1. For a linear model $Y = X\beta + \epsilon$, with $E(\epsilon) = 0$ and $Cov(\epsilon) = \sigma^2 I_n$, the residuals are $\hat{e} = (I - P)Y$, where P is the projection matrix onto the column space of X. Find the following quantities:

a.)

$$\begin{aligned}
E(\hat{e}) &= E[(I-P)Y] \\
&= (I-P)E[X\beta + \epsilon] \\
&= (I-P)X\beta + E[\epsilon] \\
&= (I - X(X'X)^{-1}X')X\beta + 0 \\
&= X\beta - X(X'X)^{-1}X'X\beta \\
&= X\beta - X\beta \\
&= 0.
\end{aligned}$$

□

b.)

$$\begin{aligned}
Cov(\hat{e}) &= Cov[(I-P)(X\beta+\epsilon)] \\
&= (I-P)^2 Cov[\epsilon + X\beta] \\
&= (I-P)Cov[\epsilon] \\
&= (I-P)\sigma^2 I \\
&= (I-P)\sigma^2.
\end{aligned}$$

c.)

$$\begin{aligned}
Cov(\hat{e}, PY) &= Cov[(I-P)Y, PY] \\
&= (I-P)\,\mathrm{Var}[Y]P' \\
&= (I-P)\,\mathrm{Var}[\epsilon + X\beta]P' \\
&= (I-P)\,\mathrm{Var}[\epsilon]P' \\
&= (I-P)\sigma^2 P \\
&= \sigma^2(IP - P^2) \\
&= \sigma^2(P - P) \\
&= 0
\end{aligned}$$

□

d.)

$$\begin{aligned}
\hat{e}'\hat{e} &= Y'(I-P)'(I-P)Y \\
&= Y'(I-P)(I-P)Y \\
&= Y'(I-P)Y
\end{aligned}$$

□

e.)

$$\begin{aligned}
E[\hat{e}'\hat{e}] &= E[Y'(I-P)Y] \\
&= E[(X\beta+\epsilon)'(I-P)(X\beta+\epsilon)] \\
&= E[(X\beta+\epsilon)'(X\beta+\epsilon-PX\beta-P\epsilon)] \\
&= E[(X\beta+\epsilon)'(X\beta+\epsilon-X\beta-P\epsilon)] \\
&= E[(X\beta+\epsilon)'(\epsilon-P\epsilon)] \\
&= E[(X\beta+\epsilon)'(I-P)\epsilon] \\
&= E[\epsilon'+\beta'X')(I-P)\epsilon] \\
&= E[(\epsilon'+\beta'X'-\epsilon'P-\beta'X'P')\epsilon] \\
&= E[(\epsilon'+\beta'X'-\epsilon'P-\beta'X')\epsilon] \\
&= E[(\epsilon'-\epsilon'P)\epsilon] \\
&= E[\epsilon'(I-P)\epsilon] \\
&= tr[(I-P)\,\mathrm{Var}[\epsilon]] + E[\epsilon]'(I-P)E[\epsilon] \\
&= tr((I-P)\sigma^2)
\end{aligned}$$

□

2. Suppose $\hat{\beta}_1 \neq \hat{\beta}_2$ are two different least squares estimates of β. Show that there are infinitely many least squares estimates of β.

Solution. Since we are given that two solutions $\hat{\beta}_1 \neq \hat{\beta}_2$ exist, then it follows that X must not have full rank and thus there must be infinitely many solutions (since their generalized inverses must be different). To show this, let $\hat{\beta} = \alpha\hat{\beta}_1 + (1-\alpha)\hat{\beta}_2$ for $\alpha \in (0,1)$. Then plugging into the normal equations, we get,

$$\begin{aligned}
X'X\hat{\beta} &= X'X(\alpha\hat{\beta}_1 + (1-\alpha)\hat{\beta}_2) \\
&= X'X(\alpha(X'X)_1^-X'Y + (1-\alpha)(X'X)_2^-X'Y) \\
&= \alpha X'X(X'X)_1^-X'Y + (1-\alpha)X'X(X'X)_2^-X'Y \\
&= \alpha X'Y + (1-\alpha)X'Y \\
&= \alpha X'Y + X'Y - \alpha X'Y \\
&= X'Y
\end{aligned}$$

Thus we have shown that the normal equations hold for any $\alpha \in (0,1)$ showing there are infinitely many solutions.

□

3. Let $Y_1, ..., Y_n$ be a random sample from a distribution with mean θ and finite variance σ^2. Find BLUE of θ. Use the definition to justify that it is, in fact, the best linear unbiased estimate.

Solution. First notice that our model is $Y = \theta 1_n + \epsilon$ Thus the vector $X = 1_n$ has full rank. Then by the corollary to theorem 3.2, we know $a'\hat{\beta}$ is the BLUE of $a\beta$ for every vector a. Thus,

$$\begin{aligned}\hat{\beta} &= (X'X)^{-1}X'\hat{\theta} \\ &= \frac{1}{n}\sum Y \\ &= \bar{Y}\end{aligned}$$

$\square$

4. Let $Y_i = B_0 + B_1 X_i + \epsilon_i$ for $i \in \{1, \dots, n\}$, where $E[\epsilon] = 0$ and $\mathrm{Var}[\epsilon] = \sigma^2 I$. Prove that the least squares of β_0 and β_1 are uncorrelated iff $\bar{x} = 0$.

Proof. First observe that $\mathrm{Var}[Y] = \mathrm{Var}[Y - X\beta] = \mathrm{Var}[\epsilon] = \sigma^2 I_n$. Thus we find the variance of $\hat{\beta}$ is,

$$\begin{aligned}
\mathrm{Var}[\hat{\beta}] &= \mathrm{Var}[(X'X)^{-1}X'Y] \\
&= (X'X)^{-1}X' \,\mathrm{Var}[Y] X(X'X)^{-1} \\
&= (X'X)^{-1}X'\sigma^2 I_n X(X'X)^{-1} \\
&= \sigma^2 I_n (X'X)^{-1}X'X(X'X)^{-1} \\
&= \sigma^2 (X'X)^{-1}.
\end{aligned}$$

Thus it follows that $\hat{\beta}_0$ and $\hat{\beta}_1$ are uncorrelated if and only if $\mathrm{Var}[\hat{\beta}]_{ij} = 0, \forall i \neq j$. We also observe that since we are given there is a β_0 and a β_1 then

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \end{bmatrix}$$

and $\mathrm{Var}[\hat{\beta}]$ is a 2x2 matrix and we find that $X'X = \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$ To find $(X'X)^{-1}$, we augment the identity matrix and row reduce as follows:

$$\begin{bmatrix} n & \sum x_i & 1 & 0 \\ \sum x_i & \sum x_i^2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{n}\sum x_i & \frac{1}{n} & 0 \\ \sum x_i & \sum x_i^2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \bar{x} & \frac{1}{n} & 0 \\ 0 & \sum x^2 - n\bar{x}^2 & -\bar{x} & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & \frac{\frac{1}{n}\bar{x}^2}{\sum x^2 - n\bar{x}^2} & -\frac{\bar{x}}{\sum x^2 - n\bar{x}^2} \\ 0 & \sum x^2 - n\bar{x}^2 & -\bar{x} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{\frac{1}{n}\bar{x}^2}{\sum x^2 - n\bar{x}^2} & -\frac{\bar{x}}{\sum x^2 - n\bar{x}^2} \\ 0 & 1 & -\frac{\bar{x}}{\sum x^2 - n\bar{x}^2} & \frac{1}{\sum x^2 - n\bar{x}^2} \end{bmatrix}$$

Thus, we find that

$$\mathrm{Var}[\beta] = \sigma^2 (X'X)^{-1} = \begin{bmatrix} \frac{\frac{1}{n}\bar{x}^2}{\sum x^2 - n\bar{x}^2} & -\frac{\bar{x}}{\sum x^2 - n\bar{x}^2} \\ -\frac{\bar{x}}{\sum x^2 - n\bar{x}^2} & \frac{1}{\sum x^2 - n\bar{x}^2} \end{bmatrix}$$

and hence, $\hat{\beta}_0$ and $\hat{\beta}_1$ are independent if and only if $\bar{x} = 0$.

■