Exercises 1b

2. If $X_1, \ldots, X_n$ are independent random variables with common mean μ and variances $\sigma_1^2, \ldots, \sigma_n^2$, show that $\sum_i \frac{(X_i - \bar{X})^2}{n(n-1)}$ is an unbiased estimator of $\mathrm{Var}[\bar{X}]$.

Solution. First observe the following $\mathrm{Var}[X] = \frac{1}{n^2}\sum_i \sigma_i^2$ and

$$\begin{aligned}
\sum_i (X_i - \bar{X})^2 &= \sum_i (X_i^2 - 2X_i\bar{X} + \bar{X}^2) \\
&= \sum_i X_i^2 - 2\bar{X}\sum X_i + n\bar{X}^2 \\
&= \sum_i X_i^2 - 2n\bar{X}^2 + n\bar{X}^2 \\
&= \sum_i X_i^2 - n\bar{X}^2
\end{aligned}$$

Thus, we find

$$\begin{aligned}
E\left[\sum_i \frac{(X_i - \bar{X})^2}{n(n-1)}\right] &= \frac{1}{n(n-1)} E\left[\sum_i (X_i - \bar{X})^2\right] \\
&= \frac{1}{n(n-1)} E\left[\sum_i X_i^2 - n\bar{X}^2\right] \\
&= \frac{1}{n(n-1)} \left(\sum_i E[X_i^2] - nE[\bar{X}^2]\right) \\
&= \frac{1}{n(n-1)} \left(\sum_i (\sigma_i^2 + \mu^2) - n(\mathrm{Var}[\bar{X}] + \mu^2)\right) \\
&= \frac{1}{n(n-1)} \left(\sum_i \sigma_i^2 + n\mu^2 - n\,\mathrm{Var}[\bar{X}] + n\mu^2\right) \\
&= \frac{1}{n(n-1)} \left(n^2\,\mathrm{Var}[\bar{X}] - n\,\mathrm{Var}[\bar{X}]\right) \\
&= \frac{1}{n(n-1)} n(n-1)\,\mathrm{Var}[\bar{X}] \\
&= \mathrm{Var}[\bar{X}]
\end{aligned}$$

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5a. Let X_is be independent $N(\mu, \sigma^2)$ random variables. Let

$$S^2 = \frac{1}{n-1}\sum_i (X_i - \bar{X})^2$$

Show that $Var[S^2] = 2\sigma^4/(n-1)$.

Solution. First, we showed previously that $\sum_{i=1}^{n-1}(X_{i+1} - X_i)^2 = X'AX$ where $A = I_n - \frac{1}{n}J_n$. Thus $S^2 = \frac{1}{n-1}X'AX$. Thus, $\mathrm{Var}[S^2] = \mathrm{Var}[\frac{1}{n-1}X'AX]$. By theorem 1.6, if we denote θ_i to be the mean of X_i and μ_i to denote the i^{th} common central moment of any of the X_is and let a be the diagonal elements of A, we get,

$$\begin{aligned}
\mathrm{Var}[\frac{1}{n-1}X'AX] &= \frac{1}{(n-1)^2}\mathrm{Var}[X'AX] \\
&= \frac{1}{(n-1)^2}(\mu_4 - 3\mu^2)a'a + 2\mu_2^2 tr(A^2) + 4\mu_2\theta'A^2\theta + 4\mu_3\theta'Aa \\
&= \frac{1}{(n-1)^2}(3\sigma^4 - 3\sigma^4)a'a + 2\sigma^4 tr(A^2) + 4\sigma^2\theta'A^2\theta + 0 \\
&= \frac{1}{(n-1)^2}2\sigma^4 tr(A^2) + 4\sigma^2\theta'A^2\theta + 0 \\
&= \frac{1}{(n-1)^2}2\sigma^4 tr\left(\begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \cdots & \ddots & \vdots \\ -\frac{1}{n} & \cdots & \cdots & 1-\frac{1}{n} \end{bmatrix} \cdot \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \cdots & \ddots & \vdots \\ -\frac{1}{n} & \cdots & \cdots & 1-\frac{1}{n} \end{bmatrix}\right) + 4\sigma^2\sum(u_i - \bar{\theta})^2 \\
&= \frac{1}{(n-1)^2}2\sigma^4\left[\underbrace{\frac{(n-1)^2}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2}}_{\text{n times}} + \cdots + \underbrace{\frac{(n-1)^2}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} + \cdots + \frac{1}{n^2}}_{\text{n times}}\right] \\
&= \frac{1}{(n-1)^2}2\sigma^4\left[\underbrace{\frac{1}{n^2}[(n-1)^2 + (n-1)] + \frac{1}{n^2}[(n-1)^2 + (n-1)] + \cdots + \frac{1}{n^2}[(n-1)^2 + (n-1)]}_{\text{n times}}\right] \\
&= 2\sigma^4\left[\frac{n(n-1)^2 + (n-1))}{(n-1)^2 n^2}\right] \\
&= 2\sigma^4\frac{(n-1)+1}{(n-1)n} \\
&= 2\sigma^4\frac{n}{(n-1)n} \\
&= \frac{2\sigma^4}{n-1}
\end{aligned}$$

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5b. Let X_is be independent $N(\mu, \sigma^2)$ random variables. Let

$$Q = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2$$

Show that Q is an unbiased estimate of σ^2.

Solution. First let

$$Y = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

Then is follows that $Y = BX$,

$$Y = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = BX.$$

Notice that $Y'Y = \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2$. Thus $(BX)'(BX) = \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2$. And $(BX)'(BX) = X'B'BX$. So if we let $A = B'B$, then $2(n-1)Q = X'AX$. To find A we get

$$B'B = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} = A$$

Then, by theorem 1.5,

$$\begin{aligned} E[Q] &= \frac{1}{2(n-1)} E[X'AX] \\ &= \frac{1}{2(n-1)} (tr(A\Sigma) + \mu' A \mu) \\ &= \frac{1}{2(n-1)} \left(tr \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma^2 & 0 \\ 0 & \cdots & \cdots & 0 & \sigma^2 \end{bmatrix} + \mu' A \mu\right) \\ &= \frac{1}{2(n-1)} (\sigma^2 + 2n\sigma^2 + \sigma^2 + 0) \qquad \text{since } \mu' A = 0 \Rightarrow \mu' A \mu = 0 \\ &= \frac{1}{2(n-1)} (2n\sigma^2 + 2\sigma^2) \\ &= \frac{\sigma^2 \cdot 2 \cdot (n-1)}{2(n-1)} \\ &= \sigma^2 \end{aligned}$$

$\blacksquare$

5c. Find the variance of Q and then show that as $n \to \infty$, the efficiency relative to S^2 is $\frac{2}{3}$.

Solution. We already proved in part A that $\text{Var}[S^2] = 2\sigma^4/(n-1)$. To find the variance of Q, we will use its quadratic form representation shown in part b and then use theorem 1.6. Thus,

$$
\begin{aligned}
\text{Var}[Q] &= \text{Var}[\frac{1}{2(n-1)}X'AX] \\
&= \frac{1}{4(n-1)^2}\,\text{Var}[X'AX] \\
&= \frac{1}{4(n-1)^2}[(\mu_4 - 3\mu^2)a'a + 2\mu_2^2 tr(A^2) + 4\mu_2\theta' A^2\theta + 4\mu_3\theta' Aa] \\
&= \frac{1}{4(n-1)^2}[0 + 2\mu_2^2 tr(A^2) + 4\mu_2\theta' A^2\theta + 0] \\
&= \frac{1}{4(n-1)^2}[2\sigma^4 tr(A^2) + 4\sigma^2\theta' A^2\theta] \\
&= \frac{1}{4(n-1)^2}[2\sigma^4 tr(\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}) + 4\sigma^2\theta' A^2\theta] \\
&= \frac{1}{4(n-1)^2}[2\sigma^4 tr(\begin{bmatrix} 2 & -3 & 1 & 0 & \cdots & 0 \\ -3 & 6 & -4 & 1 & \ddots & \vdots \\ 1 & -4 & 6 & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & -4 & 1 \\ \vdots & \ddots & \ddots & -4 & 6 & -3 \\ 0 & \cdots & 0 & 1 & -3 & 2 \end{bmatrix}) + 4\sigma^2\theta' \begin{bmatrix} 2 & -3 & 1 & 0 & \cdots & 0 \\ -3 & 6 & -4 & 1 & \ddots & \vdots \\ 1 & -4 & 6 & \ddots & \ddots & 0 \\ 0 & 1 & \ddots & \ddots & -4 & 1 \\ \vdots & \ddots & \ddots & -4 & 6 & -3 \\ 0 & \cdots & 0 & 1 & -3 & 2 \end{bmatrix}\theta] \\
&= \frac{1}{4(n-1)^2}[2\sigma^4(4 + 6(n-2))] \\
&= \frac{1}{4(n-1)^2}[2\sigma^4(6n-8)] \\
&= \frac{4\sigma^4(3n-4)}{4(n-1)^2} \\
&= \frac{\sigma^4(3n-4)}{(n-1)^2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim_{n\to\infty} \text{Var}[Q]/\,\text{Var}[S^2] &= \lim_{n\to\infty} \frac{(3n-4)(n-1)}{2(n-1)^2} \\
&= \lim_{n\to\infty} \frac{3n^2 - 7n + 4}{2n^2 - 4n + 2} \\
&= \frac{3}{2}
\end{aligned}
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Miscellaneous Exercise 2. Let $X = [X_1, X_2, X_3]'$with

$$\text{Var}[X] = \begin{bmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

First find the variance of $Y = X_1 - 2X_2 + X_3$.
Solution. Notice

$$Y = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = AX$$

Since $\text{Var}[AX] = A\,\text{Var}[X]A'$, we get

$$\begin{aligned} \text{Var}[Y] &= A\,\text{Var}[X]A' \\ &= \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\ &= 18 \end{aligned}$$

Next, find the variance of $Y = (Y_1 \;\; Y_2)'$ where $Y_1 = X_1 + X_2$, and $Y_2 = X_1 + X_2 + X_3$.
Solution. Notice $Y = AX$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Thus, the variance of Y is

$$\begin{aligned} \text{Var}[Y] &= \text{Var}[AX] \\ &= A\,\text{Var}[X]A' \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 5 & 3 \\ 10 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 12 & 15 \\ 15 & 21 \end{bmatrix}. \end{aligned}$$

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