1. Show that if $rank(A) = q$, then

$$E[RSS_H - RSS] = \sigma^2 q + (A\beta)'(A(X'X)^{-}A')^{-1}(A\beta)$$

Proof. First observe that from section 3.8.1 and using formula 4.3, $\hat{\beta}_H = \hat{\beta} + (X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c)$, we find,

$$\begin{aligned}
RSS_H - RSS &= ||\hat{Y} - \hat{Y}_H||^2 \\
&= (\hat{\beta} - \hat{\beta}_H)'X'X(\hat{\beta} - \hat{\beta}_H). \\
&= (\hat{\beta} - \hat{\beta} + (X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c))'X'X(\hat{\beta} - \hat{\beta} + (X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c)) \\
&= ((X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c))'X'X((X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c)) \\
&= (A\hat{\beta} - c)'[A(X'X)^{-1}A']^{-1}A(X'X)^{-1}X'X((X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c)) \\
&= (A\hat{\beta} - c)'[A(X'X)^{-1}A']^{-1}A((X'X)^{-1}A'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c)) \\
&= (A\hat{\beta} - c)'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c)).
\end{aligned}$$

Next, from section 2.2 in the text, we know that $\hat{\beta} \sim N_p(\beta, \sigma^2(X'X)^{-1})$ and thus, $A\hat{\beta} \sim N_p(A\beta, \sigma^2 A(X'X)^{-1}A')$. Then if we let $Z = A\hat{\beta} - c$ and $B = A(X'X)^{-1}A'$, we get that

$$E[Z] = A\beta - c, \qquad \text{Var}[Z] = \text{Var}[A\hat{\beta}] = \sigma^2 B.$$

Now we refer to theorem 1.5 which shows the form of the expected value of quadratics forms,

$$\begin{aligned}
E[RSS_H - RSS] &= E[(A\hat{\beta} - c)'[A(X'X)^{-1}A']^{-1}(A\hat{\beta} - c)) \\
&= E[Z'B^{-1}Z] \\
&= tr(\sigma^2 B^{-1}B) + (A\beta - c)'B^{-1}(A\beta - c) \\
&= tr(\sigma^2 I_q) + (A\beta - c)'B^{-1}(A\beta - c) \\
&= \sigma^2 q + (A\beta - c)'B^{-1}(A\beta - c) \\
&= \sigma^2 q + (A\beta - c)'[A(X'X)^{-1}A']^{-1}(A\beta - c)
\end{aligned}$$

□

2. Let

$$
\begin{aligned}
Y_1 &= \theta_1 + \theta_2 + \epsilon_1 \\
Y_2 &= 2\theta_2 + \epsilon_2 \\
Y_3 &= -\theta_1 + \theta_2 + \epsilon_3
\end{aligned}
$$

where the ϵ_i are iid $N(0, \sigma^2)$. Derive the F statistic for testing the hypothesis $H : \theta_1 = 2\theta_2$.

Solution.

$\square$

3. Given the two regression lines

$$Y_{ki} = \beta_k + \epsilon_{ki}, \qquad\qquad k = 1, 2, \ldots, n.$$

show that the Fstatistic for testing $H : \beta_1 = \beta_2$ can be written as

$$F = \frac{(\hat{\beta}_1 - \hat{\beta}_2)^2}{(2S^2(\sum_i x_i^2)^{-1}}.$$

Obtain RSS and RSS_H and verify that $RSS_H - RSS = \sum_i x_i^2(\hat{\beta}_1 - \hat{\beta}_2)^2/2$

Solution.

□

4. Prove that $(RSS_H - RSS)/\sigma^2$ has a non-central chi-squared distribution with non-centrality parameter,

$$\lambda = \mu'(P_\Omega \ - P_\omega)\mu/2\sigma^2.$$

Solution.

$\square$