Math 108:
Linear Algebra

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Ryerson University

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Some Course Information

About the Course

About Me

About Remote Learning

Some Course Information

There are two D2L shells for this class at my.ryerson.ca.

One is a **common course shell** for all sections and the other is **specifically for my sections** (Sections 4-6).

Use the **common course shell** to:

- find the **Course Management Form** (and **Homework Problems**),
- submit the **midterm** and **final exam.**

Use the **one for my sections** to:

- find the **Course Management Form** (and **Homework Problems**),
- submit **weekly homework** and **weekly quizzes**,
- access Zoom links to my live lectures,
- access recordings of my live lectures.

About the Course

Textbook:
Linear Algebra,
by Kunquan Lan (fourth edition), Pearson, 2020.

No lab this week! Check the **Course Management Form** for your lab time. First lab and lab quiz next week.

Midterm test: 8:10-9:50am, Tuesday, October 20, 2020.

About the Course

Week	Sections	Topics
1 (Sept. 8, 11)	1.1–1.3	Euclidean Spaces
2 (Sept. 15, 18)	1.4, 2.1–2.3	Matrices
3 (Sept. 22, 25)	2.4, 2.5	Matrices
4 (Sept. 29, Oct. 2)	2.7, 3.1–3.3	Determinants
5 (Oct. 6, 9)	3.4, 4.1, 4.2	Systems of Linear Eqns
6 (Oct. 12–16)	Reading Week	
7 (Oct. 23)	4.3–4.5	Systems of Linear Eqns
8 (Oct. 27, 30)	4.6, 5.1, 5.2	Linear transformations,
9 (Nov. 3, 6)	5.3, 6.1, 6.2	Planes and Lines in $\mathbb{R}^3$
10 (Nov. 10, 13)	6.3, 7.1	Bases and Dimension
11 (Nov. 17, 20)	8.1, 8.2	Eigenvalues
12 (Nov. 24, 27)	10.1, 10.2	Complex Numbers
13 (Dec. 1, 4)	10.3, 10.4	Complex Numbers

About the Course

- 10% **Weekly Homework** turn in on D2L
- 12% **Lab Quizzes** turn in on D2L
- 36% **Midterm test** 100 minutes,
  8:10-9:50, Tuesday, October 20, 2020.
- 42% **Final Exam** 2 hours,
  during the exam period (December 9-19).

Check the course outline (on D2L) for more details and notes, including what to do in case of illness, policy on accommodations, etc.

About Me

Dr. Michelle Delcourt
Virtual office hours: Thursdays: 10:00am to noon via Zoom
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Please email from your `@ryerson.ca` email address - otherwise your email may be ignored.

Please include MTH 108 in the subject line.

About Remote Learning

There are many resources for my sections are available:

- Discussion Boards on D2L
- Synchronous classes via Zoom (attendance required)
- Links to recordings posted on D2L

Section 1.1 Euclidean Spaces

Notation and Terminology

- $\mathbb{N}$ denotes the set of positive integers, $\{1, 2, \ldots\}$.
- $\mathbb{R}$ denotes the set of real numbers.
- for x an element of $\mathbb{R}$, we write $x \in \mathbb{R}$.
- for $n \in \mathbb{N}$, we define $I_n = \{1, 2, \ldots, n\}$.

Definition

The set containing all elements of n-ordered numbers is called $\mathbb{R}^n$ (an n-dimensional Euclidean space).

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ and } i \in I_n\}.$$

Geometrically

- $\mathbb{R}^1$ is the x-axis ($\mathbb{R}^1 = \mathbb{R}$)
- $\mathbb{R}^2$ is the xy-plane
- $\mathbb{R}^3$ is the xyz-space

What about $\mathbb{R}^n$ in general?

For $n \geqslant 4$, $\mathbb{R}^n$ exist (higher dimensions).

Definition

An element $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ is a point, and $x_1, x_2, \ldots, x_n$ are the coordinates of this point.

Notation for Points

We can write $(x_1, x_2, \ldots, x_n)$, $P(x_1, x_2, \ldots, x_n)$, or P.

The Origin

$(0, 0, \ldots, 0)$ is a special point in $\mathbb{R}^n$ called the origin.

We can write $(0, 0, \ldots, 0)$, $O(0, 0, \ldots, 0)$, or O for the origin.

Vectors

Definition

Let $P \in \mathbb{R}^n$. The directed line segment from the origin O to point P is a vector, written $\overrightarrow{OP}$.

Notation for Vectors

For $\overrightarrow{u} = (x_1, x_2, \dots, x_n)$, x_i is the ith component or ith entry.

$\overrightarrow{u}$ can be written either as a column vector

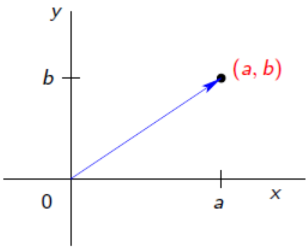
$$\overrightarrow{u} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

or a row vector

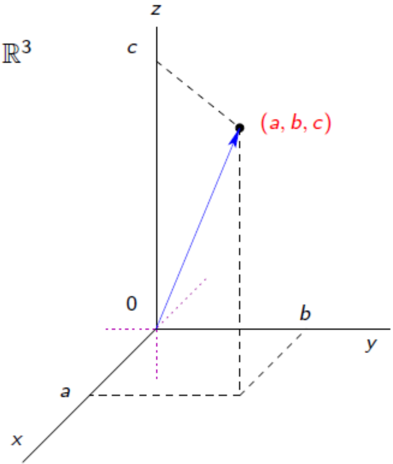
$$\overrightarrow{u} = (x_1, x_2, \dots, x_n).$$

Note that $(x_1, x_2, \dots, x_n)^T$ is a column vector.

$\mathbb{R}^2$



$\mathbb{R}^3$



Standard Vectors

$$\overrightarrow{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \overrightarrow{e_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \overrightarrow{e_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

are the standard vectors of $\mathbb{R}^n$.

The standard vectors of $\mathbb{R}^2$ are

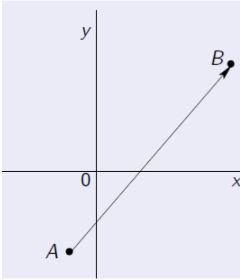
$$\overrightarrow{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \overrightarrow{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The standard vectors of $\mathbb{R}^3$ are

$$\overrightarrow{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \overrightarrow{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \overrightarrow{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Position Vectors Between Points

Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be two points in $\mathbb{R}^n$.



$$\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix}$$ is the position vector from A to B.

A is the initial point and B is the terminating point of $\overrightarrow{AB}$.

Position Vectors Between Points

Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be two points in $\mathbb{R}^n$.
Let $C = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$.

$$\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix} = \begin{pmatrix} b_1 - a_1 - 0 \\ b_2 - a_2 - 0 \\ \vdots \\ b_n - a_n - 0 \end{pmatrix} = \overrightarrow{OC}.$$

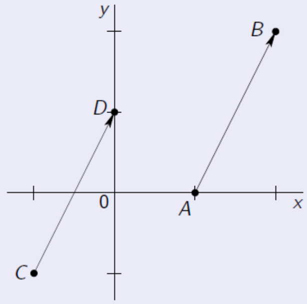
Relationship between Points and Vectors in $\mathbb{R}^n$

For each $P \in \mathbb{R}^n$, $\overrightarrow{OP}$ is the unique vector corresponding to the point P.

Conversely, for each vector $\overrightarrow{u} = (x_1, x_2, \ldots, x_n)$, there is a unique point $(x_1, x_2, \ldots, x_n)$ corresponding to $\overrightarrow{u}$.

Thus the space $\mathbb{R}^n$ can be treated as the set of vectors in $\mathbb{R}^n$.

Equality of Vectors



- $\overrightarrow{AB}$ is the vector from $A = (1, 0)$ to $B = (2, 2)$.
- $\overrightarrow{CD}$ is the vector from $C = (-1, -1)$ to $D = (0, 1)$.
- $\overrightarrow{AB} = \overrightarrow{CD}$ as the vectors have the same length and direction.

Note $\overrightarrow{AB} = \left(\begin{array}{c} 2-1 \\ 2-0 \end{array} \right) = \left(\begin{array}{c} 1 \\ 2 \end{array} \right) = \left(\begin{array}{c} 0-(-1) \\ 1-(-1) \end{array} \right) = \overrightarrow{CD}$.

(That A and B are different from C and D is not important.)

Operations on Vectors

Definition

Two vectors are said to be equal if the two following conditions are satisfied:

1. The number of components is the same.
2. The corresponding components are equal.

Notation

If $\overrightarrow{a}$ and $\overrightarrow{b}$ are equal, then $\overrightarrow{a} = \overrightarrow{b}$. Otherwise, $\overrightarrow{a} \neq \overrightarrow{b}$.

Operations on Vectors

Operations on vectors are defined in the natural way.

If $\overrightarrow{a} = (a_1, a_2, \ldots, a_n)$ and $\overrightarrow{b} = (b_1, b_2, \ldots, b_n)$ are in $\mathbb{R}^n$, then

- Addition $\overrightarrow{a} + \overrightarrow{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$
- Subtraction $\overrightarrow{a} - \overrightarrow{b} = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$
- Scalar Multiplication $k\overrightarrow{a} = (ka_1, ka_2, \ldots, ka_n)$

Operations on Vectors

Problem

Let $\overrightarrow{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *and* $\overrightarrow{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$. *What is* $\overrightarrow{u} + \overrightarrow{v}$*?*

Solution

$$\overrightarrow{u} + \overrightarrow{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1+4 \\ 2+5 \\ 3+6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}.$$

Operations on Vectors

Problem

Let $\overrightarrow{u} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ *and* $\overrightarrow{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. *What is* $\overrightarrow{u} - \overrightarrow{v}$*?*

Solution

Because $2 \neq 3$

$$\overrightarrow{u} - \overrightarrow{v} = \text{undefined}.$$

Tip-to-Tail Method for Vector Addition

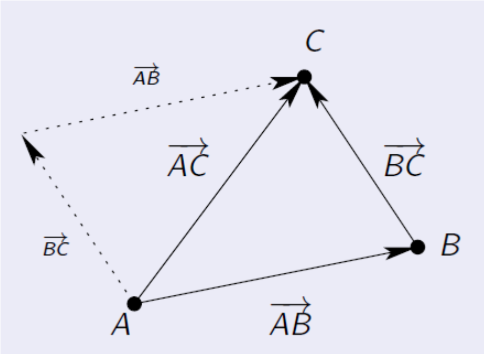
For points A, B and C,

$$
\begin{aligned}
\overrightarrow{AB} + \overrightarrow{BC} &= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ \vdots \\ b_n - a_n \end{pmatrix} + \begin{pmatrix} c_1 - b_1 \\ c_2 - b_2 \\ \vdots \\ c_n - b_n \end{pmatrix} \\
&= \begin{pmatrix} (b_1 - a_1) + (c_1 - b_1) \\ (b_2 - a_2) + (c_2 - b_2) \\ \vdots \\ (b_n - a_n) + (c_n - b_n) \end{pmatrix} = \begin{pmatrix} c_1 - a_1 \\ c_2 - a_2 \\ \vdots \\ c_n - a_n \end{pmatrix} \\
&= \overrightarrow{AC}.
\end{aligned}
$$

Thus, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

Tip-to-Tail Method for Vector Addition

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$



Note that if A is the origin, then $\overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$.

Properties of Vector Addition

Theorem

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$ and $k, c \in \mathbb{R}$ be scalars. Then the following properties hold.

1. $\vec{u} + \vec{0} = \vec{u}$ *(identity)*.
2. $0\vec{u} = \vec{0}$ *(multiplication by zero)*.
3. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ *(commutative)*.
4. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ *(associative)*.
5. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ *(distributive)*.
6. $(k + c)\vec{u} = k\vec{u} + c\vec{u}$ *(distributive)*.
7. $k(c\vec{u}) = (kc)\vec{u}$ *(associative)*.
8. $1\vec{u} = \vec{u}$ *(identity)*.
9. $\vec{u} + (-\vec{u}) = \vec{0}$ *(inverse)*.

Properties of Vector Addition

Problem

Find $\overrightarrow{a}$ if

$$\frac{1}{3}\left[5\overrightarrow{a} - \left(\begin{array}{c} 8 \\ 2 \end{array}\right)\right] = \left(\begin{array}{c} 1 \\ 3 \end{array}\right) - 2\overrightarrow{a}.$$

Solution

Rearranging,

$$5\overrightarrow{a} - \left(\begin{array}{c} 8 \\ 2 \end{array}\right) = 3\left(\begin{array}{c} 1 \\ 3 \end{array}\right) - 6\overrightarrow{a} = \left(\begin{array}{c} 3 \\ 9 \end{array}\right) - 6\overrightarrow{a}.$$

Thus

$$5\overrightarrow{a} + 6\overrightarrow{a} = \left(\begin{array}{c} 3 \\ 9 \end{array}\right) + \left(\begin{array}{c} 8 \\ 2 \end{array}\right) = \left(\begin{array}{c} 3+8 \\ 9+2 \end{array}\right) = \left(\begin{array}{c} 11 \\ 11 \end{array}\right),$$

and $11\overrightarrow{a} = \left(\begin{array}{c} 11 \\ 11 \end{array}\right)$ *and* $\overrightarrow{a} = \frac{1}{11}\left(\begin{array}{c} 11 \\ 11 \end{array}\right) = \left(\begin{array}{c} 1 \\ 1 \end{array}\right).$

Parallel Vectors

Definition

Two nonzero vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ in $\mathbb{R}^n$ are said to be parallel if there exists a scalar $k \in \mathbb{R}$ such that $\overrightarrow{a} = k\overrightarrow{b}$.

If $k > 0$, then $\overrightarrow{a}$ and $\overrightarrow{b}$ have the same direction.

If $k < 0$, then $\overrightarrow{a}$ and $\overrightarrow{b}$ are in opposite directions.

Notation If $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel, then $\overrightarrow{a} \parallel \overrightarrow{b}$.

What about Vector Multiplication?

The correct notion of vector multiplication seems less natural.

Definition

Let $\overrightarrow{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\overrightarrow{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ be vectors in $\mathbb{R}^3$.

The dot product of $\overrightarrow{u}$ and $\overrightarrow{v}$ is

$$\overrightarrow{u} \bullet \overrightarrow{v} = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

i.e., $\overrightarrow{u} \bullet \overrightarrow{v}$ is a scalar.

The Dot Product

Definition

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be vectors in $\mathbb{R}^n$.

The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

It is convenient to write

$$\vec{u} \bullet \vec{v} = (u_1, u_2, \ldots, u_n) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

The Dot Product

Problem

Find $\vec{u} \bullet \vec{v}$ for $\vec{u} = (1, 2, 0, -1)$, $\vec{v} = (0, 1, 2, 3)$.

Solution

$$\begin{aligned}
\vec{u} \bullet \vec{v} &= (1)(0) + (2)(1) + (0)(2) + (-1)(3) \\
&= 0 + 2 + 0 + -3 \\
&= -1.
\end{aligned}$$

Properties of the Dot Product

Theorem

Let $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ be vectors and let $k \in \mathbb{R}$.

1. $\overrightarrow{0} \bullet \overrightarrow{v} = 0$ (multiply by zero).
2. $\overrightarrow{u} \bullet \overrightarrow{v} = \overrightarrow{v} \bullet \overrightarrow{u}$ (commutative).
3. $\overrightarrow{u} \bullet (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \bullet \overrightarrow{v} + \overrightarrow{u} \bullet \overrightarrow{w}$
   $\overrightarrow{u} \bullet (\overrightarrow{v} - \overrightarrow{w}) = \overrightarrow{u} \bullet \overrightarrow{v} - \overrightarrow{u} \bullet \overrightarrow{w}$ (distributive).
4. $(k\overrightarrow{u}) \bullet \overrightarrow{v} = k(\overrightarrow{u} \bullet \overrightarrow{v}) = \overrightarrow{u} \bullet (k\overrightarrow{v})$ (distributive).

See you on Friday!