Balanced Trees (***Red-Black*** Trees)

Warren Kim

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(v) Graph algorithm optimizations (for AI/ML)[e.g. K-mean clustering]

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Motivation

(i) Raw binary search tree performance is highly dependant on input order

(ii) We want to ensure $\mathcal{O}(\log n)$ performance

Example

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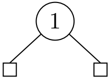

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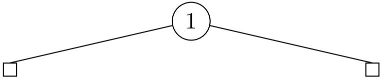

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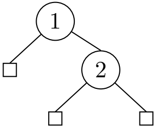

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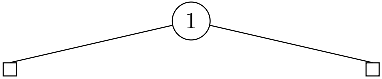

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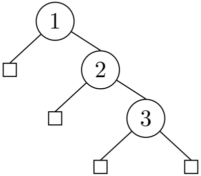

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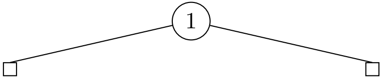

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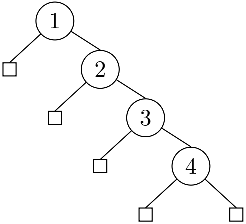

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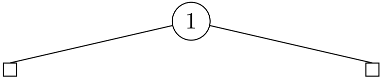

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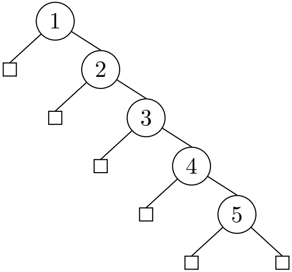

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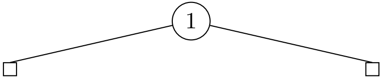

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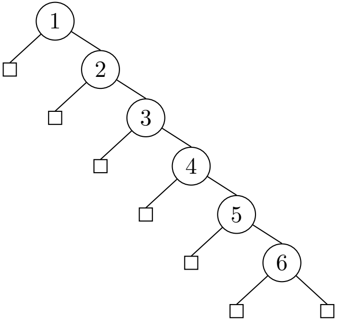

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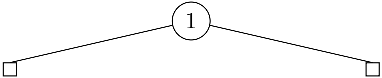

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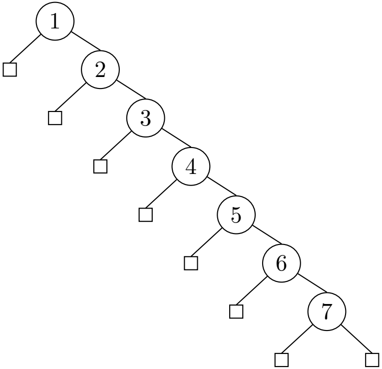

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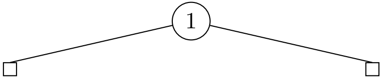

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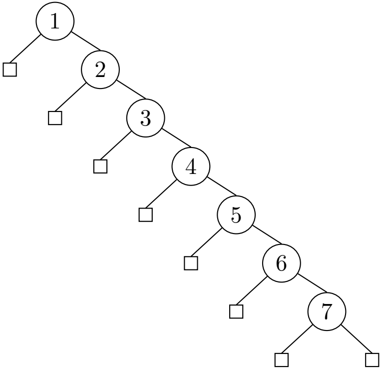

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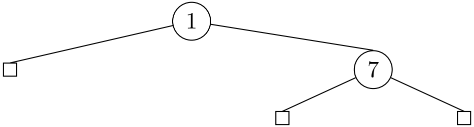

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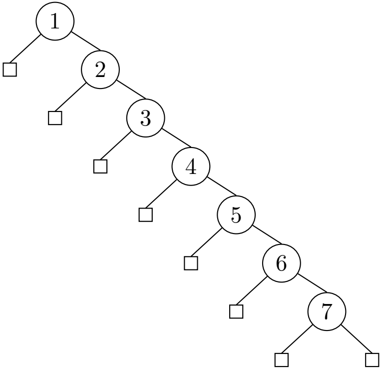

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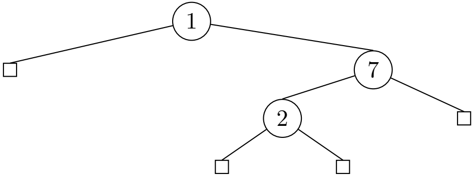

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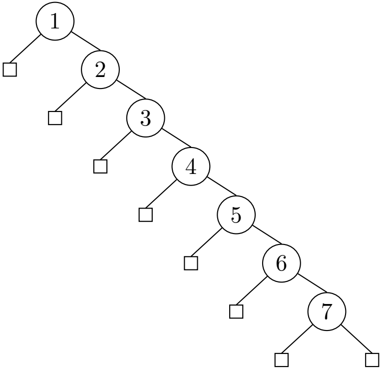

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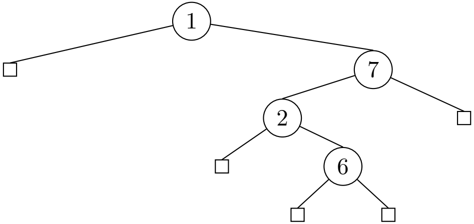

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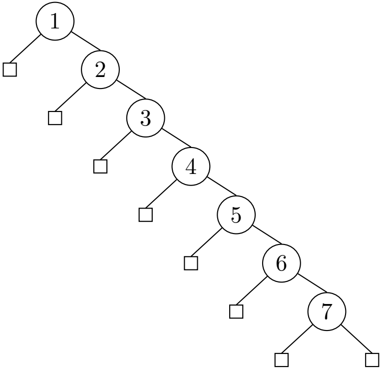

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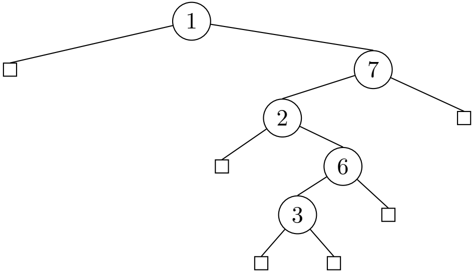

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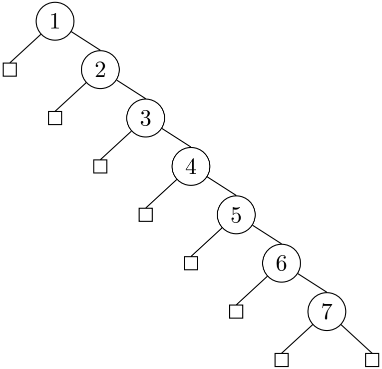

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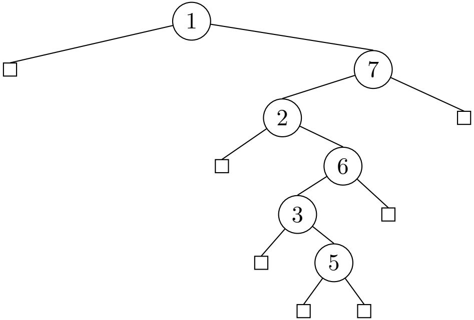

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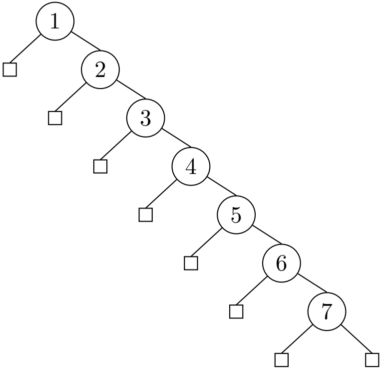

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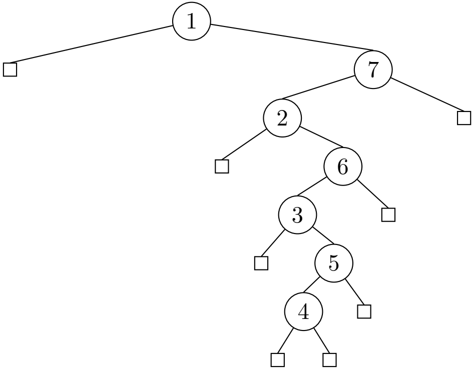

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We want to enforce a set of well-defined conditions. We can achieve this by adding additional member variables in our `Node` struct.

Definition and Properties

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(i) *Color:* Every node is either ***red*** or ***black***

```
typedef enum Color { RED, BLACK };

typedef struct Node {
    Color color;
    int data;
    Node *left;
    Node *right;
    Node *parent;
} Node;
```

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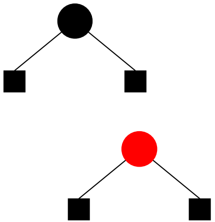

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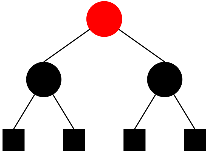

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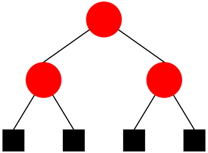

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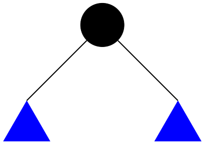

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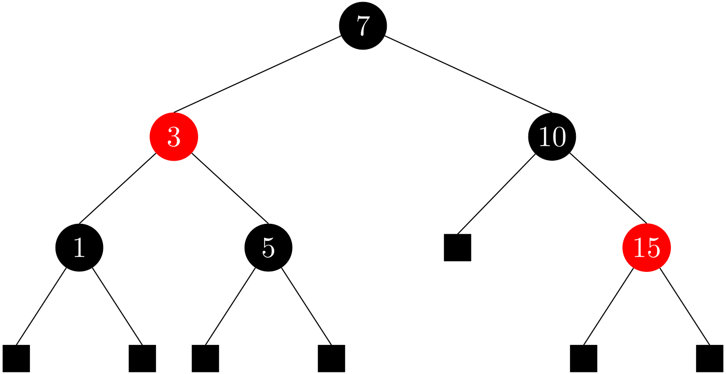

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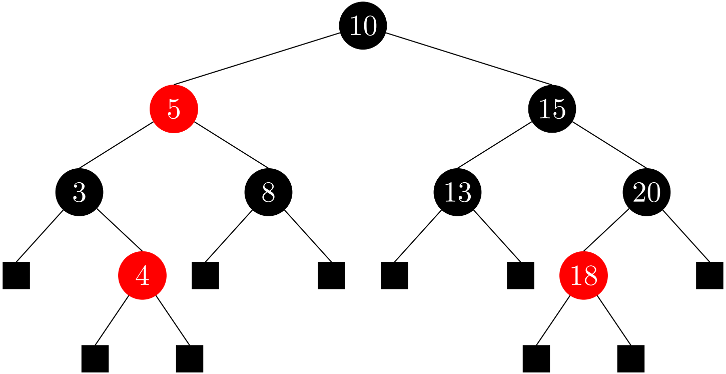

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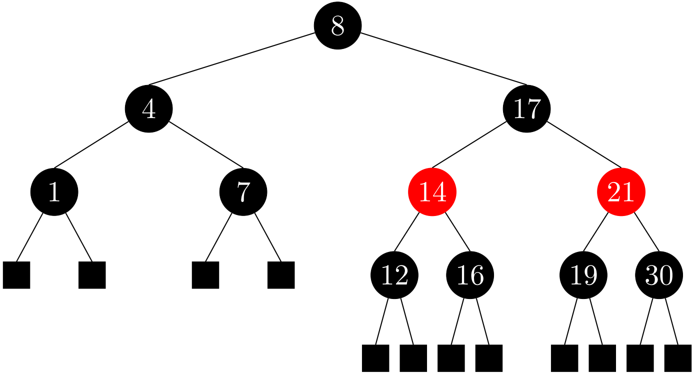

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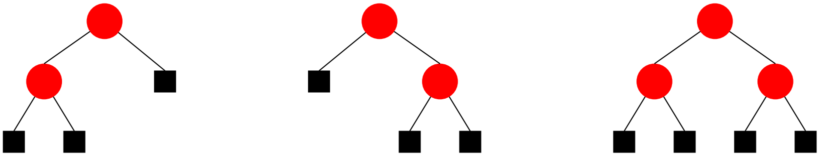
(iv) Recursively fix violations upward.

Double Red Violations

Recall *Property (iii): A **red** node does not have a **red** child.* All of the diagrams shown below are examples of ***invalid red-black*** trees.



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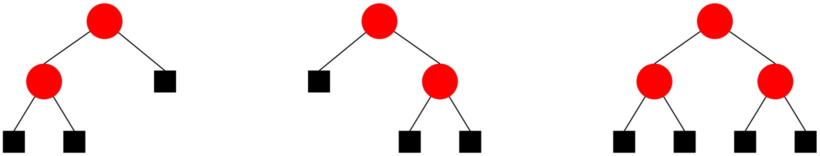

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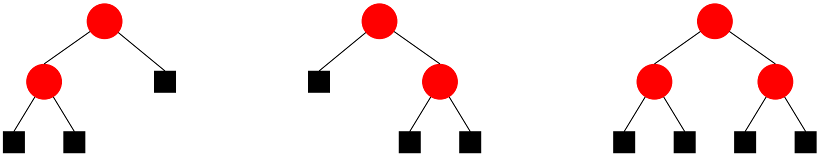
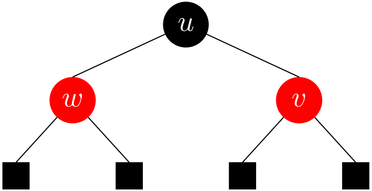


There are two cases:

(i) ***Recolor:*** If both the *parent* and *uncle* are ***red***, perform a *recolor*.

Recolor



where

z is the new node

v is z's parent

u is z's grandparent

w is z's uncle

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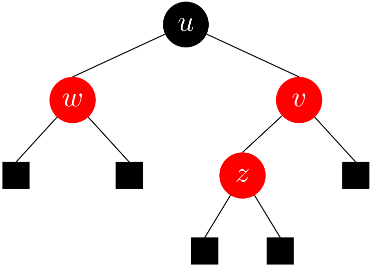

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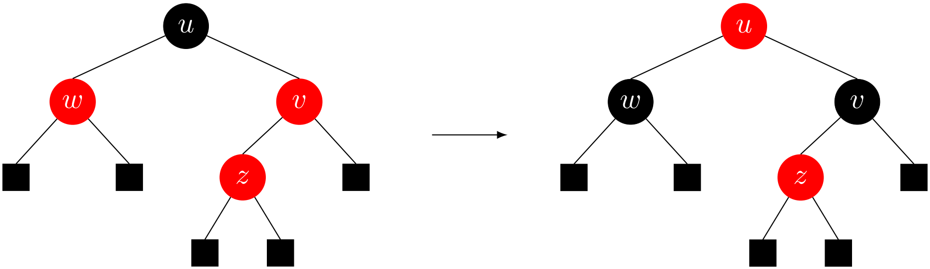

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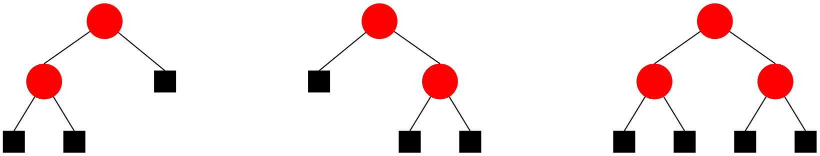


There are two cases:

(i) ***Recolor:*** If both the *parent* and *uncle* are ***red***, perform a *recolor*.

(ii) ***Restructure:*** If the *parent* is ***red*** but the *uncle* is ***black***, perform a *tri-node restructure*.

Tri-Node Restructure

There are four cases:

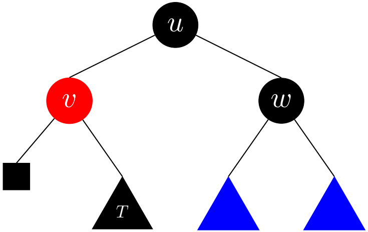
(i) Left-Left

(ii) Right-Right

(iii) Left-Right

(iv) Right-Left

Case: Left-Left



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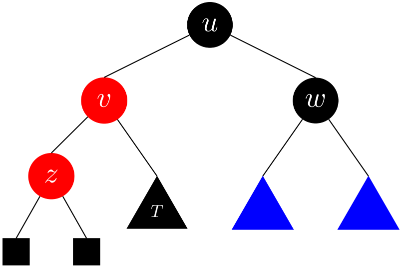

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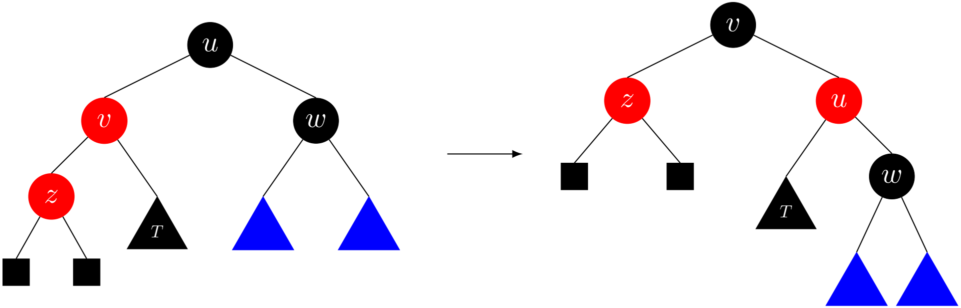

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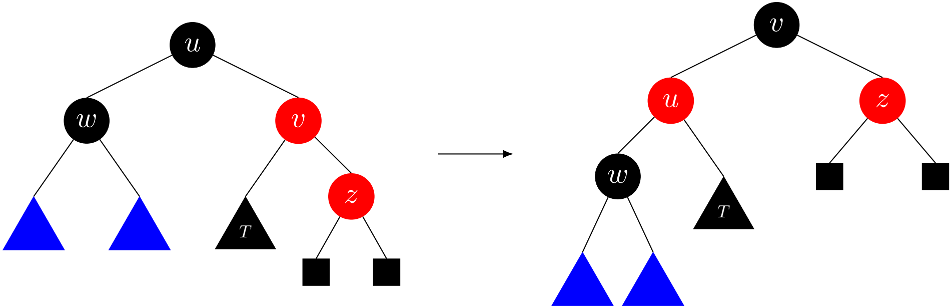

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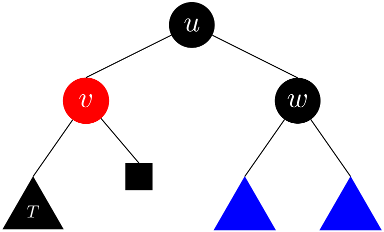

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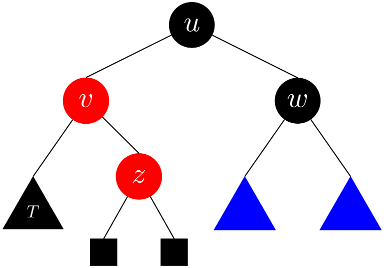

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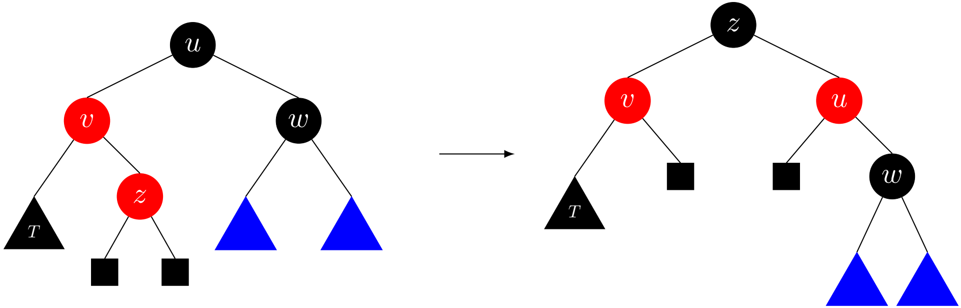
Case: Left-Right



where

z is the new node

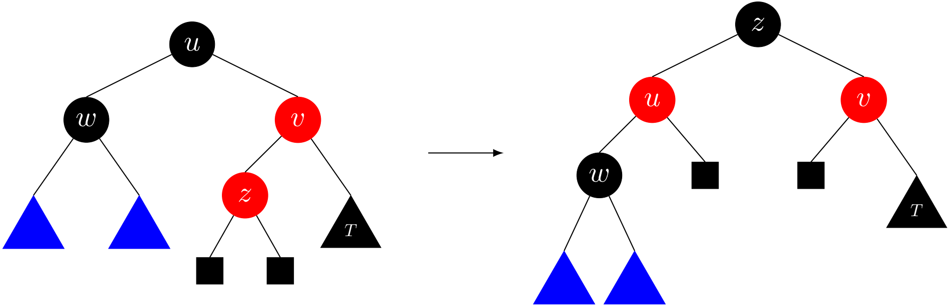
v is z's parent

u is z's grandparent

w is z's uncle

T is v's subtree

Case: Right-Left



where

z is the new node

v is z's parent

u is z's grandparent

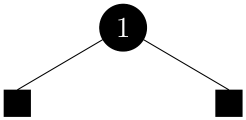
w is z's uncle

T is v's subtree

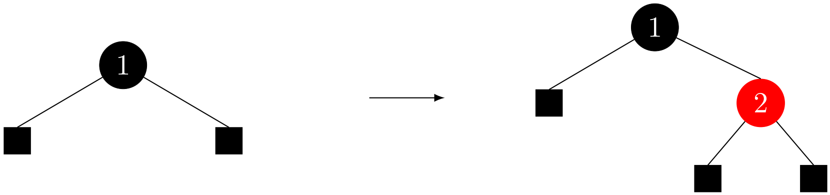
Time and Space Complexities

Insertion: $\mathcal{O}(\log n)$
Deletion: $\mathcal{O}(\log n)$
Search: $\mathcal{O}(\log n)$

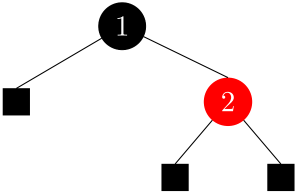
Red-Black Tree: Example



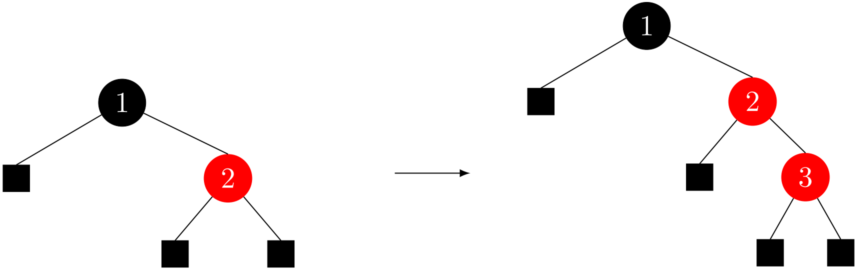
Red-Black Tree: Example



Red-Black Tree: Example

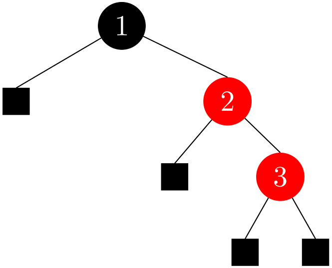


Red-Black Tree: Example



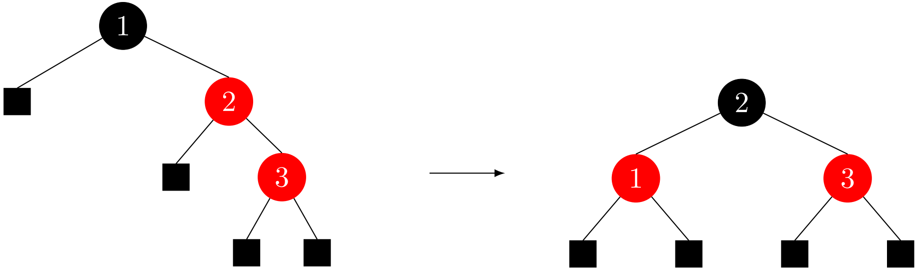
Case: Right-Right

Red-Black Tree: Example



Case: Right-Right

Red-Black Tree: Example



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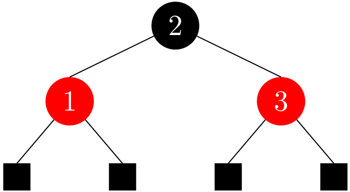

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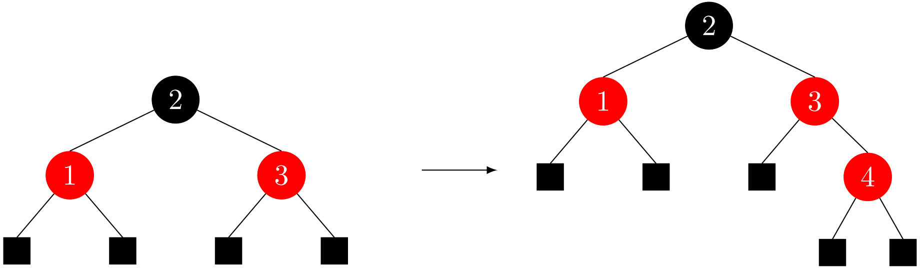
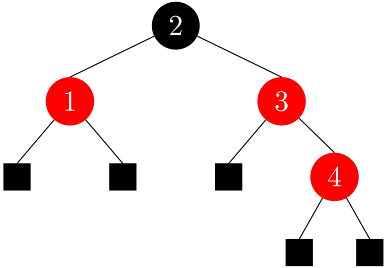


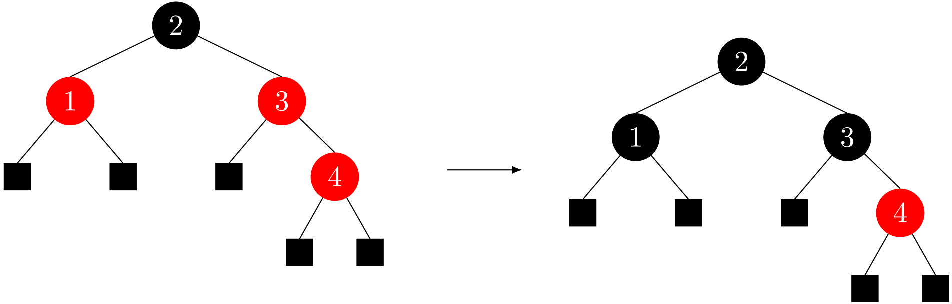
Case: Recolor

Red-Black Tree: Example



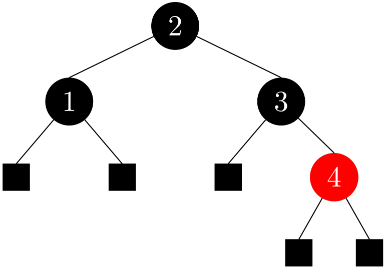
Case: Recolor

Red-Black Tree: Example



Case: Recolor

Red-Black Tree: Example



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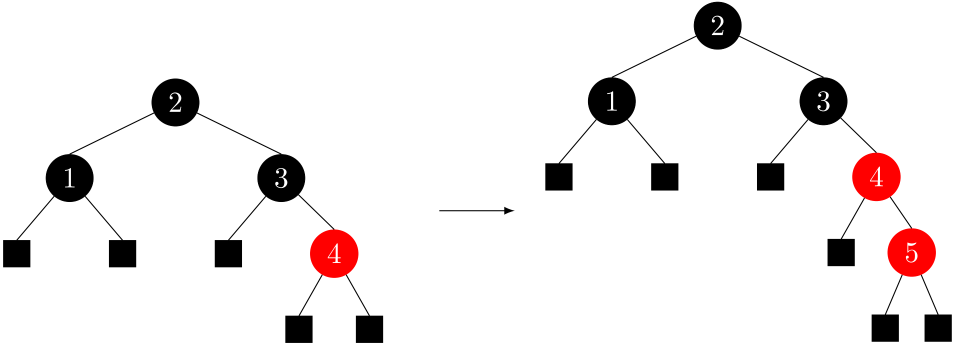
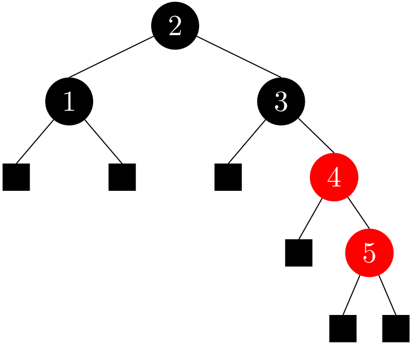


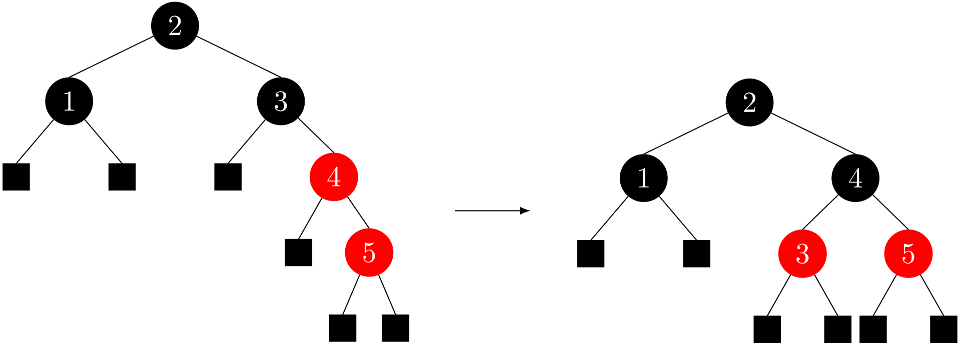
Case: Right-Right

Red-Black Tree: Example



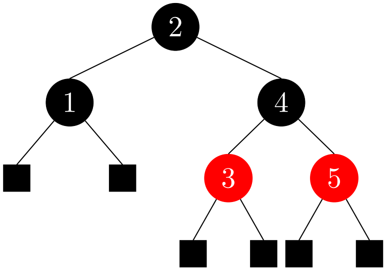
Case: Right-Right

Red-Black Tree: Example



Case: Right-Right

Red-Black Tree: Example



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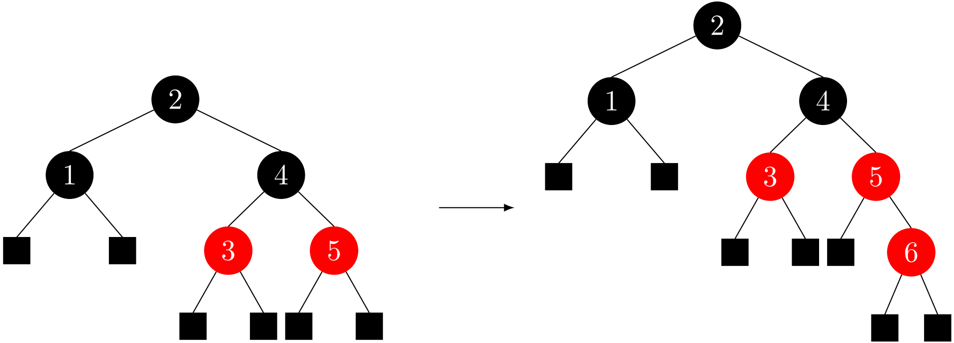

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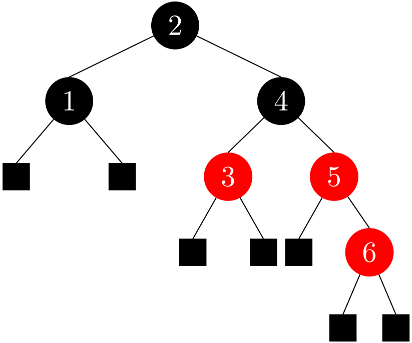
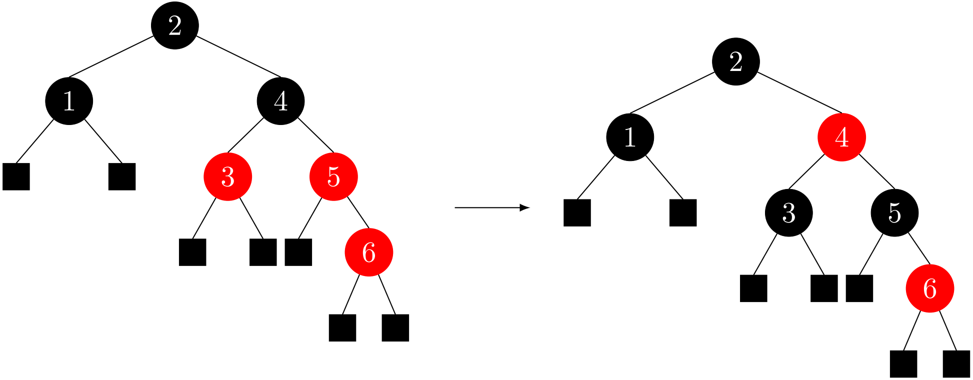
Red-Black Tree: Example



Case: Recolor

Red-Black Tree: Example



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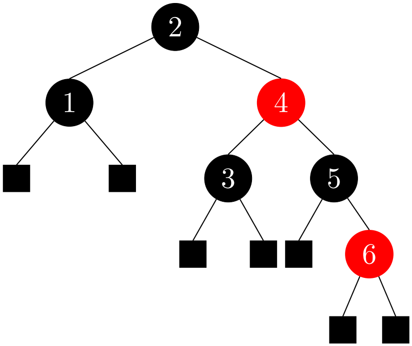
Red-Black Tree: Example



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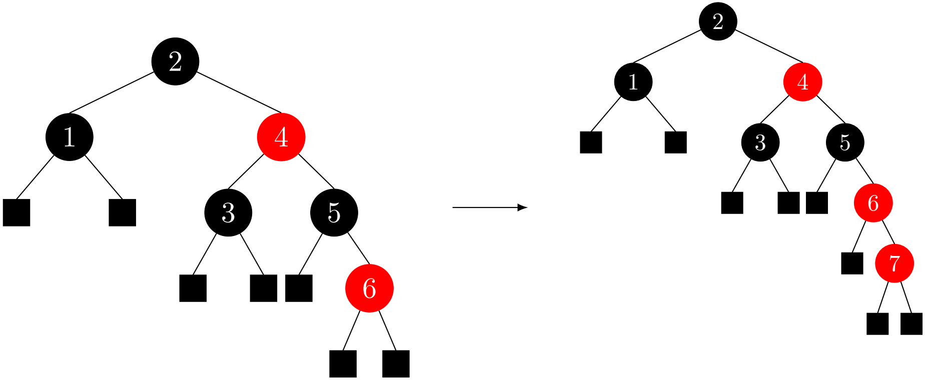
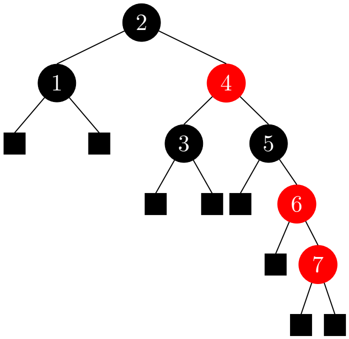
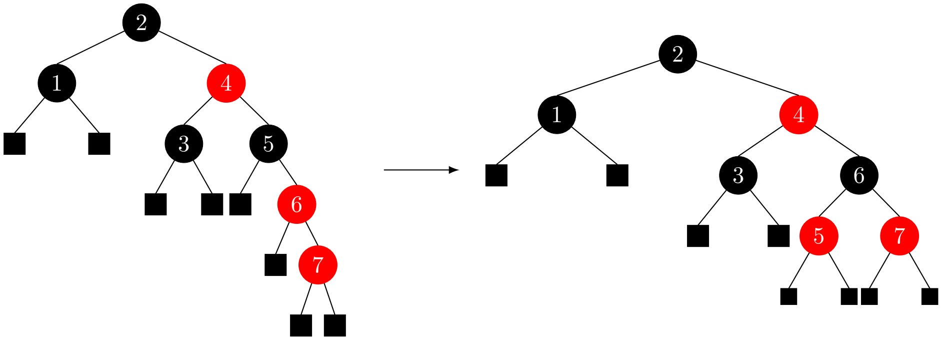


Case: Right-Right

Red-Black Tree: Example



Red-Black Tree: Example



End

Thank you!

Appendix

Below are slides that didn't make the cut :(

Corollaries

Proposition

If a node n has exactly one child, c, then **(a)** *c is* ***red*** *,* **(b)** *n is* ***black****, and* **(c)** *c has no children.*

Proof. Suppose we have a valid red-black tree. Consider a node n with exactly one child. Without loss of generality, choose n's left node to be the child and call it c.

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(a) n passes through no ***black*** nodes on the right side by assumption. If c were ***black***, then n would pass through 1 ***black*** node, a contradiction since this violates the *depth property*.

(b) By **(a)**, n's child is ***red*** and by the *internal property*, n is ***black***.

(c) Since n passes through no ***black*** nodes on the right side by assumption, n cannot pass through any ***black*** nodes on the left side by the *the depth property*. Then, since c is ***red*** by **(a)**, c has only nil nodes, which are ***black*** by the *external property*.

□