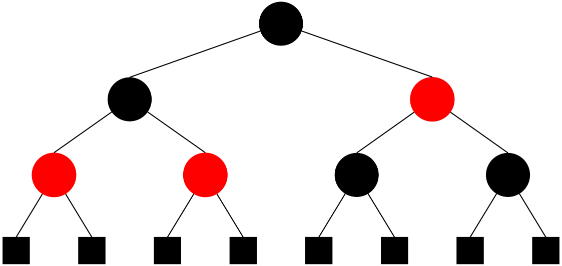
Balanced Trees (***Red*-*Black*** Trees)

Warren Kim

Quick Definition

Definition

A ***red-black*** *tree is a type of* ***self-balancing*** *binary search tree that guarantees* $\mathcal{O}(\log n)$ *performance.*



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$\rightarrow$ Graph algorithm optimizations (for AI/ML)[e.g. K-mean clustering]

$\rightarrow$ Priority Queues (e.g. Range Queries)

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$\rightarrow$ We want to ensure $\mathcal{O}(\log n)$ performance.

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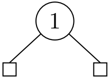

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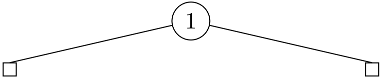

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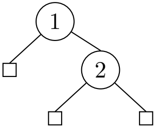

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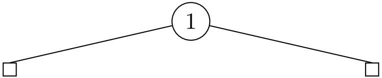

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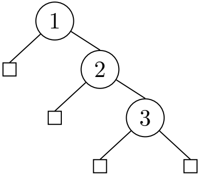

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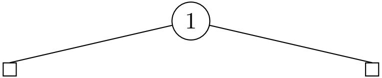

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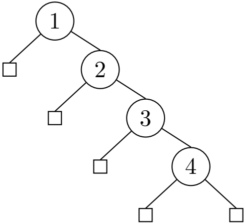

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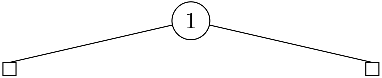

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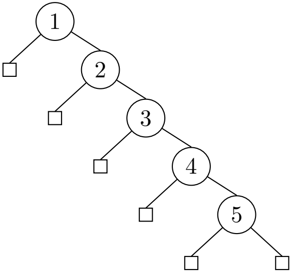

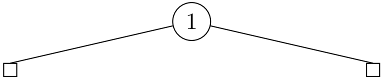

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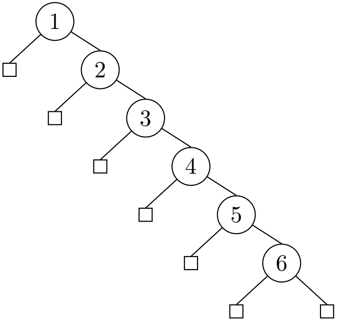

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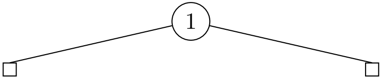

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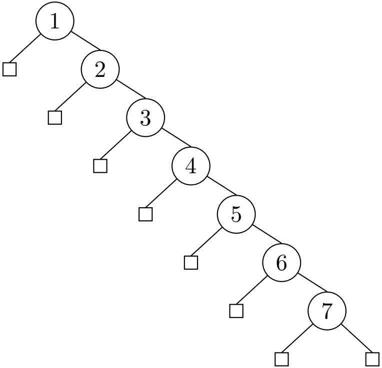

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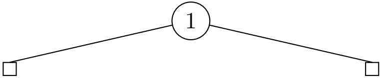

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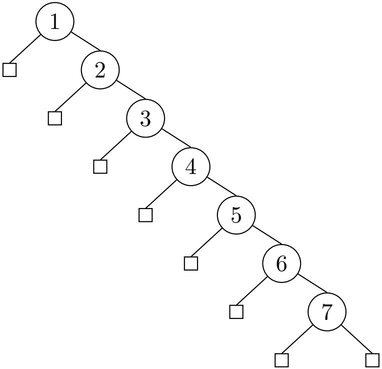

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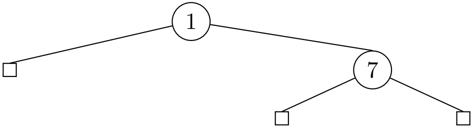

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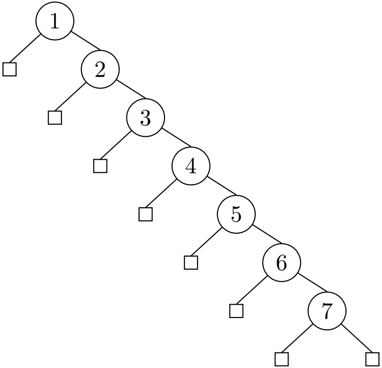

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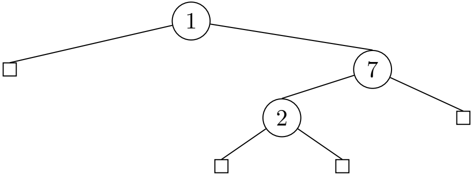

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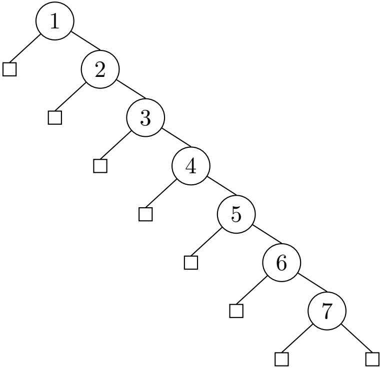

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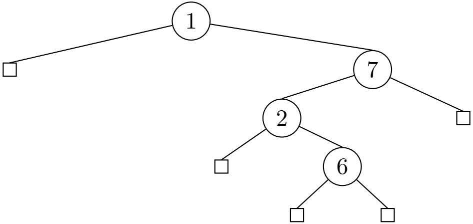

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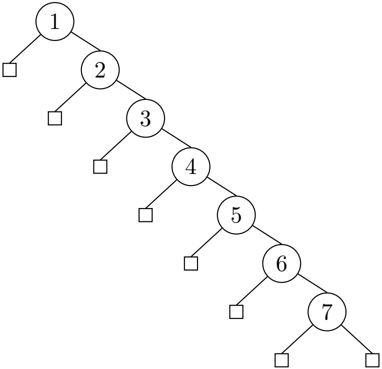

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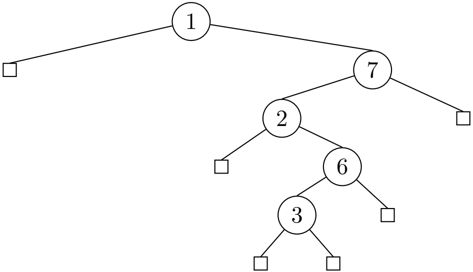

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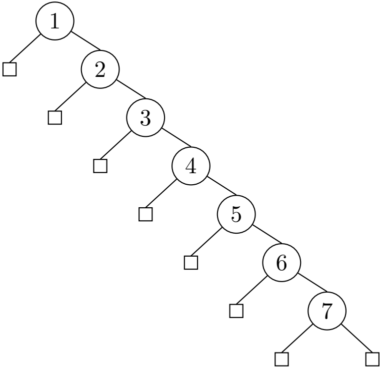

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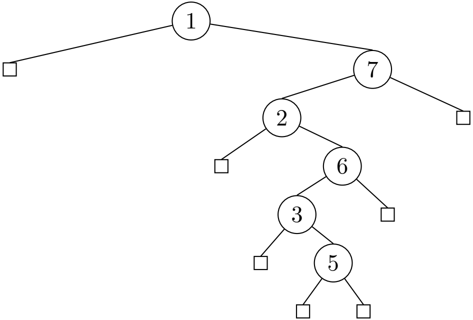

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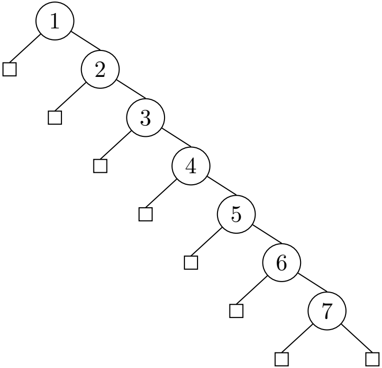

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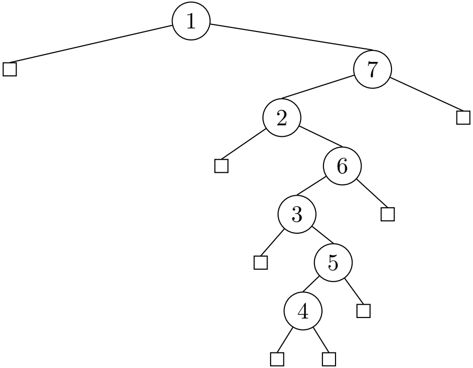

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$\rightarrow$ We can define a set of conditions that enforce balance.

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Definition and Properties

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*A **red-black** tree is a type of **self-balancing** binary search tree that guarantees $\mathcal{O}(\log n)$ search, insertion, and deletion operations with the following properties:*

(i) *Color:* Every node is either ***red*** or ***black***

```
typedef enum Color { RED, BLACK };

struct Node {
    Color color;
    int data;
    Node *left;
    Node *right;
    Node *parent;
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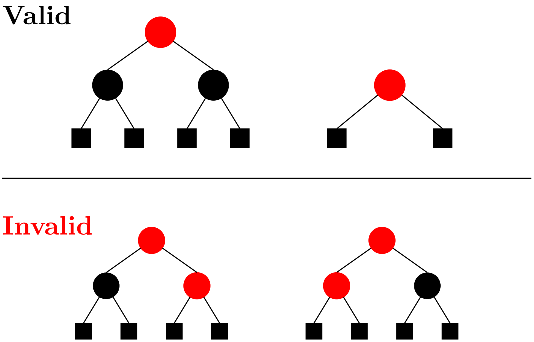

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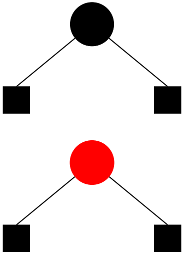

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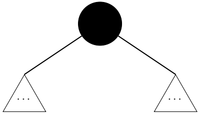

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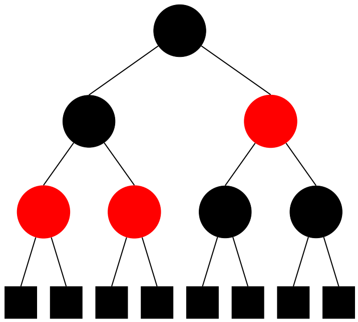

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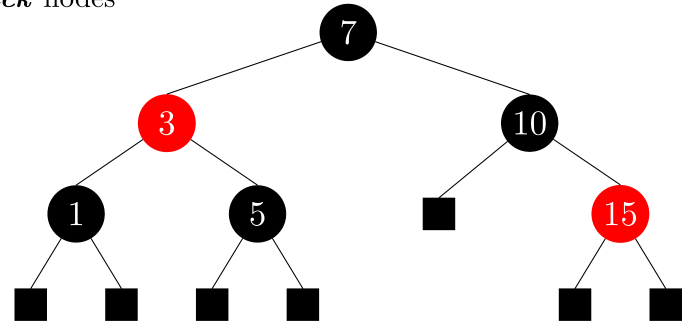

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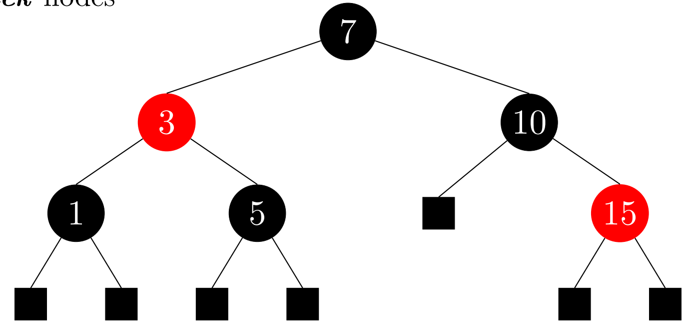

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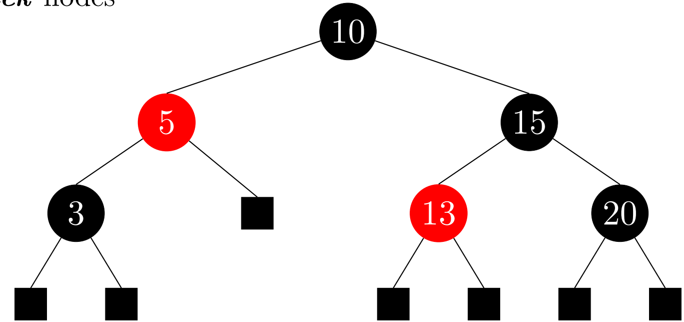

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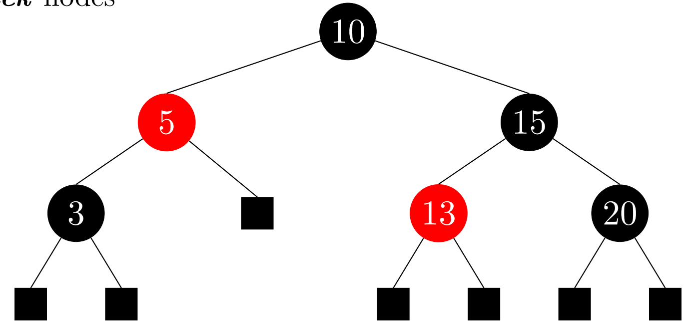

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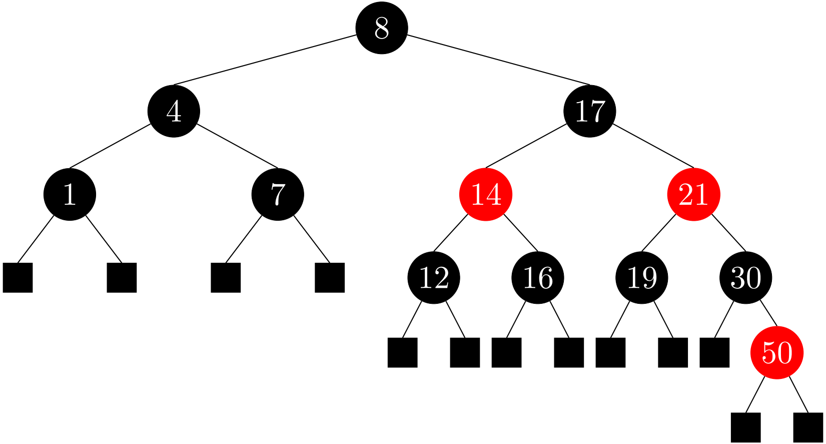

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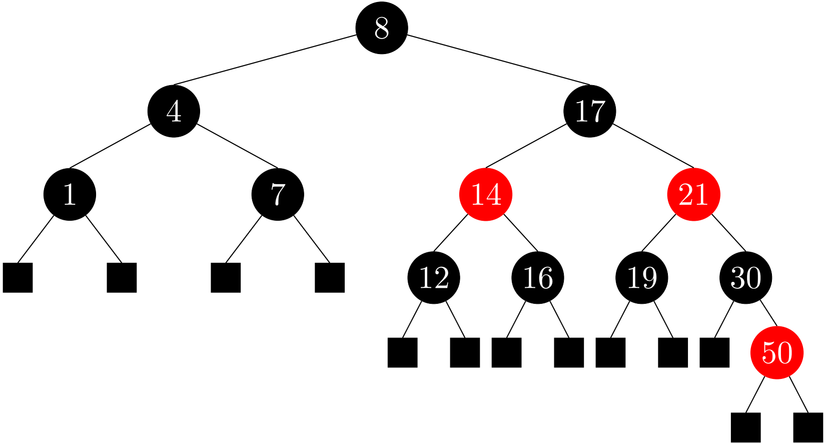

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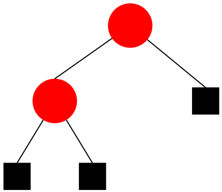
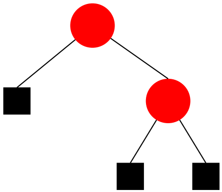
(iv) Recursively fix violations upward.

Double Red Violations

Recall *Property (ii): A* ***red*** *node does not have a* ***red*** *child.* All of the diagrams shown below are examples of **invalid** ***red-black*** trees.





Fixing Double Red Violations

Terminology

With respect to node z,

→ *Parent* (p): z's direct parent

→ *Uncle* (u): p's sibling

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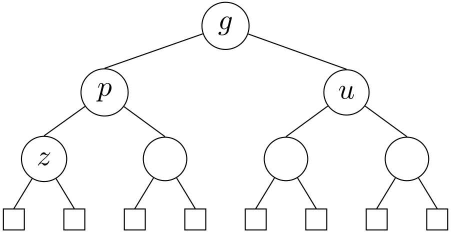

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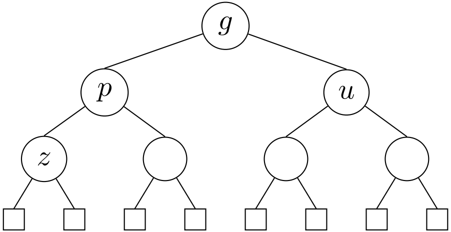

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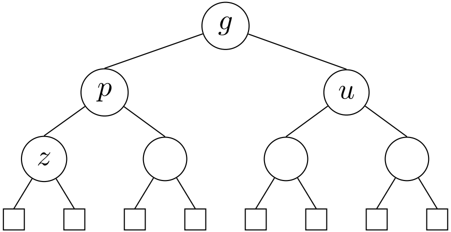

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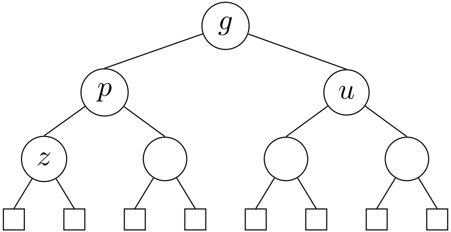

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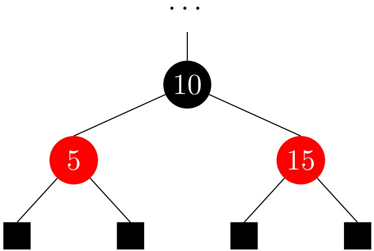
(i) ***Recolor:*** If both the *parent* and *uncle* are ***red***, perform a *recolor*.

(ii) ***Restructure:*** If the *parent* is ***red*** but the *uncle* is ***black***, perform a *tri-node restructure*.

Recolor



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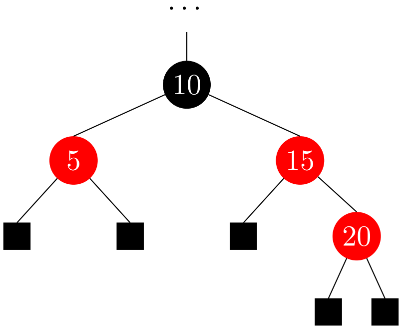

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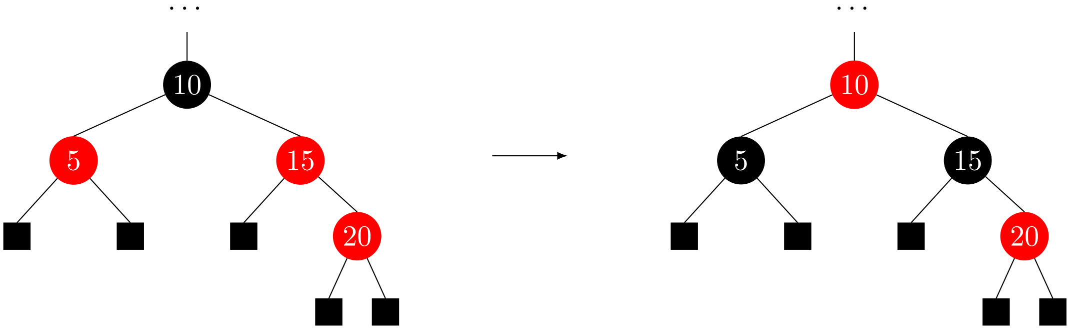
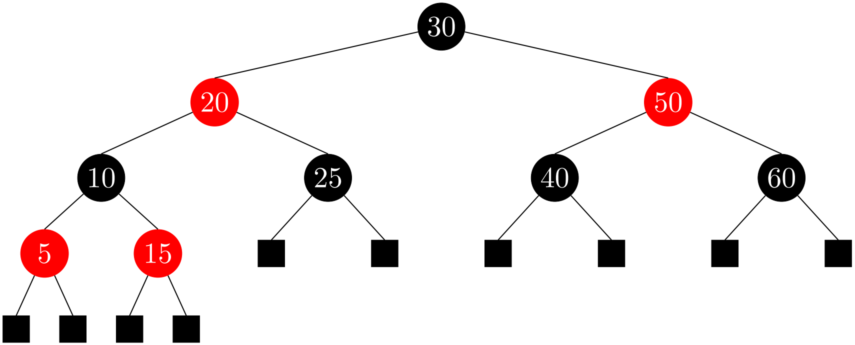
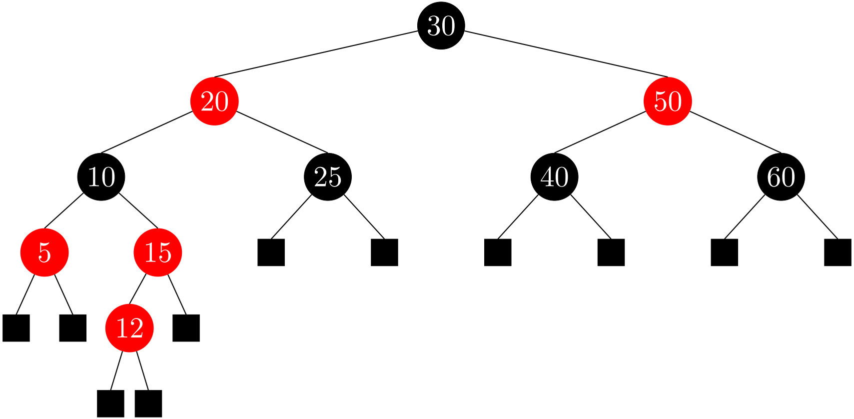


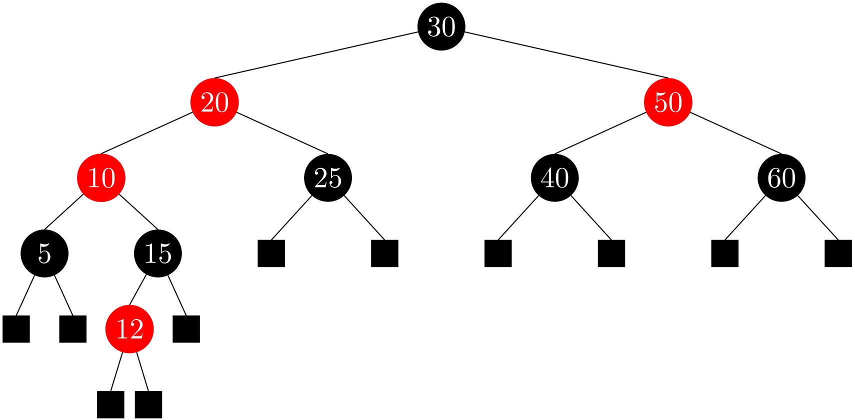
Recolor (Recursive)



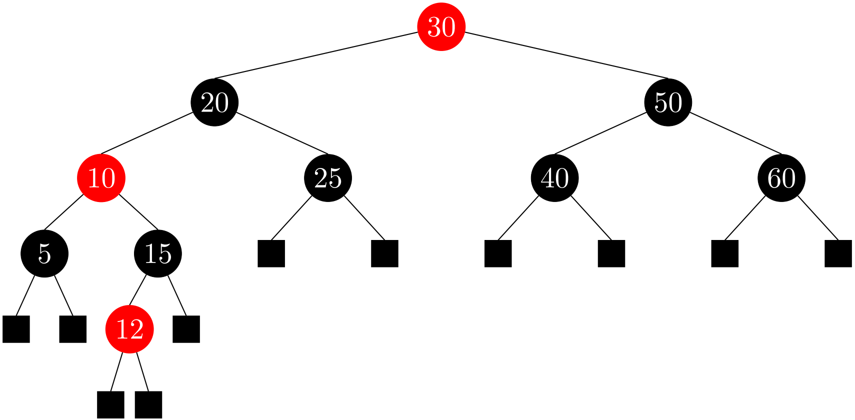
Recolor (Recursive)



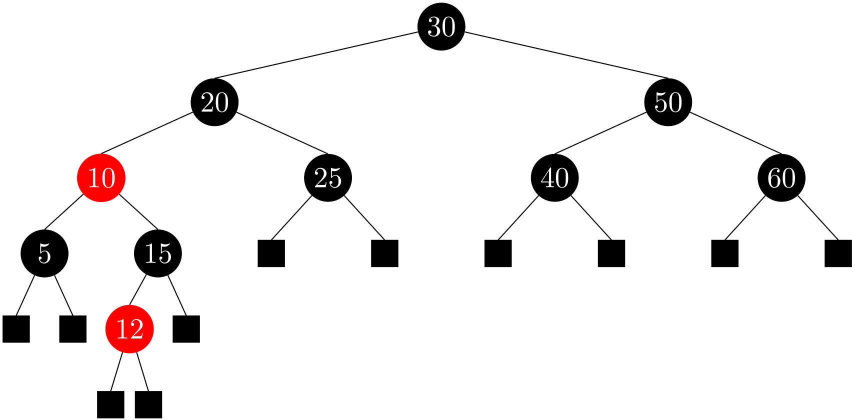
Recolor (Recursive)



Recolor (Recursive)



Recolor (Recursive)



Tri-Node Restructure

There are four cases:

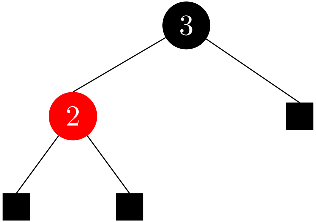
(i) Left-Left

(ii) Right-Right

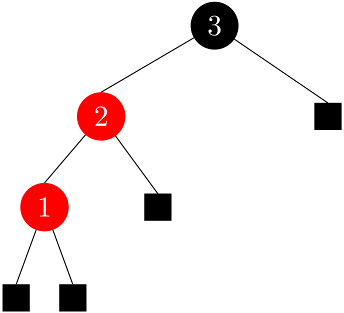
(iii) Left-Right

(iv) Right-Left

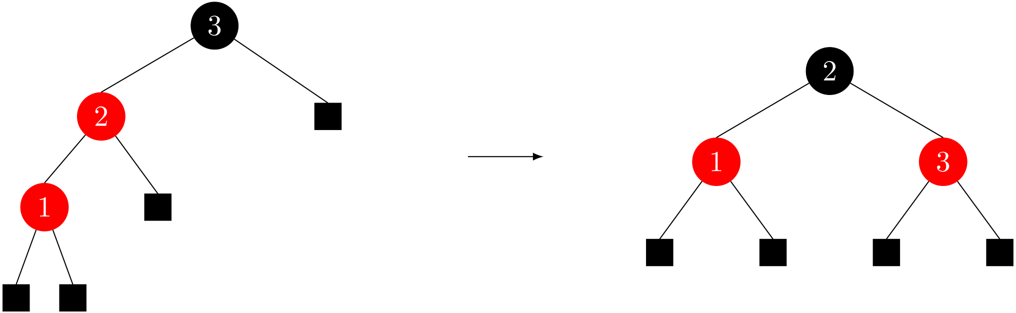
Case: Left-Left (Simple)



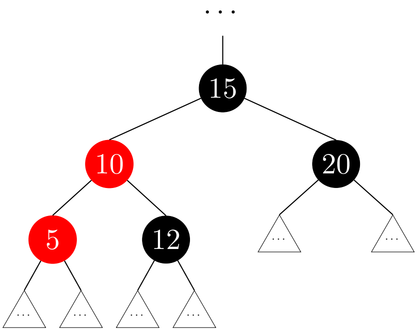
Case: Left-Left (Simple)



Case: Left-Left (Simple)

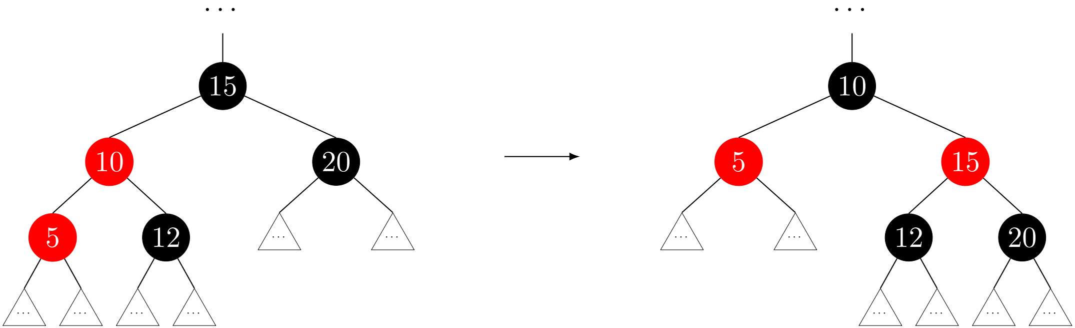


Case: Left-Left (General)



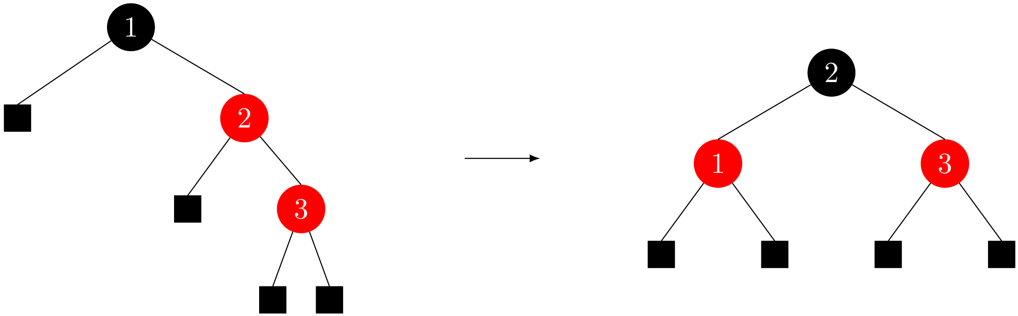
Here,  represents a subtree and $\cdots$ represents the rest of the tree.

Case: Left-Left (General)

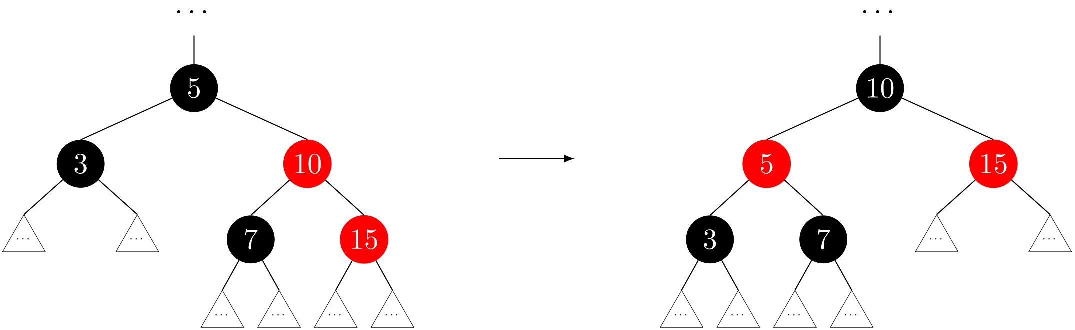


Here,  represents a subtree and $\cdots$ represents the rest of the tree.

Case: Right-Right (Simple)

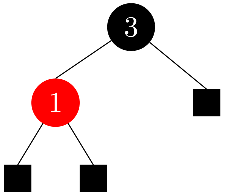


Case: Right-Right (General)

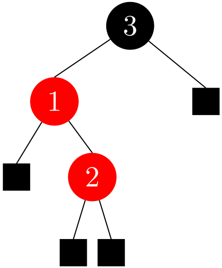


Here,  represents a subtree and $\cdots$ represents the rest of the tree.

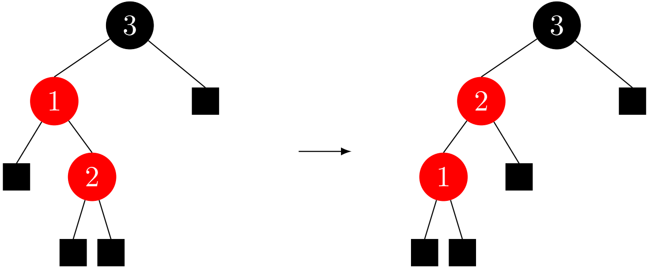
Case: Left-Right (Simple)



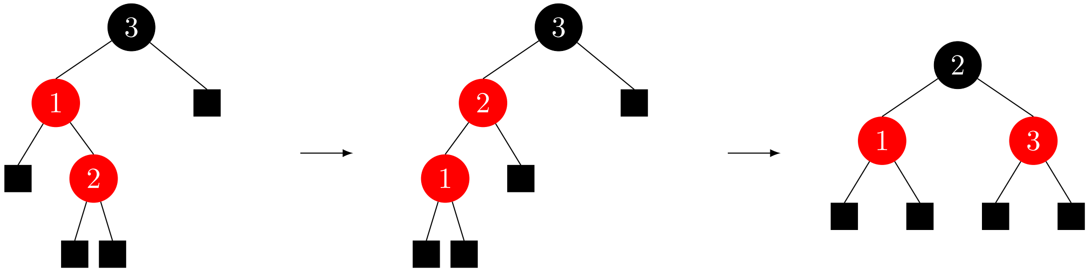
Case: Left-Right (Simple)



Case: Left-Right (Simple)

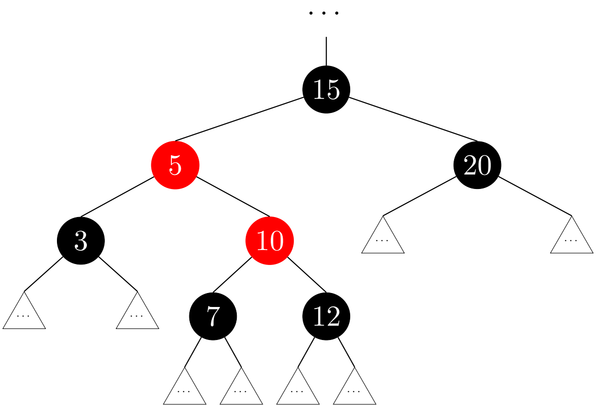


Case: Left-Right (Simple)



Case: Left-Right (General)

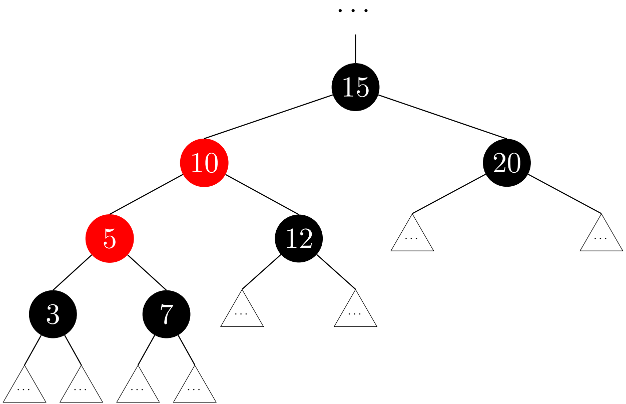
Step 1



Here,  represents a subtree and $\cdots$ represents the rest of the tree.

Case: Left-Right (General)

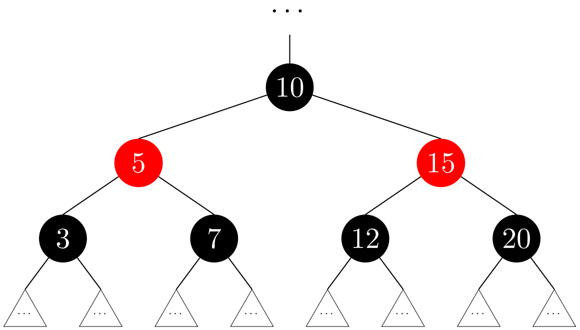
Step 2



Here,  represents a subtree and $\cdots$ represents the rest of the tree.

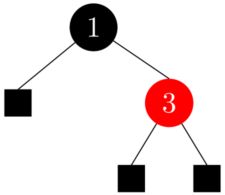
Case: Left-Right (General)

Step 3

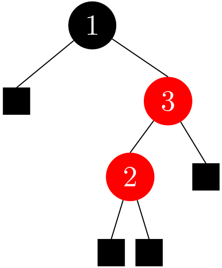


Here,  represents a subtree and $\cdots$ represents the rest of the tree.

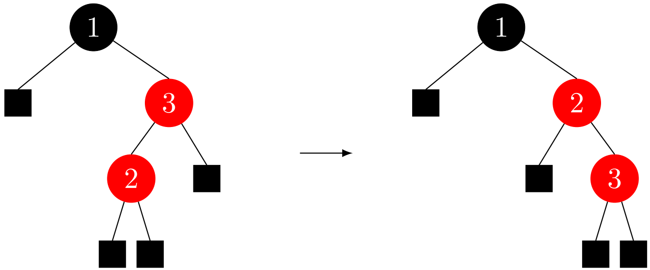
Case: Right-Left (Simple)



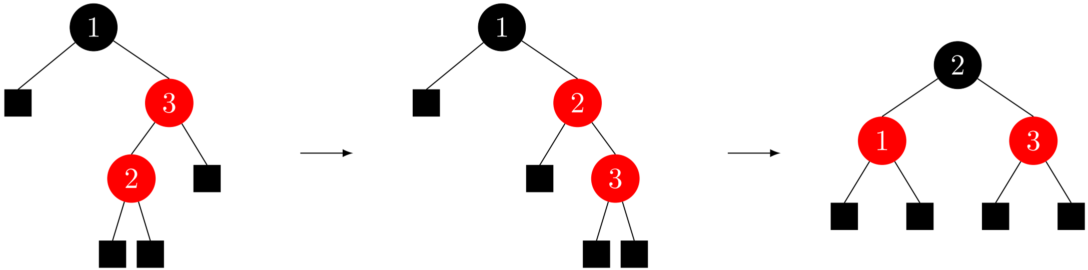
Case: Right-Left (Simple)



Case: Right-Left (Simple)

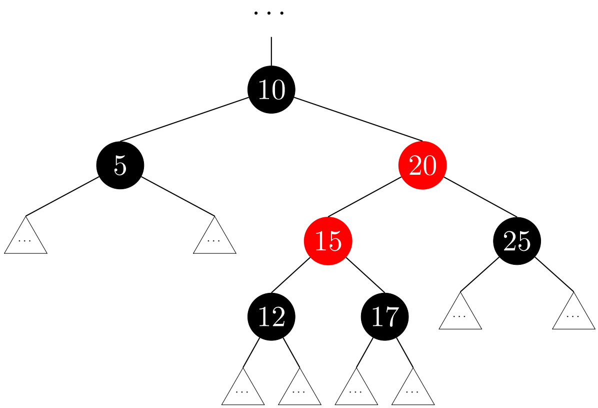


Case: Right-Left (Simple)



Case: Right-Left (General)

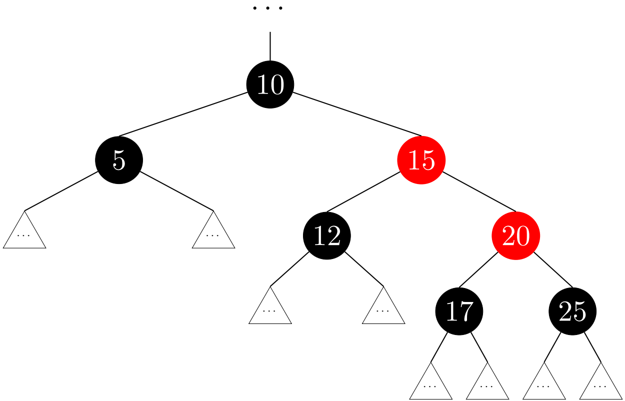
Step 1



Here, △ represents a subtree and $\cdots$ represents the rest of the tree.

Case: Right-Left (General)

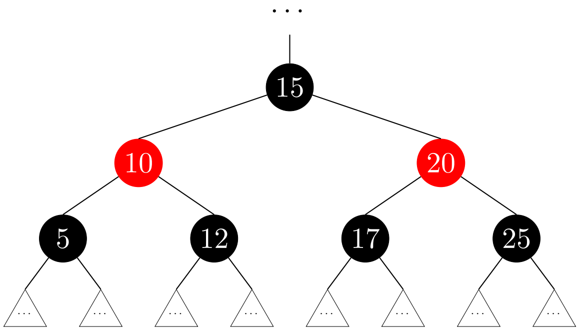
Step 2



Here,  represents a subtree and $\cdots$ represents the rest of the tree.

Case: Right-Left (General)

Step 3

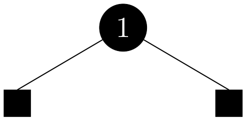


Here,  represents a subtree and $\cdots$ represents the rest of the tree.

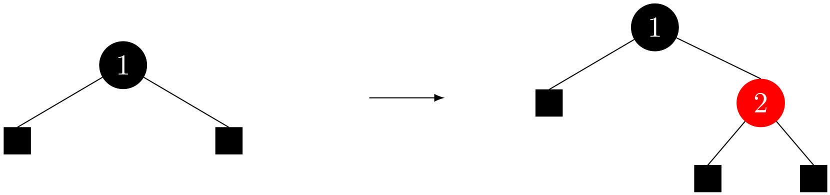
Time and Space Complexities

Insertion: $\mathcal{O}(\log n)$
Deletion: $\mathcal{O}(\log n)$
Search: $\mathcal{O}(\log n)$

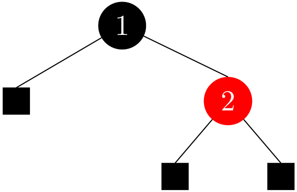
Red-Black Tree: Example



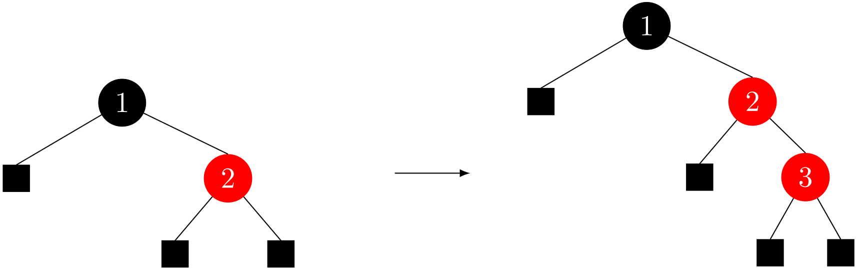
Red-Black Tree: Example



Red-Black Tree: Example

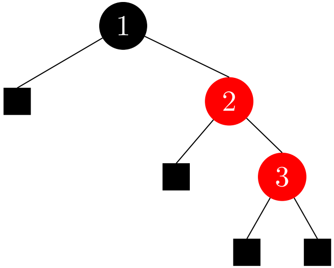


Red-Black Tree: Example



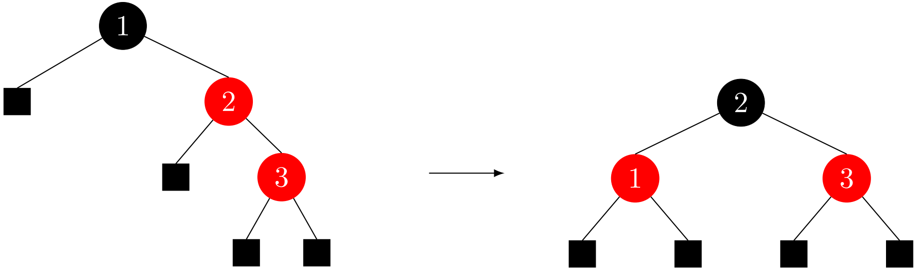
Case: Right-Right

Red-Black Tree: Example



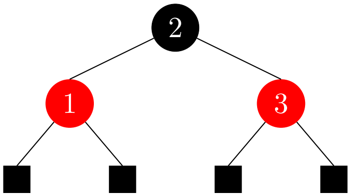
Case: Right-Right

Red-Black Tree: Example

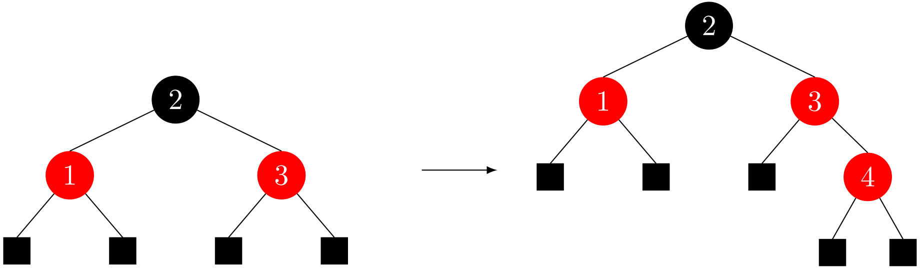


Case: Right-Right

Red-Black Tree: Example

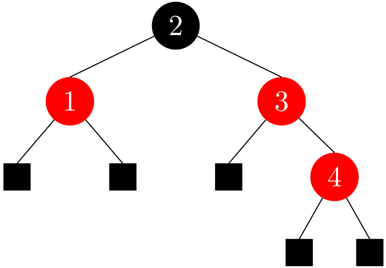


Red-Black Tree: Example



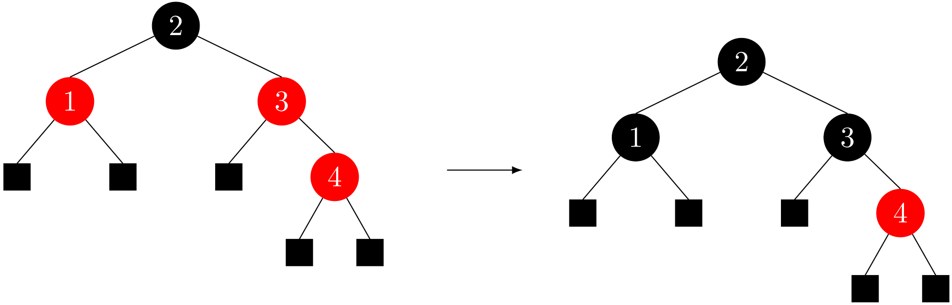
Case: Recolor

Red-Black Tree: Example



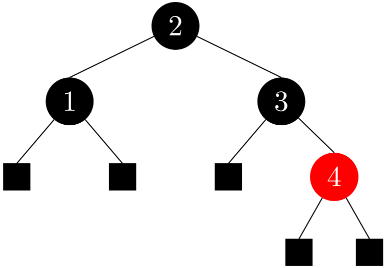
Case: Recolor

Red-Black Tree: Example

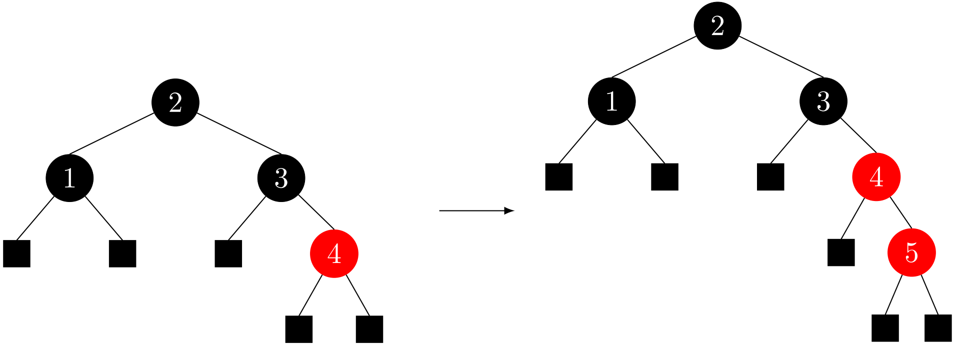


Case: Recolor

Red-Black Tree: Example

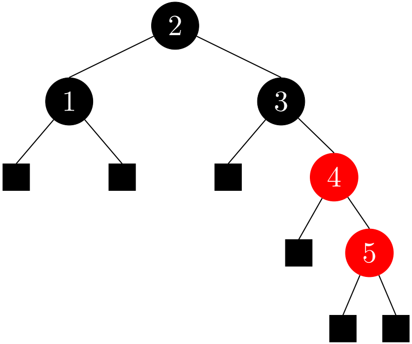


Red-Black Tree: Example



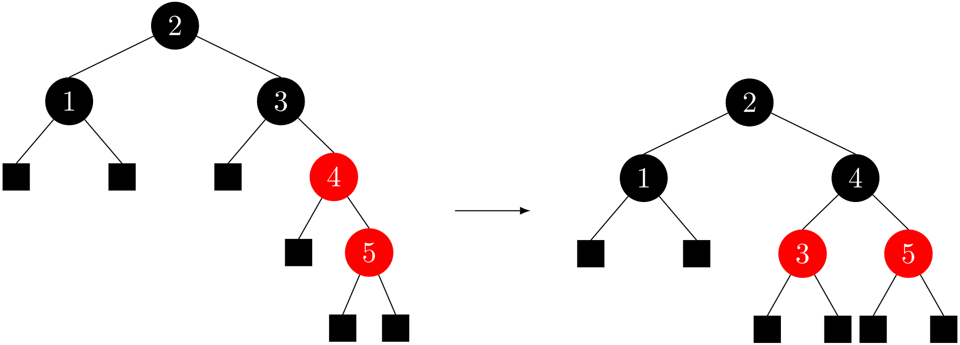
Case: Right-Right

Red-Black Tree: Example



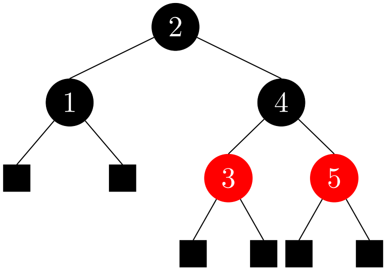
Case: Right-Right

Red-Black Tree: Example

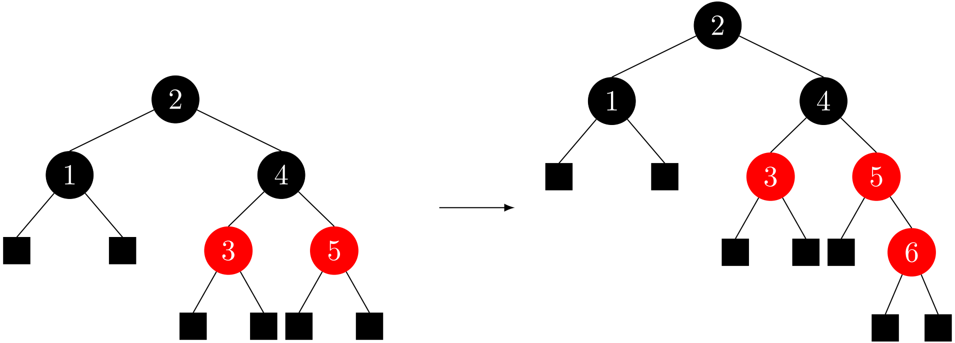


Case: Right-Right

Red-Black Tree: Example

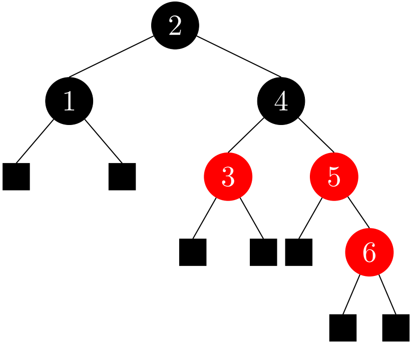


Red-Black Tree: Example



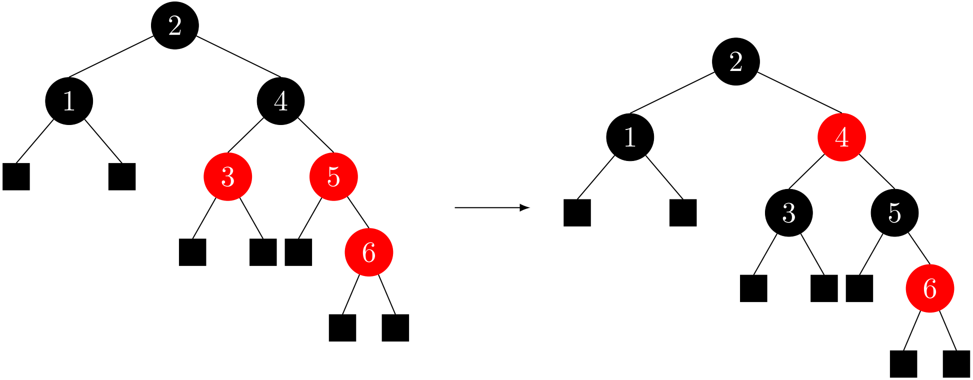
Case: Recolor

Red-Black Tree: Example



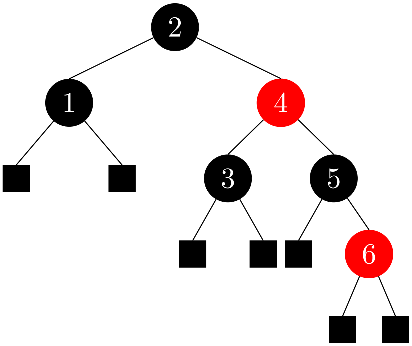
Case: Recolor

Red-Black Tree: Example

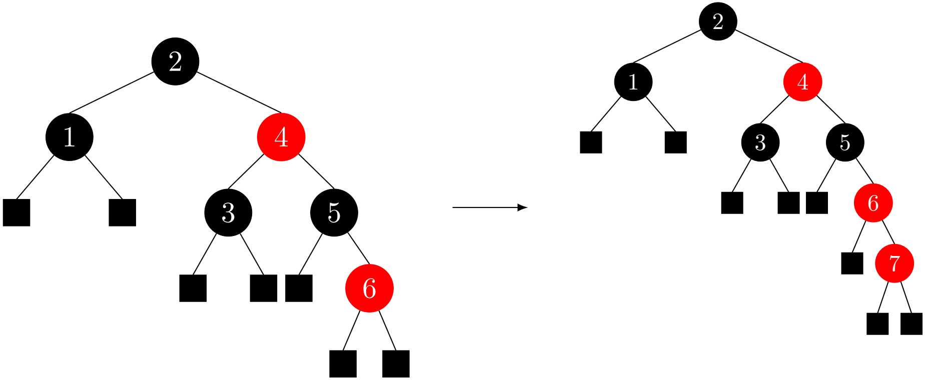


Case: Recolor

Red-Black Tree: Example

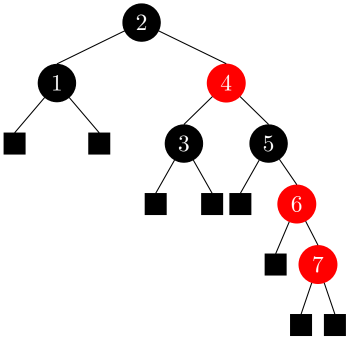


Red-Black Tree: Example

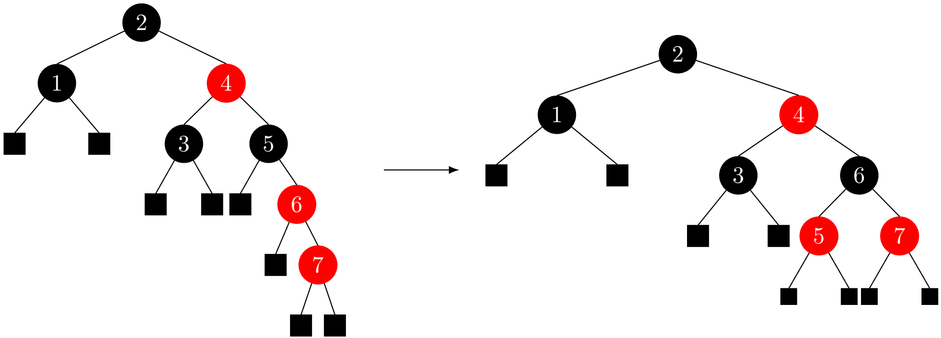


Case: Right-Right

Red-Black Tree: Example



Red-Black Tree: Example



End

Thank you!

Appendix

Below are slides that didn't make the cut.

Corollaries

Proposition

If a node z has exactly one child, c, then **(a)** *c is* ***red****,* **(b)** *z is* ***black****, and* **(c)** *c has no children.*

Proof. Suppose we have a valid ***red-black*** tree. Consider a node z with exactly one child. Without loss of generality, choose z's left node to be the child and call it c.

(a) z passes through no ***black*** nodes on the right side by assumption. If c were ***black***, then z would pass through 1 ***black*** node, a contradiction since this violates the *depth property*.

(b) By **(a)**, z's child is ***red*** and by the *internal property*, z is ***black***.

(c) Since z passes through no ***black*** nodes on the right side by assumption, z cannot pass through any ***black*** nodes on the left side by the *the depth property*. Then, since c is ***red*** by **(a)**, c has only nil nodes □

Height of a Red-Black Tree

Theorem

*A **red-black** tree with n nodes has a height h that is $\mathcal{O}(\log n)$.*

Proof. Suppose we have a ***red-black*** tree with n nodes and height h. Let b be the number of ***black*** nodes on the shortest path from root to any leaf. In the worst case, the longest path alternates between ***red*** and ***black*** nodes and thus has a height of $2b$. Then, h is bounded above by $2b$; that is, $h \leq 2b$. There are $2^b - 1 \leq n$ nodes in this tree. Solving for b, we get $b \leq \log(n+1)$. Substituting b, we get $b \leq \log(n+1) \leq h \leq 2b \leq 2\log(n+1)$ so h is bounded below by $\log(n+1)$ and above by $2\log(n+1)$; that is, $\log(n+1) \leq h \leq 2\log(n+1)$. So, h is $\mathcal{O}(\log n)$. $\square$