5 Prove that ω and α are graph invariants.

Proof that ω is invariant. Let $G = (V, E)$ and $H = (V', E')$ be graphs such that $G \cong H$. Let $f : V \to V'$ be an isomorphism. Suppose $\omega(G) = n$. Then this means that the largest clique in G has n vertices. Let $C = \{v_1, v_2, \cdots, v_n\}$ be the set of vertices for a largest clique in G. (We say *a* largest clique to imply that there could be multiple largest cliques. WLOG, proving for one still proves the theorem for all). By the definition of a clique, these vertices are all pairwise connected and isomorphic to K_n. Then let $C' = \{f(v_1), f(v_2), \cdots, f(v_n)\}$ be the set $\{f(v) : v \in C\}$. Since every $v \in C$ is pairwise adjacent to every other vertex in C, and since f is an isomorphism from G to H then it follows that every vertex in C' must be pairwise adjacent. And since f is one to one, then C' must contain precisely n vertices. Thus, the vertices of C' form a clique in H. To show that this must be a largest clique in H, we will show via contradiction that there cannot be a larger clique in H. So assume there exists a larger clique in H. Let C'' be the set of vertices in this larger clique. Then $C'' = \{v'_1, v'_2, \cdots, v'_m\} \subseteq V'$ where $m > n$. And since f is bijective, there exists a unique $u \in V$ for each $v' \in C''$. Thus we can write $C'' = \{f(u_1), \cdots f(u_m)\}$. Since each vertex in C'' is pairwise adjacent and f is an isomorphism, then every vertex in the set $\{u : f(u) \in C''\}$ must also be adjacent by the properties of a graph isomorphism. Thus the set $\{u : f(u) \in C''\}$ is a clique in G and has m vertices $\rightarrow\!\leftarrow$. But this contradicts our assumption that the largest clique in G had n vertices. Thus the largest clique in H must also have n vertices. Thus the clique number of $H, \omega(H) = n$. $\therefore \omega$ is a graph invariant.

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Proof that α is invariant. Let $G = (V, E)$ and $H = (V', E')$ be graphs such that $G \cong H$. Let $f : V \to V'$ be an isomorphism. Suppose the independence number of G, $\alpha(G) = n$. Then it follows that $\omega(G^c) = n$ since by the definition of $\alpha(G) = \omega(G^c) = n$. And since it is trivially shown that $G \cong H \iff G^c \cong H^c$, then we can use the fact that $\omega(G^c) = n \iff \omega(H^c) = n$ since we proved that ω is a graph invariant. And by the definition of independence number, $\alpha(H) = \omega(H^c) = n$. $\therefore \alpha$ is a graph invariant.

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