1a. Consider the following IVP:

$$u_{tt} = -u_{xxxx}, \qquad x \in \mathbb{R}, \qquad t > 0$$
$$u(x,0) = \phi(x), \qquad u_t(x,0) = 0, \qquad x \in \mathbb{R}.$$

Find the solution $\hat{u}(\xi, t)$ of the IVP in the transform domain.

Solution. Before we begin, we make the following observation. Even though we are told to find $\hat{u}(\xi, t)$ which suggests we are to take the Fourier transform of the PDE, we observe that the taking a Fourier transform is appropriate because the PDE is for x on the whole real line and we have a fourth derivative of u with respect to x. Fourier transform will reduce the fourth derivative with respect to x to an ODE with respect to t.

First, we take the Fourier transform of both sides of the PDE, (we notice since $\mathcal{F}[u] = \int_{-\infty}^{\infty} ue^{i\xi t}dx$ then we can pull out a constant times u $\mathcal{F}[-u] = -\int_{-\infty}^{\infty} ue^{i\xi t}dx = -\mathcal{F}[u]$)

$$\begin{aligned}
\mathcal{F}[u_{tt}] &= -\mathcal{F}[uxxxx] \\
\hat{u}_{tt} &= -(i\xi)^4\hat{u} && \text{by page 118, } \mathcal{F}[u^k] = (-i\xi)^k)\hat{u}(\xi) \\
\hat{u}_{tt} &= -(i^2 i^2 \xi^4)\hat{u} \\
\hat{u}_{tt} &= -\xi^4\hat{u} && \text{where } \hat{u} = \hat{u}(\xi, t)
\end{aligned}$$

We now have an ODE of $\hat{u}(\xi, t)$ of the form

$$\hat{u}_{tt} + \xi^4\hat{u} = 0$$

To solve this, we solve the corresponding characteristic equation $r^2 + \xi^4 = 0$. We find that $r = \pm(0 + \xi^2 i)$. From elementary differential equations, we know the solution of $\hat{u}$ has the form

$$\hat{u}(\xi, t) = a(\xi)\sin(\xi^2 t) + b(\xi)\cos(\xi^2 t)$$

Next, we take the Fourier transform of the initial condition of u to try to solve for $a(\xi)$ and $b(\xi)$. Thus, $\mathcal{F}[u(x,0)] = \mathcal{F}[\phi(x)] \Rightarrow \hat{u}(\xi, 0) = \hat{\phi}(\xi)$. Plugging this result into the expanded form of $\hat{u}$ we find,

$$\begin{aligned}
\hat{u}(\xi, 0) = \hat{\phi}(\xi) &= a(\xi)\sin(\xi^2 \cdot 0) + b(\xi)\cos(\xi^2 \cdot 0) \\
&= a(\xi) \cdot 0 + b(\xi) \cdot 1 \\
\Rightarrow \hat{\phi}(\xi) &= b(\xi).
\end{aligned}$$

Therefore, we have found the solution of $\hat{u}(\xi, t)$ in the transform domain,

$$\hat{u}(\xi, t) = \hat{\phi}(\xi)\cos(\xi^2 t).$$

$\diamondsuit$

1b. Find the solution $u(x,t)$.

Solution. To find the solution $u(x,t)$, we will use both the convolution theorem as applied to Fourier transforms and the specific inverse Fourier transform given in the problem, $\mathcal{F}^{-1}(\cos(\gamma\xi^2)) = \frac{\cos(\frac{x^2}{4\gamma}) + \sin(\frac{x^2}{4\gamma})}{\sqrt{8\pi\gamma}}$. First, the convolution theorem states that $\mathcal{F}^{-1}(\hat{u}(\xi)\hat{v}(\xi)) = \int_{-\infty}^{\infty} u(x-y)v(y)dy$. In words, this says that if we have a product of two Fourier transforms in the transform domain, then the inverse Fourier transform of this product is the convolution integral as shown above. Thus, we see that our solution $\hat{u} = \hat{\phi}(x)\cos(\xi^2 t) = \mathcal{F}[\phi(x)] \cdot \mathcal{F}[\frac{\cos(\frac{x^2}{4t}) + \sin(\frac{x^2}{4t})}{\sqrt{8\pi t}}]$. So $\hat{u}$ is a product of two Fourier transform and by the convolution theorem,

$$u(x,t) = \mathcal{F}^{-1}[\hat{u}(\xi, t)] = \mathcal{F}^{-1}[\hat{\phi}(\xi)\cos(\xi^2 t)] = \int_{-\infty}^{\infty} \phi(x-y) \cdot \frac{\cos(\frac{x^2}{4t}) + \sin(\frac{x^2}{4t})}{\sqrt{8\pi t}} dy.$$

$\diamondsuit$

2a. Consider the IVP for the heat equation with dissipation ($b > 0$) and a heat source ($\frac{1}{te^{bt}(x^2+1)}$)

$$u_t - ku_{xx} + bu = \frac{1}{te^{bt}(x^2+1)}, \quad ,x \in \mathbb{R}, \quad t > 0, \tag{1}$$
$$u(x,0) = 0, \quad x \in \mathbb{R}. \tag{2}$$

Let $u(x,t) = e^{-bt}v(x,t)$ and derive the new IVP for $v(x,t)$:

$$v_t - kv_{xx} = \frac{1}{t(x^2+1)}, \quad ,x \in \mathbb{R}, \quad t > 0 \tag{3}$$
$$v(x,0) = 0, \quad x \in \mathbb{R} \tag{4}$$

Solution. Let $u(x,t) = e^{-bt}v(x,t)$. Next, we solve for the various derivatives of u.

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}[e^{-bt}v(x,t)] \\ &= -be^{-bt}v + e^{-bt}v_t \end{aligned} \qquad \begin{aligned} u_x &= \frac{\partial}{\partial x}[e^{-bt}v(x,t)] \\ &= e^{-bt}v_x \end{aligned}$$

$$bu = be^{-bt}v \qquad \begin{aligned} u_{xx} &= \frac{\partial}{\partial x}[e^{-bt}v_x] \\ &= e^{-bt}v_{xx} \end{aligned}$$

We now observe that with our transformation from u to v transforms the equation $u_t - ku_{xx} + bu = \frac{1}{te^{bt}(x^2+1)}$, to

$$\begin{aligned} e^{-bt}v_t - be^{-bt}v - ke^{-bt}v_{xx} + be^{-bt}v &= \frac{e^{-bt}}{t(x^2+1)} \\ e^{bt}[e^{-bt}v_t - be^{-bt}v - ke^{-bt}v_{xx} + be^{-bt}v] &= e^{bt}\frac{e^{-bt}}{t(x^2+1)} && \text{multiply by } e^{bt} \\ v_t - \cancel{bv} - kv_{xx} + \cancel{bv} &= \frac{1}{t(x^2+1)} && \text{simplify} \\ v_t - kv_{xx} &= \frac{1}{t(x^2+1)} \end{aligned}$$

We also see that since $u(x,0) = 0$ for $x \in \mathbb{R}$, then $u(x,0) = 0 = e^0 v(x,0) = v(x,0)$. Now, we have a initial value problem for v.

$$v_t - kv_{xx} = \frac{1}{t(x^2+1)}, \quad x \in \mathbb{R}, \quad t > 0$$
$$v(x,0) = 0, \quad x \in \mathbb{R}.$$

◊

2b. Solve the IVP (3)-(4) from part (a.) to find the solution $v(x,t)$.

Solution. We see that the PDE of v is a diffusion equation with a source on an infinite spatial domain and with zero-valued initial condition. Thus we proceed by using the methods from section 2.5 and 2.1.
Let $w(x,t;\tau)$ be a solution to the following:

$$w_t - kw_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$
$$w(x,0;\tau) = \frac{1}{\tau(x^2+1)}, \quad x \in \mathbb{R}$$

By section 2.1, since we now have a pure heat equation with no source, and an initial condition $w(x,0;\tau) = f(x;\tau)$, then we know by the methods derived in section 2.1, that w has the form,

$$w(x,t;\tau) = \int_{-\infty}^{\infty} G(x-y,t)\frac{1}{\tau(y^2+1)}dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}}e^{-(x-y)^2/4kt}\frac{1}{\tau(y^2+1)}dy.$$

Now, by Duhamel's principal, since our function v and w have the all the required criteria outlined in section 2.5, we know that $v(x,t)$ has the form,

$$v(x,t) = \int_0^t w(x,t-\tau;\tau)d\tau = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-\tau)}}e^{-(x-y)^2/4k(t-\tau)}\frac{1}{\tau(y^2+1)}dyd\tau.$$

$\diamondsuit$

2c. Find the solution $u(x,t)$ to the original IVP (1)-(2).

Solution. Since we set $u(x,t) = e^{-bt}v(x,t)$ then it follows that

$$u(x,t) = e^{-bt}\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-\tau)}}e^{-(x-y)^2/4k(t-\tau)}\frac{1}{\tau(y^2+1)}dyd\tau.$$

$\diamondsuit$

3a. Consider the following IBVP,

$$u_{tt} = c^2 u_{xx}, \quad x > 0, \quad t > 0 \tag{5}$$
$$u_x(0,t) = 0, \quad t > 0 \tag{6}$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x > 0. \tag{7}$$

Find the solution $u(x,t)$.

Solution. We first notice that this is the wave equation on a semi infinite domain with a Neumann boundary condition and initial conditions for both u and u_t. Since we are on a semi-infinite domain, we will solve a similar PDE for $v(x,t)$ on an infinite domain and use the method of reflection to solve for u. Since we have a Neumann boundary condition, we will use *even* functions F and G to substitute for f and g. Let $v(x,t)$ be a solution for the following,

$$v_{tt} = c^2 v_{xx}, \quad x \in \mathbb{R}, \quad t > 0 \tag{8}$$
$$v_x(0,t) = 0, \quad t > 0 \tag{9}$$
$$v(x,0) = F(x), \quad u_t(x,0) = G(x), \quad x \in \mathbb{R}. \tag{10}$$

where

$$F(x) = \begin{cases} f(x) & \text{for } x > 0 \\ f(-x) & \text{for } x < 0, \end{cases} \qquad G(x) = \begin{cases} g(x) & \text{for } x > 0 \\ g(-x) & \text{for } x < 0. \end{cases}$$

Now we see that $v(x,t)$ is a Cauchy problem for the wave equation with boundary condition $v_x(0,t) = 0$ and functions for both initial conditions. Thus we know by our derivations in section 2.2 that v has the form,

$$v(x,t) = \frac{1}{2}[F(x-ct) + F(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds$$

We now make the following substitutions to change from F and G back to f and g. First, we observe that since u is only defined for $x \in (0,\infty)$, then we are only concerned about this region also. Thus we need to consider the cases when $x > ct$ and when $0 < x < ct$.

Case 1.$(x > ct)$ When $x > ct$ then $x - ct > 0$ and $x + ct > 0$. Thus $F(x) = f(x)$ and $G(x) = g(x)$ when $x > ct$.

Case 2.$(0 < x < ct)$ When $0 < x < ct$ then $x - ct < 0$ but $x + ct > 0$. So $F(x-ct) = f(ct-x)$, $G(x-ct) = g(ct-x)$, $F(x+ct) = f(x+ct)$, and $G(x+ct) = g(x+ct)$.
Solving for f and g within the function v, we find for $x > ct$

$$\begin{aligned} v(x,t) &= \frac{1}{2}[F(x-ct) + F(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds \\ &= \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds. \end{aligned}$$

And for the region $(0 < x < ct)$, we have

$$\begin{aligned} v(x,t) &= \frac{1}{2}[F(x-ct) + F(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)ds \\ &= \frac{1}{2}[f(ct-x) + f(x+ct)] + \frac{1}{2c}[\int_{x-ct}^{0} G(\hat{s})d\hat{s} + \int_{0}^{x+ct} G(s)ds] && \text{since } x-ct<0 \text{ and } x+ct>0 \\ &= \frac{1}{2}[f(ct-x) + f(x+ct)] + \frac{1}{2c}[\int_{x-ct}^{0} g(-\hat{s})d\hat{s} + \int_{0}^{x+ct} g(s)ds] && \text{since } x-ct<0 \\ &= \frac{1}{2}[f(ct-x) + f(x+ct)] + \frac{1}{2c}[\int_{ct-x}^{0} -g(s)ds + \int_{0}^{x+ct} g(s)ds] && \text{make substitution } s=-\hat{s} \\ &= \frac{1}{2}[f(ct-x) + f(x+ct)] + \frac{1}{2c}[\int_{0}^{ct-x} g(s)ds + \int_{0}^{x+ct} g(s)ds] && \text{use } -1 \text{ to flip limits of integration} \end{aligned}$$

And thus, we have found the solution for $v(x,t)$,

$$v(x,t) = \begin{cases} \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds & \text{for } x > ct \\ \frac{1}{2}[f(ct-x)+f(x+ct)] + \frac{1}{2c}[\int_0^{ct-x} g(s)ds + \int_0^{x+ct} g(s)ds] & \text{for } 0 < x < ct. \end{cases}$$

Therefore, when we restrict the spatial region to $x > 0$, then we have found the solution for $u(x,t)$,

$$u(x,t) = v(x,t) = \begin{cases} \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds & \text{for } x > ct \\ \frac{1}{2}[f(ct-x)+f(x+ct)] + \frac{1}{2c}[\int_0^{ct-x} g(s)ds + \int_0^{x+ct} g(s)ds] & \text{for } 0 < x < ct. \end{cases}$$

□

3b. Now consider the specific case with $c = 3$, $f(x) = \frac{x^2}{1+x^2}$, and $g(x) = 1 - e^{-x}$. Write the solution $u(x,t)$ in a form without integrals.

Solution. Since u is a piecewise function, we solve for each region separately. Let $c = 3$. For $x > ct$,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\frac{(x-3t)^2}{1+(x-3t)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[\int_{x-3t}^{x+3t} 1 - e^{-s}ds\right] \\ &= \frac{1}{2}\left[\frac{(x-3t)^2}{1+(x-3t)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[s + e^{-s}\Big|_{x-3t}^{x+3t}\right] \\ &= \frac{1}{2}\left[\frac{(x-3t)^2}{1+(x-3t)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[\cancel{x} + 3t + e^{-x-3t} - \cancel{x} + 3t - e^{3t-x}\right] \\ &= \frac{1}{2}\left[\frac{(x-3t)^2}{1+(x-3t)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[6t + e^{-x-3t} - e^{3t-x}\right] \end{aligned}$$

For $0 < x < ct$,

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\frac{(3t-x)^2}{1+(3t-x)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[\int_0^{3t-x} 1 - e^{-s}ds + \int_0^{x+3t} 1 - e^{-s}ds\right] \\ &= \frac{1}{2}\left[\frac{(3t-x)^2}{1+(3t-x)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[s + e^{-s}\Big|_0^{3t-x} + s + e^{-s}\Big|_0^{x+3t}\right] \\ &= \frac{1}{2}\left[\frac{(3t-x)^2}{1+(3t-x)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[(3t - x + e^{x-3t} - 1) + (x + 3t + e^{-x-3t} - 1)\right] \\ &= \frac{1}{2}\left[\frac{(3t-x)^2}{1+(3t-x)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[e^{x-3t} + e^{-x-3t} + 6t - 2\right] \end{aligned}$$

Therefore, the solution of $u(x,t)$ for the given functions f and g is

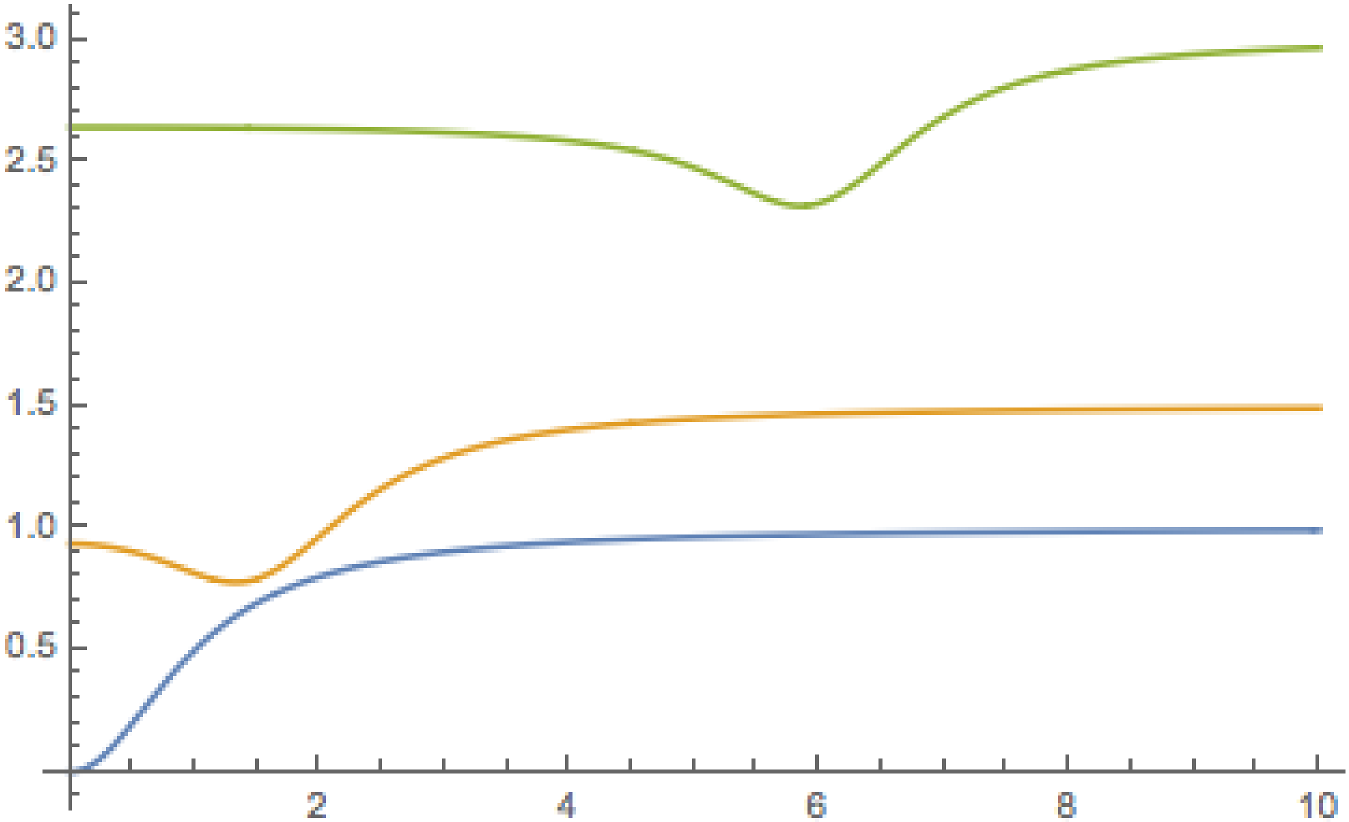
$$u(x,t) = \begin{cases} \frac{1}{2}\left[\frac{(x-3t)^2}{1+(x-3t)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[6t + e^{-x-3t} - e^{3t-x}\right] & \text{for } x > ct, \\ \frac{1}{2}\left[\frac{(3t-x)^2}{1+(3t-x)^2} + \frac{(x+3t)^2}{1+(x+3t)^2}\right] + \frac{1}{6}\left[e^{x-3t} + e^{-x-3t} + 6t - 2\right] & \text{for } 0 < x < ct. \end{cases}$$

♢

Note: We verified this result by referring to the text by Andrei D. Polyanin [1].

3c. Use technology to plot time snapshots at $t = 0, .5,$ and 2 for $0 \leq x \leq 10$.

Solution. We enter the piecewise function obtained in part 3b. in mathematica to make the following plot.



Legend: $t = 0$ $t = .5$ $t = 2$

4a. Consider the Cauchy problem for the heat equation

$$u_t = k u_{kk}, \qquad x \in \mathbb{R}, \qquad t > 0, \tag{11}$$
$$u(x,0) = \phi(x), \qquad x \in \mathbb{R} \tag{12}$$

with $k > 0$ constant. You will be guided through the steps to show that if $\phi(x)$ is odd, then for all $t > 0, u(x,t)$ is also an odd function of x. Given that the solution to the Cauchy problem is $u(x,t)$, show that

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x+y)^2/4kt} dy.$$

Solution. Let u be a solution to (11)-(12). By section 2.1, we know u is

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x-y)^2/4kt} dy.$$

We then observe the following result for $u(x,t)$ for $-x$,

$$\begin{aligned}
u(-x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(-x-y)^2/4kt} dy, \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-((-1)(x+y))^2/4kt} dy, \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-((-1)^2(x+y)^2)/4kt} dy, \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x+y)^2/4kt} dy.
\end{aligned}$$

$\diamondsuit$

4b. Assume $\phi(x)$ is an odd function. Show that $u(x,t) = -u(-x,t)$, and conclude that $u(x,t)$ is an odd function of x.

Solution. Suppose $\phi(x)$ is an odd function. Then $\phi(x) = -\phi(-x)$. Now we will use the result from (4a.) to show that $u(x,t) = -u(-x,t)$. Notice,

$$\begin{aligned}
-u(-x,t) &= -\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(\hat{y}) e^{-(x+\hat{y})^2/4kt} d\hat{y} \\
&= -\frac{1}{\sqrt{4\pi kt}} \int_{\infty}^{-\infty} -\phi(-y) e^{-(x-y)^2/4kt} dy && \text{Let } y = -\hat{y} \\
&= -\frac{1}{\sqrt{4\pi kt}} \int_{\infty}^{-\infty} \phi(y) e^{-(x-y)^2/4kt} dy && \text{since } \phi(x) = -\phi(-x) \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(y) e^{-(x-y)^2/4kt} dy && \text{flip limits of integration} \\
&= u(x,t)
\end{aligned}$$

Therefore, since have shown that $-u(-x,t) = u(x,t)$, we have shown that u is an odd function of x.

4c. Explain how this fact is useful when solving the PDE on the semi-infinite domain:

$$u_t = ku_{xx}, \qquad x > 0, \qquad t > 0, \tag{13}$$
$$u(0,t) = 0, \qquad t > 0, \tag{14}$$
$$u(x,0) = \phi(x), \qquad x > 0 \tag{15}$$

Solution. First, we will note that we know how to find the solution for Cauchy heat equation,

$$u_t = ku_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0,$$
$$u(x,0) = \phi(x), \qquad x \in \mathbb{R}$$

We will also take note of the results of 4b, that if $\phi(x)$ is an odd function, then $-u(-x,t) = u(x,t)$, where $u(x,t)$ is the solution to the Cauchy heat equation, $u(x,t) = \int_{-\infty}^{\infty} \phi(y)G(x-y,t)dy$

Now consider the PDE we get if we create a similar PDE

$$v_t = kv_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0, \tag{16}$$
$$v(x,0) = \psi(x), \qquad x \in \mathbb{R} \tag{17}$$

where

$$\psi(x) = \begin{cases} \phi(x) & \text{for } x > 0, \\ -\phi(-x) & \text{for } x < 0. \end{cases}$$

Since the equations (16)-(17) are the Cauchy problem for the heat equation, then we know that the solution $v(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\psi(y)dy$. Then we show,

$$\begin{aligned}
v(x,t) &= \int_{-\infty}^{\infty} G(x-y,t)\psi(y)dy \\
&= \int_{-\infty}^{0} G(x-y,t)\psi(y)dy + \int_{0}^{\infty} G(x-y,t)\psi(y)dy && \text{split up integral} \\
&= -\int_{-\infty}^{0} G(x-\hat{y},t)\phi(-\hat{y})d\hat{y} + \int_{0}^{\infty} G(x-y,t)\phi(y)dy && \text{since } \phi \text{ is odd} \\
&= -\int_{0}^{\infty} G(x+y,t)\phi(y)dy + \int_{0}^{\infty} G(x-y,t)\phi(y)dy && \text{let } y = -\hat{y} \\
&= \int_{0}^{\infty} (G(x-y,t) - G(x+y,t))\phi(y)dy
\end{aligned}$$

Thus we have re-derived the solution to the heat equation on a semi-infinite domain. We now have one subtlety to consider: Since $\psi(x)$ is an odd function, then we have shown that $v(x,t)$ must be an odd function of x, $-v(-x,t) = v(x,t)$. And since we are solving partial differential equations, then we have required v to be a differentiable function. Thus v should not have any jump discontinuities at $x = 0$. And thus, $-v(-x,t) = v(x,t) \iff v(0,t) = 0$. Thus, the initial condition has fallen out of our new Cauchy problem precisely due to our requirements that ϕ is odd and thus v is an odd function of x. So rewriting all we have derived about v, we have,

$$v_t = kv_{xx}, \qquad x \in \mathbb{R}, \qquad t > 0, \tag{18}$$
$$v(0,t) = 0 \qquad t > 0 \tag{19}$$
$$v(x,0) = \psi(x), \qquad x \in \mathbb{R} \tag{20}$$

And we have shown the solution to v. And now since both PDEs v and u are identical except for domain, we can just restrict the solution to $x > 0$,

$$u(x,t) = \int_{0}^{\infty} (G(x-y,t) - G(x+y,t))\phi(y)dy$$

References

[1] Polyanin, Andrei D., *Handbook of Linear Partial Differential Equations for Scientists and Engineers*. Washington D.C. Chapman & Hall, 2002. Print / PDF.