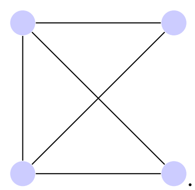
1. Is it possible to have a graph G with degree sequence $(4, 3, 3, 3, 2)$? If not, explain why not.

Solution. No, it is not possible to have a graph G to have a degree sequence $(4, 3, 3, 3, 2)$. To see why, notice that vertices with odd degree must come in pairs. This is true because every time we add an edge to a graph, then it increase the degree of exactly two vertices by one each. So the first edge will result in the degree sequence having two vertices of odd degree. And since every additional edge will change the parity of precisely two vertices, then the number of vertices with an odd degree will always be an even number.

□

2c. Find a graph in figure 1.9 isomorphic to



Solution. The graph from figure 1.9 that is isomorphic to the graph pictured above is the second graph from the left on the bottom row. To prove they are isomorphic, we will arbitrary label the vertices on the graph above and show that an isomorphism exists. The labeling of the vertices are:

Then let the original graph $G = (V = \{a, b, c, d\}, E = \{ab, ac, ad, bd, cd\})$ and the graph from figure 1.9 be $H = (V' = \{e, f, g, h\}, E' = \{ef, eg, eh, fg, gh, \})$. Let $f : V \to V'$ such that $f(a) = e, f(b) = f$, $f(c) = h, f(d) = g$. Then it is easily verified that f is both one to one and onto. And furthermore, the edge ab maps to $f(a)f(b) = ef$, ac maps to $f(a)f(c) = eh$, ad maps to $f(a)f(d) = eg$, bd maps to $f(b)f(d) = fg$, and cd maps to $f(c)f(d) = gh$. Thus, f is an isomorphism and this implies that the two graphs are isomorphic.

□

4b. Claim: If graph G_1 is isomorphic to graph G_2, then G_2 is isomorphic to G_1.

Proof. Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be graphs such that G_1 is isomorphic to G_2. Then there exists a bijective function $f : V_1 \to V_2$ such that for every edge $uv \in E_1$, the edge $f(u)f(v)$ is in E_2. To show that G_2 is isomorphic to G_1, we need to show that there exists another bijective function $g : V_2 \to V_1$ with the property that an edge $\hat{v}\hat{u}$ is in E_2 if and only if the edge $g(\hat{v})g(\hat{u})$ is in E_1. Let $g : V_2 \to V_1 : g = f^{-1}$. We know f^{-1} exists and is a bijective function since f is a bijection. Let uv be an edge in E_2. Then this edge is in E_2 if and only if there is an edge $xy \in E_1$ such that $uv = f(x)f(y)$. This is true if and only if $f^{-1}(u)f^{-1}(v) = xy = g(u)g(v)$. And since the edge we picked was arbitrary, then this is true for all edges. Thus, we have shown that an edge uv is in E_2, if and only if the function $g = f^{-1}$ maps that edge to an edge $xy \in E_1$. Therefore, g is an isomorphism from G_1 to G_2. $\therefore G_2$ is isomorphic to G_1.

■

4c. Claim: If G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, then G_1 is isomorphic to G_3.

Proof. Suppose $G_1 = (V_1, E_1)$ is isomorphic to $G_2 = (E_2, V_2)$ and G_2 is isomorphic to $G_3 = (V_3, E_3)$. Then there exists a bijection $f : V_1 \to V_2$ such that $u_1v_1 \in E_1 \iff f(u_1)f(v_1) \in E_2$. And there exists a bijection $g : V_2 \to V_3$ such that $u_2v_2 \in E_2 \iff g(u_2)g(v_2) \in E_3$. Let $h : V_1 \to V_3 : h = g \circ f$. The from elementary proofs, we know that h is a one to one correspondence between V_1 and V_3. To show that h preserves the edges between E_1 and E_3, let $u_1v_1 \in E_1$. We know $u_1v_1 \in E_1 \iff f(u_1)f(v_1) \in E_2$. And $f(u_1)f(v_1) \in E_2 \iff g(f(u_1))g(f(v_1)) \in E_3$. And since $h = g \circ f$, then we have shown

$$u_1v_1 \in E_1 \iff h(u_1)h(v_1) \in E_3.$$

$\therefore G_1$ is isomorphic to G_3.

■

7. Let G be a graph with n vertices and m edges. Prove that $\delta(G) \le 2m/n \le \Delta(G)$.

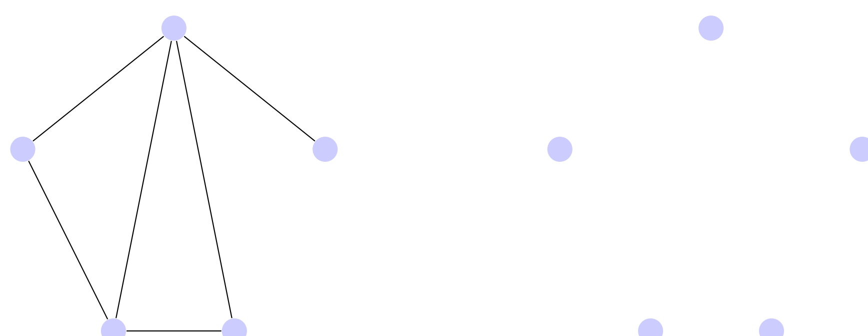
Proof. Let G be a graph with n vertices and m edges. First, observe that each edge m adds two to the total degrees of the vertices. So $\sum d(v_i) = 2m$. Next, let $[d(v_1), d(v_2), \ldots, d(v_n)]$ be the degree sequence of G. Then $\delta(G)$ is defined as $\min([d(v_1), d(v_2), \ldots, d(v_n)]$. Then it is apparent that $\delta(G)$ is maximized when the degrees of the vertices are spread out as evenly as possible. It follows the $\delta(G) \le \lfloor \frac{2m}{n} \rfloor \le \frac{2m}{n}$. Next, we notice that $\Delta(G)$ is minimized when the m edges are distributed as evenly as possible across the n vertices. It follows from this that $\Delta(G) \ge \lceil \frac{2m}{n} \rceil \ge \frac{2m}{n}$. Putting the two parts together, we have shown

$$\delta(G) \le 2m/n \le \Delta(G).$$

■

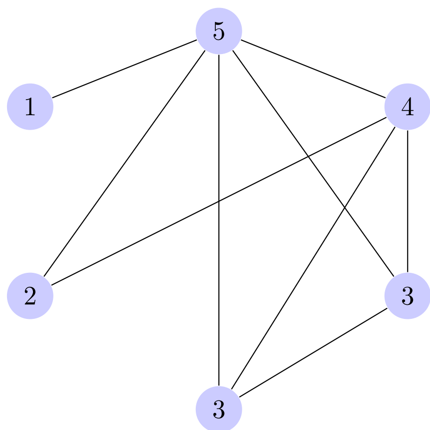
20a. Illustrate the connected antiregular graphs on 5 vertices.

Solution. First we construct the possible degree sequences for this antiregular graph to find $d(G) = [4, 3, 2, 1, 4]$ or $d(V) = [4, 3, 2, 1, 2]$. Notice these are only two possible degree sequences since we must have an even number of vertices with odd degree. Next, we notice that for the first degree sequence, however we draw the edges to satisfy the first four degrees $[4, 3, 2, 1]$, since we have a degree of one, then there will be no way to have an additional vertex with degree four. So this leaves the only possible degree sequence of $d(G) = [4, 3, 2, 1, 2,]$. After trial and error, the anitregular graph(s) are/(is):



20a. Illustrate the connected antiregular graphs on 6 vertices.

Solution. Using the same arguments as above, we come to the conclusion that the only viable degree sequence is $d(G) = [5, 4, 3, 3, 2, 1]$. Using brute force, we guess that the unique graph(s) with this degree sequence are/(is):



25. For the graphs in Example 1.22, compute $\det(xI_n - A(G_1))$.
Solution Using the mathematica, we use the following code to solve this:

```
Det[{{x, -1, -1, -1}, {-1, x, 0, 0}, {-1, 0, x, -1}, {-1, 0, -1, x}}]
1 - 2 x - 4 x^2 + x^4
```

This code is interpreted as

$$\det \begin{bmatrix} x & -1 & -1 & -1 \\ -1 & x & 0 & 0 \\ -1 & 0 & x & -1 \\ -1 & 0 & -1 & x \end{bmatrix} = x^4 - 4x^2 - 2x + 1$$