Math 5563
Graph Theory
Connectivity Invariance Proof

Warren Keil

1. Prove $\kappa(G)$ is graph invariant.

Proof. Let $G = (V, E)$ and $H = (V', E')$ be simple graphs such that $G \cong H$. Let $\kappa(G)$ be the connectivity of G. Then $\kappa(G)$ equals some integer k where k is the minimum number of vertices in any vertex cut of G. Let S be a minimum vertex cut with k vertices. Then $G - S$ is disconnected. Let $S' = \{f(u) : u \in S\}$. (First we will show that S' is a disconnecting set for H. Then we will show that it is a minimum disconnecting set).

I tried to start the proof exactly as i did in class. Here is where I started re-wording things, and fixing errors in the way I stated things

To show that $H - S'$ is disconnected, first let vertices a and b live in different components of $G - S$. Then notice that since a and b are not in S, then $f(a)$ and $f(b)$ are guaranteed to be in $H - S'$. To show these vertices $f(a)$ and $f(b)$ are disconnected in $H - S'$, suppose there exists some path of length n from $f(a)$ to $f(b)$. Then we can label the edges in this path as $p = [\{f(a), f(v_1)\}, \{f(v_1), f(v_2)\}, \ldots \{f(v_{n-2}), f(b)\}]$. Then, this is true if and only if there exists some path in $G - S$ called $\hat{p} = [\{a, v_1\}, \{v_1, v_2\}, \ldots \{v_{n-2}, b\}]$ $\rightarrow\!\leftarrow$ But this cannot happen since we assumed that a and b were vertices in different components of $G - S$. Thus S' is a disconnecting set for H.

To show that this is a smallest disconnecting set, suppose there is a smaller disconnecting set for H called T' such that $|T'| < |S'|$. Then $H - T'$ is disconnected. Let $T = \{u : f(u) \in T'\}$. Now let $f(a)$ and $f(b)$ be in different components of $H - T'$. Then with the same argument used above, we know that a and b in $G - T$ have to be in different components of $G - T$. Thus $G - T$ is disconnected. Thus T is a disconnecting set for G. And since f is a bijection, then the number of elements of T' equals the number of elements of T. And since $|T'| < |S'| \Rightarrow |T| < |S|$. $\rightarrow\!\leftarrow$ But this surely cannot happen since we assumed that S is a minimum disconnecting set for G. Thus a smaller disconnecting set cannot exist. Therefore S' is a smallest disconnecting set for H. And since $|S'| = |S| = k$, then $\kappa(H) = k$.
$\therefore \kappa(G)$ is a graph invariant

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