2b.

3. Suppose the $Y \sim N_3(\mu, \Sigma)$,, where

$$\mu = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Find the joint distribution of $Z_1 = Y_1 + Y_2 + Y_3$ and $Z_2 = Y_1 - Y_2$.

Solution. Notice $Z = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = CY$. By theorem 2.2, $Z \sim N_2(C\mu + d, C\Sigma C')$. And notice $C\mu = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and

$$\begin{aligned} C\Sigma C' &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 4 & 2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix}. \end{aligned}$$

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6. If Y_1, Y_2 are random variables such that $Y_1 + Y_2$ and $Y_1 - Y_2$ are independent $N(0,1)$ random variables, show that Y_1 and Y_2 have a bivariate normal distribution and find its mean and variance.

Solution. Let $X = X_1, X_2$ where $X_1 = Y_1 + Y_2$ and $X_2 = Y_1 - Y_2$. This means that $Y_1 = X_1 - Y_2$, and $Y_2 = Y_1 - X_2$. Thus $Y_1 = .5X_1 + .5X_2$ and $Y_2 = .5X_1 - .5X_2$. Thus by theorem 2.2,

$$Y \sim N_2(C\mu, C\Sigma C') \text{ with } C = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

And

$$C\mu = C * [0,0]'$$

and

$$\begin{aligned} C\Sigma C' &= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \end{aligned}$$

■

9. Let $X_1, ..., X_3$ be iid $N(0,1)$. Let $Y = AX$ where

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

Show the Y_is are iid standard normal.
Solution.
By theorem 2.2 , $E[Y] = 0$ and the variance of Y is

$$\begin{aligned} AA' &= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}' \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

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Exercises 2c, 3.
Let $Y \sim N_3(\mu, \Sigma)$ with

$$\Sigma = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}$$

For what value of ρ is $Y_1 + Y_2 + Y_3$ and $Y_1 - Y_2 - Y_3$ independent?

Solution. By theorem 2.2, $X = CY$. Variance of X is,

$$\begin{aligned} \operatorname{Var}[X] &= CYC'] \\ &= \begin{bmatrix} 2 + 4\rho & -1 - 2\rho \\ -1 - 2\rho & 3 \end{bmatrix} \end{aligned}$$

Thus, X is independent iff $\rho = 1/2$.

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2d, 3. If $Y \sim N_2(0, I_2)$ find the values of a, b such that

$$a(Y_1 - Y_2)^2 + b(Y_1 + Y_2)^2 \sim \chi^2$$

Solution. Using theorem 2.7, we need to express this problem as $Y'AY$ for an idempotent A. First notice that $(Y_1 - Y_2)^2 + (Y_1 + Y_2)^2 = X'X$ if $X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} Y = BY$. Thus if we let $A = [a \quad b]$, then $A'X'X = a(Y_1 - Y_2)^2 + b(Y_1 + Y_2)^2$. A quick calculation of the matrices yields,

$$\begin{aligned} a(Y_1 - Y_2)^2 + b(Y_1 + Y_2)^2 &= Y' \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} Y \\ &= Y' \begin{bmatrix} a+b & -a+b \\ -a+b & a+b \end{bmatrix} Y \end{aligned}$$

We have represented the problem with a symmetric matrix A. We now just need to find the values of $a, b \ni A^2 = A$. Setting $A^2 = A$, we get

$$A^2 = \begin{bmatrix} 2a^2 + 2b^2 & -2a^2 + 2b^2 \\ -2a^2 + 2b^2 & 2a^2 + 2b^2 \end{bmatrix}$$

And this equals A if and only if

$$\begin{aligned} 2a^2 + 2b^2 &= a + b \\ 2a^2 &= a + b - 2b^2 \\ 2a^2 - a &= \frac{1}{2} - 2 \cdot \frac{1}{4} \\ a(2a-1) &= 0 \\ a &= \frac{1}{2} \end{aligned} \qquad \text{and} \qquad \begin{aligned} -2a^2 + 2b^2 &= -a + b \\ -a - b + 2b^2 + 2b^2 &= -a + b \\ 4b^2 - 2b &= 0 \\ 2b(b-1) &= 0 \\ b &= \frac{1}{2} \end{aligned}$$

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2d 4. Suppose that $Y \sim N_3(0, I_n)$. Show that

$$\frac{1}{3}[(Y_1 - Y_2)^2 + (Y_2 - Y_3)^2 + (Y_3 - 1)^2]$$

has a χ^2_2 distribution.

Solution. First notice that we can write $\frac{1}{3}[(Y_1 - Y_2)^2 + (Y_2 - Y_3)^2 + (Y_3 - 1)^2]$ as

$$\frac{1}{3}[(Y_1 - Y_2)^2 + (Y_2 - Y_3)^2 + (Y_3 - 1)^2] = Y' \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} Y$$

Thus, by theorem 2.8, if we show that 2 of the eigenvalues of A are 1 and one is zero, then we are done. Using mathematica, we find,

```
Eigenvalues[{{2/3, -(1/3), -(1/3)}, {-(1/3), 2/
   3, -(1/3)}, {-(1/3), -(1/3), 2/3}}]

   {1, 1, 0}
```

Thus,

$$\text{Eigenvalues}\left[\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}\right] = [1, 1, 0]$$

Chapter 2 Miscellaneous Exercise 3. If $Y_1, ..., Y_n$ is random sample from $N(\mu, \sigma^2)$, prove $\bar{Y}$ is independent of $\sum_{i=1}^{n-1}(Y_i - Y_{i+1})^2$.

Solution. Notice that $\bar{Y} = n^{-1}1_n'Y$ and $\sum_{i=1}^{n-1}(Y_i - Y_{i+1})^2 = (AY)'(AY)$ where

$$A = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 & -1 \\ 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

So let $U = n^{-1}1_n'Y$ and $V = AY$ Then

$$\begin{aligned} Cov[U, V] &= Cov[n^{-1}1_n'Y, AY] \\ &= n^{-1}1_n'Cov[Y, Y]A' \\ &= \sigma^2 \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 & 0 \\ 0 & \cdots & \cdots & -1 & 1 \end{bmatrix} \\ &= \sigma^2 \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

Thus U, V are independent.

NOTE: I did not see in the theorem where this connects to showing that $\bar{Y}$ is independent from $(AY)'(AY)$. I think this is the case but have not made the connection yet.

Chapter 2 Miscellaneous Exercise 8 Let $Y \sim N_n(0, I_n)$, let A, B be symmetric idempotent matrices with $AB = BA = 0$. Show that $Y'AY, Y'BY$ and $Y'(I_n - A - B)Y$ have independent chi-square distributions.

Solution. By theorem 2.7, we get that each of these quadratic forms will have a χ^2 distribution. We only have to show independence. Thus, if we let $U = AY$, $V = BY$, and $W = (I_n - A - B)Y$, then

$$\begin{aligned} Cov(U,V) &= Cov(AY, BY) \\ &= A\,\mathrm{Var}[Y]B' \\ &= AI_nB' \\ &= AB \\ &= 0 \\ &= 0 \end{aligned} \qquad \begin{aligned} Cov(U,W) &= Cov[AY, (I_n - A - B)Y] \\ &= A\,\mathrm{Var}[Y](I_n - A - B)' \\ &= AI_n(I_n' - A' - B') \\ &= AI_n - A^2 - AB \\ &= A - A - 0 \\ &= 0 \end{aligned} \qquad \begin{aligned} Cov(V,W) &= Cov[BY, (I_n - A - B)Y] \\ &= B\,\mathrm{Var}[Y](I_n - A_B)' \\ &= BI_n(I_n' - A' - B') \\ &= BI_n - BA - B^2 \\ &= B - BA - B \\ &= 0 \end{aligned}$$

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