Section 2.5

1. Write a formula for the solution to the problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= \sin x, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_t(x,0) &= 0, \quad x \in \mathbb{R}. \end{aligned}$$

Graph the solution surface when $c = 1$.

Solution. First, let $w(x, t; \tau)$ be a solution to the following:

$$\begin{aligned} w_{tt} - c^2 w_{xx} &= 0, \quad x \in \mathbb{R} \;\; t > 0, \\ w(x, 0; \tau) &= 0, \quad x \in \mathbb{R}, \\ w_t(x, 0, \tau) &= \sin(x; \tau), \quad x \in \mathbb{R}. \end{aligned}$$

By d'Alembert's formula in section 2.2, we know that w has the form,

$$w(x, t, \tau) = \frac{1}{2c} \int_{x-ct}^{x+ct} \sin(s; \tau) ds.$$

Hence, by Duhamel's Principal, we know that u has the form,

$$\begin{aligned} u(x,t) &= \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin(s, \tau) ds d\tau \\ &= \frac{1}{2c} \int_0^t \cos(x - ct + c\tau)) - \cos(x + ct - c\tau) d\tau \\ &= \frac{1}{2c} \left(\frac{1}{c} \sin(x - ct + ct) - \frac{1}{-c} \sin(x + ct - ct) - \frac{1}{c} \sin(x - ct) + \frac{1}{-c} \sin(x + ct) \right) \\ &= \frac{1}{2c^2} (\sin(x) + \sin(x) - \sin(x + ct) - \sin(x - ct)) \\ &= \frac{1}{c^2} \sin(x) - \frac{1}{2c^2} \sin(x + ct) + \frac{1}{2c^2} \sin(x - ct). \end{aligned}$$

□

We use the following python code to plot the 3d surface of this solution when $c = 1$.

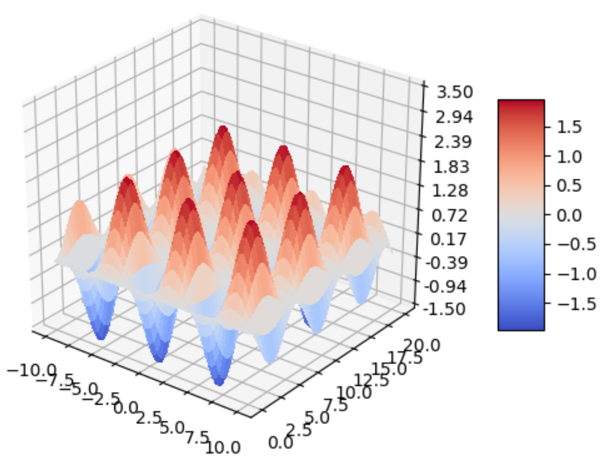
```
x = np.arange(-10, 10, .05)
t = np.arange(0, 20, .05)
x, t = np.meshgrid(x, t)
u = np.sin(x) - .5*np.sin(x+t) - .5*np.sin(x-t)

# Plot the surface.
surf = ax.plot_surface(x, t, u, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False

# Customize the z axis.
ax.set_zlim(-1.5, 3.5)
ax.zaxis.set_major_locator(LinearLocator(10))
ax.zaxis.set_major_formatter(FormatStrFormatter('%.02

# Add a color bar which maps values to colors.
fig.colorbar(surf, shrink=0.5, aspect=5)

plt.show()
```



4. Formulate Duhamel's principle and solve the initial boundary value problem

$$\begin{aligned} u_t &= ku_{xx} + f(x,t), \quad x > 0, \quad t > 0, \\ u(x,0) &= 0, \quad x > 0, \\ u(0,t) &= 0, \quad t > 0. \end{aligned}$$

The solution is

$$u(x,t) = \int_0^t \int_0^\infty (G(x-y,t-\tau) - G(x+y,t-\tau))f(y,\tau)dyd\tau.$$

Solution. We first notice that this is a non-homogeneous diffusion PDE with a source on a semi-definite spatial domain and with zero values for the initial and boundary conditions. First, we will use Duhamel's Principal to transform this PDE to a homogeneous one. Let $w(x,t;\tau)$ be a solution to the following,

$$\begin{aligned} w_t &= kw_{xx}, \quad x > 0, \quad t > 0, \\ w(x,0) &= f(x,\tau), \quad x > 0, \\ w(0,t) &= 0 \quad t > 0. \end{aligned}$$

Next, since we are on a semi-infinite domain, then by section 2.4 and on pg. 97, we know w has the form

$$w(x,t;\tau) = \int_0^\infty [G(x-y,t) - G(x+t,t)]f(y,\tau)dy, \quad x \geq 0.$$

(where G is Green's function) Now using Duhamel's Principal, we get that,

$$u(x,t) = \int_0^t \int_0^\infty [G(x-y,t-\tau) - G(x+y,t-\tau)]f(y,\tau)dyd\tau.$$

◇

Section 2.6
1.Solve the following using Laplace transforms.

$$u_{tt} = c^2 u_{xx} - g, \quad x > 0, \quad t > 0,$$
$$u(0,t) = 0, \quad t > 0,$$
$$u(x,0) = u_t(x,0) = 0, \quad x > 0.$$

The solution shows what happens to a falling cable lying on a table that is suddenly removed. Sketch some time snapshots of the solution.

Solution. We begin by taking the Laplace transform of both sides of the PDE, (for the rest of this paper, we denote $\mathcal{L}[u(x,t)] = U(x,s) = \int_0^\infty u(x,t)e^{-st}dt$). Also before we begin computation, we notice that the problem gives us just g for a non-homogenous term. Since this model describes something falling, we know that g is the acceleration due to gravity. $\frac{9.8m}{s^2}$.

$$\begin{aligned}
\mathcal{L}[u_{tt}] &= \mathcal{L}[c^2 u_{xx} - f(x,t)], \quad ,x > 0, \quad t > 0, \\
s^2 U(x,s) - su(x,0) - u_t(x,0) &= c^2 U_{xx} - \frac{g}{s} \\
s^2 U(x,s) &= c^2 U_{xx} - \frac{g}{s} && \text{by initial conditions, } u(x,0) = u_t(x,0) = 0 \\
U_{xx} - \frac{s^2}{c^2} U &= \frac{g}{sc^2} && \text{by algebra}
\end{aligned}$$

By method of undetermined coefficients (pg. 283), we find that U has the form,

$$U(x,s) = c_1(s)U_1(x,s) + c_2(s)U_2(x,s) + U_p(x),$$

where U_1, U_2 solve the associated homogenous problem. We solve the homogeneous problem by the characteristic equation method, $m^2 = \frac{s^2}{c^2}$. $\Rightarrow m = \pm s/c$. Thus the form of the solution to the homogenous problem is,

$$U(x,s) = c_1(s)e^{\frac{sx}{c}} + c_2(s)e^{\frac{-sx}{c}}$$

And since only a bounded solution will be sensible, we take $c_1(x) = 0$. Next, to solve to $U_p(x,s)$, we make the following initial guess $U_p(x,s) = A$. Then $U_{px} = 0$, and $U_{pxx} = 0$. . So $U_{pxx} - \frac{s^2}{c^2}U_p = \frac{g}{sc^2}$ becomes: $0 - \frac{s^2}{c^2}A = \frac{g}{sc^2}$. Therefore, $U_p = A = -\frac{g}{s^3}$. Now that we have the particular solution, we can take the Laplace transform of the boundary condition to solve for $c_2(s)$:

$$\begin{aligned}
\mathcal{L}[u(0,t)] &= \mathcal{L}[0], t > 0 \\
U(0,s) &= 0 = c_2(s)e^0 - g/s^3 \\
c_2(s) &= +g/s^3
\end{aligned}$$

Thus have now have our final solution in the x, s domain.

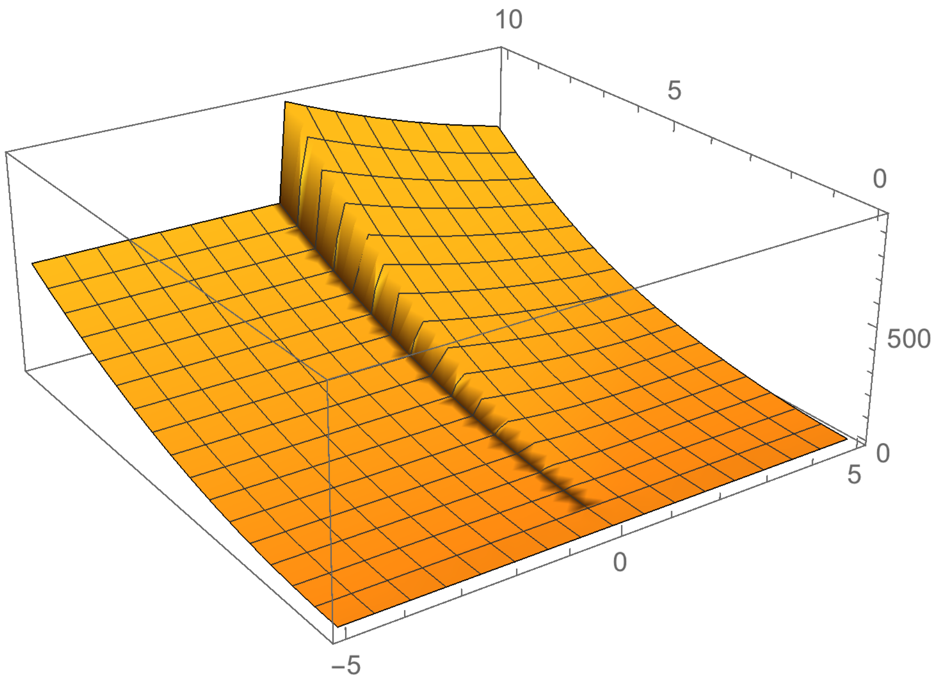
$$U(x,s) = g/s^3 e^{-sx/c} - g/s^3.$$

Thus to find u, we take the inverse Laplace transform using the convolution theorem,

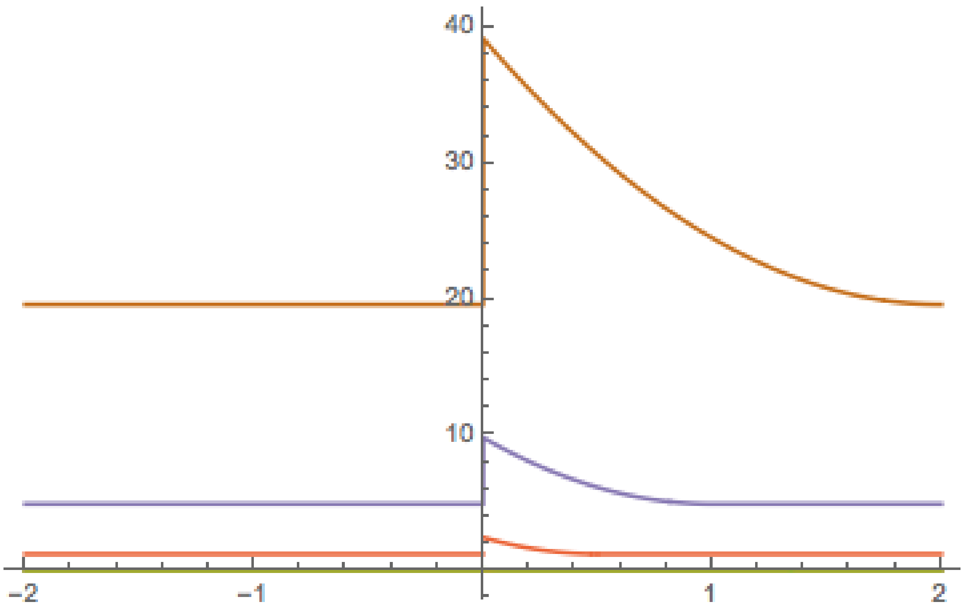
$$\begin{aligned}
u(x,t) = \mathcal{L}^{-1}[U] &= \mathcal{L}^{-1}[(\frac{2g}{2s^3})(e^{-sx/c}) - \mathcal{L}^{-1}[g/s^3] \\
&= \int_0^t (\frac{g(t-\tau)^2}{2})(\delta(\tau - x/c)d\tau - \frac{gt^2}{2}
\end{aligned}$$

We now enter our function into mathematica and observe the 3D plot to gain intuition about the solution and we also plot various time snapshots.

```
u[x_, t_] :=  Integrate[ (9.8 (t - h)^2 /2)* DiracDelta[h - x]   , {h, 0, t}] + (9.8/2)*t^2
Plot3D[u[x, t] , {x, -5, 5}, {t, 0, 10} ]
```




```
Plot[ {u[x, .001], u[x, .05], u[x, .1], u[x, .5], u[x, 1], u[x, 2]}, {x, -2, 2}, PlotRange -> All ,
 PlotLegends -> {"t=.001", "t=.0\.105", "t=.1", "t=.5", "t=1",  "t=2"}]
```



Note: The legend did not save with the rest of the image. The smallest value of t is the highest line and they follow in order.

2. In the quarter plane $x, y > 0$, where the temperature is initially zero, heat flows only in the y-direction; along the edge $y = 0$ heat is convected along the x-axis, and the termperature is constantly 1 at the point $x = y = 0$. The boundary value problem for the temperature $u(x, y, t)$ is

$$\begin{aligned} u_t &= u_{yy}, \quad x, t, y > 0, \\ u(x, y, 0) &= 0, \quad x, y > 0 \\ u(0, 0, t) &= 1, \quad t > 0, \\ u_t(x, 0, t) + u_x(x, 0, t) &= 0, \quad x, t > 0. \end{aligned}$$

Find a bounded solution using Laplace transforms.

Solution. We start by taking Laplace transforms of both sides of the equation.

$$\begin{aligned} \mathcal{L}[u_t] &= \mathcal{L}[u_{yy}] \\ sU - u(x, y, 0) &= U_{yy} \\ sU &= U_{yy} \\ U_{yy} - sU &= 0 \end{aligned}$$

Now we solve by solving the characteristic equation:

$$\begin{aligned} m^2 - s &= 0 \\ m &= \pm\sqrt{s} \\ \Rightarrow U(x, y, s) &= a(x, s)e^{\sqrt{s}y} + b(x, s)e^{-\sqrt{s}y} \end{aligned}$$

Since we only care about bounded solutions, then we can set $a(x, s) = 0$. Thus, we have

$$U(x, y, s) = b(x, s)e^{-\sqrt{s}y}.$$

We then solve for $b(x, s)$ by taking the Laplace transform of the boundary condition of y.

$$\begin{aligned} \mathcal{L}[u_t(x, 0, t)] + \mathcal{L}[u_x(x, 0, t)] &= \mathcal{L}[0] \\ sU(x, 0, s) - u(x, 0, 0) + U_x(x, 0, s) &= 0 && \text{now plug in out solution for } U \\ sb(x, 0)e^0 + \frac{\partial}{\partial x}b(x, 0)e^0 &= 0 \\ \frac{\partial}{\partial x}b(x, 0) + sb(x, 0) &= 0 && \text{rearranging} \\ \int \frac{\partial}{\partial x}b(x, 0) \cdot e^{sx}dx &= \int dx \text{ solving with ODE method, IF} \\ b(x, 0) &= e^{-sx}f(s) \end{aligned}$$

So far we have used the boundary condition to show that

$$U(x, y, s) = f(s)e^{-sx}e^{-\sqrt{s}y}.$$

We now take the Laplace transform to of $u(0, 0, t) = 1$ to solve for $f(s)$.

$$\begin{aligned} \mathcal{L}[u(0, 0, t) = 1] &= \mathcal{L}[1] \\ U(0, 0, s) &= 1/s = f(s) \end{aligned}$$

Thus our solution in terms of U is $U(x, y, s) = (1/s)e^{-sx}e^{-\sqrt{s}y}$. Thus,

$$u(x, y, t) = \mathcal{L}^{-1}[U(x, y, s)] = \mathcal{L}^{-1}[\left(s^{-1}e^{-\sqrt{s}y}\right) \cdot \left(e^{-sx}\right)] = \int_0^t \delta(t - \tau - x)(1 - \text{erf}(\frac{y}{2\sqrt{\tau}})d\tau$$