GenePair: Statistical Methods for Modeling Spatially-Referenced Paired Genetic Relatedness Data

Transmission Probabilities Model

$Y_{ij} \overset{\text{ind}}{\sim} f_{y_{ij}}(y) = (1-\pi_{ij})^{1(y=0)} \left[\frac{\pi_{ij}}{y(1-y)} f_{w_{ij}} \left\{\ln\left(\frac{y}{1-y}\right)\right\}\right]^{1(y>0)}, \ y \in [0,1), \ i = 1, ..., n, \ j = 1, ..., n, \ i \neq j$

- $$\ln\left(\frac{\pi_{ij}}{1-\pi_{ij}}\right) = \mathbf{x}_{ij}^{\text{T}}\boldsymbol{\beta}_z + \mathbf{d}_j^{\text{T}}\boldsymbol{\gamma}_z^{(g)} + \mathbf{d}_i^{\text{T}}\boldsymbol{\gamma}_z^{(r)} + \theta_{zj}^{(g)} + \theta_{zi}^{(r)} + \nu_{zi}\nu_{zj}$$
- $$f_{w_{ij}}(w) \equiv \text{N}\left(\mathbf{x}_{ij}^{\text{T}}\boldsymbol{\beta}_w + \mathbf{d}_j^{\text{T}}\boldsymbol{\gamma}_w^{(g)} + \mathbf{d}_i^{\text{T}}\boldsymbol{\gamma}_w^{(r)} + \theta_{wj}^{(g)} + \theta_{wi}^{(r)} + \nu_{wi}\nu_{wj}, \ \sigma_\epsilon^2\right)$$

$$\begin{aligned}
\theta_{zi}^{(g)} &= \eta_z^{(g)}\left\{d\left(\mathbf{s}_i\right)\right\} + \zeta_{zi}^{(g)}, \\
\theta_{zi}^{(r)} &= \eta_z^{(r)}\left\{d\left(\mathbf{s}_i\right)\right\} + \zeta_{zi}^{(r)}, \\
\theta_{wi}^{(g)} &= \eta_w^{(g)}\left\{d\left(\mathbf{s}_i\right)\right\} + \zeta_{wi}^{(g)}, \\
\theta_{wi}^{(r)} &= \eta_w^{(r)}\left\{d\left(\mathbf{s}_i\right)\right\} + \zeta_{wi}^{(r)}, \ i = 1, ..., n,
\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{\eta}|\Omega, \phi \sim \text{MVN}\left\{\mathbf{0}_{4m}, \Sigma(\phi) \otimes \Omega\right\} \text{ where} \\
&\Sigma(\phi)_{ij} = \exp\left\{-\phi\left\|\mathbf{s}_i^* - \mathbf{s}_j^*\right\|\right\}, \\
&\boldsymbol{\eta}^{\text{T}} = \left\{\boldsymbol{\eta}\left(\mathbf{s}_1^*\right)^{\text{T}}, \ldots, \boldsymbol{\eta}\left(\mathbf{s}_m^*\right)^{\text{T}}\right\}, \text{ and} \\
&\boldsymbol{\eta}\left(\mathbf{s}_i^*\right)^{\text{T}} = \left\{\eta_z^{(g)}\left(\mathbf{s}_i^*\right), \eta_z^{(r)}\left(\mathbf{s}_i^*\right), \eta_w^{(g)}\left(\mathbf{s}_i^*\right), \eta_w^{(r)}\left(\mathbf{s}_i^*\right)\right\}.
\end{aligned}$$

- $d(\mathbf{s}_i)$: Maps the spatial location of an individual to an entry within a smaller set of $m < n$ unique locations such that $d(\mathbf{s}_i) \in \{\mathbf{s}_1^*, \ldots, \mathbf{s}_m^*\}$;
- $\zeta_{zi}^{(g)}|\sigma^2_{\zeta_z^{(g)}} \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma^2_{\zeta_z^{(g)}}\right)$, $\zeta_{zi}^{(r)}|\sigma^2_{\zeta_z^{(r)}} \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma^2_{\zeta_z^{(r)}}\right)$, $\zeta_{wi}^{(g)}|\sigma^2_{\zeta_w^{(g)}} \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma^2_{\zeta_w^{(g)}}\right)$, $\zeta_{wi}^{(r)}|\sigma^2_{\zeta_w^{(r)}} \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma^2_{\zeta_w^{(r)}}\right)$;
- $\nu_{zi}|\sigma^2_{\nu_z} \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma^2_{\nu_z}\right)$, $\nu_{wi}|\sigma^2_{\nu_w} \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma^2_{\nu_w}\right)$, $i = 1, ..., n$;
- m: Number of unique spatial locations ($m \leq n$);
- n: Number of individuals.

Prior Information

$\beta_{zj}, \beta_{wj}, \gamma_{zk}^{(g)}, \gamma_{zk}^{(r)}, \gamma_{wk}^{(g)}, \gamma_{wk}^{(r)} \overset{\text{iid}}{\sim} \text{N}\left(0, \sigma_r^2\right), \ j = 1, ..., p_x, \ k = 1, ..., p_d;$

- p_x: Length of $\mathbf{x}_i$ vector (same for all i) which includes an intercept term;
- p_d: Length of $\mathbf{d}_j$ vector (same for all j) which **does not** include an intercept term;
- Default setting: $\sigma_r^2 = 10,000$.

$\sigma_\epsilon^2 \sim \text{Inverse Gamma}\left(a_{\sigma_\epsilon^2}, b_{\sigma_\epsilon^2}\right);$

- Default setting: $a_{\sigma_\epsilon^2} = 0.01$, $b_{\sigma_\epsilon^2} = 0.01$.

$\sigma^2_{\zeta_z^{(g)}} \sim \text{Inverse Gamma}\left(a_{\sigma^2_{\zeta_z^{(g)}}}, b_{\sigma^2_{\zeta_z^{(g)}}}\right)$, $\sigma^2_{\zeta_z^{(r)}} \sim \text{Inverse Gamma}\left(a_{\sigma^2_{\zeta_z^{(r)}}}, b_{\sigma^2_{\zeta_z^{(r)}}}\right)$, $\sigma^2_{\zeta_w^{(g)}} \sim \text{Inverse Gamma}\left(a_{\sigma^2_{\zeta_w^{(g)}}}, b_{\sigma^2_{\zeta_w^{(g)}}}\right)$, $\sigma^2_{\zeta_w^{(r)}} \sim \text{Inverse Gamma}\left(a_{\sigma^2_{\zeta_w^{(r)}}}, b_{\sigma^2_{\zeta_w^{(r)}}}\right);$

- Default setting: $a_{\sigma^2_{\zeta_z^{(g)}}}, a_{\sigma^2_{\zeta_z^{(r)}}}, a_{\sigma^2_{\zeta_w^{(g)}}}, a_{\sigma^2_{\zeta_w^{(r)}}} = 0.01$, $b_{\sigma^2_{\zeta_z^{(g)}}}, b_{\sigma^2_{\zeta_z^{(r)}}}, b_{\sigma^2_{\zeta_w^{(g)}}}, b_{\sigma^2_{\zeta_w^{(r)}}} = 0.01$.

$\sigma^2_{\nu_z} \sim \text{Inverse Gamma}\left(a_{\sigma^2_{\nu_z}}, b_{\sigma^2_{\nu_z}}\right)$; $\sigma^2_{\nu_w} \sim \text{Inverse Gamma}\left(a_{\sigma^2_{\nu_w}}, b_{\sigma^2_{\nu_w}}\right)$;

- Default setting: $a_{\sigma^2_{\nu_z}}, a_{\sigma^2_{\nu_w}} = 0.01$, $b_{\sigma^2_{\nu_z}}, b_{\sigma^2_{\nu_w}} = 0.01$.

$\Omega^{-1} \sim \text{Wishart}\left(\nu, \Omega^*\right)$;

- Default setting: $\nu = 5$, $\Omega^* = I_4$.

$\phi \sim \text{Gamma}\left(a_\phi, b_\phi\right)$;

- Default setting: $a_\phi = 1.00$, $b_\phi = 1.00$.

Default Initial Values

- $\beta_{zj}, \beta_{wj}, \gamma^{(g)}_{zk}, \gamma^{(r)}_{zk}, \gamma^{(g)}_{wk}, \gamma^{(r)}_{wk} = 0$ for all j, k;
- $\theta^{(g)}_{zi}, \theta^{(r)}_{zi}, \theta^{(g)}_{wi}, \theta^{(r)}_{wi} = 0$ for all i;
- $\eta^{(g)}_{zi}, \eta^{(r)}_{zi}, \eta^{(g)}_{wi}, \eta^{(r)}_{wi} = 0$ for all i;
- $\sigma^2_\epsilon = \text{variance}\left(\boldsymbol{Y}\right)$;
- $\sigma^2_{\zeta_z^{(g)}}, \sigma^2_{\zeta_z^{(r)}}, \sigma^2_{\zeta_w^{(g)}}, \sigma^2_{\zeta_w^{(r)}} = 1.00$;
- $\sigma^2_{\nu_z}, \sigma^2_{\nu_w} = 1.00$;
- $\Omega = I_4$;
- $\phi = 1.00$.