

# DLfuse: Distributed Lag Data Fusion for Estimating Ambient Air Pollution

## DLfuseST\_Example

[1] Simulate data from the proposed model:

- Setting the reproducibility seed and initializing packages for data simulation:

```
set.seed(3146)

library(DLfuse)
library(geoR) #Spatial covariance functions

## Warning: package 'geoR' was built under R version 4.1.1
## -----
## Analysis of Geostatistical Data
## For an Introduction to geoR go to http://www.leg.ufpr.br/geoR
## geoR version 1.8-1 (built on 2020-02-08) is now loaded
## -----

library(mnormt) #Multivariate normal distribution
library(matrixStats) #colMedians
```

- Setting the global data values:

```
#####
#Full CMAQ Grid
#####
m<-(52)
grid<-matrix(0,
              nrow = m,
              ncol = 2)
counter<-1
for(j in 1:sqrt(m)){
  for(k in 1:sqrt(m)){
    grid[counter,]<-c(j,k)
    counter<-counter +
      1
  }
}

neighbors<-1/as.matrix(dist(grid,
                             diag = TRUE,
                             upper = TRUE))
diag(neighbors)<-0

CAR<-diag(rowSums(neighbors)) -
  neighbors

#####
#Full AQS Locations
#####
sample_size<-list(0)
```

```

CMAQ_key<-list(0)
AQS_key<-list(0)

sample_size[[1]]<-rpois(n = m,
                        lambda = 1)
sample_size[[1]]<-pmax(sample_size[[1]],
                        rep(1,
                            times = m))

locs<-matrix(runif(n = (2*sum(sample_size[[1]])),
                  min = 0,
                  max = 1),
             ncol = 2)
spatial_dists<-as.matrix(dist(locs,
                              diag = TRUE,
                              upper = TRUE))

diag(spatial_dists)<-0

CMAQ_key[[1]]<-rep(0,
                  times = sum(sample_size[[1]]))
counter<-0
for(j in 1:length(sample_size[[1]])){

  CMAQ_key[[1]][(1 + counter):(sample_size[[1]][j] + counter)]<-j
  counter<-counter +
    sample_size[[1]][j]

}

CMAQ_key[[1]]<-c(1:nrow(neighbors))
AQS_key[[1]]<-c(1:nrow(spatial_dists))

#####
#Creating the Spatiotemporal Information
#####
d<-16 #Must be larger than one
if(d > 1){
  for(j in 2:d){

    CMAQ_key[[j]]<-sample(c(1:m),
                          size = round((4/5)*m),
                          replace = FALSE)

    sample_size[[j]]<-rpois(n = length(CMAQ_key[[j]]),
                            lambda = 1)
    sample_size[[j]]<-pmax(sample_size[[j]],
                            rep(1,
                                times=length(CMAQ_key[[j]])))

    while(sum(sample_size[[j]]) > nrow(spatial_dists)){

      sample_size[[j]]<-rpois(n = length(CMAQ_key[[j]]),
                              lambda = 1)
    }
  }
}

```

```

        sample_size[[j]]<-pmax(sample_size[[j]],
                                rep(1,
                                    times = length(CMAQ_key[[j]])))
    }

    AQS_key[[j]]<-sample(c(1:nrow(spatial_dists)),
                        size = sum(sample_size[[j]]),
                        replace = FALSE)

}
}

#####
#True Spatial Parameter Settings
#####
sigma2_epsilon_true<-0.05

beta0_true<-0.25
beta1_true<-1.75

A11_true<-0.05
A22_true<-0.05
A21_true<- -0.01

phi0_true<-1.50
Sigma0_true<-cov.spatial(spatial_dists,
                        cov.model = "exponential",
                        cov.pars=c(1, (1/phi0_true)))
w0_true<-rmnorm(n = 1,
                mean = rep(0,
                            times = sum(sample_size[[1]])),
                varcov = Sigma0_true)
w0_true<-w0_true -
            mean(w0_true)
beta0_tilde_true<-A11_true*w0_true

phi1_true<-1.50
Sigma1_true<-cov.spatial(spatial_dists,
                        cov.model = "exponential",
                        cov.pars = c(1, (1/phi1_true)))
w1_true<-rmnorm(n = 1,
                mean = rep(0,
                            times=sum(sample_size[[1]])),
                varcov = Sigma1_true)
w1_true<-w1_true -
            mean(w1_true)
beta1_tilde_true<-A21_true*w0_true +
                A22_true*w1_true

tau2_true<-1.00
rho_true<-0.99 #ICAR Model Approximation
CAR_cov_true<-tau2_true*chol2inv(chol(rho_true*CAR + (1 - rho_true)*diag(m)))

```

```

alpha_true<-rmnorm(n = 1,
                  mean = rep(0,
                             times=m),
                  varcov = CAR_cov_true)
alpha_true<-alpha_true -
  mean(alpha_true)

mu_true<-1.00

#####
#True Temporal Parameters
#####
V_true<-matrix(c(0.03, -0.01, -0.01, 0.03),
               nrow = 2,
               ncol = 2)

rho1_true<-0.90
rho2_true<-0.10
Omega_true<-matrix(0,
                  nrow = 2,
                  ncol = 2)
Omega_true[1,1]<-rho1_true
Omega_true[2,2]<-rho2_true

betat_true<-matrix(0,
                  nrow = d,
                  ncol = 2)
betat_true[1,]<-rmnorm(n = 1,
                      mean = rep(0,
                                 times = 2),
                      varcov = V_true)

rho3_true<-0.75

sigma2_delta_true<-0.01

mut_true<-rep(0,
              times = d)
mut_true[1]<-rnorm(n = 1,
                  mean = 0,
                  sd = sqrt(sigma2_delta_true))

for(j in 2:d){

  betat_true[j,]<-Omega_true%%betat_true[(j-1),] +
    rmnorm(n = 1,
          mean = rep(0,
                     times = 2),
          varcov = V_true)

  mut_true[j]<-rho3_true*mut_true[j-1] +
    rnorm(n = 1,
          mean = 0,

```

```

        sd = sqrt(sigma2_delta_true))

    }

    betat_true[,1]<-betat_true[,1] -
        mean(betat_true[,1])
    betat_true[,2]<-betat_true[,2] -
        mean(betat_true[,2])

    mut_true<-mut_true -
        mean(mut_true)

#####
#Creating Lagged Covariates and Observed Data
#####
L<-11
y<-list(0)
z<-list(0)
covars_true<-list(0)
for(j in 1:d){

    z[[j]]<-matrix(rgamma(n = (length(CMAQ_key[[j]])*L),
        shape = 1,
        rate = 1),
        nrow = length(CMAQ_key[[j]]),
        ncol = L)

    covars_true[[j]]<-construct_lagged_covars_st(
        z[[j]],
        mu_true,
        mut_true[j],
        alpha_true,
        sample_size[[j]],
        CMAQ_key[[j]],
        0)[[1]] #Probit Weights (1 for Spherical)

    y[[j]]<-rnorm(n = sum(sample_size[[j]]),
        mean = ((beta0_true + betat_true[j,1] +
            beta0_tilde_true[AQS_key[[j]]]) +
            (beta1_true + betat_true[j,2] +
            beta1_tilde_true[AQS_key[[j]]])*covars_true[[j]]),
        sd = sqrt(sigma2_epsilon_true))

}

sample_size_validation<-sample_size[[d]]
AQS_key_validation<-AQS_key[[d]]
CMAQ_key_validation<-CMAQ_key[[d]]
y_validation<-y[[d]]
z_validation<-z[[d]]

sample_size[[d]]<-NULL
AQS_key[[d]]<-NULL

```

```
CMAQ_key[[d]]<-NULL
y[[d]]<-NULL
z[[d]]<-NULL
```

[2] Fit DLfuse to a subset of the data:

```
samples<-11000

dlfuse_st_results<-DLfuse_st(mcmc_samples = samples,
                             y = y,
                             z = z,
                             sample_size = sample_size,
                             AQS_key = AQS_key,
                             CMAQ_key = CMAQ_key,
                             spatial_dists = spatial_dists,
                             AQS_unique_total = nrow(spatial_dists),
                             neighbors = neighbors,
                             CMAQ_unique_total = nrow(neighbors),
                             metrop_var_rho1_trans = (3.00^2),
                             metrop_var_rho2_trans = (3.00^2),
                             metrop_var_A11_trans = (0.40^2),
                             metrop_var_A22_trans = (0.60^2),
                             metrop_var_mu = (0.20^2),
                             metrop_var_mut = rep(0.75^2, times=length(y)),
                             metrop_var_rho3_trans = (3.00^2),
                             metrop_var_alpha = rep(1.40^2, times=nrow(neighbors)),
                             metrop_var_phi0_trans = (0.60^2),
                             metrop_var_phi1_trans = (0.62^2),
                             weights_definition_indicator = 0, #Probit (1 for Spherical)
                             model_type_indicator = 0)
```

```
## Progress: 5%
## rho1 Acceptance: 43%
## rho2 Acceptance: 46%
## A11 Acceptance: 47%
## A22 Acceptance: 50%
## mu Acceptance: 37%
## mut Acceptance (min): 31%
## mut Acceptance (max): 57%
## rho3 Acceptance: 38%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 75%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## DLfuse: ST
## *****
## Progress: 10%
## rho1 Acceptance: 42%
## rho2 Acceptance: 45%
## A11 Acceptance: 48%
## A22 Acceptance: 50%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 57%
## rho3 Acceptance: 40%
```

```

## alpha Acceptance (min): 16%
## alpha Acceptance (max): 78%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## DLfuse: ST
## *****
## Progress: 15%
## rho1 Acceptance: 42%
## rho2 Acceptance: 44%
## A11 Acceptance: 44%
## A22 Acceptance: 49%
## mu Acceptance: 36%
## mut Acceptance (min): 33%
## mut Acceptance (max): 57%
## rho3 Acceptance: 39%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 78%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## DLfuse: ST
## *****
## Progress: 20%
## rho1 Acceptance: 42%
## rho2 Acceptance: 44%
## A11 Acceptance: 47%
## A22 Acceptance: 50%
## mu Acceptance: 36%
## mut Acceptance (min): 34%
## mut Acceptance (max): 56%
## rho3 Acceptance: 39%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 78%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 25%
## rho1 Acceptance: 42%
## rho2 Acceptance: 44%
## A11 Acceptance: 48%
## A22 Acceptance: 50%
## mu Acceptance: 36%
## mut Acceptance (min): 33%
## mut Acceptance (max): 56%
## rho3 Acceptance: 40%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 78%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 30%
## rho1 Acceptance: 42%
## rho2 Acceptance: 44%

```

```

## A11 Acceptance: 47%
## A22 Acceptance: 51%
## mu Acceptance: 36%
## mut Acceptance (min): 33%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 78%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## DLfuse: ST
## *****
## Progress: 35%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 46%
## A22 Acceptance: 52%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 78%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 40%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 45%
## A22 Acceptance: 52%
## mu Acceptance: 35%
## mut Acceptance (min): 32%
## mut Acceptance (max): 57%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 77%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 45%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 46%
## A22 Acceptance: 52%
## mu Acceptance: 35%
## mut Acceptance (min): 32%
## mut Acceptance (max): 57%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 77%
## phi0 Acceptance: 45%

```



```

## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 50%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 47%
## A22 Acceptance: 52%
## mu Acceptance: 35%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 77%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 55%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 47%
## A22 Acceptance: 52%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 60%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 47%
## A22 Acceptance: 52%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 65%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 47%
## A22 Acceptance: 52%
## mu Acceptance: 36%

```

```

## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 70%
## rho1 Acceptance: 43%
## rho2 Acceptance: 43%
## A11 Acceptance: 46%
## A22 Acceptance: 53%
## mu Acceptance: 35%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 44%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 75%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 46%
## A22 Acceptance: 54%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 80%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 47%
## A22 Acceptance: 54%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 44%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****

```

```

## Progress: 85%
## rho1 Acceptance: 42%
## rho2 Acceptance: 43%
## A11 Acceptance: 48%
## A22 Acceptance: 54%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 90%
## rho1 Acceptance: 42%
## rho2 Acceptance: 42%
## A11 Acceptance: 47%
## A22 Acceptance: 53%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 44%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 95%
## rho1 Acceptance: 42%
## rho2 Acceptance: 42%
## A11 Acceptance: 47%
## A22 Acceptance: 53%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 44%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
## Progress: 100%
## rho1 Acceptance: 42%
## rho2 Acceptance: 42%
## A11 Acceptance: 47%
## A22 Acceptance: 53%
## mu Acceptance: 36%
## mut Acceptance (min): 32%
## mut Acceptance (max): 56%
## rho3 Acceptance: 41%

```

```
## alpha Acceptance (min): 16%
## alpha Acceptance (max): 76%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## DLfuse: ST
## *****
```

[3] Comparing parameter estimates to true values:

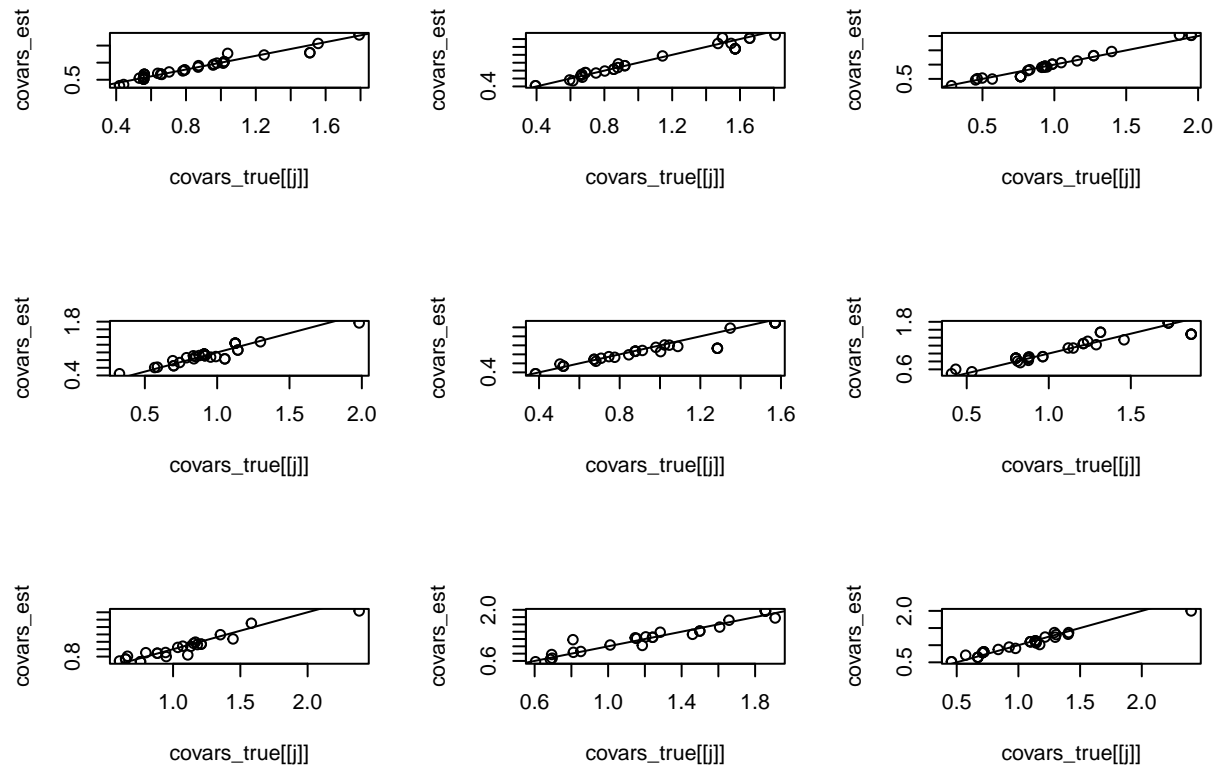
```
burnin<-1000
thin<-10
keep_set<-seq((burnin + 1),
              samples,
              thin)
par(mfrow=c(3,3))

for(j in 1:(d-1)){

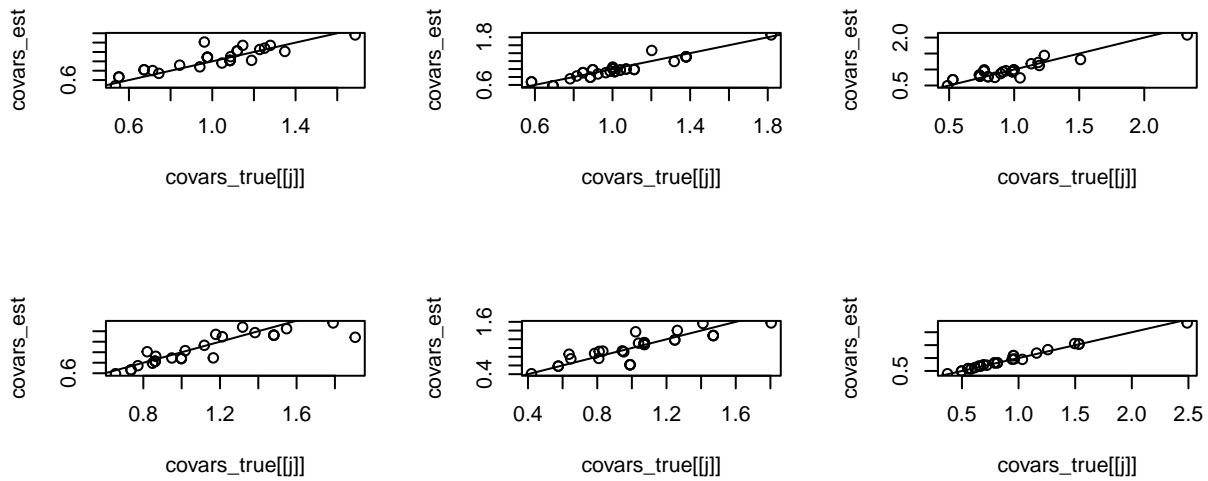
  covars_est<-construct_lagged_covars_st(
    z[[j]],
    mean(dlfuse_st_results$lag_info$mu[keep_set]),
    mean(dlfuse_st_results$lag_info$mut[j,keep_set]),
    rowMeans(dlfuse_st_results$lag_info$alpha[,keep_set]),
    sample_size[[j]],
    CMAQ_key[[j]],
    0)[[1]] #Probit Weights (1 for Spherical)

  plot(covars_true[[j]],
       covars_est)
  abline(0,1)

}
```



```
par(mfrow=c(3,3))
```



```

betat<-simplify2array(dlfuse_st_results$betat)[,keep_set]
for(j in 1:(d-1)){

  true<-c(sigma2_epsilon_true,
          (beta0_true + betat_true[j,1] + beta0_tilde_true[AQS_key[[j]]]),
          (beta1_true + betat_true[j,2] + beta1_tilde_true[AQS_key[[j]]]))

  est<-c(mean(dlfuse_st_results$sigma2_epsilon[keep_set]),
         rowMeans(matrix(dlfuse_st_results$beta0[keep_set],
                        nrow = sum(sample_size[[j]]),
                        ncol = length(keep_set),
                        byrow = TRUE) +
            matrix(betat[1,j,],
                  nrow = sum(sample_size[[j]]),
                  ncol = length(keep_set),
                  byrow = TRUE) +
            matrix(dlfuse_st_results$A11[keep_set],
                  nrow = sum(sample_size[[j]]),
                  ncol = length(keep_set),
                  byrow = TRUE)*dlfuse_st_results$w0[AQS_key[[j]],keep_set]),
         rowMeans(matrix(dlfuse_st_results$beta1[keep_set],
                        nrow = sum(sample_size[[j]]),
                        ncol = length(keep_set),
                        byrow = TRUE) +
            matrix(betat[2,j,],
                  nrow = sum(sample_size[[j]]),

```

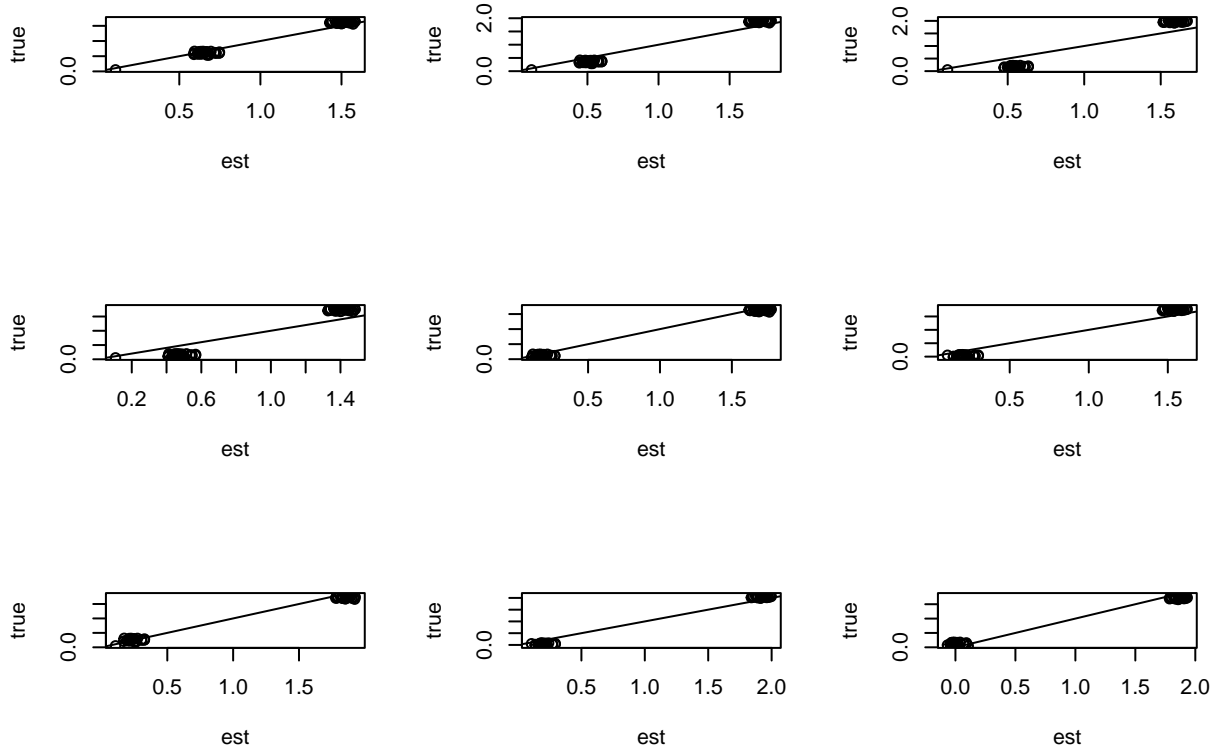
```

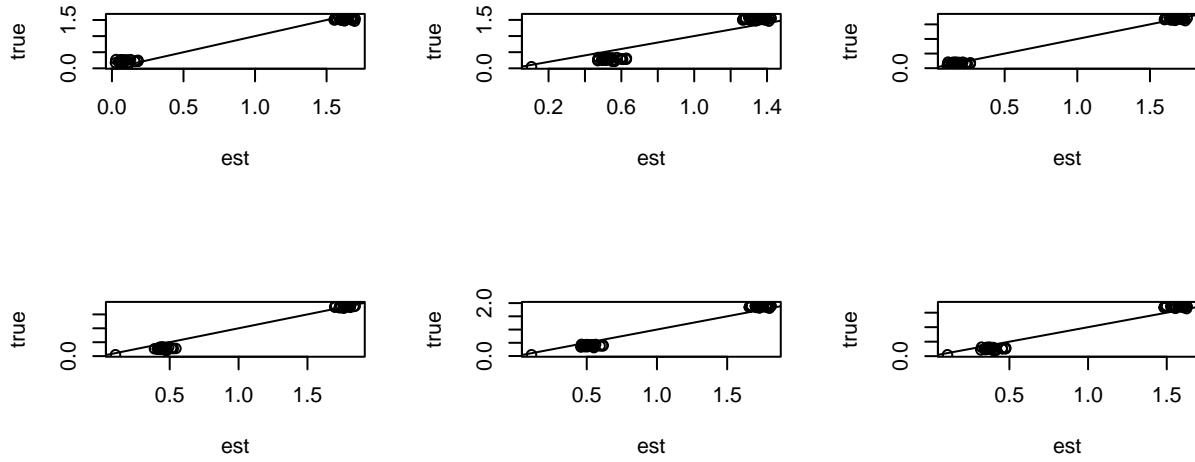
        ncol = length(keep_set),
        byrow = TRUE) +
matrix(dlfuse_st_results$A21[keep_set],
       nrow = sum(sample_size[[j]]),
       ncol = length(keep_set),
       byrow = TRUE)*dlfuse_st_results$w0[AQS_key[[j]], keep_set] +
matrix(dlfuse_st_results$A22[keep_set],
       nrow = sum(sample_size[[j]]),
       ncol = length(keep_set),
       byrow = TRUE)*dlfuse_st_results$w1[AQS_key[[j]], keep_set]))

plot(est, true)
abline(0,1)

}

```





[4] Spatiotemporal predictions of validation data:

```
spatial_dists_full<-as.matrix(dist(rbind(locs[AQS_key_validation,], locs),
                                   diag = TRUE,
                                   upper = TRUE))

diag(spatial_dists_full)<-0
loc_temp<-rbind(locs[AQS_key_validation,], locs)
for(j in 1:nrow(loc_temp)){
  for(k in 1:nrow(loc_temp)){
    if(prod(loc_temp[j,] == loc_temp[k,]) == 1){
      spatial_dists_full[j,k]<-0
    }
  }
}

neighbors_full<-1/as.matrix(dist(rbind(grid[CMAQ_key_validation,], grid),
                                   diag = TRUE,
                                   upper = TRUE))

diag(neighbors_full)<-0
loc_temp<-rbind(grid[CMAQ_key_validation,], grid)
for(j in 1:nrow(loc_temp)){
  for(k in 1:nrow(loc_temp)){
    if(prod(loc_temp[j,] == loc_temp[k,]) == 1){
      neighbors_full[j,k]<-Inf
    }
  }
}
```



```

diag(neighbors_full)<-0

dlfuse_st_pred_results<-
ppd_st(modeling_output = dlfuse_st_results,
        n_pred = length(y_validation),
        m_pred = nrow(z_validation),
        z_pred = z_validation,
        sample_size_pred = sample_size_validation,
        spatial_dists_full = spatial_dists_full,
        neighbors_full = neighbors_full,
        inference_set = keep_set,
        params_only_indicator = 0,
        weights_definition_indicator = 0, #Probit Weights (1 for Spherical)
        model_type_indicator = 0)

## Progress: 5%
## *****
## Progress: 10%
## *****
## Progress: 15%
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## Progress: 25%
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## Progress: 30%
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## Progress: 35%
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## Progress: 75%
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## Progress: 80%
## *****
## Progress: 85%
## *****
## Progress: 90%
## *****
## Progress: 95%
## *****
## Progress: 100%

```

```
## *****
```

[5] Comparison with other approaches:

```
slr_st_results<-DLfuse_st(mcmc_samples = samples,
                          y = y,
                          z = z,
                          sample_size = sample_size,
                          AQS_key = AQS_key,
                          CMAQ_key = CMAQ_key,
                          spatial_dists = spatial_dists,
                          AQS_unique_total = nrow(spatial_dists),
                          neighbors = neighbors,
                          CMAQ_unique_total = nrow(neighbors),
                          metrop_var_rho1_trans = (3.00^2),
                          metrop_var_rho2_trans = (3.00^2),
                          metrop_var_A11_trans = (0.30^2),
                          metrop_var_A22_trans = (0.30^2),
                          metrop_var_mu = (0.10^2),
                          metrop_var_mut = rep(0.75^2, times=length(y)),
                          metrop_var_rho3_trans = (3.00^2),
                          metrop_var_alpha = rep(1.40^2, times=nrow(neighbors)),
                          metrop_var_phi0_trans = (0.60^2),
                          metrop_var_phi1_trans = (0.62^2),
                          model_type_indicator = 3)
```

```
## Progress: 5%
## rho1 Acceptance: 43%
## rho2 Acceptance: 53%
## Simple Linear Regression: ST
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## rho2 Acceptance: 50%
## Simple Linear Regression: ST
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## rho2 Acceptance: 50%
## Simple Linear Regression: ST
## *****
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## Simple Linear Regression: ST
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## Simple Linear Regression: ST
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## Simple Linear Regression: ST
```

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## Simple Linear Regression: ST
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## Simple Linear Regression: ST
## *****
## Progress: 80%
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## rho2 Acceptance: 48%
## Simple Linear Regression: ST
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## Progress: 85%
## rho1 Acceptance: 44%
## rho2 Acceptance: 48%

```

```

## Simple Linear Regression: ST
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## rho1 Acceptance: 44%
## rho2 Acceptance: 48%
## Simple Linear Regression: ST
## *****
## Progress: 95%
## rho1 Acceptance: 44%
## rho2 Acceptance: 48%
## Simple Linear Regression: ST
## *****
## Progress: 100%
## rho1 Acceptance: 44%
## rho2 Acceptance: 48%
## Simple Linear Regression: ST
## *****

slr_st_pred_results<-ppd_st(modeling_output = slr_st_results,
                           n_pred = length(y_validation),
                           m_pred = nrow(z_validation),
                           z_pred = z_validation,
                           sample_size_pred = sample_size_validation,
                           spatial_dists_full = spatial_dists_full,
                           neighbors_full = neighbors_full,
                           inference_set = keep_set,
                           params_only_indicator = 0,
                           model_type_indicator = 3)

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```

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## *****
```

```
ok_st_results<-DLfuse_st(mcmc_samples = samples,
  y = y,
  z = z,
  sample_size = sample_size,
  AQS_key = AQS_key,
  CMAQ_key = CMAQ_key,
  spatial_dists = spatial_dists,
  AQS_unique_total = nrow(spatial_dists),
  neighbors = neighbors,
  CMAQ_unique_total = nrow(neighbors),
  metrop_var_rho1_trans = (3.00^2),
  metrop_var_rho2_trans = (3.00^2),
  metrop_var_A11_trans = (0.60^2),
  metrop_var_A22_trans = (0.30^2),
  metrop_var_mu = (0.10^2),
  metrop_var_mut = rep(0.75^2, times=length(y)),
  metrop_var_rho3_trans = (3.00^2),
  metrop_var_alpha = rep(1.40^2, times=nrow(neighbors)),
  metrop_var_phi0_trans = (0.60^2),
  metrop_var_phi1_trans = (0.62^2),
  model_type_indicator = 2)
```

```
## Progress: 5%
## rho1 Acceptance: 49%
## A11 Acceptance: 57%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 10%
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## A11 Acceptance: 56%
## phi0 Acceptance: 44%
## Ordinary Kriging: ST
## *****
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## A11 Acceptance: 56%
## phi0 Acceptance: 44%
## Ordinary Kriging: ST
## *****
```

```

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## phi0 Acceptance: 44%
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## phi0 Acceptance: 44%
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## Ordinary Kriging: ST
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## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
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## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 55%
## rho1 Acceptance: 45%
## A11 Acceptance: 57%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 60%
## rho1 Acceptance: 45%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****

```

```

## Progress: 65%
## rho1 Acceptance: 45%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 70%
## rho1 Acceptance: 45%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 75%
## rho1 Acceptance: 46%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 80%
## rho1 Acceptance: 46%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 85%
## rho1 Acceptance: 46%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 90%
## rho1 Acceptance: 46%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 95%
## rho1 Acceptance: 46%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****
## Progress: 100%
## rho1 Acceptance: 46%
## A11 Acceptance: 56%
## phi0 Acceptance: 45%
## Ordinary Kriging: ST
## *****

```

```

ok_st_pred_results<-ppd_st(modeling_output = ok_st_results,
                           n_pred = length(y_validation),
                           m_pred = nrow(z_validation),
                           z_pred = z_validation,
                           sample_size_pred = sample_size_validation,

```

```

spatial_dists_full = spatial_dists_full,
neighbors_full = neighbors_full,
inference_set = keep_set,
params_only_indicator = 0,
model_type_indicator = 2)

```

```

## Progress: 5%
## *****
## Progress: 10%
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## Progress: 85%
## *****
## Progress: 90%
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## Progress: 95%
## *****
## Progress: 100%
## *****

```

```

ds_st_results<-DLfuse_st(mcmc_samples = samples,
                          y = y,
                          z = z,
                          sample_size = sample_size,
                          AQS_key = AQS_key,
                          CMAQ_key = CMAQ_key,
                          spatial_dists = spatial_dists,

```



```

AQS_unique_total = nrow(spatial_dists),
neighbors = neighbors,
CMAQ_unique_total = nrow(neighbors),
metrop_var_rho1_trans = (3.00^2),
metrop_var_rho2_trans = (3.00^2),
metrop_var_A11_trans = (0.60^2),
metrop_var_A22_trans = (0.60^2),
metrop_var_mu = (0.10^2),
metrop_var_mut = rep(0.75^2, times=length(y)),
metrop_var_rho3_trans = (3.00^2),
metrop_var_alpha = rep(1.40^2, times=nrow(neighbors)),
metrop_var_phi0_trans = (0.60^2),
metrop_var_phi1_trans = (0.62^2),
model_type_indicator = 1)

```

```

## Progress: 5%
## rho1 Acceptance: 43%
## rho2 Acceptance: 46%
## A11 Acceptance: 63%
## A22 Acceptance: 60%
## phi0 Acceptance: 42%
## phi1 Acceptance: 41%
## Original: ST
## *****
## Progress: 10%
## rho1 Acceptance: 43%
## rho2 Acceptance: 47%
## A11 Acceptance: 61%
## A22 Acceptance: 59%
## phi0 Acceptance: 43%
## phi1 Acceptance: 44%
## Original: ST
## *****
## Progress: 15%
## rho1 Acceptance: 43%
## rho2 Acceptance: 49%
## A11 Acceptance: 59%
## A22 Acceptance: 57%
## phi0 Acceptance: 44%
## phi1 Acceptance: 44%
## Original: ST
## *****
## Progress: 20%
## rho1 Acceptance: 44%
## rho2 Acceptance: 49%
## A11 Acceptance: 58%
## A22 Acceptance: 56%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 25%
## rho1 Acceptance: 45%
## rho2 Acceptance: 49%

```

```

## A11 Acceptance: 57%
## A22 Acceptance: 56%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## Original: ST
## *****
## Progress: 30%
## rho1 Acceptance: 45%
## rho2 Acceptance: 49%
## A11 Acceptance: 56%
## A22 Acceptance: 56%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## Original: ST
## *****
## Progress: 35%
## rho1 Acceptance: 45%
## rho2 Acceptance: 49%
## A11 Acceptance: 57%
## A22 Acceptance: 55%
## phi0 Acceptance: 45%
## phi1 Acceptance: 44%
## Original: ST
## *****
## Progress: 40%
## rho1 Acceptance: 45%
## rho2 Acceptance: 48%
## A11 Acceptance: 56%
## A22 Acceptance: 55%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 45%
## rho1 Acceptance: 45%
## rho2 Acceptance: 47%
## A11 Acceptance: 55%
## A22 Acceptance: 56%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 50%
## rho1 Acceptance: 45%
## rho2 Acceptance: 48%
## A11 Acceptance: 55%
## A22 Acceptance: 56%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 55%
## rho1 Acceptance: 45%
## rho2 Acceptance: 48%

```

```

## A11 Acceptance: 55%
## A22 Acceptance: 55%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 60%
## rho1 Acceptance: 45%
## rho2 Acceptance: 48%
## A11 Acceptance: 55%
## A22 Acceptance: 55%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 65%
## rho1 Acceptance: 45%
## rho2 Acceptance: 48%
## A11 Acceptance: 56%
## A22 Acceptance: 55%
## phi0 Acceptance: 45%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 70%
## rho1 Acceptance: 45%
## rho2 Acceptance: 48%
## A11 Acceptance: 55%
## A22 Acceptance: 55%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 75%
## rho1 Acceptance: 45%
## rho2 Acceptance: 48%
## A11 Acceptance: 55%
## A22 Acceptance: 55%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 80%
## rho1 Acceptance: 45%
## rho2 Acceptance: 48%
## A11 Acceptance: 56%
## A22 Acceptance: 55%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 85%
## rho1 Acceptance: 44%
## rho2 Acceptance: 48%

```

```

## A11 Acceptance: 55%
## A22 Acceptance: 55%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## Original: ST
## *****
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## rho1 Acceptance: 44%
## rho2 Acceptance: 48%
## A11 Acceptance: 55%
## A22 Acceptance: 55%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## Original: ST
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## rho1 Acceptance: 44%
## rho2 Acceptance: 48%
## A11 Acceptance: 55%
## A22 Acceptance: 55%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## Original: ST
## *****
## Progress: 100%
## rho1 Acceptance: 44%
## rho2 Acceptance: 48%
## A11 Acceptance: 55%
## A22 Acceptance: 55%
## phi0 Acceptance: 44%
## phi1 Acceptance: 43%
## Original: ST
## *****

```

```

ds_st_pred_results<-ppd_st(modeling_output = ds_st_results,
                           n_pred = length(y_validation),
                           m_pred = nrow(z_validation),
                           z_pred = z_validation,
                           sample_size_pred = sample_size_validation,
                           spatial_dists_full = spatial_dists_full,
                           neighbors_full = neighbors_full,
                           inference_set = keep_set,
                           params_only_indicator = 0,
                           model_type_indicator = 1)

```

```

## Progress: 5%
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## Progress: 85%
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## Progress: 90%
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## Progress: 95%
## *****
## Progress: 100%
## *****
```

#### *#Results Matrix*

```
results<-matrix(0,
                 nrow = 3,
                 ncol = 4)
colnames(results)<-c("dlfuse_st", "slr", "ok", "ds")
rownames(results)<-c("mse", "cover", "length")

#DLfuse_st
cover<-rep(0,
           times=length(y_validation))
len<-rep(0,
         times=length(y_validation))
for(j in 1:length(y_validation)){

  ci<-quantile(dlfuse_st_pred_results[[1]][j,], c(0.025, 0.975))
  if((ci[1] <= y_validation[j]) & (ci[2] >= y_validation[j])){
    cover[j]<-1
  }
  len[j]<-ci[2] -
    ci[1]

}
results[1,1]<-mean((y_validation -
```

```

                                rowMedians(dlfuse_st_pred_results[[1]]))^2)
results[2,1]<-mean(cover)
results[3,1]<-mean(len)

#SLR_st
cover<-rep(0,
            times=length(y_validation))
len<-rep(0,
         times=length(y_validation))
for(j in 1:length(y_validation)){

  ci<-quantile(slr_st_pred_results[[1]][j,], c(0.025, 0.975))
  if((ci[1] <= y_validation[j]) & (ci[2] >= y_validation[j])){
    cover[j]<-1
  }
  len[j]<-ci[2] -
          ci[1]

}
results[1,2]<-mean((y_validation -
                    rowMedians(slr_st_pred_results[[1]]))^2)
results[2,2]<-mean(cover)
results[3,2]<-mean(len)

#OK_st
cover<-rep(0,
            times=length(y_validation))
len<-rep(0,
         times=length(y_validation))
for(j in 1:length(y_validation)){

  ci<-quantile(ok_st_pred_results[[1]][j,], c(0.025, 0.975))
  if((ci[1] <= y_validation[j]) & (ci[2] >= y_validation[j])){
    cover[j]<-1
  }
  len[j]<-ci[2] -
          ci[1]

}
results[1,3]<-mean((y_validation -
                    rowMedians(ok_st_pred_results[[1]]))^2)
results[2,3]<-mean(cover)
results[3,3]<-mean(len)

#DS_st
cover<-rep(0,
            times=length(y_validation))
len<-rep(0,
         times=length(y_validation))
for(j in 1:length(y_validation)){

  ci<-quantile(ds_st_pred_results[[1]][j,], c(0.025, 0.975))

```

```

    if((ci[1] <= y_validation[j]) & (ci[2] >= y_validation[j])){
      cover[j]<-1
    }
    len[j]<-ci[2] -
      ci[1]

  }
results[1,4]<-mean((y_validation -
  rowMedians(ds_st_pred_results[[1]]))^2)
results[2,4]<-mean(cover)
results[3,4]<-mean(len)

results

```

```

##          dlfuse_st          slr          ok          ds
## mse      0.07232132 0.3608366 0.3750603 0.3814659
## cover    1.00000000 0.9696970 0.9696970 0.9393939
## length   1.98989554 2.3808350 2.8806060 2.4060989

```