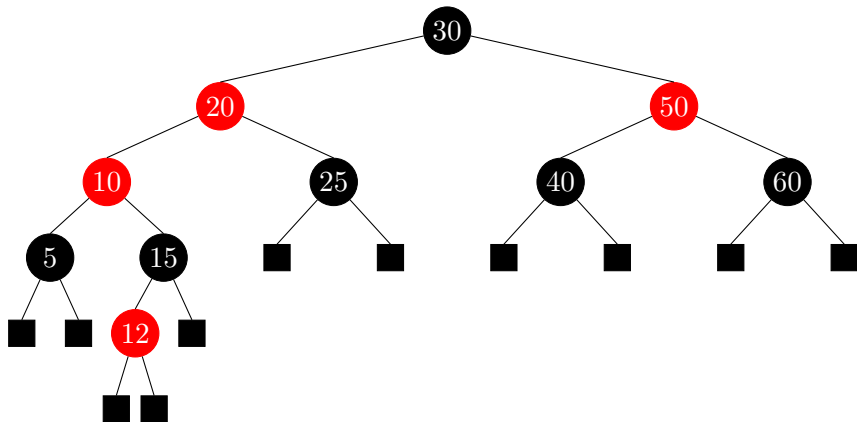
## Recolor (Recursive)



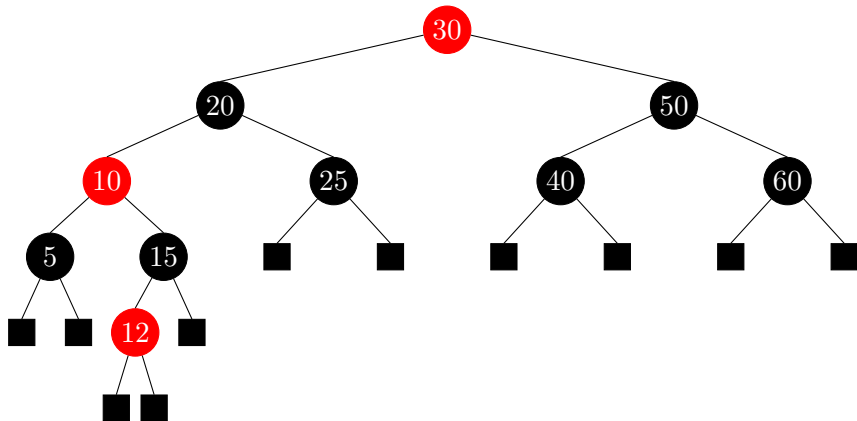
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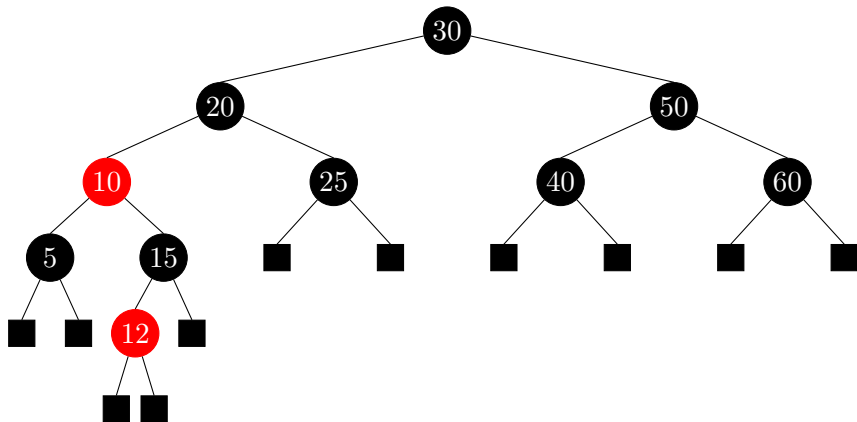
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# Fixing Double Red Violations

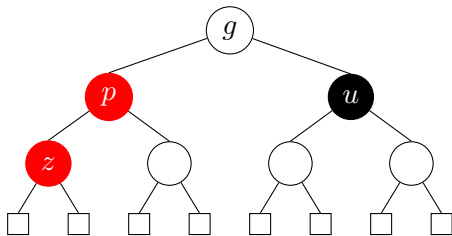
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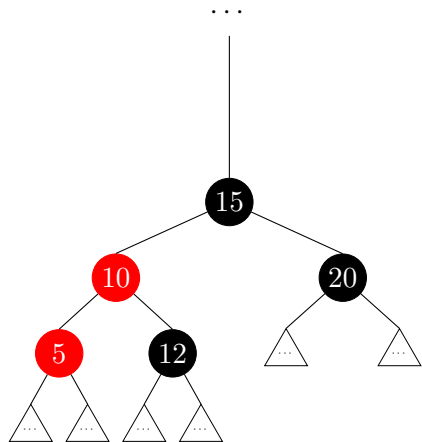



# Tri-Node Restructure

There are four cases:

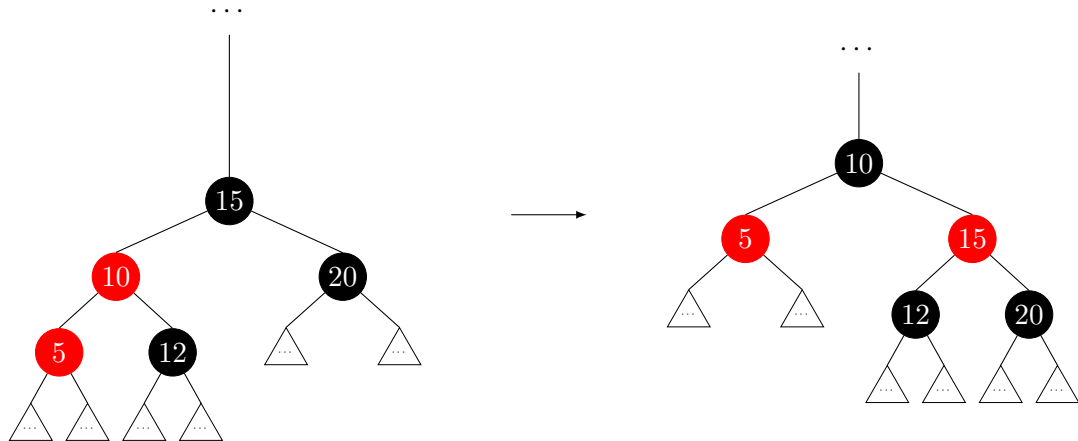
- (i)* Left-Left
- (ii)* Right-Right
- (iii)* Left-Right
- (iv)* Right-Left


## Case: Left-Left (*Right* Rotation)



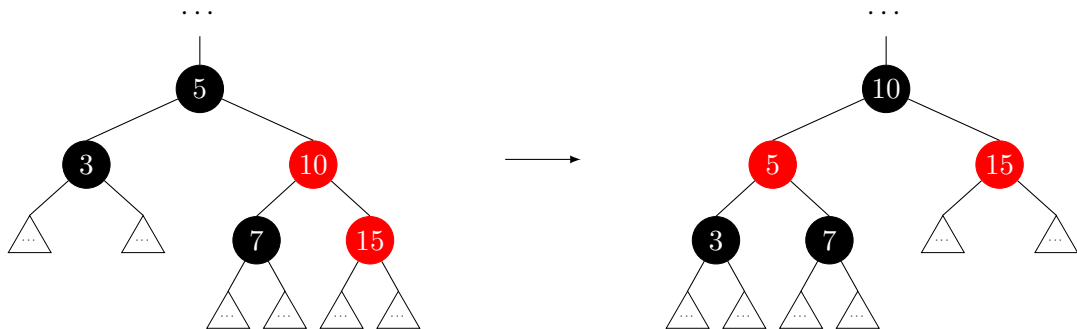
Here,  represents a subtree and ... represents the rest of the tree.


## Case: Left-Left (*Right* Rotation)



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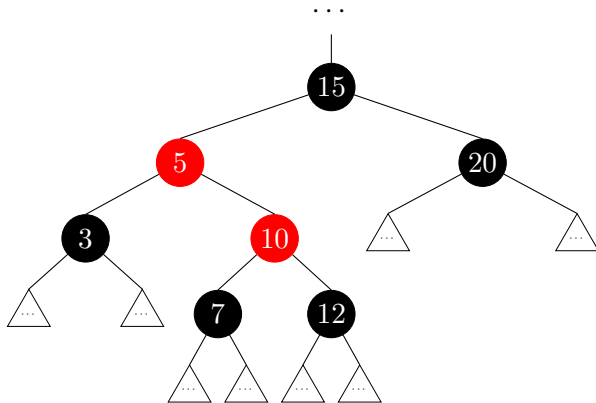
## Case: Right-Right (*Left* Rotation)




Here,  represents a subtree and  $\dots$  represents the rest of the tree.

# Case: Left-Right

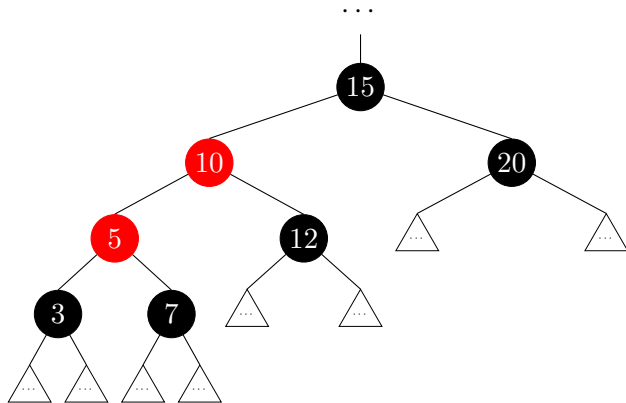
## Left Rotation



Here,  represents a subtree and  $\dots$  represents the rest of the tree.

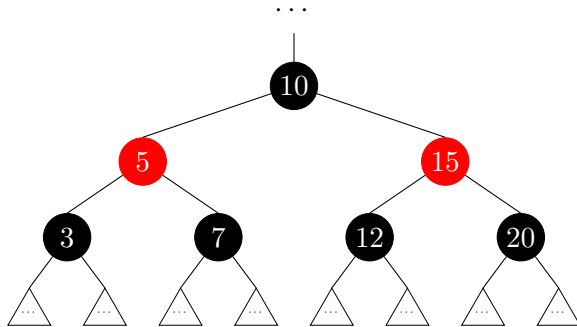
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## Right Rotation



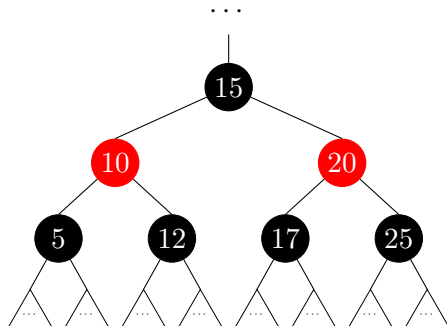
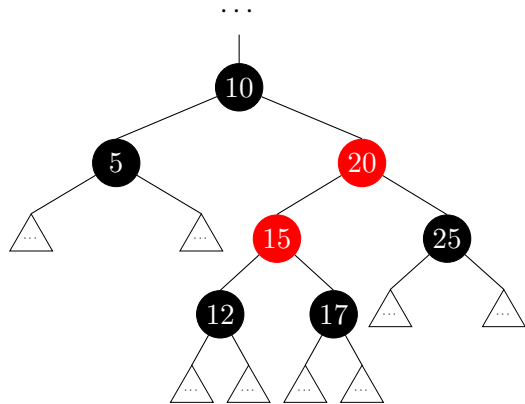
Here,  $\triangle$  represents a subtree and  $\dots$  represents the rest of the tree.


## Case: Left-Right



Here,  $\triangle$  represents a subtree and  $\dots$  represents the rest of the tree.

## Case: Right-Left



Here,  represents a subtree and ... represents the rest of the tree.



# Time and Space Complexities

→ *Insertion*:  $\mathcal{O}(\log n)$

→ *Deletion*:  $\mathcal{O}(\log n)$

→ *Search*:  $\mathcal{O}(\log n)$

→ *Space*:  $\mathcal{O}(n)$

# Resources

A code demo and these lecture slides (insertion only) can be found here:

<https://github.com/warrenjkim/rbtree-lecture>

**End**

Thank you!

# Appendix

Below are slides that didn't make the cut.

# Height of a Red-Black Tree

## Theorem

*A **red-black** tree with  $n$  nodes has a height  $h$  that is  $\mathcal{O}(\log n)$ .*

*Proof.* Suppose we have a **red-black** tree with  $n$  nodes and height  $h$ . Let  $b$  be the number of **black** nodes on the shortest path from root to any nil node. In the worst case, the longest path alternates between **red** and **black** nodes and thus has a height of  $2b$ . Then,  $h$  is bounded above by  $2b$ ; that is,  $h \leq 2b$ . There are  $2^b - 1 \leq n$  nodes in this tree. Solving for  $b$ , we get  $b \leq \log(n + 1)$ . Substituting  $b$ , we get  $b \leq \log(n + 1) \leq h \leq 2b \leq 2 \log(n + 1)$  so  $h$  is bounded below by  $\log(n + 1)$  and above by  $2 \log(n + 1)$ ; that is,  $\log(n + 1) \leq h \leq 2 \log(n + 1)$ . So,  $h$  is  $\mathcal{O}(\log n)$ .  $\square$

# Corollaries

## Proposition

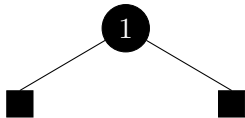
*If a node  $z$  has exactly one child,  $c$ , then (a)  $c$  is **red**, (b)  $z$  is **black**, and (c)  $c$  has no children.*

*Proof.* Suppose we have a valid **red-black** tree. Consider a node  $z$  with exactly one child. Without loss of generality, choose  $z$ 's left node to be the child and call it  $c$ .

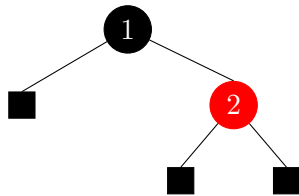
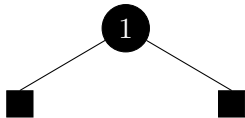
- (a)  $z$  passes through no **black** nodes on the right side by assumption. If  $c$  were **black**, then  $z$  would pass through 1 **black** node, a contradiction since this violates the *depth property*.
- (b) By (a),  $z$ 's child is **red** and by the *internal property*,  $z$  is **black**.
- (c) Since  $z$  passes through no **black** nodes on the right side by assumption,  $z$  cannot pass through any **black** nodes on the left side by the *the depth property*. Then, since  $c$  is **red** by (a),  $c$  has only nil nodes

□

## Red-Black Tree: Example

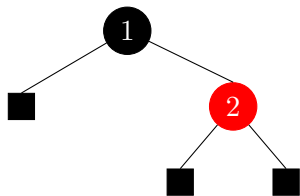


# Red-Black Tree: Example

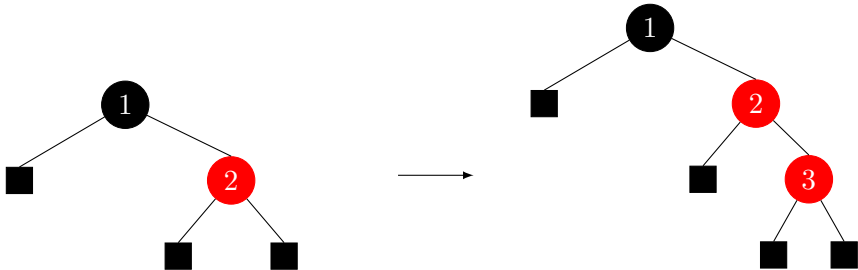




## Red-Black Tree: Example

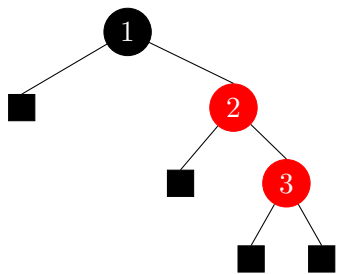


# Red-Black Tree: Example



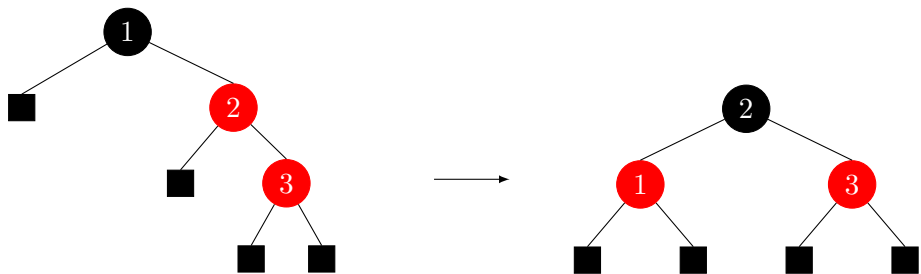
*Case: Right-Right*

# Red-Black Tree: Example



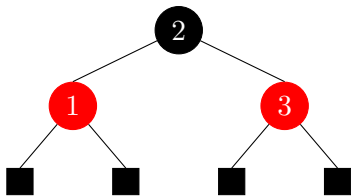
*Case: Right-Right*

# Red-Black Tree: Example

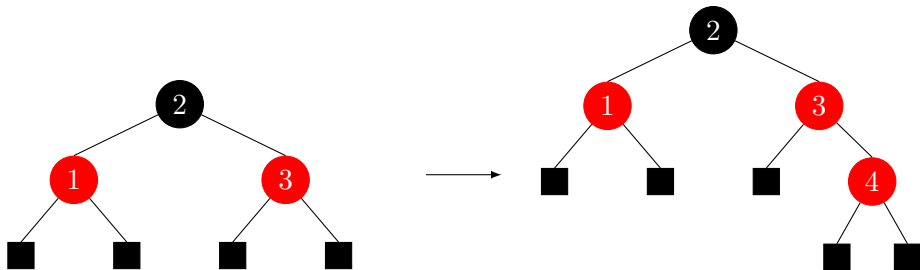


*Case: Right-Right*

## Red-Black Tree: Example

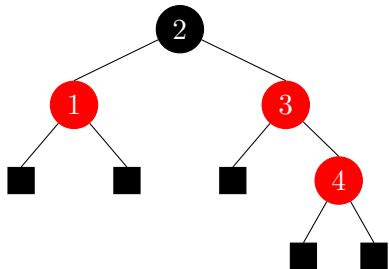


# Red-Black Tree: Example



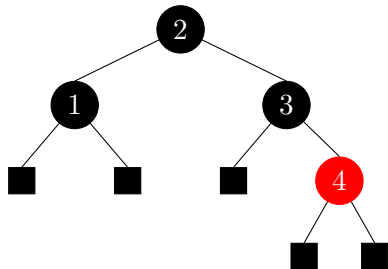
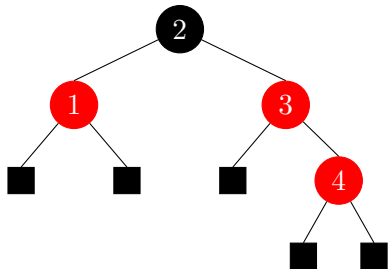
*Case: Recolor*

# Red-Black Tree: Example



*Case: Recolor*

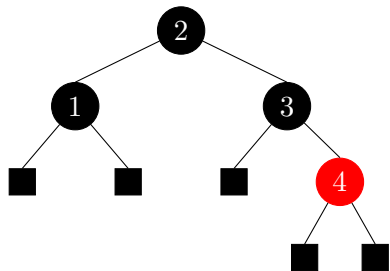
# Red-Black Tree: Example



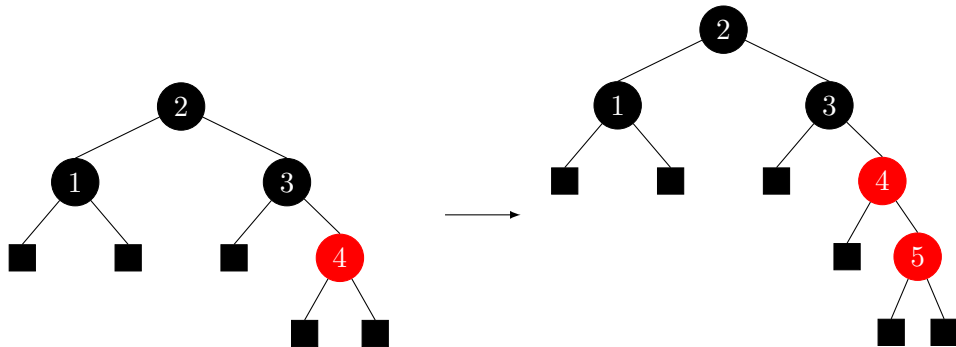
*Case: Recolor*



## Red-Black Tree: Example

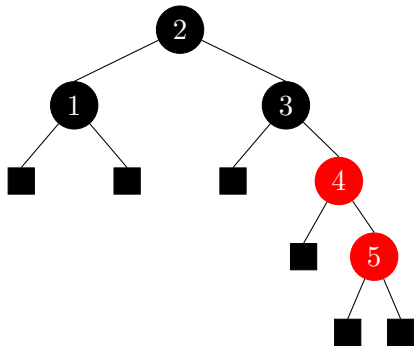


# Red-Black Tree: Example



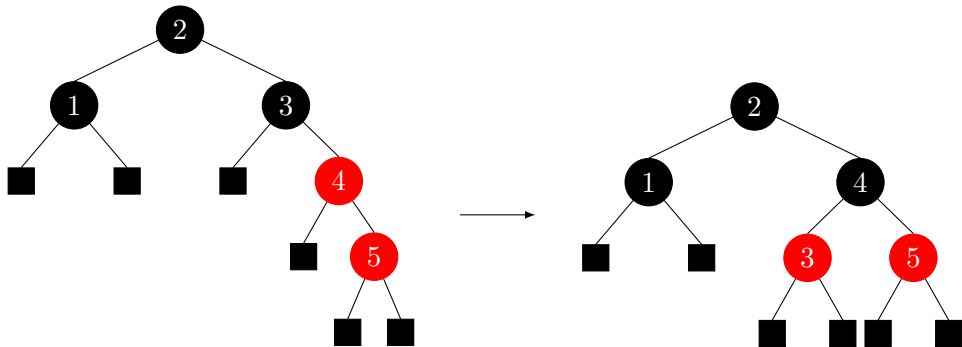
*Case: Right-Right*

# Red-Black Tree: Example



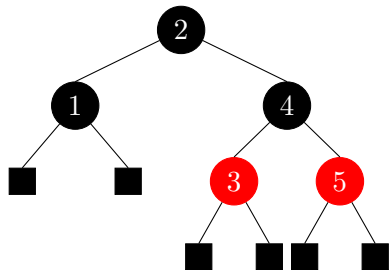
*Case: Right-Right*

## Red-Black Tree: Example

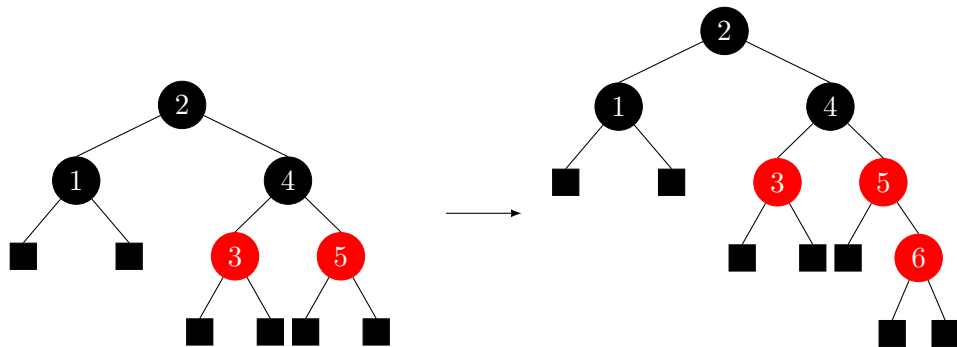


*Case: Right-Right*

## Red-Black Tree: Example

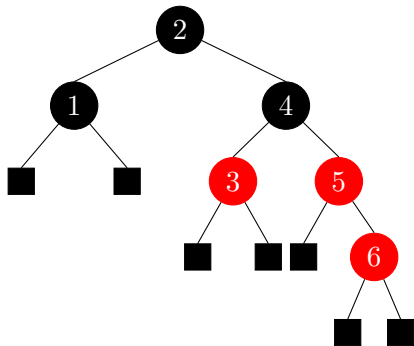


# Red-Black Tree: Example



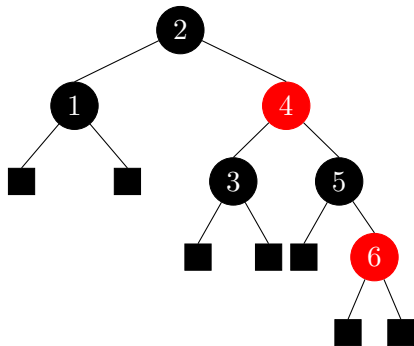
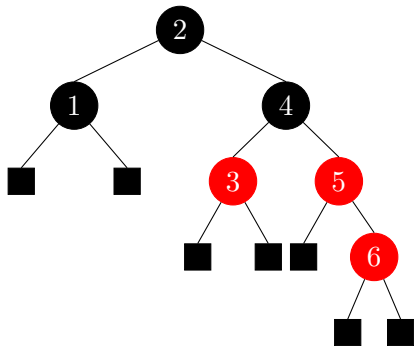
*Case: Recolor*

# Red-Black Tree: Example



*Case: Recolor*

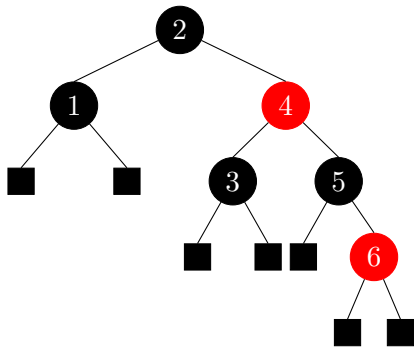
# Red-Black Tree: Example



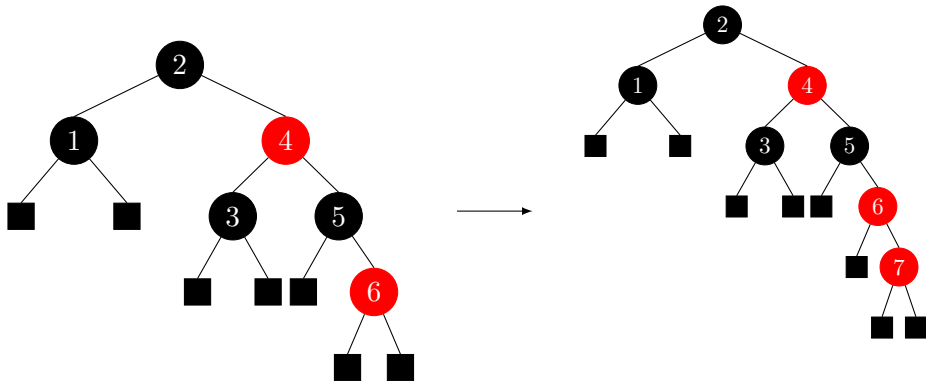
*Case: Recolor*



## Red-Black Tree: Example

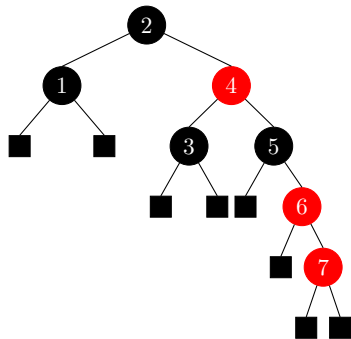


# Red-Black Tree: Example



*Case: Right-Right*

# Red-Black Tree: Example



# Red-Black Tree: Example

