Section 1.7

3. Suppose $u = u(x, y, z)$ is a solution of the Nuemann problem,

$$-K\Delta u = f, (x, y, z) \in \Omega$$
$$-K\nabla u \cdot \mathbf{n} = g(x, y, z), (x, y, z) \in \partial\Omega$$

where f and g are functions of x, y, z. Show that f and g must satisfy the relation

$$\int_\Omega f dV = \int_{\partial\Omega} g dA$$

In terms of heat flow, what is the physical meaning of this relation?

Solution. Let f and g be described as above. To show that $\int_\Omega f dV = \int_{\partial\Omega} g dA$, we observe,

$$\begin{aligned}
\int_\Omega f dV &= \int_\Omega -K\Delta u dV \\
&= \int_\Omega -K\nabla^2 u dV \\
&= \int_\Omega -K(\nabla \cdot \nabla u) dV \\
&= \int_\Omega -K(div \nabla u) dV \\
&= \int_{\partial\Omega} -K\nabla u \cdot \mathbf{n} dA \\
&= \int_{\partial\Omega} g dA
\end{aligned}$$

The interpretation of these equations is that for a system to achieve a steady state, then f, the heat generated or lost within the region, must equal the heat gained or lost through the boundary of the region.

◇

4. Let w be a scalar field and ϕ a vector filed. Verify the vector identity

$$\text{div}(w\phi) = \phi \cdot \nabla w + w\text{div}\phi.$$

Integrate this equation over Ω and take $\phi = \nabla u$, where u is a scalar fielf, to prove Green's identity

$$\int_\Omega w\Delta u dV = -\int_\Omega \nabla u \cdot \nabla w dV + \int_{\partial\Omega} w\nabla u \cdot \mathbf{n} dA$$

Solution. First, to verify the vector identity,

$$\begin{aligned}
\text{div}(w\phi) &= \nabla \cdot (w\phi_1, w\phi_2, w\phi_3) \\
&= (\frac{\partial}{\partial x} w\phi_1 + \frac{\partial}{\partial y} w\phi_2 + \frac{\partial}{\partial z} w\phi_3) \\
&= w_x\phi_1 + w\phi_{1x} + w_y\phi_1 + w\phi_{1y} + w_z\phi_1 + w\phi_{1z} \\
&= \phi \cdot \nabla w + w(\phi_{1x} + \phi_{2y} + \phi_{3z} \\
&= \phi \cdot \nabla w + w \text{ div } \phi
\end{aligned}$$

To prove Green's identity, we will use the previous result starting with a slight rearrangement and substituting $\phi = \nabla u$, and by using Gauss's Theorem (aka divergence theorem).

$$
\begin{aligned}
w \text{ div } \nabla u &= -\nabla u \cdot \nabla w + \text{ div } (w \nabla u) \\
w(\nabla \cdot \nabla u) &= -\nabla u \cdot \nabla w + \text{ div } (w \nabla u) \\
w \Delta u &= -\nabla u \cdot \nabla w + \text{ div } (w \nabla u) \\
\int_\Omega w \Delta u dV &= \int_\Omega -\nabla u \cdot \nabla w + \text{ div } (w \nabla u) dV \\
\int_\Omega w \Delta u dV &= -\int_\Omega \nabla u \cdot \nabla w dV + \int_{\partial\Omega} w \nabla u \cdot \mathbf{n} dA
\end{aligned}
$$

$\diamondsuit$

5. Show that if the Dirishlet problem

$$
\begin{aligned}
\Delta u &= \lambda u, (x, y, x) \in \Omega \\
u &= 0, (x, y, z) \in \partial\Omega
\end{aligned}
$$

has a nontrivial solution $u = u(x, y, z)$, the λ must be negative.

Solution. We will use Green's identity as shown above and substitute in u in place of w. We take note that the reason we are able to make this substitution is because u and w are both scalar fields. Making the substitution, we get,

$$
\begin{aligned}
u \Delta u &= \lambda u^2 \\
\int_\Omega u \Delta u = \int_\Omega \lambda u^2 &= -\int_\Omega \nabla u \cdot \nabla u dV + \int_{\partial\Omega} u \nabla u \cdot \mathbf{n} dA \\
\lambda \int_\Omega u^2 &= -\int_\Omega \nabla u \cdot \nabla u dV + \int_{\partial\Omega} u \nabla u \cdot \mathbf{n} dA \\
\lambda \int_\Omega u^2 &= -\int_\Omega (u_x^2 + u_y^2 + u_z^2) dV + \int_{\partial\Omega} u \nabla u \cdot \mathbf{n} dA
\end{aligned}
$$

Now, we make a few observations before continuing here. First, we notice that the boundary condition $u = 0, (x, y, z) \in \partial\Omega$ implies that the integra $\int_{\partial\Omega} u \nabla u \cdot \mathbf{n} dA = 0$ since u is zero everywhere on the boundary. Next, we also notice that since $(u_x^2 + u_y^2 + u_z^2) \geq 0$ and the problem stated that there is a nontrivial solution, then we have $(u_x^2 + u_y^2 + u_z^2) > 0$ and $u^2 > 0$. So let $k = \int_\Omega u^2 dV$ and $j = \int_\Omega (u_x^2 + u_y^2 + u_z^2) dV$ Then $k, j > 0$. So finishing our calculation, we have,

$$
\begin{aligned}
\lambda \int_\Omega u^2 &= -\int_\Omega (u_x^2 + u_y^2 + u_z^2) dV + \int_{\partial\Omega} u \nabla u \cdot \mathbf{n} dA \\
\lambda k &= -j + 0 \\
\lambda &= -\frac{j}{k} \\
\lambda &< 0
\end{aligned}
$$

$\therefore \lambda$ must be negative.

$\square$

Section 1.8

1. In two dimensions suppose $u = u(r, \theta)$ satisfies Laplace's equation $\Delta u = 0$ in the disk $0 \leq r < 2$, and on the boundary it satisfies $u(2, \theta) = 3\sin 2\theta + 1, (0 \leq \theta < 2$. What is the value of u at the origin? Where do the maximum and minimum of u occur in the closed domain $0 \leq r \leq 2$?

Solution. Using the method described in the text, we will take an average of all temperatures on the boundary to do this. Using the equation for line integrals, we have

$$
\begin{aligned}
\int_C f(r,\theta)ds &= \frac{1}{2\pi r}\int_0^{2\pi}(3\sin 2\theta + 1)\sqrt{\left(\frac{\partial r\cos\theta}{\partial\theta}\right)^2 + \left(\frac{\partial r\sin\theta}{\partial\theta}\right)^2}\,d\theta \\
&= \frac{1}{2\pi r}\int_0^{2\pi}(3\sin 2\theta + 1)\cdot r\sqrt{\sin^2\theta + \cos^2\theta} \\
&= \frac{r}{2\pi r}\int_0^{2\pi}(3\sin 2\theta + 1)\cdot 1 \\
&= \frac{1}{2\pi}|3/2\cos 2\theta + \theta|_0^{2\pi} \\
&= \frac{1}{2\pi}\left[3/2 + 2\pi - (3/2 + 0)\right] \\
&= \frac{2\pi}{2\pi} \\
&= 1
\end{aligned}
$$

To find where the maximums and minimums occur, we use Theorem 1.23 (the maximum principal), which says if u satisfies Laplace's equation on an open, bounded, connected region, and if u is not a constant function, then the max and min of u are attained on the boundary of Ω. Thus, since a have a function for the values of u on the boundary, we take its derivative and set it to zero: $\frac{\partial}{\partial\theta}3\sin 2\theta + 1 = 6\cos 2\theta = 0$. If we sketch the graph, we easily see that the zeros are found at $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. We also can tell by the sign of the graph around these points that the max are found at $\pi/4$ and $5\pi/4$ and the mins are found at $3\pi/4, 7\pi/4$.

$\square$

4. Find all radial solutions to the two-dimensional Laplace's equation. That is, find all solutions of the form $u = u(r)$, where $r = \sqrt{x^2 + y^2}$. Find the steady=state temperature distribution in the annular domain$1 \leq r \leq 2$ if the inner circle $r = 1$ is held at 0 degrees and the outer circle $r = 2$ is held at 10 degrees.

Solution. To find the general solution to the 2D Laplace equation, we first need to find u_{xx} and u_{yy}. As stated above, let $u = u(r)$, with $r = \sqrt{x^2 + y^2}$. Then $u_x = u'(r)\frac{x}{(x^2+y^2)^{\cdot 5}}$ Thus, $u_{xx} = u''(r)\frac{x}{(x^2+y^2)^{\cdot 5}} \cdot \frac{x}{(x^2+y^2)^{\cdot 5}} + \frac{y^3}{(x^2+y^2)^{-1.5}}$. Using the symmetry of the functions u and r, we know that $u_{yy} = u''(r)\frac{y}{(x^2+y^2)^{\cdot 5}} \cdot \frac{y}{(x^2+y^2)^{\cdot 5}} + \frac{x^3}{(x^2+y^2)^{-1.5}}$. Thus, since $r = \sqrt{x^2 + y^2}$, we substitute back in to get,

$$u_{xx} = \frac{x^2}{r^2}u''(r) + \frac{y^2}{r^3}, u_{yy} = \frac{y^2}{r^2}u''(r) + \frac{x^2}{r^3}$$

Thus, the Laplace equation $u_{xx} + u_{yy} = 0$ becomes

$$\Delta u = u'' + \frac{1}{r}u' = 0 \textit{ (from cylindrical coordinate in polar part of chapter)}$$

We then notice that that this equation is an expansion of the product rule for $\frac{d}{du}(ru') = 0$. Thus

$$\begin{aligned}\int \frac{d}{dr}(ru')dr &= \int 0dr\\ ru' &= c\\ u' &= \frac{c}{r}\\ \int u'dr &= \int \frac{c}{r}dr\\ u(r) &= c_1 \ln r + c_2\end{aligned}$$

Thus, we have found our general form of $u(r)$. Now using the conditions, we get $u(1) = 0 = c_2$. And $u(2) = 10 = c_1 \ln 2 \Rightarrow c_1 = \frac{10}{\ln 2}$

$$\therefore u(r) = \frac{10}{\ln 2} \cdot \ln r$$

□

Section 1.9
1. Classify the PDE

$$u_{xx} + 2ku_{xt} + k^2 u_{tt} = 0, k \neq 0$$

Then find a transformation $\xi = x + bt, \tau = x + dt$ of the independent variables that transforms the equation into a simpler equation of the form $U_{\xi\xi} = 0$. Find the solution to the given equation in terms of two arbitrary functions.

Solution. To classify this PDE, we take the A, B, C coefficients and find $B^2 - 4AC = 4k^2 - 4k^2 = 0$. Thus, this PDE has a parabolic form. As suggested in the problem, we let $\xi = x + bt, \tau = x + dt$ and solve for b, d. Our original PDE , $u_{xx} + 2ku_{xt} + k^2 u_{tt} = 0$ becomes,

$$Au_{xx} + Bu + xt + Cu_{tt} = (Aa^2 + Bab + Cb^2)U_{\xi\xi} + (2acA + B(ad + bc) + 2Cbd)U_{\xi\tau} + (Ac^2 + Bcd + Cd^2)U_{\tau\tau}$$

From our notes in class, we know to let $a = b = c = 1$ and $d = -\frac{B}{2C}$. This will make our coefficients for $U_{\tau\tau}$ and $U_{\xi\tau}$ disappear. Thus, we are left with $U_{\xi\xi} = 0$. We integrate to find, $U_\xi = \phi(\tau)$. And integrate again to find $U = \xi\phi(\tau) + \psi$. So transforming back to our x and t variables find our solution, $u = x \cdot \phi(x + dt) + \psi(x + dt)$. And since $d = -\frac{B}{2C} = \frac{1}{k}$,

$$\therefore u = x \cdot \phi(x - \frac{t}{k}) + \psi(x - \frac{t}{k})$$

□