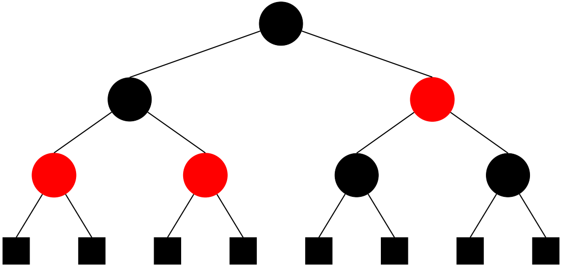
Balanced Trees (***Red*-*Black*** Trees)

Warren Kim

Quick Definition

Definition

A ***red-black*** *tree is a type of* ***self-balancing*** *binary search tree that guarantees* $\mathcal{O}(\log n)$ *performance.*



Applications

Red*-*Black Trees have a variety of applications. Some include:

→ Implementing 2-4 trees[1]. 2-4 Trees $\simeq$ ***Red*-*Black*** Trees.

→ Linux CPU scheduler (Completely Fair Scheduler)

→ Linux Virtual Memory Areas (VMA)

→ STL Data Structures (e.g. C++'s `std::map`, Java's `HashMap`)

→ Graph algorithm optimizations (for AI/ML)[e.g. K-mean clustering]

→ Priority Queues (e.g. Bounded Queries)

[1] 2-4 (sometimes called 2-3-4) trees are a subset of B^+-trees.

Motivation

Why do we want balanced binary trees?

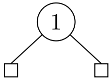
$\rightarrow$ Raw binary search tree performance is highly dependant on input order.

$\rightarrow$ We want to ensure $\mathcal{O}(\log n)$ performance.

Example

Suppose we have the input set $\{1, 2, 3, 4, 5, 6, 7\}$ and consider two input orders:

$\{1, 2, 3, 4, 5, 6, 7\}$



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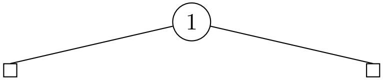

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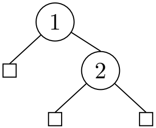

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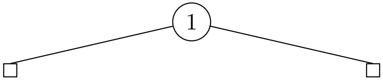

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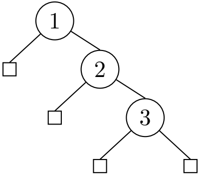

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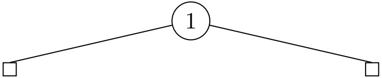

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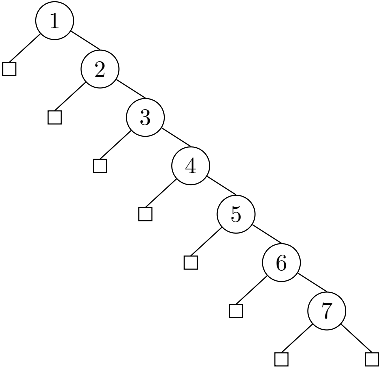

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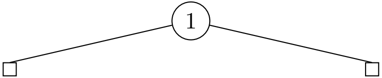

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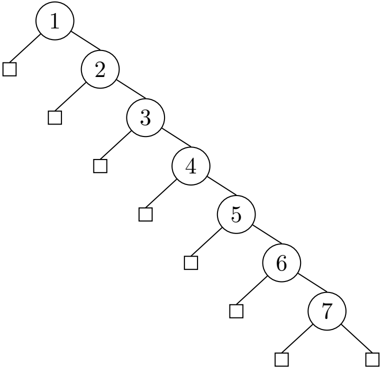

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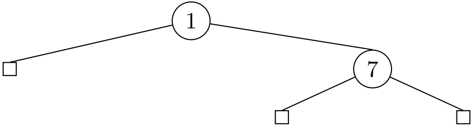

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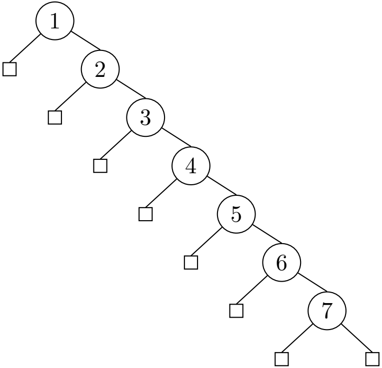

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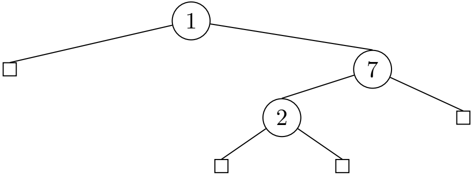

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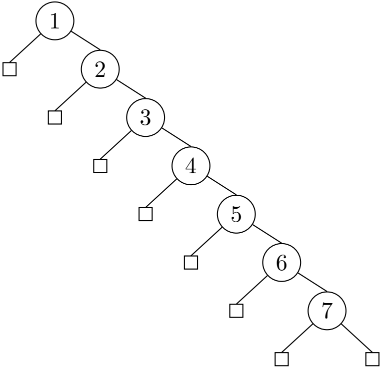

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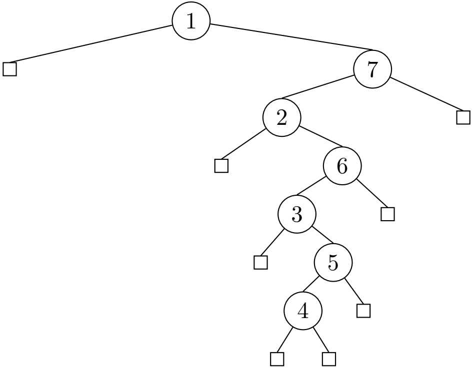

Intuition

How do we balance binary trees?

We can *dynamically* balance the tree!

$\rightarrow$ We can add metadata[2] to our `Node` struct.

$\rightarrow$ We can define a set of conditions that enforce balance.

[2] Metadata: Additional member variables

Definition and Properties

Definition

*A **red-black** tree is a type of **self-balancing** binary search tree that guarantees* $\mathcal{O}(\log n)$ *search, insertion, and deletion operations with the following properties:*

(i) *Color:* Every node is either ***red*** or ***black***

```
enum Color { RED, BLACK };

struct Node {
    Color color;
    Node *left;
    Node *right;
    Node *parent;

    int data;
};
```

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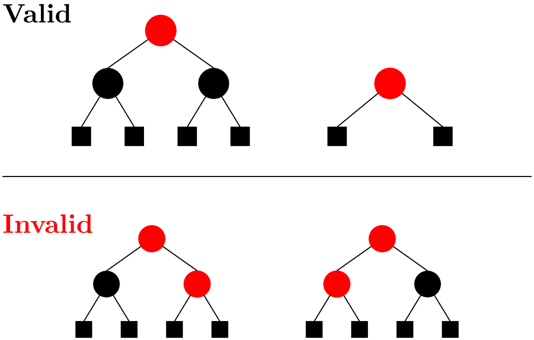

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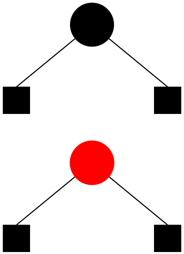

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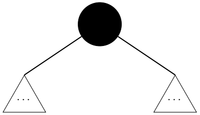

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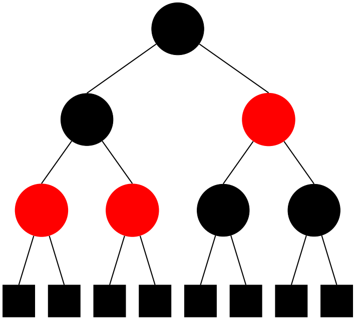

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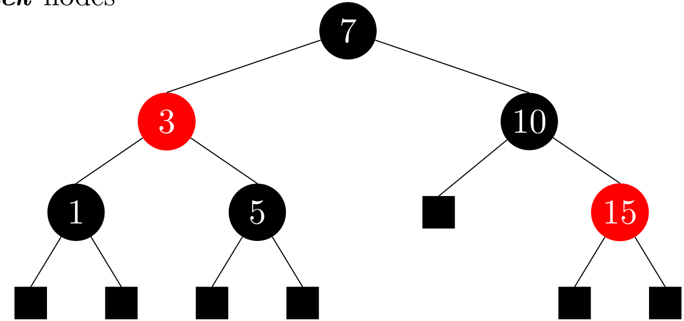

Valid

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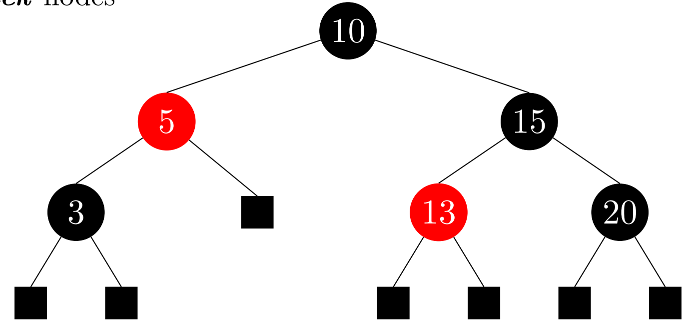

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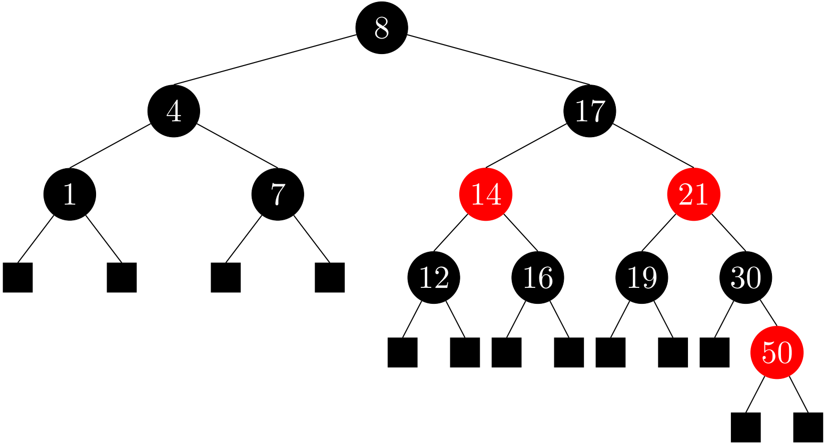

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(i) Like a BST, insert z.

(ii) Color z ***red***.

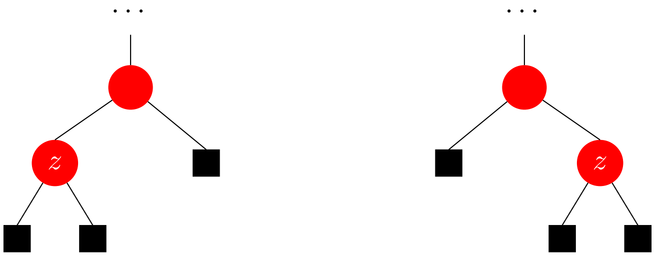
(iii) Fix double ***red*** violations, if any.

(iv) Recursively fix violations upward.

Double Red Violations

Recall *Property (ii): A **red** node does not have a **red** child.*

When we insert our node z (***red*** by definition), its parent may be ***red***. Below are examples of such cases.



Fixing Double Red Violations

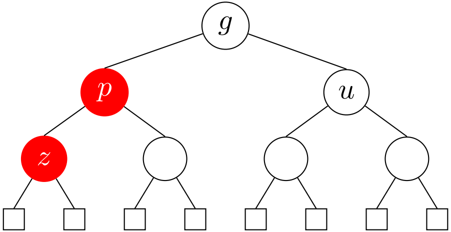
Terminology

With respect to inserted node z,

→ *Parent* (p): z's direct parent

→ *Uncle* (u): p's sibling

→ *Grandparent* (g): p's parent



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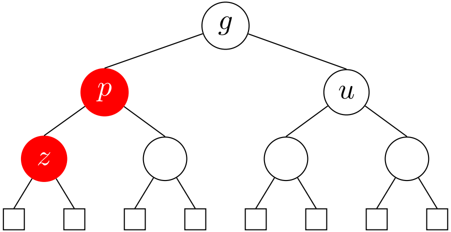

There are two cases:

Fixing Double Red Violations

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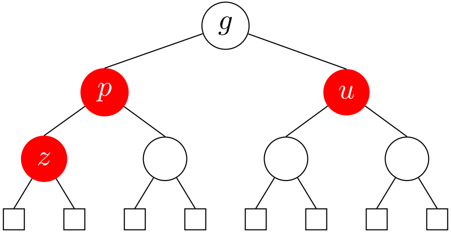

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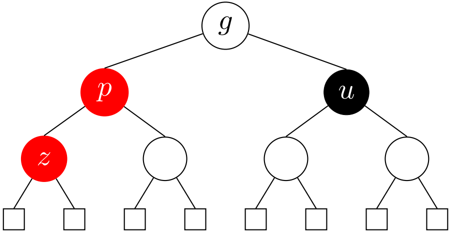

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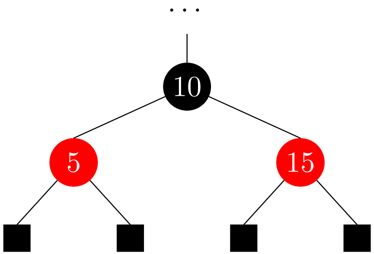
(ii) ***Restructure:*** If the *parent* is ***red*** but the *uncle* is ***black***, perform a *tri-node restructure*.

Recolor



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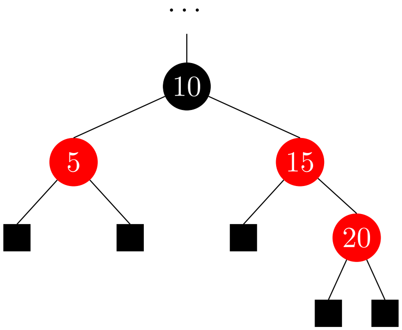

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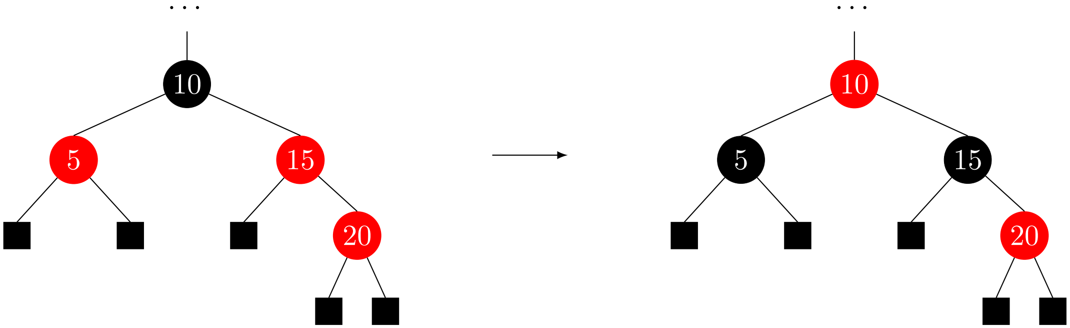

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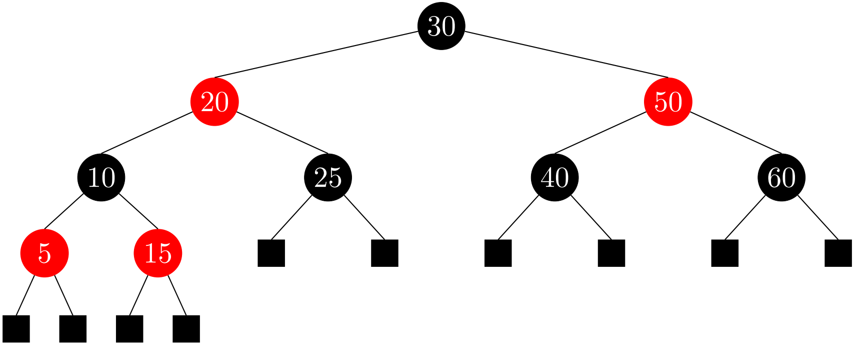

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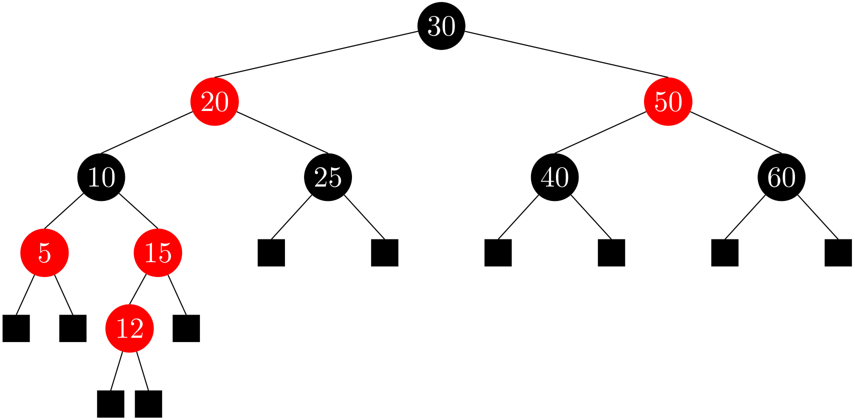

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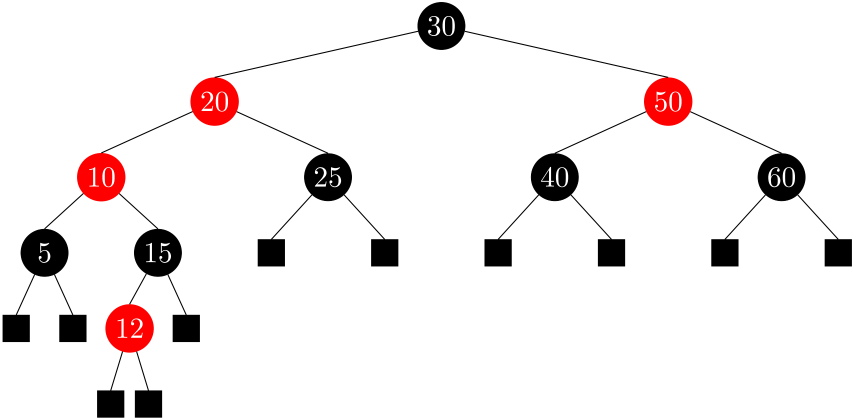

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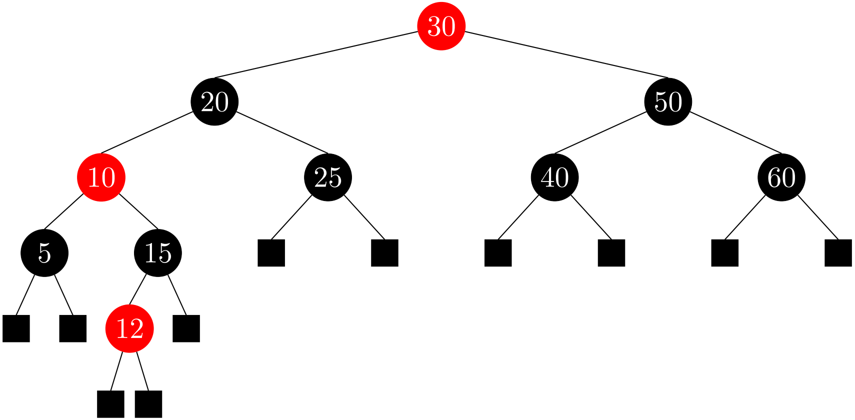

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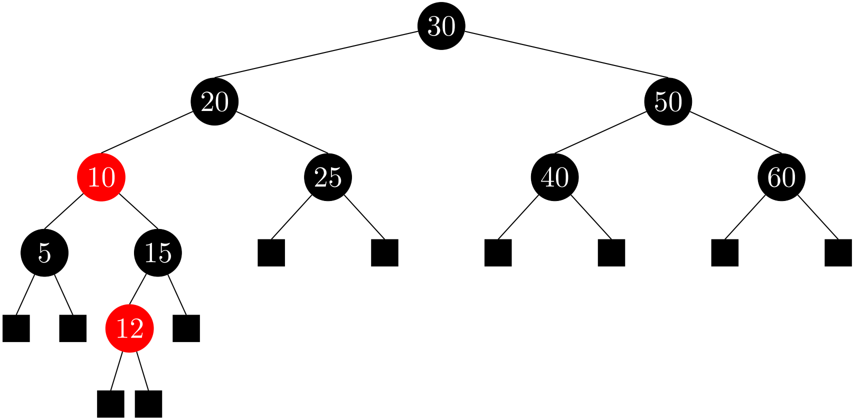

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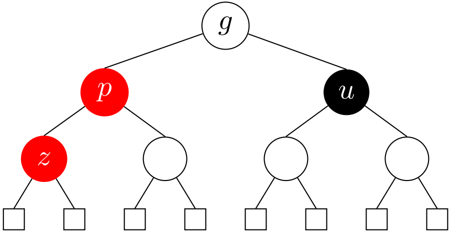

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(ii) ***Restructure:*** If the *parent* is ***red*** but the *uncle* is ***black***, perform a *tri-node restructure*.

Tri-Node Restructure

There are four cases:

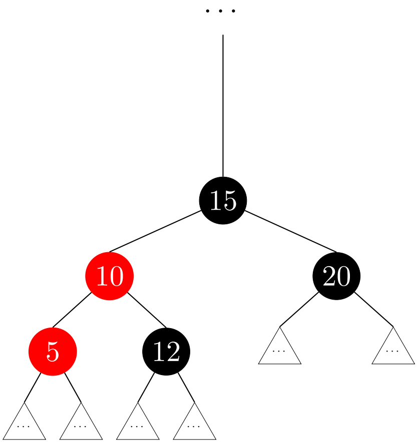
(i) Left-Left

(ii) Right-Right

(iii) Left-Right

(iv) Right-Left

Case: Left-Left (*Right* Rotation)



Here, △ represents a subtree and $\cdots$ represents the rest of the tree.

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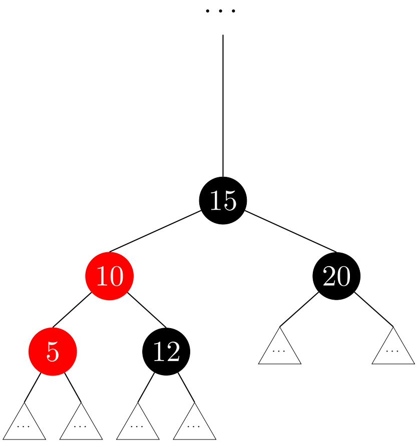

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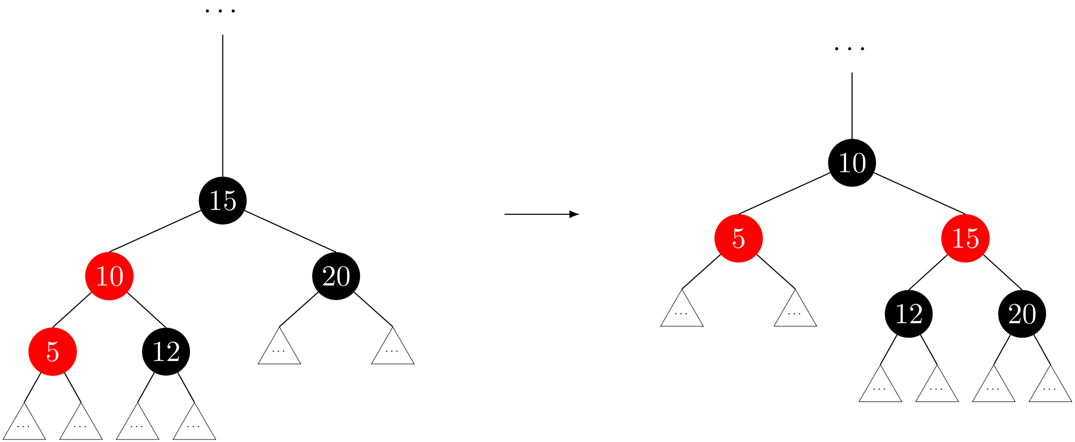

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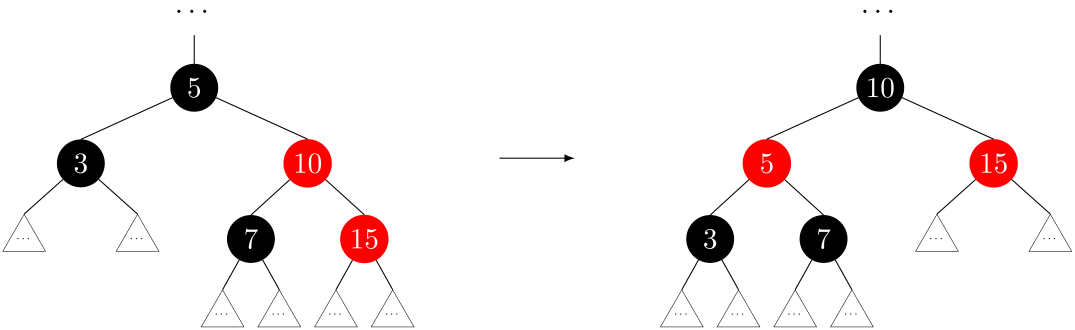

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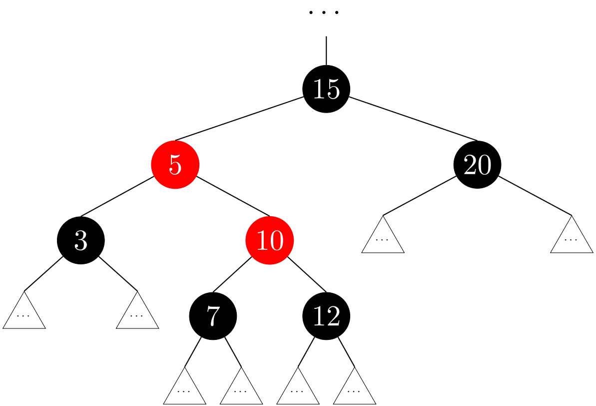

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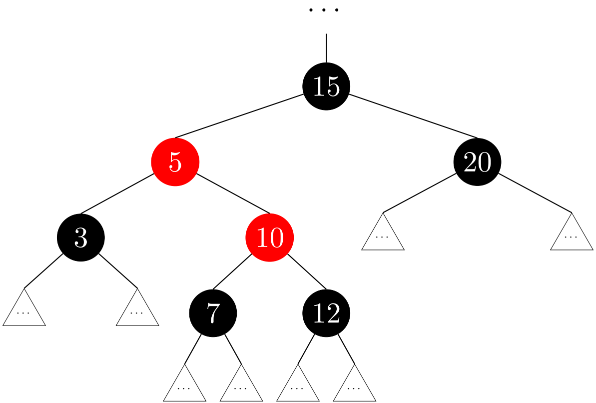

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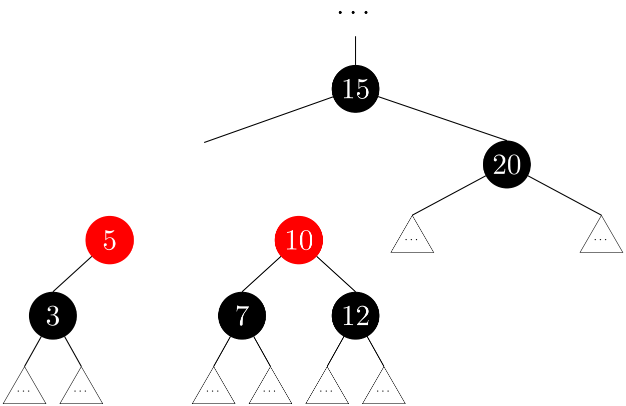

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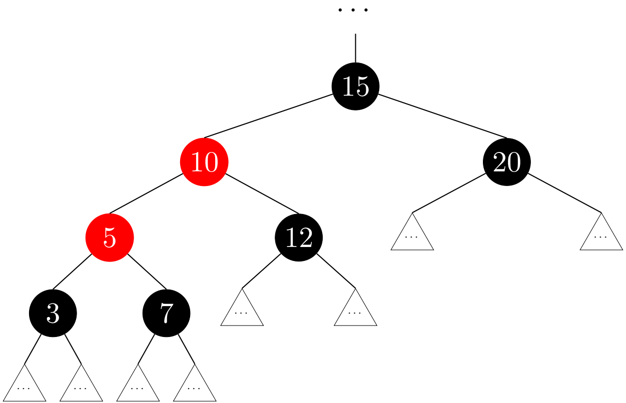

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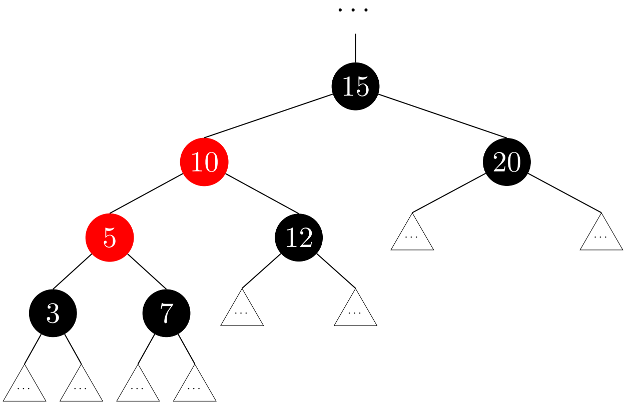
Right Rotation



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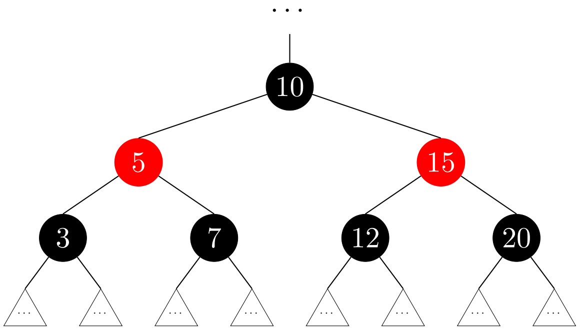

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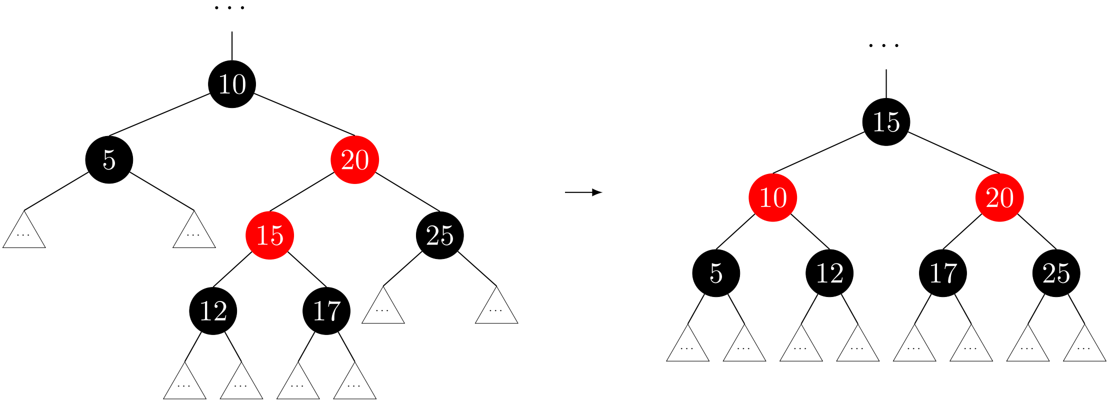

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Time and Space Complexities

- $\rightarrow$ ***Insertion***: $\mathcal{O}(\log n)$
- $\rightarrow$ ***Deletion***: $\mathcal{O}(\log n)$
- $\rightarrow$ ***Search***: $\mathcal{O}(\log n)$
- $\rightarrow$ ***Space***: $\mathcal{O}(n)$

Resources

A code demo and these lecture slides (insertion only) can be found here:

https://github.com/warrenjkim/rbtree-lecture

End

Thank you!

Appendix

Below are slides that didn't make the cut.

Height of a Red-Black Tree

Theorem

A ***red-black*** *tree with n nodes has a height h that is $\mathcal{O}(\log n)$.*

Proof. Suppose we have a ***red-black*** tree with n nodes and height h. Let b be the number of ***black*** nodes on the shortest path from root to any nil node. In the worst case, the longest path alternates between ***red*** and ***black*** nodes and thus has a height of $2b$. Then, h is bounded above by $2b$; that is, $h \leq 2b$. There are $2^b - 1 \leq n$ nodes in this tree. Solving for b, we get $b \leq \log(n+1)$. Substituting b, we get $b \leq \log(n+1) \leq h \leq 2b \leq 2\log(n+1)$ so h is bounded below by $\log(n+1)$ and above by $2\log(n+1)$; that is, $\log(n+1) \leq h \leq 2\log(n+1)$. So, h is $\mathcal{O}(\log n)$. $\square$

Corollaries

Proposition

If a node z has exactly one child, c, then **(a)** *c is* ***red****,* **(b)** *z is* ***black****, and* **(c)** *c has no children.*

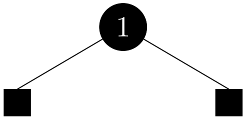
Proof. Suppose we have a valid ***red-black*** tree. Consider a node z with exactly one child. Without loss of generality, choose z's left node to be the child and call it c.

(a) z passes through no ***black*** nodes on the right side by assumption. If c were ***black***, then z would pass through 1 ***black*** node, a contradiction since this violates the *depth property*.

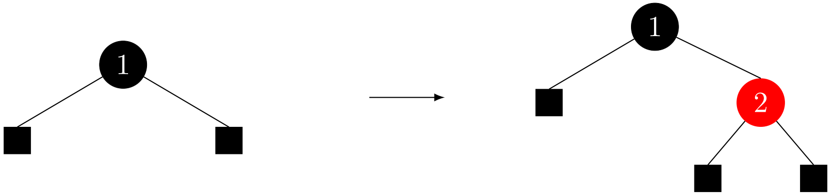
(b) By **(a)**, z's child is ***red*** and by the *internal property*, z is ***black***.

(c) Since z passes through no ***black*** nodes on the right side by assumption, z cannot pass through any ***black*** nodes on the left side by the *the depth property*. Then, since c is ***red*** by **(a)**, c has only nil nodes □

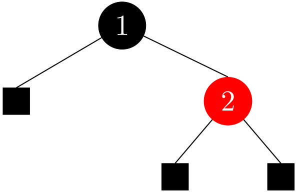
Red-Black Tree: Example



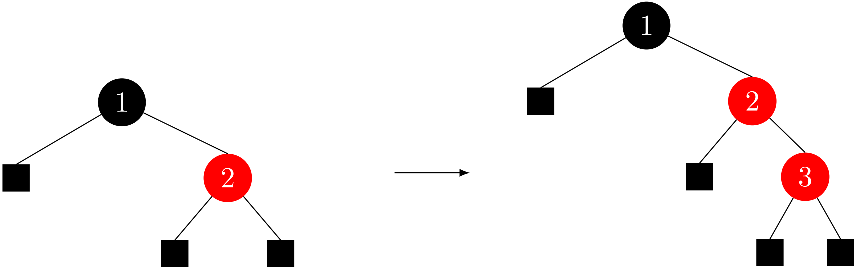
Red-Black Tree: Example



Red-Black Tree: Example

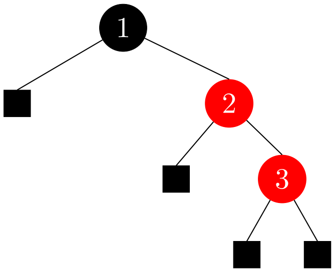


Red-Black Tree: Example



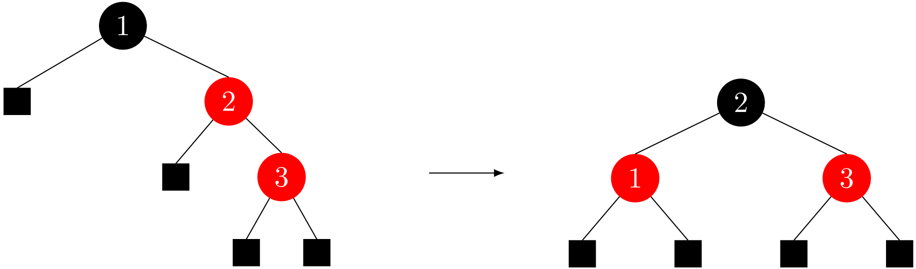
Case: Right-Right

Red-Black Tree: Example



Case: Right-Right

Red-Black Tree: Example



Case: Right-Right

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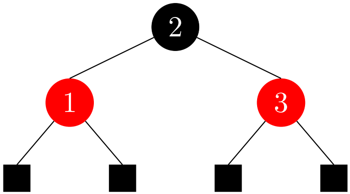
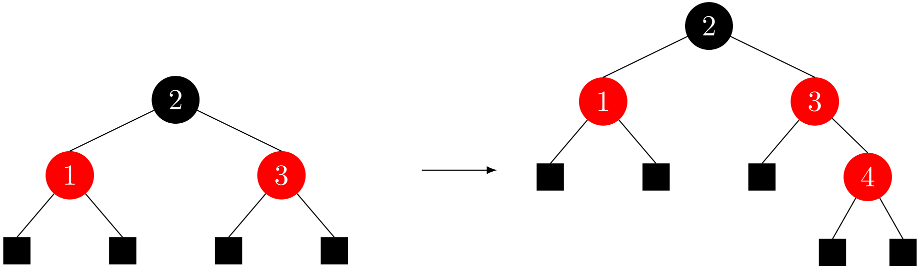
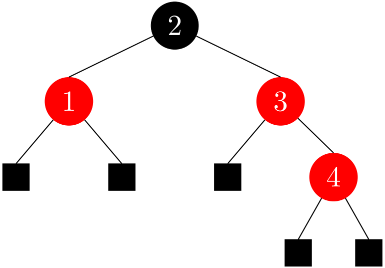


Red-Black Tree: Example



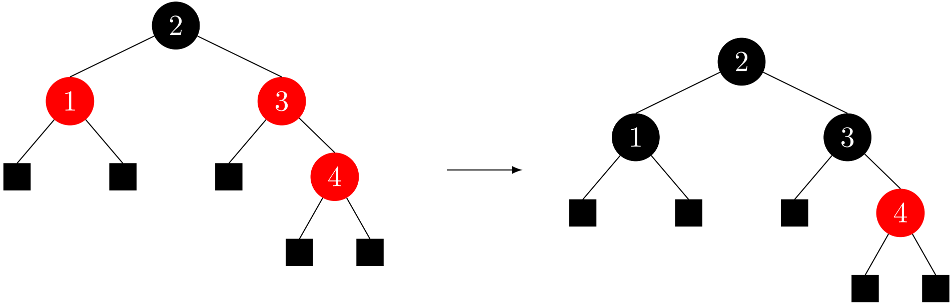
Case: Recolor

Red-Black Tree: Example



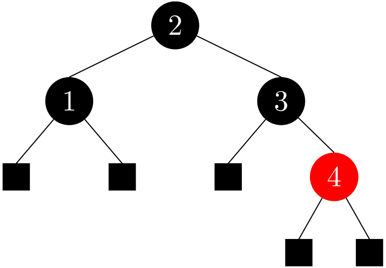
Case: Recolor

Red-Black Tree: Example



Case: Recolor

Red-Black Tree: Example



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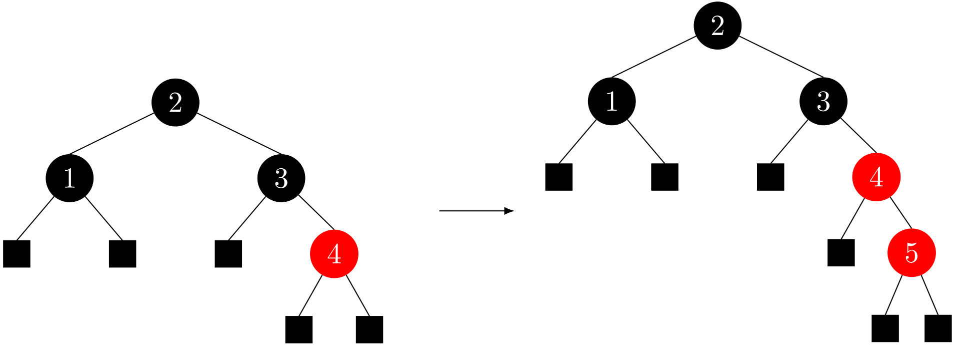
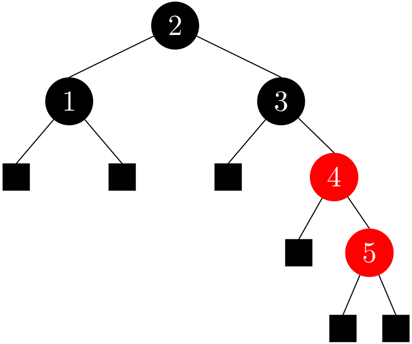


Case: Right-Right

Red-Black Tree: Example



Case: Right-Right

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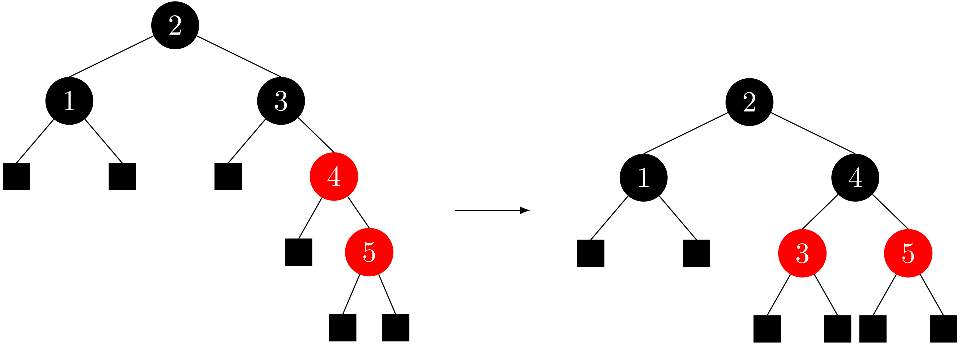

Case: Right-Right

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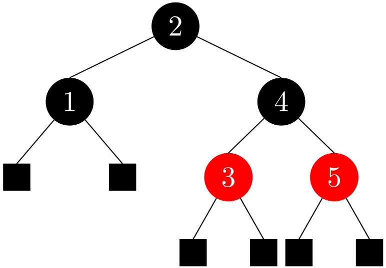

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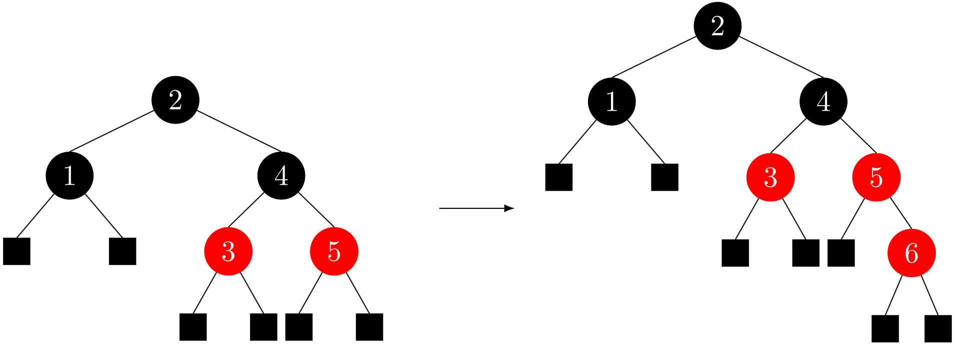
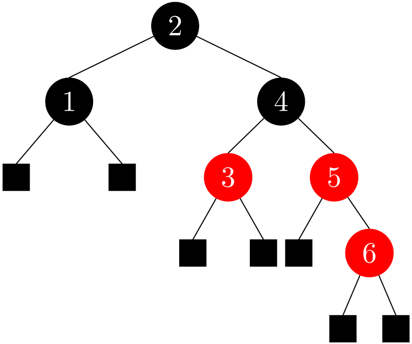


Case: Recolor

Red-Black Tree: Example



Case: Recolor

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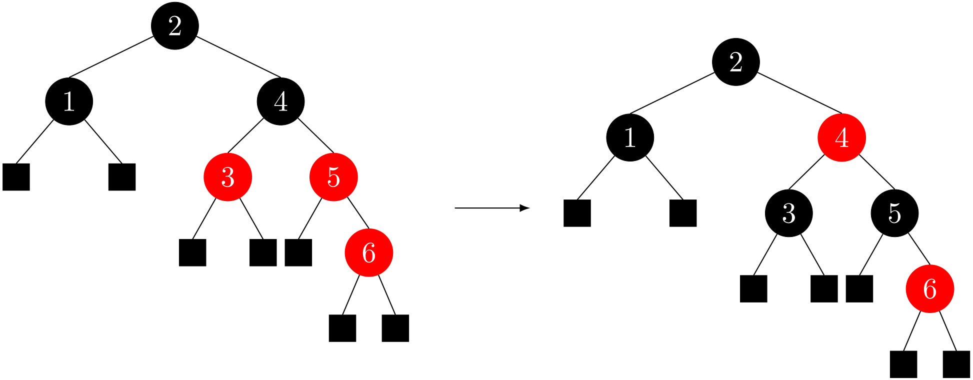

Case: Recolor

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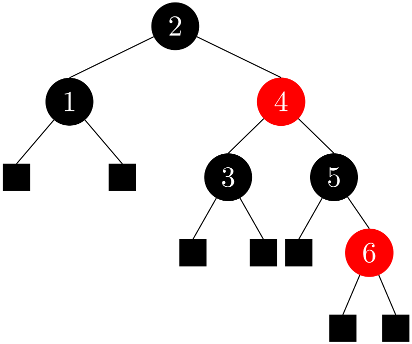

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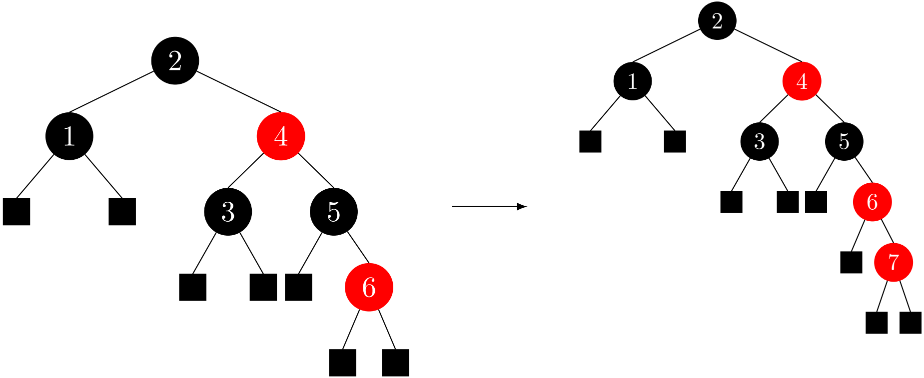

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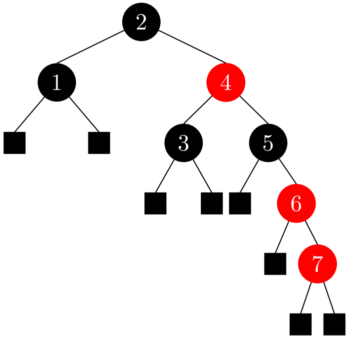

Red-Black Tree: Example