Highway Games on Weakly Cyclic Graphs

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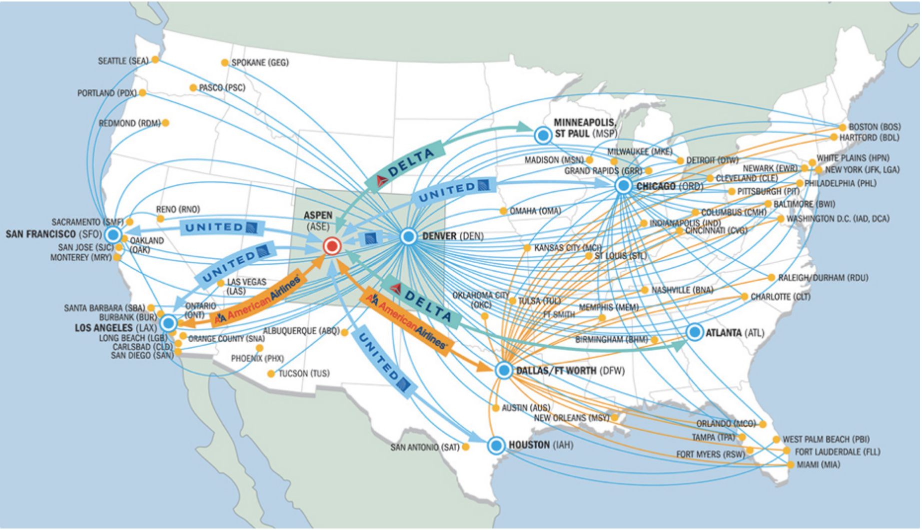
Introduction

A highway problem is a problem that can be modeled with a graph such that there is a set of required paths between specific vertices. Examples of highway problems include:

- Choosing the locations of the construction of highways
- Scheduling flights between airports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Definitions

Cooperative Cost Game- a pair (N, c), of a set of N players and a cost function c

Coalition - is a set $S \subset N$ of players.

Minimum joint cost of S - $c(S)$ is the minimum cost of S.

Definition

Highway Problem $\Gamma = (N, G, \{s_i\}_{i \in N}, \{t_i\}_{i \in N}, w)$

- $N = \{1, 2, \ldots, n\}$ - the indexing set of players
- $G = (V, E)$ - a graph
- $\{s_i\}_{i \in N}$ - the starting vertex for player i
- $\{t_i\}_{i \in N}$ - the terminating vertex for player i
- $w : E \to \mathbb{R}_{\geq 0}$ - a cost function that assigns a cost to each edge

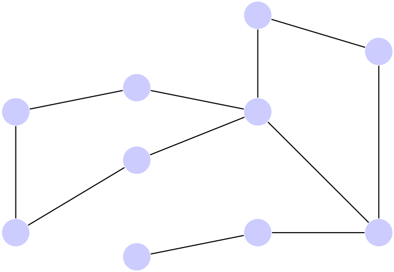
Definition

Highway Game (N, c_Γ)- of a highway problem Γ: (optimization problem)

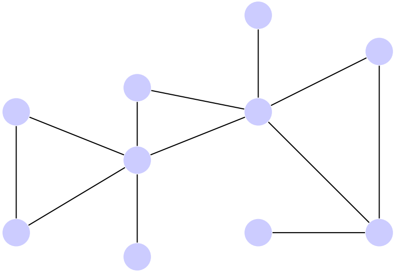
$$c_\Gamma = \min_{E' \subseteq E} \{w(E') | s_i \text{ and } t_i \text{ are connected in } (V, E') \forall i \in S\} \text{ for } S \subseteq N\}$$

Definition

Weakly cyclic- A graph G is weakly cyclic is every edge in G is contained in at most one cycle.
Weakly triangular- A weakly cyclic graph G is weakly triangular if every cycle in G has length 3.



Weakly Cyclic



Weakly Triangular

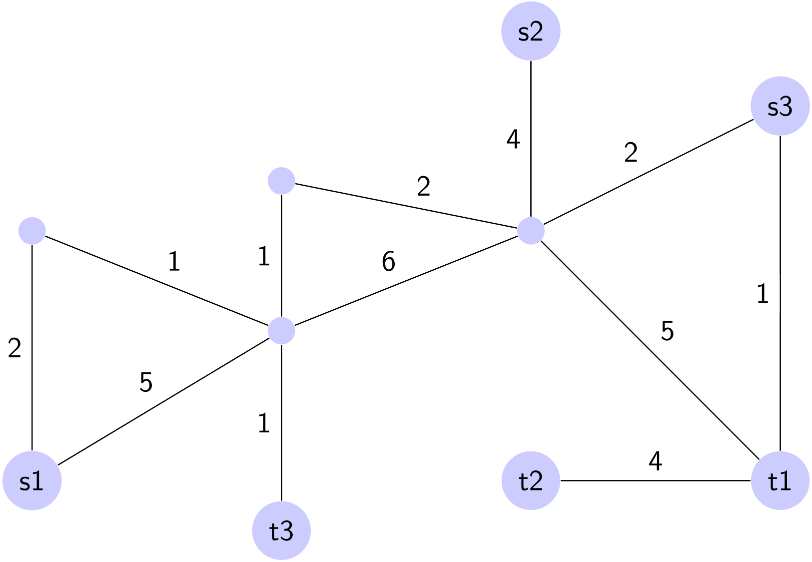
Definition

Game Concavity- A (cooperative cost) game (N, c) is concave if

$$c(T \cup S) + c(T \cap S) \leq c(T) + c(S) \qquad \forall S, T \subseteq N$$

Highway Game Concavity- A graph G is **HG concave** if for every highway problem $\Gamma = (N, G, \{s_i\}_{i \in N}, \{t_i\}_{i \in N}, w)$, the corresponding highway game (N, c_Γ) is concave.

Example of Highway Game Concavity



Example of Highway Game Concavity

If we have the set of players $N = \{1, 2, 3\}$ and subsets $T = \{1, 3\}$ and $S = \{2, 3\}$, then notice:

$$\begin{aligned}
c(T \cup S) &= c(\{1,2,3\}) = 2+1+1+1+2+4+2+1+4 = 18 \\
c(T \cap S) &= c(\{3\}) = 2+2+1+1 = 6 \\
c(T) &= 2+1+1+2+2+1+1 = 10 \\
c(S) &= 4+2+1+4+2+1+1 = 15
\end{aligned}$$

$$24 = c(T \cup S) + c(T \cap S) \leq c(T) + c(S) = 25$$

(for this example)

Lemmas

Lemma 1: Let $\Gamma = (N, C, \{s_i\}_{i\in N}, \{t_i\}_{i\in N}, w)$ be a highway problem where C is a cycle of length 3. Then the corresponding highway game (N, C_Γ) is concave.

Idea of Proof of Lemma 1:

- For every coalition S, it is optimal to either construct one or two edges.
- If optimal to construct one edge $\{u, v\} \Rightarrow \{s_i, t_i\} = \{u, v\} \forall i \in N$.
- If optimal to construct two edges for a coalition then it must be optimal to construct these same.two edges for any other coalition

Lemmas

Lemma 2: Let $\Gamma = (N, G, \{s_i\}_{i \in N}, \{t_i\}_{i \in N}, w)$ be a highway problem where G is a weakly cyclic graph. Let $\mathscr{C}(G)$ denote the set of cycles in G and $\mathscr{BE}(G)$ denote the set of bridge edges in G. Then,

$$c_\Gamma(S) = \sum_{i \in \mathscr{C}(G)} c_{\Gamma^i}(S) + \sum_{j \in \mathscr{BE}(G)} c_{\Gamma^j}(S) \text{ for every } S \subseteq N$$

Proof: Straightforward.

Theorem 1

Theorem 1: A connected graph G is HG-concave if and only if it is weakly triangular.

Proof. [$\Leftarrow$] Suppose G is weakly triangular. Then G can be deconstructed as a bunch of bridge edges and cycles of length 3. Lemma 1 guarantees that every cycle of length 3 is HG-concave. It is trivially shown that every tree graph is HG concave and thus, every bridge edge is concave. Lemma 2 shows that the cost of any coalition of a weakly cyclic graph is equal to the sum of the costs of its individual cycles and bridge edges. Thus, G must be HG concave since it is a linear combination of HG concave graphs.

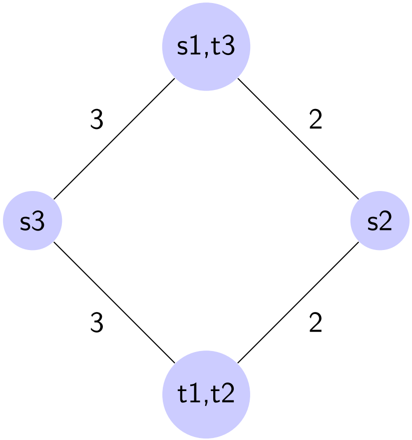
Theorem 1

Proof continued. [⇒] Suppose G is not weakly triangular. Then there exists a cycle in of length k such that $k \geq 4$. Let,

$$w(e) = \begin{cases} 2 & \text{if } e \in \{\{v_1, v_2\}, \{v_2, v_3\}\} \\ 0 & \text{if } e \in \{\{v_3, v_4\}, \ldots, \{v_{k-2}, v_{k-1}\}\} \\ 3 & \text{if } e \in \{\{v_k, v_1\}, \ldots, \{v_k, v_{k-1}\}\} \\ 100 & \text{if } e \notin C_k \end{cases}$$

Theorem 1

Proof continued. $[\Rightarrow]$ $N = \{1, 2, 3\}$. Let $S = \{1, 2\}$ and $T = \{1, 3\}$.



Theorem 1

Proof continued. $[\Rightarrow]$

$$c_\Gamma(S) = c_\Gamma(\{1,2\}) = 4$$
$$c_\Gamma(T) = c_\Gamma(\{1,3\}) = 4$$
$$c_\Gamma(S \cup T) = c_\Gamma(\{1,2,3\}) = 7$$
$$c_\Gamma(S \cap T) = c_\Gamma(\{1\}) = 4$$

$$\Rightarrow 11 = c_\Gamma(S \cup T) + c_\Gamma(S \cap T) > c_\Gamma(S) + c_\Gamma(T) = 8$$

$\therefore$ G is not HG concave ■

Implications of Theorem 1

This theorem can be used to:

- Guarantee optimality of algorithms
- Help determine the complexity of algorithms
- Use in further study of properties of weakly cyclic graphs and/or HG concave graphs

Relation to Lecture

- Approaches the problem by breaking graph down into smaller induced subgraphs
- Greatly utilizes idea and properties of paths and cycles in proof
- Uses properties of trees in proof
- Optimization problem, similar to lots of other problems we have discussed in class.

Reference

Baris Ciftci, Peter Borm, Herbert Hamers. *Highway Games on Weakly Cyclic Graphs*. 2009. *Stochastics and Statistics*. Elsevier. European Journal of Operational Research