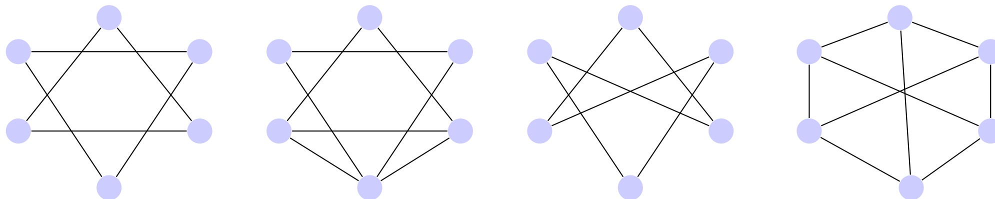
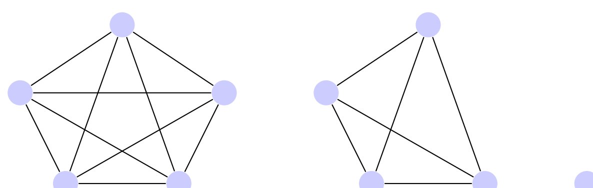
2. Determine the connectivity $\kappa(G)$ of the graphs in Exercise 1.



Solution Let G_i for $i \in \{1,2,3,4\}$ be the graphs in order as illustrated above. We see that $\kappa(G_1) = 0$ since G_1 already has two components. Next, we see that $\kappa(G_2) = 1$ since G_2 becomes disconnected by removing the bottom vertex. Next, we observe that G_3 is really just C_6, and so $\kappa(G_3) = \kappa(C_6) = 2$. And lastly, we show by exhausting all cases that $\kappa G_4 = 3$. We first remove one vertex and see that G_4 is connected. Notice it does not matter which vertex we remove since this graph is symmetric. Next, we check the result of removing each additional vertex and find that the graph is still connected. So $\kappa(G_4) > 2$. But observe that if we remove every other vertex of G_4 we are left with a disconnected set. Thus $\kappa(G_4) = 3$.

4. Find the flaw in the following proof: If $\varepsilon(G) = 0$, then $G = K_1$ or G is disconnected, in which cases $\kappa(G) = \varepsilon(G)$. Otherwise, suppose $\varepsilon(G) = t > 1$ and let $\{e_1, e_2, \ldots, e_t\}$ be a disconnecting set of edges of G. If $e_i = \{u_i, v_i\}, 1 \leq i \leq t$, then $S = \{u_i : 1 \leq i \leq t\}$ is a vertex cut. Hence, $\varepsilon(G) = t \geq o(S) \geq \kappa(G)$.

Solution. The first minor error in this proof is that it does not cover the case when $\varepsilon(G) = 1$. It states that fact about the case when $\varepsilon(G) = 0$ and then moves on to the cases when $\varepsilon(G) > 1$. The main flaw of this proof is the statement, *If* $e_i = \{u_i, v_i\}, 1 \leq i \leq t$ *is a disconnecting set, then* $S = \{u_i : 1 \leq i \leq t\}$ *is a vertex cut.* To see why this statement is false, consider K_5. Let $e_i = \{u_i, v_i\}, 1 \leq i \leq 4$ be the smallest disconnecting set of edges, $\varepsilon(G)$. Since our graph is K_5, then it is trivially shown that the smallest disconnecting set consists of all the edges required to isolate a single vertex. Then the set S as defined as $S = \{u_i : 1 \leq i \leq t\}$ either contains one vertex or four. In either case the graph $G - S$ is still connected as illustrated below. Another way to state this is that $G - S$ either becomes the middle graph or the single vertex on the right of the graphs below.



In either case, since we have shown that $G - S$ is connected, then S is not a vertex cut.

□

5. Corollary 3.4 *Suppose G is a connected graph. Let $e = uv \in E(G)$. Then e is a bridge if and only if no cycle of G contains contains both u and v.*

Proof. Let $G = (V, E)$ be a connected graph. Let $e = uv \in E(G)$. First, suppose that e is a bridge. Then by theorem 3.3, we know that the only path from u to v is $P = [u, v]$. Then this clearly cannot be a cycle since it is a path of length 1. And furthermore, the theorem shows that this must be the only path from u to v which means these vertices cannot possibly be in another path and thus a cycle.
Next, we show the opposite direction by the contrapositive. So assume that e is not a bridge and note that the term bridge is synonymous with cut-edge. Then by theorem 3.3, we know that the path $P = [u, v]$ is not the only path from u to v. Thus, there exist another path, call it P', such that $P' = [u, x_1, x_2, \ldots, x_k, v]$ for some $k \in \mathbb{N}$. We know that $k \geq 1$ since if $k = 0$, then G would no longer be a simple graph. So since these two distinct paths exist, and they both have endpoints of u and v, then we can construct a cycle by starting with P' and then attaching the path P to connect u and v. Thus we have shown there must exist a cycle that connects u to v. Hence, if e is not a bridge, then there always exists a cycle of G that connects u to v. Therefore, if no cycle of G contains both u and v, then e must be a bridge.

□

7 a. Prove that $\kappa(G)$ is invariant.

Proof. Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be simple graphs such that $G \cong H$. Let $f : V_1 \to V_2$ be the corresponding isomorphism. Suppose $\kappa(G) = k$. Then there exists a separating set $S = \{v_1, v_2, \ldots, v_k\} \subseteq V_1$ such that $|S| = k$ and $G - S$ is disconnected. And since this is the smallest separating set, then any other separating set will have at least as many elements as S. Next, partition V_1 into the two components we are guaranteed to have with $G - S$. Name these two components of $G - S$, $\hat{V}_1$ and $\bar{V}_1$. Next, let $S' = f[S] = [f(v_1), f(v_2), \ldots, f(v_k)]$, let $\hat{V}_2 = \{f(v) : v \in \hat{V}_1\}$ and let $\bar{V}_2 = \{f(v) : v \in \bar{V}_1\}$. To show that S' is a separating set by contradiction, let $v_2 \in \hat{V}_2$ and $u_2 \in \bar{V}_2$, and suppose there is some path in $H - S$ from v_2 to u_2. Then this path consists of vertices with that property that it starts at $v_2 \in \hat{V}_2$ and end with the vertex $u_2 \in \bar{V}_2$. Then there must exist a path in $G - S$ that starts at $f^{-1}(v_2) \in \hat{V}_1$ and ends at $f^{-1}(u_2) \in \bar{V}_2$. But this cannot happen since $(G - S)[\hat{V}_1]$ and $(G - S)[\bar{V}_1]$ are different components of $(G - S)$. $\rightarrow\!\leftarrow$ Thus, $H - S'$ must be disconnected. Thus, S' must be a separating set. To show that S' is the smallest separating set by contradiction, suppose there is a smaller separating set of H called S'' such that $|S''| < |S'|$. Then using the same argument we used to show that S' is a separating set with k elements, there must exist another separating set in G with the same number of elements as S''. But this breaks are statement that the smallest separating set of G was of order k. $\rightarrow\!\leftarrow$ Therefore S' is the smallest separating set of H. $\therefore \kappa(H) = k$. $\therefore \kappa(G)$ is invariant.

□

7b. Prove that $\varepsilon(G)$ is invariant.

Proof. Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be simple graphs such that $G \cong H$. Let $f : V_1 \to V_2$ be the isomorphism. Let the edge connectivity of G, $\varepsilon(G) = k$. The the smallest disconnecting set of edges D has order k. Partition V_1 into the two components we are guaranteed to have with $G - D$. Name these two components of $G - D$, $\hat{V}_1$ and $\bar{V}_1$. Let $\hat{V}_2 = \{f(v) : v \in \hat{V}_1\}$ and let $\bar{V}_2 = \{f(v) : v \in \bar{V}_1\}$. Let $D' = \{f(u)f(v) : uv \in D\}$. Then we will show that D' is a smallest disconnecting set in H and $\hat{V}_2$ and $\bar{V}_2$ are the two disconnected components. To first show that $H - D$ is disconnected, suppose there exist a past from some vertex in $\hat{V}_2$ to $\bar{V}_2$. Then since f is a bijection, there must also exist corresponding path from $\hat{V}_1$ to $\bar{V}_1$ $\rightarrow\!\leftarrow$. This surely cannot happen since $\hat{V}_1$ and $\bar{V}_1$ are disconnected. To show that D' is a smallest disconnecting set, suppose there is a smaller disconnecting set, D''. Then by the same argument we used to show that $D' = \{f(u)f(v) : uv \in D\}$ is a disconnecting set in H, there must exist another disconnecting set $D''' = \{f^{-1}(u)f^{-1}(v) : uv \in D''\}$. But this set must have the same number of edges as D'' which is less then D $\rightarrow\!\leftarrow$. This surely cannot happen since D is the smallest disconnecting set. Thus, D' must be a smallest disconnecting set in H. $\therefore \varepsilon(G)$ is invariant.

□

24 a. Find the girth of the graph in Figure 3.12.

Solution. By inspection, we see that the girth of this graph is three.

24 b. Find the curcumference of the graph in Figure 3.12.

Solution. By inspection, we see that the circumference of this graph is seven. We know it is seven because there is an easy path to see that goes around the outside of the graph. And it uses every vertex. So there cannot be a longer path since a path cannot be longer than the number of vertices.

24 c. Prove that $g \leq 2d + 1$, where $d = \mathrm{diam}(G)$ is the diameter of G.

Proof. Let G be a two-connected graph with girth g, and circumference c. First observe that for any cycle with n vertices or edges, the maximum distance between any two vertices is $\lfloor \frac{n}{2} \rfloor$. Suppose that $g > 2d + 1$. Then g has more than $2d + 1$ vertices. Then it follows that the diameter of g is

$$\begin{aligned} \mathrm{diam}(g) &> \lfloor \frac{2d+1}{2} \rfloor \\ &> \lfloor d + \frac{1}{2} \rfloor \\ &> d \\ &\rightarrow\!\leftarrow \end{aligned}$$

This surely cannot be true since d is the maximum distance in G. $\therefore g \leq 2d + 1$.

□

24 d. Prove or disprove that any two cycles of G of length c must have at least two vertices in common if G is two connected.

Proof. Let G be a two connected graph with circumference c. Let C_1, C_2 be two cycles in G such that they both have length c. To show that they must share at least two vertices in common, we will show by contradiction that they cannot share zero or only one vertex.

Case 1: C_1 and C_2 share zero vertices. Then since G is a simple graph and two connected, then there must exist at least two distinct paths from C_1 to C_2 (or else we could easily show that G is not 2-connected) Let C' be the cycle constructed by these two paths from C_1 to C_1 as well as the longest distance around each of C_1 and C_2 to the other path. Then if n is the number of vertices in c, then it follows that

$$\begin{aligned} C' &\geq 2 + 2(c - \lfloor \frac{n}{2} \rfloor) \\ &\geq 2 + 2c - n \\ &\geq 2 + c \\ &> c \end{aligned}$$

But this cannot happen since c is the longest cycle of G. $\rightarrow\!\leftarrow$

Case 2: C_1 and C_2 share one vertex. Since G is two-connected, then there must exist another path from C_1 to C_2. The shortest this path can be is length one. Let C' be the maximum cycle that consist of the two cycles C_1 and C_2 but also connecting at this new path of at least length one. Then if n is the number of vertices in c, we have

$$\begin{aligned} C' &\geq 1 + 2(c - \lfloor \frac{n}{2} \rfloor) \\ &\geq 1 + 2c - n \\ &\geq 1 + c \\ &> c \end{aligned}$$

Again, we know this cannot happen since c is the longest cycle of G. $\rightarrow\!\leftarrow$ So we have shown that the number of vertices shared by C_1 and C_2 cannot be equal to zero or one. Therefore, any two cycles of length c a 2-connected simple graph must share two or more vertices.

$\square$

25. Find $\kappa(G), \psi_G(u, v)$ and $\psi_G(u, w)$.

Solution. By inspection, we see that $\kappa(G) = 2$. $\psi_G(u, v) = 3$. $\psi_G(u, w) = 2$.