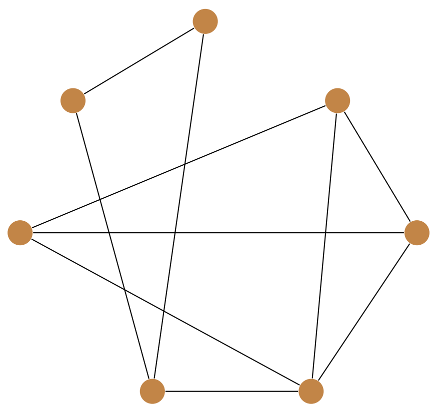
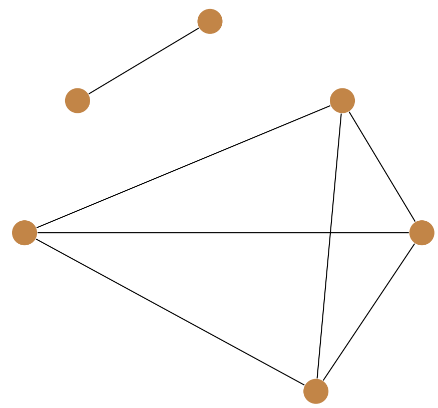
1a. Determine whether G is hamiltonian and justify your answer.

Solution. Recall that Theorem 5.2 shows us that *every hamiltonian graph is 2-connected.* We then observe what happens if we remove the following single vertex from the graph:

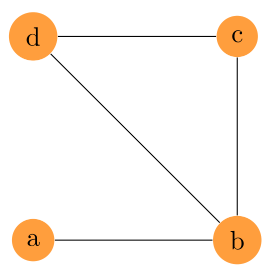


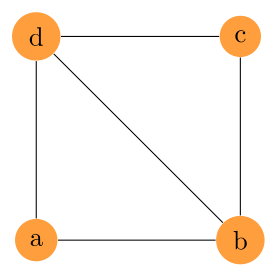


Thus since we showed that the removal of a single vertex resulted in an increase in the number components of G then G is not 2-connected. $\therefore G$ cannot be hamiltonian.

2. Show that the conclusion of Ore's Lemma does not follow from the weaker hypothesis $d(u)+d(v) \geq n-1$.

Solution. To show that Ore's Lemma does not hold for the weaker hypothesis, $d(u)+d(v) \geq n-1$, consider the two graphs below called G and G_1 respectively.

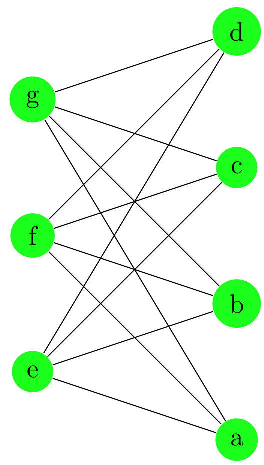




We see that $n = 4$ for these graphs. Also notice that the degree of vertex d in G is 2 and the degree of vertex a in G is 1. So $d(d) + d(a) \geq 4 - 1 = 3$. Thus we obtain the graph on the right G_1 by adding the edge ad. And clearly, G_1 is hamiltonian but G is not. Thus, Ore's Lemma does not hold for the weaker hypothesis, $d(u) + d(v) \geq n - 1$.

4a. Show that $K_{3,4}$ does not have any hamiltonian cycles.

Solution. To show that $K_{3,4}$ does not have any hamiltonian cycles, recall theorem 5.3 which states, *if G is hamiltonian, and S is a vertex cut, then the number of components of G-S is at most the cardinality of S.* Using the contrapositive of this theorem, let S be the vertex cut consisting of the vertices $\{g, f, e\}$. Then the number of components, $\xi(G - S) = 4$ and the cardinality of S is 3. Thus by theorem 5.3, $K_{3,4}$ cannot be hamiltonian.



18a. Show that C_4 is hamiltonian but not hamiltonian connected.

Solution. The definition of a hamiltonian cycle of a graph is a cycle that contains every vertex of the graph. A graph is called hamiltonian if it contains a hamiltonian cycle. So it follows that the graph, C_4 clearly contains a cycle that has every vertex of itself in it. Thus C_4 is hamiltonian. A *hamiltonian path* is a path that contains every vertex of the graph. A graph is *hamiltonian connected* if there exists a hamiltonian path from every two vertices in the graph. To show that C_4 is not hamiltonian, consider the graph of C_4 drawn below. Consider the vertices A and C. The only two paths that exist between A and C are $[A, D, C]$, $[A, B, C]$, $[C, B, A]$, and $[C, D, A]$. Since none of these paths are hamiltonian paths then C_4 is not hamiltonian connected. $\square$