Exercises 1a

1. Prove that if $\mathbf{a}$ is a vector of constants with the same dimension as the random vector $\mathbf{X}$, then

$$E[(\mathbf{X}-\mathbf{a})(\mathbf{X}-\mathbf{a})'] = \text{Var}[\mathbf{X}] + (E[\mathbf{X}]-\mathbf{a})(E[\mathbf{X}]-\mathbf{a})'$$

If $\text{Var}[\mathbf{X}] = \Sigma = (\sigma_{ij})$, deduce that

$$E[||\mathbf{X}-\mathbf{a}||^2] = \sum_i \sigma_{ii} + ||E[\mathbf{X}]-\mathbf{a}||^2$$

Solution. For the first part of the problem, let $\mathbf{Y} = \mathbf{X}$-$\mathbf{a}$. Then $\mathbf{X} = \mathbf{Y}$+$\mathbf{a}$. Thus,

$$\begin{aligned}
E[(\mathbf{X}-\mathbf{a})(\mathbf{X}-\mathbf{a})'] &= E[\mathbf{Y}\mathbf{Y}'] \\
&= \text{Var}[\mathbf{Y}] + E[\mathbf{Y}]E[\mathbf{Y}]' \\
&= \text{Var}[\mathbf{Y}] + E[\mathbf{Y}+\mathbf{a}-\mathbf{a}]E[\mathbf{Y}+\mathbf{a}-\mathbf{a}]' \\
&= \text{Var}[\mathbf{Y}] + (E[\mathbf{Y}+\mathbf{a}]-\mathbf{a})(E[\mathbf{Y}+\mathbf{a}]-\mathbf{a}') \\
&= \text{Var}[\mathbf{X}\text{+}\mathbf{a}] + (E[\mathbf{X}]-\mathbf{a})(E[\mathbf{X}]-\mathbf{a}') \\
&= \text{Var}[\mathbf{X}] + (E[\mathbf{X}]-\mathbf{a})(E[\mathbf{X}]-\mathbf{a}')
\end{aligned}$$

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Next, to show that $E[||\mathbf{X}-\mathbf{a}||^2] = \sum_i \sigma_{ii} + ||E[\mathbf{X}]-\mathbf{a}||^2$, notice,

$$\begin{aligned}
E[||\mathbf{X}-\mathbf{a}||^2] &= E[(\mathbf{X}-\mathbf{a})'(\mathbf{X}-\mathbf{a})] \\
&= E[(\mathbf{X}'-\mathbf{a}')(\mathbf{X}'-\mathbf{a}')'] \\
&= \text{Var}[\mathbf{X}'] + (E[\mathbf{X}]'-\mathbf{a}')(E[\mathbf{X}]'-\mathbf{a}')' && \text{by the identity proven above} \\
&= \text{Var}[\mathbf{X}'] + (E[\mathbf{X}]-\mathbf{a})'(E[\mathbf{X}]-\mathbf{a}) \\
&= \text{Var}[\mathbf{X}'] + ||E[\mathbf{X}]-\mathbf{a}||^2 \\
&= E[\mathbf{X}'\mathbf{X}] - E[\mathbf{X}]'E[\mathbf{X}] + ||E[\mathbf{X}]-\mathbf{a}||^2 \\
&= \sum_{i=1}^{n} E[\mathbf{X}_i^2] - E[\mathbf{X}_i]^2 + ||E[\mathbf{X}]-\mathbf{a}||^2 \\
&= \sum_{i=1}^{n} \sigma_{ii} + ||E[\mathbf{X}]-\mathbf{a}||^2
\end{aligned}$$

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3. Let $\mathbf{X}$ be a vector of random variables, and let $Y_1 = X_1, Y_i = X_i - X_{i-1}, i \in \{2, 3, \dots, n\}$. If the Y_i are mutually independent random variables, each with unit variance, find $\text{Var}[\mathbf{X}]$.

Solution. First, we observe that for any $X_i \in \mathbf{X}, X_i = \sum_{j=1}^{i} Y_i$. We also see that

$$\text{Var}[\mathbf{X}_i] = \text{Var}[\mathbf{Y}_i + \mathbf{Y}_{i-1} + \cdots + \mathbf{Y}_2 + \mathbf{Y}_1] = \text{Var}[\mathbf{Y}_i + \text{Var}[\mathbf{Y}_{i-1}] + \cdots + \text{Var}[\mathbf{Y}_2] + \text{Var}[\mathbf{Y}_1] = i$$

This follows since each of the pairwise covariances between the $\mathbf{Y}$s is zero and their coefficients are equal to one. Thus, we expand the covariance matrix of $\mathbf{X}$ to find,

$$\text{Var}[\mathbf{X}] = \text{Cov}[X_i, X_j]$$

$$= \begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1, X_2] & \dots & \text{Cov}[X_1, X_n] \\ \text{Cov}[X_2, X_1] & \text{Var}[X_2] & \dots & \text{Cov}[X_2, X_n] \\ \vdots & \dots & \ddots & \vdots \\ \text{Cov}[X_n, X_1] & \text{Cov}[X_n, X_2] & \dots & \text{Var}[X_n] \end{bmatrix}$$

$$= \begin{bmatrix} \text{Var}[Y_1] & \text{Cov}[Y_1, Y_2 + Y_1] & \dots & \text{Cov}[Y_1, Y_n + \cdots + Y_2 + Y_1 \\ \text{Cov}[Y_2 + Y_1, Y_1] & \text{Var}[Y_2 + Y_1] & \dots & \text{Cov}[Y_2 + Y_1, Y_n + \cdots + Y_2 + Y_1 \\ \vdots & \dots & \ddots & \vdots \\ \text{Cov}[Y_n + \cdots + Y_2 + Y_1, Y_1] & \text{Cov}[Y_n + \cdots + Y_2 + Y_1, Y_2 + Y_1] & \dots & \text{Var}[Y_n + \cdots + Y_2 + Y_1] \end{bmatrix}$$

Upon expansion of an arbitrary $\text{Cov}[X_i, X_j] = \text{Cov}[Y_i + Y_{i-1} + \cdots + Y_2 + Y_1, Y_j + Y_{j-1} + \cdots + Y_2 + Y_1]$, we quickly find the all of the covariance terms for $i \neq j$ go to zero and we are left with the $\min\{i, j\}$ variance terms which are each equal to one. Thus,

$$\text{Var}[\mathbf{X}] = \Sigma = \sigma_{ij} = \min\{i, j\}$$

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4. If $X_1, X_2, \ldots, X_n$ are random variables satisfying $X_{i+1} = \rho X_i$, where ρ is constant, and $\mathrm{Var}[X_i] = \sigma^2$, find $\mathrm{Var}[\mathbf{X}]$.

Solution. First, observe that each $X_i = \rho^{i-1} X_1$. We also get that $\mathrm{Var}[X_i] = \mathrm{Var}[\rho^{i-1} X_1] = \rho^{2i-2}\,\mathrm{Var}[X_1] = \rho^{(2i-2)}\sigma^2$. Next we look at an arbitrary covariance term and find $\mathrm{Cov}[X_i, X_j] = \mathrm{Cov}[\rho^{i-1}X_1, \rho^{j-1}X_1] = \rho^{i+j-2}\sigma^2$. Thus, when expanding the covariances matrix of $\mathbf{X}$, it is easy to see that,

$$
\begin{aligned}
\mathrm{Var}[\mathbf{X}] &= \mathrm{Cov}[X_i, X_j] \\
&= \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] & \ldots & \mathrm{Cov}[X_1, X_n] \\ \mathrm{Cov}[X_2, X_1] & \mathrm{Var}[X_2] & \ldots & \mathrm{Cov}[X_2, X_n] \\ \vdots & \ldots & \ddots & \vdots \\ \mathrm{Cov}[X_n, X_1] & \mathrm{Cov}[X_n, X_2] & \ldots & \mathrm{Var}[X_n] \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, \rho X_1] & \ldots & \mathrm{Cov}[X_1, \rho^{n-1} X_1] \\ \mathrm{Cov}[\rho X_1, X_1] & \mathrm{Var}[\rho X_1] & \ldots & \mathrm{Cov}[\rho X_1, \rho^{n-1} X_1] \\ \vdots & \ldots & \ddots & \vdots \\ \mathrm{Cov}[\rho^{n-1} X_1, X_1] & \mathrm{Cov}[\rho^{n-1} X_1, \rho X_1] & \ldots & \mathrm{Var}[\rho^{n-1} X_1] \end{bmatrix} \\
&= \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \ldots & \rho^{n-1}\sigma^2 \\ \rho\sigma^2 & \rho^2\sigma^2 & \ldots & \rho^n\sigma^2 \\ \vdots & \ldots & \ddots & \vdots \\ \rho^{n-1}\sigma^2 & \rho^n\sigma^2 & \ldots & \rho^{2n-2}\sigma^2 \end{bmatrix}
\end{aligned}
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Thus, we find that $\mathrm{Var}[\mathbf{X}] = \Sigma = \sigma_{ij} = \rho^{i+j-2}\sigma^2$

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