A.1 Euclidean spaces

Section 1.1

1. Write a three-column zero vector, and a three-column nonzero vector, and a five-column zero vector and a five-column nonzero vector.
2. Suppose that the buyer for a manufacturing plant must order different quantities of oil, paper, steel, and plastics. He will order 40 units of oil, 50 units of paper, 80 units of steel, and 20 units of plastics. Write the quantities in a single vector.
3. Suppose that a student's course marks for quiz 1, quiz 2, test 1, test 2, and the final exam are 70, 85, 80, 75, and 90, respectively. Write his marks as a column vector.
4. Let $\overrightarrow{a} = \begin{pmatrix} x-2y \\ 2x-y \\ 2z \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Find all $x,\ y,\ z \in \mathbb{R}$ that $\overrightarrow{a} = \overrightarrow{b}$.
5. $\overrightarrow{a} = \begin{pmatrix} |x| \\ y^2 \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Find all $x,\ y \in \mathbb{R}$ such that $\overrightarrow{a} = \overrightarrow{b}$.
6. $\overrightarrow{a} = \begin{pmatrix} x-y \\ 4 \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} 2 \\ x+y \end{pmatrix}$. Find all $x,\ y \in \mathbb{R}$ such that $\overrightarrow{a} - \overrightarrow{b}$ is a nonzero vector.
7. Let $P(1,\ x^2),\ Q(3,\ 4),\ P_1(4,\ 5),$ and $Q_1(6,\ 1)$. Find all possible values of $x \in \mathbb{R}$ such that $\overrightarrow{PQ} = \overrightarrow{P_1Q_1}$.
8. A company with 553 employees lists each employee's salary as a component of a vector $\overrightarrow{a}$ in $\mathbb{R}^{553}$. If a 6% salary increase has been approved, find the vector involving $\overrightarrow{a}$ that gives all the new salaries.
9. Let $\overrightarrow{a} = \begin{pmatrix} 110 \\ 88 \\ \end{pmatrix}$ denote the current prices of three items at a store. Suppose that the store

$\left.\begin{matrix} \\ 40 \end{matrix}\right)$

announces a sale so that the price of each item is reduced by 20%.

a. Find a three-vector that gives the price changes for the three items.
b. Find a three-vector that gives the new prices of the three items.

10. Let $\vec{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, $\alpha \in \mathbb{R}$. Compute

i. $2\vec{a} - \vec{b} + 5\vec{c}$;
ii. $4\vec{a} + \alpha\vec{b} - 2\vec{c}$

11. Find *x*, *y*, and *z* such that $\begin{pmatrix} 9 \\ 4y \\ 2z \end{pmatrix} + \begin{pmatrix} 3x \\ 8 \\ -6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

12. Let $\vec{u} = (1,\ -1,\ 0,\ 2)$, $\vec{v} = (-2,\ -1,\ 1,\ -4)$, and $\vec{w} = (3,\ 2,\ -1,\ 0)$.

a. Find a vector $\vec{a} \in \mathbb{R}^4$ such that $2\vec{u} - 3\vec{v} - \vec{a} = \vec{w}$.
b. Find a vector $\vec{a}$

$$\frac{1}{2}[2\vec{u} - 3\vec{v} + \vec{a}] = 2\vec{w} + \vec{u} - 2\vec{a}.$$

13. Determine whether $\vec{a} \parallel \vec{b}$,

1. $\vec{a} = (1,\ 2,\ 3)$ and $\vec{b} = (-2,\ -4,\ -6)$.
2. $\vec{a} = (-1,\ 0,\ 1,\ 2)$ and $\vec{b} = (-3,\ 0,\ 3,\ 6)$.
3. $\vec{a} = (1,\ -1,\ 1,\ 2)$ and $\vec{b} = (-2,\ 2,\ 2,\ 3)$.

14. Let $x,\ y,\ a,\ b \in \mathbb{R}$ with $x \neq y$. Let $P(x,\ 2x)$, $Q(y,\ 2y)$, $P_1(a,\ 2a)$, and $Q_1(b,\ 2b)$ be points in

$\mathbb{R}^2$. Show that $\overrightarrow{PQ} \parallel \overrightarrow{P_1Q_1}$.

15. Let $P(x,\ 0),\ Q(3,\ x^2),\ P_1(2x,\ 1),$ and $Q_1(6,\ x^2)$. Find all possible values of $x \in \mathbb{R}$ such that $\overrightarrow{PQ} \parallel \overrightarrow{P_1Q_1}$.
16. Let $\overrightarrow{a} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix},\ \overrightarrow{b} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix},\ \overrightarrow{c} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix},$ and $\alpha \in \mathbb{R}$.

    Compute

    a. $\overrightarrow{a} \cdot \overrightarrow{b}$;
    b. $\overrightarrow{a} \cdot \overrightarrow{c}$;
    c. $\overrightarrow{b} \cdot \overrightarrow{c}$;
    d. $\overrightarrow{a} \cdot (\overrightarrow{b} + \overrightarrow{c})$.

17. Let $\overrightarrow{u} = (1,\ -1,\ 0,\ 2),\ \overrightarrow{v} = (-2,\ -1,\ 1,\ -4),$ and $\overrightarrow{w} = (3,\ 2,\ -1,\ 0)$. Compute $\overrightarrow{u} \cdot \overrightarrow{v},\ \overrightarrow{u} \cdot \overrightarrow{w}$, and $\overrightarrow{w} \cdot \overrightarrow{v}$.
18. Assume that the percentages for homework, test 1, test 2, and the final exam for a course are 10%, 25%, 25%, and 40%, respectively. The total marks for homework, test 1, test 2, and the final exam are 10, 50, 50, and 90, respectively. A student's corresponding marks are 8, 46, 48, and 81, respectively. What is the student's final mark out of 100?
19. Assume that the percentages for homework, test 1, test 2, and the final exam for a course are 14%, 20%, 20%, and 46%, respectively. The total marks for homework, test 1, test 2, and the final exam are 14, 60, 60, and 80, respectively. A student's corresponding marks are 12, 45, 51, and 60, respectively. What is the student's final mark out of 100?
20. A manufacturer produces three different types of products. The demand for the products is denoted by the vector $\overrightarrow{a} = (10,\ 20,\ 30)$. The price per unit for the products is given by the vector $\overrightarrow{b} = (\$200,\ \$150,\ \$100)$. If the demand is met, how much money will the manufacturer receive?
21. A company pays four groups of employees a salary. The numbers of the employees for the four

groups are expressed by a vector $\overrightarrow{a} = (5,\ 20,\ 40)$. The payments for the groups are expressed by a vector $\overrightarrow{b} = (\$100,\ 000,\ \$80,\ 000,\ \$60,\ 000)$. Use the dot product to calculate the total amount of money the company paid its employees.

22. There are three students who may buy Calculus or algebra books. Use the dot product to find the total number of students who buy both calculus and algebra books.
23. There are *n* students who may buy a calculus or algebra books $(n \geq 2)$. Use the dot product to find the total number of students who buy both calculus and algebra books.
24. Assume that a person *A* has contracted a contagious disease and has direct contacts with four people: *P*1, *P*2, *P*3, and *P*4. We denote the contacts by a vector $\overrightarrow{a} := (a_{11},\ a_{12},\ a_{13},\ a_{14})$, where if the person *A* has made contact with the person *Pj*, then $a_{1j} = 1$, and if the person *A* has made no contact with the person *Pj*, then $a_{1j} = 0$. Now we suppose that the four people then have had a variety of direct contacts with another individual *B*, which we denote by a vector $\overrightarrow{b} := (b_{11},\ b_{21},\ b_{31},\ b_{41})$, where if the person *Pj* has made contact with the person *B*, then $b_{j1} = 1$, and if the person *Pj* has made no contact with the person *B*, then $b_{j1} = 0$.
    i. Find the total number of indirect contacts between *A* and *B*.
    ii. If $\overrightarrow{a} = (1,\ 0,\ 1,\ 1)$ and $\overrightarrow{a} = (1,\ 1,\ 1,\ 1)$, find the total number of indirect contacts between *A* and *B*.

Solution

1. **1.** $(0,\ 0,\ 0)^T,\ (1,\ 0,\ 1)^T,\ (0,\ 0,\ 0,\ 0,\ 0)^T,\ (-1,\ 0,\ 1,\ 0,\ 1)^T.$
2. **2.** $(\text{oil},\ \text{paper},\ \text{steel},\ \text{plastics})^T = (40,\ 50,\ 80,\ 20)^T.$
3. **3.** $(\text{quiz } 1,\ \text{quiz } 2,\ \text{test } 1,\ \text{test } 2,\ \text{final exam})^T = (70,\ 85,\ 80,\ 75,\ 90)^T.$
4. **4.** $\overrightarrow{a} = \overrightarrow{b}$ if and only if

$$x - 2y = 2 \qquad (1)$$

$$2x - y = -2 \qquad (2)$$

$$2z = 1. \qquad (3)$$

$(1) - (2) \times 2:\ (x - 2y) - 2(2x - y) = 2 - 2(-2)$ and $x = -2$. By (2), we get $y = 2x + 2 = 2(-2) + 2 = -2$ and by (3), we have $z = \frac{1}{2}$. Therefore, when $x = -2,\ y = -2$ and $z = \frac{1}{2}$, $\overrightarrow{a} = \overrightarrow{b}$.

5. **5.** $\overrightarrow{a} = \overrightarrow{b}$ if and only if $|x| = 1$ and $y^2 = 4$. Hence, $x = 1$ or $x = -1$ and $y = 2$ or $y = -2$. Hence, $\overrightarrow{a} = \overrightarrow{b}$ if $x = 1$ and $y = 2$; $x = 1$ and $y = -2$; $x = -1$ and $y = 2$; or $x = -1$ and $y = -2$.
6. **6.** We first find all *x*, *y* such that $\overrightarrow{a} - \overrightarrow{b} = \overrightarrow{0}$. Because

$$\overrightarrow{a} - \overrightarrow{b} = \begin{pmatrix} x - y \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ x + y \end{pmatrix} = \begin{pmatrix} x - y - 2 \\ 4 - x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we obtain

$$x - y - 2 = 0 \tag{1}$$

$$4 - x - y = 0. \tag{2}$$

$(1) + (2):\ (x - y - 2) + (4 - x - y) = 0$ and $y = 1$. By (1), we get $x = y + 2 = 1 + 2 = 3$. Hence when $x = 3$ and $y = 1$, $\vec{a} - \vec{b} = \vec{0}$. Now, we can find all *x*, *y* such that $\vec{a} - \vec{b} \neq \vec{0}$. When $x \neq 3$ or $y \neq 1$, $\vec{a} - \vec{b} \neq \vec{0}$, that is, $\vec{a} - \vec{b}$ is a nonzero vector.

7. **7.** Because $\overrightarrow{PQ} = (3 - 1,\ 4 - x^2) = (2,\ 4 - x^2)$ and

$$\overrightarrow{P_1Q_1} = (6 - 4,\ 1 - 5) = (2,\ -4),$$

then $\overrightarrow{PQ} = \overrightarrow{P_1Q_1}$ if and only if $4 - x^2 = -4$ if and only if $x^2 = 8$ if and only if $x = -2\sqrt{2}$ or $x = 2\sqrt{2}$.

8. **8.** $1.06\vec{a}$.

9. **9.**

a. $-0.2\vec{a} = \begin{pmatrix} -22 \\ -17.6 \\ -8 \end{pmatrix}.$

b. $0.8\vec{a} = \begin{pmatrix} 88 \\ 70.4 \\ 32 \end{pmatrix}.$

10. **10.**

i. $$2\vec{a} - \vec{b} + 5\vec{c} = 2\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 15 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} -2-2+15 \\ 4+2+0 \\ 6-0-5 \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \\ 1 \end{pmatrix}.$$

ii.

$$4\overrightarrow{a} + \alpha\overrightarrow{b} - 2\overrightarrow{c} = 4\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \alpha\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 8 \\ 12 \end{pmatrix} - \begin{pmatrix} 2\alpha \\ -2\alpha \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -10+2\alpha \\ 8-2\alpha \\ 14 \end{pmatrix}.$$

11. **11.** Because $\begin{pmatrix} 9+3x \\ 4y+8 \\ 2z-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, we have

$$9+3x=0, \quad 4y+8=0, \quad 2z-6=0.$$

This implies $x=-3,\ y=-2,\ z=3.$

12. **12.**

a. Because $2\overrightarrow{u} - 3\overrightarrow{v} - \overrightarrow{a} = \overrightarrow{w}$, we have

$$\begin{aligned} \overrightarrow{a} &= 2\overrightarrow{u} - 3\overrightarrow{v} - \overrightarrow{w} \\ &= 2(1,\ -1,\ 0,\ 2) - 3(-2,\ -1,\ 1,\ -4) - (3,\ 2,\ -1,\ 0) \\ &= (2,\ -2,\ 0,\ 4) - (-6,\ -3,\ 3,\ -12) - (3,\ 2,\ -1,\ 0) \\ &= (2+6-3,\ -2+3-2,\ 0-3+1,\ 4+12-0) = (5,\ -1,\ -2,\ 16). \end{aligned}$$

b. Because $\frac{1}{2}[2\overrightarrow{u} - 3\overrightarrow{v} + \overrightarrow{a}] = 2\overrightarrow{w} + \overrightarrow{u} - 2\overrightarrow{a}$, we have

$$2\overrightarrow{u} - 3\overrightarrow{v} + \overrightarrow{a} = 2[2\overrightarrow{w} + \overrightarrow{u} - 2\overrightarrow{a}] = 4\overrightarrow{w} + 2\overrightarrow{u} - 4\overrightarrow{a}.$$

Hence,

$$\begin{aligned} 5\overrightarrow{a} &= 3\overrightarrow{v} + 4\overrightarrow{4} = 3(-2,\ -1,\ 1,\ -4) + 4(3,\ 2,\ -1,\ 0) \\ &= (-6,\ -3,\ 3,\ -12) + (12,\ 8,\ -4,\ 0) = (6,\ 5,\ -1,\ -12) \end{aligned}$$

and

$$\overrightarrow{a} = \frac{1}{5}(6,\ 5,\ -1,\ -12) = \left(\frac{6}{5},\ 1,\ -\frac{1}{5},\ -\frac{12}{5}\right).$$

13. **13.**

    1. Because $\overrightarrow{b} = -2\overrightarrow{a}$, $\overrightarrow{b} \,||\, \overrightarrow{a}$.
    2. Because $\overrightarrow{b} = 3\overrightarrow{a}$, $\overrightarrow{b} \,||\, \overrightarrow{a}$.
    3. Because $\overrightarrow{b} \neq k\overrightarrow{a}$ for any $k \in \mathbb{R}$, $\overrightarrow{b} \not\parallel \overrightarrow{a}$.

14. **14.** Because $\overrightarrow{PQ} = (y - x,\ 2y - 2x) = (y - x)(1,\ 2)$,

$$\overrightarrow{P_1Q_1} = (b - a,\ 2b - 2a) = (b - a)(1,\ 2) \quad \text{and } x \neq y,$$

we have

$$\overrightarrow{P_1Q_1} = \frac{a - b}{x - y}\overrightarrow{PQ}.$$

Hence, $\overrightarrow{PQ} \,||\, \overrightarrow{P_1Q_1}$.

15. **15.** Because $\overrightarrow{PQ} = (3 - x,\ x^2 - 0) = (3 - x,\ x^2)$ and $\overrightarrow{P_1Q_1} = (6 - 2x,\ x^2 - 1)$, then

$\overrightarrow{PQ}||\overrightarrow{P_1Q_1}$ if and only if there exists $k \in \mathbb{R}$ such that $\overrightarrow{P_1Q_1} = k\overrightarrow{PQ}$, that is,

$$(6-2x,\ x^2-1) = k(3-x,\ x^2) = (k(3-x),\ kx^2).$$

if and only if $6-2x = k(3-x)$ and $x^2-1 = kx^2$ if and only if $3(2-k) = (2-k)x$ and $(1-k)x^2 = 1$. These last two equations imply $k \neq 1$ and $k \neq 2$ because if $k=1$ or $k=2$, the last equation implies $0=1$ or $x^2=-1$, respectively, and it is impossible. If $k \neq 1$ and $k \neq 2$, then $x=3$ and $k=\frac{8}{9}$. Hence, only when $x=3$, $\overrightarrow{PQ}||\overrightarrow{P_1Q_1}$.

16. **16.**

a. $\vec{a} \cdot \vec{b} = (-1,\ 2,\ 3)\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} = (-1)(2) + 2(-2) + (3)(0) = -6.$

b. $\vec{a} \cdot \vec{c} = (-1,\ 2,\ 3)\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = (-1)(3) + 2(0) + (3)(-1) = -6.$

c.
$$\vec{b} \cdot \vec{c} = (2,\ -2,\ 0)\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = (2)(3) + (-2)(0) + (0)(-1) = 6.$$

d.
$$\vec{a} \cdot (\vec{b} + \vec{c}) = (-1,\ 2,\ 3)\left(\begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}\right)$$
$$= (-1,\ 2,\ 3)\begin{pmatrix} 5 \\ -2 \\ -1 \end{pmatrix} = (-1)(5) + 2(-2) + 3(-1) = -12.$$

$\Big(-2 \Big)$

$$\overrightarrow{u} \cdot \overrightarrow{v} = (1,\ -1,\ 0,\ 2)\begin{pmatrix} -1 \\ 1 \\ -4 \end{pmatrix} = 1(-2) + (-1)(-1) + 0(1) + 2(-4)$$

$$= -2 + 1 + 0 - 8 = -9.$$

17. **17.**

$$\overrightarrow{u} \cdot \overrightarrow{w} = (1,\ -1,\ 0,\ 2)\begin{pmatrix} 3 \\ 2 \\ -1 \\ 0 \end{pmatrix} = 1(3) + (-1)2 + 0(-1) + 2(0)$$

$$= 3 - 2 + 0 + 0 = 1.$$

$$\overrightarrow{w} \cdot \overrightarrow{v} = (3,\ 2,\ -1,\ 0)\begin{pmatrix} -2 \\ -1 \\ 1 \\ -4 \end{pmatrix} = 3(-2) + 2(-1) + (-1)(1) + 0(-4)$$

$$= -6 - 2 - 1 + 0 = -9.$$

18. **18.** Let

$$\overrightarrow{a} = (10/10,\ 25/50,\ 25/50,\ 40/90) = (1,\ 1/2,\ 1/2,\ 4/9)$$

and $\overrightarrow{b} = (8,\ 46,\ 48,\ 81)$. Then the final mark out of 100 for the student is

$$\begin{aligned} \overrightarrow{a} \cdot \overrightarrow{b} &= (1,\ 1/2,\ 1/2,\ 4/9) \cdot (8,\ 46,\ 48,\ 81) \\ &= (1)(8) + (1/2)(46) + (1/2)(48) + (4/9)(81) = 8 + 23 + 24 + 36 \\ &= 91. \end{aligned}$$

19. **19.** Let

$$\overrightarrow{a} = (14/14,\ 20/60,\ 20/60,\ 46/80) = (1,\ 1/3,\ 1/3,\ 23/40)$$

and $\overrightarrow{b} = (12,\ 45,\ 51,\ 60)$. Then the final mark out of 100 for the student is

$$\begin{aligned}\overrightarrow{a}\cdot\overrightarrow{b} &= (1,\ 1/3,\ 1/3,\ 23/40)\cdot(12,\ 45,\ 51,\ 60)\\ &= (1)(12)+(1/3)(45)+(1/3)(51)+(23/40)(60)\\ &= 12+15+17+34.5=78.5\end{aligned}$$

20. **20.** The total cash received is $\overrightarrow{a}\cdot\overrightarrow{b} = 10(200)+20(150)+30(100)=\$8000.$

21. **21.** The total amount of money the company paid is the dot product

$$\overrightarrow{a}\cdot\overrightarrow{b} = 5(100,000)+20(80,000)+40(60,000)=\$4,500,000$$

because there are five employees with salary $100,000, 20 with $80,000, and 40 with $60,000.

22. **22.** We denote the calculus book by book 1 and the algebra book by book 2. If student $i(i=1,\ 2,\ 3)$ wants to buy book $j(j=1,\ 2),$ then we denote by $a_{ij}=1.$ If student *i* doesn't want to buy book *j*, then we denote by $a_{ij}=0.$ We write $\overrightarrow{a} = (a_{11},\ b_{21},\ a_{31})$ and $\overrightarrow{b} = (a_{12},\ b_{22},\ a_{32}).$ Then for $i=1,\ 2,\ 3,$ if $a_{i1}a_{i2}=1,$ then student *i* wants to buy both calculus and algebra books, and if $a_{i1}a_{i2}=0,$ then student *i* doesn't want to buy both calculus and algebra books. Hence, the total number of students who buy both calculus and algebra books is equal to

$$\overrightarrow{a}\cdot\overrightarrow{b} = a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32}.$$

23. **23.** We denote by $a_{ij}=1$ if student $i(i=1,\ 2,\ \cdots,\ n)$ wants to buy book $j(j=1,2)$ and by $a_{ij}=0$ if student *i* doesn't want to buy book *j*. We write $\overrightarrow{a} = (a_{11},\ a_{21},\ \cdots,\ a_{n1})$ and $\overrightarrow{b} = (a_{12},a_{22},\cdots,a_{n2}).$ Then the total number of students who buy both calculus and algebra books is equal to the dot product

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_{11}a_{12} + a_{21}a_{21} + \cdots + a_{n1}a_{n2}).$$

24. **24.**

   i. The total number of indirect contacts between *A* and *B* is given by

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}.$$

   ii. The total number of indirect contacts between *A* and *B* is

$$\overrightarrow{a} \cdot \overrightarrow{b} = (1)(1) + (0)(1) + (1)(1) + (1)(1) = 3. \qquad \text{(A.1.1)}$$

   From the first term of (A.1.1), we see that *A* has contact with *P*1 and *P*1 has been in contact with *B*, so *A* and *B* have one indirect contact via *P*1, which is shown by $(1)(1) = 1.$ Similarly, *A* and *B* have one indirect contact via each of *P*3 and *P*4. From the second term of (A.1.1), we see that *A* has no contact with *P*2 and *P*2 has been in contact with *B*, so *A* and *B* have no indirect contact, which is shown by $(0)(1) = 0.$

Section 1.2

1. For each of the following vectors, find its norm and normalize it to a unit vector.
   i. $\overrightarrow{a} = (1,\ 0,\ -2);$
   ii. $\overrightarrow{b} = (1,\ 1,\ -2);$
   iii. $\overrightarrow{c} = (2,\ 2,\ -2).$

2. Let $\overrightarrow{a},\ \overrightarrow{b} \in \mathbb{R}^n$. Show that the following identities hold.
   i. $\left\|\overrightarrow{a} + \overrightarrow{b}\right\|^2 + \left\|\overrightarrow{a} - \overrightarrow{b}\right\|^2 = 2\left\|\overrightarrow{a}\right\|^2 + 2\left\|\overrightarrow{b}\right\|^2.$

ii. $\|\vec{a} + \vec{b}\| - \|\vec{a} - \vec{b}\| = 4(\vec{a} \cdot \vec{b})$.

3. Evaluate the determinant of each of the following matrices.

$$A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \; B = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \; C = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \; D = \begin{pmatrix} -2 & 1 \\ -6 & 3 \end{pmatrix}$$

4. For each pair of vectors, verify that Cauchy-Schwarz and triangle inequalities hold.
   i. $\vec{a} = (1,\ 2),\ \vec{b} = (2,\ -1)$
   ii. $\vec{a} = (1,\ 2,\ 1),\ \vec{b} = (0,\ 2,\ -2)$
   iii. $\vec{a} = (1,\ -1,\ 0,\ 1),\ \vec{b} = (0,\ -1,\ 2,\ 1)$

5. Let $\vec{a} = (2,\ 1)$ and $\vec{b} = (1,\ x)$. Use **Theorem 1.2.3** to find $x \in \mathbb{R}$ such that $\vec{a} \parallel \vec{b}$.
6. For each pair of points P_1 and P_2, find the distance between the two points.
   1. $P_1(1,\ -2),\ P_2(-3,\ -1)$
   2. $P_1(3,\ -1),\ P_2(-2,\ 1)$
   3. $P_1(1,\ -1,\ 5),\ P_2(1,\ -3,\ 1)$
   4. $P_1(0,\ -1,\ 2),\ P_2(1,\ -3,\ 1)$
   5. $P_1(-1,\ -1,\ 5,\ 3),\ P_2(0,\ -2,\ -3,\ -1)$

7. Let $A(-1,\ -2),\ B(2,\ 3),\ C(-1,\ 1),$ and *D*(0, 1) be four different points in $\mathbb{R}^2$. Find the distance between the midpoints of the segments *AB* and *CD*.
8. For each pair of vectors, determine whether they are orthogonal.
   i. $\vec{a} = (1,\ 2),\quad \vec{b} = (2,\ -1)$
   ii. $\vec{a} = (1,\ 1,\ 1),\quad \vec{b} = (0,\ 2,\ -2)$
   iii. $\vec{a} = (1,\ -1,\ -1),\quad \vec{b} = (1,\ 2,\ -2)$
   iv. $\vec{a} = (1,\ -1,\ 1),\quad \vec{b} = (0,\ 2,\ -1)$

v. $\overrightarrow{a} = (2,\ -1,\ 0,\ 1),\qquad \overrightarrow{b} = (0,\ -1,\ 3,\ -1)$

9. For each pair of vectors, find all values of $x \in \mathbb{R}$ such that they are orthogonal.

   i. $\overrightarrow{a} = (-1,\ 2),\ \ \overrightarrow{b} = (2,\ x^2)$
   ii. $\overrightarrow{a} = (1,\ 1,\ 1),\ \ \overrightarrow{b} = (x,\ 2,\ -2)$
   iii. $\overrightarrow{a} = (5,\ x,\ 0,\ 1),\ \ \overrightarrow{b} = (0,\ -1,\ 3,\ -1)$

10. For each pair of vectors, find $\cos\ \theta$, where θ is the angle between them.

    1. $\overrightarrow{a} = (1,\ -1,\ 1)$ and $\overrightarrow{b} = (1,\ 1,\ -2)$;
    2. $\overrightarrow{a} = (1,\ 0,\ 1)$ and $\overrightarrow{b} = (-1,\ -1,\ -1)$.

11. Let $\overrightarrow{a} = (6,\ 8)$ and $\overrightarrow{b} = (1,\ x)$. Find $x \in \mathbb{R}$ such that

    i. $\overrightarrow{a} \perp \overrightarrow{b}$
    ii. $\overrightarrow{a} \parallel \overrightarrow{b}$
    iii. the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\frac{\pi}{4}$.

12. Three vertices of a triangle are

$$P_1(6,\ 6,\ 5,\ 8),\ P_2(6,\ 8,\ 6,\ 5),\ \text{and}\ P_3(5,\ 7,\ 4,\ 6).$$

    a. Show that the triangle is a right triangle.
    b. Find the area of the triangle.

13. Suppose three vertices of a triangle in $\mathbb{R}^5$ are

$$P(4,\ 5,\ 5,\ 6,\ 7),\ O(3,\ 6,\ 4,\ 8,\ 4),\ Q(5,\ 7,\ 7,\ 13,\ 6).$$

a. Show that the triangle is a right triangle.
b. Find the area of the triangle.

14. Four vertices of a quadrilateral are

$$A(4,\ 5,\ 5,\ 7),\ B(6,\ 3,\ 3,\ 9),\ C(5,\ 2,\ 2,\ 8),\ D(3,\ 4,\ 4,\ 6).$$

a. Show that the quadrilateral is a rectangle.
b. Find the area of the quadrilateral.

Solution

1. **1.**

i. $||\overrightarrow{a}|| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{5}.$

$$\frac{\overrightarrow{a}}{||\overrightarrow{a}||} = \frac{1}{\sqrt{5}}(1,\ 0,\ -2) = \left(\frac{1}{\sqrt{5}},\ 0,\ -\frac{2}{\sqrt{5}}\right).$$

ii. $||\overrightarrow{b}|| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}.$

$$\frac{\overrightarrow{b}}{||\overrightarrow{b}||} = \frac{1}{\sqrt{6}}(1,\ 1,\ -2) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}}\right).$$

iii. $||\overrightarrow{c}|| = \sqrt{2^2 + 2^2 + (-2)^2} = \sqrt{12} = 2\sqrt{3}.$

$$\frac{\overrightarrow{c}}{||\overrightarrow{c}||} = \frac{1}{2\sqrt{3}}(2,\ 2,\ -2) = \left(\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right).$$

2. **2.** By Theorem 1.2.1 (*iii*)and (*iv*),

$$\begin{aligned}
&||\overrightarrow{a} + \overrightarrow{b}||^2 + ||\overrightarrow{a} - \overrightarrow{b}||^2 \\
&= (||\overrightarrow{a}||^2 + 2(\overrightarrow{a} \cdot \overrightarrow{b}) + ||\overrightarrow{b}||^2) + (||\overrightarrow{a}||^2 - 2(\overrightarrow{a} \cdot \overrightarrow{b}) + ||\overrightarrow{b}||^2) \\
&= 2||\overrightarrow{a}||^2 + 2||\overrightarrow{b}||^2.
\end{aligned}$$

$$\begin{aligned}&||\vec{a}+\vec{b}||^2-||\vec{a}-\vec{b}||^2\\&=(||\vec{a}||^2+2(\vec{a}\cdot\vec{b})+||\vec{b}||^2)-(||\vec{a}||^2-2(\vec{a}\cdot\vec{b})+||\vec{b}||^2)\\&=4(\vec{a}\cdot\vec{b}).\end{aligned}$$

3. **3.**

$$|A|=\begin{vmatrix}1&1\\4&-2\end{vmatrix}=-2-4=-6.\quad |B|=\begin{vmatrix}-2&2\\1&-3\end{vmatrix}=6-2=4.$$
$$|C|=\begin{vmatrix}2&-1\\3&0\end{vmatrix}=0-(-3)=3.\quad |D|=\begin{vmatrix}-2&1\\-6&3\end{vmatrix}=-6-(-6)=0.$$

4. **4.**

i. By computation, $||\vec{a}||=\sqrt{1^2+2^2}=\sqrt{5}$, $||\vec{b}||=\sqrt{2^2+(-1)^2}=\sqrt{5}$ and
$\vec{a}\cdot\vec{b}=(1)(2)+(2)(-1)=0.$
Hence,

$$|\vec{a}\cdot\vec{b}|=0\leq\sqrt{5}\sqrt{5}=||\vec{a}||||\vec{b}||$$

and the Cauchy-Schwarz inequality holds. By computation,

$$||\vec{a}+\vec{b}||=\sqrt{10}\leq\sqrt{5}+\sqrt{5}=||\vec{a}||+||\vec{b}||$$

and the triangle inequality holds.

ii. By computation, $||\vec{a}||=\sqrt{6}$, $||\vec{b}||=2\sqrt{2}$, and $\vec{a}\cdot\vec{b}=2$. Hence,

$$|\vec{a}\cdot\vec{b}|=2\leq\sqrt{6}(2\sqrt{2})=||\vec{a}||||\vec{b}||$$

and the Cauchy-Schwarz inequality holds. By computation,

$$||\overrightarrow{a} + \overrightarrow{b}|| = 3\sqrt{2} \leq \sqrt{6} + 2\sqrt{2} = ||\overrightarrow{a}|| + ||\overrightarrow{b}||$$

and the triangle inequality holds.

iii. By computation, $||\overrightarrow{a}|| = \sqrt{3}$, $||\overrightarrow{b}|| = \sqrt{6}$, and $\overrightarrow{a} \cdot \overrightarrow{b} = 2$. Hence,

$$|\overrightarrow{a} \cdot \overrightarrow{b}| = 2 \leq \sqrt{3}\sqrt{6} = ||\overrightarrow{a}||||\overrightarrow{b}||$$

and the Cauchy-Schwarz inequality holds. By computation,

$$||\overrightarrow{a} + \overrightarrow{b}|| = \sqrt{13} \leq \sqrt{3} + \sqrt{6} = ||\overrightarrow{a}|| + ||\overrightarrow{b}||$$

and the triangle inequality holds.

5. **5.** By computation, $|\overrightarrow{a} \cdot \overrightarrow{b}| = |2 + x|$, $||\overrightarrow{a}|| = \sqrt{2^2 + 1^2} = \sqrt{5}$, and

$$||\overrightarrow{b}|| = \sqrt{1^2 + x^2} = \sqrt{1 + x^2}.$$

If $|\overrightarrow{a} \cdot \overrightarrow{b}| = ||\overrightarrow{a}||||\overrightarrow{b}||$, then

$$|2 + x| = \sqrt{5}\sqrt{1 + x^2} \qquad \text{and } (2 + x)^2 = 5(1 + x^2).$$

This implies

$$4 + 4x + x^2 = 5 + 5x^2 \text{ and } 4x^2 - 4x + 1 = (2x - 1)^2 = 0.$$

Hence, $2x - 1 = 0$ and $x = \frac{1}{2}$.

6. **6.**

1. $||\overrightarrow{P_1P_2}|| = \sqrt{(-3-1)^2 + (-1-(-2))^2} = \sqrt{4^2+1^2} = \sqrt{17}.$
2. $||\overrightarrow{P_1P_2}|| = \sqrt{(-2-3)^2 + (1-(-1))^2} = \sqrt{(-5)^2+2^2} = \sqrt{29}.$
3. $$\begin{aligned}||\overrightarrow{P_1P_2}|| &= \sqrt{(1-1)^2 + (-3-(-1))^2 + (1-5)^2}\\ &= \sqrt{0^2 + (-2)^2 + (-4)^2} = 2\sqrt{5}.\end{aligned}$$
4. $$\begin{aligned}||\overrightarrow{P_1P_2}|| &= \sqrt{(1-0)^2 + (-3-(-1))^2 + (1-2)^2}\\ &= \sqrt{1^2 + (-2)^2 + (-1)^2} = \sqrt{6}\end{aligned}$$
5. $$\begin{aligned}||\overrightarrow{P_1P_2}|| &= \sqrt{(0-1)^2 + (-2-(-1))^2 + (-3-5)^2 + (-1-3)^2}\\ &= \sqrt{1^2 + (-1)^2 + (-8)^2 + (-4)^2} = \sqrt{82}.\end{aligned}$$

7. **7.** By (1.2.14), two midpoints of the segments *AB* and *CD* are

$$\left(\frac{-1+2}{2}, \frac{-2+3}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ and } \left(\frac{-1+0}{2}, \frac{1+1}{2}\right) = \left(-\frac{1}{2}, 1\right).$$

By (1.2.10) with $n = 2,$ we obtain the distance between two midpoints

$$\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)^2 + \left(\frac{1}{2}-1\right)^2} = \frac{5}{2}.$$

8. **8.**

i. $\vec{a} \perp \vec{b}$ because $\vec{a} \cdot \vec{b} = (1)(2) + (2)(-1) = 0.$
ii. $\vec{a} \perp \vec{b}$ because $\vec{a} \cdot \vec{b} = (1)(0) + (1)(2) + (1)(-2) = 0.$

iii. $\vec{a} \not\perp \vec{b}$ because $\vec{a} \cdot \vec{b} = (1)(1) + (-1)(2) + (-1)(-2) = 1 \neq 0$

iv. $\vec{a} \not\perp \vec{b}$ because $\vec{a} \cdot \vec{b} = (1)(0) + (-1)(2) + (1)(-1) = -3 \neq 0.$

v. $\vec{a} \perp \vec{b}$ because $\vec{a} \cdot \vec{b} = (2)(0) + (-1)(-1) + (0)(3) + (1)(-1) = 0.$

9. **9.**

i. If $\vec{a} \cdot \vec{b} = (-1)(2) + (2)(x^2) = -2 + 2x^2 = 0,$ then $x^2 = 1$. Hence, when $x = 1$ or $x = -1,$ $\vec{a} \perp \vec{b}$.

ii. If $\vec{a} \cdot \vec{b} = (1)(x) + (1)(2) + (1)(-2) = x = 0,$ then $x = 0$. Hence, when $x = 0,$ $\vec{a} \perp \vec{b}$.

iii. If $\vec{a} \cdot \vec{b} = (5)(0) + (x)(-1) + (0)(3) + (1)(-1) = -x - 1 = 0,$ then $x = -1$. Hence, when $x = -1,$ $\vec{a} \perp \vec{b}$.

10. **10.**

1. By computation, $\vec{a} \cdot \vec{b} = (1)(1) + (-1)(1) + (1)(-2) = -2,$
$||\vec{a}|| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$ and $||\vec{b}|| = \sqrt{1^2 + (1)^2 + (-2)^2} = \sqrt{6}.$
By Theorem 1.2.15, we have

$$\cos\ \theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||} = \frac{-2}{\sqrt{3}\sqrt{6}} = -\frac{2}{3\sqrt{2}} = -\frac{\sqrt{2}}{3}.$$

2. By computation, $\vec{a} \cdot \vec{b} = (1)(-1) + (0)(-1) + (1)(-1) = -2,$

$$||\overrightarrow{a}|| = \sqrt{1^2 + (0)^2 + 1^2} = \sqrt{2} \text{ and}$$
$$||\overrightarrow{b}|| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}.$$

By Theorem 1.2.15,

$$\cos\ \theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{||\overrightarrow{a}||||\overrightarrow{b}||} = \frac{-2}{\sqrt{2}\sqrt{3}} = -\frac{2}{\sqrt{6}} = -\frac{\sqrt{6}}{3}.$$

11. **11.**

i. If $\overrightarrow{a} \cdot \overrightarrow{b} = 6 + 8x = 0,$ then $x = -\frac{3}{4}$. Hence, when $x = -\frac{3}{4}$, $\overrightarrow{a} \perp \overrightarrow{b}$.

ii. If $\frac{6}{1} = \frac{8}{x}$, then $x = \frac{4}{3}$. Hence, when $x = \frac{4}{3}$, $\overrightarrow{a} \parallel \overrightarrow{b}$.

iii. By computation, $\overrightarrow{a} \cdot \overrightarrow{b} = 6 + 8x,$ $||\overrightarrow{a}|| = \sqrt{6^2 + 8^2} = 10,$ and $||\overrightarrow{b}|| = \sqrt{1 + x^2}$ Hence,

$$\cos\ \theta = \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{||\overrightarrow{a}||||\overrightarrow{b}||} = \frac{6 + 8x}{10\sqrt{1 + x^2}} = \frac{3 + 4x}{5\sqrt{1 + x^2}}.$$

If $\theta = \frac{\pi}{4}$, then

$$\frac{3 + 4x}{5\sqrt{1 + x^2}} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}. \qquad \text{(A.1.2)}$$

This implies

$$2(3 + 4x) = \sqrt{2}\left(5\sqrt{1 + x^2}\right).$$

Squaring both sides of the last equation implies

$$4(3+4x)^2 = 2(25)(1+x^2).$$

Hence,

$$2(9+24x+16x^2) = 25+25x^2.$$

This implies that $7x^2+48x-7=0$. Solving the equation, we obtain $x=\frac{1}{7}$ and $x=-7$. Note that $x=-7$ is not a solution of the equation (A.1.2) because it does not satisfy $3+4x>0$. Hence, when $x=\frac{1}{7}$, the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\frac{\pi}{4}$.

12. **12.** By computation, we have

$$||\overrightarrow{P_1P_2}||^2 = ||(0,\ 2,\ 1,\ -3)||^2 = 0^2+2^2+1^2+(-3)^2 = 14.$$
$$||\overrightarrow{P_2P_3}||^2 = ||(-1,\ -1,\ -2,\ 1)||^2 = (-1)^2+(-1)^2+(-2)^2+1^2 = 7 \text{ and}$$
$$||\overrightarrow{P_3P_1}||^2 = ||(1,\ -1,\ 1,\ 2)||^2 = 1^2+(-1)^2+1^2+2^2 = 7,$$
$$||\overrightarrow{P_1P_2}||^2 = ||\overrightarrow{P_2P_3}||^2 + ||\overrightarrow{P_3P_1}||^2$$

By the Pythagorean theorem, the triangle is a right triangle and

$$\overrightarrow{P_2P_3} \perp \overrightarrow{P_3P_1}.$$

b. Because $\overrightarrow{P_2P_3} \perp \overrightarrow{P_3P_1}$, the area of the triangle is

$$\frac{1}{2}||\overrightarrow{P_2P_3}||||\overrightarrow{P_3P_1}|| = \frac{1}{2}\sqrt{7}\sqrt{7} = \frac{7}{2}.$$

13. **13.**

a. By computation, we have

$$\begin{aligned}\overrightarrow{AB} &= (6,\ 3,\ 3,\ 9) - (4,\ 5,\ 5,\ 7) = (2,\ -2,\ -2,\ 2),\\ \overrightarrow{CD} &= (3,\ 4,\ 4,\ 6) - (5,\ 2,\ 2,\ 8) = (-2,\ 2,\ 2,\ -2),\\ \overrightarrow{AD} &= (3,\ 4,\ 4,\ 6) - (4,\ 5,\ 5,\ 7) = (1,\ 1,\ 1,\ 1),\\ \overrightarrow{BC} &= (5,\ 2,\ 2,\ 8) - (6,\ 3,\ 3,\ 9) = (-1,\ -1,\ -1,\ -1).\end{aligned}$$

It is easy to see that $\overrightarrow{AB} = -\overrightarrow{CD},\ \overrightarrow{AD} = -\overrightarrow{BC}$. Hence $\overrightarrow{AB}||\overrightarrow{CD}$ and $\overrightarrow{AD}||\overrightarrow{BC}$ and the quadrilateral is a parallelogram. Because

$$\overrightarrow{AD}\cdot\overrightarrow{CD} = (1,\ 1,\ 1,\ 1)\cdot(-2,\ 2,\ 2,\ -2) = -2+2+2-2 = 0,$$

$\overrightarrow{AD} \perp \overrightarrow{CD}$ and the quadrilateral is a rectangle.

b. The area of the rectangle is

$$||\overrightarrow{AD}||||\overrightarrow{CD}|| = \sqrt{1+1+1+1}\cdot\sqrt{4+4+4+4} = 8.$$

Section 1.3

1. For each pair of vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, find $\mathrm{Proj}_{\overrightarrow{a}}\ \overrightarrow{b}$ and verify that $(\overrightarrow{b} - \mathrm{proj}_{\overrightarrow{a}}\ \overrightarrow{b}) \perp \overrightarrow{a}$.
   1. $\overrightarrow{a} = (-1,\ 2),\ \overrightarrow{b} = (1,\ 1);$
   2. $\overrightarrow{a} = (2,\ -1,\ 1),\ \overrightarrow{b} = (1,\ -1,\ 2);$
   3. $\overrightarrow{a} = (-1,\ -2,\ 2),\ \overrightarrow{b} = (1,\ 0,\ -1);$

4. $\vec{a} = (0, \ -1, \ 1, \ -1), \ \vec{b} = (1, \ 1, \ -1, \ 2).$

2. Let $\vec{a} = (1,1,1)$ and $\vec{b} = (1,-1,1)$.
   1. Use **Theorem 1.3.2** (*i*) to find $\left\| \mathrm{Pr}\,oj_{\vec{a}} \ \vec{b} \right\|$.
   2. Use **Theorem 1.3.2** (*ii*) to find cos θ, where θ is the angle between $\vec{a}$ and $\vec{b}$.

3. Let $\vec{a} = (1,0,1,2)$ and $\vec{b} = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{3}{2})$.
   1. Use **Theorem 1.3.2** (*i*) to find $\left\| \mathrm{Pr}\,oj_{\vec{a}} \ \vec{b} \right\|$.
   2. Use **Theorem 1.3.2** (*ii*) to find θ, where θ is the angle between $\vec{a}$ and $\vec{b}$.

4. Find the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.
   1. $\vec{a} = (1{,}0), \ \vec{b} = (0, -1);$
   2. $\vec{a} = (1{,}0), \ \vec{b} = (-1, \ 1);$
   3. $\vec{a} = (0, 2), \ \vec{b} = (1, \ 1);$
   4. $\vec{a} = (1, \ -1), \ \vec{b} = (-1, \ 2).$

5. Find the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.
   1. $\vec{a} = (1, \ 0, \ 1), \ \vec{b} = (-1, \ 0, \ 1);$
   2. $\vec{a} = (1, \ 0, \ 0), \ \vec{b} = (-1, \ 1, \ -2);$
   3. $\vec{a} = (0, \ 2, \ -1), \ \vec{b} = (-5, \ 1, \ 1);$
   4. $\vec{a} = (1, \ -1, \ -1), \ \vec{b} = (-1, \ 1, \ 2);$
   5. $\vec{a} = (1, \ 0, \ 2, \ 1), \ \vec{b} = (2, \ 1, \ 0, \ 3);$

6. $\overrightarrow{a} = (-2,\ -1,\ 0,\ 3),\ \overrightarrow{b} = (3,\ 1,\ -1,\ 0).$

6. Find $G(\overrightarrow{a},\ \overrightarrow{b})$ and $\left\|\overrightarrow{b} - \mathrm{Porj}_{\overrightarrow{a}}\ \overrightarrow{b}\right\|$ for each pair of vectors.

a. $\overrightarrow{a} = (-1,\ 2),\ \overrightarrow{b} = (1,\ -1);$

b. $\overrightarrow{a} = (2,\ -1,\ 1),\ \overrightarrow{b} = (1,\ -1,\ 2);$

c. $\overrightarrow{a} = (1,\ 2,\ 0,\ 1),\ \overrightarrow{b} = (1,\ -1,\ 2,\ 3);$

d. $\overrightarrow{a} = (2,\ -1,\ 1,\ 0),\ \overrightarrow{b} = (3,\ 0,\ 1,\ 3).$

Solution

1. **1.**

1. $\text{Proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a} = \frac{(1)(-1)+(1)(2)}{(-1)^2+2^2}(-1,\ 2) = \left(-\frac{1}{5},\ \frac{2}{5}\right).$

   Because $\vec{b} - \text{Proj}_{\vec{a}}\vec{b} = (1,\ 1) - \left(-\frac{1}{5},\ \frac{2}{5}\right) = \left(\frac{6}{5},\ \frac{3}{5}\right),$

   $$(\vec{b} - \text{Proj}_{\vec{a}}\vec{b})\cdot\vec{a} = \left(\frac{6}{5},\ \frac{3}{5}\right)\cdot(-1,\ 2) = -\frac{6}{5}+\frac{6}{5} = 0$$

   and $(\vec{b} - \text{Proj}_{\vec{a}}\vec{b}) \perp \vec{a}$.

2. $\text{Proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a} = \frac{(1)(2)+(-1)(-1)+(2)(1)}{2^2+(-1)^2+1^2}(2,\ -1,\ 1) = \left(\frac{5}{3},\ -\frac{5}{6},\ \frac{5}{6}\right).$

   Because $\vec{b} - \text{Proj}_{\vec{a}}\vec{b} = (1,\ -1,\ 2) - \left(\frac{5}{3},\ -\frac{5}{6},\ \frac{5}{6}\right) = \left(-\frac{2}{3},\ -\frac{1}{6},\ \frac{7}{6}\right),$

   $$(\vec{b} - \text{Proj}_{\vec{a}}\vec{b})\cdot\vec{a} = \left(-\frac{2}{3},\ -\frac{1}{6},\ \frac{7}{6}\right)\cdot(2,\ -1,\ 1) = -\frac{4}{3}+\frac{1}{6}+\frac{7}{6} = 0$$

   and $(\vec{b} - \text{Proj}_{\vec{a}}\vec{b}) \perp \vec{a}$.

3. $\text{Proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{b}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a} = \frac{(1)(-1)+(0)(-2)+(-1)(2)}{(-1)^2+(-2)^2+2^2}(-1,\ -2,\ 2) = \left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right).$

   Because $\vec{b} - \text{Proj}_{\vec{a}}\vec{b} = (1,\ 0,\ -1) - \left(\frac{1}{3},\ \frac{2}{3},\ -\frac{2}{3}\right) = \left(\frac{2}{3},\ -\frac{2}{3},\ -\frac{1}{3}\right),$

$$\left(\overrightarrow{b} - \text{Proj}_{\overrightarrow{a}}\overrightarrow{b}\right) \cdot \overrightarrow{a} = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right) \cdot (-1, -2, 2) = -\frac{2}{3} + \frac{4}{3} - \frac{2}{3} = 0$$

and $\left(\overrightarrow{b} - \text{Proj}_{\overrightarrow{a}}\overrightarrow{b}\right) \perp \overrightarrow{a}$.

4. $$\begin{aligned} \text{Proj}_{\overrightarrow{a}}\overrightarrow{b} &= \left(\frac{\overrightarrow{b} \cdot \overrightarrow{a}}{\|\overrightarrow{a}\|^2}\right)\overrightarrow{a} \\ &= \frac{(1)(0)+(1)(-1)+(-1)(1)+2(-1)}{0^2+(-1)^2+1^2+(-1)^2}(0, -1, 1, -1) \\ &= \frac{-4}{3}(0, -1, 1, -1) = \left(0, \frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right). \end{aligned}$$

Because $\overrightarrow{b} - \text{Proj}_{\overrightarrow{a}}\overrightarrow{b} = (1, 1, -1, 2) - \left(0, \frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right) = \left(1, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$,

$$\left(\overrightarrow{b} - \text{Proj}_{\overrightarrow{a}}\overrightarrow{b}\right) \cdot \overrightarrow{a} = \left(1, -\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right) \cdot (0, -1, 1, -1) = 0 + \frac{1}{3} + \frac{1}{3} - \frac{2}{3} = 0$$

and $\left(\overrightarrow{b} - \text{Proj}_{\overrightarrow{a}}\overrightarrow{b}\right) \perp \overrightarrow{a}$.

2. **2.**

1. Because $\overrightarrow{a} \cdot \overrightarrow{b} = (1)(1) + (1)(-1) + (1)(1) = 1$ and $\|\overrightarrow{a}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$, by Theorem 1.3.2 (*i*),

$$\|\text{Proj}_{\overrightarrow{a}}\overrightarrow{b}\| = \frac{\left|\overrightarrow{a} \cdot \overrightarrow{b}\right|}{\|\overrightarrow{a}\|} = \frac{|1|}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

2. Because $\|\overrightarrow{b}\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3}$, by Theorem 1.3.2 (*ii*),

$$|\cos\ \theta| = \frac{||\text{Proj}_{\overrightarrow{a}}\overrightarrow{b}\,||}{||\overrightarrow{b}\,||} = \frac{\frac{\sqrt{3}}{3}}{\sqrt{3}} = \frac{1}{3}.$$

Because $\overrightarrow{a} \cdot \overrightarrow{b} = 1 > 0,\ \cos\ \theta > 0$ and $\cos\ \theta = \left|\cos\ \theta\right| = \frac{1}{3}.$

3. **3.**

1. Because

$$\overrightarrow{a} \cdot \overrightarrow{b} = (1)\left(-\frac{1}{2}\right) + (0)\left(-\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right) + (2)\left(-\frac{3}{2}\right) = -3$$

and $||\overrightarrow{a}\,|| = \sqrt{1^2 + 0^2 + 1^2 + 2^2} = \sqrt{6}$, by Theorem 1.3.2 (*i*),

$$||\text{Proj}_{\overrightarrow{a}}\overrightarrow{b}\,|| = \frac{\left|\overrightarrow{a} \cdot \overrightarrow{b}\right|}{\left|\left|\overrightarrow{a}\right|\right|} = \frac{|-3|}{\sqrt{6}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}.$$

2. Because $||\overrightarrow{b}\,|| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{3}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4}} = \sqrt{3}.$ by Theorem 1.3.2 (*ii*),

$$|\cos\ \theta| = \frac{||\text{Proj}_{\overrightarrow{a}}\overrightarrow{b}\,||}{||\overrightarrow{b}\,||} = \frac{\frac{\sqrt{6}}{2}}{3}\sqrt{3} = \frac{\sqrt{2}}{2}.$$

Because $\overrightarrow{a} \cdot \overrightarrow{b} = -3 < 0,\ \cos\ \theta < 0.$ Hence, $\cos\ \theta = -\left|\cos\ \theta\right| = -\frac{\sqrt{2}}{2}.$ Hence, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$

4. **4.**

   1. $A(\overrightarrow{a},\ \overrightarrow{b}) = \left|\begin{vmatrix}1 & 0\\ 0 & -1\end{vmatrix}\right| = |-1| = 1;$
   2. $A(\overrightarrow{a},\ \overrightarrow{b}) = \left|\begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}\right| = |1| = 1;$
   3. $A(\overrightarrow{a},\ \overrightarrow{b}) = \left|\begin{vmatrix}0 & 2\\ 1 & 1\end{vmatrix}\right| = |-2| = 2;$
   4. $A(\overrightarrow{a},\ \overrightarrow{b}) = \left|\begin{vmatrix}1 & -1\\ -1 & 2\end{vmatrix}\right| = |1| = 1;$

5. **5.**

   1. Because

   $$G(\overrightarrow{a},\ \overrightarrow{b}) = \begin{vmatrix}1 & 0\\ -1 & 0\end{vmatrix}^2 + \begin{vmatrix}1 & 1\\ -1 & 1\end{vmatrix}^2 + \begin{vmatrix}0 & 1\\ 0 & 1\end{vmatrix}^2 = 0^2 + 2^2 + 0^2 = 4,$$
   $$A(\overrightarrow{a},\ \overrightarrow{b}) = \sqrt{G(\overrightarrow{a},\ \overrightarrow{b})} = \sqrt{4} = 2.$$

   2. $$G(\overrightarrow{a},\ \overrightarrow{b}) = \begin{vmatrix}1 & 0\\ -1 & 1\end{vmatrix}^2 + \begin{vmatrix}1 & 0\\ -1 & -2\end{vmatrix}^2 + \begin{vmatrix}0 & 0\\ 1 & -2\end{vmatrix}^2 = 1^2 + (-2)^2 + 0^2 = 5,$$
   $$A(\overrightarrow{a},\ \overrightarrow{b}) = \sqrt{G(\overrightarrow{a},\ \overrightarrow{b})} = \sqrt{5}.$$

   3. $$G(\overrightarrow{a},\ \overrightarrow{b}) = \begin{vmatrix}0 & 2\\ -5 & 1\end{vmatrix}^2 + \begin{vmatrix}0 & -1\\ -5 & 1\end{vmatrix}^2 + \begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix}^2 = 10^2 + 5^2 + 3^2 = 134,$$
   $$A(\overrightarrow{a},\ \overrightarrow{b}) = \sqrt{G(\overrightarrow{a},\ \overrightarrow{b})} = \sqrt{134}.$$

4. $G(\overrightarrow{a}, \overrightarrow{b}) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}^2 + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}^2 + \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix}^2 = 0^2 + 1^2 + (-1)^2 = 2,$
   $A(\overrightarrow{a}, \overrightarrow{b}) = \sqrt{G(\overrightarrow{a}, \overrightarrow{b})} = \sqrt{2}.$
5. Because $G(\overrightarrow{a}, \overrightarrow{b}) = ||\overrightarrow{a}||^2||\overrightarrow{b}||^2 - |(\overrightarrow{a} \cdot \overrightarrow{b}) = (6)(14) - 25 = 39,$ by Theorem 1.3.3,
   $A(\overrightarrow{a}, \overrightarrow{b}) = \sqrt{G(\overrightarrow{a}, \overrightarrow{b})} = \sqrt{39}.$
6. Because

$$G(\overrightarrow{a}, \overrightarrow{b}) = ||\overrightarrow{a}||^2||\overrightarrow{b}||^2 - |(\overrightarrow{a} \cdot \overrightarrow{b}) = (14)(11) - (-7)^2 = 105,$$

   by Theorem 1.3.3, $A(\overrightarrow{a}, \overrightarrow{b}) = \sqrt{G(\overrightarrow{a}, \overrightarrow{b})} = \sqrt{105}.$

6. **6.**

1. By Theorem 1.3.3 and Corollary 1.3.1,

$$G(\overrightarrow{a}, \overrightarrow{b}) = [A(\overrightarrow{a}, \overrightarrow{b})]^2 = \left(\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}\right)^2 = 1.$$

$$\left\|\overrightarrow{b} - \mathrm{Proj}_{\overrightarrow{a}}\overrightarrow{b}\right\| = \frac{\sqrt{G(\overrightarrow{a}, \overrightarrow{b})}}{\|\overrightarrow{a}\|} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}.$$

2. By Theorem 1.3.3 and Corollary 1.3.1,

$$G(\overrightarrow{a}, \overrightarrow{b}) = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix}^2 + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}^2 + \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix}^2$$
$$= (-1)^2 + 3^2 + (-1)^2 = 11$$

and $||\overrightarrow{b} - \mathrm{Proj}_{\overrightarrow{a}} \overrightarrow{b}|| = \frac{\sqrt{G(\overrightarrow{a},\ \overrightarrow{b})}}{||\overrightarrow{a}||} = \frac{\sqrt{11}}{\sqrt{2^2+(-1)^2+1^2}} = \frac{\sqrt{66}}{6}$.

3. By Theorem 1.3.3,

$$G(\overrightarrow{a},\ \overrightarrow{b}) = ||\overrightarrow{a}||^2||\overrightarrow{b}||^2 - |(\overrightarrow{a} \cdot \overrightarrow{b})|^2 = (6)(15) - 4 = 86.$$

$$||\overrightarrow{b} - \mathrm{Proj}_{\overrightarrow{a}} \overrightarrow{b}|| = \frac{\sqrt{G(\overrightarrow{a},\ \overrightarrow{b})}}{||\overrightarrow{a}||} = \frac{\sqrt{86}}{\sqrt{1^2+2^2+0+1^2}} = \frac{\sqrt{86}}{2}.$$

4. By Theorem 1.3.3,

$$G(\overrightarrow{a},\ \overrightarrow{b}) = ||\overrightarrow{a}||^2||\overrightarrow{b}||^2 - |(\overrightarrow{a} \cdot \overrightarrow{b})|^2 = (6)(19) - 49 = 65.$$

$$\begin{aligned} ||\overrightarrow{b} - \mathrm{Proj}_{\overrightarrow{a}} \overrightarrow{b}|| &= \frac{\sqrt{G(\overrightarrow{a},\ \overrightarrow{b})}}{||\overrightarrow{a}||} = \frac{\sqrt{65}}{\sqrt{2^2+(-1)^2+1^2+0}} \\ &= \frac{\sqrt{65}}{\sqrt{6}} = \frac{\sqrt{390}}{6}. \end{aligned}$$

Section 1.4

1. Let $\overrightarrow{a_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\overrightarrow{a_2} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, and $\overrightarrow{a_3} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$.

i. Let $\overrightarrow{b} = (-1,\ -9,\ -4)^T$. Verify whether $2\overrightarrow{a_1} - 3\overrightarrow{a_2} = \overrightarrow{b}$.

ii. Let $\overrightarrow{b} = (3,\ 6,\ 6)^T$. Verify whether $\overrightarrow{a_1} - 2\overrightarrow{a_2} = \overrightarrow{b}$.

iii. Let $\overrightarrow{b} = (2,\ 7,\ 7)^T$. Verify whether $\overrightarrow{a_1} - 2\overrightarrow{a_2} + \overrightarrow{a_3} = \overrightarrow{b}$.

2. Let $\overrightarrow{a_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{a_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\overrightarrow{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Determine whether $\overrightarrow{b}$ is a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}$.

3. Let $\overrightarrow{a_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\overrightarrow{a_2} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Show whether $\overrightarrow{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}$.

4. Let $\overrightarrow{a_1} = (1, -1, 1)^T$, $\overrightarrow{a_2} = (0, 1, 1)^T$, $\overrightarrow{a_3} = (1, 3, 2)^T$, and $\overrightarrow{a_4} = (0, 0, 1)^T$.
   1. Is $\overrightarrow{0} = (0, 0, 1)^T$ a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}, \overrightarrow{a_4}$?
   2. Is $\overrightarrow{b} = (1, 0, 2)^T$ a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}, \overrightarrow{a_4}$?

5. Let $\overrightarrow{a_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{a_2} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, and $\overrightarrow{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Show that $\overrightarrow{b} \in \operatorname{span}\left\{\overrightarrow{a_1}, \overrightarrow{a_2}\right\}$.

Solution

1. **1.**

i.
$$2\overrightarrow{a_1} - 3\overrightarrow{a_2} = 2\begin{pmatrix}1\\0\\1\end{pmatrix} - 3\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}2\\0\\2\end{pmatrix} - \begin{pmatrix}3\\9\\6\end{pmatrix}$$
$$= \begin{pmatrix}2-3\\0-9\\2-6\end{pmatrix} = \begin{pmatrix}-1\\-9\\-4\end{pmatrix} = \overrightarrow{b}.$$

ii. $$\overrightarrow{a_1} - 2\overrightarrow{a_2} = \begin{pmatrix}1\\0\\1\end{pmatrix} - 2\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}1-2\\0-6\\1-4\end{pmatrix} = \begin{pmatrix}-1\\-6\\-3\end{pmatrix} \neq \overrightarrow{b}.$$

iii.
$$\overrightarrow{a_1} + 2\overrightarrow{a_2} + \overrightarrow{a_3} = \begin{pmatrix}1\\0\\1\end{pmatrix} + 2\begin{pmatrix}1\\3\\2\end{pmatrix} + \begin{pmatrix}-1\\1\\3\end{pmatrix}$$
$$= \begin{pmatrix}1+2-1\\0+6+1\\1+4+3\end{pmatrix} = \begin{pmatrix}2\\7\\8\end{pmatrix} \neq \overrightarrow{b}.$$

2. **2.** Let $x\overrightarrow{a_1} + y\overrightarrow{a_2} = \overrightarrow{b}$. Then $x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ and

$$\begin{pmatrix}x\\2x\end{pmatrix} + \begin{pmatrix}y\\y\end{pmatrix} = \begin{pmatrix}x+y\\2x+y\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

Hence,

$$x + y = 1 \quad (1)$$

$$2x + y = 0. \quad (2)$$

$(1) - (2): \ (x+y) - (2x+y) = 1 - 0$ and $x = -1$. By (1),

$$y = 1 - x = 1 - (-1) = 1 + 1 = 2.$$

Hence, $-\overrightarrow{a_1} + 2\overrightarrow{a_2} = \overrightarrow{b}$ and $\overrightarrow{b}$ is a linear combination of $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$.

3. **3.** Let $x\overrightarrow{a_1} + y\overrightarrow{a_2} = \overrightarrow{b}$. Then

$$x\begin{pmatrix}1\\1\end{pmatrix} + y\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Hence,

$$x + 2y = 0 \quad (1)$$

$$x + y = 1. \quad (2)$$

$(1) - (2)$ implies that $(x+2y) - (x+y) = 0 - 1$ and $y = -1$. By (1), $x = -2y = -2(-1) = 2$. Hence, $2\overrightarrow{a_1} - \overrightarrow{a_2} = \overrightarrow{b}$ and $\overrightarrow{b}$ is a linear combination of $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$.

4. **4.** The result (1) follows from Theorem 1.4.1. Because $\overrightarrow{b} = \overrightarrow{a_1} + \overrightarrow{a_2}$, $\overrightarrow{b}$ is a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}$. By Theorem 1.4.2, $\overrightarrow{b}$ is a linear combination of $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}, \overrightarrow{a_4}$.

5. **5.** Let $x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} = \overrightarrow{b}$. Then

$$x_1\begin{pmatrix}1\\2\end{pmatrix} + x_2\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}4\\3\end{pmatrix}.$$

This implies

$$x_1 + 2x_2 = 4 \tag{1}$$

$$2x_1 - x_2 = 3. \tag{2}$$

Equation $(1) + (2) \times 2$ implies $5x_1 = 10$ and $x_1 = 2$. Substituting $x_1 = 2$ into the second equation implies $x_2 = 2x_1 - 3 = 2(2) - 3 = 1$. Hence, $\overrightarrow{b} = 2\overrightarrow{a_1} + \overrightarrow{a_2}$ and $\overrightarrow{b} \in \text{span}\left\{\overrightarrow{a_1},\ \overrightarrow{a_2}\right\}$.

A.2 Matrices

Section 2.1

1. Find the sizes of the following matrices:

$$A = (2)\ B = \begin{pmatrix} 5 & 8 \\ 1 & 4 \end{pmatrix}\ \ C = \begin{pmatrix} 10 & 3 & 8 \\ 10 & 3 & 4 \end{pmatrix}\ \ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 10 & 2 \\ 0 & 0 & 1 \\ 6 & 9 & 10 \end{pmatrix}$$

2. Use column vectors and row vectors to rewrite each of the following matrices.

$$A = \begin{pmatrix} 3 & 3 & 2 \\ 1 & 8 & 1 \\ 5 & 4 & 10 \end{pmatrix}\ \ B = \begin{pmatrix} 1 & 9 & 7 & 2 \\ 3 & 10 & 8 & 7 \\ 4 & 3 & 10 & 4 \end{pmatrix}\ \ C = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \\ 0 & -1 & -2 \end{pmatrix}$$

3. Find the transposes of the following matrices:

$$C = (4)\ B = \begin{pmatrix} 3 & 11 & 2 \end{pmatrix}\ \ C = \begin{pmatrix} 33 \\ 8 \\ 12 \end{pmatrix}\ \ D = \begin{pmatrix} 3 & 9 & -19 \\ -2 & 8 & -7 \\ 5 & 3 & -9 \end{pmatrix}$$

4. Let $A = \begin{pmatrix} 9 & 3 \\ 6 & x^2 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & 3 \\ 6 & 4 \end{pmatrix}$. Find all $x \in \mathbb{R}$ such that $A = B$.

5. Let $C = \begin{pmatrix} 12 & 5 & 25 \\ 19 & 4 & 6 \end{pmatrix}$ and $D = \begin{pmatrix} 12 & 5 & x^2 \\ 19 & 4 & 6 \end{pmatrix}$. Find all $x \in \mathbb{R}$ such that $C = D$.

6. Let $E = \begin{pmatrix} 120 & 25 & 122 \\ 123 & 124 & 125 \\ 126 & 127 & 128 \end{pmatrix}$ and $F = \begin{pmatrix} 120 & x^2 & 122 \\ 123 & 124 & x^3 \\ 26x - 4 & 127 & 128 \end{pmatrix}$. Find all $x \in \mathbb{R}$ such that $E = F$.

7. Let $A = \begin{pmatrix} a & b \\ 3x+2y & -x+y \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 3 & 6 \end{pmatrix}$. Find all $a,\ b,\ x,\ y \in \mathbb{R}$ such that $A = B$.

8. Let

$$C = \begin{pmatrix} a+b & 2b-a \\ x-2y & 5x+3y \end{pmatrix} \text{ and } D = \begin{pmatrix} 6 & 0 \\ 8 & 14 \end{pmatrix}.$$

Find all $a,\ b,\ x,\ y \in \mathbb{R}$ such that $C = D$.

9. Let $A = \begin{pmatrix} -2 & 3 & 4 \\ 6 & -1 & -8 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & -8 & 9 \\ 0 & -1 & 0 \end{pmatrix}$. Compute
   i. $A+B$;
   ii. $-A$;
   iii. $4A-2B$;
   iv. $100A+B$.

10. Let $A = \begin{pmatrix} 9 & 5 & 1 \\ 8 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & 8 \end{pmatrix}$, and $C = \begin{pmatrix} 4 & 7 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

    Compute
    1. $3A-2B+C$,
    2. $[3(A+B)]^T$,
    3. $\left(4A+\frac{1}{2}B-C\right)^T$.

11. Find the matrix *A* if $\left[\left(3A^T\right) - \begin{pmatrix} -7 & -2 \\ -6 & 9 \end{pmatrix}^T\right]^T = \begin{pmatrix} -5 & -10 \\ 33 & 12 \end{pmatrix}$.

12. Find the matrix *B* if

$$\left[\frac{1}{2}B+\begin{pmatrix}6 & 3\\ 8 & 3\\ 1 & 4\end{pmatrix}\right]^T-3\begin{pmatrix}-5 & 6\\ 8 & -9\\ -4 & 2\end{pmatrix}^T=\begin{pmatrix}23 & -16 & 17\\ -16 & 26.5 & 2\end{pmatrix}.$$

Solution

1. **1.** The sizes of *A*, *B*, *C*, and *D* are $1 \times 1, 2 \times 2, 2 \times 3,$ and $4 \times 3,$ respectively.

2. **2.** Let $\overrightarrow{c_1} = \begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}, \overrightarrow{c_2} = \begin{pmatrix} 3 \\ 8 \\ 4 \end{pmatrix}, \overrightarrow{c_3} = \begin{pmatrix} 2 \\ 1 \\ 10 \end{pmatrix}$. Then $A = (\overrightarrow{c_1} \quad \overrightarrow{c_2} \quad \overrightarrow{c_3})$.

   Let $\overrightarrow{r_1} = (3 \quad 3 \quad 2), \overrightarrow{r_2} = (1 \quad 8 \quad 1), \overrightarrow{r_3} = (5 \quad 4 \quad 10)$. Then $A = \begin{pmatrix} \overrightarrow{r_1} \\ \overrightarrow{r_2} \\ \overrightarrow{r_3} \end{pmatrix}$.

   Let $\overrightarrow{c_1} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \overrightarrow{c_2} = \begin{pmatrix} 9 \\ 10 \\ 3 \end{pmatrix}, \overrightarrow{c_3} = \begin{pmatrix} 7 \\ 8 \\ 10 \end{pmatrix}, \overrightarrow{c_4} = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}$. Then $B = (\overrightarrow{c_1} \quad \overrightarrow{c_2} \quad \overrightarrow{c_3} \quad \overrightarrow{c_4})$.

   Let $\overrightarrow{r_1} = (1 \quad 9 \quad 7 \quad 2), \overrightarrow{r_2} = (3 \quad 10 \quad 8 \quad 7), \overrightarrow{r_3} = (4 \quad 3 \quad 10 \quad 4)$. Then $B = \begin{pmatrix} \overrightarrow{r_1} \\ \overrightarrow{r_2} \\ \overrightarrow{r_3} \end{pmatrix}$.

   Let $\overrightarrow{c_1} = \begin{pmatrix} 9 \\ 6 \\ 3 \\ 0 \end{pmatrix}, \overrightarrow{c_2} = \begin{pmatrix} 8 \\ 5 \\ 2 \\ -1 \end{pmatrix}$, and $\overrightarrow{c_3} = \begin{pmatrix} 7 \\ 4 \\ 1 \\ -2 \end{pmatrix}$. Then $C = (\overrightarrow{c_1} \quad \overrightarrow{c_2} \quad \overrightarrow{c_3})$.

Let $\overrightarrow{r_1} = (9 \quad 8 \quad 7), \overrightarrow{r_2} = (6 \quad 5 \quad 4), \overrightarrow{r_3} = (3 \quad 2 \quad 1),$ and $\overrightarrow{r_4} = (0 \quad -1 \quad -2).$ Then

$$C = \begin{pmatrix} \overrightarrow{r_1} \\ \overrightarrow{r_2} \\ \overrightarrow{r_3} \\ \overrightarrow{r_4} \end{pmatrix}.$$

3. **3.** $A^T = (4), B^T = \begin{pmatrix} 3 \\ 11 \\ 2 \end{pmatrix}, C^T = (33 \quad 8 \quad 12),$ and

$$D^T = \begin{pmatrix} 3 & -2 & 5 \\ 9 & 8 & 3 \\ -19 & -7 & -9 \end{pmatrix}.$$

4. **4.** Let $A = B$. Then $x^2 = 4$. This implies $x = 2$ or $x = -2$.
5. **5.** Let $C = D$. Then $x^2 = 25$. Solving the equation, we obtain $x = 5$ or $x = -5$.
6. **6.** Let $E = F$. Then $x^2 = 25, x^3 = 125,$ and $26x - 4 = 126.$ Solving the three equations implies $x = 5$ or $x = -5; x = 5;$ and $x = \frac{130}{26} = 5$. Hence, when $x = 5, E = F$.
7. **7.** Let $A = B$. Then $a = 1$ and $b = 1$. Hence,

(1)

$$3x + 2y = 3$$

(2)

$$-x + y = 6.$$

$(1) - (2) \times 2$ implies that $3x + 2y - 2(-x + y) = 3 - 12$ and $x = -\frac{9}{5}$. By (2), we obtain $y = x + 6 = -\frac{9}{5} + 6 = \frac{21}{5}$. Hence, when $a = 1, b = 1, x = -\frac{9}{5}, y = \frac{21}{5}, A = B$.

8. **8.** Let $C = D$. Then

(1)

$$a + b = 6$$

(2)

$$2b - a = 0$$

(3)

$$x - 2y = 8$$

(4)

$$5x + 3y = 14.$$

$(1) + (2)$ implies that $(a + b) + (2b - a) = 6 + 0 = 6$ and $b = 2$. By (1), we have $a = 6 - b = 6 - 2 = 4.$
$(3) \times 5 - (4)$ implies that $5(x - 2y) - (5x + 3y) = 40 - 14$ and $-13y = 26$. Hence, $y = -2$.
By (3), we get $x = 8 + 2y = 8 + 2(-2) = 4.$ Hence, when $a = 4, b = 2, x = 4,$ and $y = -2, C = D.$

9. **9.**

i.
$$\begin{aligned} A + B &= \begin{pmatrix} -2 & 3 & 4 \\ 6 & -1 & -8 \end{pmatrix} + \begin{pmatrix} 7 & -8 & 9 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2+7 & 3-8 & 4+9 \\ 6+0 & -1-1 & -8+0 \end{pmatrix} = \begin{pmatrix} 5 & -5 & 13 \\ 6 & -2 & -8 \end{pmatrix}. \end{aligned}$$

ii. $-A = -\begin{pmatrix} -2 & 3 & 4 \\ 6 & -1 & -8 \end{pmatrix} = \begin{pmatrix} 2 & -3 & -4 \\ -6 & 1 & 8 \end{pmatrix}.$

iii. $4A - 2B = 4\begin{pmatrix} -2 & 3 & 4 \\ 6 & -1 & -8 \end{pmatrix} - 2\begin{pmatrix} 7 & -8 & 9 \\ 0 & -1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} -8 & 12 & 16 \\ 24 & -4 & -32 \end{pmatrix} - \begin{pmatrix} 14 & -16 & 18 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} -22 & 28 & -2 \\ 24 & -2 & -32 \end{pmatrix}.$$

iv. $100A + B = 100\begin{pmatrix} -2 & 3 & 4 \\ 6 & -1 & -8 \end{pmatrix} + \begin{pmatrix} 7 & -8 & 9 \\ 0 & -1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} -200 & 300 & 400 \\ 600 & -100 & -800 \end{pmatrix} + \begin{pmatrix} 7 & -8 & 9 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -200+7 & 300-8 & 400+9 \\ 600+0 & -100-1 & -800+0 \end{pmatrix} = \begin{pmatrix} -193 & 292 & 409 \\ 600 & -101 & -800 \end{pmatrix}.$$

10. **10.**

1. Let $D = 3A - 2B + C$. Then

$$D = 3\begin{pmatrix} 9 & 5 & 1 \\ 8 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix} - 2\begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & 8 \end{pmatrix} + \begin{pmatrix} 4 & 7 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 27 & 15 & 3 \\ 24 & 0 & 0 \\ 0 & 9 & 6 \end{pmatrix} - \begin{pmatrix} 4 & 4 & 4 \\ 0 & 0 & 0 \\ 8 & 12 & 16 \end{pmatrix} + \begin{pmatrix} 4 & 7 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 27-4 & 15-4 & 3-4 \\ 24-0 & 0-0 & 0-0 \\ 0-8 & 9-12 & 6-16 \end{pmatrix} + \begin{pmatrix} 4 & 7 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 23 & 11 & -1 \\ 24 & 0 & 0 \\ -8 & -3 & -10 \end{pmatrix} + \begin{pmatrix} 4 & 7 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 23+4 & 11+7 & -1+0 \\ 24+0 & 0+3 & 0+1 \\ -8+0 & -3+0 & -10+2 \end{pmatrix} = \begin{pmatrix} 27 & 18 & -1 \\ 24 & 3 & 1 \\ -8 & -3 & -8 \end{pmatrix}.$$

2. $[3(A+B)]^T = 3(A+B)^T = 3(A^T + B^T)$

$$= 3\left[\begin{pmatrix} 9 & 8 & 0 \\ 5 & 0 & 3 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 4 \\ 2 & 0 & 6 \\ 2 & 0 & 8 \end{pmatrix}\right] = 3\begin{pmatrix} 11 & 8 & 4 \\ 7 & 0 & 9 \\ 3 & 0 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 33 & 24 & 12 \\ 21 & 0 & 27 \\ 9 & 0 & 30 \end{pmatrix}.$$

3. Let $D = \left(4A + \frac{1}{2}B - C\right)^T$. Then

$$D = 4\begin{pmatrix} 9 & 8 & 0 \\ 5 & 0 & 3 \\ 1 & 0 & 2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 & 0 & 4 \\ 2 & 0 & 6 \\ 2 & 0 & 8 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 7 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 36 & 32 & 0 \\ 20 & 0 & 12 \\ 4 & 0 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 7 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 36+1 & 32+0 & 0+2 \\ 20+1 & 0+0 & 12+3 \\ 4+1 & 0+0 & 8+4 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 7 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 37 & 32 & 2 \\ 21 & 0 & 15 \\ 5 & 0 & 12 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 7 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 37-4 & 32-0 & 2-0 \\ 21-7 & 0-3 & 15-0 \\ 5-0 & 0-1 & 12-2 \end{pmatrix} = \begin{pmatrix} 33 & 32 & 2 \\ 14 & -3 & 15 \\ 5 & -1 & 10 \end{pmatrix}.$$

11. **11.**

$$\left[(3A^T) - \begin{pmatrix} -7 & -2 \\ -6 & 9 \end{pmatrix}^T\right]^T = (3A^T)^T - \left[\begin{pmatrix} -7 & -2 \\ -6 & 9 \end{pmatrix}^T\right]^T$$
$$= 3(A^T)^T - \begin{pmatrix} -7 & -2 \\ -6 & 9 \end{pmatrix} = 3A - \begin{pmatrix} -7 & -2 \\ -6 & 9 \end{pmatrix} = \begin{pmatrix} -5 & -10 \\ 33 & 12 \end{pmatrix}.$$

Hence,

$$3A = \begin{pmatrix} -5 & -10 \\ 33 & 12 \end{pmatrix} + \begin{pmatrix} -7 & -2 \\ -6 & 9 \end{pmatrix} = \begin{pmatrix} -12 & -12 \\ 27 & 21 \end{pmatrix}.$$

This implies that

$$A = \frac{1}{3}\begin{pmatrix} -12 & -12 \\ 27 & 21 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ 9 & 7 \end{pmatrix}.$$

12. **12.**

$$\left[\tfrac{1}{2}B + \begin{pmatrix} 6 & 3 \\ 8 & 3 \\ 1 & 4 \end{pmatrix}\right]^T - 3\begin{pmatrix} -5 & 6 \\ 8 & -9 \\ -4 & 2 \end{pmatrix}^T$$

$$= \tfrac{1}{2}B^T + \begin{pmatrix} 6 & 8 & 1 \\ 3 & 3 & 4 \end{pmatrix} - 3\begin{pmatrix} -5 & 8 & -4 \\ 6 & -9 & 2 \end{pmatrix}$$

$$= \tfrac{1}{2}B^T + \begin{pmatrix} 6+15 & 8-24 & 1+12 \\ 3-18 & 3+27 & 4-6 \end{pmatrix} = \tfrac{1}{2}B^T + \begin{pmatrix} 21 & -16 & 13 \\ -15 & 30 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 23 & -16 & 17 \\ -16 & 26.5 & 2 \end{pmatrix}.$$

This implies that

$$\tfrac{1}{2}B^T = \begin{pmatrix} 23 & -16 & 17 \\ -16 & 26.5 & 2 \end{pmatrix} - \begin{pmatrix} 21 & -16 & 13 \\ -15 & 30 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 4 \\ -1 & -3.5 & 4 \end{pmatrix},$$

$$B^T = 2\begin{pmatrix} 2 & 0 & 4 \\ -1 & -3.5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 8 \\ -2 & -7 & 8 \end{pmatrix} \text{ and } B = \begin{pmatrix} 4 & -2 \\ 0 & -7 \\ 8 & 8 \end{pmatrix}.$$

Section 2.2

1. Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$, $\overrightarrow{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $\overrightarrow{X_1} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, $\overrightarrow{X_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Compute $A\overrightarrow{X}$, $A\overrightarrow{X_1}$, and $A\overrightarrow{X_2}$.

2. 2. $A = \begin{pmatrix} -2 & -1 \\ 3 & 1 \\ 0 & 1 \end{pmatrix}$, $\overrightarrow{X} = \begin{pmatrix} -a \\ 2a \end{pmatrix}$, $\overrightarrow{X_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Compute $A\overrightarrow{X}$ and $A\overrightarrow{X_1}$.

3. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$, $\overrightarrow{X_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\overrightarrow{X_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\overrightarrow{X_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}$.

   Determine whether $A\overrightarrow{X_i}$. is defined for each $i = 1,\ 2,\ 3.$

4. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}$, $\overrightarrow{X} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\overrightarrow{Y} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$. Compute $A(\overrightarrow{X} - 3\overrightarrow{Y})$.

5. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 2 & 1 & -1 \end{pmatrix}$ and $\overrightarrow{X} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Write $A\overrightarrow{X}$ as a linear combination of the column vectors of *A*.

6. We are interested in predicting the population change among three cities in a country. It is known that in the current year, 20% and 30% of the populations of City 1 will move to Cities 2 and 3, respectively; 10% and 20% of the populations of City 2 will move to Cities 1 and 3, respectively; and 25% and 10% of the populations of City 3 will move to Cities 1 and 2, respectively. Assume that 200,000 600,000, and 500,000 thousand people live in Cities, 1, 2, and 3, respectively, in the current year. Find the populations in the three cities in the following year.

7. Let *A*, *B*, and *C* be matrices such that $AB = C$. Assume that the sizes of *B* and *C* are 6×8 and 3×8, respectively. Find the size of *A*.

8. Let $A = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 4 & 1 \end{pmatrix}$. Find *AB* and the sizes of *A*, *B*, and *AB*.

9. Let $A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 2 & 4 \end{pmatrix}$. Compute *AB* and find the sizes of *A*, *B*, and *AB*.

10. Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 & 2 & 1 \\ -1 & 2 & 1 & -2 \\ 2 & 0 & 0 & 1 \end{pmatrix}$. Compute *AB* and find the sizes of *A*, *B*, and *AB*.
11. 10. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$. Are *AB* and *BA* defined? If so, compute them. If not, explain why.
12. Let $A = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -2 \\ 4 & 2 \end{pmatrix}$. Are *AB* and *BA* defined? If so, compute them. If not, explain why. Is *AB* equal to *BA*?
13. Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ -2 & 0 & 3 \end{pmatrix}$. Show that $A(BC) = (AB)C$ and $(AB)^T = B^T A^T$.
14. Two students buy three items in a bookstore. Students 1 and 2 buy 3,2,4 and 6,2,3, respectively. The unit prices and unit taxes are 5, 4, 6 (dollars) and 0.08, 0.07, 0.05, respectively. Use a matrix to show the total prices and taxes paid by each student.
15. Assume that three individuals have contracted a contagious disease and have had direct contacts with four people in a second group. The direct contacts can be expressed by a matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Now, assume that the second group has had a variety of direct contacts with five people in a third group. The direct contacts between groups 2 and 3 are expressed by a matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

i. Find the total number of indirect contacts between groups 1 and 3.
ii. How many indirect contacts are between the second individual in group 1 and the fourth individual in group 3?

16. There are three product items, $P1$, $P2$, and $P3$, for sale from a large company. In the first day, 5, 10, and 100 are sought out. The corresponding unit profits are 500, 400, and 20 (in hundreds of dollars) and the corresponding unit taxes are 3, 2, and 1. Find the total profits and taxes in the first day.

Solution

$$A\overrightarrow{X} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 + x_3 \\ x_1 - x_2 + 2x_3 \end{pmatrix}.$$

1. **1.** $$A\overrightarrow{X_1} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2(0) + 1(1) + 1(2) \\ 1(0) + (-1)(1) + 2(2) \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

$$A\overrightarrow{X_2} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2(2) + 1(1) + 1(1) \\ 1(2) - 1(1) + 2(1) \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}.$$

2. **2.** $$A\overrightarrow{X} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a \\ 2a \end{pmatrix} = \begin{pmatrix} 2a - 2a \\ -3a + 2a \\ 0 + 2a \end{pmatrix} = \begin{pmatrix} 0 \\ -a \\ 2a \end{pmatrix}.$$

$$A\overrightarrow{X_1} = \begin{pmatrix} -2 & -1 \\ 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 2 \end{pmatrix}.$$

3. **3.** $A\overrightarrow{X_1}$ is not defined, $A\overrightarrow{X_2}$ is defined, and $A\overrightarrow{X_3}$ is not defined.

4. **4.** $$\overrightarrow{X} - 3\overrightarrow{Y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 - 9 \\ 0 - 0 \\ -1 + 3 \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \\ 2 \end{pmatrix}.$$

$$A(\overrightarrow{X}-3\overrightarrow{Y})=\begin{pmatrix}0&1&0\\1&0&-1\\1&1&2\end{pmatrix}\begin{pmatrix}-8\\0\\2\end{pmatrix}=\begin{pmatrix}0\\-10\\-4\end{pmatrix}.$$

5. **5.** $$A\overrightarrow{X}=\begin{pmatrix}1&1&-1\\1&0&2\\2&1&-1\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}+0\begin{pmatrix}-1\\2\\-1\end{pmatrix}.$$

6. **6.** We denote by a_{ij} the percentage of populations moving from City *j* to City *i*, where $i\neq j$ and $i,j=1,2,3,$ and by a_{ii} the percentage of populations staying in City *i*. Let x_i and y_i denote the populations in Cities *i* in the current and following year, respectively. Then

$$\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}50\%&10\%&25\%\\20\%&70\%&10\%\\30\%&20\%&65\%\end{pmatrix}\begin{pmatrix}200\\600\\500\end{pmatrix}$$
$$=\begin{pmatrix}100+60+125\\40+420+50\\60+120+325\end{pmatrix}=\begin{pmatrix}285\\510\\505\end{pmatrix}.$$

7. **7.** Because $AB=C$ and the sizes of *B* and *C* are 6×8 and $3\times 8,$ respectively, it follows from the formula $(m\times n)(n\times r)=m\times r$ that the size of *A* is $3\times 6.$

8. **8.** $$AB=\begin{pmatrix}2&1\\-2&3\end{pmatrix}\begin{pmatrix}1&-2&1\\3&4&1\end{pmatrix}$$
$$=\begin{pmatrix}2(1)+1(3)&2(-2)+1(4)&2(1)+(1)\\-2(1)+3(3)&-2(-2)+3(4)&-2(1)+3(1)\end{pmatrix}=\begin{pmatrix}5&0&3\\7&16&1\end{pmatrix}.$$

The sizes of *A*, *B*, *AB* are $2\times 2, 2\times 3, 2\times 3,$ respectively.

9. **9.**

$$\begin{aligned} AB &= \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0(2)+1(1)-1(-1) & 0(1)+1(0)-1(2) & 0(-1)+1(1)-1(4) \\ 2(2)+3(1)+1(-1) & 2(1)+3(0)+1(2) & 2(-1)+3(1)+1(4) \end{pmatrix} \\ &= \begin{pmatrix} 2 & -2 & -3 \\ 6 & 4 & 5 \end{pmatrix}. \end{aligned}$$

The sizes of *A*, *B*, *AB* are $2 \times 3, 3 \times 3, 2 \times 3,$ respectively.

10. **10.** $$AB = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 & 1 \\ -1 & 2 & 1 & -2 \\ 2 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -1 & 2 & -1 \\ -4 & 1 & 8 & -2 \end{pmatrix}.$$

The sizes of *A*, *B*, *AB* are $2 \times 3, 3 \times 4, 2 \times 4,$ respectively.

11. **11.**

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 1 \\ 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1(2)+1(-2) & 1(1)+1(3) \\ 1(2)+3(-2) & 1(1)+3(3) \\ 3(2)+1(-2) & 3(1)+1(3) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 4 \\ -4 & 10 \\ 4 & 6 \end{pmatrix}. \end{aligned}$$

BA is not defined because the size of *B* is 2×2 and the size of *A* is 3×2.

$$AB = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2(0) - 1(4) & 2(-2) - 1(2) \\ -2(0) + 3(4) & -2(-2) + 3(2) \end{pmatrix}$$
$$= \begin{pmatrix} -4 & -6 \\ 12 & 10 \end{pmatrix}.$$

12. **12.**

$$BA = \begin{pmatrix} 0 & -2 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0(2) - 2(-2) & 0(-1) - 2(3) \\ 4(2) + 2(-2) & 4(-1) + 2(3) \end{pmatrix}$$
$$= \begin{pmatrix} 4 & -6 \\ 4 & 2 \end{pmatrix}.$$

$$BC = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ -2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -4 & -3 & 4 \\ -5 & 2 & 12 \end{pmatrix}.$$

$$A(BC) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -4 & -3 & 4 \\ -5 & 2 & 12 \end{pmatrix} = \begin{pmatrix} 1 & -5 & -8 \\ -6 & 7 & 20 \end{pmatrix}.$$

13. **13.**

$$AB = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -1 \\ 2 & 3 & 4 \end{pmatrix}.$$

$$(AB)C = \begin{pmatrix} -1 & -2 & -1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 2 \\ -2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -5 & -8 \\ -6 & 7 & 20 \end{pmatrix}.$$

Hence, $A(BC) = (AB)C$.

$$B^T A^T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0-1 & 0+2 \\ -1-1 & 1+2 \\ 2-3 & -2+6 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \\ -1 & 4 \end{pmatrix}.$$

$$(AB)^T = \begin{pmatrix} -1 & -2 & -1 \\ 2 & 3 & 4 \end{pmatrix}^T = \begin{pmatrix} -1 & 2 \\ -2 & 3 \\ -1 & 4 \end{pmatrix}.$$

Hence, $(AB)^T = B^T A^T$.

14. **14.**

$$\begin{pmatrix} \text{Student 1} \\ \text{Student 2} \end{pmatrix} = \begin{pmatrix} 3 & 2 & 4 \\ 6 & 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0.08 \\ 4 & 0.07 \\ 6 & 0.05 \end{pmatrix}$$

$$= \begin{pmatrix} 3(5)+2(4)+4(6) & 0.24+0.14+0.2 \\ 6(4)+2(4)+3(6) & 0.48+0.14+0.15 \end{pmatrix} = \begin{pmatrix} 47 & 0.58 \\ 56 & 0.77 \end{pmatrix}.$$

15. **15.**

i. The total number of indirect contacts between groups 1 and 3 is given by

$$AB = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 3 & 1 \\ 3 & 1 & 0 & 4 & 1 \\ 2 & 0 & 0 & 3 & 0 \end{pmatrix}.$$

ii. From the last matrix, we see that $c_{21} = 4,$ so there are four indirect contacts between the second individual in group 1 and the fourth in group 3.

16. 16. Let $A = (5, 10, 100)$ and $B = \begin{pmatrix} 500 & 3 \\ 400 & 2 \\ 20 & 1 \end{pmatrix}$. Then the total profits and taxes in the first day can be computed by

$$AB = (5, 10, 100) \begin{pmatrix} 500 & 3 \\ 400 & 2 \\ 20 & 1 \end{pmatrix} = (8500, 135)\,.$$

Section 2.3

1. For each of the following matrices, find its trace and determine whether it is symmetric.

$$A_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 4 & 5 \\ 4 & -3 & -1 \\ 5 & 0 & 7 \end{pmatrix}, A_3 = \begin{pmatrix} y & x^3 & 1 \\ x^3 & y & x \\ 1 & x & z \end{pmatrix}.$$

2. Identify which of the following matrices are upper triangular, diagonal, triangular, or an identity matrix.

$$A_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \; A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 0 & x \\ 1 & 0 & 0 \\ x & 0 & 0 \end{pmatrix} \; A_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; A_6 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \; A_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix} \; A_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \; A_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \; A_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

3. Let $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Find A^0, A^2, A^3, and A^4.

4. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Find A^6.

5. Let $P(x) = 1 - 2x - x^2$ and $A = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$. Compute $P(A)$.

6. Let $P(x) = x^2 - x - 4$. Compute $P(A)$ and $P(B)$, where

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 3 & -2 & 3 \\ 4 & 5 & -1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 6 & -2 & -1 \\ 3 & 0 & 2 \\ -1 & 3 & 4 \end{pmatrix}.$$

7. Assume that the population of a species has a life span of less than 3 years. The females are divided into three age groups: group 1 contains those of ages less than 1, group 2 contains those of ages between 1 to 2, and group 3 contains those of age 2. Suppose that the females in group 1 do not give birth, and the average numbers of newborn females produced by one female in group 2 and 3, respectively, are 3 and 2. Suppose that 50% of the females in group 1 survive to age 1 and 80% of the females in group 2 live to age 2. Assume that the numbers of females in groups 1, 2, and 3 are $X_1,\ X_2,$ and $X_3,$ respectively.
   i. Find the number of females in the three groups in the following year.
   ii. Find the number of females in the three groups in *n* years.
   iii. If the current population distribution $(x_1, x_2, x_3) = \ (1000{,}500{,}100),$ then use result 2 to predict the population distribution in 2 years.

Solution

1. **1.**

   i. $\operatorname{tr}(A_1) = 0 + 0 = 0, \operatorname{tr}(A_2) = 1 + (-3) + 7 = 5$ and

   $$\operatorname{tr}(A_3) = y + y + z = 2y + z.$$

   ii. A_1 is symmetric, A_2 is not symmetric, and A_3 is symmetric.

2. **2.** Lower triangular: A_2, A_3, A_7, A_8, A_9.
   Upper triangular: A_1, A_3, A_6, A_7.
   Diagonal: A_3, A_7.
   Triangular: $A_1, A_2, A_3, A_6, A_7, A_8, A_9$.
   There are no identity matrices.

3. **3.**

$$A^0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

$$A^4 = A^3 \cdot A = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}.$$

4. **4.**

$$A^6 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^6 = \begin{pmatrix} 2^6 & 0 & 0 \\ 0 & (-2)^6 & 0 \\ 0 & 0 & 3^6 \end{pmatrix} = \begin{pmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 729 \end{pmatrix}.$$

5. **5.**

$$\begin{aligned} P(A) &= I - 2A - A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}^2 \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -2 \\ -2 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -3 & -8 \end{pmatrix}. \end{aligned}$$

6. **6.**

$$\begin{aligned} P(A) &= A^2 - A - 4I \\ &= \begin{pmatrix} 3 & 1 & 0 \\ 3 & -2 & 3 \\ 4 & 5 & -1 \end{pmatrix}^2 - \begin{pmatrix} 3 & 1 & 0 \\ 3 & -2 & 3 \\ 4 & 5 & -1 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 1 & 3 \\ 15 & 22 & -9 \\ 23 & -11 & 16 \end{pmatrix} - \begin{pmatrix} 3 & 1 & 0 \\ 3 & -2 & 3 \\ 4 & 5 & -1 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 0 & 3 \\ 12 & 20 & -12 \\ 19 & -16 & 13 \end{pmatrix}. \end{aligned}$$

$$P(B) = B^2 - B - 4I$$
$$= \begin{pmatrix} 6 & -2 & -1 \\ 3 & 0 & 2 \\ -1 & 3 & 4 \end{pmatrix}^2 - \begin{pmatrix} 6 & -2 & -1 \\ 3 & 0 & 2 \\ -1 & 3 & 4 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 31 & -15 & -14 \\ 16 & 0 & 5 \\ -1 & 14 & 23 \end{pmatrix} - \begin{pmatrix} 6 & -2 & -1 \\ 3 & 0 & 2 \\ -1 & 3 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 21 & -13 & -13 \\ 13 & -4 & 3 \\ 0 & 11 & 15 \end{pmatrix}.$$

7. **7.**

i. We denote by y_1, y_2, and y_3 the numbers of females in groups 1, 2, and 3, respectively. Then the first group in the following year consists of the newborn females from groups 2 and 3 because the females in group 1 do not give birth, so $y_1 = 3x_2 + 2x_3$. The Group 2 in the following year comes from group 1 only and $y_2 = 50\%x_1 = \frac{1}{2}x_1$. The Group 3 in the following year comes from group 2 only and $y_3 = 80\%x_2 = \frac{4}{5}x_2$. Hence, the populations for the three groups in the following year can be expressed as

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{4}{5} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

.

ii. We write $\overrightarrow{X_0} = (x_1, x_2, x_3)^T$. The vector $\overrightarrow{X_0}$ is called the population distribution. We denote by $\overrightarrow{X_n}$ the population distribution for the female population in the *n*th year. Let

$$A = \begin{pmatrix} 0 & 3 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{4}{5} & 0 \end{pmatrix}.$$

Then we have

$$\overrightarrow{X_1} = A\overrightarrow{X_0},\ \overrightarrow{X_2} = A\overrightarrow{X_1} = A^2\overrightarrow{X_0}, \cdots, \overrightarrow{X_n} = A^n\overrightarrow{X_0}.$$

iii.

$$\overrightarrow{X_2} = A^2\overrightarrow{X_0} = \begin{pmatrix} 0 & 3 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{4}{5} & 0 \end{pmatrix} \begin{pmatrix} 0 & 3 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{4}{5} & 0 \end{pmatrix} \begin{pmatrix} 1000 \\ 500 \\ 100 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2} & \frac{8}{5} & 0 \\ 0 & \frac{3}{2} & 1 \\ \frac{2}{5} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1000 \\ 500 \\ 100 \end{pmatrix} = \begin{pmatrix} 2300 \\ 850 \\ 400 \end{pmatrix}.$$

Section 2.4

1. Which of the following matrices are row echelon matrices?

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B = \begin{pmatrix} 9 & 4 & 3 & 5 & 9 \\ 0 & 0 & 0 & 7 & 4 \end{pmatrix} C = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 0 & 1 & 6 & 0 \\ 0 & 0 & 8 & 8 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} F = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & 6 & 6 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 & 8 & 1 & 7 \\ 2 & 9 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 5 & 8 \\ 0 & 9 & 6 & 1 \end{pmatrix}$$

2. Which of the following matrices are reduced row echelon matrices?

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} 1 & 5 & 0 & 6 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} D = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} F = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{pmatrix}$$

3. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & -2 \end{pmatrix}$. Use the second row operation to change the numbers 2 and −3 in the first column of *A* to zero.
4. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 3 & 1 \end{pmatrix}$. Use the second row operation to change the number 3 in *A* to 0.
5. For each of the following matrices, find its row echelon matrix. Clearly mark every leading entry *a* by using the symbol ⓐ and use the symbol ↓ to show the direction of eliminating nonzero entries below leading entries.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 3 & 0 & 3 & 2 \\ 5 & -2 & -1 & 3 \end{pmatrix}$$

6. For each of the following matrices, find its reduced row echelon matrix. Clearly mark every leading entry *a* by using the symbol ⓐ and use the symbols ↓ and ↑ to show the directions of eliminating nonzero entries below or above leading entries.

$$A = \begin{pmatrix} 2 & -4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & -6 \end{pmatrix} \; B = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 6 & -3 & -6 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 6 & -3 \\ 2 & -2 & 6 \\ 2 & 0 & 1 \end{pmatrix} \; D = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 3 & 0 & 3 & 2 \\ 5 & -2 & -1 & 3 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -2 & 1 & 1 \\ 2 & -2 & 1 & 2 \end{pmatrix} \; F = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & -2 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ -3 & 1 & 2 & -2 & -5 \\ -2 & -1 & 4 & 8 & 0 \\ 3 & 1 & -3 & -7 & 0 \end{pmatrix}$$

Solution

1. **1.** *A, B, C, D* are row echelon matrices.
2. **2.** *A, B, E* are reduced row echelon matrices.
3. **3.** $A=\begin{pmatrix}1&0&1\\2&1&-1\\-3&2&-2\end{pmatrix}\xrightarrow[R_1(3)+R_3]{R_1(2)+R_2}\begin{pmatrix}1&0&1\\0&1&-3\\0&2&1\end{pmatrix}.$
4. **4.** $A=\begin{pmatrix}1&0&1\\0&1&-3\\0&3&1\end{pmatrix}\xrightarrow{R_2(-3)(R_3)}\begin{pmatrix}1&0&1\\0&1&-3\\0&0&10\end{pmatrix}.$
5. **5.** $A=\begin{pmatrix}1&0&1\\2&1&-1\\-3&2&-2\end{pmatrix}\xrightarrow[R_1(3)+R_3]{R_1(-2)R_2}\begin{pmatrix}1&0&1\\0&1&-3\\0&2&1\end{pmatrix}\xrightarrow{R_2(-2)+R_3}\begin{pmatrix}1&0&1\\0&1&-3\\0&0&7\end{pmatrix}.$

$B=\begin{pmatrix}0&1&-2\\1&-1&1\\-2&1&0\end{pmatrix}\xrightarrow{R_{1,2}}\begin{pmatrix}1&-1&1\\0&1&-2\\-2&1&0\end{pmatrix}\xrightarrow{R_1(2)+R_2}\begin{pmatrix}1&-1&1\\0&1&-2\\0&-1&2\end{pmatrix}\xrightarrow{R_2(1)+R_3}\begin{pmatrix}1&-1&1\\0&1&-2\\0&0&0\end{pmatrix}.$

$$C=\begin{pmatrix}2&1&0&-1\\3&0&3&2\\5&-2&-1&3\end{pmatrix}\xrightarrow{R_2(-1)+R_1}\begin{pmatrix}-1&1&-3&-3\\3&0&3&6\\5&-2&9&12\end{pmatrix}$$

$$\xrightarrow[R_1(5)+R_3]{R_1(3)+R_2}\begin{pmatrix}-1&1&-3&-3\\0&3&-6&-3\\0&3&-6&-3\end{pmatrix}\xrightarrow{R_2(-1)+R_3}$$

$$\begin{pmatrix}-1&1&-3&-3\\0&3&-6&-3\\0&0&0&0\end{pmatrix}.$$

$$E=\begin{pmatrix}1&0&0&-1\\1&-2&1&1\\2&-2&1&2\end{pmatrix}\xrightarrow[R_1(-2)+R_3]{R_1(-1)+R_2}\begin{pmatrix}1&0&0&-1\\0&-2&1&2\\0&-2&1&4\end{pmatrix}\xrightarrow{R_2(-1)}$$

$$\begin{pmatrix}1&0&0&-1\\0&2&-1&-2\\0&-2&1&4\end{pmatrix}\xrightarrow{R_2(1)+R_3}\begin{pmatrix}1&0&0&-1\\0&2&-1&-2\\0&0&0&2\end{pmatrix}\xrightarrow{R_3(\frac{1}{2})}$$

$$\begin{pmatrix}1&0&0&-1\\0&2&-1&-2\\0&0&0&1\end{pmatrix}\xrightarrow[R_3(1)+R_1]{R_3(2)+R_2}\begin{pmatrix}1&0&0&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}\xrightarrow{R_2(\frac{1}{2})}$$

$$\begin{pmatrix}1&0&0&0\\0&1&-\frac{1}{2}&0\\0&0&0&1\end{pmatrix}.$$

$$A=\begin{pmatrix}2&-4&2\\0&1&-3\\0&0&-6\end{pmatrix}\xrightarrow[R_3\left(-\frac{1}{6}\right)]{R_1\left(\frac{1}{2}\right)}\begin{pmatrix}1&-2&1\\0&1&-3\\0&0&1\end{pmatrix}\xrightarrow[R_3(-1)+R_1]{R_3(3)+R_2}$$

6. **6.**

$$\begin{pmatrix}1&-2&0\\0&1&0\\0&0&1\end{pmatrix}\xrightarrow{R_2(2)+R_1}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

$$B=\begin{pmatrix}1&1&0&1&0\\0&3&6&-3&-6\\0&0&0&-2&2\\0&0&0&0&2\end{pmatrix}\xrightarrow[R_4\left(\frac{1}{2}\right)]{R_2\left(\frac{1}{3}\right),R_3\left(-\frac{1}{2}\right)}\begin{pmatrix}1&1&0&1&0\\0&1&2&-1&-2\\0&0&0&1&-1\\0&0&0&0&1\end{pmatrix}$$

$$\xrightarrow[R_4(2)+R_2]{R_4(1)+R_3}\begin{pmatrix}1&1&0&1&0\\0&1&2&-1&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}\xrightarrow[R_3(-1)+R_1]{R_3(1)+R_2}\begin{pmatrix}1&1&0&0&0\\0&1&2&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

$$\xrightarrow{R_2(-1)+R_1}\begin{pmatrix}1&0&-2&0&0\\0&1&2&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

$$C = \begin{pmatrix} 0 & 6 & -3 \\ 2 & -2 & 6 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow[R_2\left(\frac{1}{2}\right)]{R_1\left(\frac{1}{3}\right)} \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1(-2)+R_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -5 \end{pmatrix} \xrightarrow{R_2(-1)+R_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{pmatrix} \xrightarrow{R_3\left(-\frac{1}{4}\right)}$$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3(-3)+R_1]{R_3(1)+R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2\left(\frac{1}{2}\right)} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2(1)+R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$D=\begin{pmatrix}2&1&0&-1\\3&0&3&2\\5&-2&-1&3\end{pmatrix}\xrightarrow{R_2(-1)+R_1}\begin{pmatrix}-1&1&-3&-3\\3&0&3&6\\5&-2&9&12\end{pmatrix}$$

$$\xrightarrow[R_1(5)+R_3]{R_1(3)+R_2}\begin{pmatrix}-1&1&-3&-3\\0&3&-6&-3\\0&3&-6&-3\end{pmatrix}\xrightarrow{R_2(-1)+R_3}$$

$$\begin{pmatrix}-1&1&-3&-3\\0&3&-6&-3\\0&0&0&0\end{pmatrix}\xrightarrow[R_2\left(\frac{1}{3}\right)]{R_1(-1)}\begin{pmatrix}1&-1&3&3\\0&1&-2&-1\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{R_2(1)+R_1}\begin{pmatrix}1&0&1&2\\0&1&-2&-1\\0&0&0&0\end{pmatrix}.$$

$$E=\begin{pmatrix}1&0&0&-1\\1&-2&1&1\\2&-2&1&2\end{pmatrix}\begin{matrix}\\ \xrightarrow[R_1(-2)+R_3]{R_1(-1)+R_2}\\ \\ \end{matrix}\begin{pmatrix}1&0&0&-1\\0&-2&1&2\\0&-2&1&4\end{pmatrix}\xrightarrow{R_2(-1)}$$

$$\begin{pmatrix}1&0&0&-1\\0&2&-1&-2\\0&-2&1&4\end{pmatrix}\xrightarrow{R_2(1)+R_3}\begin{pmatrix}1&0&0&-1\\0&2&-1&-2\\0&0&0&2\end{pmatrix}\xrightarrow{R_3(\frac{1}{2})}$$

$$\begin{pmatrix}1&0&0&-1\\0&2&-1&-2\\0&0&0&1\end{pmatrix}\xrightarrow[R_3(1)+R_1]{R_3(2)+R_2}\begin{pmatrix}1&0&0&0\\0&2&-1&0\\0&0&0&1\end{pmatrix}\xrightarrow{R_2(\frac{1}{2})}$$

$$\begin{pmatrix}1&0&0&0\\0&1&-\frac{1}{2}&0\\0&0&0&1\end{pmatrix}.$$

$$F = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & -2 \\ 2 & 2 & 1 & 1 \end{pmatrix} \xrightarrow[R_1(-2)+R_3]{R_1(-1)+R_2} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & -3 & -1 \end{pmatrix} \xrightarrow{R_2(-1)}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & -1 \end{pmatrix} \xrightarrow{R_2(3)+R_3} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 8 \end{pmatrix} \xrightarrow{R_3(\frac{1}{8})}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3(-1)+R_1]{R_3(-3)+R_2} \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2(-2)+R_1}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last matrix is the reduced row echelon matrix of F.

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ -3 & 1 & 2 & -2 & -5 \\ -2 & -1 & 4 & 8 & 0 \\ 3 & 1 & -3 & -7 & 0 \end{pmatrix} \xrightarrow[R_1(-3)+R_4]{\substack{R_1(3)+R_2 \\ R_1(2)+R_3}} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 5 & 1 & -2 \\ 0 & -1 & 6 & 10 & 2 \\ 0 & 1 & -6 & -10 & -3 \end{pmatrix}$$

$$\xrightarrow[R_2(-1)+R_4]{R_2(1)+R_3} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 5 & 1 & -2 \\ 0 & 0 & 11 & 11 & 0 \\ 0 & 0 & -11 & -11 & -1 \end{pmatrix} \xrightarrow{R_3(\frac{1}{11})}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 5 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -11 & -11 & -1 \end{pmatrix} \xrightarrow{R_3(11)+R_4} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 5 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xrightarrow{R_4(-1)} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 5 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_4(-1)+R_1]{R_4(2)+R_2} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_3(-1)+R_1]{R_3(-5)+R_2} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The last matrix is the reduced row echelon matrix of G.

Section 2.5

1. For each of the following matrices, find its rank, nullity, and pivot columns. Clearly mark every leading entry *a* by using the symbol ⓐ and use the symbol ↓ to show the direction of eliminating nonzero entries below leading entries.

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & -1 \\ 2 & -1 & 3 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 4 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad E = \begin{pmatrix} 1 & -2 & -1 & 0 \\ -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & -1 & 5 \end{pmatrix} \quad G = \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 3 & 2 & 7 \\ 3 & 0 & 1 & 2 \\ 5 & -1 & 1 & 1 \end{pmatrix}$$

Solution

Because *A* is a row echelon matrix, $r(A)=2$ and null $(A)=4-2=2.$ Because columns 1 and 2 of *A* contain the leading entries, the column vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\-1\\0\end{pmatrix}$ are the pivot column vectors of *A*.

$$B=\begin{pmatrix}0 & 2 & -1\\ 2 & -1 & 3\end{pmatrix}\xrightarrow{R_{1,2}}\begin{pmatrix}2 & -1 & 3\\ 0 & 2 & -1\end{pmatrix}:=B^*.$$

$r(B)=2$ and null $(B)=3-2=1.$ Because the columns 1 and 2 of *B*∗ contain the leading entries, the columns 1 and 2 of $B:\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}$ are the pivot column of *B*.

$$C=\begin{pmatrix}1 & 4 & 2 & 1\\ 0 & 2 & 1 & 1\\ 1 & 0 & 1 & 4\\ 2 & 0 & 1 & -3\end{pmatrix}\xrightarrow[R_1(-2)+R_4]{R_1(-1)+R_3}\begin{pmatrix}1 & 4 & 2 & 1\\ 0 & 2 & 1 & 1\\ 0 & -4 & -1 & 3\\ 0 & -8 & -3 & -5\end{pmatrix}$$

$$\xrightarrow[R_2(4)+R_4]{R_2(2)+R_3}\begin{pmatrix}1 & 4 & 2 & 1\\ 0 & 2 & 1 & 1\\ 0 & 0 & 1 & 5\\ 0 & 0 & 1 & -1\end{pmatrix}\xrightarrow{R_3(-1)+R_4}\begin{pmatrix}1 & 4 & 2 & 1\\ 0 & 2 & 1 & 1\\ 0 & 0 & 1 & 5\\ 0 & 0 & 0 & -6\end{pmatrix}=C^*.$$

$r(C)=4$ and null $(C)=4-4=0.$ Because the columns 1, 2, 3, 4 of *C*∗ contain the leading entries, the columns 1, 2, 3, 4 of *C* are the pivot columns of *C*.

$$D = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow[R_1(2)+R_4]{R_1(1)+R_2} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & -1 & 0 \end{pmatrix} \xrightarrow[R_2(-1)+R_4]{R_2(3)+R_3}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} := D^*.$$

$r(D) = 3$ and null $(D) = 3 - 3 = 0.$ Because the columns 1, 2, 3 of $D*$ contain the leading entries, the columns 1, 2, 3 of *D* are the pivot columns of *D*.

$$E = \begin{pmatrix} 1 & -2 & -1 & 0 \\ -2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1(2)+R_2} \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = E^*.$$

$r(E) = 3$ and null $(E) = 4 - 3 = 1.$ Because the columns 1, 2, 4 of $E*$ contain $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are the pivot columns of *E*.

$$F = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 1 & 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & -1 & 5 \end{pmatrix} \xrightarrow{R_1(-1)R_3} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & -1 & -1 & 5 \end{pmatrix}$$

$$\xrightarrow{R_2(-1)+R_3} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 5 \end{pmatrix} \xrightarrow{R_{3,4}} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & -1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = F^*.$$

$r(F) = 3$ ansd null $(3) = 5 - 3 = 2.$ Because the columns 1, 2, 3 of F^* contain the leading entries, the columns 1, 2, 3 of F: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ are the pivot columns of F.

$$G = \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 3 & 2 & 7 \\ 3 & 0 & 1 & 2 \\ 5 & -1 & 1 & 7 \end{pmatrix} \xrightarrow{R_3(-1)+R_1} \begin{pmatrix} -1 & -1 & -1 & -3 \\ 0 & 3 & 2 & 7 \\ 3 & 0 & 1 & 2 \\ 5 & -1 & 1 & 1 \end{pmatrix} \xrightarrow[R_1(5)+R_4]{R_1(3)+R_3}$$

$$\begin{pmatrix} -1 & -1 & -1 & -3 \\ 0 & 3 & 2 & 7 \\ 0 & -3 & -2 & -7 \\ 0 & -6 & -4 & -14 \end{pmatrix} \xrightarrow[R_2(2)+R_4]{R_2(1)+R_3} \begin{pmatrix} -1 & -1 & -1 & -3 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = G^*.$$

$r(G) = 2$ and null $(G) = 4 - 2 = 2.$ Because the columns 1, 2 of G^* contain the leading entries, the columns 1, 2, 4 of G: $\begin{pmatrix} 2 \\ 0 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \\ -1 \end{pmatrix}$ are the pivot columns of G.

Section 2.6

1. Determine which of the following matrices are elementary matrices.
   1. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
   2. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
   3. $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$
   4. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
   5. $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$
   6. $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$
   7. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
   8. $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$
10. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
11. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
12. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

2. Use row operations to change the above matrices (1), (2), (3), (6), (7), (8), (10), and (12) to an identity matrix.
3. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Verify the following assertions.
    1. $E_2(5)E_2\left(\frac{1}{5}\right) = I$;
    2. $E_{1(2)+2}E_{1\left(\frac{1}{2}\right)+2} = I$;
    3. $E_{1,\,2}E_{1,\,2} = I$.

4. Find a matrix *E* such that $B = EA$, where

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{pmatrix}.$$

Solution

1. **1.**

   1. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ is an elementary matrix because

   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1(2)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

   2. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an elementary matrix because $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
   3. $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ is an elementary matrix because

   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2(5)+R_1} \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}.$$

   4. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is not an elementary matrix because we need to use two row operations to change the 2×2 identity matrix *I* to the matrix:

   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1(2)} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2(2)} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

   (6), (7), (8), (10), (12) are elementary matrices, but (5), (9), and (11) are not elementary matrices.

2. **2.**

1. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1\left(\frac{1}{2}\right)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
2. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
3. $\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2(-5)+R_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

6. $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \xrightarrow{R_1(2)+R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The row operations used in (7), (8), (10), (12) are $R_{1,3}, R_1(1)+R_2, R_{\frac{1}{5}}, R_{2,4}$, respectively.

3. **3.** Because

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2(5)} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} = E_2(5), \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2\left(\frac{1}{5}\right)} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} = E_2\left(\frac{1}{5}\right),$$

$$E_2(5)E_2\left(\tfrac{1}{5}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Because

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1(2)+R_2} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = E_{1(2)+2}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_1(-2)+R_2} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = E_{1\left(\frac{1}{2}\right)+2},$$

we have

$$E_{1(2)+2}E_{1(\frac{1}{2})+2} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Because

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_{1,2},$$

$$E_{1,2}E_{1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

4. **4.**

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_1(-2)+R_3}$$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -5 \end{pmatrix} \xrightarrow{R_2(1-)+R_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{pmatrix} = B.$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_1.$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1(-2)+R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = E_2.$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2(-1)+R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = E_3.$$

Let $E = E_3E_2E_1$. Then

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\right] \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -2 & 1 \end{pmatrix}$$

$$EA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{pmatrix} = B.$$

Section 2.7

1. Show that *B* is an inverse of *A* if

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

2. Use **Definition 2.7.1** to show that $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ is not invertible.
3. For each of the following matrices, determine whether it is invertible. If so, find its inverse.

$$A = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & -4 \\ 2 & 3 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 4 \\ -1 & -2 \end{pmatrix}$$

4. Let $A = \begin{pmatrix} 1 & x^2 \\ 1 & 1 \end{pmatrix}$. Find all $x \in \mathbb{R}$ such that *A* is invertible.
5. Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$. Find $A^{-1},\ B^{-1}$, and $(AB)^{-1}$ and verify $(AB)^{-1} = B^{-1}A^{-1}$.
6. For each of the following matrices, use its rank to determine whether it is invertible.

$$A = \begin{pmatrix} 0 & 2 & -5 \\ 1 & 5 & -5 \\ 1 & 0 & 8 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & -1 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ -2 & -1 & -4 \end{pmatrix}$$

7. For each of the following matrices, use row operations to determine if it is invertible. If so, find its inverse.

$$A = \begin{pmatrix} -1 & 2 \\ 4 & 7 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -3 \\ -4 & 5 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & -5 \\ 3 & 2 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 0 & -8 & -9 \\ 2 & 4 & -1 \\ 1 & -2 & -5 \end{pmatrix}$$

8. Let $A=\begin{pmatrix}-1 & 2\\ 1 & 3\end{pmatrix}$. Find $A^3,\ A^{-1},\ \left(A^{-1}\right)^3$, and verify $\left(A^3\right)^{-1}=\left(A^{-1}\right)^3$.

9. Let $A=\begin{pmatrix}-4 & 1\\ 3 & 1\end{pmatrix}$. Find $A^{-1},\ \left(A^T\right)^{-1}$ and verify that $\left(A^T\right)^{-1}=\left(A^{-1}\right)^T$.

10. Let $A=\begin{pmatrix}1 & 3\\ 3 & 2\end{pmatrix}$. Find A^{-1}. Is A^{-1} symmetric?

Solution

1. **1.** Because

$$AB = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2,$$

   by Definition 2.7.1, *B* is the inverse of *A*.

2. **2.** Let $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ be a 2×2 matrix. Because

   $$\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} - b_{21} & b_{12} - b_{22} \\ -2b_{11} + 2b_{21} & -2b_{12} + 2b_{22} \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

   by Definition 2.7.1, $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ is not invertible.

3. **3.** Because *A* and *B* are diagonal matrices, by Example (2.7.2),

$$A^{-1} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

   Because $|C| = \begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix} \neq 17 \equiv 0,$ so *C* is invertible and

$$C^{-1} = \frac{1}{17} \begin{pmatrix} 3 & 4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{17} & \frac{4}{17} \\ -\frac{2}{17} & \frac{3}{17} \end{pmatrix}.$$

$|D| = \begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} = 0,$ so D is not invertible.

4. **4.** Because $|A| = \begin{vmatrix} 1 & x^2 \\ 1 & 1 \end{vmatrix} = 1 - x^2 = 0.$ Then $x = -1$ or $x = 1.$ Hence when $x \neq 1$ and $x \neq 1, |A| = 1 - x^2 \neq 0$ and A is invertible.

5. **5.** Because $|A| = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}.$

Because $|B| = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2, B^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{pmatrix}.$

$$B^{-1}A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{pmatrix}.$$

Because $AB = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 6 \\ 9 & 8 \end{pmatrix}, |AB| = \begin{vmatrix} 7 & 6 \\ 9 & 8 \end{vmatrix} = 2,$

$$(AB)^{-1} = \frac{1}{2}\begin{pmatrix} 8 & -6 \\ -9 & 7 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{pmatrix}.$$

Hence, $(AB)^{-1} = B^{-1}A^{-1}.$

6. **6.** Because

$$A = \begin{pmatrix} 0 & 2 & -5 \\ 1 & 5 & -5 \\ 1 & 0 & 8 \end{pmatrix} \xrightarrow{R_{1,3}} \begin{pmatrix} 1 & 0 & 8 \\ 1 & 5 & -5 \\ 0 & 2 & -5 \end{pmatrix} \xrightarrow{R_1(-1)+R_2} \begin{pmatrix} 1 & 0 & 8 \\ 0 & 5 & -13 \\ 0 & 2 & -5 \end{pmatrix}$$

$$\xrightarrow{R_3(-2)+R_2} \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & -3 \\ 0 & 2 & -5 \end{pmatrix} \xrightarrow{R_2(2)+R_3} \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & -3 \\ 0 & 0 & -11 \end{pmatrix},$$

$r(A) = 3$ and A is invertible.

$$B^T = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \xrightarrow{R_2(-2)+R_3} \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3(\frac{2}{7})+R_4} \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & \frac{4}{7} \end{pmatrix}.$$

This implies that $r(B) = r(B^T) = 4$ and *B* is invertible. Because

$$C = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ -2 & -1 & -4 \end{pmatrix} \xrightarrow{R_1(1)+R_3} \begin{pmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 1 & -2 & 4 \\ 2 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1(-2)+R_2} \begin{pmatrix} 1 & -2 & 4 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{pmatrix},$$

$r(C) = 2$ and *C* is not invertible.

7. **7.**

$$(A|I) = \left(\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right) \xrightarrow{R_1(4)+R_2} \left(\begin{array}{cc|cc} -1 & 2 & 1 & 0 \\ 0 & 15 & 4 & 1 \end{array}\right)$$

$$\xrightarrow[R_2\left(\frac{1}{15}\right)]{R_1(-1)} \left(\begin{array}{cc|cc} 1 & -2 & -1 & 0 \\ 0 & 1 & \frac{4}{15} & \frac{1}{15} \end{array}\right) \xrightarrow{R_2(2)+R_1} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{7}{15} & \frac{2}{15} \\ 0 & 1 & \frac{4}{15} & \frac{1}{15} \end{array}\right).$$

Hence, $A^{-1} = \begin{pmatrix} -\frac{7}{15} & \frac{2}{15} \\ \frac{4}{15} & \frac{1}{15} \end{pmatrix}.$

$$(B|I)=\left(\begin{array}{cc|cc}1&-3&1&0\\-4&5&0&1\end{array}\right)\xrightarrow{R_1(4)R_2}\left(\begin{array}{cc|cc}1&-3&1&0\\0&-7&4&1\end{array}\right)$$

$$\xrightarrow{R_2(-\frac{1}{7})}\left(\begin{array}{cc|cc}1&-3&1&0\\0&1&-\frac{4}{7}&-\frac{1}{7}\end{array}\right)\xrightarrow{R_2(3)+R_1}\left(\begin{array}{cc|cc}1&0&-\frac{5}{7}&-\frac{3}{7}\\0&1&-\frac{4}{7}&-\frac{1}{7}\end{array}\right).$$

Hence, $B^{-1}=\begin{pmatrix}-\frac{5}{7}&-\frac{3}{7}\\-\frac{4}{7}&-\frac{1}{7}\end{pmatrix}$.

$$(C|I)=\left(\begin{array}{cc|cc}1&-2&1&0\\-2&4&0&1\end{array}\right)\xrightarrow{R_1(2)+R_2}\left(\begin{array}{cc|cc}1&-2&1&0\\0&0&2&1\end{array}\right).$$

Because $\begin{pmatrix}1&-2\\0&0\end{pmatrix}$ contains a row of zeros, the matrix *C* is not invertible.

$$(D|I)=\left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&-1&0&1&0\\2&0&-1&0&0&1\end{array}\right)\xrightarrow{R_1(-2)+R_3}\left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&-1&0&1&0\\0&2&-3&-2&0&1\end{array}\right)$$

$$\xrightarrow{R_2(-2)+R_3}\left(\begin{array}{ccc|ccc}1&-1&1&1&0&0\\0&1&-1&0&1&0\\0&0&-1&-2&-2&1\end{array}\right)\xrightarrow[R_3(-1)+R_2]{R_3(1)+R_1}$$

$$\left(\begin{array}{ccc|ccc}1&-1&0&-1&-2&1\\0&1&0&2&3&-1\\0&0&-1&-2&-2&1\end{array}\right)\xrightarrow[R_3(-1)]{R_2(1)+R_1}\left(\begin{array}{ccc|ccc}1&0&0&1&1&0\\0&1&0&2&3&-1\\0&0&1&2&2&-1\end{array}\right).$$

Hence, $D^{-1}=\begin{pmatrix}1&1&0\\2&3&-1\\2&2&-1\end{pmatrix}$

$$(E|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & -5 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_1(-2)+R_2]{R_1(3)+R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & 9 & -2 & 1 & 0 \\ 0 & -1 & -5 & -3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_{2,3}} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -5 & -3 & 0 & 1 \\ 0 & -2 & 9 & -2 & 1 & 0 \end{array}\right) \xrightarrow{R_2(-1)} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -1 \\ 0 & -2 & 9 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R_2(2)+R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & -1 \\ 0 & 0 & 1 & 4 & 1 & -2 \end{array}\right) \xrightarrow[R_3(-5)+R_2]{R_3(-2)+R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -7 & -2 & 4 \\ 0 & 1 & 0 & -17 & -5 & 9 \\ 0 & 0 & 1 & 4 & 1 & -2 \end{array}\right) \xrightarrow{R_2(-1)+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & 3 & -5 \\ 0 & 1 & 0 & -17 & -5 & 9 \\ 0 & 0 & 1 & 4 & 1 & -2 \end{array}\right).$$

Hence, $E^{-1} = \begin{pmatrix} 10 & 3 & -5 \\ -17 & -5 & 9 \\ 4 & 1 & -2 \end{pmatrix}$

$$(F|I) = \left(\begin{array}{ccc|ccc} 0 & -8 & -9 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ 1 & -2 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_{1,3}} \left(\begin{array}{ccc|ccc} 1 & -2 & -5 & 0 & 0 & 1 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ 0 & -8 & -9 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1(-2)+R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & -5 & 0 & 0 & 1 \\ 0 & 8 & 9 & 0 & 1 & -2 \\ 0 & -8 & -9 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_2(1)+R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -5 & 0 & 0 & 1 \\ 0 & 8 & 9 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right).$$

Because the left of the last matrix contains a row of zeros, F is not invertible.

8. **8.**

$$A^2 = \begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 11 \end{pmatrix}$$

$$A^3 = A^2 \cdot A = \begin{pmatrix} 3 & 4 \\ 2 & 11 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 18 \\ 9 & 37 \end{pmatrix}$$

Because $|A| = \begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix} = -5,$

$$A^{-1} = \frac{1}{-5}\begin{pmatrix} 3 & -2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}.$$

$$\left(A^{-1}\right)^2 = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{11}{25} & -\frac{4}{25} \\ -\frac{2}{25} & \frac{3}{25} \end{pmatrix}.$$

$$\left(A^{-1}\right)^3 = \left(A^{-1}\right)^2 A^{-1} = \begin{pmatrix} \frac{11}{25} & -\frac{4}{25} \\ -\frac{2}{25} & \frac{3}{25} \end{pmatrix}\begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} -\frac{37}{125} & \frac{18}{125} \\ \frac{9}{125} & -\frac{1}{125} \end{pmatrix}.$$

$|A^3| = \begin{vmatrix} 1 & 18 \\ 9 & 37 \end{vmatrix} = -125$ and

$$\left(A^3\right)^{-1} = -\frac{1}{125}\begin{pmatrix} 37 & -18 \\ -9 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{37}{125} & \frac{18}{125} \\ \frac{9}{125} & -\frac{1}{125} \end{pmatrix}.$$

Hence, $\left(A^3\right)^{-1} = \left(A^{-1}\right)^3$.

9. **9.** $|A| = \begin{vmatrix} -4 & 1 \\ 3 & 1 \end{vmatrix} = -7; A^{-1} = -\frac{1}{7}\begin{pmatrix} 1 & -1 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{4}{7} \end{pmatrix}$

$A^T = \begin{pmatrix} -4 & 3 \\ 1 & 1 \end{pmatrix}, (A^T)^{-1} = -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -1 & -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{4}{7} \end{pmatrix}$ and

$(A^{-1})^T = \begin{pmatrix} -\frac{1}{7} & \frac{3}{7} \\ \frac{1}{7} & \frac{4}{7} \end{pmatrix}$. Hence, $(A^T)^{-1} = (A^{-1})^T$.

10. **10.** $A^{-1} = \frac{1}{|A|}\begin{pmatrix} 2 & -3 \\ -3 & 1 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} 2 & -3 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{7} & \frac{3}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{pmatrix}, A^{-1}$ is symmetric.

A.3 Determinants

Section 3.1

1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a,\ b,\ c,\ d \in \mathbb{R}$. Show that

$$\left|\lambda I - A\right| = \lambda^2 - \operatorname{tr}(A)\lambda + \left|A\right|$$

for each $\lambda \in \mathbb{R}$.

2. Let $A_1 = \begin{pmatrix} 2 & 4 \\ 3 & -2 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$, $A_3 = \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix}$. For each $i = 1,\ 2,\ 3,$ find all $\lambda \in \mathbb{R}$ such that $|\lambda I - A_i| = 0.$
3. Use **(3.1.3)** and **(3.1.4)** to calculate each of the following determinants.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ 3 & -1 & 4 \end{vmatrix} \quad |B| = \begin{vmatrix} 2 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} \quad |C| = \begin{vmatrix} 0 & -1 & 3 \\ 4 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

4. Compute the determinant $|A|$ by using its expansion of cofactors corresponding to each of its rows, where

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ 3 & -1 & 4 \end{vmatrix}.$$

5. Compute each of the following determinants.

$$|A| = \begin{vmatrix} 2 & 3 & -4 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{vmatrix} \qquad |B| = \begin{vmatrix} -2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 8 \end{vmatrix}$$

Solution

1. **1.** Because

$$\lambda I - A = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix},$$

we have

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda - a)(\lambda - d) + bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - \mathrm{tr}(A)\lambda + |A|. \end{aligned}$$

2. **2.** Because $|A_1| = \begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix} = -16$ and $\mathrm{tr}(A_1) = 2 + (-2) = 0$,

$$|\lambda I - A_1| = \lambda^2 - \mathrm{tr}(A_1)\lambda + |A_1| = \lambda^2 - 16 = (\lambda - 4)(\lambda + 4) = 0.$$

Hence, $\lambda = 4$ or $\lambda = -4$.
Because $|A_2| = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 6$ and $\mathrm{tr}(A_1) = 5$,

$$|\lambda I - A_2| = \lambda^2 - \mathrm{tr}(A_2)\lambda + |A_2| = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0.$$

Hence, $\lambda = 2$ or $\lambda = 3$.
Because $|A_3| = \begin{vmatrix} -1 & 1 \\ -1 & -3 \end{vmatrix} = 4$ and $\mathrm{tr}(A_1) = -4$,

$$|\lambda I - A_3| = \lambda^2 - \mathrm{tr}(A_3)\lambda + |A_3| = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0.$$

Hence, $\lambda = -2$.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ 3 & -1 & 4 \end{vmatrix} \begin{matrix} 1 & 2 \\ -2 & 1 \\ 3 & -1 \end{matrix} = (4+12+6) - (9-2-16) = 31.$$

3. **3.**

$$|B| = \begin{vmatrix} 2 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 1 & 2 \end{vmatrix} \begin{matrix} 2 & -2 \\ 4 & 3 \\ 0 & 1 \end{matrix} = (12+0+16) - (0+2-16) = 42.$$

$$|C| = \begin{vmatrix} 0 & -1 & 3 \\ 4 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} 0 & -1 \\ 4 & 1 \\ 0 & 0 \end{matrix} = (0+0+0) - (0+0-4) = 4.$$

4. **4.**

1.
$$M_{11} = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 6.\ A_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = M_{11} = 6.$$
$$M_{12} = \begin{vmatrix} -2 & 2 \\ 3 & 4 \end{vmatrix} = -14.\ A_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -M_{12} = -(-14) = 14.$$
$$M_{13} = \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix} = -1.\ A_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13} = M_{13} = -1.$$

$$\begin{aligned} |A| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = A_{11} + 2A_{12} + 3A_{13} \\ &= 6 + 2(14) + 3(-1) = 31. \end{aligned}$$

2.

$$M_{21}=\begin{vmatrix}2&3\\-1&4\end{vmatrix}=8-(-3)=11.\ A_{11}=(-1)^{2+1}M_{21}=-11.$$

$$M_{22}=\begin{vmatrix}1&3\\3&4\end{vmatrix}=4-9=-5.\ A_{22}=(-1)^{2+2}M_{22}=M_{22}=-5.$$

$$M_{23}=\begin{vmatrix}1&2\\3&-1\end{vmatrix}=-1-6=-7.\ A_{23}=(-1)^{2+3}M_{13}=-M_{23}=7.$$

$$|A|=a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}=-2(-11)+(-5)+2(7)=31.$$

3.

$$M_{31}=\begin{vmatrix}2&3\\1&2\end{vmatrix}=4-3=1.\ A_{31}=(-1)^{3+1}M_{31}=M_{31}=1.$$

$$M_{32}=\begin{vmatrix}1&3\\-2&2\end{vmatrix}=2-(-6)=8.\ A_{32}=(-1)^{3+2}M_{32}=-M_{32}=-8.$$

$$M_{33}=\begin{vmatrix}1&2\\-2&1\end{vmatrix}=-1-(-4)=5.\ A_{33}=(-1)^{3+3}M_{33}=M_{33}=5.$$

$$|A|=a_{31}A_{31}+a_{32}A_{32}+a_{33}A_{33}=3(1)-(-8)+4(5)=31.$$

5. **5.** By Theorem 3.1.3, $|A|=2(6)(5)=60$; $|B|=(-2)(1)(8)=-16$.

Section 3.2

1. For each of the following matrices, find $M_{32},\ M_{24},\ A_{32},\ A_{24},\ M_{41},\ M_{43},\ A_{41},\ A_{43}$

$$A=\begin{pmatrix}1&-1&2&3\\2&2&0&3\\1&2&3&-1\\-3&0&1&6\end{pmatrix}\quad B=\begin{pmatrix}1&0&2&-1\\2&0&0&1\\1&0&1&-1\\-1&0&1&2\end{pmatrix}.$$

2. Compute each of the following determinants by using its expansion of cofactors of each row.

$$|A| = \begin{vmatrix} 1 & 0 & 2 & -1 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 3 \end{vmatrix} \qquad |B| = \begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 0 & 3 \\ 1 & 2 & 3 & -1 \\ -3 & 0 & 1 & 6 \end{vmatrix}.$$

3. Evaluate $|A|$ by using the expansion of cofactors of a suitable row, where

$$|A| = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{vmatrix}.$$

4. Evaluate each of the following determinants by inspection.

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{vmatrix} \quad |B| = \begin{vmatrix} 2 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 1 & 2 \end{vmatrix} \quad |C| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix}$$

$$|D| = \begin{vmatrix} 1 & -1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 5 & 6 & 3 & -4 & 6 \\ 2 & -2 & 6 & 10 & 0 \end{vmatrix} \quad |E| = \begin{vmatrix} 1 & -1 & 3 & 5 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 7 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

$$|F| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ -1 & 6 & -6 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} \quad |G| = \begin{vmatrix} -2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

Solution

1. **1.**

$$M_{32} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ -3 & 1 & 6 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 0 \\ -3 & 1 \end{matrix} = (0 - 18 + 6) - (0 + 3 + 24) = -39.$$

$$M_{32} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ -3 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & -1 \\ 1 & 2 \\ -3 & 0 \end{matrix} = (2 + 9 + 0) - (-12 + 0 - 1) = -24.$$

$A_{32} = (-1)^{3+2} M_{32} = -M_{32} = 39.$ $A_{24} = (-1)^{2+4} M_{24} = M_{24} = 24.$

$$M_{41} = \begin{vmatrix} -1 & 2 & 3 \\ 2 & 0 & 3 \\ 2 & 3 & -1 \end{vmatrix} \begin{matrix} -1 & 2 \\ 2 & 0 \\ 2 & 3 \end{matrix} = (0 + 12 + 18) - (0 - 9 - 4) = 43.$$

$$M_{43} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 2 & 3 \\ 1 & 2 & -1 \end{vmatrix} \begin{matrix} 1 & -1 \\ 2 & 2 \\ 1 & 2 \end{matrix} = (-2 - 3 + 12) - (6 + 6 + 2) = -7.$$

$A_{41} = (-1)^{4+1} M_{41} = -M_{41} = -43.$ $A_{43} = (-1)^{4+3} M_{43} = -M_{43} = 7.$

$$M_{32} = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 0 \\ -1 & 1 \end{matrix} = (0 - 2 - 2) - (0 + 1 + 8) = -13.$$

$A_{32} = (-1)^{3+2} M_{32} = -M_{32} = 13.$

$$M_{24} = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix} \begin{matrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{matrix} = (0 + 0 + 0) - (0 + 0 + 0) = 0.$$

$A_{24} = (-1)^{2+4} M_{24} = M_{24} = 0.$

$$M_{41}=\begin{vmatrix}0&2&-1\\0&0&1\\0&1&-1\end{vmatrix}\begin{matrix}0&2\\0&0\\0&1\end{matrix}=(0+0+0)-(0+0+0)=0.$$

$A_{41}=(-1)^{4+1}M_{41}=-M_{41}=0.$

$$M_{43}=\begin{vmatrix}1&0&-1\\2&0&1\\1&0&-1\end{vmatrix}\begin{matrix}1&0\\2&0\\1&0\end{matrix}=(0+0+0)-(0+0+0)=0.$$

$A_{43}=(-1)^{4+3}M_{43}=-M_{43}=0.$

2. **2.**

i. Use row 1 to compute $|A|$.

$|A|=a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}+a_{14}A_{14}=A_{11}+0+2A_{13}-A_{14}.$

$$M_{11}=\begin{vmatrix}0&0&1\\0&1&-1\\0&1&2\end{vmatrix}=0;\ A_{11}=(-1)^{1+1}M_{11}=M_{11}=0.$$

$$M_{13}=\begin{vmatrix}2&0&1\\1&0&-1\\-1&0&2\end{vmatrix}=0;\ A_{13}=(-1)^{1+3}M_{13}=M_{13}=0.$$

$$M_{14}=\begin{vmatrix}2&0&0\\1&0&1\\-1&0&1\end{vmatrix}=0;\ A_{14}=(-1)^{1+4}M_{14}=M_{14}=0.$$

Hence, $|A|=0+2(0)-(0)=0$.

ii. Use row 2 to compute $|A|$.

$|A|=a_{21}A_{21}+a_{22}A_{22}+a_{23}A_{23}+a_{24}A_{24}=2A_{21}+0A_{22}+0A_{23}+A_{24}.$

$$M_{21}=\begin{vmatrix}0&2&-1\\0&1&-1\\0&1&2\end{vmatrix}=0;\ A_{21}=(-1)^{2+1}M_{21}=0.$$

$$M_{24} = \begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 0;\ A_{24} = (-1)^{2+4} M_{24} = 0.$$

Hence, $|A| = 0 + 2(0) - (0) = 0$

iii. Use row 3 to compute $|A|$.

$$|A| = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} + a_{34}A_{34} = 2A_{31} + 0A_{32} + 0A_{33} + A_{34}.$$

$$M_{31} = \begin{vmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 0;\ A_{31} = (-1)^{3+1} M_{31} = 0.$$

$$M_{33} = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 0;\ A_{33} = (-1)^{3+3} M_{33} = 0.$$

$$M_{34} = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0;\ A_{34} = (-1)^{3+4} M_{34} = 0.$$

$|A| = 0 + 0 - 0 = 0.$

iv. Use row 4 to compute $|A|$.

$$|A| = -A_{41} + 0A_{42} + A_{43} + 2A_{44} = -A_{41} + A_{43} + 2A_{44}.$$

$$M_{41} = \begin{vmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 0;\ A_{41} = (-1)^{4+1} M_{41} = 0.$$

$$M_{43} = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 0;\ A_{43} = (-1)^{4+3} M_{43} = 0.$$

$$M_{44} = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0;\ A_{44} = (-1)^{4+4} M_{44} = 0.$$

Hence, $|A| = -0 + 0 + 2(0) = 0$.

i. Use row 1 to compute $|B|$.

$|B| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14} = A_{11} - A_{12} + 2A_{13} + 3A_{14}.$

$$M_{11} = \begin{vmatrix} 2 & 0 & 3 \\ 2 & 3 & -1 \\ 0 & 1 & 6 \end{vmatrix} \begin{matrix} 2 & 0 \\ 2 & 3 \\ 0 & 1 \end{matrix} = (36 + 0 + 6) - (0 - 2 + 0) = 44;$$

$A_{11} = (-1)^{1+1} M_{11} = M_{11} = 44.$

$$M_{12} = \begin{vmatrix} 2 & 0 & 3 \\ 1 & 3 & -1 \\ -3 & 1 & 6 \end{vmatrix} \begin{matrix} 2 & 0 \\ 1 & 3 \\ -3 & 1 \end{matrix} = (36 + 0 + 3) - (-27 - 2 + 0) = 68;$$

$A_{12} = (-1)^{1+2} M_{12} = -M_{12} = -68.$

$$M_{13} = \begin{vmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ -3 & 0 & 6 \end{vmatrix} \begin{matrix} 2 & 2 \\ 1 & 2 \\ -3 & 0 \end{matrix} = (24 + 6 + 0) - (-18 + 0 + 12) = 36;$$

$A_{13} = (-1)^{1+3} M_{13} = M_{13} = 36.$

$$M_{14} = \begin{vmatrix} 2 & 2 & 0 \\ 1 & 2 & 3 \\ -3 & 0 & 1 \end{vmatrix} \begin{matrix} 2 & 2 \\ 1 & 2 \\ -3 & 0 \end{matrix} = (4 - 18 + 0) - (0 + 0 + 2) = -16;$$

$A_{14} = (-1)^{1+4} M_{14} = -M_{14} = 16.$

Hence, $|B| = 44 - (-68) + 2(36) + 3(16) = 44 + 68 + 72 + 48 = 232.$.

ii. Use row 2 to compute $|B|$.

$$\begin{aligned} |B| &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + a_{24}A_{24} \\ &= 2A_{21} + 2A_{22} + 0A_{23} + 3A_{24} = 2A_{21} + 2A_{22} + 3A_{24}. \end{aligned}$$

$$M_{21} = \begin{vmatrix} -1 & 2 & 3 \\ 2 & 3 & -1 \\ 0 & 1 & 6 \end{vmatrix} \begin{matrix} -1 & 2 \\ 2 & 3 \\ 0 & 1 \end{matrix} = (-18 + 0 + 6) - (0 + 1 + 24) = -37;$$

$A_{21} = (-1)^{2+1} M_{21} = -(-37) = 37.$

$$M_{22} = \begin{array}{|ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 1 & 3 & -1 & 1 & 3 \\ -3 & 1 & 6 & -3 & 1 \end{array} = (18+6+3)-(-27-1-12)=43;$$

$A_{22}=(-1)^{2+2}M_{22}=43.$

$$M_{24} = \begin{array}{|ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 1 & 2 & 3 & 1 & 2 \\ -3 & 0 & 1 & -3 & 0 \end{array} = (2+9+0)-(-12+0-1)=24;$$

$A_{24}=(-1)^{2+4}M_{24}=24.$

Hence, $|B|=2(37)+2(43)+3(24)=74+86+72=232.$

iii. Use row 3 to compute $|B|$.

$|B|=a_{31}A_{31}+a_{32}A_{32}+a_{33}A_{33}+a_{34}A_{34}=A_{31}+2A_{32}+3A_{23}-A_{34}.$

$$M_{31} = \begin{array}{|ccc|cc} -1 & 2 & 3 & -1 & 2 \\ 2 & 0 & 3 & 2 & 0 \\ 0 & 1 & 6 & 0 & 1 \end{array} = (0+0+6)-(0-3+24)=-15;$$

$A_{31}=(-1)^{3+1}M_{31}=-15.$

$$M_{32} = \begin{array}{|ccc|cc} -1 & 2 & 3 & -1 & 2 \\ 2 & 0 & 3 & 2 & 0 \\ -3 & 1 & 6 & -3 & 1 \end{array} = (0-18+6)-(0+3+24)=-39;$$

$A_{32}=(-1)^{3+2}M_{32}=-M_{32}=-(-39)=39.$

$$M_{33} = \begin{array}{|ccc|cc} 1 & -1 & 3 & 1 & -1 \\ 2 & 2 & 3 & 2 & 2 \\ -3 & 0 & 6 & -3 & 0 \end{array} = (12+9+0)-(-18+0-12)=51;$$

$A_{33}=(-1)^{3+3}M_{33}=51.$

$$M_{34} = \begin{array}{|ccc|cc} 1 & -1 & 2 & 1 & -1 \\ 2 & 2 & 0 & 2 & 2 \\ -3 & 0 & 1 & -3 & 0 \end{array} = (2+0+0)-(-12+0-2)=16;$$

$A_{34}=(-1)^{3+4}M_{34}=-M_{34}=-16.$

$|B|=-15+2(39)+3(51)-(-16)=-15+78+153+16=232.$

iv. Use row 4 to compute $|B|$.

$$\begin{aligned}|B| &= a_{41}A_{41} + a_{42}A_{42} + a_{43}A_{43} + a_{44}A_{44}\\ &= -3A_{41} + 0A_{42} + A_{43} + 6A_{44} = -3A_{41} + A_{43} + 6A_{44}.\end{aligned}$$

$$M_{41} = \begin{vmatrix} -1 & 2 & 3 \\ 2 & 0 & 3 \\ 2 & 3 & -1 \end{vmatrix} \begin{matrix} -1 & 2 \\ 2 & 0 \\ 2 & 3 \end{matrix} = (0+12+18)-(0-9-4) = 43;$$

$A_{41} = (-1)^{4+1} = -M_{41} = -43.$

$$M_{43} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 2 & 3 \\ 1 & 2 & -1 \end{vmatrix} \begin{matrix} 1 & -1 \\ 2 & 2 \\ 1 & 2 \end{matrix} = (-2-3+12)-(6+6+2) = -7;$$

$A_{43} = (-1)^{4+3}M_{43} = -M_{43} = -(-7) = 7.$

$$M_{44} = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} \begin{matrix} 1 & -1 \\ 2 & 2 \\ 1 & 2 \end{matrix} = (6+0+8)-(4+0-6) = 16;$$

$A_{44} = (-1)^{4+4}M_{44} = 0.$

Hence, $|B| = -3(-43) + 7 + 6(16) = 129 + 7 + 96 = 232.$.

3. **3.** We use the first row to compute $|A|$ because it contains more zeros than the other rows.

$$\begin{aligned}|A| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + a_{14}A_{14}\\ &= 0A_{11} + 0A_{12} + 0A_{13} + A_{14} = A_{14}.\end{aligned}$$

$$M_{14} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{matrix} = (0+1+0)-(0+0+0) = 1;\ A_{14} = (-1)^{1+4}M_{14} = -M_{14} = -1.$$

Hence, $|A| = -1$.

4. **4.** $|A| = 1(2)(6) = 12$ because *A* is a diagonal matrix.
$|B| = 2(-2)(2) = -8$ because *B* is a lower triangular matrix.
$|C| = 1(1)(-1) = -1$ because *C* is an upper triangular matrix.
$|D| = 0$ because row 2 is a zero row.
$|E| = 1(3)(-1)(-4)(2) = 24.$
$|F| = 1(2)(-6)(-1) = 12.$
$|G| = (-2)(3)(-1)(2) = 12.$

Section 3.3

1. Evaluate each of the following determinants by inspection.

$$|A| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -2 & -4 & 6 \end{vmatrix} \quad |B| = \begin{vmatrix} 2 & -2 & 4 \\ 2 & -2 & 4 \\ 0 & 1 & 2 \end{vmatrix} \quad |C| = \begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 2 \\ 1 & -\frac{1}{2} & -1 \end{vmatrix}$$

2. Use row operations to evaluate each of the following determinants.

$$|A| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ 0 & 2 & 4 \end{vmatrix} \quad |B| = \begin{vmatrix} 0 & 2 & 3 \\ -2 & 1 & 2 \\ 3 & -1 & -1 \end{vmatrix} \quad |C| = \begin{vmatrix} 1 & 2 & -3 & -2 \\ 2 & 1 & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & -1 & 2 & 0 \end{vmatrix}$$

$$|D| = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 6 & -6 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} \quad |E| = \begin{vmatrix} -2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix}$$

$$|F| = \begin{vmatrix} 0 & -1 & 3 & 5 & 0 \\ 1 & -1 & 2 & 4 & -1 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 2 & -2 & 6 & 10 & 0 \end{vmatrix} \quad |G| = \begin{vmatrix} 1 & -1 & 3 & 5 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ -2 & 0 & -1 & 2 & 1 \\ 3 & 0 & 0 & -6 & 6 \\ 2 & 0 & 0 & 0 & 1 \end{vmatrix}$$

3. Evaluate each of the following determinants by using its transpose.

$$|A| = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 7 & 0 & 2 \\ 0 & 6 & 3 & 0 \\ 5 & 3 & 1 & 2 \end{vmatrix} \quad |B| = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 7 & 0 & 2 & 1 \\ 0 & 6 & 3 & 0 & 0 \\ 1 & 3 & 1 & 2 & -1 \\ 0 & 6 & 1 & 2 & 1 \end{vmatrix}.$$

4. Use **Theorem 3.3.7** to find the inverse of the following matrix

$$A = \begin{pmatrix} 0 & 2 & -5 \\ 1 & 5 & -5 \\ 1 & 0 & 8 \end{pmatrix}$$

Solution

1. **1.** $|A| = 0$ because row 1 is proportional to row 3. $|B| = 0$ because row 1 equals row 2. $|C| = 0$ because row 2 is proportional to row 3.
2. **2.**

$$|A| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ 0 & 2 & 4 \end{vmatrix} \xrightarrow{R_{1,2}} - \begin{vmatrix} 1 & -2 & 4 \\ 2 & 1 & 4 \\ 0 & 2 & 4 \end{vmatrix} \xrightarrow{R_1(-2)+R_2} - \begin{vmatrix} 1 & -2 & 4 \\ 0 & 5 & -4 \\ 0 & 2 & 4 \end{vmatrix}$$

$$\xrightarrow{R_3\left(\frac{1}{3}\right)} (-2)\begin{vmatrix} 1 & -2 & 4 \\ 0 & 5 & -4 \\ 0 & 1 & 2 \end{vmatrix} \xrightarrow{R_{2,3}} 2\begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 5 & -4 \end{vmatrix} \xrightarrow{R_2(-5)+R_3}$$

$$2\begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -14 \end{vmatrix} = 2(-14) = -28.$$

$$|B| = \begin{vmatrix} 0 & 2 & 3 \\ -2 & 1 & 2 \\ 3 & -1 & 1 \end{vmatrix} \xrightarrow{R_{1,2}} -\begin{vmatrix} -2 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} \xrightarrow{R_3(1)+R_1}$$

$$-\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} \xrightarrow{R_1(-3)+R_3} -\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & -1 & -4 \end{vmatrix} \xrightarrow{R_{2,3}}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 2 & 3 \end{vmatrix} \xrightarrow{R_2(2)+R_3} \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & -5 \end{vmatrix}$$

$$= (1)(-1)(-5) = 5.$$

$$|C| = \begin{vmatrix} 1 & 2 & -3 & -2 \\ 2 & 1 & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & -1 & 2 & 0 \end{vmatrix} = \text{doubt} = \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -3 & 8 & 4 \\ 0 & 1 & -5 & -6 \\ 0 & -3 & 5 & 2 \end{vmatrix}$$

$$R_{2,\,3}{-}{-} \quad - \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & -5 & -6 \\ 0 & -3 & 8 & 4 \\ 0 & -3 & 5 & 2 \end{vmatrix} = \text{doubt} = \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & -7 & -14 \\ 0 & 0 & -10 & -16 \end{vmatrix}$$

$$= \text{doubt} = -(-7)(2) \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -5 & -8 \end{vmatrix} R_3(5) + R_4{-}{-}$$

$$14 \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (14)(1)(-1)(1)(2) = -28.$$

$$|D| = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 6 & -6 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} = \text{doubt} = (-2)(6)\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} R_{1,\,2}\text{— —}$$

$$12\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} R_{2,\,4}\text{— —} \; -12\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix} R_1(-4) + R_2\text{— —}$$

$$-12\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} R_{2,\,3}\text{— —} \; 12\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$R_2(4) + R_3\text{— —} \; 12\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = (1)(1)(-2)(0) = 0.$$

$$|E| = \begin{vmatrix} -2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} R_1\left(-\frac{1}{2}\right){--}\,(-2)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} = \text{doubt} =$$

$$-2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} R_{2,\,3}{--}\,2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} R_2(-3) + R_3{--}$$

$$2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 12.$$

$$\left|F\right| = \begin{vmatrix} 0 & -1 & 3 & 5 & 0 \\ 1 & -1 & 2 & 4 & -1 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 2 & -2 & 6 & 10 & 0 \end{vmatrix} R_{1,\,2}{--}\,-\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 2 & -2 & 6 & 10 & 0 \end{vmatrix}$$

$$R_5\left(\frac{1}{2}\right) \quad -2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \end{vmatrix} = \text{doubt} =$$

$$\begin{vmatrix} 1 & 1 & -2 & 3 & 1 \\ 1 & -1 & 3 & 5 & 0 \end{vmatrix}$$

$$-2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 2 & -4 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{vmatrix} R_2(2) + R_4 \to -2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 2 & 9 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

$$= \text{doubt} = -2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 5 & -12 \\ 0 & 0 & 0 & -1 & -6 \end{vmatrix} R_{4,\,5} \to$$

$$2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & 5 & -12 \end{vmatrix} R_4(5) + R_5 \to 2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & 0 & -42 \end{vmatrix}$$

$$= 2(-1)(1)(-1)(-42) = -84.$$

3. **3.** $$\left|A\right| = \left|A^T\right| = \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 7 & 6 & 3 \\ 0 & 0 & 3 & 1 \\ 1 & 2 & 0 & 2 \end{vmatrix} R_1(-1)+R_{4}{-\,-} \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 7 & 6 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = -63.$$

$$|B| = \left|B^T\right| = \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 7 & 6 & 3 & 6 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 1 \end{vmatrix} R_{2,\,5}{-\,-} \; - \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 7 & 6 & 3 & 6 \end{vmatrix}$$

$$= \text{doubt} = \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 6 & 10 & -1 \end{vmatrix} R_3(-2)+R_{5}{-\,-}$$

$$- \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 & -3 \end{vmatrix} R_4(-2)+R_{5}{-\,-} \; - \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix} = 36$$

4. **4.** By (3.1.9), we have

$$A_{11} = \begin{vmatrix} 5 & -5 \\ 0 & 8 \end{vmatrix} = 40,\ A_{12} = -\begin{vmatrix} 1 & -5 \\ 1 & 8 \end{vmatrix} = -13,\ A_{13} = \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} = -5,$$

$$A_{21} = -\begin{vmatrix} 2 & -5 \\ 0 & 8 \end{vmatrix} = -16,\ A_{22} = \begin{vmatrix} 0 & -5 \\ 1 & 8 \end{vmatrix} = 5,\ A_{23} = -\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 2,$$

$$A_{31} = \begin{vmatrix} 2 & -5 \\ 5 & -5 \end{vmatrix} = 15,\ A_{32} = -\begin{vmatrix} 0 & -5 \\ 1 & -5 \end{vmatrix} = -5,\ A_{33} = \begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = -2.$$

This, together with (3.3.11), implies

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 40 & -16 & 15 \\ -13 & 5 & -5 \\ -5 & 2 & -2 \end{pmatrix}.$$

By computation, $|A| = -1$. It follows from Theorem 3.3.7 that

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = -\operatorname{adj}(A) = \begin{pmatrix} -40 & 16 & -15 \\ 13 & -5 & 5 \\ 5 & -2 & 2 \end{pmatrix}.$$

Section 3.4

1. For each of the following matrices, use its determinant to determine whether it is invertible.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & -1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 & 3 & 5 & 2 \\ 1 & -1 & 2 & 4 & -1 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 0 & -2 & 6 & 10 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ -2 & -1 & -4 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 & 3 \\ -2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & -1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 2 & 1 \\ 3 & 0 & 0 & -6 & 6 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 6 & -6 & 0 \\ 4 & 0 & 2 & -1 \end{pmatrix}$$

2. For each of the following matrices, find all real numbers *x* such that it is invertible.

$$A = \begin{pmatrix} x & 0 & 0 & 0 \\ 2 & x & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & x & 4 \\ 1 & -2 & 4 \\ -2 & -1 & x \end{pmatrix}$$

3. Let *A* be a 4×4 matrix. Assume that $|A| = 2$. Compute each of the following determinants.

1. $|-3A|$
2. $\left|A^{-1}\right|$
3. $\left|A^{T}\right|$
4. $\left|A^{3}\right|$
5. $\left|(2A^{-1})^{T}\right|$
6. $\left|(2(-A)^{T})^{-1}\right|$

4. Compute $|A| + |B|$ if $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 1 & 4 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$.

5. Compute $|A| + |B|$ if

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 4 & 3 \\ -2 & 2 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 1 & 4 & 3 \\ -2 & 2 & 0 & 1 \end{pmatrix}.$$

Solution

1. **1.** $|A| = 0$ because row 1 is proportional to row 3. $|B| = 0$ because row 1 equals row 2. $|C| = 0$ because row 2 is proportional to row 3.
2. **2.**

$$\left|A\right| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ 0 & 2 & 4 \end{vmatrix} \underline{\underline{R_{1,2}}} - \begin{vmatrix} 1 & -2 & 4 \\ 2 & 1 & 4 \\ 0 & 2 & 4 \end{vmatrix} \underline{\underline{R_1(-2)+R_2}} - \begin{vmatrix} 1 & -2 & 4 \\ 0 & 5 & -4 \\ 0 & 2 & 4 \end{vmatrix}$$

$$\underline{\underline{R_3\left(\frac{1}{3}\right)}} \quad (-2)\begin{vmatrix} 1 & -2 & 4 \\ 0 & 5 & -4 \\ 0 & 1 & 2 \end{vmatrix} \underline{\underline{R_{2,3}}} \quad 2\begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 5 & -4 \end{vmatrix} \underline{\underline{R_2(-5)+R_3}}$$

$$2\begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -14 \end{vmatrix} = 2(-14) = -28.$$

$$|B| = \begin{vmatrix} 0 & 2 & 3 \\ -2 & 1 & 2 \\ 3 & -1 & 1 \end{vmatrix} \xrightarrow{R_{1,2}} - \begin{vmatrix} -2 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} \xrightarrow{R_3(1)+R_1}$$

$$- \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix} \xrightarrow{R_1(-3)+R_3} - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & -1 & -4 \end{vmatrix} \xrightarrow{R_{2,3}}$$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 2 & 3 \end{vmatrix} \xrightarrow{R_2(2)+R_3} \begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -4 \\ 0 & 0 & -5 \end{vmatrix}$$

$$= (1)(-1)(-5) = 5.$$

$$|C| = \begin{vmatrix} 1 & 2 & -3 & -2 \\ 2 & 1 & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & -1 & 2 & 0 \end{vmatrix} = \text{doubt} = \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -3 & 8 & 4 \\ 0 & 1 & -5 & -6 \\ 0 & -3 & 5 & 2 \end{vmatrix}$$

$$R_{2,\,3}\text{— —} \quad - \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & -5 & -6 \\ 0 & -3 & 8 & 4 \\ 0 & -3 & 5 & 2 \end{vmatrix} = \text{doubt} = \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & -7 & -14 \\ 0 & 0 & -10 & -16 \end{vmatrix}$$

$$= \text{doubt} = -(-7)(2) \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -5 & -8 \end{vmatrix} R_3(5) + R_4\text{— —}$$

$$14 \begin{vmatrix} 1 & 2 & -3 & -2 \\ 0 & -1 & -5 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} = (14)(1)(-1)(1)(2) = -28.$$

$$|D| = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 6 & -6 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} = \text{doubt} = (-2)(6)\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} R_{1,\,2}\text{— —}$$

$$12\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} R_{2,\,4}\text{— —} \; -12\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix} R_1(-4) + R_2\text{— —}$$

$$-12\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & -4 & 2 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} R_{2,\,3}\text{— —} \; 12\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

$$R_2(4) + R_3\text{— —} \; 12\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = (1)(1)(-2)(0) = 0.$$

$$|E| = \begin{vmatrix} -2 & 0 & 0 & 0 \\ 3 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} \xrightarrow{R_1\left(-\frac{1}{2}\right)} (-2)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} = \text{doubt} =$$

$$-2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} \xrightarrow{R_{2,\,3}} 2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} \xrightarrow{R_2(-3)+R_3}$$

$$2\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 12.$$

$$\left|F\right| = \begin{vmatrix} 0 & -1 & 3 & 5 & 0 \\ 1 & -1 & 2 & 4 & -1 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 2 & -2 & 6 & 10 & 0 \end{vmatrix} \xrightarrow{R_{1,\,2}} -\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 2 & -2 & 6 & 10 & 0 \end{vmatrix}$$

$$\xrightarrow{R_5\left(\frac{1}{2}\right)} -2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \end{vmatrix} = \text{doubt} =$$

$$\begin{vmatrix} 1 & 1 & -2 & 3 & 1 \\ 1 & -1 & 3 & 5 & 0 \end{vmatrix}$$

$$-2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 2 & -4 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{vmatrix} R_2(2)+R_{4}\text{——} -2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 2 & 9 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{vmatrix}$$

$$= \text{doubt} = -2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & 5 & -12 \\ 0 & 0 & 0 & -1 & -6 \end{vmatrix} R_{4,\,5}\text{——}$$

$$2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & 5 & -12 \end{vmatrix} R_4(5)+R_{5}\text{——} 2\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 0 & -1 & -6 \\ 0 & 0 & 0 & 0 & -42 \end{vmatrix}$$

$$= 2(-1)(1)(-1)(-42) = -84.$$

3. **3.** $$\left|A\right| = \left|A^T\right| = \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 7 & 6 & 3 \\ 0 & 0 & 3 & 1 \\ 1 & 2 & 0 & 2 \end{vmatrix} \underline{\underline{R_1(-1)+R_4}} \begin{vmatrix} 1 & 2 & 0 & 5 \\ 0 & 7 & 6 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix} = -63.$$

$$|B| = \left|B^T\right| = \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 7 & 6 & 3 & 6 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 1 \end{vmatrix} \underline{\underline{R_{2,\,5}}} - \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 7 & 6 & 3 & 6 \end{vmatrix}$$

$$= \text{doubt} = \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 6 & 10 & -1 \end{vmatrix} \underline{\underline{R_3(-2)+R_5}}$$

$$- \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 & -3 \end{vmatrix} \underline{\underline{R_4(-2)+R_5}} - \begin{vmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix} = 36$$

4. **4.** By (3.1.9), we have

$$A_{11} = \begin{vmatrix} 5 & -5 \\ 0 & 8 \end{vmatrix} = 40,\ A_{12} = -\begin{vmatrix} 1 & -5 \\ 1 & 8 \end{vmatrix} = -13,\ A_{13} = \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} = -5,$$

$$A_{21} = -\begin{vmatrix} 2 & -5 \\ 0 & 8 \end{vmatrix} = -16,\ A_{22} = \begin{vmatrix} 0 & -5 \\ 1 & 8 \end{vmatrix} = 5,\ A_{23} = -\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} = 2,$$

$$A_{31} = \begin{vmatrix} 2 & -5 \\ 5 & -5 \end{vmatrix} = 15,\ A_{32} = -\begin{vmatrix} 0 & -5 \\ 1 & -5 \end{vmatrix} = -5,\ A_{33} = \begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = -2.$$

This, together with (3.3.11), implies

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 40 & -16 & 15 \\ -13 & 5 & -5 \\ -5 & 2 & -2 \end{pmatrix}.$$

By computation, $|A| = -1$. It follows from Theorem 3.3.7 that

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = -\operatorname{adj}(A) = \begin{pmatrix} -40 & 16 & -15 \\ 13 & -5 & 5 \\ 5 & -2 & 2 \end{pmatrix}.$$

Section 3.4

1. For each of the following matrices, use its determinant to determine whether it is invertible.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & -1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & -1 & 3 & 5 & 2 \\ 1 & -1 & 2 & 4 & -1 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 0 & -2 & 6 & 10 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ -2 & -1 & -4 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 & 3 \\ -2 & 1 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & -1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 2 & 1 \\ 3 & 0 & 0 & -6 & 6 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 6 & -6 & 0 \\ 4 & 0 & 2 & -1 \end{pmatrix}$$

2. For each of the following matrices, find all real numbers *x* such that it is invertible.

$$A = \begin{pmatrix} x & 0 & 0 & 0 \\ 2 & x & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & x & 4 \\ 1 & -2 & 4 \\ -2 & -1 & x \end{pmatrix}$$

3. Let *A* be a 4×4 matrix. Assume that $|A| = 2$. Compute each of the following determinants.

1. $|-3A|$
2. $\left|A^{-1}\right|$
3. $\left|A^{T}\right|$
4. $\left|A^{3}\right|$
5. $\left|(2A^{-1})^{T}\right|$
6. $\left|(2(-A)^{T})^{-1}\right|$

4. Compute $|A| + |B|$ if $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 1 & 4 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}$.

5. Compute $|A| + |B|$ if

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 4 & 3 \\ -2 & 2 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 1 & 4 & 3 \\ -2 & 2 & 0 & 1 \end{pmatrix}.$$

Solution

1. **1.**

$$\left|A\right| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & -1 & 0 & 0 \end{vmatrix} = \left|A^T\right| = \begin{vmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{vmatrix} \underline{\underline{R_2(-2)+R_3}} \begin{vmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & -2 & 0 \end{vmatrix} \underline{\underline{R_4(3)+R_3}} \begin{vmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 0 \end{vmatrix} \underline{\underline{R_3(2)+R_4}} \begin{vmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \end{vmatrix} = (1)(1)(1)(4) = 4 \neq 0,$$

so *A* is invertible.

$$|B| = \begin{vmatrix} 0 & -1 & 3 & 5 & 2 \\ 1 & -1 & 2 & 4 & -1 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 0 & -2 & 6 & 10 & 4 \end{vmatrix} \underline{\underline{R_{1,2}}} - \begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 2 \\ 0 & 0 & 1 & 2 & 7 \\ 1 & 1 & -2 & 3 & 1 \\ 0 & -2 & 6 & 10 & 4 \end{vmatrix}$$

$$\underline{\underline{R_2(-1)+R_4}} - \begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 2 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 2 & -4 & -1 & 2 \\ 0 & 0 & 2 & 9 & 6 \end{vmatrix} \underline{\underline{R_2(2)+R_4}}$$

$$-\begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 2 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 2 & 9 & 6 \\ 0 & 0 & 2 & 9 & 6 \end{vmatrix} \underline{\underline{R_3(-1)+R_4}} - \begin{vmatrix} 1 & -1 & 2 & 4 & -1 \\ 0 & -1 & 3 & 5 & 2 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 0 & 2 & 9 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0,$$

so B is not invertible.

C is not invertible because

$$|C| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ -2 & -1 & -4 \end{vmatrix} \underline{\underline{R_1(1)+R_3}} - \begin{vmatrix} 2 & 1 & 4 \\ 1 & -2 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

$$\left|D\right| = \begin{vmatrix} 0 & 1 & 3 \\ -2 & 1 & 2 \\ 1 & -1 & -1 \end{vmatrix} \underline{\underline{R_{1,\,3}}} \quad - \begin{vmatrix} 1 & -1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix} \underline{\underline{R_1(2) + R_2}}$$

$$- \begin{vmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 3 \end{vmatrix} \underline{\underline{R_2(1) + R_3}} \quad - \begin{vmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3 \neq 0,$$

so *D* is invertible.

Because it contains a zero row, $|E| = 0$ and *E* is not invertible.

$|F| = 1(2)(-6)(-1) = 12 \neq 0,$ and *F* is invertible.

2. **2.**

$$|A| = \begin{vmatrix} x & 0 & 0 & 0 \\ 2 & x & 2 & 0 \\ -2 & -3 & 1 & -2 \\ 1 & 0 & 0 & 1 \end{vmatrix} = A^T \begin{vmatrix} x & 2 & -2 & 1 \\ 0 & x & -3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{vmatrix} = x \begin{vmatrix} x & -3 & 0 \\ 2 & 1 & 0 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= x[x - (-6)] = x(x+6).$$

Let $|A| = 0.$ Then $x = 0$ or $x = -6.$ Hence, when $x \neq 0$ and $x \neq -6,$ $|A| \neq 0$ and *A* is invertible.

$$|B| = \begin{vmatrix} 2 & x & 4 \\ 1 & -2 & 4 \\ -2 & -1 & x \end{vmatrix} \underline{\underline{R_{1,2}}} - \begin{vmatrix} 1 & -2 & 4 \\ 2 & x & 4 \\ -2 & -1 & x \end{vmatrix} = \text{doubt} =$$

$$- \begin{vmatrix} 1 & -2 & 4 \\ 0 & x+4 & -4 \\ 0 & -5 & x+8 \end{vmatrix} = -(1) \begin{vmatrix} x+4 & -4 \\ -5 & x+8 \end{vmatrix}$$

$$= -[(x+4)(x+8) - (-4)(-5)] = -(x^2 + 12x + 12)$$

$$= -[(x+6)^2 - 24] = (2\sqrt{6})^2 - (x+6)^2$$

$$= [2\sqrt{6} - (x+6)][2\sqrt{6} + (x+6)].$$

Let $|B| = 0$. Then $[2\sqrt{6} - (x+6)][2\sqrt{6} + (x+6)] = 0$. This implies

$$x = -6 + 2\sqrt{6} \quad \text{or } x = -6 - 2\sqrt{6}.$$

Hence when $x \neq -6 - 2\sqrt{6}$ and $x \neq -6 + 2\sqrt{6}$, $|B| \neq 0$ and B is invertible.

3. **3.**

1. $\left|-3A\right| = (-3)^4 \left|A\right| = 162.$
2. $\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{2}.$
3. $\left|A^{-1}\right| = \left|A\right| = 2.$
4. $\left|A^3\right| = \left|A\right|^3 = 8.$
5. $\left|(2A^{-1})^T\right| = \left|2A^{-1}\right| = 2^4 \left|A^{-1}\right| = \frac{16}{|A|} = 8.$
6. $$\left|\left(2(-A)^T\right)^{-1}\right| = \frac{1}{\left|2(-A)^T\right|} = \frac{1}{2^4\left|(-A)^T\right|} = \frac{1}{16|-A|}$$ $$= \frac{1}{16(-1)^4\left|A\right|} = \frac{1}{32}.$$

4. **4.**

$$|A|+|B|=\begin{vmatrix}2&1&3\\1&0&3\\1&4&7\end{vmatrix}+\begin{vmatrix}2&1&3\\1&0&3\\0&1&-1\end{vmatrix}$$

$$=\begin{vmatrix}2&1&3\\1&0&3\\1+0&4+1&7-1\end{vmatrix}$$

$$=\begin{vmatrix}2&1&3\\1&0&3\\1&5&6\end{vmatrix}\underline{\underline{R_{1,\,2}}}-\begin{vmatrix}1&0&3\\2&1&3\\1&5&6\end{vmatrix}$$

$$=\text{doubt}=-\begin{vmatrix}1&0&3\\0&1&-3\\0&5&3\end{vmatrix}$$

$$\underline{\underline{R_2(-5)+R_3}}-\begin{vmatrix}1&0&3\\0&1&-3\\0&0&18\end{vmatrix}=-18.$$

5. **5.**

$$\left|A\right| + \left|B\right| = \begin{vmatrix} 1 & 0 & 0 & 2 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 4 & 3 \\ -2 & 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 1 & 4 & 3 \\ -2 & 2 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 2 \\ -1+3 & 0+0 & 1+0 & 2+1 \\ 0 & 1 & 4 & 3 \\ -2 & 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & 3 \\ 0 & 1 & 4 & 3 \\ -2 & 2 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 4 & 3 \\ 2 & 3 & 3 & 1 \end{vmatrix} \xlongequal{R_1(-2)+R_4} \begin{vmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & -1 & 3 & 5 \end{vmatrix} \xlongequal{R_3(1)+R_4}$$

$$\begin{vmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 7 & 5 \end{vmatrix} \xlongequal{R_{2,\,3}} - \begin{vmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 7 & 5 \end{vmatrix} \xlongequal{R_3(-7)+R_4}$$

$$- \begin{vmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -9 \end{vmatrix} = 9.$$

A.4 Systems of linear equations

Section 4.1

1. Determine which of the following equations are linear.
   a. $x + 2y + z = 6$
   b. $2x - 6y + z = 1$
   c. $-\sqrt{2}x + 6^{-\frac{2}{3}}y = 4 - 3z$
   d. $3x_1 + 2x_2 + 4x_3 + 5x_4 = 1$
   e. $2xy + 3yz + 5z = 8$

2. For each of the following systems of linear equations, find its coefficient and augmented matrices
   1. $\begin{cases} 2x_1 - x_2 = 6 \\ 4x_1 + x_2 = 3 \end{cases}$
   2. $\begin{cases} -x_1 + 2x_2 + 3x_3 = 4 \\ 3x_1 + 2x_2 - 3x_3 = 5 \\ 2x_1 + 3x_2 - x_3 = 1 \end{cases}$
   3. $\begin{cases} x_1 - 3x_3 - 4 = 0 \\ 2x_2 - 5x_3 - 8 = 0 \\ 3x_1 + 2x_2 - x_3 = 4 \end{cases}$

3. For each of the following augmented matrices, find the corresponding linear system.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & 1 & 0 & -2 \\ 0 & -1 & 1 & 0 \end{array}\right) \qquad \left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 3 & -1 \end{array}\right)$$

4. For each of the following linear systems, write it into the form $A\overrightarrow{X} = \overrightarrow{b}$ and express $\overrightarrow{b}$ as a linear combination of the column vectors of *A*.

   a. $\begin{cases} -x_1 + x_2 + 2x_3 = 3 \\ 2x_1 + 6x_2 - 5x_3 = 2 \\ -3x_1 + 7x_2 - 5x_3 = -1 \end{cases}$

   b. $\begin{cases} x_1 - x_2 + 4x_3 = 0 \\ -2x_1 + 4x_2 - 3x_3 = 0 \\ 3x_1 + 6x_2 - 8x_3 = 0, \end{cases}$

   c. $\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ -x_1 + 3x_2 - 4x_3 = 0 \end{cases}$

5. For each of the following points, $P_1\left(6,\ -\frac{1}{2}\right),\ P_2(8,\ 1),\ P_3(2,3),$ and $P_4(0,\ 0)$, verify whether it is a solution of the system

$$\begin{cases} x - 2y = 7, \\ x + 2y = 5. \end{cases}$$

6. Consider the system of linear equations

$$\begin{cases} -x_1 + x_2 + 2x_3 = 3 \\ 2x_1 + 6x_2 - 5x_3 = 2 \\ -3x_1 + 7x_2 - 5x_3 = -1 \end{cases}$$

   1. Verify that $(1,\ 3,\ -1)$ is a solution of the system.
   2. Verify that $\left(\frac{5}{6},\ \frac{7}{6},\ \frac{4}{3}\right)$ is a solution of the system.
   3. Determine if the vector $\overrightarrow{b} = (3,\ 2,\ -1)^T$ is a linear combination of the column vectors of the coefficient matrix of the system.

4. Determine if the vector $\vec{b} = (3,\ 2,\ -1)^T$ belongs to the spanning space of the column vectors of the coefficient matrix of the system.

7. Solve each of the following systems.
   1. $\begin{cases} x_1 - x_2 = 6 \\ x_1 + x_2 = 5 \end{cases}$
   2. $\begin{cases} x_1 + 2x_2 = 1 \\ 2x_1 + x_2 = 2 \end{cases}$
   3. $\begin{cases} x_1 + 2x_2 = 1 \\ x_1 + 2x_2 = 2 \end{cases}$

Solution

1. **1.** *a*), *b*), *c*), *d*) are linear.
2. **2.**

   1. $A = \begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}$, $(A \mid B) = \begin{pmatrix} 2 & -1 & 6 \\ 4 & 1 & 3 \end{pmatrix}$

   2. $A = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 2 & -3 \\ 2 & 3 & -1 \end{pmatrix}$, $(A \mid \vec{b}) = \begin{pmatrix} -1 & 2 & 3 & 4 \\ 3 & 2 & -3 & 5 \\ 2 & 3 & -1 & 1 \end{pmatrix}$

   3. $A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & -5 \\ 3 & 2 & -1 \end{pmatrix}$, $(A \mid \vec{b}) = \begin{pmatrix} 1 & 0 & -3 & 4 \\ 0 & 2 & -5 & 8 \\ 3 & 2 & -1 & 4 \end{pmatrix}$

3. **3.**

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ -x_1 + x_2 = -2 \\ -x_2 + x_3 = 0 \end{cases} \qquad \begin{cases} 2x + y = 1 \\ -y = -2 \\ 3y = -1 \end{cases}$$

4. **4.**

a. Let $A = \begin{pmatrix} -1 & 1 & 2 \\ 2 & 6 & -5 \\ -3 & 7 & -5 \end{pmatrix}$, $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$. Then $A\vec{X} = \vec{b}$. Let $\overrightarrow{c_1} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$, $\overrightarrow{c_2} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}$,

and $\overrightarrow{c_3} = \begin{pmatrix} 2 \\ -5 \\ -5 \end{pmatrix}$. Then $x_1\overrightarrow{c_1} + x_2\overrightarrow{c_2} + x_3\overrightarrow{c_3} = \vec{b}$.

b. Let $A = \begin{pmatrix} 1 & -1 & 4 \\ -2 & 4 & -3 \\ 3 & 6 & -8 \end{pmatrix}$, $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then $A \,\Big|\, \vec{X} = \vec{b}$. Let $\overrightarrow{c_1} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $\overrightarrow{c_2} = \begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix}$,

and $\overrightarrow{c_3} = \begin{pmatrix} 4 \\ -3 \\ -8 \end{pmatrix}$. Then $x_1\overrightarrow{c_1} + x_2\overrightarrow{c_2} + x_3\overrightarrow{c_3} = \vec{b}$.

c. Let $A = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & -4 \end{pmatrix}$, $\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then the system can be rewritten as

$$A\vec{X} = \vec{0}.$$

Let $\overrightarrow{a_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\overrightarrow{a_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, and $\overrightarrow{a_3} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Then the system can be rewritten as

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + x_3\overrightarrow{a_3} = \vec{0}.$$

5. **5.** Because $6-2\left(-\frac{1}{2}\right)=7$ and $6+2\left(-\frac{1}{2}\right)=5,\ \left(6,\ -\frac{1}{2}\right)$ is a solution. Because $8-2(2)=6\neq 7,\ (8,\ 1)$ is not a solution. Because $2-2(3)=2-6=\ -4\neq 7,\ (2,\ 3)$ is not a solution. Because $0-2(0)=0\neq 7,\ (0,\ 0)$ is not a solution.

6. **6.**
   1. Because $-1+3+2(-1)=0\neq 3,\ (1,\ 3,\ -1)$ is not a solution.
   2. Because

$$\begin{cases} -\frac{5}{6}+\frac{7}{6}+2\left(\frac{4}{3}\right)=\frac{-5+7+16}{6}=\frac{18}{6}=3,\\ 2\left(\frac{5}{6}\right)+6\left(\frac{7}{6}\right)-5\left(\frac{4}{3}\right)=\frac{5}{3}+7-\frac{20}{3}=7-5=2,\\ -3\left(\frac{5}{6}\right)+7\left(\frac{7}{6}\right)-5\left(\frac{4}{3}\right)=-\frac{15}{6}+\frac{49}{6}-\frac{40}{6}=-1,\end{cases}$$

   $\left(\frac{5}{6},\ \frac{7}{6},\ \frac{4}{3}\right)$ is a solution of the system.

   3. Let $\overrightarrow{c_1}=\begin{pmatrix}-1\\2\\-3\end{pmatrix},\ \overrightarrow{c_2}=\begin{pmatrix}1\\6\\7\end{pmatrix}$, and $\overrightarrow{c_3}=\begin{pmatrix}2\\-5\\-5\end{pmatrix}$. Because $\left(\frac{5}{6},\ \frac{7}{6},\ \frac{4}{3}\right)$ is a solution of the system, by Theorem 4.1.1, $\vec{b}=(3,\ 2,\ -1)^T$ is a linear combination of $c_1,\ c_2,\ c_3$.

4. Because $\vec{b} = \left(\frac{5}{6}, \frac{7}{6}, \frac{4}{3}\right)$ is a solution of the system, by Theorem 4.1.1, $\vec{b} \in \operatorname{span}\left\{\overrightarrow{c_1}, \overrightarrow{c_2}, \overrightarrow{c_3}\right\}$.

7. **7.**

1. Adding the two equations implies $2x_1 = 11$ and $x_1 = \frac{11}{2}$. By the first equation, we have $x_2 = x_1 - 6 = \frac{11}{2} - 6 = -\frac{1}{2}$, so $(x_1, x_2) = \left(\frac{11}{2}, -\frac{1}{2}\right)$ is a solution.
2. The first equation minus 2 times the second equation implies

$$(x_1 + 2x_2) - 2(2x_1 + x_2) = 1 - 2(2)$$

and $x_1 = 1$. Substituting $x_1 = 1$ into the second equation, $x_2 = 2 - 2x_1 = 2 - 2(1) = 0$. Hence $(x_1, x_2) = (1, 0)$ is a solution.
3. This system has no solutions.

Section 4.2

1. Consider each of the following systems.

i. $\begin{cases} x_1 + x_2 - 3x_3 = 2 \\ -x_2 - 6x_3 = 4 \\ -x_3 = 1 \end{cases}$

ii. $\begin{cases} x_1 - 2x_2 + 3x_3 + x_4 = -1 \\ x_2 - x_3 + x_4 = 4 \\ -x_4 = 2 \end{cases}$

iii. $\begin{cases} -x_1 - 2x_2 - 2x_3 + x_4 - 3x_5 = 4 \\ -2x_3 + 3x_4 = 1 \\ -4x_4 + 2x_5 = 5 \end{cases}$

iv. $\begin{cases} -3x_1 - 2x_2 - 2x_3 = 2 \\ x_2 - x_3 = 2 \\ 0x_3 = 4 \end{cases}$

a. Find the coefficient and augmented matrices of the system.
b. Find the basic variables and free variables of the system.
c. Solve the system by using back-substitution if it is consistent, and express its solution as a linear combination.
d. Determine if the vector $\vec{b}$ on the right side of the system is a linear combination of the column vectors of the coefficient matrix of the system.

2. Solve each of the following systems by using Gaussian elimination and express the solutions as linear combinations if they have infinitely many solutions.

a. $\begin{cases} x_1 - 2x_2 + 3x_3 = 6 \\ 2x_1 + x_2 + 4x_3 = 5 \\ -3x_1 + x_2 - 2x_3 = 3 \end{cases}$

b. $\begin{cases} x_1 + x_2 - 2x_3 = 3 \\ 2x_1 + 3x_2 - x_3 = -4 \\ -2x_1 - 3x_2 - x_3 = 4 \end{cases}$

c. $\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 4x_1 + 7x_2 + 6x_3 = 17 \\ 2x_1 + 5x_2 + 12x_3 = 7 \end{cases}$

d. $\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 5 \\ 3x_1 - x_2 + 4x_3 + 2x_4 = 1 \\ -x_1 + 5x_2 - 6x_3 + 4x_4 = 9 \end{cases}$

e. $\begin{cases} x_1 - 2x_2 = 3 \\ 2x_1 - x_2 = 0 \\ -x_1 + 4x_2 = -1 \end{cases}$

f. $\begin{cases} -x_1 + 2x_2 - 3x_3 = -4 \\ 2x_1 - x_2 + 2x_3 = 2 \\ x_1 + x_2 - x_3 = 2 \end{cases}$

Solution

1. **1.**

i.

a. The coefficient and the augmented matrices are

$$A = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -1 & -6 \\ 0 & 0 & -1 \end{pmatrix},\ (A \,|\, \vec{b}) = \begin{pmatrix} 1 & 1 & -3 & 2 \\ 0 & -1 & -6 & 4 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

b. Because *A* is a row echelon matrix, the basic variables are $x_1,\ x_2,$ and x_3. There are no free variables.

c. By the third equation, $x_3 = -1$. By the second equation,

$$x_2 = -6x_3 - 4 = -6(-1) - 4 = 2$$

and by the first equation,

$$x_1 = 2 - x_2 + 3x_3 = 2 - 2 + 3(-1) = -3.$$

Hence, $(x_1,\ x_2,\ x_3) = (-3,\ 2,\ -1)$ is a solution.

d. Because the system has a solution $(-3,\ 2,\ -1)$, $\vec{b} = (2,\ 4,\ 1)^T$ is a linear combination of the column vectors of *A*.

ii.

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a. The coefficient and the augmented matrices are

$$A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ (A \,|\, \vec{b}) = \begin{pmatrix} 1 & -2 & 3 & 1 & -1 \\ 0 & 1 & -1 & 1 & 4 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

b. $x_1, \ x_2, \ x_4$ are basic variables and x_3 is a free variable.

c. By the third equation, $x_4 = -2$. Let $x_3 = t$. Then by the second equation, $x_2 = 4 - x_4 + x_3 = 4 + 2 + x_3 = 6 + t$. By the first equation,

$$x_1 = -1 + 2x_2 - 3x_3 - x_4 = -1 + 2(6 + t) - 3t - (-2) = 13 - t.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 13 \\ 6 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} -t \\ t \\ t \\ 0 \end{pmatrix} = \begin{pmatrix} 13 \\ 6 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

d. Because the system has infinitely many solutions, $\vec{b} = (-1, \ 4, \ 2)^T$ is a linear combination of the column vectors of *A*.

iii.

a. The coefficient and the augmented matrices are

$$A = \begin{pmatrix} -1 & -2 & -2 & 1 & -3 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 0 & -4 & 2 \end{pmatrix}, \ (A \,|\, \vec{b}) = \begin{pmatrix} -1 & -2 & -2 & 1 & -3 & 4 \\ 0 & 0 & -2 & 3 & 0 & 1 \\ 0 & 0 & 0 & -4 & 2 & 5 \end{pmatrix}.$$

b. $x_1,\ x_2,\ x_4$ are basic variables and x_2 and x_5 are free variables.

c. Let $x_2 = s$ and $x_5 = t$. By the third equation,

$$-4x_4 = 5 - 2x_5 \quad \text{and } x_4 = -\frac{5}{4} + \frac{1}{2}t.$$

By the second equation, we have

$$-2x_3 = 1 - 3x_4 = 1 - 3\left(-\frac{5}{4} + \frac{1}{2}t\right) = 1 + \frac{15}{4} - \frac{3}{2}t = \frac{19}{4} - \frac{3}{2}t$$

and $x_3 = -\frac{19}{8} + \frac{3}{4}t$. By the first equation,

$$\begin{aligned} x_1 &= -4 - 2x_2 - 2x_3 + x_4 - 3x_5 \\ &= -4 - 2s - 2\left(-\frac{19}{8} + \frac{3}{4}t\right) + \left(-\frac{5}{4} + \frac{1}{2}t\right) - 3t \\ &= -4 - 2s + \frac{19}{4} - \frac{3}{2}t - \frac{5}{4} + \frac{1}{2}t - 3t = -\frac{1}{2} - 2s - 4t. \end{aligned}$$

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} - 2s - 4t \\ s \\ -\frac{19}{8} + \frac{3}{4}t \\ -\frac{5}{4} + \frac{1}{2}t \\ t \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{19}{8} \\ -\frac{5}{4} \\ 0 \end{pmatrix} + \begin{pmatrix} -2s \\ s \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4t \\ 0 \\ \frac{3}{4}t \\ \frac{1}{2}t \\ t \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\frac{19}{8} \\ -\frac{5}{4} \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ \frac{3}{4} \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

d. Because the system has infinitely many solutions, $\vec{b} = (4,\ 1,\ 5)^T$ is a linear combination of the column vectors of *A*.

iv.

a. The coefficient and the augmented matrices are

$$A = \begin{pmatrix} -3 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \ (A \mid \vec{b}) = \begin{pmatrix} -3 & -2 & -2 & 2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

b. $x_1, \ x_2$ are basic variables and x_3 is a free variable.

c. By the last equation of the system, we see that the system has no solutions.

2. **2.**

a. *a*)

$$(A \mid \vec{b}) = \begin{pmatrix} 1 & -2 & 3 & 6 \\ 2 & 1 & 4 & 5 \\ -3 & 1 & -2 & 3 \end{pmatrix} \xrightarrow[R_1(3)+R_3]{R_1(-2)+R_2}$$

$$\begin{pmatrix} 1 & -2 & 3 & 6 \\ 0 & 5 & -2 & -7 \\ 0 & -5 & 7 & 21 \end{pmatrix} \xrightarrow{R_2(1)+R_3}$$

$$\begin{pmatrix} 1 & -2 & 3 & 6 \\ 0 & 5 & -2 & -7 \\ 0 & 0 & 5 & 14 \end{pmatrix}.$$

Hence, we obtain

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$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 6 \qquad (1)\\ 5x_2 - 2x_3 &= -7 \qquad (2)\\ 5x_3 &= 14. \qquad (3) \end{aligned}$$

By (3), $x_3 = \frac{14}{5}$, by (2),

$$5x_2 = -7 + 2x_3 = -7 + 2\left(\frac{14}{5}\right) = -\frac{7}{5},$$

and $x_2 = -\frac{7}{25}$. By (1),

$$x_1 = 6 + 2x_2 - 3x_3 = 6 + 2\left(-\frac{7}{25}\right) - 3\left(\frac{14}{5}\right) = -\frac{74}{25}.$$

b. *b*)

$$(A\,|\,\vec{b}) = \begin{pmatrix} 1 & 1 & -2 & 3 \\ 2 & 3 & -1 & -4 \\ -2 & -3 & -1 & 4 \end{pmatrix} \xrightarrow[R_1(2)+R_3]{R_1(-2)+R_2}$$

$$\begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 3 & -10 \\ 0 & -1 & -5 & 10 \end{pmatrix} \xrightarrow{R_2(1)+R_3}$$

$$\begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 1 & 3 & -10 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$

Hence, we obtain

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 3 && (1) \\ x_2 + 3x_3 &= 10 && (2) \\ -2x_3 &= 0. && (3) \end{aligned}$$

By (3), $x_3 = 0$. By (2), $x_2 = -10 - 3x_3 = -10$. By (1), $x_1 = 3 - x_2 + 2x_3 = 3 - (-10) = 13$. Hence, $(x_1,\ x_2,\ x_3) = (13,\ -10,\ 0)$ is a solution.

c. *c*)

$$(A \mid \vec{b}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 7 & 6 & 17 \\ 2 & 5 & 12 & 7 \end{pmatrix} \begin{array}{c} R_1(-4)+R_2 \\ \longrightarrow \\ R_1(-2)+R_3 \end{array}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -6 & 1 \\ 0 & 1 & 6 & -1 \end{pmatrix} \xrightarrow{R_2(1)+R_3}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -6 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, we have

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \qquad (1) \\ -x_2 - 6x_3 &= 1. \qquad (2) \end{aligned}$$

Let $x_3 = t$. By (2), $x_2 = -1 - 6x_3 = -1 - 6t$. By (1),

$$x_1 = 4 - 2x_2 - 3x_3 = 4 - 2(-1 - 6t) - 3t = 6 + 9t.$$

Hence,

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$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6+9t \\ -1-6t \\ t \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 9t \\ -6t \\ t \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 9 \\ -6 \\ 1 \end{pmatrix}.$$

d. *d*)

$$(A \mid \vec{b}) = \begin{pmatrix} 1 & 2 & -1 & 3 & 5 \\ 3 & -1 & 4 & 2 & 1 \\ -1 & 5 & -6 & 4 & 9 \end{pmatrix} \xrightarrow[R_1(1)+R_3]{R_1(-3)+R_2}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 & 5 \\ 0 & -7 & 7 & -7 & -14 \\ 0 & 7 & -7 & 7 & 14 \end{pmatrix} \xrightarrow{R_2(1)+R_3}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 & 5 \\ 0 & -7 & 7 & -7 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2\left(-\frac{1}{7}\right)}$$

$$\begin{pmatrix} 1 & 2 & -1 & 3 & 5 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

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Hence, we obtain

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 5 \qquad (1) \\ x_2 - x_3 + x_4 &= 2 \qquad (2) \end{aligned}$$

x_1 and x_2 are the basic variables and x_3 and x_4 are free variables. Let $x_3 = s$ and $x_4 = t,$ where $s,\ t \in \mathbb{R}.$ By (2), we get $x_2 = 2 + x_3 - x_4 = 2 + s - t$ and by (1),

$$x_1 = 5 - 2x_2 + x_3 - 3x_4 = 5 - 2(2 + s - t) + s - 3t = 1 - s - t.$$

We write the solution as a linear combination:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - s - t \\ 2 + s - t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

e. *e*)

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$$(A \mid \vec{b}) = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 0 \\ -1 & 4 & -1 \end{pmatrix} \begin{array}{c} R_1(-2)+R_2 \\ \longrightarrow \\ R_1(1)+R_3 \end{array} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -6 \\ 0 & 2 & 2 \end{pmatrix}$$

$$\begin{array}{c} R_2\left(\frac{1}{3}\right) \\ \longrightarrow \\ R_3\left(\frac{1}{2}\right) \end{array} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix} \begin{array}{c} R_2(-1)+R_3 \\ \longrightarrow \end{array} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}.$$

Hence, we have

$$\begin{cases} x_1 - 2x_2 = 3 \\ 0x_1 + x_2 = -2 \\ 0x_1 + 0x_2 = 3 \end{cases}$$

From the last equation, we see that the system has no solutions.

f. *f*)

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$$(A \mid \vec{b}) = \begin{pmatrix} -1 & 2 & -3 & -4 \\ 2 & -1 & 2 & 2 \\ 1 & 1 & -1 & 2 \end{pmatrix} \xrightarrow[R_1(1)+R_3]{R_1(2)+R_2}$$

$$\begin{pmatrix} -1 & 2 & -3 & -4 \\ 0 & 3 & -4 & -6 \\ 0 & 3 & -4 & -2 \end{pmatrix} \xrightarrow{R_2(-1)+R_3} \begin{pmatrix} -1 & 2 & -3 & -4 \\ 0 & 3 & -4 & -6 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Hence, we obtain

$$\begin{cases} -x_1 + 2x_2 - 3x_3 = -4 \\ 0x_1 + 3x_2 - 4x_3 = -6 \\ 0x_1 + 0x_2 + 0x_3 = 4 \end{cases}$$

From the last equation we see that the system has no solutions.

Section 4.3

1. Solve each of the following systems using Gauss-Jordan elimination.

a. $\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases}$

$\begin{cases} 3x_1 - x_2 + 3x_3 = -1 \\ -4x_1 + 2x_2 - 4x_3 = 2 \end{cases}$

c. $\begin{cases} x_1 + 3x_2 - x_3 = 2 \\ 4x_1 - 6x_2 + 6x_3 = 14 \\ -3x_1 - x_2 - 2x_3 = 3 \end{cases}$

d. $\begin{cases} x_1 - 2x_2 + 2x_3 = 2 \\ -2x_1 + 3x_2 - 4x_3 = -2 \\ -3x_1 + 4x_2 + 6x_3 = 0 \end{cases}$

e. $\begin{cases} x_2 - x_3 = 1 \\ -x_1 + 3x_2 - x_3 = -2 \\ 3x_1 + 3x_2 - 2x_3 = 4 \end{cases}$

f. $\begin{cases} x_1 + x_2 + 2x_3 = 1 \\ -2x_1 - 4x_2 - 5x_3 = -1 \\ 3x_1 + 6x_2 + 5x_3 = 0 \end{cases}$

g. $\begin{cases} 2x_2 + 3x_3 - 2x_4 = 3 \\ 2x_1 + 3x_3 + 2x_4 = 1 \\ -3x_1 + x_2 - 2x_3 = 3 \end{cases}$

h. $\begin{cases} x_1 - x_2 + 2x_3 = -1 \\ -x_1 + 3x_2 - 4x_3 = -3 \\ x_2 - x_3 = -2 \\ x_1 - 2x_2 + 3x_3 = 1 \end{cases}$

2. For each of the following homogeneous systems, solve it by using Gauss-Jordan elimination, write its solution as a linear combination, and find its solution space N_A and $\dim(N_A)$.

a. $\begin{cases} x + 2y - z = 0 \\ 2x - y + 3y = 0 \end{cases}$

b. $\begin{cases} 2x - y + 3z = 0 \\ 4x - 2y + 6z = 0 \\ -6x + 3y - 9z = 0 \end{cases}$

c. $\begin{cases} x_1 + 4x_2 + 6x_3 = 0 \\ 2x_1 + 6x_2 + 3x_3 = 0 \\ -3x_1 + x_2 + 8x_3 = 0. \end{cases}$

d. $\begin{cases} -x_1 + 2x_2 + x_3 = 0 \\ 3x_1 - 3x_2 - 9x_3 = 0 \\ -2x_1 - x_2 + 12x_3 = 0. \end{cases}$

e. $$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$$

f. $$\begin{cases} x_1 + x_2 - 2x_3 = 0 \\ -3x_1 + 2x_2 - x_3 = 0 \\ -2x_1 + 3x_2 - 3x_3 = 0 \\ 4x_1 - x_2 - x_3 = 0. \end{cases}$$

Solution

1. **1.**

a.

$$(A\,|\,\vec{b}) = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 2 & -3 & 1 & -1 \end{pmatrix} \xrightarrow{R_1(-2)+R_2} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & -1 & -1 & -5 \end{pmatrix}$$

$$\xrightarrow{R_2(-1)} \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & 5 \end{pmatrix} \xrightarrow{R_2(1)+R_1} \begin{pmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 1 & 5 \end{pmatrix} = (B\,|\,\vec{c}).$$

The system of linear equations corresponding to $(B\,|\,\vec{c})$ is

$$\begin{cases} x_1 + 2x_3 = 7 \\ x_2 + x_3 = 5 \end{cases}$$

x_1 and x_2 are basic variables and x_3 is a free variable. Let $x_3 = t.$ Then $x_1 = 7 - 2x_3 = 7 - 2t$ and $x_2 = 5 - x_3 = 5 - t.$ Hence,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -7 - 2t \\ 5 - t \\ t \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ -t \\ t \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}.$$

b.

$$(A \,|\, \vec{b}) = \begin{pmatrix} 3 & -1 & 3 & -1 \\ -4 & 2 & -4 & 2 \end{pmatrix} \xrightarrow{R_2(1)+R_1} \begin{pmatrix} -1 & 1 & -1 & 1 \\ -4 & 2 & -4 & 2 \end{pmatrix}$$

$$\xrightarrow{R_1(-1)} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -4 & 2 & -4 & 2 \end{pmatrix} \xrightarrow{R_1(4)+R_2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & -2 & 0 & -2 \end{pmatrix}$$

$$\xrightarrow{R_2\left(-\frac{1}{2}\right)} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_2(1)+R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = (B \,|\, \vec{c}).$$

The system of linear equations corresponding to $(B \,|\, \vec{c})$ is

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 = 1 \end{cases}$$

x_1 and x_2 are basic variables and x_3 is a free variable. Let $x_3 = t$. Then $x_1 = -x_3 = -t$ and $x_2 = 1$. Hence,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

c.

$$\left(A \,\middle|\, \vec{b}\right) = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 4 & -6 & 6 & 14 \\ -3 & -1 & -2 & 3 \end{pmatrix}$$

$$\underset{R_1(3)+R_3}{\overset{R_1(-4)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & -18 & 10 & 6 \\ 0 & 8 & -5 & 9 \end{pmatrix}$$

$$\overset{R_2\left(-\frac{1}{2}\right)}{\longrightarrow} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 9 & -5 & -3 \\ 0 & 8 & -5 & 9 \end{pmatrix} \overset{R_3(-1)+R_2}{\longrightarrow} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 0 & -12 \\ 0 & 8 & -5 & 9 \end{pmatrix}$$

$$\overset{R_2(-8)+R_3}{\longrightarrow} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & -5 & 105 \end{pmatrix} \overset{R_3\left(-\frac{1}{5}\right)}{\longrightarrow} \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & -21 \end{pmatrix}$$

$$\overset{R_3(1)+R_1}{\longrightarrow} \begin{pmatrix} 1 & 3 & 0 & -19 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & -21 \end{pmatrix} \overset{R_2(-3)+R_1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 17 \\ 0 & 1 & 0 & -12 \\ 0 & 0 & 1 & -21 \end{pmatrix}.$$

Hence, $(x_1,\ x_2,\ x_3) = (7,\ -12,\ -21)$ is a solution of the system.

d.

$$\left(A \,\middle|\, \vec{b}\right) = \begin{pmatrix} 1 & -2 & 2 & 2 \\ -2 & 3 & -4 & -2 \\ -3 & 4 & 6 & 0 \end{pmatrix}$$

$$\xrightarrow[R_1(3)+R_3]{R_1(2)+R_2}\begin{pmatrix}1 & -2 & 2 & 2\\ 0 & -1 & 0 & 2\\ 0 & -2 & 12 & 6\end{pmatrix}$$

$$\xrightarrow[R_3\left(-\frac{1}{2}\right)]{R_2(-1)}\begin{pmatrix}1 & -2 & 2 & 2\\ 0 & 1 & 0 & -2\\ 0 & 1 & -6 & -3\end{pmatrix}$$

$$\xrightarrow{R_2(-1)+R_3}\begin{pmatrix}1 & -2 & 2 & 2\\ 0 & 1 & 0 & -2\\ 0 & 0 & -6 & -1\end{pmatrix}$$

$$\xrightarrow{R_3\left(-\frac{1}{6}\right)}\begin{pmatrix}1 & -2 & 2 & 2\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & \frac{1}{6}\end{pmatrix}\xrightarrow{R_3(-2)+R_1}\begin{pmatrix}1 & -2 & 0 & \frac{5}{3}\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & \frac{1}{6}\end{pmatrix}$$

$$\xrightarrow{R_2(2)+R_1}\begin{pmatrix}1 & 0 & 0 & -\frac{7}{3}\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & \frac{1}{6}\end{pmatrix}.$$

Hence, $(x_1,\ x_2,\ x_3) = \left(-\frac{7}{3},\ \ -2,\ \frac{1}{6}\right)$ is a solution of the system.

e.

$$\left(A \,\middle|\, \vec{b}\right) = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 3 & -1 & -2 \\ 3 & 3 & -2 & 4 \end{pmatrix} \xrightarrow{R_{1,\,2}} \begin{pmatrix} -1 & 3 & -1 & -2 \\ 0 & 1 & -1 & 1 \\ 3 & 3 & -2 & 4 \end{pmatrix}$$

$$\xrightarrow{R_1(3)+R_3} \begin{pmatrix} -1 & 3 & -1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 12 & -5 & -2 \end{pmatrix} \xrightarrow{R_2(-12)+R_3}$$

$$\begin{pmatrix} -1 & 3 & -1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 7 & -14 \end{pmatrix} \xrightarrow{R_3\left(\frac{1}{7}\right)} \begin{pmatrix} -1 & 3 & -1 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow[R_3(1)+R_1]{R_3(1)+R_2} \begin{pmatrix} -1 & 3 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{R_2(-3)+R_1}$$

$$\begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \xrightarrow{R_1(-1)} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Hence, $(x_1,\ x_2,\ x_3) = (1,\ \ -1,\ \ -2)$ is a solution of the system.

f.

$$(A \mid \vec{b}) = \begin{pmatrix} 1 & 1 & 2 & 1 \\ -2 & -4 & -5 & -1 \\ 3 & 6 & 5 & 0 \end{pmatrix} \underset{R_1(-3)+R_3}{\overset{R_1(2)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 3 & -1 & -3 \end{pmatrix}$$

$$\overset{R_3(1)+R_2}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 3 & -1 & -3 \end{pmatrix} \overset{R_2(-3)+R_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 3 \end{pmatrix}$$

$$\overset{R_3\left(\frac{1}{5}\right)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & \frac{3}{5} \end{pmatrix} \underset{R_3(-2)+R_1}{\overset{R_3(2)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & 1 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & \frac{3}{5} \end{pmatrix}$$

$$\overset{R_2(-1)+R_1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & \frac{3}{5} \\ 0 & 1 & 0 & -\frac{4}{5} \\ 0 & 0 & 1 & \frac{3}{5} \end{pmatrix} = (D \mid \vec{d}).$$

Hence, the solution is $(x_1,\ x_2,\ x_3) = \left(\frac{3}{5},\ -\frac{4}{5},\ \frac{3}{5}\right)$.

g.

$$(A \mid \vec{b}) = \begin{pmatrix} 0 & 2 & 3 & -2 & 3 \\ 2 & 0 & 3 & 2 & 1 \\ -3 & 1 & -2 & 0 & 3 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 2 & 0 & 3 & 2 & 1 \\ 0 & 2 & 3 & -2 & 3 \\ -3 & 1 & -2 & 0 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3(1)R_1} \begin{pmatrix} -1 & 1 & 1 & 2 & 4 \\ 0 & 2 & 3 & -2 & 3 \\ -3 & 1 & -2 & 0 & 3 \end{pmatrix} \xrightarrow{R_1(-3)+R_3}$$

$$\begin{pmatrix} -1 & 1 & 1 & 2 & 4 \\ 0 & 2 & 3 & -2 & 3 \\ 0 & -2 & -5 & -6 & -9 \end{pmatrix} \xrightarrow{R_2(1)R_3} \begin{pmatrix} -1 & 1 & 1 & 2 & 4 \\ 0 & 2 & 3 & -2 & 3 \\ 0 & 0 & -2 & -8 & -6 \end{pmatrix}$$

$$\xrightarrow{R_3\left(-\frac{1}{2}\right)} \begin{pmatrix} -1 & 1 & 1 & 2 & 4 \\ 0 & 2 & 3 & -2 & 3 \\ 0 & 0 & 1 & 4 & 3 \end{pmatrix} \underset{R_3(-3)+R_2}{\xrightarrow{R_3(-1)+R_1}}$$

$$\begin{pmatrix} -1 & 1 & 0 & -2 & 1 \\ 0 & 2 & 0 & -14 & -6 \\ 0 & 0 & 1 & 4 & 3 \end{pmatrix} \xrightarrow{R_2\left(\frac{1}{2}\right)} \begin{pmatrix} -1 & 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -7 & -3 \\ 0 & 0 & 1 & 4 & 3 \end{pmatrix}$$

$$\xrightarrow{R_2(-1)+R_1} \begin{pmatrix} -1 & 0 & 0 & 5 & 4 \\ 0 & 1 & 0 & -7 & -3 \\ 0 & 0 & 1 & 4 & 3 \end{pmatrix} \xrightarrow{R_1(-1)}$$

$$\begin{pmatrix} 1 & 0 & 0 & -5 & -1 \\ 0 & 1 & 0 & -7 & -3 \\ 0 & 0 & 1 & 4 & 3 \end{pmatrix} = (B \mid \vec{c}).$$

The system of linear equations corresponding to $(B \mid \vec{c})$ is

$$\begin{cases} x_1 - 5x_4 &= -4 \\ x_2 - 7x_4 &= -3 \\ x_3 + 4x_4 &= 3. \end{cases}$$

$x_1,\ x_2$ and x_3 are basic variables and x_4 is a free variable. Let $x_4 = t$. Then $x_1 = -4 + 5x_4 = -4 + 5t,\ x_2 = -3 + 7x_4 = -3 + 7t,$ and $x_3 = 3 - 4x_4 = 3 - 4t$. Hence,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 + 5t \\ -3 + 7t \\ 3 - 4t \\ t \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ 7 \\ -4 \\ 1 \end{pmatrix}.$$

h.

$$(A \mid \vec{b}) = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -1 & 3 & -4 & -3 \\ 0 & 1 & -1 & -2 \\ 1 & -2 & 3 & 1 \end{pmatrix} \underset{R_1(-1)+R_4}{\overset{R_1(1)+R_2}{\longrightarrow}}$$

$$\begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & -2 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & 1 & 2 \end{pmatrix} \overset{R_2\left(\frac{1}{2}\right)}{\longrightarrow} \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

$$\underset{R_2(1)+R_4}{\overset{R_2(-1)+R_3}{\longrightarrow}} \begin{pmatrix} 1 & -1 & 2 & -1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_2(1)+R_1}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = (B \mid \vec{c}).$$

The system of linear equations corresponding to $(B \mid \vec{c})$ is

$$\begin{cases} x_1 + x_3 = -3 \\ x_2 - x_3 = -2. \end{cases}$$

x_1 and x_2 are basic variables and x_3 is a free variable. Let $x_3 = t$. Then $x_1 = -3 - x_4 = -3 - t$ and $x_2 = -2 + x_3 = -2 + t$. Hence,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3-t \\ -2+t \\ t \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

2. **2.**

a.

$$(A \mid \vec{0}) = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & -1 & 3 & 0 \end{pmatrix} \xrightarrow{R_1(-2)+R_2} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{pmatrix} \xrightarrow{R_2\left(-\frac{1}{5}\right)}$$

$$\begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \xrightarrow{R_2(-2)+R_1} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

The system of linear equations corresponding to the last augmented matrix is

$$\begin{cases} x + z = 0 \\ y - z = 0 \end{cases}$$

Let $z = t$. Then $x = -t$ and $y = t$. We write the solution into a linear combination as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $\overrightarrow{v_1} = (-1, 1, 1)^T$. $N_A = \left\{ t\overrightarrow{v_1} : t \in \mathbb{R} \right\} = \text{span}\left\{ \overrightarrow{v_1} \right\}$. Because $r(A) = 2$, $\text{null}(A) = 3 - 2 = 1$ and $\dim(N_A) = 1$.

b.

$$(A \mid \vec{0}) = \begin{pmatrix} 2 & -1 & 3 & 0 \\ 4 & -2 & 6 & 0 \\ -6 & 3 & -9 & 0 \end{pmatrix} \xrightarrow[R_1(3)+R_3]{R_1(-2)+R_2} \begin{pmatrix} 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system of linear equations corresponding to the last augmented matrix is

$$2x - y + 3z = 0.$$

Let $y = s$ and t. Then $x = \frac{1}{2}s - \frac{3}{2}t$. We write the solution into a linear combination as follows:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2}s - \frac{3}{2}t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{3}{2} \\ 0 \\ 1 \end{pmatrix}.$$

Let $\overrightarrow{v_1} = \left(\frac{1}{2}, 1, 0\right)^T$ and $\overrightarrow{v_2} = \left(-\frac{3}{2}, 0, 1\right)^T$. Then and

$N_A = \left\{ s\overrightarrow{v_1} + t\overrightarrow{v_2} : s, t \in \mathbb{R} \right\} = \text{span}\left\{ \overrightarrow{v_1}, \overrightarrow{v_2} \right\}$. Because $r(A) = 1$, $\text{null}(A) = 3 - 1 = 2$ and $\dim(N_A) = 2$.

c.

$$\left(A \,\middle|\, \vec{0}\right) = \begin{pmatrix} 1 & 4 & 6 & 0 \\ 2 & 6 & 3 & 0 \\ -3 & 1 & 8 & 0 \end{pmatrix} \overset{R_1(-2)+R_2}{\underset{R_1(3)+R_3}{\longrightarrow}} \begin{pmatrix} 1 & 4 & 6 & 0 \\ 0 & -2 & -9 & 0 \\ 0 & 13 & 26 & 0 \end{pmatrix}$$

$$\overset{R_3\left(\frac{1}{13}\right)}{\longrightarrow} \begin{pmatrix} 1 & 4 & 6 & 0 \\ 0 & -2 & -9 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \overset{R_{2,3}}{\longrightarrow} \begin{pmatrix} 1 & 4 & 6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -9 & 0 \end{pmatrix}$$

$$\overset{R_2(2)+R_3}{\longrightarrow} \begin{pmatrix} 1 & 4 & 6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -5 & 0 \end{pmatrix} \overset{R_3\left(\frac{1}{5}\right)}{\longrightarrow} \begin{pmatrix} 1 & 4 & 6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\overset{R_3(-2)+R_2}{\underset{R_3(-6)+R_1}{\longrightarrow}} \begin{pmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \overset{R_2(-4)+R_1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Hence, $x_1 = x_2 = x_3 = 0$ is a solution of the system. $N_A = \left\{\vec{0}\right\}$ and $\dim(N_A) = 0$.

d.

$$(A \mid \vec{0}) = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 3 & -3 & -9 & 0 \\ -2 & -1 & 12 & 0 \end{pmatrix} \xrightarrow[R_1(-2)+R_3]{R_1(3)+R_2} \begin{pmatrix} -1 & 2 & 1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & -5 & 10 & 0 \end{pmatrix}$$

$$\xrightarrow[R_3\left(-\frac{1}{5}\right)]{R_1(-1),\ R_2\left(\frac{1}{3}\right)} \begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{pmatrix} \xrightarrow{R_2(-1)+R_3}$$

$$\begin{pmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2(2)+R_1} \begin{pmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system corresponding to the last augmented matrix is

$$\begin{cases} x_1 - 5x_3 = 0 \\ x_2 - 2x_3 = 0. \end{cases}$$

Let $x_3 = t$. Then $x_1 = 5t$ and $x_2 = 2t$. We write the solution into a linear combination as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5t \\ 2t \\ t \end{pmatrix} = t\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}.$$

Let $\overrightarrow{v_1} = (5,\ 2,\ 1)^T$. Then $N_A = \left\{t\overrightarrow{v_1} : t \in \mathbb{R}\right\} = \operatorname{span}\left\{\overrightarrow{v_1}\right\}$. Because $r(A) = 2,\ \operatorname{null}(A) = 3 - 2 = 1$ and $\dim(N_A) = 1$.

e.

$$(A \mid \vec{0}) = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_{1,2}}$$

$$\begin{pmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[R_1(1)+R_3]{R_1(2)+R_2}$$

$$\begin{pmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[R_3\left(-\frac{1}{3}\right)]{R_2\left(\frac{1}{3}\right)}$$

$$\begin{pmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_2(-1)+R_4}$$

$$\begin{pmatrix} -1 & -1 & 2 & -3 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \end{pmatrix} \xrightarrow{R_3(-3)+R_4}$$

$$\begin{pmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \underset{R_3(3)+R_1}{\xrightarrow{R_3(2)+R_2}}$$

$$\begin{pmatrix} -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2(-2)+R_1}$$

$$\begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1(-1)}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The system corresponding to the last augmented matrix is

$$\begin{cases} x_1 + x_2 + x_5 = 0 \\ x_3 + x_5 = 0 \\ x_4 = 0. \end{cases}$$

Let $x_2 = s$ and $x_5 = t$. Then $x_1 = -s - t$, $x_3 = -t$ and $x_4 = 0$. We can write the solution into a linear combination as follows:

$$(x_1, x_2, x_3, x_4, x_5)^T = s(-1, 1, 0, 0, 0)^T + t(-1, 0, -1, 0, 1)^T.$$

Let $\overrightarrow{v_1} = (-1, 1, 0, 0, 0)^T$ and $\overrightarrow{v_2} = (-1, 0, -1, 0, 1)^T$. Then

$$N_A = \left\{ s\overrightarrow{v_1} + t\overrightarrow{v_2} : s, t \in \mathbb{R} \right\} = \text{span}\left\{ \overrightarrow{v_1}, \overrightarrow{v_2} \right\}.$$

Because $r(A) = 3$, $\text{null}(A) = 5 - 3 = 2$ and $\dim(N_A) = 2$.

f.

$$\left(A\,\middle|\,\vec{0}\right)=\begin{pmatrix}1&1&-2&0\\-3&2&-1&0\\-2&3&-3&0\\4&-1&-1&0\end{pmatrix}\begin{matrix}R_1(3)+R_2\\R_1(2)+R_3\\\longrightarrow\\R_1(-4)+R_4\end{matrix}\begin{pmatrix}1&1&-2&0\\0&5&-7&0\\0&5&-7&0\\0&-5&7&0\end{pmatrix}$$

$$\begin{matrix}R_2(-1)+R_3\\\longrightarrow\\R_2(1)+R_4\end{matrix}\begin{pmatrix}1&1&-2&0\\0&5&-7&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\xrightarrow{R_2\left(-\frac{1}{5}\right)}\begin{pmatrix}1&1&-2&0\\0&1&-\frac{7}{5}&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\xrightarrow{R_2(-1)+R_1}\begin{pmatrix}1&0&-\frac{3}{5}&0\\0&1&-\frac{7}{5}&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

The system of linear equations corresponding to the last augmented matrix is

$$\begin{cases} x_1 - \frac{3}{5}x_3 = 0 \\ x_2 - \frac{7}{5}x_3 = 0 \end{cases}$$

Let $x_3 = 5t$. Then $x_1 = 3t$ and $x_2 = 7t$. We write the solution into a linear combination as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3t \\ 7t \\ 5t \end{pmatrix} = t\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}.$$

Let $\overrightarrow{v_1} = (3,\ 7,\ 5)^T$. Then $N_A = \left\{ t\overrightarrow{v_1} : t \in \mathbb{R} \right\} = \text{span}\left\{ \overrightarrow{v_1} \right\}$. Because $r(A) = 2,\ \text{null}(A) = 3 - 2 = 1$ and $\dim(N_A) = 1$.

Section 4.4

1. Solve each of the following systems using the inverse matrix method.

a. $\begin{cases} 3x_1 + 4x_2 = 7 \\ x_1 - x_2 = 14 \end{cases}$

b. $\begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + 3x_2 + x_3 = 2 \\ x_1 - x_3 = 1 \end{cases}$

c. $\begin{cases} x_1 - 2x_2 + 2x_3 = 1 \\ 2x_1 + 3x_2 + 3x_3 = -1 \\ x_1 + 6x_3 = -4 \end{cases}$

d. $\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 1 \\ x_1 + 8x_3 = -1 \end{cases}$

e. $\begin{cases} x_1 - 2x_3 + x_4 = 0 \\ x_2 + x_3 + 2x_4 = 1 \\ -x_1 - x_2 + x_3 = -1 \\ 2x_1 + 9x_4 = 1 \end{cases}$

2. Show that if $a \neq 3,$ then the following system has a unique solution for $(b_1,\ b_2) \in \mathbb{R}^2$.

$$\begin{cases} x - y = b_1 \\ ax - 3y = b_2 \end{cases}$$

3. For each $(b_1,\ b_2,\ b_3) \in \mathbb{R}^3,$ show that the following system has a unique solution.

$$\begin{cases} x_1 + 2x_3 = b_1 \\ -3x_1 + 4x_2 + 6x_3 = b_2 \\ -x_1 - 2x_2 + 3x_3 = b_3 \end{cases}$$

4. Solve each of the following systems by using Cramer's rule.

a. $\begin{cases} x - 3y = 6 \\ 2x + 5y = -1 \end{cases}$

b. $\begin{cases} x_1 + 2x_3 = 2 \\ -2x_1 + 3x_2 + x_3 = 16 \\ -x_1 - 2x_2 + 4x_3 = 5 \end{cases}$

c. $\begin{cases} x_1 + x_2 - x_3 = 1 \\ 2x_1 + x_2 - x_3 = 0 \\ x_2 - 2x_3 = -2 \end{cases}$

d. $\begin{cases} x_1 + x_2 + x_3 = 0 \\ 2x_1 - x_2 - x_3 = 1 \\ x_2 + 2x_3 = 2 \end{cases}$

e. $\begin{cases} x_1 - x_4 = 0 \\ x_2 + x_3 = 0 \\ x_1 - x_2 = 0 \\ x_3 - x_4 = 1 \end{cases}$

Solution

1. **1.**

a. Because $|A| = \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} = -7,\ A^{-1} = -\frac{1}{7}\begin{pmatrix} -1 & -4 \\ -1 & 3 \end{pmatrix}$. By Theorem 4.4.1, the solution of the system is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}\begin{pmatrix} 7 \\ 14 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -1 & -4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 7 \\ 14 \end{pmatrix} = \begin{pmatrix} 9 \\ -5 \end{pmatrix}.$$

b.

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{R_1(-1)+R_3}{\overset{R_1(-2)+R_2}{\longrightarrow}}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{pmatrix} \overset{R_2(1)+R_3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 1 & 1 \end{pmatrix}$$

$$\overset{R_3\left(-\frac{1}{3}\right)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix} \underset{R_3(-1)+R_1}{\overset{R_3(1)+R_2}{\longrightarrow}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right) \overset{R_2(-1)+R_1}{\longrightarrow}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & -1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 1 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right) = (I_3 \,|\, A^{-1}).$$

By Theorem 4.4.1, the solution of the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{2}{3} \\ -1 & \frac{2}{3} & -\frac{1}{3} \\ 1 & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

c.

$$(A \,|\, I) = \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 & 0 & 1 \end{array}\right) \underset{R_1(-1)+R_3}{\overset{R_1(-2)+R_2}{\longrightarrow}}$$

$$\left(\begin{array}{cccccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 7 & -1 & -2 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3(-3)+R_2}$$

$$\left(\begin{array}{cccccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -13 & 1 & 1 & -3 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2(-2)+R_3}$$

$$\left(\begin{array}{cccccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -13 & 1 & 1 & -3 \\ 0 & 0 & 30 & -3 & -2 & 7 \end{array}\right) \xrightarrow[R_3\left(\frac{1}{30}\right)]{R_2(2)+R_1}$$

$$\left(\begin{array}{cccccc} 1 & 0 & -24 & 3 & 2 & -6 \\ 0 & 1 & -13 & 1 & 1 & -3 \\ 0 & 0 & 1 & -\frac{1}{10} & -\frac{1}{15} & -\frac{7}{30} \end{array}\right) \xrightarrow[R_3(24)+R_1]{R_3(13)+R_2}$$

$$= \left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{3}{10} & \frac{2}{15} & \frac{1}{30} \\ 0 & 0 & 1 & -\frac{1}{10} & -\frac{1}{15} & \frac{7}{30} \end{array}\right) = (I_3 \mid A^{-1}).$$

By Theorem 4.4.1, the solution of the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ -\frac{3}{10} & \frac{2}{15} & \frac{1}{30} \\ -\frac{1}{10} & -\frac{1}{15} & \frac{7}{30} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} \\ -\frac{17}{30} \\ -\frac{29}{30} \end{pmatrix}.$$

d.

$$(A \mid I_3) = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1(-1)+R_3]{R_1(-2)+R_2}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{pmatrix} \xrightarrow{R_2(2)+R_3}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{pmatrix} \xrightarrow{R_3(-1)} \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3(-3)+R_1]{R_3(3)+R_2} \begin{pmatrix} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{pmatrix} \xrightarrow{R_2(-2)+R_1}$$

$$\begin{pmatrix} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{pmatrix} = (I_3 \mid A^{-1}).$$

By Theorem 4.4.1, the solution of the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & 3 \\ 5 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -1 \end{pmatrix}.$$

e.

$$(A \mid I_4) = \begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 9 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1(-2)+R_4]{R_1(1)+R_3}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 7 & -2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2(1)+R_3}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 7 & -2 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3(-2)+R_4}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & -2 & -2 & 1 \end{pmatrix} \xrightarrow[R_4(-2)+R_2,\ R_4(-1)+R_1]{R_4(-3)+R_3}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 5 & 2 & 2 & -1 \\ 0 & 1 & 1 & 0 & 8 & 5 & 4 & -2 \\ 0 & 0 & 1 & 0 & 13 & 7 & 7 & -3 \\ 0 & 0 & 0 & 1 & -4 & -2 & -2 & 1 \end{pmatrix} \xrightarrow[R_3(1)+R_1]{R_3(-1)+R_2}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 18 & 9 & 9 & -4 \\ 0 & 1 & 0 & 0 & -5 & -2 & -3 & 1 \\ 0 & 0 & 1 & 0 & 13 & 7 & 7 & -3 \\ 0 & 0 & 0 & 1 & -4 & -2 & -2 & 1 \end{array}\right) = (I_4 \,|\, A^{-1}).$$

By Theorem 4.4.1, the solution of the system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 18 & 9 & 9 & -4 \\ -5 & -2 & -3 & 1 \\ 13 & 7 & 7 & -3 \\ -4 & -2 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \\ 2 \end{pmatrix}.$$

2. **2.** Because $|A| = \begin{vmatrix} 1 & -1 \\ a & -3 \end{vmatrix} = -3 + a$, $|A| \neq 0$ if $a \neq 3$. By Corollary 4.4.1, the system has a unique solution for each $(b_1, b_2) \in \mathbb{R}^2$.

3. **3.**

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{pmatrix} \underset{R_1(3)+R_2}{\overset{R_1(1)+R_3}{\longrightarrow}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 12 \\ 0 & -2 & 5 \end{pmatrix} \overset{R_2\left(\frac{1}{2}\right)}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 6 \\ 0 & -2 & 5 \end{pmatrix} \overset{R_2(1)+R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 11 \end{pmatrix}.$$

Hence, $r(A) = 3$. By Corollary 4.4.1, the system has a unique solution for each $(b_1,\ b_2,\ b_3) \in \mathbb{R}^3$.

4. **4.**

a. Because

$$|A| = \begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix} = 11, \quad |A_1| = \begin{vmatrix} 6 & -3 \\ -1 & 5 \end{vmatrix} = 27,$$

$$|A_2| = \begin{vmatrix} 1 & 6 \\ 2 & -1 \end{vmatrix} = -13.$$

This implies

$$x = \frac{|A_1|}{|A|} = \frac{27}{11} \quad \text{and } y = \frac{|A_2|}{|A|} = -\frac{13}{11}.$$

Hence, $\left(\frac{27}{11},\ -\frac{13}{11}\right)$ is the solution of the system.

b.

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ -2 & 3 & 1 \\ -1 & -2 & 4 \end{vmatrix} \begin{matrix} 1 & 0 \\ -2 & 3 \\ -1 & -2 \end{matrix} = (12 + 8) - (-6 - 2) = 28.$$

$$|A_1| = \begin{vmatrix} 2 & 0 & 2 \\ 16 & 3 & 1 \\ 5 & -2 & 4 \end{vmatrix} \begin{matrix} 2 & 0 \\ 16 & 3 \\ 5 & -2 \end{matrix} = (24 - 64) - (30 - 4) = -66.$$

$$|A_2| = \begin{vmatrix} 1 & 2 & 2 \\ -2 & 16 & 1 \\ -1 & 5 & 4 \end{vmatrix} \begin{matrix} 1 & 2 \\ -2 & 16 \\ -1 & 5 \end{matrix} = (64 - 2 - 20) - (-32 + 5 - 16) = 85.$$

$$|A_3| = \begin{vmatrix} 1 & 0 & 2 \\ -2 & 3 & 16 \\ -1 & -2 & 5 \end{vmatrix} \begin{matrix} 1 & 0 \\ -2 & 3 \\ -1 & -2 \end{matrix} = (15 + 8) - (-6 - 32) = 61.$$

This implies

$$x_1 = \frac{|A_1|}{|A|} = -\frac{66}{28} = -\frac{33}{14}, \ x_2 = \frac{|A_2|}{|A|} = \frac{85}{28} \quad \text{and } x_3 = \frac{|A_3|}{|A|} = \frac{61}{28}.$$

Hence, $\left(-\frac{33}{14}, \frac{85}{28}, \frac{61}{28}\right)$ is the solution of the system.

c.

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$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & -1 \\ 0 & 1 & -2 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{matrix} = (-2-2)-(-1-4) = 1,$$

$$|A_1| = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ -2 & 1 & -2 \end{vmatrix} \begin{matrix} 1 & 1 \\ 0 & 1 \\ -2 & 1 \end{matrix} = (-2+2)-(2-1) = -1,$$

$$|A_2| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 0 & -2 & -2 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 0 \\ 0 & -2 \end{matrix} = 4-(2-4) = 6,$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & -2 \end{vmatrix} \begin{matrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{matrix} = (-2+2)-(-4) = 4.$$

This implies

$$x_1 = \frac{|A_1|}{|A|} = \frac{-1}{1} = -1, \quad x_2 = \frac{|A_2|}{|A|} = \frac{6}{1} = 6 \quad \text{and } x_3 = \frac{|A_3|}{|A|} = \frac{4}{1} = 4.$$

Hence, $(-1, 6, 4)$ is the solution of the system.

d. The coefficient matrix is $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$. Then

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix}, \text{ and}$$

$$A_3 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

By computation, we obtain $\left|A\right| = -3$, $\left|A_1\right| = -1$, $\left|A_2\right| = 8$, $\left|A_3\right| = -7$. By (4.4.3), $x_1 = \frac{1}{3}$, $x_2 = -\frac{8}{3}$, and $x_3 = \frac{7}{3}$. Hence, $\left(\frac{1}{3}, -\frac{8}{3}, \frac{7}{3}\right)$ is a solution of the system.

e.

$$|A| = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -2.$$

$$|A_1| = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{vmatrix} = 0 \text{ because columns 1 and 3 are the same.}$$

$$|A_2| = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{vmatrix} = 0 \text{ because columns 2 and 3 are the same.}$$

$$|A_3| = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix} = |A| = -2$$

$$|A_4| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 0 \text{ because columns 3 and 4 are the same.}$$

Hence, $x_1 = \frac{|A_1|}{|A|} = 0,\ x_2 = \frac{|A_2|}{|A|} = 0,\ x_3 = \frac{|A_3|}{|A|} = \frac{-2}{-2} = 1,\ x_4 = 0,$ and (0, 0, 1, 0) is a solution of the system.

Section 4.5

1. For each of the following systems, determine whether it has a unique solution, infinitely many solutions, or no solutions.

a. $$\begin{cases} x_1 - 2x_2 + 3x_3 = 6 \\ 2x_1 + x_2 + 4x_3 = 5 \\ -3x_1 + x_2 - 2x_3 = 3 \end{cases}$$

b. $$\begin{cases} x_1 + x_2 - 2x_3 = 3 \\ 2x_1 + 3x_2 - x_3 = -4 \\ -2x_1 - 3x_2x_3 = 4 \end{cases}$$

c. $$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 4x_1 + 7x_2 + 6x_3 = 17 \\ 2x_1 + 5x_2 + 12x_3 = 7 \end{cases}$$

d. $$\begin{cases} x_1 - 2x_2 - x_3 + 2x_4 = 0 \\ -x_2 - 2x_3 + x_4 = 0 \end{cases}$$

e. $\begin{cases} x_1 - x_2 + 3x_3 = 2 \\ -2x_1 + 2x_2 - 6x_3 = 5 \\ 2x_1 - 2x_2 + 6x_3 = 4 \end{cases}$

f. $\begin{cases} x_1 - x_2 - 3x_3 - 4x_4 = 1 \\ -x_1 + x_2 - x_3 = -1 \end{cases}$

2. For each of the following systems, determine if it has a unique solution.

a. $\begin{cases} 2x_1 - 3x_2 - x_3 - 5x_4 = -1 \\ -3x_1 + 6x_2 - 2x_3 + x_4 = 3 \\ x_1 + x_4 = -2 \end{cases}$

b. $\begin{cases} x_1 - 3x_2 + 2x_3 = -1 \\ x_1 + 4x_2 - x_3 = 2 \end{cases}$

3. For each of the following homogeneous systems, determine whether it has a unique solution or infinitely many solutions.

a. $\begin{cases} x_1 - 2x_2 - x_3 = 0 \\ -x_1 + 3x_2 + 2x_3 = 0 \\ -x_1 + x_2 = 0 \\ x_2 + x_3 = 0. \end{cases}$

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b. $$\begin{cases} x_1 + x_2 - 2x_3 = 0 \\ 2x_1 - x_2 + 4x_3 = 0 \\ -x_1 - 2x_2 - 3x_3 = 0. \end{cases}$$

c. $$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_2 - 4x_3 + x_3 = 0. \end{cases}$$

d. $$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ -2x_1 - 2x_2 + x_3 = 0 \\ -x_1 - 3x_2 + 3x_3 = 0 \\ 3x_1 + x_2 + x_3 = 0. \end{cases}$$

4. Consider the following system

$$\begin{cases} 6x_1 - 4x_2 = b_1 \\ 3x_1 - 2x_2 = b_2 \end{cases}$$

1. Find b_1 and b_2 such that the system is consistent.
2. Find b_1 and b_2 such that the system is inconsistent.
3. Determine whether the system with $\vec{b} = (4,\ 1)^T$ is inconsistent.

5. Determine if the following system is consistent for all $b_1,\ b_2,\ b_3 \in \mathbb{R}$.

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$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = b_1 \\ 2x_1 + 4x_2 - 4x_3 = b_2 \\ -3x_1 + 4x_3 - x_4 = b_3 \end{cases}$$

6. Find conditions on $b_1,\ b_2,\ b_3$ such that the following system is consistent.

$$\begin{cases} x_1 - x_2 + 2x_3 = b_1 \\ 2x_1 + 4x_2 - 4x_3 = b_2 \\ -3x_1 - 2x_3 = b_3 \end{cases}$$

7. Determine whether the following system has infinitely many solutions for all $b_1,\ b_2 \in \mathbb{R}$.

$$\begin{cases} x_1 - 2x_2 + 2x_3 = b_1 \\ -3x_1 + 3x_2 + x_3 = b_2 \end{cases}$$

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Solution

1. **1.**

a.

$$(A\,|\,\vec{b}) = \left(\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 2 & 1 & 4 & 5 \\ -3 & 1 & -2 & 3 \end{array}\right) \underset{R_1(3)+R_3}{\overset{R_1(-2)+R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 5 & -2 & -7 \\ 0 & -5 & 7 & 21 \end{array}\right)$$

$$\overset{R_2(1)+R_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 5 & -2 & -7 \\ 0 & 0 & 5 & 14 \end{array}\right).$$

Hence, $r(A) = r(A\,|\,\vec{b}) = 3$. By Theorem 4.5.1 (1), the system has a unique solution.

b.

$$(A\,|\,\vec{b}) = \left(\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 2 & 3 & -1 & -4 \\ -2 & -3 & -1 & 4 \end{array}\right) \underset{R_1(2)+R_3}{\overset{R_1(-2)+R_2}{\longrightarrow}}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 6 \\ 0 & 1 & 3 & -10 \\ 0 & -1 & -5 & -2 \end{array}\right) \overset{R_2(1)+R_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 6 \\ 0 & 1 & 3 & -10 \\ 0 & 0 & -2 & -12 \end{array}\right).$$

Hence, $r(A) = r(A\,|\,\vec{b}) = 3$. By Theorem 4.5.1 (1), the system has a unique solution.

c.

$$(A \mid \vec{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 4 & 7 & 6 & 17 \\ 2 & 5 & 12 & 7 \end{array}\right) \underset{R_1(-2)+R_3}{\overset{R_1(-4)+R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -6 & 1 \\ 0 & 1 & 6 & -1 \end{array}\right)$$

$$\overset{R_2(1)+R_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & 6 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence, $r(A) = r(A \mid \vec{b}) = 2 < 3,$. By Theorem 4.5.1 (2), the system has infinitely many solutions.

d. Because $n = 4$ and $m = 2$, $r(A) = r(A \mid \vec{0}) = r(A \mid \vec{b}) \leq m < n$. By Theorem 4.5.1 (2), the system has infinitely many solutions.

e.

$$(A \mid \vec{b}) = \left(\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ -2 & 2 & -6 & 5 \\ 2 & -2 & 6 & 4 \end{array}\right) \underset{R_1(-2)+R_3}{\overset{R(2)+R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Hence, $r(A) = 1$ and $r(A \mid \vec{b}) = 2$. By Theorem 4.5.1 (3), the system has no solutions.

f.

$$(A \mid \vec{b}) = \left(\begin{array}{cccc|c} 1 & -1 & -3 & -4 & 1 \\ -1 & 1 & -1 & 0 & -1 \end{array}\right) \underset{R_1(-2)+R_3}{\overset{R_1(1)+R_2}{\longrightarrow}}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & -3 & -4 & 1 \\ 0 & 0 & -4 & -4 & 0 \end{array}\right).$$

Hence, $r(A) = r(A\,|\,\vec{b}) = 2 < 4 = n.$ By Theorem 4.5.1 (2), the system has infinitely many solutions.

2. **2.**

a. Because $m = 3 < 4 = n,$ by Corollary 4.5.2, the system doesn't have a unique solution

b. Because $m = 3 < 3 = n,$ by Corollary 4.5.2, the system doesn't have a unique solution

3. **3.**

a. Because

$$A = \begin{pmatrix} 1 & -2 & -1 \\ -1 & 3 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \underset{R_1(1)+R_3}{\overset{R_1(1)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\underset{R_2(-1)+R_4}{\overset{R_2(1)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$r(A) = 2.$ Note that $m = 4$ and $n = 3.$ Hence, $n = 3 < 4 = m$ and $r(A) = 2 < n$ and by Theorem 4.5.2 (2), the system has infinitely many solutions.

b. Because

$$|A| = \begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 4 \\ -1 & -2 & -3 \end{vmatrix}\begin{matrix} 1 & 1 \\ 2 & -1 \\ -1 & -2 \end{matrix} = (3-4+8)-(-2-8-6) = 23 \neq 0,$$

it follows from Theorem 4.5.3 that the system has a unique solution.

c. Because $m = 2 < 3 = n$, it follows from Theorem 4.5.2 (*ii*) that the system has infinitely many solutions.

d. Because

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & -2 & 1 \\ -1 & -3 & 3 \\ 3 & 1 & 1 \end{pmatrix} \xrightarrow[R_1(-3)+R_4]{\substack{R_1(2)+R_2 \\ R_1(1)+R_3}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -4 & 5 \\ 0 & -4 & 5 \\ 0 & 4 & -5 \end{pmatrix}$$

$$\xrightarrow[R_2(1)+R_4]{R_2(-1)+R_3} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -4 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$r(A) = 2$. Hence, $n = 3 < 4 = m$ and $r(A) = 2 < 3 = n$. It follows from Theorem 4.5.2 (2) that the system has infinitely many solutions.

4. **4.**

1. If $b_2 - \frac{b_1}{2} = 0$, then $r(A) = r(A\,|\,\vec{b}) = 1 < 2$. It follows from Theorem 4.5.1 (2) that the system has infinitely many solutions and is consistent.

2. If $b_2 - \frac{b_1}{2} \neq 0,$ then $r(A) = 1$ and $r(A \mid \vec{b}) = 2$. Hence, $r(A) < r(A \mid \vec{b})$. By Theorem 4.5.1 (3), the system has no solutions and is inconsistent.
3. By $\vec{b} = (4,\ 1)^T,$ we have $b_1 = 4$ and $b_1 \neq 2b_2,$ it follows from result (2) that the system is inconsistent.

5. **5.**

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 4 & -4 & 0 \\ -3 & 0 & 4 & -1 \end{pmatrix} \xrightarrow[R_1(3)+R_3]{R_1(-2)+R_2} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 6 & -8 & 0 \\ 0 & -3 & 10 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2(\frac{1}{2})+R_3} \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 6 & -8 & 0 \\ 0 & 0 & 6 & 2 \end{pmatrix}.$$

Because $r(A) = 3 = m,$ by Theorem 4.5.4 (1), the system is consistent for any $b_1,\ b_2,\ b_3.$

6. **6.**

$$(A\,|\,\vec{b}) = \left(\begin{array}{ccc|c} 1 & -1 & 2 & b_1 \\ 2 & 4 & -4 & b_2 \\ -3 & 0 & -2 & b_3 \end{array}\right) \underset{R_1(3)+R_3}{\overset{R_1(-2)+R_2}{\longrightarrow}}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & b_1 \\ 0 & 6 & -8 & b_2 - 2b_1 \\ 0 & -3 & 4 & b_3 + 3b_1 \end{array}\right) \overset{R_2(\frac{1}{2})+R_3}{\longrightarrow}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & b_1 \\ 0 & 6 & -8 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + \frac{1}{2}(b_2 + 2b_1) \end{array}\right).$$

If $b_3 + \frac{1}{2}(b_2 + 2b_1) = 0,$ then the system is consistent.

7. **7.**

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 4 & -4 \end{pmatrix} \overset{R_1(-2)+R_2}{\longrightarrow} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 6 & -8 \end{pmatrix}.$$

Because $r(A) = 2 = m$ and $m = 2 < 3 = n,$ by Theorem 4.5.5, the system has infinitely many solutions for any $b_1,\ b_2 \in \mathbb{R}$.

Section 4.6

1. Let $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\ \overrightarrow{a_4}\right\}$ and $A = (\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\ \overrightarrow{a_4})$, where

$$\overrightarrow{a_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \overrightarrow{a_2} = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}, \quad \overrightarrow{a_3} = \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \quad \overrightarrow{a_1} = \begin{pmatrix} -1 \\ -4 \\ -5 \end{pmatrix},$$

$$\vec{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}.$$

1. Determine whether $A\vec{X} = \vec{b}$ is consistent, where $\vec{X} = (x_1,\ x_2,\ x_3,\ x_4)^T$.
2. Determine whether the vector $\vec{b}$ is a linear combination of S.
3. Determine whether the vector $\vec{b} \in \operatorname{span} S$.

2. Let $\overrightarrow{a_1} = (1,\ -2,\ 2)^T,\ \overrightarrow{a_2} = (-1,\ 0,\ -10)^T,\ \overrightarrow{a_3} = (1,\ -1,\ 6)^T$, and $\vec{b} = (b_1,\ b_2,\ b_3)^T$.

1. Find conditions on $b_1,\ b_2,\ b_3$ such that $\vec{b} \in \operatorname{span}\left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$.
2. Find conditions on $b_1,\ b_2,\ b_3$ such that $\vec{b} \notin \operatorname{span}\left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$.
3. Find a vector $\vec{b} = (b_1,\ b_2,\ b_3)^T$ such that $\vec{b} \notin \operatorname{span}\left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$.

3. Let $S = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \end{pmatrix} \right\}$.

1. Determine whether $\text{span}\, S = \mathbb{R}^2$.
2. Determine whether $\vec{b} = (4,\ 2)^T$ is in span S.
3. Determine whether $\vec{b} = (6,\ 1)^T$ is in span S.

4. Determine whether $\text{span}\, S = \mathbb{R}^2$, where

i. $S = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix} \right\}$.

ii. $S = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -5 \\ 5 \end{pmatrix} \right\}$.

iii. $S = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$.

5. Determine whether $\text{span}\, S = \mathbb{R}^3$, where

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \right\}.$$

6. Determine whether $\text{span}\, S = \mathbb{R}^4$, where

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Solution

1. **1.** We find *r*(*A*) and $r(A|\overrightarrow{b})$.

$$(A|\overrightarrow{b}) = \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 2 \\ 2 & 1 & 4 & -4 & 2 \\ 3 & -1 & 7 & -5 & 4 \end{array}\right) \xrightarrow[R_1(-3)+R_3]{R_1(-2)+R_2}$$
$$\left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 2 \\ 0 & 5 & -2 & -2 & -2 \\ 0 & 5 & -2 & -2 & -2 \end{array}\right) \xrightarrow{R_2(-1)+R_3} \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 2 \\ 0 & 5 & -2 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Hence, $r(A) = r(A|\overrightarrow{b}) = 2.$ By Theorem 4.6.1, we have the following conclusions.

1. The system $A\overrightarrow{X} = \overrightarrow{b}$ is consistent.
2. The vector $\overrightarrow{b}$ is a linear combination of *S*.
3. The vector $\overrightarrow{b} \in \operatorname{span} S.$

2. **2.** Let $\overrightarrow{b} = (b_1, b_2, b_3)$ and $A = (\overrightarrow{a_1}\,\overrightarrow{a_2}\,\overrightarrow{a_3}).$

$$(A|\overrightarrow{b}) = \left(\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ -2 & 0 & -1 & b_2 \\ 2 & -10 & 6 & b_3 \end{array}\right) \xrightarrow[R_1(-2)+R_3]{R_1(2)+R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 0 & -2 & 1 & b_2+2b_1 \\ 0 & -8 & 4 & b_3-2b_1 \end{array}\right) \xrightarrow{R_2(-4)+R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 0 & -2 & 1 & b_2+2b_1 \\ 0 & 0 & 0 & b_3-2b_1-4(b_2+2b_1) \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -1 & 1 & b_1 \\ 0 & -2 & 1 & b_2+2b_1 \\ 0 & 0 & 0 & b_3-4b_2-10b_1 \end{array}\right).$$

1. If $b_3 - 4b_2 - 10b_1 = 0,$ then $r(A) = (A|\overrightarrow{b}).$ By Theorem 4.6.1, $\overrightarrow{b} \in \operatorname{span}\left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}\right\}.$
2. If $b_3 - 4b_2 - 10b_1 \neq 0,$ then $r(A) < (A|\overrightarrow{b}).$ By Theorem 4.6.1, $\overrightarrow{b} \in \operatorname{span}\left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}\right\}.$
3. Let $b_1 = b_2 = 0$ and $b_3 = 1.$ Then, $b_3 - 4b_2 - 10b_1 \neq 0.$ Hence, $\overrightarrow{b} = (0, 0, 1) \notin \operatorname{span}\left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}\right\}.$

3. **3.**

   1. Because

$$A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \xrightarrow{R_1(\frac{4}{2})} \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \xrightarrow{R_1(-1)+R_2} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix},$$

$r(A) = 1 < 2 = m.$ By Theorem 4.6.2, $\operatorname{span} S \neq \mathbb{R}^2.$

2. $$(A|\overrightarrow{b}) = \left(\begin{array}{cc|c} 2 & -4 & b_1 \\ 1 & -2 & b_2 \end{array}\right) \xrightarrow{R_1(\frac{1}{2})} \left(\begin{array}{cc|c} 1 & -2 & \frac{b_1}{2} \\ 1 & -2 & b_2 \end{array}\right) \xrightarrow{R_1(-1)+R_2} \left(\begin{array}{cc|c} 1 & -2 & \frac{b_1}{2} \\ 0 & 0 & b_2 - \frac{b_1}{2} \end{array}\right).$$

1. If $b_2 - \frac{b_1}{2} \neq 0$, then $r(A) = (A|\overrightarrow{b})$ and by Theorem 4.6.1, $\overrightarrow{b} \in \operatorname{span} S$. Hence, if $\overrightarrow{b} = (b_1, b_2) = (4, 2)$, then $b_2 - \frac{b_1}{2} = 2 - \frac{4}{2} = 0$ and $\overrightarrow{b} = (4, 2) \in \operatorname{span} S$.
2. If $b_2 - \frac{b_1}{2} \neq 0$, then $r(A) < (A|\overrightarrow{b})$ and by Theorem 4.6.1, $\overrightarrow{b} \notin \operatorname{span} S$. Hence, if $\overrightarrow{b} = (b_1, b_2) = (6, 1)$ then $b_2 - \frac{b_1}{2} = 1 - \frac{6}{2} = -2 \neq 0$ and $\overrightarrow{b} = (6, 1) \notin \operatorname{span} S$.

4. **4.**

i. Let $A = \begin{pmatrix} -1 & 1 & 1 & 3 \\ 0 & 0 & -2 & 5 \end{pmatrix}$. Then $m = 2$ and $n = 4$. Because $r(A) = 2 = m < n = 4,$ by Theorem 4.6.2, $\operatorname{span} S = \mathbb{R}^2$.

ii. Because $A = \begin{pmatrix} 1 & 2 & -5 \\ -1 & -2 & 5 \end{pmatrix} \xrightarrow{R_1(1)+R_2} \begin{pmatrix} 1 & 2 & -5 \\ 0 & 0 & 0 \end{pmatrix}$, $r(A) = 1 < 2 = m.$ By Theorem 4.6.2, $\operatorname{span} S \neq \mathbb{R}^2$.

iii. Let $A = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \end{pmatrix}$. Then $m = 2$ and $n = 4$. Because $r(A) = 2 = m < n = 3,$ by Theorem 4.6.2, $\operatorname{span} S = \mathbb{R}^2$.

5. **5.** $$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 0 & 5 \end{pmatrix} \xrightarrow[R_1(-1)+R_3]{R_1(-2)+R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & -2 & 5 \end{pmatrix} \xrightarrow{R_2(2)+R_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence, $r(A) = 3.$ By Theorem 4.6.2, $\operatorname{span} S = \mathbb{R}^3$.

6. **6.** Because $m = 4$ and $n = 3,$ by Theorem 4.6.2, $\operatorname{span} S \neq \mathbb{R}^4$.

A.5 Linear transformations

Section 5.1

1. For each of the following matrices,

$$A_1 = \begin{pmatrix} 2 & -2 & 4 & -1 \\ -1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} -2 & 2 & 1 \\ 1 & 1 & -3 \\ 1 & 0 & -2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & 1 \\ 3 & -2 & 3 \\ 1 & 2 & -1 \end{pmatrix} \qquad A_4 = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & -2 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

determine the domain and codomain of $T_{A_i}, i = 1, 2, 3, 4.$ Moreover, compute $T_{A_3}(0, 1, 1, 2), T_{A_2}(1, -2, 1), T_{A_3}(-1, 0, 1), T_{A_4}(-1, 1, 1).$

2. Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$. Find $T_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $T_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
3. Let $A = \begin{pmatrix} -1 & 2 & 1 \\ 3 & 0 & 6 \end{pmatrix}$. Find $T_A(\overrightarrow{e_1}), T_A(\overrightarrow{e_2}), T_A(\overrightarrow{e_3})$, where $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ are the standard vectors in $\mathbb{R}^3$.
4. For each of the following linear transformations, express it in **(5.1.1)** .
   a. $T(x_1, x_2) = (x_1 - x_2, 2x_1 + x_2, 5x_1 - 3x_2).$
   b. $T(x_1, x_2, x_3, x_4) = (6x_1 - x_2 - x_3 + x_4, x_2 + x_3 - 2x_4, 3x_3 + x_4, 5x_4).$
   c. $T(x_1, x_2, x_3) = (4x_1 - x_2 + 2x_3, 3x_1 - x_3, 2x_1 + x_2 + 5x_3).$
   d. $T(x_1, x_2, x_3) = (8x_1 - x_1 + x_3, x_2 - x_3, 2x_1 + x_2, x_2 + 3x_3).$

5. Let $\overrightarrow{X} = (2, -2, -2, 1), \overrightarrow{Y} = (1, -2, 2, 3)$ and

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}.$$

Compute $T_A(\overrightarrow{X} - 3\overrightarrow{Y})$.

6. Let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be linear operators such that $T(1,2,3) = (1,-4)$ and $S(1,2,3) = (4,9)$. Compute $(5T + 2S)(1,2,3)$.
7. Let $S, T : \mathbb{R}^3 \to \mathbb{R}^4$ be linear transformations such that $T(1,2,3) = (1,1,0,-1)$ and $S(1,2,3) = (2,-1,2,2)$. Compute $(3T - 2S)(1,2,3)$.
8. Compute $(T + 2S)(x_1, x_2, x_3, x_4)$, where

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3, x_1 - x_2 + 2x_3 - x_4),$$
$$S(x_1, x_2, x_3, x_4) = (x_2 - x_3 + 2x_4, x_1 + x_3 - x_4).$$

9. For each pair of linear transformations T_A and T_B, find $T_B T_A$.
   a. $T_A(x_1, x_2) = (x_1 + x_2, x_1 - x_2), T_B(x_1, x_2) = (x_1 - x_2, 2x_1 - 3x_2, 3x_1 + x_2)$.
   b. $T_A(x_1, x_2, x_3) = (-x_1 - x_2 + x_3, x_1 + x_2), T_B(x_1, x_2) = (x_1 - x_2, 2x_1 + 3x_2)$.
   c. $T_A(x_1, x_2, x_3) = (-x_1 + x_2 - x_3, x_1 - 2x_2, x_3), T_B(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + 3x_2 + x_3)$.

Solution

1. **1.** Because the size of A_1 is $3 \times 4,$ the domain and codomain of T_{A_1} are $\mathbb{R}^4$ and $\mathbb{R}^3,$ respectively.

$$T_{A_1}\begin{pmatrix}0\\1\\1\\2\end{pmatrix} = \begin{pmatrix}2 & -1 & 4 & -1\\-1 & 2 & 3 & 0\\0 & 1 & 2 & -1\end{pmatrix}\begin{pmatrix}0\\1\\1\\2\end{pmatrix} = \begin{pmatrix}0\\5\\1\end{pmatrix}.$$

Because the size of A_2 is $3 \times 3,$ the domain and codomain of T_{A_2} are $\mathbb{R}^3$.

$$T_{A_2}\begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}-2 & 2 & 1\\1 & 1 & -3\\1 & 0 & -2\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}-5\\-4\\-1\end{pmatrix}.$$

Because the size of A_3 is $4 \times 3,$ the domain and codomain of T_{A_3} are $\mathbb{R}^3$ and $\mathbb{R}^4,$ respectively.

$$T_{A_3}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1 & -2 & 0\\-2 & 3 & 1\\3 & -2 & 3\\1 & 2 & -1\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\0\\-2\end{pmatrix}.$$

Because the size of A_4 is $4 \times 3,$ the domain and codomain of T_{A_4} are $\mathbb{R}^3$ and $\mathbb{R}^4,$ respectively.

$$T_{A_4}\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}-1 & 1 & 1\\2 & 0 & -2\\-1 & -1 & 1\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\-4\\1\\1\end{pmatrix}.$$

2. **2.**

$$\begin{aligned} T\begin{pmatrix}1\\0\end{pmatrix} &= \begin{pmatrix}1 & 2\\-1 & 3\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix},\\ T\begin{pmatrix}0\\1\end{pmatrix} &= \begin{pmatrix}1 & 2\\-1 & 3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}. \end{aligned}$$

3. **3.**

$$\begin{aligned} T(\vec{e}_1) &= \begin{pmatrix}-1 & 2 & 1\\3 & 0 & 6\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix},\\ T(\vec{e}_2) &= \begin{pmatrix}-1 & 2 & 1\\3 & 0 & 6\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix},\\ T(\vec{e}_3) &= \begin{pmatrix}-1 & 2 & 1\\3 & 0 & 6\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\6\end{pmatrix}. \end{aligned}$$

4. **4.**

a. $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

b. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 & -1 & -1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$.

c. $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 \\ 3 & 0 & -1 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

d. $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

5. **5.** By Proposition 5.1.1, we have

$$T(\vec{X}+3\vec{Y}) = T(\vec{X}) + 3T(\vec{Y})$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 6 \\ -6 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 36 \\ -3 \end{pmatrix}.$$

6. **6.** $(5T+2S)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5T\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2S\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5\begin{pmatrix} 1 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 13 \\ -2 \end{pmatrix}.$

7. **7.**
$(3T-2S)(1,\ 2,\ 3) = 3T(1,\ 2,\ 3) - 2S(1,\ 2,\ 3) = 3(1,\ 1,\ 0,\ -1) - 2(2,\ -1,\ 2,\ 2) = (-1,\ 5,\ -4,\ -7).$

8. **8.** Let $\vec{a} = (T+2S)(x_1,\ x_2,\ x_3,\ x_4)$. By Definition 5.1.3, we have

$$\begin{aligned} \vec{a} &= (x_1 - x_2 + x_3,\ x_1 - x_2 + 2x_3 - x_4) + 2(x_2 - x_3 + 2x_4,\ x_1 + x_3 - x_4) \\ &= (x_1 + x_2 - x_3 + 4x_4,\ 3x_1 - x_2 + 4x_3 - 3x_4). \end{aligned}$$

9. **9.**

a. Because

$$(ST)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & 5 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

b.

$$(ST)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} -1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

c.

$$(ST)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 3 & 0 \\ 1 & -4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Section 5.2

1. For each of the following linear transformations, determine whether it is onto.

a. $T(x_1, x_2, x_3) = (2x_1 - 2x_2 + x_3, 3x_1 - 2x_2 + x_3, x_1 + 2x_2 - 4x_3, 2x_1 - 3x_2 + 2x_3)$.

b. $T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, \ -x_1 - 2x_2, \ -x_1 + x_2 + 2x_3)$.

c. $T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + x_4, 3x_1 + 2x_4, x_1 - 2x_3 + 3x_4)$.

d. $T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3, x_1 - x_4, 2x_1 - x_2 + x_3 - x_4)$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x, y) = (x - y, 2x - 2y).$$

3. Show that T is not onto and find a vector $\vec{b} \in \mathbb{R}^2$ such that $\vec{b} \notin T(\mathbb{R}^2)$.

4. For each of the following linear transformations, determine whether it is one to one.

a. $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, \ -x_1 - x_2, x_1 + x_2 - x_3, \ -2x_1 - x_2 + x_3)$.

b. $T(x_1, x_2, x_3) = (-x_1 + x_2 + x_3, x_1 - x_2 - 2x_3, \ +x_1 + x_2)$.

c. $T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3, 2x_1 + x_2 + x_3, x_1 + x_2 + 2x_4)$.

d. $T(x_1, x_2, x_3) = (x_1 - x_2 + 3x_3, x_1 - 2x_3, 2x_1 - x_2 + x_3)$.

5. For each of the following linear operators, determine whether it is invertible. If so, find its inverse.

a. $T(x_1, x_2) = (x_1 + x_2, 6x_1 + 7x_2)$.

b. $T(x_1, x_2) = (3x_1 - 9x_2, 4x_1 - 12x_2)$.

c. $T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3, x_2 - x_3, x_3)$.

d. $T(x_1, x_2, x_3) = (x_1 + 3x_2, 3x_1 + 7x_2 - 4x_3, x_1 + 5x_2 + 5x_3)$.

Solution

1. **1.**

   a. The standard matrix *A* of *T* is a 4×3 matrix, By Theorem 5.2.3, *T* is not onto because $r(A) \leq 3 = \min\{4, 3\} < 4 = m$.

   b. Because $A = \left| \begin{matrix} 1 & 2 & -1 \\ -1 & -2 & 0 \\ -1 & 1 & 2 \end{matrix} = 3 \neq 0, \right|$ by Theorem 5.2.5, *T* is onto.

   c. Because

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 1 & 0 & -2 & 3 \end{pmatrix} \underset{R_1(-1)+R_3}{\overset{R_1(-3)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 6 & 0 & -1 \\ 0 & 2 & -2 & 2 \end{pmatrix} \overset{R_{2,3}}{\longrightarrow}$$

$$\begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 6 & 0 & -1 \end{pmatrix} \overset{R_2(-3)+R_3}{\longrightarrow} \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & 6 & -7 \end{pmatrix},$$

   $r(A) = 3 = m$. By Theorem 5.2.3, *T* is onto.

   d. Because

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 2 & -1 & 1 & -1 \end{pmatrix} \underset{R_1(-2)+R_3}{\overset{R_1(-1)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

$$\overset{R_2(-1)+R_3}{\longrightarrow} \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

2. **2.** Because $|A| = \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = 0,$ by Theorem 5.2.5, *T* is not onto. Consider the system

(A.5.1)

$$\begin{cases} x - y = b_1 \\ 2x - 2y = b_2. \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & -1 & b_1 \\ 2 & -2 & b_2 \end{array}\right) \overset{R_1(-2)+R_2}{\longrightarrow} \left(\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right).$$

If $b_2 - 2b_1 \neq 0,$ the system has no solutions. We choose

$$\vec{b} = (b_1,\ b_2) = (1,\ 1).$$

3. **3.**

   a. Because

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & -1 & 0 \\ 1 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix} \xrightarrow[\substack{R_1(-1)+R_3 \\ R_1(2)+R_4}]{R_1(1)+R_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & -3 & 5 \end{pmatrix}$$

$$\xrightarrow[R_2\left(-\frac{3}{2}\right)+R_4]{R_1(1)+R_3} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{R_3(2)+R_3} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$r(A) = 3 = n.$ By Theorem 5.2.4, T is one to one.

b. Because $|A| = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = -2 \neq 0,$ by Theorem 5.2.5, T is one to one.

c. The standard matrix of T is a 3×4 matrix. By Theorem 5.2.4, T is not one to one.

d. Because $|A| = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 0 & -2 \\ 2 & -1 & 1 \end{vmatrix} = 0,$ by Theorem 5.2.5, T is not one to one.

4. **4.**

a. Because $|A| = \begin{vmatrix} 1 & 1 \\ 6 & 7 \end{vmatrix} = 1 \neq 0,$ T is invertible. Because

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} 7 & -1 \\ -6 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ -6 & 1 \end{pmatrix},$$

$$T^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ -6 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

b. Because $|A| = \begin{vmatrix} 3 & -9 \\ 4 & -12 \end{vmatrix} = 0$, by Theorem 5.2.5, T is not invertible.

c. Because $|A| = \begin{vmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$, T is invertible. Because

$$(A \mid I) = \left(\begin{array}{cccccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \underset{R_3(-2)+R_1}{\overset{R_3(1)+R_2}{\longrightarrow}} \left(\begin{array}{cccccc} 1 & 3 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\overset{R_2(-3)+R_1}{\longrightarrow} \left(\begin{array}{cccccc} 1 & 0 & 0 & 1 & -3 & -5 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = (I_3 \mid A^{-1}),$$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -3 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

d. Because $|A| = \begin{vmatrix} 1 & 3 & 0 \\ 3 & 7 & -4 \\ 1 & 5 & 5 \end{vmatrix} = -2 \neq 0,\ T$ is invertible.

$$(A \mid I) = \begin{pmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 3 & 7 & -4 & 0 & 1 & 0 \\ 1 & 5 & 5 & 0 & 0 & 1 \end{pmatrix} \underset{R_1(-1)+R_3}{\overset{R_1(-3)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -2 & -4 & -3 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{pmatrix}$$

$$\overset{R_2(1)+R_3}{\longrightarrow} \begin{pmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -2 & -4 & -3 & 1 & 0 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{pmatrix} \overset{R_2\left(-\frac{1}{2}\right)}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{pmatrix} \overset{R_3(-2)+R_2}{\longrightarrow} \begin{pmatrix} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{19}{2} & -\frac{5}{2} & -2 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{pmatrix}$$

$$\overset{R_2(-3)+R_1}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & -\frac{55}{2} & \frac{15}{2} & 6 \\ 0 & 1 & 0 & \frac{19}{2} & -\frac{5}{2} & -2 \\ 0 & 0 & 1 & -4 & 1 & 1 \end{pmatrix} = (I_3 \mid A^{-1}).$$

Hence, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{55}{2} & \frac{15}{2} & 6 \\ \frac{19}{2} & -\frac{5}{2} & -2 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Section 5.3

1. In **Theorem 5.3.1**, if $\theta = \frac{\pi}{6}, \frac{\pi}{3}$, find $T\begin{pmatrix} 4 \\ -4 \end{pmatrix}$.
2. Find the linear operator $T = T_2T_1$ on $\mathbb{R}^2$, where T_1 is a dilation with factor $k = 2$ on $\mathbb{R}^2$ and T_2 is a projection on the line $y = x$.
3.
    1. Find the linear operator on $\mathbb{R}^2$ that contracts with factor $\frac{1}{2}$ and then reflects about the line $y = x$.
    2. Find the linear operator on $\mathbb{R}^2$ that reflects about the *y*-axis and then reflects about the *x*-axis, and the linear operator on $\mathbb{R}^2$ that reflects about the *x*-axis and then reflects about the *y*-axis.
4. Find the rotation operator with the rotation angle of $\frac{\pi}{4}$ and the image of $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ under the operator.
5. Find the linear operator on $\mathbb{R}^2$ that rotates by $\frac{\pi}{3}$ radians about the origin.
6. Find the linear operator on $\mathbb{R}^3$ that rotates by $\frac{\pi}{4}$ radians about the *z*-axis and then reflects about the *yz*-plane.
7. Prove **Theorem 5.3.1** by using **Theorem 5.1.1**.
8. Prove **Theorem 5.3.2** by using a method similar to the proof of **Theorem 5.3.1**.
9. Prove **Theorem 5.3.2** by using **Theorem 5.1.1**.

Solution

1. **1.** By Theorem 5.3.1,

$$T\begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} \cos^2 \frac{\pi}{6} & \sin \frac{\pi}{6} \cos \frac{\pi}{6} \\ \cos \frac{\pi}{6} \sin \frac{\pi}{6} & \sin^2 \frac{\pi}{6} \end{pmatrix}\begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 3-\sqrt{3} \\ \sqrt{3}-1 \end{pmatrix}.$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$$T\begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} \cos^2 \frac{\pi}{3} & \sin \frac{\pi}{3} \cos \frac{\pi}{3} \\ \cos \frac{\pi}{3} \sin \frac{\pi}{3} & \sin^2 \frac{\pi}{3} \end{pmatrix}\begin{pmatrix} 4 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 1+\sqrt{3} \\ \sqrt{3}+3 \end{pmatrix}.$$

2. **2.** By Definition 5.1.2, $T_1(x,\ y) = (2x,\ 2y)$ for $(x,\ y) \in \mathbb{R}^2$ and by Corollary 5.3.1 (3), $T_2(x,\ y) = \left(\frac{x+y}{2},\ \frac{x+y}{2}\right)$. It follows that

$$T(x,\ y) = T_2(T_1(x,\ y)) = T_2(2x,\ 2y) = \left(\frac{2x+2y}{2},\ \frac{2x+2y}{2}\right) = (x+y,\ x+y).$$

3. **3.** By Definition 5.1.2 (2), we have

$$T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = T_2T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

4. **4.**

1. By (5.3.6) and $\theta = \frac{\pi}{4}$,

$$T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{2}\\0\end{pmatrix}.$$

2. Because the reflection operator about the *y*-axis is $T_1(x,\ y) = (-x,\ y)$ and the reflection operator about the *x*-axis is $T_2(x,\ y) = (x,\ -y),$ we have

$$T_1T_2(x,\ y) = T_1(T_2(x,\ y)) = T_1(x,\ -y) = (-x,\ -y)$$

and

$$T_2T_1(x,\ y) = T_2(-x,\ y) = (-x,\ -y).$$

5. **5.**

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

6. **6.**

$$T_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$T_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and}$$

$$T_2T_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

7. **7.** $T(\vec{e}1) = T(1,\ 0)^T = (b_1,\ b_2)^T.$ Then $b_1 = r\cos\theta$ and $b_2 = r\sin\theta$ (see Figure 6 (I)). Because $r = \cos\theta,\ b_1 = \cos^2\theta,\ b_2 = \cos\theta\sin\theta.$ Hence,

$$T(\vec{e}_1) = (\cos^2\theta,\ \cos\theta\sin\theta)^T.$$

Let $T(\vec{e}_2) = T(0,\ 1)^T = (u_1,\ u_2)^T.$ Then $u_1 = r\cos\theta$ and $u_2 = r\sin\theta$ (see Figure 6 (II)). Because $r = \sin\theta,\ u_1 = \sin\theta\cos\theta,$ and $u_2 = \sin^2\theta,$

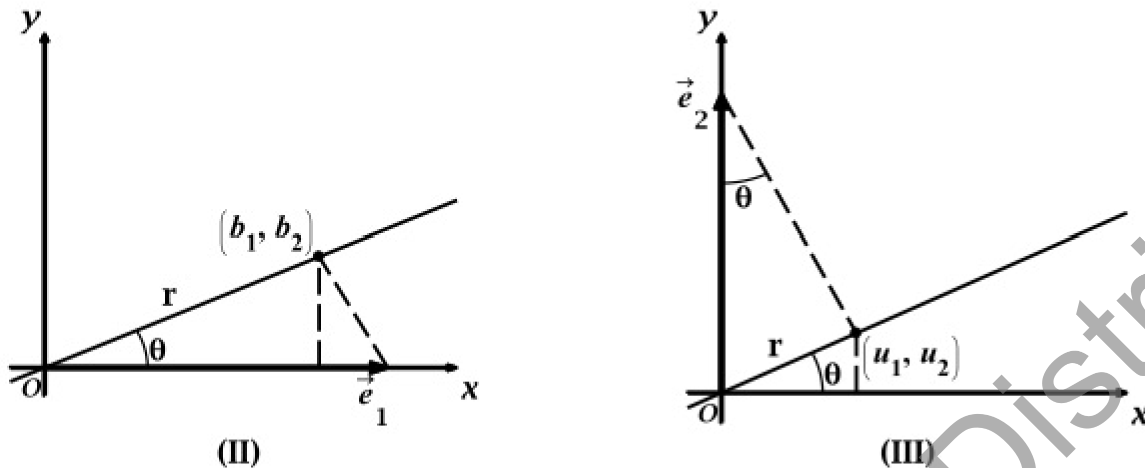
$$T(\vec{e}_2) = (\sin\theta\cos\theta,\ \sin^2\theta)^T.$$

By Theorem 5.1.1, the standard matrix for *T* is

$$(T(\vec{e}_1),\ T(\vec{e}_2)) = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

and (5.3.1) holds.

Figure A.1: (I), (II)



8. **8.**

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} r\cos(\theta+\beta) \\ r\sin(\theta+\beta) \end{pmatrix} = \begin{pmatrix} r\cos\beta\cos\theta - r\sin\beta\sin\theta \\ r\cos\beta\sin\theta + r\sin\beta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r\cos\beta \\ r\sin\beta \end{pmatrix}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\beta-\theta) \\ r\sin(\beta-\theta) \end{pmatrix} = \begin{pmatrix} r\cos\beta\cos\theta + r\sin\beta\sin\theta \\ r\cos\beta\sin\theta - r\sin\beta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} r\cos\beta \\ r\sin\beta \end{pmatrix}.$$

Solving the above equation, we have

$$\begin{pmatrix} r\cos\beta \\ r\sin\beta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{pmatrix} \mathrm{w1} \\ \mathrm{w2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \mathrm{r}\cos\beta \\ \mathrm{r}\sin\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin 2\,\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and (5.3.5) holds.

9. **9.** By the first figure in 8, since $OB = OA = 1,$ we have

$$(w_1,\ w_2) = T(1,\ 0) = (\cos 2\theta,\ \sin 2\theta).$$

A.6 Lines and planes in $\mathbb{R}^3$

Section 5.4

1. Find $\vec{a} \times \vec{b}$ and verify $(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp \vec{b}$.
    1. $\vec{a} = (1,\ 2,\ 3);\ \vec{b} = (1,\ 0,\ 1),$
    2. $\vec{a} = (1,\ 0,\ -3),\ \vec{b} = (-1,\ 0,\ -2),$
    3. $\vec{a} = (0,\ 2,\ 1);\ \vec{b} = (-5,\ 0,\ 1),$
    4. $\vec{a} = (1,\ 1,\ -1);\ \vec{b} = (-1,\ 1,\ 0),$

2. Let $\vec{a} = (1,\ -2,\ 3),\ \vec{b} = (-1,\ -4,\ 0),$ and $\vec{c} = (2,\ 3,\ 1)$. Find the scalar triple product of $\vec{a},\ \vec{b},\ \vec{c}$.

3. Find the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.
    1. $\vec{a} = (1,\ 0,\ 1),\ \vec{b} = (-1,\ 0,\ 1);$
    2. $\vec{a} = (1,\ 0,\ 0),\ \vec{b} = (-1,\ 1,\ -2).$

4. Find the volume of the parallelpiped determined by $\vec{a},\ \vec{b},$ and $\vec{c}$.
    1. $\vec{a} = (1,\ -1,\ 0),\ \vec{b} = (-1,\ 0,\ 2),\ \vec{c} = (0,\ -1,\ -1).$
    2. $\vec{a} = (-1,\ -1,\ 0),\ \vec{b} = (-1,\ 1,\ -2),\ \vec{c} = (-1,\ 1,\ 1).$

Solution

1. **1.**

1. $\vec{a} \times \vec{b} = \left(\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}, \ - \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}, \ \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \right) = (2, \ 2, \ -2).$

   Because

   $$(\vec{a} \times \vec{b}) \cdot \vec{a} = (2)(1) + (2)(2) + (-2)(3) = 0,$$

   and

   $$(\vec{a} \times \vec{b}) \cdot \vec{a} = (2)(1) + (2)(0) + (-2)(1) = 0,$$

   $(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp \vec{b}$.

2. $(\vec{a} \times \vec{b}) = \left(\begin{vmatrix} 0 & -3 \\ 0 & -2 \end{vmatrix}, \ - \begin{vmatrix} 1 & -3 \\ -1 & -2 \end{vmatrix}, \ \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} \right) = (0, \ 5, \ 0).$

   Because

   $$(\vec{a} \times \vec{b}) \cdot \vec{a} = (0)(1) + (5)(0) + (0)(-3) = 0$$

   and

   $$(\vec{a} \times \vec{b}) \cdot \vec{a} = (0)(-1) + (5)(0) + (0)(-2) = 0,$$

   $(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp \vec{b}$.

3. $\vec{a} \times \vec{b} = \left(\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}, \ - \begin{vmatrix} 0 & 1 \\ -5 & 1 \end{vmatrix}, \ \begin{vmatrix} 0 & 2 \\ -5 & 0 \end{vmatrix} \right) = (2, \ -5, \ 10).$

Because

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (2)(0) + (-5)(2) + (10)(1) = 0$$

and

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (2)(-5) + (-5)(0) + (10)(1) = 0,$$

$(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp \vec{b}$.

4. $\vec{a} \times \vec{b} = \left(\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}, \ - \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix}, \ \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \right) = (1, \ 1, \ 2).$

Because

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (1)(1) + (1)(1) + (2)(-1) = 0$$

and

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (1)(-1) + (1)(1) + (2)(0) = 0,$$

$(\vec{a} \times \vec{b}) \perp \vec{a}$ and $(\vec{a} \times \vec{b}) \perp \vec{b}$.

2. **2.** By (6.1.3), we have

$$\vec{a} \times \vec{b} = \left(\begin{vmatrix} -2 & 3 \\ -4 & 0 \end{vmatrix}, \ - \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}, \ \begin{vmatrix} 1 & -2 \\ -1 & -4 \end{vmatrix} \right) = (12, \ -3, \ -6).$$

Hence,

$$\vec{a} \cdot (\vec{b}) \times \vec{c}) = 24 - 9 - 6 = 9.$$

3. **3.**
    1. Because

$$\vec{a} \times \vec{b} = \left(\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}, \; - \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}, \; \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} \right) = (0, \; -2, \; 0),$$

    by Theorem 6.1.1, the area equals

$$\left\| \vec{a} \times \vec{b} \, \right\| = \sqrt{0^2 + 2^2 + 1^2} = \sqrt{5}.$$

4. **4.**
    1. Because $\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & -1 \end{vmatrix} = 3, \; V = |3| = 3.$
    2. Because $\begin{vmatrix} -1 & -1 & 0 \\ -1 & 1 & -2 \\ -1 & 1 & 1 \end{vmatrix} = -6, \; V = |1-6| = 6.$

Section 5.5

1. a. Find the parametric equation of the line passing through the point $P_0(-1, \; 2, \; 0)$ that is parallel to the vector $\vec{v} = (1, \; -1, \; 1)$.
   b. Where does the line intersect the *xy*-plane?

2. Find the parametric equation of the line that passes through $P_1(2, \; 1, \; -1)$ and $P_2(-2, \; 3, \; -5)$.
3. Find the distance from the point $Q(-1, \; 2, \; 1)$ to the line

$$x = 1 + t, \quad y = 2 - 4t, \quad z = -3 + 2t.$$

4. Find the distance from the point $Q(-1,\ 2)$ to the line $y = -2x + 4$.

Solution

1. **1.**
   a. By (6.2.1), $x = -1 + t,\ y = 2 - t,\ z = t.$
   b. When the line intersects the *xy*-plane $z = 0$ and $t = 0.$ When $t = 0,\ x = -1$ and $y = 2.$ Hence, the intersection point is $(-1,\ 2,\ 0).$

2. **2.** Let $P_0 = P_1(2,\ 1,\ -1)$ and $\vec{v} = \overrightarrow{P_1P_2} = (-4,\ 2,\ -4).$ By (6.2.1),

$$x = 2 - 4t, \quad y = 1 + 2t, \quad z = -1 - 4t.$$

3. **3.** Take $P(1,\ 2,\ -3)$ on the line. Then $\overrightarrow{PQ} = (-2,\ 0,\ 4).$ Let $\vec{v} = (1,\ -4,\ 2).$ Then $\vec{v}$ is parallel to the line. By computation, $\|\vec{v}\| = \sqrt{21}$ and

$$\vec{v} \times \overrightarrow{PQ} = \left(\begin{vmatrix} -4 & 2 \\ 0 & 4 \end{vmatrix},\ -\begin{vmatrix} 1 & 2 \\ -2 & 4 \end{vmatrix},\ \begin{vmatrix} 1 & -4 \\ -2 & 0 \end{vmatrix} \right) = (-16,\ -8,\ -8),$$

By Theorem 6.2.2, $D = \dfrac{\left\| \vec{v} \times \overrightarrow{PQ} \right\|}{\|\vec{v}\|} = \dfrac{8\sqrt{14}}{7}.$

4. **4.** Rewrite $y = -2x + 4$ as $2x + y = 4.$

$$D = \frac{\left| Ax_1 + By_1 - C \right|}{\sqrt{A^2 + B^2}} = \frac{|2(-1) + 2 - 4|}{\sqrt{(-1)^2 + 2^2}} = \frac{4\sqrt{5}}{5}.$$

Section 5.6

1. Find a normal vector for each of the following planes.
   a. $x + 2y - z = 1,$
   b. $4x - 2y - 3z = 5,$
   c. $-x - 6y + z - 1 = 0,$
   d. $2x + 3z - 4 = 0.$

2. Determine whether the following planes are parallel or orthogonal.
   a. $x - y + z = 2, \quad 2x - 2y + 2z = 5,$
   b. $2x - y + z = 1, \quad x + y - z = 6,$
   c. $2x - y + 3z = 2, \quad x + 2y + z = 4,$
   d. $4x - 2y + 6z = 3, \quad -2x + y - 3z = 5$

3. For each pair of the following planes, find a parametric equation of the line of intersection of the two planes.
   a. $x - y + z = 2, \quad 2x - 3y + z = -1;$
   b. $3x - y + 3z = -1, \quad -4x + 2y - 4z = 2,$
   c. $x - 3y - 2z = 4, \quad 3x + 2y - 4z = 6.$

4. Find an equation for the line through $P(2, -3, 0)$ that is parallel to the planes $2x + 2y + z = 2$ and $x - 3y = 5$.
5. Find an equation for the line through $P(2, -3, 0)$ that is perpendicular to the two lines $x = -1 + y,\ y = 2 - 2t,\ z = 1 - t$ and $x = 1 - t,\ y = 1 - t,\ z = 2 + t$.
6. Find an equation of the plane passing through the point *P* and perpendicular to the vector $\vec{n}$.
   a. $P(1, -1, 0),\ \vec{n} = (1, 3, 5)$
   b. $P(1, -1, 3),\ \vec{n} = (-1, 0, -1)$
   c. $P(0, -1, 2),\ \vec{n} = (-1, 1, -1)$
   d. $P(1, 0, 0),\ \vec{n} = (1, 1, -1)$

7. Find an equation for the plane through $P(2,\ 7,\ -1)$ that is parallel to the plane $4x - y + 3z = 3$.
8. Find an equation of the plane passing through the following given three points:
   a. $P_1(1,\ 2,\ -1),\quad P_2(2,\ 3,\ 1),\quad P_3(3,\ -1,\ 2)$
   b. $P_1(1,\ 0,\ 1)\quad P_2(0,\ 1,\ -1)\quad P_3(1,\ 1,\ -2)$

9. Find an equation for the plane containing the line $x = 3 + 6t,\ y = 4,\ z = t$ and that is parallel to the line of intersection of the planes $2x + y + z = 1$ and $x - 2y + 3z = 2$.
10. Find an equation for the plane through $P(1, 4, 4)$ that contains the line of intersection of the planes $x - y + 3z = 5$ and $2x + 2y + 7z = 0$.
11. Find an equation of the plane passing through the point $Q(1,\ 2,\ -1)$ and perpendicular to the planes $x + y + z = 2$ and $-x + 2y + 3z = 5$.
12. Find an equation for the plane through $P(1, 1, 3)$ that is perpendicular to the line

$$x = 2 - 3t,\quad y = 1 + t,\quad z = 2t.$$

13. Find an equation for the plane that contains the line $x = 3 + t,\ y = 5,\ z = 5 + 2t,$ and is perpendicular to the plane $x + y + z = 4$.
14. Find the distance from the point to the plane.
   a. $P(1,\ -2,\ -1),\ x - y + 2z = -1;$
   b. $P(0,\ 1,\ 2),\ 2x + y + 3z = 4;$
   c. $P(1,\ 0,\ 1),\ 2x - 5y + z = 3;$
   d. $P(-1,\ -1,\ 1),\ -x + 2y + 3z = 1.$

15. Find the distance between the given parallel planes.
   a. $x - y + 2z = -3,\quad 3x - 3y + 6z = 1$
   b. $x + y - z = 1,\quad 2x + 2y - 2z = -5$

Solution

1. **1.** By (6.3.1), we have
   a. $\vec{n} = (1, 2, -1)$,
   b. $\vec{n} = (4, -2, -3)$,
   c. $\vec{n} = (-1, -6, 1)$, and
   d. $\vec{n} = (2, 0, 3)$.

2. **2.**
   a. Let $\overrightarrow{n_1} = (1, -1, 1)$ and $\overrightarrow{n_2} = (2, -2, 2)$. Then $\overrightarrow{n_1} = 2\overrightarrow{n_1}$ and $\overrightarrow{n_1} \parallel \overrightarrow{n_2}$.
   b. Let $\overrightarrow{n_1} = (2, -1, 1)$ and $\overrightarrow{n_2} = (1, 1, -1)$. Then $\overrightarrow{n_1} \cdot \overrightarrow{n_2} = 0$ and $\overrightarrow{n_1} \perp \overrightarrow{n_2}$.
   c. Let $\overrightarrow{n_1} = (2, -1, 3)$ and $\overrightarrow{n_2} = (1, 2, 1)$. Then $\overrightarrow{n_1}$ is not parallel to $\overrightarrow{n_2}$ because they are not proportional. Because $\overrightarrow{n_1} \cdot \overrightarrow{n_2} = 3 \neq 0$, $\overrightarrow{n_2}$ is not orthogonal to $\overrightarrow{n_1}$.
   d. $\overrightarrow{n_1} = (4, -2, 6)$ and $\overrightarrow{n_2} = (-2, 1, -3)$. Then $\overrightarrow{n_1} = -2\overrightarrow{n_2}$ and $\overrightarrow{n_1} \parallel \overrightarrow{n_2}$.

3. **3.** For a) and b), see the solutions to the exercises a) and b) in Section 4.3.

$$c)\ (A \mid \vec{b}) = \begin{pmatrix} 1 & -3 & -2 & 4 \\ 3 & 2 & -4 & 6 \end{pmatrix} R_1(-3) + R_2 \begin{pmatrix} 1 & -3 & -2 & 4 \\ 0 & 11 & 2 & -6 \end{pmatrix}$$

$$R_2\left(\frac{1}{11}\right)\rightarrow \begin{pmatrix} 1 & -3 & -2 & 4 \\ 0 & 1 & \frac{2}{11} & -\frac{6}{11} \end{pmatrix} R_2(2) + R_1 \rightarrow \begin{pmatrix} 1 & 0 & -\frac{16}{11} & \frac{26}{11} \\ 0 & 1 & \frac{2}{11} & -\frac{6}{11} \end{pmatrix}.$$

The system corresponding to the last augmented matrix is

$$\begin{cases} x - \frac{16}{11}z = \frac{26}{11} \\ y + \frac{2}{11}z = -\frac{6}{11}, \end{cases}$$

where *x* and *y* are basic variables and *z* is a free variable. Let $z = t$. Then $x = \frac{26}{11} + \frac{16}{11}z = \frac{26}{11} + \frac{16}{11}t$ and $y = -\frac{6}{11} - \frac{2}{11}z = -\frac{6}{11} - \frac{2}{11}t$ Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{26}{11} + \frac{16}{11}t \\ -\frac{6}{11} - \frac{2}{11}t \\ t \end{pmatrix} = \begin{pmatrix} \frac{26}{11} \\ -\frac{6}{11} \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{16}{11} \\ -\frac{2}{11} \\ 1 \end{pmatrix}.$$

4. **4.** Let $\vec{n_1} = (2, 2, 1)$ and $\vec{n_2} = (1, -3, 0)$. Then $\vec{n_1} \times \vec{n_2} = (3, 1, -8)$.

$$x = 2 + 3t, \qquad y = -3 + t, \qquad z = -8t.$$

5. **5.** Let $\vec{v_1} = (1, -2, -1)$ and $\vec{v_2} = (-1, 1, 1)$. Then

$$\vec{v_1} \times \vec{v_2} = \left(\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} \right) = (-1, 0, -1).$$

The equation of the line is $x = 2 - t,\ y = -3,\ z = -t.$

6. **6.** By (6.3.1), we have
   a. $x + 3y + 5z = (1)(1) + 3(-1) + (5)(0) = -2.$
   b. $-x + 0y - z = (-1)(1) + 0(-1) - 3 = -4.$
   c. $-x + y - z = (-1)(0) + (1)(-1) - 2 = -3.$
   d. $x + y - z = (1)(1) + (1)(0) - 0 = 1.$

7. **7.** Let $\vec{n} = (4, -1, 3)$. Then $4x - y + 3z = (4, -1, 3) \cdot (2, 7, -1) = -2.$

8. **8.**

   a. $\overrightarrow{P_1P_2} = (1, 1, 2)$ and $\overrightarrow{P_1P_3} = (2, -3, 3)$. By computation, we obtain

$$\vec{n} := \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (9, 1, -5).$$

   By (6.3.1), we have $9x + y - 5z = (9, 1, -5) \cdot (1, 2, -1) = 16.$

   b. $\overrightarrow{P_1P_2} = (-1, 1, -2)$ and $\overrightarrow{P_1P_3} = (0, 1, -3)$. By computation, we obtain

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-1,\ -3,\ -1).$$

By (6.3.1), we have $-x - 3y - z = (-1,\ -3,\ -1)\ \cdot\ (1,\ 0,\ 1) = -2.$

9. **9.** Let $\overrightarrow{n_1} = (2,\ 1,\ 1)$ and $\overrightarrow{n_1} = (1,\ -2,\ 3).$ Then

$$\begin{aligned}\overrightarrow{n_1} \times \overrightarrow{n_2} &= \left(\begin{vmatrix}1 & 1\\ -2 & 3\end{vmatrix},\ -\begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix},\ \begin{vmatrix}2 & 1\\ 1 & -2\end{vmatrix}\right)\\ &= (5,\ -5,\ -5) = 5(1,\ -1,\ -1).\end{aligned}$$

Let $\overrightarrow{v_1} = (1,\ -1,\ -1)$ and $\overrightarrow{v_2} = (6,\ 0,\ 1).$ Then

$$\overrightarrow{v_1} \times \overrightarrow{v_2} = \left(\begin{vmatrix}-1 & -1\\ 0 & 1\end{vmatrix},\ -\begin{vmatrix}1 & -1\\ 6 & 1\end{vmatrix},\ \begin{vmatrix}1 & -1\\ 6 & 0\end{vmatrix}\right) = (-1,\ -7,\ 6).$$

Hence, $-x - 7y + 6z = (-1,\ -7,\ 6)\ \cdot\ (3,\ 4,\ 0) = -31.$

10. **10.** Solving the following system

$$\begin{cases} x - y + 3z = 5\\ 2x + 2y + 7z = 0,\end{cases}$$

we obtain $x = \frac{5}{2} - \frac{13}{4}t,\ y = -\frac{5}{2} - \frac{1}{4}t$ and $z = t.$ Let $t = 2.$ Then $(x,\ y,\ z) = (-4,\ -3,\ 2)$ is a point on the above line, denoted by Q. Let $\vec{v} = (-4)\left(-\frac{13}{4},\ -\frac{1}{4},\ 1\right) = (13,\ 1,\ -4).$ Then $\vec{v}$ is parallel to the above line, so it is parallel to the plane we seek. Because P and Q are in the plane we seek, so $\overrightarrow{PQ} = (-4, -3, 2) - (1, 4, 4) = (-5, -7, -2)$ is parallel to the plane we seek. Therefore,

$$\vec{v} \times \overrightarrow{PQ} = \left(\begin{vmatrix} 1 & -4 \\ -7 & -2 \end{vmatrix}, - \begin{vmatrix} 13 & -4 \\ -5 & -2 \end{vmatrix}, \begin{vmatrix} 13 & -1 \\ -5 & -7 \end{vmatrix} \right) = -2(15, -23, 43).$$

Let $\vec{n} = (15, -23, 43)$. Then

$$15x - 23y + 43z = (15, -23, 43) \cdot (1, 4, 4) = 95.$$

11. **11.** Let $\overrightarrow{n_1} = (1, 1, 1)$ and $\overrightarrow{n_2} = (-1, 2, 3)$ be the normal vectors of the two given planes. By computation, we have

$$\overrightarrow{n_1} \times \overrightarrow{n_2} = \left(\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}, - \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} \right) = (1, -4, 3).$$

The equation of the plane is $x - 4y + 3z = (1, -4, 3) \cdot (1, 2, -1) = -10$.

12. **12.** Let $\vec{n} = (-3, 1, 2)$. Then $-3x + y + 2z = (-3, 1, 2) \cdot (1, 1, 3) = 4$.

13. **13.** Let $\overrightarrow{v_1} = (1, 0, 2)$ and $\overrightarrow{v_1} = (1, 1, 1)$. Then

$$\overrightarrow{v_1} \times \overrightarrow{v_2} = (1, 0, 2) \times (1, 1, 1) = (-2, 1, 1)$$

and $-2x + y + z = (-2, 1, 1) \cdot (3, 5, 5) = 4$.

14. **14.**

a. $D = \dfrac{|(1)(1) + (-1)(-2) + (2)(-1) + 1|}{\sqrt{1^2 + (-1)^2 + 2^2}} = \dfrac{\sqrt{6}}{3}$.

b. $D = \dfrac{|(2)(0)+(1)(1)+(3)(2)-4|}{\sqrt{2^2+1^2+1^2}} = \dfrac{3\sqrt{14}}{14}.$

c. $D = \dfrac{|(2)(1)+(-5)(0)+(1)(1)-3|}{\sqrt{2^2+(-5)^2+1^2}} = 0.$

d. $D = \dfrac{|(-1)(-1)+(2)(-1)+(3)(1)-1|}{\sqrt{(-1)^2+2^2+3^2}} = \dfrac{\sqrt{14}}{14}.$

15. **15.**

a. In the second equation, let $y = z = 0$. Then we have $x = \frac{1}{3}$. So $P_0\left(\frac{1}{3},\ 0,\ 0\right)$ is a point in the second plane. The distance between the point $P_0\left(\frac{1}{3},\ 0,\ 0\right)$ and the first plane equals the distance between the two parallel planes. By (6.3.6), we have

$$D = \frac{\left|(1)\left(\frac{1}{3}\right) + (-1)(0) + (2)(0) + 3\right|}{\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{5\sqrt{6}}{9}.$$

b. In the second equation, let $y = z = 0$. Then we have $x = -\frac{5}{2}$. So $P_0\left(-\frac{5}{2},\ 0,\ 0\right)$ is a point in the second plane. By (6.3.6), we have

$$D = \frac{\left|(1)\left(-\frac{5}{2}\right) + (1)(0) + (-1)(0) - 1\right|}{\sqrt{1^2 + 1^2 + (-1)^2}} = \frac{7\sqrt{3}}{6}.$$

A.7 Bases and dimensions

Section 7.1

1. Let $\overrightarrow{a_1} = (2,\ 4,\ 6)^T$, $\overrightarrow{a_2} = (0,\ 0,\ 0)^T$, and $\overrightarrow{a_3} = (1,\ 2,\ 3)^T$. Determine whether $\left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$ is linearly dependent.
2. Determine whether $\left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\ \overrightarrow{a_4}\right\}$ is linearly dependent, where

$$\overrightarrow{a_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix},\quad \overrightarrow{a_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad \overrightarrow{a_3} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix},\quad \overrightarrow{a_4} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

3. Determine whether $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\ \overrightarrow{a_4}\right\}$ is linearly independent, where

$$\overrightarrow{a_1} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix},\quad \overrightarrow{a_2} = \begin{pmatrix} 7 \\ 9 \\ 8 \end{pmatrix},\quad \overrightarrow{a_3} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix},\quad \overrightarrow{a_4} = \begin{pmatrix} 6 \\ -2 \\ 5 \end{pmatrix}.$$

4. Determine that $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$ is linearly independent, where

$$\overrightarrow{a_1} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix},\quad \overrightarrow{a_2} = \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix},\quad \overrightarrow{a_3} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Solution

1. **1.** Because $\left\{\overrightarrow{a}_1,\ \overrightarrow{a}_2,\ \overrightarrow{a}_3\right\}$ contains a zero vector $\overrightarrow{a}_2$, $\left\{\overrightarrow{a}_1,\ \overrightarrow{a}_2,\ \overrightarrow{a}_1\right\}$ is linearly dependent.
2. **2.** Because $\overrightarrow{a}_3 = \overrightarrow{a}_1 - \overrightarrow{a}_2$, $\left\{\overrightarrow{a}_1,\ \overrightarrow{a}_2,\ \overrightarrow{a}_3\right\}$ is linearly dependent. By Corollary 7.1.2, $\left\{\overrightarrow{a}_1,\ \overrightarrow{a}_2,\ \overrightarrow{a}_3,\ \overrightarrow{a}_4\right\}$ is linearly dependent.
3. **3.** Because $n = 4 > 3 = m$, by Theorem 7.1.1, *S* is linearly independent.
4. **4.** Because

$$|A| = \begin{vmatrix} -1 & 3 & 1 \\ 0 & -1 & 2 \\ -1 & -6 & 8 \end{vmatrix} \underline{\underline{R_1(-1)+R_3}} \begin{vmatrix} -1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & -9 & 7 \end{vmatrix} \underline{\underline{R_2(-9)+R_3}}$$
$$\begin{vmatrix} -1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -11 \end{vmatrix} = -11 \neq 0,$$

by Theorem 7.1.5, *S* is linearly independent.

Section 7.2

1. Let $\overrightarrow{a_1} = (1,\ -1,\ 1,\ 0)$, and $\overrightarrow{a_2} = (1,\ 0,\ 1,\ 1)$. Determine whether $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2}\right\}$ is a basis of span *S*. If so, find dim(span *S*).
2. Determine whether $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$ is a basis of span *S*, where
   i. $\overrightarrow{a_1} = (1,\ 0,\ -1,\ 1)$, $\overrightarrow{a_2} = (0,\ 0,\ 1,\ 1)$ and $\overrightarrow{a_3} = (-2,\ 1,\ 0,\ 1)$.
   ii. $\overrightarrow{a_1} = (1,\ 0,\ 1,\ 1)$, $\overrightarrow{a_2} = (0,\ 0,\ 1,\ -1)$ and $\overrightarrow{a_3} = (1,\ 0,\ 2,\ 0)$.

3. Let $\overrightarrow{a_1} = (1,\ -1)$, $\overrightarrow{a_2} = (0,\ 1)$, $\overrightarrow{a_3} = (-3,\ 2)$ and $\overrightarrow{a_4} = (1,\ 0)$. Determine whether $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3},\ \overrightarrow{a_4}\right\}$ is a basis of span *S*.
4. Let $\overrightarrow{a_1} = (1,\ 0,\ 1)$ and $\overrightarrow{a_2} = (1,\ 1,\ 1)$. Find a vector $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{b}\right\}$ is a basis of span *S*.
5. Use **Theorem 7.1.1** to prove **Theorem 7.2.2** .
6. Use **Theorem 7.1.1** to prove **Theorem 7.2.4** .

Solution

1. **1.** Because $\vec{a}_1$ is not parallel to $\overrightarrow{a_2}$, *S* is linearly independent. Hence, *S* is basis of span *S* and $\dim(\text{span S}) = 2$.
2. **2.** Let $A = (\overrightarrow{a_1}\overrightarrow{a_2}\overrightarrow{a_3})$. Then

$$A^T = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ -2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_1(-2)+R_3} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{R_{2,3}} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $r(A^T) = 3$. By Theorem 2.5.5, $r(A) = r(A^T) = 3$. It follows from Theorem 7.2.1 that *S* is not a basis of span *S* and by Definition 7.2.1, $\dim(\text{span } S) = 3$.
3. **3.**

   i. Let $A = (\overrightarrow{a_1}\overrightarrow{a_2}\overrightarrow{a_3})$. Then

$$A^T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $r(A^T) = 2$. By Theorem 2.5.5, $r(A) = r(A^T) = 2 < 3 = n$. It follows from Theorem 7.2.1 that *S* is not a basis of span *S*.

ii. Because $n = 4 > 2 = m$, by Theorem 7.2.1, *S* is not a basis of span *S*.

4. **4.** Let $A = (\vec{a}_1 \vec{a}_2)$ and $\vec{b} = (\vec{b}_1, \vec{b}_2, \vec{b}_3)^T$. Then

$$(A \mid \vec{b}) = \begin{pmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 \\ 1 & 1 & b_3 \end{pmatrix} \xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 - b_1 \end{pmatrix} = (B \mid \vec{c}).$$

If $b_3 - b_1 \neq 0$, and $r(A) = 2$ and $r(A \mid \vec{b}) = 3$. Hence, $r(A) < r(A \mid \vec{b})$. By Theorem 4.5.1 (3), we see that if $b_1 \neq b_3$ and b_2 is a real number, then the system has no solutions. Hence we can choose $b_1 = b_2 = 1$ and $b_3 = 0$, that is, $\vec{b} = (1,\ 1,\ 0)^T$ By Theorem 7.2.4, $S = \left\{ \vec{a}_1, \vec{a}_2 \vec{b} \right\}$ is a basis of span *S*.

5. **5.** Let $S = \left\{\overrightarrow{a_1}, \overrightarrow{a_2} \cdots, \overrightarrow{a_n}\right\}$ be a basis of span S. We will prove that for each $\vec{b} \in \text{span } S$, there exists a unique vector $(x_1, x_2 \cdots, x_n)$ such that

(A.7.1)

$$\vec{b} = x_1 \overrightarrow{a_1} + x_2 \overrightarrow{a_2} + \cdots x_n \overrightarrow{a_n}.$$

Because $\vec{b} \in \text{span } S$, there exists a vector $(x_1, x_2 \cdots x_n)$ such that (A.7.1) holds. Assume that there exists another vector $(y_1, y_2, \cdots, y_n)$ such that

(A.7.2)

$$\vec{b} = y_1 \overrightarrow{a_1} + y_2 \overrightarrow{a_2} + \cdots y_n \overrightarrow{a_n}.$$

Then

$$x_1 \overrightarrow{a_1} + x_2 \overrightarrow{a_2} + \cdots x_n \overrightarrow{a_n} = y_1 \overrightarrow{a_1} + y_2 \overrightarrow{a_2} + \cdots y_n \overrightarrow{a_n}$$

and

(A.7.3)

$$(x_1 - y_1) \overrightarrow{a_1} + (x_2 - y_2) \overrightarrow{a_2} + \cdots (x_n - y_n) \overrightarrow{a_n} = \vec{0}.$$

Because S is a basis of span S, S is linearly independent. By (A.7.3) and Theorem 7.1.1, we obtain $x_i - y_i = 0$ for $i \in I_n$. This implies that $x_i = y_i$ for $i \in I_n$. Hence, the vectors satisfying (A.7.1) are unique.

6. **6.** Let $S = \left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots, \overrightarrow{a_n}\right\}$ be a basis of span *S*. Assume that $S \neq \mathbb{R}^m$. Let $\vec{b} \in \text{ span } S$. We prove that

$$S_1 = \left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots \overrightarrow{a_n}, \vec{b}\right\}$$

is linearly independent. Indeed, assume that there exists a vector $(x_1, x_2, \cdots, x_n, x_{n+1})$ such that (A.7.4)

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots x_n\overrightarrow{a_n} + x_{n+1}\vec{b} = \vec{0}.$$

Because $\vec{b} \in \text{ span } S,$ we have $x_{n+1} = 0.$ If not, then $x_{n+1} \neq 0$ and by (A.7.4), $\vec{b}$ is a linear combination of *S*. Hence, we obtain $\vec{b} \in \text{span } S,$ a contradiction. Because $x_{n+1} = 0,$ it follows from (A.7.4) that (A.7.5)

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots x_n\overrightarrow{a_n} = \vec{0}.$$

Because *S* is a basis of span *S*, *S* is linearly independent and by (A.7.5), $x_i = 0$ for $i \in I_n.$ Hence, (A.7.4) only has the zero solution. By Theorem 7.1.1, S is linearly independent.

Section 7.3

1. Find a basis for the column space of *C*, where

$$C = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. For each of the following matrices, find a set of its column vectors that forms a basis for its column space.

$$A_1 = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix}.$$

3. For each of the following row echelon matrices, find the basis and dimension of its row space and use the basis vectors to denote the row space.

$$D_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad D_2 = \begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad D_3 = \begin{pmatrix} 1 & 2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. Find a set of basis vectors for R_B and $\dim R_B$, where

$$B = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{pmatrix}.$$

5. Let $S = \left\{ \vec{a}_1,\ \vec{a}_2,\ \vec{a}_3,\ \vec{a}_4,\ \vec{a}_5 \right\}$, where

$$\vec{a}_1 = (1,\ -2,\ 0,\ 3)^T,\ \vec{a}_2 = (2,\ -5,\ -3,\ 6)^T,\ \vec{a}_3 = (0,\ 1,\ 3,\ 0)^T,$$
$$\vec{a}_4 = (2,\ -1,\ 4,\ -7)^T,\ \vec{a}_5 = (5,\ -8,\ 1,\ 2)^T.$$

a. Find a subset of *S* that forms a basis for the spanning space span *S*.
b. Find another basis for span *S*.

Solution

1. **1.** Because $\vec{a}_1$ is not parallel to $\vec{a}_2$, S is linearly independent. Hence, S is basis of span S and $\dim(\text{span S}) = 2$.
2. **2.** Let $A = (\vec{a}_1 \vec{a}_2 \vec{a}_3)$. Then

$$A^T = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ -2 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_1(-2)+R_3} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{R_{2,3}} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $r(A^T) = 3$. By Theorem 2.5.5, $r(A) = r(A^T) = 3$. It follows from Theorem 7.2.1 that S is not a basis of span S and by Definition 7.2.1, $\dim(\text{span } S) = 3$.

3. **3.**

   i. Let $A = (\vec{a}_1 \vec{a}_2 \vec{a}_3)$. Then

$$A^T = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 \end{pmatrix} \xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $r(A^T) = 2$. By Theorem 2.5.5, $r(A) = r(A^T) = 2 < 3 = n$. It follows from Theorem 7.2.1 that *S* is not a basis of span *S*.

ii. Because $n = 4 > 2 = m$, by Theorem 7.2.1, *S* is not a basis of span *S*.

4. **4.** Let $A = (\vec{a}_1 \vec{a}_2)$ and $\vec{b} = (\vec{b}_1, \vec{b}_2, \vec{b}_3)^T$. Then

$$(A\,|\,\vec{b}) = \begin{pmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 \\ 1 & 1 & b_3 \end{pmatrix} \xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & b_3 - b_1 \end{pmatrix} = (B\,|\,\vec{c}).$$

If $b_3 - b_1 \neq 0$, and $r(A) = 2$ and $r(A \,\big|\, \vec{b}) = 3$. Hence, $r(A) < r(A \,\big|\, \vec{b})$. By Theorem 4.5.1 (3), we see that if $b_1 \neq b_3$ and b_2 is a real number, then the system has no solutions. Hence we can choose $b_1 = b_2 = 1$ and $b_3 = 0$, that is, $\vec{b} = (1,\ 1,\ 0)^T$ By Theorem 7.2.4, $S = \left\{ \vec{a}_1, \vec{a}_2 \vec{b} \right\}$ is a basis of span *S*.

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5. **5.** Let $S = \left\{\overrightarrow{a_1}, \overrightarrow{a_2} \cdots, \overrightarrow{a_n}\right\}$ be a basis of span S. We will prove that for each $\vec{b} \in \text{span } S,$ there exists a unique vector $(x_1, x_2 \cdots, x_n)$ such that

(A.7.1)

$$\vec{b} = x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots x_n\overrightarrow{a_n}.$$

Because $\vec{b} \in \text{ span } S,$ there exists a vector $(x_1, x_2 \cdots x_n)$ such that (A.7.1) holds. Assume that there exists another vector $(y_1,\ y_2,\ \cdots,\ y_n)$ such that

(A.7.2)

$$\vec{b} = y_1\overrightarrow{a_1} + y_2\overrightarrow{a_2} + \cdots y_n\overrightarrow{a_n}.$$

Then

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots x_n\overrightarrow{a_n} = y_1\overrightarrow{a_1} + y_2\overrightarrow{a_2} + \cdots y_n\overrightarrow{a_n}$$

and

(A.7.3)

$$(x_1 - y_1)\overrightarrow{a_1} + (x_2 - y_2)\overrightarrow{a_2} + \cdots (x_n - y_n)\overrightarrow{a_n} = \vec{0}.$$

Because S is a basis of span S, S is linearly independent. By (A.7.3) and Theorem 7.1.1, we obtain $= 0$ for $i \in I_n.$ This implies that $x_i = y_i$ for $i \in I_n.$ Hence, the vectors satisfying (A.7.1) are unique.

6. **6.** Let $S = \left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots, \overrightarrow{a_n}\right\}$ be a basis of span *S*. Assume that $S \neq \mathbb{R}^m$. Let $\vec{b} \in \text{ span } S$. We prove that

$$S_1 = \left\{\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots \overrightarrow{a_n}, \vec{b}\right\}$$

is linearly independent. Indeed, assume that there exists a vector $(x_1, x_2, \cdots, x_n, x_{n+1})$ such that (A.7.4)

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots x_n\overrightarrow{a_n} + x_{n+1}\vec{b} = \vec{0}.$$

Because $\vec{b} \in \text{ span } S,$ we have $x_{n+1} = 0.$ If not, then $x_{n+1} \neq 0$ and by (A.7.4), $\vec{b}$ is a linear combination of *S*. Hence, we obtain $\vec{b} \in \text{span } S,$ a contradiction. Because $x_{n+1} = 0,$ it follows from (A.7.4) that (A.7.5)

$$x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots x_n\overrightarrow{a_n} = \vec{0}.$$

Because *S* is a basis of span *S*, *S* is linearly independent and by (A.7.5), $x_i = 0$ for $i \in I_n.$ Hence, (A.7.4) only has the zero solution. By Theorem 7.1.1, S is linearly independent.

Section 7.3

1. Find a basis for the column space of *C*, where

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$$C = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. For each of the following matrices, find a set of its column vectors that forms a basis for its column space.

$$A_1 = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{pmatrix}.$$

3. For each of the following row echelon matrices, find the basis and dimension of its row space and use the basis vectors to denote the row space.

$$D_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad D_2 = \begin{pmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad D_3 = \begin{pmatrix} 1 & 2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. Find a set of basis vectors for R_B and $\dim R_B,$ where

$$B = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{pmatrix}.$$

5. Let $S = \left\{ \vec{a}_1,\ \vec{a}_2,\ \vec{a}_3,\ \vec{a}_4,\ \vec{a}_5 \right\},$ where

$$\vec{a}_1 = (1,\ -2,\ 0,\ 3)^T,\ \vec{a}_2 = (2,\ -5,\ -3,\ 6)^T,\ \vec{a}_3 = (0,\ 1,\ 3,\ 0)^T,$$
$$\vec{a}_4 = (2,\ -1,\ 4,\ -7)^T,\ \vec{a}_5 = (5,\ -8,\ 1,\ 2)^T.$$

a. Find a subset of *S* that forms a basis for the spanning space span *S*.
b. Find another basis for span *S*.

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Solution

1. **1.** We denote the column vectors of *C* by $\vec{C_i},\ i = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$ Then

$$B = (\vec{C_1}\vec{C_2}\vec{C_3}\vec{C_4}\vec{C_5}\vec{C_6}).$$

The column vectors containing the leading entries are $\vec{C_1}, \vec{C_3}, \vec{C_5}$. Hence, $\left\{\vec{C_1}, \vec{C_3}, \vec{C_5}\right\}$ forms a basis for the column space of *C*.

2. **2.**

$$A_1 = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix} \begin{matrix} R_1(-2)+R_3 \\ \longrightarrow \\ \longrightarrow \end{matrix} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -5 \end{pmatrix}$$

$$\xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{pmatrix} \underset{R_3\left(-\frac{1}{4}\right)}{\xrightarrow{R_3\left(\frac{1}{2}\right)}} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = C_1^*.$$

Because the column vectors $\vec{C_1}, \vec{C_2}, \vec{C_3}$ of C_1^* contain the leading entries, by Theorem 7.3.1, the pivot

 of A_1 form a basis for C_{A_1}.

$$A_2 = \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix} \underset{R_1(-2)+R_3}{\overset{R_1(-1)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & -2 & 1 & -5 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}$$

$$\overset{R_1(-1)+R_3}{\longrightarrow} \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix} \overset{R_1(1)+R_4}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \underset{R_3\left(-\frac{1}{3}\right)}{\overset{R_2\left(-\frac{1}{2}\right)}{\longrightarrow}} \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = C_2^*.$$

Because the column vectors $\vec{C}_1, \vec{C}_2, \vec{C}_4, \vec{C}_5$ of C_2^* contain the leading entries, by Theorem 7.3.1, the pivot column vectors $\vec{a}_1, \vec{a}_2, \vec{a}_4, \vec{a}_5,$ of A_2 form a basis for C_{A_2}.

We denote the column vectors of A_3 by $\vec{a}_i,\ i = 1, 2, 3, 4, 5, 6.$ Then

$$A_3 = (\vec{a}_1 \vec{a}_2 \vec{a}_3 \vec{a}_4 \vec{a}_5 \vec{a}_6).$$

By using row operations, we can change A_3 into a row echelon matrix C_3^*, which is the same as Exercise 1 above. Hence, the column vectors $\vec{C_1}, \vec{C_3}, \vec{C_5}$ of C_3^* contain leading entries.

By Theorem 7.3.1, the pivot column vectors $\vec{a_1}, \vec{a_3}, \vec{a_5}$ of A_3 form a basis of C_{A_3}.

3. **3.** The basis vectors for R_{D_1} are $\vec{d_1} = (1,\ 1,\ 0)$ and $\vec{d_2} = (0,\ 1,\ 0)$. Hence, $\dim(R_{D_1}) = 2$ and

$$R_{D_1} = \operatorname{span}\left\{\vec{d_1}, \vec{d_2}\right\}.$$

The basis vectors for R_{D_2} are $\vec{d_1} = (0,\ 1,\ -2,\ 0,\ 1)$ and $\vec{d_2} = (0,\ 0,\ 0,\ 1,\ 3)$. Hence, $\dim(R_{D_2}) = 2$ and

$$R_{D_1} = \operatorname{span}\left\{\vec{d_1}, \vec{d_2}\right\}.$$

$$\vec{d_1} = (1,\ 2,\ 5,\ 0,\ 3), \quad \vec{d_2} = (0,\ 1,\ 3,\ 0,\ 0), \ \vec{d_3} = (0,\ 0,\ 0,\ 1,\ 0).$$

Hence, $\dim(R_{D_3}) = 3$ and $R_{D_3} = \operatorname{span}\{\vec{d_1}, \vec{d_2}, \vec{d_3}\}$.

4. **4.**

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$$B = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{pmatrix} \underset{R_1(1)+R_3}{\xrightarrow{R_1(-2)+R_2}} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & -4 & 4 \end{pmatrix} \xrightarrow{R_2(2)+R_3}$$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2\left(\frac{1}{2}\right)} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = D.$$

Let $\vec{d}_1 = (1,\ -1,\ 3)$, $\vec{d}_2 = (0,\ 1,\ -1)$. Then $\vec{d}_1, \vec{d}_2$ are basis vectors for $\mathbb{R}_B$. $\mathbb{R}_B = \text{span}\left\{\vec{d}_1, \vec{d}_2\right\}$ and

$\dim(R_B) = 2$.

5. **5.**

a. To find a subset of *S* that forms a basis for span *S*, we need to put these vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$ as the column vectors of a matrix *A*, that is,

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{pmatrix}.$$

We use row operations to change *A* into a row echelon matrix.

$$A=\begin{pmatrix}1&2&0&2&5\\-2&-5&1&-1&-8\\0&-3&3&4&1\\3&6&0&-7&2\end{pmatrix}\xrightarrow[R_1(-3)+R_4]{R_1(2)+R_2}\begin{pmatrix}1&2&0&2&5\\0&-1&1&3&2\\0&-3&3&4&1\\0&0&0&-13&13\end{pmatrix}$$

$$\xrightarrow{R_1(-3)+R_3}\begin{pmatrix}1&2&0&2&5\\0&-1&1&3&2\\0&0&0&-5&-5\\0&0&0&-13&13\end{pmatrix}\xrightarrow{R_3\left(-\frac{1}{5}\right)}$$

$$\begin{pmatrix}1&2&0&2&5\\0&-1&1&3&2\\0&0&0&1&1\\0&0&0&-13&13\end{pmatrix}\xrightarrow{R_3\left(-\frac{1}{13}\right)+R_4}\begin{pmatrix}1&2&0&2&5\\0&-1&1&3&2\\0&0&0&1&1\\0&0&0&0&0\end{pmatrix}=C.$$

The column vectors $\vec{C_1}, \vec{C_2}, \vec{C_4}$ of *C* contain the leading entries of *C*. By Theorem 7.3.1, the pivot column vectors $\left\{\vec{a_1}, \vec{a_2}, \vec{a_4}\right\}$ of *A* form a basis of span *S*.

b. We can use Theorem 7.3.2 to find a basis for span *S*. We need to put these vectors $\vec{a_1}, \vec{a_2}, \vec{a_3}, \vec{a_4}, \vec{a_5}$ as row vectors of a matrix *B*, that is,

$$B = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -5 & -3 & 6 \\ 0 & 1 & 3 & 0 \\ 2 & -1 & 4 & -7 \\ 5 & -8 & 1 & 2 \end{pmatrix}.$$

We use row operations to change *B* into a row echelon matrix *D*.

$$B \underset{\substack{R_1(-2)+R_4 \\ R_1(-5)+R_5}}{\overset{R_1(-2)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \end{pmatrix} \overset{R_2(1)+R_3}{\longrightarrow}$$

$$\begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \end{pmatrix} \underset{R_{4,5}}{\overset{R_{3,4}}{\longrightarrow}} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{R_2(-2)+R_4}{\overset{R_2(-3)+R_3}{\longrightarrow}}$$

$$\begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -5 & -13 \\ 0 & 0 & 5 & -13 \end{pmatrix} \overset{R_3(1)+R_4}{\longrightarrow} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -5 & -13 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix}0 & 0 & 0 & 0\end{matrix} \qquad \begin{matrix}0 & 0 & 0 & 0\end{matrix}$$

$$\begin{matrix} R_2(-1) \\ \longrightarrow \\ R_3\left(-\frac{1}{5}\right) \end{matrix} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{13}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = D.$$

Let $\vec{d}_1 = (1,\ -2,\ 0,\ 3)$, $\vec{d}_2 = (0,\ 1,\ 3,\ 0)$, and $\vec{d}_3 = \left(0,\ 0,\ 1,\ \frac{13}{5}\right)$. By Theorem 7.3.2, $\left\{\vec{d}_1, \vec{d}_2, \vec{d}_3\right\}$ forms a basis for span *S*.

Section 7.4

1. For each of the following pairs of vectors, verify that it is a basis of $\mathbb{R}^2$ and find the coordinates of the vector $\vec{b} = (x,\ y)^T$.

$$S_1 = \left\{\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\} \quad S_2 = \left\{\begin{pmatrix}-3\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right\}$$

$$S_3 = \left\{\begin{pmatrix}5\\-6\end{pmatrix}, \begin{pmatrix}4\\-1\end{pmatrix}\right\} \quad S_4 = \left\{\begin{pmatrix}3\\-2\end{pmatrix}, \begin{pmatrix}-2\\5\end{pmatrix}\right\}$$

2. Let $\overrightarrow{a_1} = (1,\ 0,\ 1)^T$, $\overrightarrow{a_2} = (0,\ 1,\ 1)^T$, and $\overrightarrow{a_3} = (1,\ 1,\ 0)^T$.

i. Show that $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$ is a basis of $\mathbb{R}^3$.

ii. Let $\vec{b} = (1,\ 1,\ 1)^T$. Find $(\vec{b})_S$.

3. Let $\overrightarrow{a_1} = (1,\ 0,\ 0)^T$, $\overrightarrow{a_2} = (2,\ 4,\ 0)^T$, and $\overrightarrow{a_3} = (3,\ -1,\ 5)^T$.

i. Show that $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$ is a basis of $\mathbb{R}^3$.

ii. Let $\vec{b} = (2,\ -1,\ -2)$. Find $(\vec{b})_S$.

4. Let $S_1 = \left\{\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\}$ and $S_2 = \left\{\begin{pmatrix}-3\\2\end{pmatrix}, \begin{pmatrix}2\\3\end{pmatrix}\right\}$.

i. If $(\vec{b})_{S_2} = (2,\ -1)$, find $(\vec{b})_{S_1}$.

ii. If $(\vec{b})_{S_1} = (13,\ -26)$, find $(\vec{b})_{S_2}$.

5. If $(\vec{b})_T = (1,\ 2,\ 3)$, find $(\vec{b})_S$, where

$$S = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

$$T = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

6. Let $\overrightarrow{b_1} = (0,\ 1,\ 0)$, $\overrightarrow{b_2} = \left(-\frac{4}{5},\ 0,\ \frac{3}{5}\right)$, and $\overrightarrow{b_3} = \left(\frac{3}{5},\ 0,\ \frac{4}{5}\right)$.

   i. Show that $S := \left\{\overrightarrow{b_1},\ \overrightarrow{b_2},\ \overrightarrow{b_3}\right\}$ is an orthonormal basis of $\mathbb{R}^3$.

   ii. Let $\vec{b} = (1,\ -1,\ 1)$. Find $(\vec{b})_S$.

7. Let $\overrightarrow{a_1} = (1,\ 1,\ 1)$, $\overrightarrow{a_2} = (0,\ 1,\ 1)$, $\overrightarrow{a_3} = (0,\ 0,\ 1)$, and $\vec{b} = (-1,\ 1,\ -1)^T$.

   1. Find an orthogonal basis $S_1 := \left\{\overrightarrow{b_1},\ \overrightarrow{b_2},\ \overrightarrow{b_3}\right\}$ of $\mathbb{R}^3$ by using the Gram-Schmidt process and $(\vec{b})_{S_1}$.

   2. Find an orthonormal basis $S_2 := \left\{\overrightarrow{q_1},\ \overrightarrow{q_2},\ \overrightarrow{q_3}\right\}$ of $\mathbb{R}^3$ by normalizing S_1 and $(\vec{b})_{S_2}$.

Solution

1. **1.** We denote the column vectors of *C* by $\overrightarrow{C_i},\ i = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$ Then

$$B = (\overrightarrow{C_1}\overrightarrow{C_2}\overrightarrow{C_3}\overrightarrow{C_4}\overrightarrow{C_5}\overrightarrow{C_6}).$$

   The column vectors containing the leading entries are $\overrightarrow{C_1}, \overrightarrow{C_3}, \overrightarrow{C_5}$. Hence, $\left\{\overrightarrow{C_1}, \overrightarrow{C_3}, \overrightarrow{C_5}\right\}$ forms a basis for the column space of *C*.
2. **2.**

$$A_1 = \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_{1,2}} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 1 \end{pmatrix} \xrightarrow{R_1(-2)+R_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -5 \end{pmatrix}$$

$$\xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{pmatrix} \xrightarrow[R_3\left(-\frac{1}{4}\right)]{R_3\left(\frac{1}{2}\right)} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = C_1^*.$$

   Because the column vectors $\overrightarrow{C_1}, \overrightarrow{C_2}, \overrightarrow{C_3}$ of C_1^* contain the leading entries, by Theorem 7.3.1, the pivot column vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \overrightarrow{a_3}$ of A_1 form a basis for C_{A_1}.

$$A_2 = \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix} \xrightarrow[R_1(-2)+R_3]{R_1(-1)+R_2} \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & -2 & 1 & -5 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 3 & 1 \end{pmatrix} \xrightarrow{R_1(1)+R_4}$$

$$\begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3\left(-\frac{1}{3}\right)]{R_2\left(-\frac{1}{2}\right)} \begin{pmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = C_2^*.$$

Because the column vectors $\vec{C_1}, \vec{C_2}, \vec{C_4}, \vec{C_5}$ of C_2^* contain the leading entries, by Theorem 7.3.1, the pivot column vectors $\vec{a_1}, \vec{a_2}, \vec{a_4}, \vec{a_5}$, of A_2 form a basis for C_{A_2}.

We denote the column vectors of A_3 by $\vec{a_i}$, $i = 1, 2, 3, 4, 5, 6$. Then

$$A_3 = (\vec{a_1}\vec{a_2}\vec{a_3}\vec{a_4}\vec{a_5}\vec{a_6}).$$

By using row operations, we can change A_3 into a row echelon matrix C_3^*, which is the same as Exercise 1 above. Hence, the column vectors $\vec{C_1}, \vec{C_3}, \vec{C_5}$ of C_3^* contain leading entries.

By Theorem 7.3.1, the pivot column vectors $\vec{a_1}, \vec{a_3}, \vec{a_5}$ of A_3 form a basis of C_{A_3}.

3. **3.** The basis vectors for R_{D_1} are $\vec{d_1} = (1, 1, 0)$ and $\vec{d_2} = (0, 1, 0)$. Hence, $\dim(R_{D_1}) = 2$ and

$$R_{D_1} = \text{span}\left\{\vec{d_1}, \vec{d_2}\right\}.$$

The basis vectors for R_{D_2} are $\vec{d_1} = (0, 1, -2, 0, 1)$ and $\vec{d_2} = (0, 0, 0, 1, 3)$. Hence, $\dim(R_{D_2}) = 2$ and

$$R_{D_1} = \text{span}\left\{\vec{d_1}, \vec{d_2}\right\}.$$

$$\vec{d_1} = (1, 2, 5, 0, 3), \quad \vec{d_2} = (0, 1, 3, 0, 0), \ \vec{d_3} = (0, 0, 0, 1, 0).$$

Hence, $\dim(R_{D_3}) = 3$ and $R_{D_3} = \text{span}\{\vec{d_1}, \vec{d_2}, \vec{d_3}\}$.

4. **4.**

$$B = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 4 \\ -1 & -3 & 1 \end{pmatrix} \underset{R_1(1)+R_3}{\overset{R_1(-2)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & -4 & 4 \end{pmatrix} \overset{R_2(2)+R_3}{\longrightarrow}$$

$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_2\left(\frac{1}{2}\right)}{\longrightarrow} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = D.$$

Let $\vec{d}_1 = (1, \ -1, \ 3)$, $\vec{d}_2 = (0, \ 1, \ -1)$. Then $\vec{d}_1, \vec{d}_2$ are basis vectors for $\mathbb{R}_B$. $\mathbb{R}_B = \text{span}\left\{\vec{d}_1, \vec{d}_2\right\}$ and $\dim(R_B) = 2$.

5. **5.**

a. To find a subset of *S* that forms a basis for span *S*, we need to put these vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$ as the column vectors of a matrix *A*, that is,

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{pmatrix}.$$

We use row operations to change *A* into a row echelon matrix.

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{pmatrix} \underset{R_1(-3)+R_4}{\overset{R_1(2)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & -3 & 3 & 4 & 1 \\ 0 & 0 & 0 & -13 & 13 \end{pmatrix}$$

$$\overset{R_1(-3)+R_3}{\longrightarrow} \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & -5 & -5 \\ 0 & 0 & 0 & -13 & 13 \end{pmatrix} \overset{R_3\left(-\frac{1}{5}\right)}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -13 & 13 \end{pmatrix} \overset{R_3\left(-\frac{1}{13}\right)+R_4}{\longrightarrow} \begin{pmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & -1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = C.$$

The column vectors $\vec{C}_1, \vec{C}_2, \vec{C}_4$ of *C* contain the leading entries of *C*. By Theorem 7.3.1, the pivot column vectors $\left\{\vec{a}_1, \vec{a}_2, \vec{a}_4\right\}$ of *A* form a basis of span *S*.

b. We can use Theorem 7.3.2 to find a basis for span *S*. We need to put these vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5$ as row vectors of a matrix *B*, that is,

$$B = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -5 & -3 & 6 \\ 0 & 1 & 3 & 0 \\ 2 & -1 & 4 & -7 \\ 5 & -8 & 1 & 2 \end{pmatrix}.$$

We use row operations to change B into a row echelon matrix D.

$$B \underset{\substack{R_1(-2)+R_4 \\ R_1(-5)+R_5}}{\overset{R_1(-2)+R_2}{\longrightarrow}} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \end{pmatrix} \overset{R_2(1)+R_3}{\longrightarrow}$$

$$\begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \end{pmatrix} \underset{R_{4,5}}{\overset{R_{3,4}}{\longrightarrow}} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 3 & 4 & -13 \\ 0 & 2 & 1 & -13 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underset{R_2(-2)+R_4}{\overset{R_2(-3)+R_3}{\longrightarrow}}$$

$$\begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -5 & -13 \\ 0 & 0 & 5 & -13 \end{pmatrix} \overset{R_3(1)+R_4}{\longrightarrow} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & -5 & -13 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 0 & 0 \end{matrix}$$

$$\xrightarrow[R_3\left(-\frac{1}{5}\right)]{R_2(-1)} \begin{pmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{13}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = D.$$

Let $\vec{d}_1 = (1, \ -2, \ 0, \ 3)$, $\vec{d}_2 = \ (0, \ 1, \ 3, \ 0)$, and $\vec{d}_3 = \left(0, \ 0, \ 1, \ \frac{13}{5}\right)$. By Theorem 7.3.2,

$\left\{\vec{d}_1, \vec{d}_2, \vec{d}_3\right\}$ forms a basis for span *S*.

Section 7.4

1. For each of the following pairs of vectors, verify that it is a basis of $\mathbb{R}^2$ and find the coordinates of the vector $\vec{b} = (x, \ y)^T$.

$$S_1 = \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \quad S_2 = \left\{\begin{pmatrix} -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$$

$$S_3 = \left\{\begin{pmatrix} 5 \\ -6 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix}\right\} \quad S_4 = \left\{\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix}\right\}$$

2. Let $\overrightarrow{a_1} = (1,\ 0,\ 1)^T,\ \overrightarrow{a_2} = (0,\ 1,\ 1)^T,$ and $\overrightarrow{a_3} = (1,\ 1,\ 0)^T.$

   i. Show that $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$ is a basis of $\mathbb{R}^3$.

   ii. Let $\vec{b} = (1,\ 1,\ 1)^T.$ Find $(\vec{b})_S$.

3. Let $\overrightarrow{a_1} = (1,\ 0,\ 0)^T,\ \overrightarrow{a_2} = (2,\ 4,\ 0)^T,$ and $\overrightarrow{a_3} = (3,\ -1,\ 5)^T.$

   i. Show that $S = \left\{\overrightarrow{a_1},\ \overrightarrow{a_2},\ \overrightarrow{a_3}\right\}$ is a basis of $\mathbb{R}^3$.

   ii. Let $\vec{b} = (2,\ -1,\ -2).$ Find $(\vec{b})_S$.

4. Let $S_1 = \left\{\begin{pmatrix}-1\\1\end{pmatrix},\ \begin{pmatrix}1\\0\end{pmatrix}\right\}$ and $S_2 = \left\{\begin{pmatrix}-3\\2\end{pmatrix},\ \begin{pmatrix}2\\3\end{pmatrix}\right\}$.

   i. If $(\vec{b})_{S_2} = (2,\ -1),$ find $(\vec{b})_{S_1}$.

   ii. If $(\vec{b})_{S_1} = (13,\ -26),$ find $(\vec{b})_{S_2}$.

5. If $(\vec{b})_T = (1,\ 2,\ 3),$ find $(\vec{b})_S,$ where

$$S = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

$$T = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

6. Let $\overrightarrow{b_1} = (0, 1, 0)$, $\overrightarrow{b_2} = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$, and $\overrightarrow{b_3} = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$.

   i. Show that $S: = \left\{\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3}\right\}$ is an orthonormal basis of $\mathbb{R}^3$.

   ii. Let $\vec{b} = (1, -1, 1)$. Find $(\vec{b})_S$.

7. Let $\overrightarrow{a_1} = (1, 1, 1)$, $\overrightarrow{a_2} = (0, 1, 1)$, $\overrightarrow{a_3} = (0, 0, 1)$, and $\vec{b} = (-1, 1, -1)^T$.

   1. Find an orthogonal basis $S_1: = \left\{\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3}\right\}$ of $\mathbb{R}^3$ by using the Gram-Schmidt process and $(\vec{b})_{S_1}$.

   2. Find an orthonormal basis $S_2: = \left\{\overrightarrow{q_1}, \overrightarrow{q_2}, \overrightarrow{q_3}\right\}$ of $\mathbb{R}^3$ by normalizing S_1 and $(\vec{b})_{S_2}$.

Solution

1. **1.** Let $A_1 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $|A_1| = -1 \neq 0$. It follows from Theorem (*ii*) that S_1 is a basis of $\mathbb{R}^2$. By Theorem 2.7.5,

$$A_1^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

By (7.4.5), $(\vec{b})_{S_1} = A_1^{-1}\vec{b} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x+y \end{pmatrix}$.

Let $A_2 = \begin{pmatrix} -3 & 2 \\ 2 & 3 \end{pmatrix}$. Then $|A_2| = -9-4 = -13 \neq 0$. By Theorem 7.2.1 (*ii*), S_2 is a basis of $\mathbb{R}^2$. By Theorem 2.7.5 and (7.4.5),

$$A_2^{-1} = \frac{1}{13}\begin{pmatrix} 3 & -2 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{3}{13} \end{pmatrix}$$

and $(\vec{b})_{S_2} = A_2^{-1}\vec{b} = \begin{pmatrix} -\frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{3}{13} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{3x}{13} + \frac{2y}{13} \\ \frac{2x}{13} + \frac{3y}{13} \end{pmatrix}$,

Let $A_3 = \begin{pmatrix} 5 & 4 \\ -6 & -1 \end{pmatrix}$. Then $|A_3| = -5+24 = 19 \neq 0$. By Theorem 7.2.1 (*ii*), S_3 is a basis of $\mathbb{R}^2$. By Theorem 2.7.5 and (7.4.5),

$$A_3^{-1} = \frac{1}{19}\begin{pmatrix} -1 & -4 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{19} & -\frac{4}{19} \\ \frac{6}{19} & \frac{5}{19} \end{pmatrix}.$$

and $(\vec{b})_{S_3} = A_3^{-1}\vec{b} = \begin{pmatrix} -\frac{1}{19} & -\frac{4}{19} \\ \frac{6}{19} & \frac{5}{19} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{x}{19} - \frac{4y}{19} \\ \frac{6x}{19} + \frac{5y}{19} \end{pmatrix},$

Let $A_4 = \begin{pmatrix} 3 & -2 \\ -2 & 5 \end{pmatrix}$. Then $|A_4| = 15 - 4 = 11 \neq 0$. By Theorem 7.2.1 (*ii*), S_4 is a basis of $\mathbb{R}^2$. By Theorem 2.7.5 and (7.4.5),

$$A_4^{-1} = \frac{1}{11}\begin{pmatrix} 5 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{3}{11} \end{pmatrix}.$$

and $(\vec{b})S_4 = A_4^{-1}\vec{b} = \begin{pmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{3}{11} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5x}{11} + \frac{2y}{11} \\ \frac{2x}{11} + \frac{3y}{11} \end{pmatrix}.$

2. **2.**

i. Because $|A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2 \neq 0$, by Theorem 7.2.1 (*ii*), $\left\{\vec{a}_1, \vec{a}_2, \vec{a}_3\right\}$ is a basis of $\mathbb{R}^3$.

ii.

$$(A \mid \vec{0}) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow{R_1(-1)+R_3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2(-1)+R_3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \end{pmatrix} \xrightarrow{R_3\left(-\frac{1}{2}\right)} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}$$

$$\underset{R_3(-1)+R_1}{\xrightarrow{R_2(-1)+R_2}} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Hence, $(\vec{b})_S = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

3. **3.**

i. Because $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 0 & 0 & 5 \end{vmatrix} = 20 \neq 0$, $\left\{\vec{a_1}, \vec{a_2}, \vec{a_3}\right\}$ is a basis of $\mathbb{R}^3$.

ii. Because $|A| = \begin{vmatrix} 2 & 2 & 3 \\ -1 & 4 & -1 \\ -2 & 0 & 5 \end{vmatrix} = 78$, $\left|A_2\right| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & 5 \end{vmatrix} = -7$ and $|A_3| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 4 & -1 \\ 0 & 0 & -2 \end{vmatrix} = -8$, by Cramer's rule

we have

$$x = \frac{|A_1|}{|A|} = \frac{78}{20} = \frac{39}{10},\ y = \frac{|A_2|}{|A|} = -\frac{7}{20} \quad \text{and } z = \frac{|A_3|}{|A|} = \frac{-8}{20} = -\frac{2}{5}.$$

Hence, $(\vec{b})_S = (x,\ y,\ z) = \left(\frac{39}{10},\ -\frac{7}{20},\ -\frac{2}{5}\right)$.

4. **4.**

i. Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ 2 & 3 \end{pmatrix}$. Then by Theorem 2.7.5, $A^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Hence,

$$C = A^{-1}B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}$$

and $(\vec{b})_S = C(\vec{b})_T = \begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \end{pmatrix}$.

ii. Let $A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 2 \\ 2 & 3 \end{pmatrix}$. Then by Theorem 2.7.5, $B^{-1} = \frac{1}{-13}\begin{pmatrix} 3 & -2 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{3}{13} \end{pmatrix}$. Hence,

$$C = B^{-1}A = \begin{pmatrix} -\frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{3}{13} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & -\frac{3}{13} \\ \frac{1}{13} & \frac{2}{13} \end{pmatrix}.$$

By Theorem 7.4.3, we have

$$(\vec{b})_S = C(\vec{b})_T = \begin{pmatrix} \frac{5}{13} & -\frac{3}{13} \\ \frac{1}{13} & \frac{2}{13} \end{pmatrix} \begin{pmatrix} 13 \\ -26 \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \end{pmatrix}.$$

5. **5.** Let $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$.

$$(A \mid I) = \begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1(-1)+R_2} \begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1(-1)} \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2(1)+R_1]{R_2(-1)+R_3} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}.$$

$$C = A^{-1}B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & -2 & -1 \end{pmatrix}.$$

By Theorem 7.4.3, we have

$$(\vec{b})_S = C(\vec{b})_T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ -8 \end{pmatrix}.$$

6. **6.** Because $|A| = \begin{vmatrix} 0 & -\frac{4}{5} & \frac{3}{5} \\ 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \end{vmatrix} = 1 \neq 0,\ \overrightarrow{b_1} \cdot \overrightarrow{b_2} = 0,\ \overrightarrow{b_1} \cdot \overrightarrow{b_3} = 0,\ \overrightarrow{b_2} \cdot \overrightarrow{b_3} = 0,\ \| \overrightarrow{b_1} \| = 1,\ \| \overrightarrow{b_2} \| = 1,\ \| \overrightarrow{b_3} \| = 1,$ S is an orthonormal basis of $\mathbb{R}^3$. Because

$$x = \frac{\vec{b} \cdot \overrightarrow{b_1}}{\| \overrightarrow{b_1} \|^2} = (1,\ -1,\ 1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -1,$$

$$y = \frac{\vec{b} \cdot \overrightarrow{b_2}}{\| \overrightarrow{b_2} \|^2} = (1,\ -1,\ 1)\begin{pmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{pmatrix} = -\frac{4}{5} + \frac{3}{5} = -\frac{1}{5} \text{ and}$$

$$z = \frac{\vec{b} \cdot \overrightarrow{b_3}}{\| \overrightarrow{b_3} \|^2} = (1,\ -1,\ 1)\begin{pmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix} = \frac{3}{5} + \frac{4}{5} = \frac{7}{5},$$

we have $(\vec{b})_S = (x, y, z) = \left(-1, -\frac{1}{5}, \frac{7}{5}\right)$.

7. **7.** Because $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, $\left\{\vec{a}_1, \vec{a}_2, \vec{a}_3\right\}$ is a basis of $\mathbb{R}^3$. We use Theorem 7.4.5 to find S_1 and S_2.

1. Let $\vec{b}_1 = \vec{a}_1 = (1, 1, 1)$ and let

$$\vec{b}_2 = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{b}_1}{\|\vec{b}_1\|^2}\vec{b}_1.$$

To find $\vec{b}_2$, we need to compute $\vec{a}_2 \cdot \vec{b}_1$ and $\|\vec{b}_2\|^2$. Because $\vec{a}_2 \cdot \vec{b}_1 = (0, 1, 1) \cdot (1, 1, 1) = 2$, $\|\vec{b}_1\|^2 = 3$, and

$$\vec{b}_2 = (0, 1, 1) - \frac{2}{3}(1, 1, 1) = (0, 1, 1) - \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Let

$$\vec{b}_3 = \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{b}_1}{\|\vec{b}_1\|^2}\vec{b}_1 - \frac{\vec{a}_3 \cdot \vec{b}_2}{\|\vec{b}_2\|^2}\vec{b}_2.$$

To find $\overrightarrow{b_3}$, we need to compute $\overrightarrow{a_3} \cdot \overrightarrow{b_1}$, $\overrightarrow{a_3} \cdot \overrightarrow{b_2}$ and $\left\|\overrightarrow{b_2}\right\|^2$. Because

$\overrightarrow{a_3} \cdot \overrightarrow{b_1} = (0,\ 0,\ 1) \cdot (1,\ 1,\ 1) = 1$, $\overrightarrow{a_3} \cdot \overrightarrow{b_2} = (0,\ 0,\ 1) \cdot \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$, we obtain

$$\overrightarrow{b_3} = (0,\ 0,\ 1) - \frac{1}{3}(1,\ 1,\ 1) - \frac{\frac{1}{3}}{\frac{1}{3}}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(0,\ -\frac{1}{2}, \frac{1}{2}\right).$$

Hence, $\overrightarrow{b_1} = (1,\ 1,\ 1)$, $\overrightarrow{b_2} = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\overrightarrow{b_3} = \left(0,\ -\frac{1}{2}, \frac{1}{2}\right)$ form an orthogonal basis of $\mathbb{R}^3$.

By computation,

$\left\|\overrightarrow{b_1}\right\| = \sqrt{3}$, $\left\|\overrightarrow{b_2}\right\| = \frac{\sqrt{6}}{3}$, $\left\|\overrightarrow{b_3}\right\| = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$, $\vec{b} \cdot \overrightarrow{b_1} = (-1,\ 1,\ -1) \cdot (1,\ 0,\ 1) = -2$, $\vec{b} \cdot \overrightarrow{b_2} = (-1,\ 1,\ -1) \cdot \left(-\frac{1}{2},\ 1,\ \frac{1}{2}\right) = 1$

and $\vec{b} \cdot \overrightarrow{b_3} = (-1,\ 1,\ -1) \cdot \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right) = \frac{2}{3}$. By Theorem 7.4.4,

$$(\vec{b})_{S_1} = \left(\frac{\vec{b} \cdot \overrightarrow{b_1}}{\left\|\overrightarrow{b_1}\right\|^2}, \frac{\vec{b} \cdot \overrightarrow{b_2}}{\left\|\overrightarrow{b_2}\right\|^2}, \frac{\vec{b} \cdot \overrightarrow{b_3}}{\left\|\overrightarrow{b_3}\right\|^2}\right) = \left(-\frac{2}{3}, \frac{1}{\frac{2}{3}}, \frac{\frac{2}{3}}{\frac{1}{2}}\right) = \left(-\frac{2}{3}, \frac{3}{2}, \frac{4}{3}\right).$$

2. By computation, we obtain

$$\vec{q}_1 = \frac{\vec{b}_1}{\left\|\vec{b}_1\right\|} = \frac{1}{\sqrt{3}}(1,\ 1,\ 1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\vec{q}_2 = \frac{\vec{b}_2}{\left\|\vec{b}_2\right\|} = \frac{3}{\sqrt{6}}\left(-\frac{2}{3}, \frac{1}{3}. \frac{1}{3}\right) = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\vec{q}_3 = \frac{\vec{b}_3}{\left\|\vec{b}_3\right\|} = \sqrt{2}(0,\ -\frac{1}{2}, \frac{1}{2}) = (0,\ -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

Hence, $S_2 = \left\{\vec{q}_1, \vec{q}_2, \vec{q}_3\right\}$ forms an orthonormal basis of $\mathbb{R}^3$. By Theorem 7.4.4,

$$(\vec{b})_{S_2} = (\vec{b} \cdot \vec{q}_1, \vec{b} \cdot \vec{q}_2, \vec{b} \cdot \vec{q}_3) = \left(-\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{6}}, -\sqrt{2}\right).$$

A.8 Eigenvalues and Diagonalizability

Section 8.1

1. For each of the following matrices, find its eigenvalues and determine which eigenvalues are repeated eigenvalues.

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & -4 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2. For each of the following matrices, find its eigenvalues and eigenspaces.

$$A = \begin{pmatrix} 5 & 7 \\ 3 & 1 \end{pmatrix} \; B = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \; C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \; D = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$$

3. Let *A* be a 3×3 matrix. Assume that $2I - A,$ $3I - A,$ and $4I - A$ are not invertible.
    1. Find $|A|$ and tr(*A*).
    2. Prove that $I + 5A$ is invertible.

4. Assume that the eigenvalues of a 3×3 matrix *A* are 1, 2, 3. Compute $\left|A^3 - 2A^2 + 3A + I\right|$.

Solution

1. **1.** By Theorem 8.1.3, the eigenvalues of *A* are $\lambda_1 = \lambda_2 = 1,$ which is a repeated eigenvalue of *A*.
   The eigenvalues of *B* are $\lambda_1 = -2,\ \lambda_2 = 1,$ and $\lambda_3 = -4.$
   The eigenvalues of *C* are $\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = -2,$ and $\lambda_4 = 0.$ The eigenvalue -1 is a repeated eigenvalue of *C*.

2. **2.** Let $A = \begin{pmatrix} 5 & 7 \\ 3 & 1 \end{pmatrix}.$ By computation, we have

$$\begin{aligned} p(\lambda) &= |\lambda I - A| = \begin{vmatrix} \lambda - 5 & -7 \\ -3 & \lambda - 1 \end{vmatrix} = (\lambda - 5)(\lambda - 1) - 21 \\ &= \lambda^2 - 6\lambda + 5 - 21 = \lambda^2 - 6\lambda - 16 = (\lambda + 2)(\lambda - 8). \end{aligned}$$

   Solving $p(\lambda) = 0,$ we obtain $\lambda_1 = -2$ and $\lambda_2 = 8.$
   For $\lambda_1 = -2,\ (\lambda_1 I - A)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} -2-5 & -7 \\ -3 & -2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & -7 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   The above system is equivalent to the equation

$$x + y = 0,$$

   where *x* is a basic variable and *y* is a free variable. Let $y = t.$ Then $x = -t.$ Hence,

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} = t\overrightarrow{v_1},$$

where $\overrightarrow{v_1} = (-1,\ 1).$ Then $E_{\lambda_1} = \left\{ t\overrightarrow{v_1} : t \in \mathbb{R} \right\}.$

For $\lambda_2 = 8,\ (\lambda_2 I - A)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} 8-5 & -7 \\ -3 & 8-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above system is equivalent to the equation

$$3x - 7y = 0,$$

where *x* is a basic variable and *y* is a free variable. Let $y = t.$ Then $x = \frac{7}{3}t.$ Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{3}t \\ t \end{pmatrix} = t\begin{pmatrix} \frac{7}{3} \\ 1 \end{pmatrix} = t\overrightarrow{v_2},$$

where $\vec{v}_2 = \begin{pmatrix} \frac{7}{3} \\ 1 \end{pmatrix}.$ Then $E_{\lambda_2} = \left\{ t\overrightarrow{v_2} : t \in \mathbb{R} \right\}.$

$\begin{pmatrix} 3 & -1 \end{pmatrix}$ … mputation,

$$p(\lambda) = |\lambda I - B| = \begin{vmatrix} \lambda - 3 & 1 \\ -1 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 5) + 1$$
$$= \lambda^2 - 8\lambda + 15 + 1 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

Solving $p(\lambda) = 0$ we obtain two eigenvalues $\lambda_1 = \lambda_2 = 4$.

For $\lambda_1 = \lambda_2 = 4,\ (\lambda_1 I - B)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} 4-3 & 1 \\ -1 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above system is equivalent to the equation

$$x + y = 0,$$

where *x* is a basic variable and *y* is a free variable. Let $y = t$. Then $x = -t$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} = t\overrightarrow{v_1},$$

where $\overrightarrow{v_1} = (-1,\ 1)^T$. Then $E_{\lambda_1} = \left\{t\overrightarrow{v_1} : t \in \mathbb{R}\right\}$.

Let $C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. By computation,

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$$p(\lambda) = |\lambda I - C| = \begin{vmatrix} \lambda - 3 & 0 \\ 0 & \lambda - 3 \end{vmatrix} = (\lambda - 3)^2 = 0.$$

Hence, $\lambda_1 = \lambda_2 = 3.$

For $\lambda_1 = \lambda_2 = 3,\ (\lambda_1 I - C)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The corresponding system is $0x + 0y = 0,$ where *x* and *y* are free variables. Let $x = s$ and $y = t.$ Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = s\vec{v_1} + t\vec{v_2},$$

where $\vec{v_1} = (1,\ 0)^T$ and $\vec{v_2} = (0,\ 1)^T.$ Moreover,

$$E_{\lambda_1} = E_{\lambda_2} = \left\{ s\vec{v_1} + t\vec{v_2} : s, t \in \mathbb{R} \right\}.$$

Let $D = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$ By computation,

$$\begin{aligned} p(\lambda) &= |\lambda I - D| = \begin{vmatrix} \lambda - 1 & -2 & -3 \\ -2 & \lambda - 1 & -3 \\ -1 & -1 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 1)^2(\lambda - 2) - 6 - 6 - 3(\lambda - 1) - 3(\lambda - 1) - 4(\lambda - 2) \\ &= \lambda^3 - 4\lambda^2 - 5\lambda = \lambda(\lambda^2 - 4\lambda - 5) = \lambda(\lambda + 1)(\lambda - 5). \end{aligned}$$

Solving $p(\lambda) = \lambda(\lambda + 1)(\lambda - 5) = 0,$ we get three eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = 0, \text{ and } \lambda_3 = 5.$$

For $\lambda_1 = -1,\ (\lambda_1 I - E)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} -2 & -2 & -3 \\ -2 & -2 & -3 \\ -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we use the row operations to solve the above system.

$$\left(\begin{array}{ccc|c} -2 & -2 & -3 & 0 \\ -2 & -2 & -3 & 0 \\ -1 & -1 & -3 & 0 \end{array}\right) \xrightarrow{R_{1,3}} \left(\begin{array}{ccc|c} -1 & -1 & -3 & 0 \\ -2 & -2 & -3 & 0 \\ -2 & -2 & -3 & 0 \end{array}\right) \xrightarrow[R_1(-2)+R_2]{R_1(-2)+R_3}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow{R_2(-1)+R_3} \left(\begin{array}{ccc|c} -1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3(1)+R_1}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1(-1)]{R_2(\frac{1}{3})} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix is

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$$\begin{cases} x_1 + x_2 = 0 \\ x_3 = 0, \end{cases}$$

where x_1 and x_3 are basic variables and x_2 is a free variable. Let $x_2 = t$. Then $x_1 = -t$. Let $\vec{v}_1 = (-1,\ 1,\ 0)^T$. Then

For $\lambda_2 = 0$, $(\lambda_2 I - E)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} -1 & -2 & -3 \\ -2 & -1 & -3 \\ -1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we use row operations to solve the above system.

$$\left(\begin{array}{ccc|c} -1 & -2 & -3 & 0 \\ -2 & -1 & -3 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right) \xrightarrow{R_1(-1)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -2 & -1 & -3 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[R_1(1)+R_3]{R_1(2)+R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2(\frac{1}{3})} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_2(-2)+R_1]{R_2(-1)+R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

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The system corresponding to the last augmented matrix is

$$\begin{cases} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0, \end{cases}$$

where x_1 and x_2 are basic variables and x_3 is a free variable. Let $x_3 = t$. Then $x_1 = x_2 = -t$. Hence, let $\vec{v}_2 = (-1, -1, 1)^T$. Then

$$E_{\lambda_2} = \left\{ t\vec{v}_2 : t \in \mathbb{R} \right\}.$$

For $\lambda_3 = 5$, $(\lambda_3 I - E)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} 4 & -2 & -3 \\ -2 & 4 & -3 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we use row operations to solve the above system.

$$\left(\begin{array}{ccc|c} 4 & -2 & -3 & 0 \\ -2 & 4 & -3 & 0 \\ -1 & -1 & 3 & 0 \end{array}\right) \xrightarrow{R_{1,3}} \left(\begin{array}{ccc|c} -1 & -1 & 3 & 0 \\ -2 & 4 & -3 & 0 \\ 4 & -2 & -3 & 0 \end{array}\right) \xrightarrow{R_1(-1)}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ -2 & 4 & -3 & 0 \\ 4 & -2 & -3 & 0 \end{array}\right) \underset{R_1(-4)+R_3}{\xrightarrow{R_1(2)+R_3}} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 6 & -9 & 0 \\ 0 & -6 & 9 & 0 \end{array}\right)$$

$$\xrightarrow{R_2(1)+R_3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 6 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2(\frac{1}{6})} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2(-1)+R_1} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix is

$$\begin{cases} x_1 - \frac{3}{2}x_3 &= 0 \\ x_2 - \frac{3}{2}x_3 &= 0, \end{cases}$$

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where x_1 and x_3 are basic variables and x_3 is a free variable. Let $x_3 = 2t$. Then $x_1 = 3t$ and $x_2 = 3t$.

Hence, Let $\overrightarrow{v_3} = (3,\ 3,\ 2)^T$ Then

$$E_{\lambda_3} = \left\{ t\overrightarrow{v_3} : t \in \mathbb{R} \right\}$$

3. **3.**

    1. Because $2I - A,\ 3I - A$ and $4I - A$ are not invertible, $|2I - A| = 0,\ |3I - A| = 0,$ and $|4I - A| = 0$. Hence, 2, 3, and 4 are the eigenvalues of *A* By Theorem 8.1.4, $|A| = 2 \times 3 \times 4 = 24$ and $\mathrm{tr}(A) = 2 + 3 + 4 = 9$.

    2. Because 2, 3, and 4 are the eigenvalues of *A*, $-\frac{1}{5}$ is not an eigenvalue of *A* and $\left| -\frac{1}{5}I - A \right| \neq 0$.

       Hence,

       $$|I + 5A| = |-5(-\frac{1}{5}I - A)| = (-5)^3 |-\frac{1}{5}I - A| \neq 0.$$

       By Theorem 3.4.1, $I + 5A$ is invertible.

4. **4.** Let $\phi(x) = x^3 - 2x^2 + 3x + 1$. Then

$$\phi(A) = A^3 - 2A^2 + 3A + 1.$$

Because 1, 2, 3 are the eigenvalues of *A*, it follows from Theorem 8.1.5 that $\phi(1),\ \phi(2),$ and $\phi(3)$ are all eigenvalues of $\phi(A)$. By computation,

$\phi(1) = 1 - 2 + 3 + 1 = 3,\ \ \phi(2) = 2^3 - 2(2)^2 + 3(2) + 1 = 7,$ and

$\phi(3) = 3^3 - 2(3)^2 + 3(3) + 1 = 19.$

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By Theorem 8.1.4, $|\phi(A)| = |A^3 - 2A^2 + 3A + I| = 3 \times 7 \times 19 = 399.$

Section 8.2

1. For each of the following matrices, determine whether it is diagonalizable.

$$A = \begin{pmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 4 & 3 \\ 0 & -1 & 1 \\ 0 & -4 & 3 \end{pmatrix}.$$

2. Find $x \in \mathbb{R}$ such that *A* is diagonalizable, where

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -4 & 1 & x \\ 1 & 0 & 2 \end{pmatrix}.$$

3. Assume that $\vec{v} = (1,\ 1,\ 0)^T$ is an eigenvector of the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & x & 2 \\ y & 0 & 1 \end{pmatrix}.$$

   i. Find $x,\ y \in \mathbb{R}$ R and the eigenvalue corresponding to $\vec{v}$.
   ii. Is A diagonalizable?

4. Let

$$A = \begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.
2. Find an inverse matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.

5. Let

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.
2. Find an inverse matrix *P* and a diagonal matrix *D* such that

$$A = PDP^{-1}.$$

3. Compute A^2 by using the result (2).

6. Let

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.

2. Find an inverse matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.
3. Compute A^2 by using the result (2).

7. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.
2. Find an inverse matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.
3. Compute A^3 by using the result (2).

8. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.
2. Find an inverse matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.
3. Compute A^2 by using the result (2).

9. Two $n \times n$ matrices *A* and *B* are said to be similar (or *A* is similar to *B*) if there exists an invertible $n \times n$ matrix *P* such that $B = P^{-1}AP$. We write $A \sim B$. Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}, \text{ and } P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Show that *A* is similar to *B*.

10. Assume that $A \sim B$. Show that the following assertions hold.
    i. $|\lambda I - A| = |\lambda I - B|$.
    ii. *A* and *B* have the same eigenvalues.
    iii. $\mathrm{tr}(A) = \mathrm{tr}(B)$.
    iv. $|A| = |B|$.
    v. $r(A) = r(B)$.

11. Assume that *A, B, C* are $n \times n$ matrices. Show that the following assertions hold
    i. $A \sim A$.
    ii. If $A \sim B$, then $B \sim A$.
    iii. If $A \sim B$ and $B \sim C$, then $A \sim C$.

12. Assume that *A* and *B* are $n \times n$ matrices and $A \sim B$. Prove that the following assertions hold.
    a. $A^k \sim B^k$ for every positive integer *k*.
    b. Let $\phi(\lambda) = a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0$. Then $\phi(A) \sim \phi(B)$.

Solution

1. **1.** By Theorem 8.1.3, the eigenvalues of *A* are $\lambda_1 = \lambda_2 = 1,$ which is a repeated eigenvalue of *A*.
   The eigenvalues of *B* are $\lambda_1 = -2,\ \lambda_2 = 1,$ and $\lambda_3 = -4.$
   The eigenvalues of *C* are $\lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = -2,$ and $\lambda_4 = 0.$ The eigenvalue -1 is a repeated eigenvalue of *C*.

2. **2.** Let $A = \begin{pmatrix} 5 & 7 \\ 3 & 1 \end{pmatrix}.$ By computation, we have

$$\begin{aligned} p(\lambda) &= |\lambda I - A| = \begin{vmatrix} \lambda - 5 & -7 \\ -3 & \lambda - 1 \end{vmatrix} = (\lambda-5)(\lambda-1) - 21 \\ &= \lambda^2 - 6\lambda + 5 - 21 = \lambda^2 - 6\lambda - 16 = (\lambda+2)(\lambda-8). \end{aligned}$$

   Solving $p(\lambda) = 0,$ we obtain $\lambda_1 = -2$ and $\lambda_2 = 8.$
   For $\lambda_1 = -2,\ (\lambda_1 I - A)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} -2-5 & -7 \\ -3 & -2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & -7 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   The above system is equivalent to the equation

$$x + y = 0,$$

   where *x* is a basic variable and *y* is a free variable. Let $y = t.$ Then $x = -t.$ Hence,

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$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} = t\overrightarrow{v_1},$$

where $\overrightarrow{v_1} = (-1, 1)$. Then $E_{\lambda_1} = \left\{ t\overrightarrow{v_1} : t \in \mathbb{R} \right\}$.

For $\lambda_2 = 8$, $(\lambda_2 I - A)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} 8-5 & -7 \\ -3 & 8-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above system is equivalent to the equation

$$3x - 7y = 0,$$

where *x* is a basic variable and *y* is a free variable. Let $y = t$. Then $x = \frac{7}{3}t$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{3}t \\ t \end{pmatrix} = t\begin{pmatrix} \frac{7}{3} \\ 1 \end{pmatrix} = t\overrightarrow{v_2},$$

where $\vec{v}_2 = \begin{pmatrix} \frac{7}{3} \\ 1 \end{pmatrix}$. Then $E_{\lambda_2} = \left\{ t\overrightarrow{v_2} : t \in \mathbb{R} \right\}$.

Processing math: 100% $= \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}$. By computation,

$$p(\lambda) = |\lambda I - B| = \begin{vmatrix} \lambda - 3 & 1 \\ -1 & \lambda - 5 \end{vmatrix} = (\lambda - 3)(\lambda - 5) + 1$$
$$= \lambda^2 - 8\lambda + 15 + 1 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2.$$

Solving $p(\lambda) = 0$ we obtain two eigenvalues $\lambda_1 = \lambda_2 = 4$.
For $\lambda_1 = \lambda_2 = 4$, $(\lambda_1 I - B)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} 4-3 & 1 \\ -1 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above system is equivalent to the equation

$$x + y = 0,$$

where *x* is a basic variable and *y* is a free variable. Let $y = t$. Then $x = -t$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix} = t\overrightarrow{v_1},$$

where $\overrightarrow{v_1} = (-1, 1)^T$. Then $E_{\lambda_1} = \left\{ t\overrightarrow{v_1} : t \in \mathbb{R} \right\}$.

Let $C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. By computation,

$$p(\lambda) = |\lambda I - C| = \begin{vmatrix} \lambda - 3 & 0 \\ 0 & \lambda - 3 \end{vmatrix} = (\lambda - 3)^2 = 0.$$

Hence, $\lambda_1 = \lambda_2 = 3.$

For $\lambda_1 = \lambda_2 = 3,\ (\lambda_1 I - C)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The corresponding system is $0x + 0y = 0,$ where *x* and *y* are free variables. Let $x = s$ and $y = t.$ Then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix} = s\overrightarrow{v_1} + t\overrightarrow{v_2},$$

where $\overrightarrow{v_1} = (1,\ 0)^T$ and $\overrightarrow{v_2} = (0,\ 1)^T.$ Moreover,

$$E_{\lambda_1} = E_{\lambda_2} = \left\{ s\overrightarrow{v_1} + t\overrightarrow{v_2} : s, t \in \mathbb{R} \right\}.$$

Let $D = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$ By computation,

$$\begin{aligned} p(\lambda) &= |\lambda I - D| = \begin{vmatrix} \lambda - 1 & -2 & -3 \\ -2 & \lambda - 1 & -3 \\ -1 & -1 & \lambda - 2 \end{vmatrix} \\ &= (\lambda - 1)^2(\lambda - 2) - 6 - 6 - 3(\lambda - 1) - 3(\lambda - 1) - 4(\lambda - 2) \\ &= \lambda^3 - 4\lambda^2 - 5\lambda = \lambda(\lambda^2 - 4\lambda - 5) = \lambda(\lambda + 1)(\lambda - 5). \end{aligned}$$

Solving $p(\lambda) = \lambda(\lambda + 1)(\lambda - 5) = 0,$ we get three eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = 0, \text{ and } \lambda_3 = 5.$$

For $\lambda_1 = -1,\ (\lambda_1 I - E)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} -2 & -2 & -3 \\ -2 & -2 & -3 \\ -1 & -1 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we use the row operations to solve the above system.

$$\left(\begin{array}{ccc|c} -2 & -2 & -3 & 0 \\ -2 & -2 & -3 & 0 \\ -1 & -1 & -3 & 0 \end{array}\right) \xrightarrow{R_{1,3}} \left(\begin{array}{ccc|c} -1 & -1 & -3 & 0 \\ -2 & -2 & -3 & 0 \\ -2 & -2 & -3 & 0 \end{array}\right) \xrightarrow[R_1(-2)+R_2]{R_1(-2)+R_3}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow{R_2(-1)+R_3} \left(\begin{array}{ccc|c} -1 & -1 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3(1)+R_1}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1(-1)]{R_2(\frac{1}{3})} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix is

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$$\begin{cases} x_1 + x_2 = 0 \\ x_3 = 0, \end{cases}$$

where x_1 and x_3 are basic variables and x_2 is a free variable. Let $x_2 = t$. Then $x_1 = -t$. Let $\vec{v}_1 = (-1,\ 1,\ 0)^T$. Then

For $\lambda_2 = 0$, $(\lambda_2 I - E)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} -1 & -2 & -3 \\ -2 & -1 & -3 \\ -1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, we use row operations to solve the above system.

$$\left(\begin{array}{ccc|c} -1 & -2 & -3 & 0 \\ -2 & -1 & -3 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right) \xrightarrow{R_1(-1)} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -2 & -1 & -3 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[R_1(1)+R_3]{R_1(2)+R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2(\frac{1}{3})} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_2(-2)+R_1]{R_2(-1)+R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

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The system corresponding to the last augmented matrix is

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0, \end{cases}$$

where x_1 and x_2 are basic variables and x_3 is a free variable. Let $x_3 = t$. Then $x_1 = x_2 = -t$. Hence, let $\vec{v_2} = (-1, -1, 1)^T$. Then

$$E_{\lambda_2} = \left\{ t\vec{v_2} : t \in \mathbb{R} \right\}.$$

For $\lambda_3 = 5$, $(\lambda_3 I - E)\vec{x} = \vec{0}$ becomes

$$\begin{pmatrix} 4 & -2 & -3 \\ -2 & 4 & -3 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we use row operations to solve the above system.

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$$\left(\begin{array}{ccc|c} 4 & -2 & -3 & 0 \\ -2 & 4 & -3 & 0 \\ -1 & -1 & 3 & 0 \end{array}\right) \xrightarrow{R_{1,3}} \left(\begin{array}{ccc|c} -1 & -1 & 3 & 0 \\ -2 & 4 & -3 & 0 \\ 4 & -2 & -3 & 0 \end{array}\right) \xrightarrow{R_1(-1)}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ -2 & 4 & -3 & 0 \\ 4 & -2 & -3 & 0 \end{array}\right) \underset{R_1(-4)+R_3}{\xrightarrow{R_1(2)+R_3}} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 6 & -9 & 0 \\ 0 & -6 & 9 & 0 \end{array}\right)$$

$$\xrightarrow{R_2(1)+R_3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 6 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2(\frac{1}{6})} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2(-1)+R_1} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix is

$$\begin{cases} x_1 - \frac{3}{2}x_3 = 0 \\ x_2 - \frac{3}{2}x_3 = 0, \end{cases}$$

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where x_1 and x_3 are basic variables and x_3 is a free variable. Let $x_3 = 2t$. Then $x_1 = 3t$ and $x_2 = 3t$.

Hence, Let $\vec{v_3} = (3,\ 3,\ 2)^T$ Then

$$E_{\lambda_3} = \left\{ t\vec{v_3} : t \in \mathbb{R} \right\}$$

3. **3.**

   1. Because $2I - A,\ 3I - A$ and $4I - A$ are not invertible, $|2I - A| = 0,\ |3I - A| = 0,$ and $|4I - A| = 0.$ Hence, 2, 3, and 4 are the eigenvalues of *A* By Theorem 8.1.4, $|A| = 2 \times 3 \times 4 = 24$ and $\mathrm{tr}(A) = 2 + 3 + 4 = 9.$

   2. Because 2, 3, and 4 are the eigenvalues of *A*, $-\frac{1}{5}$ is not an eigenvalue of *A* and $\left| -\frac{1}{5}I - A \right| \neq 0.$

      Hence,

      $$|I + 5A| = |-5(-\frac{1}{5}I - A)| = (-5)^3 |-\frac{1}{5}I - A| \neq 0.$$

      By Theorem 3.4.1, $I + 5A$ is invertible.

4. **4.** Let $\phi(x) = x^3 - 2x^2 + 3x + 1$. Then

$$\phi(A) = A^3 - 2A^2 + 3A + 1.$$

Because 1, 2, 3 are the eigenvalues of *A*, it follows from Theorem 8.1.5 that $\phi(1),\ \phi(2),$ and $\phi(3)$ are all eigenvalues of $\phi(A)$. By computation,

$$\phi(1) = 1 - 2 + 3 + 1 = 3,\quad \phi(2) = 2^3 - 2(2)^2 + 3(2) + 1 = 7,\quad \text{and}$$

$$\phi(3) = 3^3 - 2(3)^2 + 3(3) + 1 = 19.$$

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By Theorem 8.1.4, $|\phi(A)| = |A^3 - 2A^2 + 3A + I| = 3 \times 7 \times 19 = 399.$

Section 8.2

1. For each of the following matrices, determine whether it is diagonalizable.

$$A = \begin{pmatrix} -1 & 2 & 4 & 0 \\ 0 & 3 & 1 & 7 \\ 0 & 0 & 5 & 8 \\ 0 & 0 & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 4 & 3 \\ 0 & -1 & 1 \\ 0 & -4 & 3 \end{pmatrix}.$$

2. Find $x \in \mathbb{R}$ such that *A* is diagonalizable, where

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -4 & 1 & x \\ 1 & 0 & 2 \end{pmatrix}.$$

3. Assume that $\vec{v} = (1,\ 1,\ 0)^T$ is an eigenvector of the matrix

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & x & 2 \\ y & 0 & 1 \end{pmatrix}.$$

   i. Find $x,\ y \in \mathbb{R}$ R and the eigenvalue corresponding to $\vec{v}$.
   ii. Is *A* diagonalizable?

4. Let

$$A = \begin{pmatrix} 1 & -1 \\ -4 & 1 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.
2. Find an inverse matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.

5. Let

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.
2. Find an inverse matrix *P* and a diagonal matrix *D* such that

$$A = PDP^{-1}.$$

3. Compute A^2 by using the result (2).

6. Let

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.

2. Find an inverse matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.
3. Compute A^2 by using the result (2).

7. Let

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.
2. Find an inverse matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.
3. Compute A^3 by using the result (2).

8. Let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1. Find the eigenvalues and eigenspaces of *A*.
2. Find an inverse matrix *P* and a diagonal matrix *D* such that $A = PDP^{-1}$.
3. Compute A^2 by using the result (2).

9. Two $n \times n$ matrices *A* and *B* are said to be similar (or *A* is similar to *B*) if there exists an invertible $n \times n$ matrix *P* such that $B = P^{-1}AP$. We write $A \sim B$. Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}, \text{ and } P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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Show that *A* is similar to *B*.

10. Assume that $A \sim B$. Show that the following assertions hold.
    i. $|\lambda I - A| = |\lambda I - B|$.
    ii. *A* and *B* have the same eigenvalues.
    iii. $\mathrm{tr}(A) = \mathrm{tr}(B)$.
    iv. $|A| = |B|$.
    v. $r(A) = r(B)$.

11. Assume that *A, B, C* are $n \times n$ matrices. Show that the following assertions hold
    i. $A \sim A$.
    ii. If $A \sim B$, then $B \sim A$.
    iii. If $A \sim B$ and $B \sim C$, then $A \sim C$.

12. Assume that *A* and *B* are $n \times n$ matrices and $A \sim B$. Prove that the following assertions hold.
    a. $A^k \sim B^k$ for every positive integer *k*.
    b. Let $\phi(\lambda) = a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0$. Then $\phi(A) \sim \phi(B)$.

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Solution

1. **1.** Because *A* is a 4×4 upper triangular matrix, by Theorem 8.1.3, all the eigenvalues of *A* are the entries on the main diagonal. Hence, $\lambda_1 = -1,\ \lambda_2 = 3,\ \lambda_3 = 5,$ and $\lambda_4 = -2$ are the eigenvalues of *A* and *A* has four distinct eigenvalues. It follows from Corollary 8.2.1 (1) that *A* is diagonalizable.

$$\left|\lambda I - B\right| = \begin{vmatrix} \lambda+1 & -4 & -3 \\ 0 & \lambda+1 & -1 \\ 0 & 4 & \lambda-3 \end{vmatrix} = (\lambda+1)(\lambda-1)^2 = 0.$$

Then $\lambda_1 = -1,\ \lambda_2 = \lambda_3 = 1.$
For $\lambda_2 = \lambda_3 = 1,$ the system $\lambda_2 I - A = 0$ becomes

$$\begin{pmatrix} 2 & 4 & -3 \\ 0 & 2 & -1 \\ 0 & 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We solve the above system.

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$$(A\,|\,\vec{0}): \quad = \left(\begin{array}{ccc|c} 2 & 4 & -3 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right) \underset{R_2(-2)+R_1}{\overset{R_{-2}+R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{R_1(\frac{1}{2})}{\overset{R_1(\frac{1}{2})}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix has one free variable and thus, $\dim(E_{\lambda_2}) = 1 < 2$ (algebraic multiplicity). By Corollary 8.2.1 (*ii*), *A* is not diagonalizable.

2. **2.** $$\left|\lambda I - A\right| = \begin{vmatrix} \lambda-2 & 0 & -1 \\ 4 & \lambda-1 & -x \\ -1 & 0 & \lambda-2 \end{vmatrix} = (\lambda-1)^2(\lambda-3) = 0.$$

Then $\lambda_1 = 1$ (with algebraic multiplicity 2), $\lambda_2 = 3$.

We find $\dim(E\lambda_1)$. For $\lambda_1 = 1$, the system $\lambda_1 I - A = 0$ becomes

$$\begin{pmatrix} -1 & 0 & -1 \\ 4 & 0 & -x \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We solve the above system.

$$(A\,|\,\vec{0}) = \left(\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 4 & 0 & -x & 0 \\ -1 & 0 & -1 & 0 \end{array}\right) \xrightarrow[R_1(-1)+R_3]{R_1(4)+R_2} \left(\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & 0 & -x-4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

If $x = -4,$ then the system corresponding to the last augmented matrix has two free variables and $\dim(E\lambda_1) = 2$ (algebraic multiplicity). By Corollary 8.2.1 (*i*), *A* is diagonalizable.

3. **3.**

i. Let $\lambda \in \mathbb{R}$ be the eigenvalue of *A* corresponding to $\vec{v}$. Then $(\lambda I - A)\vec{v} = \vec{0},$ that is,

$$\begin{pmatrix} \lambda-1 & 1 & -1 \\ -3 & \lambda-x & -2 \\ -y & 0 & \lambda-1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} \lambda \\ \lambda-x-3 \\ -y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and $\lambda = 0,$ $\lambda - x - 3 = 0,$ and $-y = 0.$ Hence, $x = -3,$ $y = 0,$ and $\lambda = 0.$

ii. When $x = -3,$ $y = 0$ and $\lambda = 0,$ A becomes

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 3 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

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Hence,

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & -1 \\ -3 & \lambda + 3 & -2 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)(\lambda + 2) = 0.$$

Then $\lambda_1 = -2,\ \lambda_2 = 0,$ and $\lambda_3 = 1$. By Corollary 8.2.1 (1), *A* is diagonalizable.

4. **4.**

1.

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 \\ 4 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3).$$

Solving $p(\lambda) = (\lambda + 1)(\lambda - 3) = 0$ implies $\lambda_1 = -1$ and $\lambda_2 = 3$.
For $\lambda_1 = -1,\ (\lambda_1 I - A)\vec{X} = \vec{0}$ becomes

$$\begin{pmatrix} -1-1 & 1 \\ 4 & -1-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above system is equivalent to the equation

$$-2x + y = 0,$$

where *x* is a basic variable and *y* is a free variable. Let $y = 2t$. Then $x = t$. Hence,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ 2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} = t\overrightarrow{v_1},$$

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where $\vec{v_1} = (1,\ 2)^T$. Then $E_{\lambda_1} = \left\{ t\vec{v_1} : t \in \mathbb{R} \right\}$.

For $\lambda_2 = 3,\ (\lambda_2 I - A)\vec{X} = \vec{0}$ becomes

$$\begin{pmatrix} 3-1 & 1 \\ 4 & 3-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above system is equivalent to

$$2x + y = 0,$$

where *x* is a basic variable and *y* is a free variable. Let $y = -2t$. Then $x = t$ and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ -2t \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix} = tv_2,$$

where $\vec{v_2} = (1,\ -2)^T$. Then $E_{\lambda_2} = \left\{ t\vec{v_2} : t \in \mathbb{R} \right\}$.

2. Let

$$D = \operatorname{diag}(\lambda_1,\ \lambda_2) = \operatorname{diag}(-1,\ 3) \ \text{ and } P = (\vec{v_1}\vec{v_2}) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Then by Theorem 8.2.2, $A = P^{-1}DP$.

5.

1. Because

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 3 & -2 & -4 \\ -2 & \lambda & -2 \\ -4 & -2 & \lambda - 3 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 15\lambda - 8$$

$$= (\lambda + 1)^2(\lambda - 8) = 0.$$

Then $\lambda_1 = -1$ (with algebraic multiplicity 2), $\lambda_2 = 8$.
We find $\dim(E\lambda_1)$. For $\lambda_1 = 1,$ the system $\lambda_1 I - A = 0$ becomes

$$\begin{pmatrix} -4 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We solve the above system.

$$(A \,|\, \vec{0}) = \left(\begin{array}{ccc|c} -4 & -2 & -4 & 0 \\ -2 & -1 & -2 & 0 \\ -4 & -2 & -4 & 0 \end{array}\right) \underset{R_1(-1)+R_3}{\xrightarrow{R_1(-\frac{1}{2})+R_2}} \left(\begin{array}{ccc|c} -4 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1(-\tfrac{1}{4}) \rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

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The system with the last augmented matrix is equivalent to the equation

$$x + \frac{1}{2}y + z = 0,$$

where *x* is a basic variable and *y*, *z* are free variables. Let $y = 2s$ and $z = t$. Then $x = -s - t$. Hence,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s-t \\ 2s \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = t\overrightarrow{v}_1 + s\overrightarrow{v}_2,$$

where $\overrightarrow{v}_1 = (-1,\ 2,\ 0)^T$ and $\overrightarrow{v}_2 = (-1,\ 0,\ 1)^T$. Then

$$E_{\lambda_1} = \left\{ s\overrightarrow{v}_1 + t\overrightarrow{v}_2 : s,\ t \in \mathbb{R} \right\}.$$

For $\lambda_2 = 8$, the system $\lambda_2 I - A = 0$ becomes

$$\begin{pmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We solve the above system.

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$$(A\,|\,\vec{0}) \quad = \left(\begin{array}{ccc|c} 5 & -2 & -4 & 0 \\ -2 & 8 & -2 & 0 \\ -4 & -2 & 5 & 0 \end{array}\right) R_2(2) + R_1 \to \left(\begin{array}{ccc|c} 1 & 14 & -8 & 0 \\ -2 & 8 & -2 & 0 \\ -4 & -2 & 5 & 0 \end{array}\right)$$

$$\underset{R_1(4)+R_3}{\overset{R_1(2)+R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 14 & -8 & 0 \\ 0 & 36 & -18 & 0 \\ 0 & 54 & -27 & 0 \end{array}\right) \underset{R_3(\frac{1}{27})}{\overset{R_2(\frac{1}{18})}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 14 & -8 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right)$$

$$R_2(-1) + R_3 \to \left(\begin{array}{ccc|c} 1 & 14 & -8 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_3(-7) + R_1 \to$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system with the last augmented matrix is equivalent to the system

$$\begin{cases} x - z = 0 \\ 2y - z = 0, \end{cases}$$

where *x* and *y* are basic variables and *z* is a free variable.

Let $z = 2t$. Then $x = 2t$ and $y = t$. Let $\vec{v_3} = (2,\ 1,\ 2)^T$. Then

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$$E_{\lambda_2} = \left\{ t\overrightarrow{v_3} : t \in \mathbb{R} \right\}.$$

2. Let $D = \mathrm{diag}(\lambda_1,\ \lambda_1,\ \lambda_2) = \mathrm{diag}(-1,\ -1,\ 8)$ and

$$P = (\overrightarrow{v_1}\overrightarrow{v_2}\overrightarrow{v_3}) = \begin{pmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Then by Theorem 8.2.2, $A = PDP^{-1}$.

3.

$$(P \,|\, I) = \left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) R_1(2) + R_3 \rightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -2 & 5 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) R_{2,\ 3} \rightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & -2 & 5 & 2 & 1 & 0 \end{array}\right) R_2(2) + R_3 \rightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 9 & 2 & 1 & 2 \end{array}\right) R_3(\tfrac{1}{9}) \rightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \end{array}\right) \underset{R_3(-2)+R_1}{\overset{R_3(-2)+R_2}{\longrightarrow}}$$

$$\left(\begin{array}{ccc|ccc} -1 & -1 & 0 & \frac{5}{9} & -\frac{2}{9} & -\frac{4}{9} \\ 0 & 1 & 0 & -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \end{array}\right) R_2(1)+R_1 \longrightarrow$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & \frac{1}{9} & -\frac{4}{9} & \frac{1}{9} \\ 0 & 1 & 0 & -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \end{array}\right) R_1(-1) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{9} & \frac{4}{9} & -\frac{1}{9} \\ 0 & 1 & 0 & -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \\ 0 & 0 & 1 & \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \end{array}\right)$$

$$= (I \mid P^{-1}).$$

Hence,

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$$P^{-1} = \begin{pmatrix} -\frac{1}{9} & \frac{4}{9} & -\frac{1}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \end{pmatrix}.$$

By Theorem 8.2.6,

$$A^2 = PD^2P^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix}^2 \begin{pmatrix} -\frac{1}{9} & \frac{4}{9} & -\frac{1}{9} \\ -\frac{4}{9} & -\frac{2}{9} & \frac{5}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \end{pmatrix}$$

$$= [\begin{pmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix}][\frac{1}{9} \begin{pmatrix} -1 & 4 & -1 \\ -4 & -2 & 5 \\ 2 & 1 & 2 \end{pmatrix}]$$

$$= \frac{1}{9} \begin{pmatrix} -1 & -1 & 128 \\ 2 & 0 & 64 \\ 0 & 1 & 128 \end{pmatrix} \begin{pmatrix} -1 & 4 & -1 \\ -4 & -2 & 5 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 261 & 126 & 252 \\ 126 & 72 & 126 \\ 252 & 126 & 261 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & 14 & 28 \\ 14 & 8 & 14 \\ 28 & 14 & 29 \end{pmatrix}.$$

6. **6.**

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$$
\begin{aligned}
p(\lambda) &= |\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{vmatrix} = \lambda(\lambda-2)(\lambda-3) + 2(\lambda-2) \\
&= (\lambda-2)(\lambda^2 - 3\lambda + 2) = (\lambda-2)(\lambda-2)(\lambda-1) \\
&= (\lambda-1)(\lambda-2)^2 = 0.
\end{aligned}
$$

This implies $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 2$.
For $\lambda_1 = 1$, the system $\lambda_1 I - A = 0$ becomes

$$
\begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

$$
\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right) \underset{R_1(1)+R_3}{\overset{R_1(1)+R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)
$$

$$
R_2(-1) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).
$$

The system corresponding to the last augmented matrix is

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0, \end{cases}$$

where x_1 and x_2 are basic variables and x_3 is a free variable. Let $x_3 = t.$ Then $x_1 = -2t$ and $x_2 = t.$ Let $\vec{v}_1 = (-2, 1, 1)^T.$ We have

$$E_{\lambda_1} = \left\{ t\vec{v}_1 : t \in \mathbb{R} \right\} = \text{span}\{\vec{v}_1\}.$$

For $\lambda_2 = \lambda_3 = 2,$ the system $\lambda_2 I - A = 0$ becomes

$$\begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right) R_1(\tfrac{1}{2}) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right)$$

$$\underset{R_1(1)+R_3}{\overset{R_1(1)+R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix is

$$x_1 + x_3 = 0,$$

where x_1 is a basic variable, and x_2 and x_3 are free variables. Let $x_2 = s$ and $x_3 = t.$ Then $x_1 = -t.$ Then

$$(x_1,\ x_2,\ x_3) = (-t,\ s,\ t) = (-t,\ 0,\ t) + (0,\ s,\ 0) = t(-1,\ 0,\ 1) + s(0,\ 1,\ 0).$$

Hence, we get two basic vectors $\overrightarrow{v_2} = (-1,\ 0,\ 1)^T$ and $\overrightarrow{v_3} = (0,\ 1,\ 0)^T.$
Hence,

$$E_{\lambda_2} = \left\{ s\overrightarrow{v_2} + t\overrightarrow{v_3} : s,\ t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \overrightarrow{v_2},\ \overrightarrow{v_3} \right\}.$$

2. Let

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$P = (\overrightarrow{v_1}\overrightarrow{v_2}\overrightarrow{v_3}) = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then P diagonalizes A and by Theorem 8.2.2, $A = PDP^{-1}$.

3.

$$(P\,|\,I) = \left(\begin{array}{ccc|ccc} -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) R_{1,\,2}\!\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1(-1)+R_3]{R_1(2)+R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right) R_2(1)+R_3\!\rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow[R_3(-1)+R_1]{R_3(-2)+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$R_2(-1)\!\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) = (I\,|\,P^{-1})$$

Hence,

$$P^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

By Theorem 8.2.6,

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$$A^2 = PD^2P^{-1} = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}^2\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= [\begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}]\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -4 & 0 \\ 1 & 0 & 4 \\ 1 & 4 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -6 \\ 3 & 4 & 3 \\ 3 & 0 & 7 \end{pmatrix}.$$

7. **7.**

i.

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & -1 \\ -3 & \lambda + 3 & -2 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = \lambda - 1\begin{vmatrix} \lambda - 1 & 1 \\ -3 & \lambda + 3 \end{vmatrix}$$

$$= (\lambda - 1)[(\lambda - 1)(\lambda + 3) - (-3)] = (\lambda - 1)(\lambda^2 + 2\lambda) = (\lambda + 2)\lambda(\lambda - 1).$$

This implies $\lambda_1 = -2, \ \lambda_2 = 0,$ and $\lambda_3 = 1.$
For $\lambda_1 = -2,$ the system $\lambda_1 I - A = 0$ becomes

$$\begin{pmatrix} -3 & 1 & -1 \\ -3 & 1 & -2 \\ 0 & 0 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c}-3 & 1 & -1 & 0\\ -3 & 1 & -2 & 0\\ 0 & 0 & -3 & 0\end{array}\right) R_1(-1)+R_2 \rightarrow \left(\begin{array}{ccc|c}-3 & 1 & -1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & -3 & 0\end{array}\right)$$

$$R_2(-3)+R_3 \rightarrow \left(\begin{array}{ccc|c}-3 & 1 & -1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right) R_3(-1)+R_1 \rightarrow \left(\begin{array}{ccc|c}-3 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right).$$

The system corresponding to the last augmented matrix is

$$\begin{cases}-3x_1 + x_2 = 0\\ -x_3 = 0,\end{cases}$$

where x_1 and x_2 are basic variables and x_2 is a free variable. Let $x_2 = 3t$. Then $x_1 = t$ and $x_3 = 0$. Let $\vec{v}_1 = (1,\ 3,\ 0)^T$. We have

$$E_{\lambda_1} = \left\{t\vec{v}_1 : t \in \mathbb{R}\right\} = \operatorname{span}\left\{\vec{v}_1\right\}.$$

For $\lambda_2 = 0$, the system $\lambda_2 I - A = 0$ becomes

$$\begin{pmatrix}-1 & 1 & -1\\ -3 & 3 & -2\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -3 & 3 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) R_1(-3) + R_2 \rightarrow \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\underset{R_2(1)+R_1}{\overset{R_2(1)+R_2}{\longrightarrow}} \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix is

$$\begin{cases} -x_1 + x_2 = 0, \\ x_3 = 0, \end{cases}$$

where x_1 and x_3 are basic variables and x_2 is a free variable. Let $x_2 = t.$ Then $x_1 = t.$ Then

$(x_1,\ x_2,\ x_3) = (t,\ t,\ 0) = t(1,\ 1,\ 0).$ Let $\vec{v}_2 = (1,\ 1,\ 0)^T.$
We have

$E_{\lambda_2} = \left\{ t\vec{v}_2 : t \in \mathbb{R} \right\} = \text{span}\left\{ \vec{v}_2 \right\}.$

For $\lambda_3 = 1,$ the system $\lambda_3 I - A = 0$ becomes

$$\begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -3 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_{1,\,2} \rightarrow \left(\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_2(-4) + R_1 \rightarrow$$

$$\left(\begin{array}{ccc|c} -3 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix is

$$\begin{cases} -3x_1 + 2x_3 = 0, \\ x_2 - x_3 = 0, \end{cases}$$

where x_1 and x_2 are basic variables and x_{23} is a free variable. Let $x_3 = 3t$. Then $x_1 = 2t$ and $x_2 = 3t$. Then $(x_1,\ x_2,\ x_3) = (2t,\ 3t,\ 3) = t(2,\ 3,\ 3)$. Let $\vec{v}_2 = (2,\ 3,\ 3)^T$. We have

$$E_{\lambda_3} = \left\{ t\vec{v}_3 : t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \vec{v}_3 \right\}.$$

ii. Let

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$P = (\overrightarrow{v_1}\,\overrightarrow{v_2}\,\overrightarrow{v_3}) = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then *P* diagonalizes *A* and by Theorem 8.2.2, $A = PDP^{-1}$.

iii.

$$(P\,|\,I) \quad = \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right) R_1(-3) + R_2 \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -3 & -3 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$R_3(\tfrac{1}{3}) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -3 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) \underset{R_3(-2)+R_1}{\overset{R_3(3)+R_2}{\longrightarrow}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -\frac{2}{3} \\ 0 & -2 & 0 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) R_2(-\tfrac{1}{2}) \rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -\frac{2}{3} \end{array}\right.$$

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) R_2(-1) + R_1 \rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right) = (I \mid P^{-1})$$

Hence,

$$P^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{6} \\ \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

By Theorem 8.2.6,

$$A^3 = PD^3P^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{6} \\ \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$= [\begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}][\frac{1}{6} \begin{pmatrix} -3 & 3 & -1 \\ 9 & -3 & -3 \\ 0 & 0 & 2 \end{pmatrix}]$$

$$= \frac{1}{6} \begin{pmatrix} -8 & 0 & 2 \\ -24 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -3 & 3 & -1 \\ 9 & -3 & -3 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 24 & -24 & 12 \\ 72 & -72 & 54 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 & 2 \\ 12 & -12 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. **8.**

1. Because $|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0,$ the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$.

   For $\lambda_1 = i,\ (\lambda_1 I - A)\vec{X} = \vec{0}$ becomes

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$$\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now we use row operations to solve the above system.

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) R_{1,2}\rightarrow \left(\begin{array}{cc|c} 1 & i & 0 \\ i & -1 & 0 \end{array}\right) R_1(-i)+R_2\rightarrow \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right).$$

The system corresponding to the last augmented matrix is $x + iy = 0,$ where *x* is a basic variable and *y* is a free variable. Let $y = t.$ Then $x = -it$ and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -it \\ t \end{pmatrix} = t\begin{pmatrix} -i \\ 1 \end{pmatrix} = tv_1,$$

where $\vec{v_2} = (i,\ 1)^T.$ Moreover, $E_{\lambda_2} = \left\{ t\vec{v_2} : t \in \mathbb{R} \right\}.$

2. Let $D = \operatorname{diag}(\lambda_1,\ \lambda_2) = \operatorname{diag}(i - i)$ and

$$P = (\vec{v_1}\vec{v_2}) = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$

Then $A = PDP^{-1}.$

3. Because

$$D^{100} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^{100} = \begin{pmatrix} i^{100} & 0 \\ 0 & (-i)^{100} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

and

$$A^{100} = PD^{100}P^{-1} = PIP^{-1} = PP^{-1} = I_2.$$

Remark A.8.1.

You can check $A = PDP^{-1}$. Indeed,

$$AP = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \text{ and}$$

$$PD = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}.$$

9. **9.** By (1.2.5), $|P| = 1 \neq 0$. By computation, we have

$$PB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

and

$$AP = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

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Hence, $PB = AP$ and A is similar to B.

10. **10.** Because A and B are similar, $B = P^{-1}AP$.

i. Noting that $\lambda I = P^{-1}(\lambda I)P,$ we have

$$|\lambda I - B| = \left|P^{-1}(\lambda I)P - P^{-1}AP\right| = \left|P^{-1}(\lambda I - A)P\right|$$
$$= \left|P^{-1}\right|\left|\lambda I - A\right|\left|P\right| = \left|\lambda I - A\right|.$$

ii. Because $|\lambda I - A| = |\lambda I - B|,$ the equations $|\lambda I - A| = 0$ and $|\lambda I - B| = 0$ have the same roots. Hence, *A* and *B* have the same eigenvalues.

iii. Because *A* and *B* have the same eigenvalues, the result (*ii*) follows from Theorem 8.1.4.

iv. $\left|B\right| = \left|P^{-1}AP\right| = \left|P^{-1}\right|\left|A\right|\left|P\right| = \left|A\right|.$

v. Because $B = P^{-1}AP$ and P^{-1} is invertible, by Corollary 2.7.3,

$$r(B) = r(P^{-1}AP) = r(AP).$$

By Theorem 2.5.5 (1) $r(AP) = r((AP)^T).$ By Theorem 2.2.3 (3), $r((AP)^T) = r(P^TA^T).$ Because P^T is invertible, by Corollaries 2.7.3 and 2.7.2,
$r(P^TA^T) = r(A^T) = r(A).$

11. **11.**

i. Let $P = I.$ Then *P* is invertible and $P^{-1} = I.$ It is obvious that $A = IAI = P^{-1}AP$ and $A \sim A.$

ii. Because $A \sim B,$ there exists an invertible $n \times n$ matrix *P* such that $B = P^{-1}AP.$ This implies that

$$A = PBP^{-1} = (P^{-1})^{-1}B(P^{-1})$$

and $B \sim A.$

iii. Because $A \sim B$ and $B \sim C,$ there exist invertible $n \times n$ matrices P_1 and P_2 such that$B = P_1^{-1}AP_1$ and $C = P_2^{-1}BP_2.$ Then

$$\begin{aligned} C &= P_2^{-1}BP_2 = C = P_2^{-1}(P_1^{-1}AP_1)P_2 = (P_2^{-1}P_1^{-1})A(P_1P_2) \\ &= (P_1P_2)^{-1}A(P_1P_2) \end{aligned}$$

and $A \sim C$.

12. **12.** Because $A \sim B$, there exists an invertible *P* such that $P^{-1}AP = B$.

a.

$$\begin{aligned} B^2 &= (P^{-1}AP)(P^{-1}AP) = (P^{-1}A(PP^{-1})AP) = (P^{-1}A(I)AP) \\ &= (P^{-1}AAP) = P^{-1}A^2P. \end{aligned}$$

Repeating the process implies that $B^k = P^{-1}A^kP$.

b.

$$\begin{aligned} \phi(B) &= a_mB^m + a_{m-1}B^{m-1} + \cdots + a_1B + a_0I \\ &= a_mP^{-1}A^mP + a_{m-1}P^{-1}A^{m-1}P + \cdots + a_1P^{-1}AP + a_0P^{-1}IP \\ &= P^{-1}(a_mA^m + a_{m-1}A^{m-1} + \cdots + a_1A + a_0I)P = P^{-1}\phi(A)P. \end{aligned}$$

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A.9 Vector spaces

1. Let $W = \{(x,\ y) \in \mathbb{R}^2 : x - y = 0\}$. Show that *W* is a subspace of $\mathbb{R}^2$.
2. Let $W = \{(x,\ y) \in \mathbb{R}^2_+ : x + y \leq 1\}$. Show that *W* is not a subspace of $\mathbb{R}^2$.
3. Let

$$\mathbb{R}^2_+ = \{(x,\ y) : x \geq 0 \quad \text{and } y \geq 0\}.$$

   i. Show that if $\overrightarrow{a} \in \mathbb{R}^2_+$ and $\overrightarrow{b} \in \mathbb{R}^2_+$, then $\overrightarrow{a} + \overrightarrow{b} \in \mathbb{R}^2_+$.
   ii. Show that $\mathbb{R}^2_+$ is not a subspace of $\mathbb{R}^2$ by using **Definition 9.1.1** .

4. Let

$$W = \mathbb{R}^2_+ \cup (-\mathbb{R}^2_+).$$

   i. Show that if $\overrightarrow{a} \in W$ and *k* is a real number, then $k\overrightarrow{a} \in W$.
   ii. Show that *W* is not a subspace of $\mathbb{R}^2$ by using **definition 9.1.1** .

5. Let $W = \{(x,\ y,\ z) \in \mathbb{R}^3 : x - 2y - z = 0\}$. Show that *W* is a subspace of $\mathbb{R}^3$ by **Definition 9.1.1** and **Theorem 9.1.1** , respectively.
6. Show that *W* is a subspace of $\mathbb{R}^4$, where

$$W = \left\{(x,\ y,\ z,\ w) \in \mathbb{R}^4 : \left\{ \begin{array}{l} x - 2y + 2z - w = 0, \\ -x + 3y + 4z + 2w = 0 \end{array} \right. \right\}.$$

7. Show that *W* is a subspace of $\mathbb{R}^3$, where

$$W = \left\{ (x,\ y,\ z) \in \mathbb{R}^3 : \begin{cases} x + 2y + 3z = 0, \\ -x + y - 4z = 0, \\ x - 2y + 5z = 0 \end{cases} \right\}.$$

8. Let $W = \{(x,\ y,\ z) \in \mathbb{R}^3 : x - 2y - z = 3\}$. Show that *W* is not a subspace of $\mathbb{R}^3$ by **Definition 9.1.1** .
9. Let $W = [0,\ 1]$. Show that *W* is not a subspace of *R*.
10. Show that all the subspaces of $\mathbb{R}$ are {0} and $\mathbb{R}$.
11. Let $W = \{(0,\ 0,\ \cdots,\ 0)\} \in \mathbb{R}^n$. Show that *W* is a subspace of $\mathbb{R}^n$.
12. Let $\overrightarrow{v} \in \mathbb{R}^n$ be a given nonzero vector. Show that

$$W = \left\{ t\overrightarrow{v} : t \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^n$ by **Definition 9.1.1** .

A.9 Vector spaces

1. Let $W=\{(x,\ y)\in\mathbb{R}^2: x-y=0\}$. Show that *W* is a subspace of $\mathbb{R}^2$.
2. Let $W=\{(x,\ y)\in\mathbb{R}^2_+: x+y\le 1\}$. Show that *W* is not a subspace of $\mathbb{R}^2$.
3. Let

$$\mathbb{R}^2_+=\{(x,\ y): x\ge 0 \quad \text{and } y\ge 0\}.$$

   i. Show that if $\vec{a}\in\mathbb{R}^2_+$ and $\vec{b}\in\mathbb{R}^2_+$, then $\vec{a}+\vec{b}\in\mathbb{R}^2_+$.
   ii. Show that $\mathbb{R}^2_+$ is not a subspace of $\mathbb{R}^2$ by using **Definition 9.1.1** .

4. Let

$$W=\mathbb{R}^2_+\cup(-\mathbb{R}^2_+).$$

   i. Show that if $\vec{a}\in W$ and *k* is a real number, then $k\vec{a}\in W$.
   ii. Show that *W* is not a subspace of $\mathbb{R}^2$ by using **definition 9.1.1** .

5. Let $W=\{(x,\ y,\ z)\in\mathbb{R}^3: x-2y-z=0\}$. Show that *W* is a subspace of $\mathbb{R}^3$ by **Definition 9.1.1** and **Theorem 9.1.1** , respectively.
6. Show that *W* is a subspace of $\mathbb{R}^4$, where

$$W=\left\{(x,\ y,\ z,\ w)\in\mathbb{R}^4: \begin{cases} x-2y+2z-w=0, \\ -x+3y+4z+2w=0 \end{cases}\right\}.$$

7. Show that *W* is a subspace of $\mathbb{R}^3$, where

$$W=\left\{(x,\ y,\ z)\in\mathbb{R}^3:\left\{\begin{array}{l}x+2y+3z=0,\\-x+y-4z=0,\\x-2y+5z=0\end{array}\right.\right\}.$$

8. Let $W=\{(x,\ y,\ z)\in\mathbb{R}^3 : x-2y-z=3\}$. Show that *W* is not a subspace of $\mathbb{R}^3$ by **Definition 9.1.1** .
9. Let $W=[0,\ 1]$. Show that *W* is not a subspace of *R*.
10. Show that all the subspaces of $\mathbb{R}$ are {0} and $\mathbb{R}$.
11. Let $W=\{(0,\ 0,\ \cdots,\ 0)\}\in\mathbb{R}^n$. Show that *W* is a subspace of $\mathbb{R}^n$.
12. Let $\overrightarrow{v}\in\mathbb{R}^n$ be a given nonzero vector. Show that

$$W=\left\{t\overrightarrow{v}:t\in\mathbb{R}\right\}$$

is a subspace of $\mathbb{R}^n$ by **Definition 9.1.1** .

Solution

1. **1.** Note that $\vec{a} = (x_{1,}\ y_1) \in W$ if and only if $x_1 - y_1 = 0$. Let $\vec{a} = (x_1, y_1) \in W, \vec{b} = (x_2,\ y_2) \in W,$ and $k \in \mathbb{R}$ is a real number. Then $x_1 - y_1 = 0$ and $x_2 - y_2 = 0,$

$$\vec{a} + \vec{b} = (x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2) \quad \mathit{and}\ k\vec{a} = (kx_1,\ ky_1).$$

Because

$$(x_1 + x_2) - (y_1 + y_2) = (x_1 - y_1) + (x_2 - y_2) = 0 + 0 = 0,$$

and $kx_1 - ky_1 = k(x_1 - y_1) = k(0) = 0,$ we have

$$\vec{a} + \vec{b} = (x_1 + x_2,\ y_1 + y_2) \in W$$

and $k\vec{a} = (kx_1,\ ky_1) \in W.$ Hence, *W* is a subspace of $\mathbb{R}^2$.

2. **2.** Let $(x,\ y) = (0,\ 1)$ and $k = 2.$ Then $k(x,\ y) = 2(0,\ 1) = (0,\ 2)$ and $kx + ky = 0 + 2 > 1.$ This implies that $k(x,\ y) \in W.$

3. **3.**

   i. Let $\vec{a} = (x1,\ y1) \in \mathbb{R}^2_+$ and $\vec{b} = (x_2,\ y_2) \in \mathbb{R}^2_+.$ Then $x_1 \geq 0,\ y_1 \geq 0,\ x_2 \geq 0,$ and $y_2 \geq 0.$ Then

$$\vec{a} + \vec{a} = (x_1,\ y_1) + (x_2,\ y_2) = (x_1 + x_2,\ y_1 + y_2) \in \mathbb{R}^2_+$$

   because $x_1 + x_2 \geq 0$ and $y_1 + y_2 \geq 0.$

   ii. Let $\vec{a} = (1,\ 1)$ and $k = \ -1,$ then $\vec{a} \in \mathbb{R}^2_+$ and $k\vec{a} = (-1,\ -1) \in W.$ Hence, $\mathbb{R}^2_+$ is not a subspace of $\mathbb{R}^2$.

4. **4.** Note that W is the union of the first and third quadrants. Also, $(x, y) \in W$ if and only if either $x \geq 0$ and $y \geq 0$ or $x \leq 0$ and $y \leq 0$.

i. Let $\vec{a} = (x, y) \in W$. Then either $x \geq 0$ and $y \geq 0$ or $x \leq 0$ and $y \leq 0$. This implies that $k\vec{a} = (kx, ky)$ satisfies either $kx \geq 0$ and $ky \geq 0$ or $kx \leq 0$ and $ky \leq 0$. Hence, $k\vec{a} \in W$.

ii. Let $\vec{a} = (1, 2)$ and $\vec{b} = (-2, -1)$, then $\vec{a} \in W$ and $\vec{b} \in W$. But

$$\vec{a} + \vec{b} = (1, 2) + (-2, -1) = (-1, 1) \in W.$$

Hence, W is not a subspace of $\mathbb{R}^2$.

5. **5.** Let $\vec{a} = (x_1, y_1, z_1) \in W$, $\vec{b} = (x_2, y_2, z_2) \in W$, and $k \in \mathbb{R}$ is a real number. Then

$$x_1 - 2y_1 - z_1 = 0 \quad \text{and } x_2 - 2y_2 - z_2 = 0,$$

$$\vec{a} + \vec{b} = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2).$$

and $k\vec{a} = (kx_1, ky_1, kz_1)$. Because

$$(x_1 + x_2) - 2(y_1 + y_2) - (z_1 + z_2) = (x_1 - 2y_1 - z_1) + (x_2 - 2y_2 - z_2) = 0 + 0 = 0,$$

and $kx_1 - 2ky_1 - kz_1 = k(x_1 - 2y_1 - z_1) = k(0) = 0$, we have

$$\vec{a} + \vec{b} \in W$$

and $k\vec{a} \in W$. Hence, by Definition 9.1.1, W is a subspace of $\mathbb{R}^3$. Let $A = 1 - 2 \ -1$, $\vec{X} = (x, y, z)^T$, and $\vec{0} = (0, 0, 0)^T$.

$$W = \left\{(x,\ y,\ z) \in \mathbb{R}^3 : x - 2y - z = 0\right\}$$
$$= \left\{(x,\ y,\ z) \in \mathbb{R}^3 :\ A\vec{X} = \vec{0}\right\}.$$

By Theorem 9.1.1, *W* is a subspace of $\mathbb{R}^3$.

6. **6.** Let $A = \begin{pmatrix} 1 & -2 & 2 & -1 \\ -1 & 3 & 4 & 2 \end{pmatrix}$, $\vec{X} = (x,\ y,\ z)^T$, and $\vec{0} = (0,\ 0,\ 0)^T$. Then

$$W = \left\{(x,\ y,\ z, w) \in \mathbb{R}^4 : \begin{cases} x - 2y + 2z - w = 0, \\ -x + 3y + 4z + 2w = 0 \end{cases}\right\}$$
$$= \left\{(x,\ y,\ z, w) \in \mathbb{R}^4 : A\vec{X} = \vec{0}\right\}.$$

By Theorem 9.1.1, *W* is a subspace of $\mathbb{R}^4$.

7. **7.** Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & -4 \\ 3 & -2 & 5 \end{pmatrix}$, $\vec{X} = (x,\ y,\ z)^T$, and $\vec{0} = (0,\ 0,\ 0)^T$. Then

$$W = \left\{(x,\ y,\ z) \in \mathbb{R}^3 : \begin{cases} x + 2y + 3z = 0, \\ -x + y - 4z = 0, \\ x - 2y + 5z = 0 \end{cases}\right\}$$
$$= \left\{(x,\ y,\ z) \in \mathbb{R}^3 : A\vec{X} = \vec{0}\right\}.$$

Processing math: 100% eorem 9.1.1, *W* is a subspace of $\mathbb{R}^3$.

8. **8.** Note that $\vec{a} = (x_1, y_1, z_1) \in W$ if and only if $x_1 - 2y_1 - z_1 = 3$. Because $0 + 2(0) + 3(0) \neq 3$, so $(0, 0, 0) \in W$ and *W* is not a subspace of $\mathbb{R}^3$.

9. **9.** Let $x = 1$ and $k = 2$. Then $x \in [0, 1] = W$ and $kx = 2(1) = 2 > 1$. This implies that $kx \in W$ and *W* is not a subspace of *R*.

10. **10.** It is easy to show that $\{0\}$ and $\mathbb{R}$ are subspaces of *R*. Let *W* be a subset of *R* satisfying $W \neq \{0\}$ and $W \neq \mathbb{R}$. Then there exist two points x_1 satisfying $x_1 \in W$ and $x_1 \neq 0$ and $x_2 \in \mathbb{R}$ satisfying $x_2 \in W$. Let $k = \frac{x_2}{x_1}$. Then $x_1 \in W$ but $kx_1 = x_2 \in W$. Hence, *W* is not a subspace of *R*.

11. **11.** Note that *W* only contains the origin $(0, 0, \cdots, 0)$, so

$$\vec{a} = (x_1, x_2, \cdots, x_n) \in W \quad \text{if and only if } x_1 = x_1 = \cdots = x_n = 0.$$

Let $\vec{a} = (x_1, x_2, \cdots, x_n) \in W$, $\vec{b} = (y_1, y_2, \cdots, y_n) \in W$, and $k \in \mathbb{R}$ is a real number. Then $x_i = y_i = 0$ for $i = 1, 2, \cdots, n$. It follows that

$$\begin{aligned} \vec{a} + \vec{b} &= (x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) \\ &= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n) = (0 + 0, 0 + 0, \cdots, 0 + 0) \\ &= (0, 0, \cdots, 0) \in W. \end{aligned}$$

and

$$\begin{aligned} k\vec{a} &= (kx_1, kx_2, \cdots, kx_n) = (k(0), k(0), \cdots, k(0)) \\ &= (0, 0, \cdots, 0) \in W. \end{aligned}$$

Hence, *W* is a subspace of $\mathbb{R}^n$.

12. **12.** Note that Note that $\vec{a} \in W$ if and only if there exists $t \in \mathbb{R}$ such that $\vec{a} = t\vec{v}$. Let $\vec{a} \in W, \vec{b} \in W$, and $k \in \mathbb{R}$. Then there exist $t_1, t_2 \in \mathbb{R}$ such that $\vec{a} = t_1\vec{v}$ and $\vec{b} = t_2\vec{v}$. Hence,

$$\vec{a} + \vec{b} = t_1\vec{v} = (t_1 + t_2)\vec{v} \in W$$

and

$$k\vec{a} = k(t_1\vec{v}) = (kt_1)\vec{v} \in W.$$

Section 8.3

1. Let

$$C(\mathbb{R}) = \{f: f: \mathbb{R} \to \mathbb{R} \quad \text{is continuous on } \mathbb{R}\}.$$

Show that $C(\mathbb{R})$ is a vector space under the following operations.

   1. $+ : (f+g)(x) = f(x) + g(x)$ for $x \in \mathbb{R}$.
   2. $(kf)(x) = k(f(x))$ for $x \in \mathbb{R}$.

2. Let

$$C[a,\ b] = \{f: f: [a,\ b] \to \mathbb{R} \text{ is continuous}\}.$$

Show that $C[a, b]$ is a vector space under standard addition and scalar multiplication:

   1. For $f,\ g \in C[a,\ b],\ (f+g)(x) = f(x) + g(x)$ for $x \in [a,\ b]$.
   2. For $k \in \mathbb{R}$ and $f \in C[a,\ b],\ (k \cdot f)(x) = kf(x)$ for $x \in [a,\ b]$

3. Let n be a nonnegative integer and let

$$P_n = \left\{p: p(x) = a_0 + a_1x + \cdots + a_nx^n \quad \text{is a polynomial of degree } n\right\}.$$

Show that P_n is a subspace of $C(\mathbb{R})$.

4. Let

$$V = \{A: A \quad \text{is a } 3 \times 3 \text{ matrix}\}.$$

We define the following addition and scalar multiplication.

1. Addition: For $A,\ B \in V,\ A+B=AB,$ where *AB* is the standard product of *A* and *B* in **Section 2.5** .
2. Scalar multiplication: For $A \in V$ and $k \in \mathbb{R},\ k \cdot A = kA,$ where *kA* is the standard scalar multiplication given in **Definition 2.1.4** .

Then *V* is not a vector space under the above addition and scalar multiplication.

5. Let

$$C^1[a,\ b] = \left\{f{:}f' \in C[a,\ b]\right\},$$

where $f'(x)$ denotes the derivative of *f* at *x*. Show that $C^1[a,\ b]$ is a vector subspace of *C*[*a*, *b*].

Solution

1. **1.** If $f,\ g \in C(\mathbb{R})$, then $f+g \in C(\mathbb{R})$, and if $k \in \mathbb{R}$ and $f \in C(\mathbb{R})$, then $kf \in C(\mathbb{R})$. Hence, the axioms (1) and (6) hold. It is easy to verify that the other eight axioms given in Definition 9.2.2 hold. Hence, $C(\mathbb{R})$ is a vector space.
2. **2.** If $f,\ g \in C[a,\ b]$, then $f+g \in C[a,\ b]$, and if $k \in \mathbb{R}$ and $f \in C[a,\ b]$, than $kf \in C[a,\ b]$. Hence, the axioms (1) and (6) hold. It is easy to verify that the other eight axioms given in Definition 9.2.2 hold. Hence, *C*[*a*, *b*] is a vector space.
3. **3.** For the set *V*, the axioms (2) and (4) are not true in general. For example, let

$$A = \begin{pmatrix} 1 & 0 & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $A+B = AB = B \neq A,$ so the axiom does not hold. Also, Let

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } BA = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $A+B = AB \neq BA = B+A$ and axiom (2) does not hold. It is easy to verify that other axioms hold. Hence, *V* is not a vector space under the above addition (1) and scalar multiplication (2).

4. **4.** Let $p \in P_n$. Because *p* is polynomial, *p* is continuous on $\mathbb{R}$ and thus, $p \in C(\mathbb{R})$. Hence, $P_n \subset C(\mathbb{R})$. Let $p,\ q \in P_n$ and $k \in \mathbb{R}$. Then $p + q \in P_n$ and $kp \in P_n$. Hence, P_n is a subspace of $C(\mathbb{R})$.
5. **5.** Let $f, g \in C^1[a,\ b]$ and $k \in \mathbb{R}$, then $f', g' \in C[a,\ b]$. This implies $(f + g)' = f' + g' \in C^1[a,\ b]$. Hence, $C^1[a,\ b]$ is a vector subspace of *C*[*a*, *b*].

A.10 Complex numbers

Section 10.1

1. Solve each of the following equations.
   a. $z^2 + z + 1 = 0$
   b. $z^2 - 2z + 5 = 0$
   c. $z^2 - 8z + 25 = 0$

2. Use the following information to find *x* and *y*.
   a. $2x - 4i = 4 + 2yi$
   b. $(x - y) + (x + y)i = 2 - 4i$

3. Let $z_1 = 2 + i$ and $z_2 = 1 - i$. Express each of the following complex numbers in the form $x + yi$, calculate its modulus, and find its conjugate.
   1. $z_1 + 2z_2$
   2. $2z_1 - 3z_2$
   3. $z_1 z_2$
   4. $\frac{z_1}{z_2}$

4. Express each of the following complex numbers in the form $x + yi$.
   a. $\overline{\left(\frac{i}{1-i}\right)}$
   b. $\frac{2-i}{(1-i)(2+i)}$
   c. $\frac{1+i}{i(1+i)(2-i)}$
   d. $\frac{1+i}{1-i} - \frac{2-i}{1+i}$

5. Find $i^2,\ i^3,\ i^4,\ i^5,\ i^6$ and compute $\left(-i - i^4 + i^5 - i^6\right)^{1000}$.

6. Let $z = \left(\frac{1-i}{1+i}\right)^8$. Calculate $z^{66} + 2z^{33} - 2$.

7. Find $z \in \mathbb{C}$

   a. $iz = 4 + 3i$

   b. $(1+i)\bar{z} = (1-i)z$

8. Find real numbers *x, y* such that

$$\frac{x + 1 + i(y-3)}{5+3i} = 1 + i.$$

9. Let $z = \frac{(3+4i)(1+i)^{12}}{i^5(2+4i)^2}$. Find $|z|$.

Solution

1. **1.**

   a. $z_1 = \frac{-1+i\sqrt{3}}{2}$ and $z_2 = \frac{-1+i\sqrt{3}}{2}$.

   b. $z_1 = \frac{2+i\sqrt{16}}{2} = 1 + 2i$ and $z_2 = \frac{2+i\sqrt{16}}{2} = 1 - 2i$.

   c. $z_1 = \frac{8+i\sqrt{36}}{2} = 4 + 3i$ and $z_2 = \frac{8+i\sqrt{36}}{2} = 4 - 3i$.

2. **2.**

   a. By Definition 10.1.2, we have $2x = 4$ and $-4 = 2y$. This implies that $x = 2$ and $y = -2$.

   b. By Definition 10.1.2, we have $x - y = 2$ and $x + y = -4$. Solving the system, we obtain $x = -1$ and $y = -3$.

3. **3.**

   1.
   $$z_1 + 2z_2 = (2+i) + 2(1-i) = (2+i) + (2-2i) = 4 - i,$$
   $$\left|z_1 + 2z_2\right| = |4 - i| = \sqrt{4^2 + (-1)^2} = \sqrt{17},$$
   and $z_1 + 2z_2 = 4 - i = 4 + i$.

   2.
   $$2z_1 - 3z_2 = 2(2+i) - 3(1-i) = (4+2i) - (3-3i) = 1 + 5i,$$
   $$\left|2z_1 - 3z_2\right| = |1 + 5i| = \sqrt{1^2 + 5^2} = \sqrt{26}$$
   and $2z_1 - 3z_2 = 1 + 5i = 1 - 5i$.

$$
\begin{aligned}
z_1 z_2 &= (2+i)(1-i) = 2(1-i) + i(1-i) = 2 - 2i + i - i^2 \\
&= 2 - i + 1 = 3 - i,
\end{aligned}
$$

3. $$
\begin{aligned}
\left| z_1 z_2 \right| &= |3-i| = \sqrt{3^2 + (-1)^2 =} \sqrt{10} \quad \text{and} \\
\overline{z_1 z_2} &= \overline{3-i} = 3 + i.
\end{aligned}
$$

4. $$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{2+i}{1-i} = \frac{(2+i)(1+i)}{(1-i)(1+i)} = \frac{2(1+i)+i(1+i)}{1-i^2} \\
&= \frac{2+2i+i+i^2}{1+1} = \frac{2+3i-1}{2} = \frac{1}{2} + \frac{3}{2}i, \\
\left| \frac{z_1}{z_2} \right| &= \left| \frac{1}{2} + \frac{3}{2}i \right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \frac{\sqrt{10}}{2} \text{ and} \\
\overline{\left(\frac{z_1}{z_2}\right)} &= \overline{\left(\frac{1}{2} + \frac{3}{2}i\right)} = \frac{1}{2} - \frac{3}{2}i.
\end{aligned}
$$

4. **4.**

a. $$
\begin{aligned}
\frac{i}{1-i} &= \frac{i(1+i)}{(1-i)(1+i)} = \frac{i-1}{2} = -\frac{1}{2} + \frac{1}{2}i, \\
\overline{\left(\frac{i}{1-i}\right)} &= \overline{\left(-\frac{1}{2} + \frac{1}{2}i\right)} = -\frac{1}{2} - \frac{1}{2}i.
\end{aligned}
$$

b. $$\frac{2-i}{(1-i)(2+i)} = \frac{(2-i)^2(1+i)}{(1-i)(1+i)(2+i)(2-i)} = \frac{(3-4i)(1+i)}{(2)(5)}$$
$$= \frac{7-i}{10} = \frac{7}{10} - \frac{1}{10}i.$$

c. $$\frac{1+i}{i(1+i)(2-i)} = \frac{i(1+i)(1-i)(2+i)}{i^2(1+i)(1-i)(2-i)(2+i)} = \frac{2i(2+i)}{-(2)(5)}$$
$$= -\frac{4i-2}{10} = \frac{1}{5} - \frac{2}{5}i.$$

d. $$\frac{1+i}{1-i} - \frac{2-i}{1+i} = \frac{(1+i)^2}{(1-i)(1+i)} - \frac{(2-i)(1-i)}{(1+i)(1-i)}$$
$$= \frac{2i}{2} - \frac{1-3i}{2} = i - \frac{1}{2} + \frac{3}{2}i = -\frac{1}{2} + \frac{5}{2}i.$$

5. **5.** Because $i^2 = -1,\ i^3 = i^2 \cdot i = -i,\ i^4 = i^2 \cdot i^2 = (-1)(-1) = 1,\ i^5 = i^4 \cdot i = i,$ and $i^6 = i^5 \cdot i = i \cdot i = -1,$

$$-i - i^4 + i^5 - i^6 = -i - 1 + i - (-1) = 0$$

and

$$(-i - i^4 + i^5 - i^6)^{1000} = 0^{1000} = 0.$$

6. **6.** Because $\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1-2i-1}{2} = -i,$

$$z = \left(\frac{1-i}{1+i}\right)^8 = (-i)^8 = [(-i)^2]^4 = (-1)^4 = 1$$

and $z^{66} + 2z^{33} - 2 = 1 + 2 - 2 = 1.$

7. **7.**

a. Because $iz = 4 + 3i,\ i(iz) = i(4 + 3i).$ This implies that $i^2 z = 4i + 3i^2$ and $-z = 4i - 3$. Hence, $z = 3 - 4i.$

b. Let $z = x + yi.$ Then $\bar{z} = x - yi$ and

$$(1+i)(x-yi)=(1-i)(x+yi).$$

Hence, $x - yi + xi + y = x + yi - xi + y$ and $2(x-y)i = 0$. This implies $x = y$ and $z = x(1+i)$ for $x \in \mathbb{R}$.

8. **8.** Because

$$\begin{aligned}\frac{x+1+i(y-3)}{5+3i} &= \frac{[x+1+i(y-3)](5-3i)}{(5+3i)(5-3i)}\\ &= \frac{[(x+1)(5-3i)+i(y-3)(5-3i)}{25+9}\\ &= \frac{(5x+3y-4)+i[-3(x-1)+5(y-3)]}{34}\\ &= i+1,\end{aligned}$$

comparing the real and imaginary parts implies that

$$\begin{cases}\dfrac{5x+3y-4}{34}=1\\ \dfrac{-3(x-1)+5(y-3)}{34}=1.\end{cases}$$

Simplifying the above equation, we obtain

$$\begin{cases}5x+3y=38\\ -3x+5y=52.\end{cases}$$

Solving the above equation, we have $x = 1$ and $y = 11$.

9. **9.** $|z| = \dfrac{|3+4i|\left|(1+i)^{12}\right|}{\left|i^5\right|\left|(2+4i)^2\right|} = \dfrac{|3+4i|\,|1+i|^{12}}{|i|^5|2+4i|^2} = \dfrac{5(\sqrt{2})^{12}}{(1)^5(20)} = \dfrac{2^6}{4} = 16.$

Section 10.2

1. For each of the following numbers, find its principal argument and argument.
   a. -4
   b. $-i$
   c. $\frac{1}{2} - \frac{\sqrt{3}}{2}i$
   d. $1 + \sqrt{3}i$
   e. $-1 + \sqrt{3}i$
   f. $-\sqrt{3} - i$

2. Express each of the following complex numbers in polar and exponent forms.
   a. 1
   b. i
   c. -2
   d. $-3i$
   e. $1 + \sqrt{3}i$
   f. $-1 - i$

Solution

1. **1.** By Theorem 10.2.1, we have

   a. $\operatorname{Arg}(-4) = \pi$ and $\arg(-4) = \pi + 2k\pi$ for $k \in \mathbb{Z}$.

   b. $\operatorname{Arg}(-i) = -\frac{\pi}{2}$ and $\arg(-i) = -\frac{\pi}{2} + 2k\pi$ for $k \in \mathbb{Z}$.

   c. Let $z = x + yi = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$ Then $x = \frac{1}{2}$ and $y = -\frac{\sqrt{3}}{2}.$ Let $\psi \in (0, \frac{\pi}{2})$ satisfy

   $$\tan\psi = \left|\frac{y}{x}\right| = \sqrt{3}.$$

   Then $\psi = \frac{\pi}{3}.$ Because $x > 0$ and $y < 0,$ z is in the fourth quadrant and by Theorem 10.2.2 (4),

   $$\operatorname{Arg} z = -\psi = -\frac{\pi}{3} \quad \text{and } \arg z = -\frac{\pi}{3} + 2k\pi \text{ for } k \in \mathbb{Z}.$$

   d. Let $z = x + yi = 1 + \sqrt{3}i.$ Then $x = 1$ and $y = \sqrt{3}.$ Let $\psi \in (0, \frac{\pi}{2})$ satisfy

   $$\tan\psi = \left|\frac{y}{x}\right| = \sqrt{3}.$$

   Then $\psi = \frac{\pi}{3}.$ Because $x > 0$ and $y > 0,$ z is in the first quadrant and by Theorem 10.2.2 (1),

   $$\operatorname{Arg} z = \psi = \frac{\pi}{3} \quad \text{and } \arg z = \frac{\pi}{3} + 2k\pi\ k \in \mathbb{Z}.$$

e. Let $z = x + yi = -1 + \sqrt{3i}$. Then $x = -1$ and $y = \sqrt{3}$. Let $\psi \in (0, \frac{\pi}{2})$ satisfy

$$\tan\psi = \left|\frac{y}{x}\right| = \sqrt{3}.$$

Then $\psi = \frac{\pi}{3}$. Because $x < 0$ and $y > 0$, z is in the second quadrant and by Theorem 10.2.2 (2),

$$\text{Arg } z = \pi - \psi = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \text{and } \arg z = \frac{2\pi}{3} + 2k\pi \text{ for } k \in \mathbb{Z}.$$

f. Let $z = x + yi = -\sqrt{3} - i$. Then $x = -\sqrt{3}$ and $y = -1$. Let $\psi \in (0, \frac{\pi}{2})$ satisfy

$$\tan\psi = \left|\frac{y}{x}\right| = \frac{\sqrt{3}}{3}.$$

Then $\psi = \frac{\pi}{6}$. Because $x < 0$ and $y < 0$, z is in the third quadrant and by Theorem 10.2.2 (3),

$$\text{Arg } z = -\pi + \psi = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$

and $\arg z = -\frac{5\pi}{6} + 2k\pi$ for $k \in \mathbb{Z}$.

2. **2.**

a. $1 = 1(\cos 0 + i \sin 0) = e^{0 \cdot i}$.

b. $i = 1\left(\cos\frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = e^{\frac{\pi}{2}i}$.

c. $-2 = 2(\cos\pi + i \sin\pi) = 2e^{\pi i}$.

d. $-3i = 3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = 3e^{-\frac{\pi}{2}i}.$

e. $r = |z| = \sqrt{x^2 + y^2} = \sqrt{1 + 3} = 2.$ Let $z = x + yi = 1 + \sqrt{3}i.$ Then $x = 1$ and $y = \sqrt{3}.$ Let $\psi \in \left(0,\ \frac{\pi}{2}\right)$ satisfy

$$\tan\psi = \left|\frac{y}{x}\right| = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

Then $\psi = \frac{\pi}{3}.$ Because $x > 0$ and $y < 0,$ z is in the first quadrant and by Theorem 10.2.2 (1), $\operatorname{Arg} z = \psi = \frac{\pi}{3}.$ Hence,

$$1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\frac{\pi}{3}}.$$

f. $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}.$ Let $z = x + yi = -1 - i.$ Then $x = -1$ and $y = -1.$ Let $\psi \in \left(0,\ \frac{\pi}{2}\right)$ satisfy

$$\tan\psi = \left|\frac{-1}{-1}\right| = 1.$$

Then $\psi = \frac{\pi}{4}.$ Because $x < 0$ and $y < 0,$ z is in the third quadrant and by Theorem 10.2.2 (3),

$$\operatorname{Arg} z = -\pi + \psi = -\pi + \frac{\pi}{4} = -\frac{3}{4}\pi.$$

Hence,

$$-1-i=\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)=\sqrt{2}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)=e^{-i\frac{3\pi}{4}}.$$

Section 10.3

1. Express the following numbers in $x+yi$ form.
    1. $(1+\sqrt{3}i)^3$
    2. $(-1-i)^4$
    3. $(1+i)^{-8}$.

2. Let $z_1=\frac{1+i}{\sqrt{2}}$ and $z_2=\sqrt{3}-i$. Find the exponential form of z_1z_2.
3. Let $z=\frac{\sqrt{2}}{2}(1-i)$. Find $z^{100}+z^{50}+1$.
4. Compute the following value

$$z=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{600}+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{60}.$$

Solution

1. **1.**

1. Because $1 + \sqrt{3i} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$,

$$\begin{aligned}(1 + \sqrt{3i})^3 &= 2^3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3 = 8\left(\cos 3\cdot\frac{\pi}{3} + i\sin 3\cdot\frac{\pi}{3}\right)\\ &= 8(\cos\pi + i\sin\pi) = 8(-1+0) = -8.\end{aligned}$$

2. Because $(-1-i) = \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$,

$$(-1-i)^4 = 4(\cos 3\pi - i\sin 3\pi) = -4.$$

3. Because $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\frac{\pi}{4}}$,

$$\begin{aligned}(1+i)^{-8} &= \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{-8} = \left(\sqrt{2}\right)^{-8}\left(e^{i\frac{\pi}{4}}\right)^{-8} = 2^{-4}e^{i\frac{-8\pi}{4}}\\ &= 2^{-4}e^{-i2\pi} = 2^{-4}\end{aligned}$$

2. **2.** Because

$$z_1 = \frac{1+i}{\sqrt{2}} = \cos\frac{\pi}{4} = +i\sin\frac{\pi}{4} = e^{i\frac{\pi}{4}} \text{ and}$$

$$z_2 = \sqrt{3} - i = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2e^{i\frac{\pi}{6}},$$

we have

$$z_1 z_2 = 2\, e^{\,i\frac{\pi}{4}}\, e^{\,i\frac{\pi}{6}} = 2e^{i}\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = 2e^{i\frac{5\pi}{12}}.$$

3. **3.** Because $z = \frac{\sqrt{2}}{2}(1 - i) = e^{-\frac{\pi}{4}}$,

$$z^{100} = \left(e^{-i\frac{\pi}{4}}\right)^{100} = e^{-i\frac{100\pi}{4}} = e^{2(-12)\pi i + (-\pi)i} = e^{2(-12)\pi i} e^{(-\pi)i}$$

$$= e^{(-\pi)i} = \cos(-\pi) + i\sin(-\pi) = \cos\pi = -1$$

and

$$z^{50} = \left(e^{-i\frac{\pi}{4}}\right)^{50} = e^{2(-6)\pi i(-\frac{\pi}{2})i} = e^{2(-6)\pi i} e^{(-\frac{\pi}{2})i} = e^{(-\frac{\pi}{2})i}$$

$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = i\sin\left(-\frac{\pi}{2}\right) = -i.$$

Hence,

$$z^{100} + z^{50} + 1 = -1 - i + 1 = -i.$$

4. **4.** Let $z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Then

$$z_1 = \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = e^{i\frac{2\pi}{3}}$$

and

$$z_2 = \cos\left(-\pi + \frac{\pi}{3}\right) + i\sin\left(-\pi + \frac{\pi}{3}\right) = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right) = e^{-i\frac{2\pi}{3}}.$$

Hence,

$$z_1^{600} = \left(e^{i\frac{2\pi}{3}}\right)^{600} = e^{i400\pi} = 1 \text{ and } z_2^{60} = \left(e^{-i\frac{2\pi}{3}}\right)^{60} = e^{-i40} = 1.$$

This implies that $z = z_1^{600} + z_2^{60} = 1 + 1 = 2.$

Section 10.4

1. Find all roots for each of the following complex numbers.
   a. $\sqrt[4]{1}$
   b. $\sqrt[6]{-1}$
   c. $\sqrt[3]{1 - i}$
   d. $\sqrt[3]{-i}$
   e. $(-1 + \sqrt{3}i)^{\frac{1}{4}}$

2. Find all solutions of $z^3 = -8$.
3. Solve the following equations.
   a. $(1-i)z^2 + 2z + (1+i) = 0$
   b. $z^2 + 2z + (1-i) = 0$

Solution

1. **1.**

   a. Because $1 = 1 \cdot (\cos 0 + i\sin 0)$,

$$(\sqrt[4]{1})_k = \sqrt[4]{1}\left(\cos\frac{0+2k\pi}{4} + i\sin\frac{0+2k\pi}{4}\right) = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}$$

$$= \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2} \text{ for } k = 0,\ 1,\ 2,\ 3.$$

   Hence, we obtain

$$(\sqrt[4]{1})_0 = \cos 0 + i\sin 0 = 1, \qquad (\sqrt[4]{1})_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i,$$

$$(\sqrt[4]{1})_2 = \cos\pi + i\sin\pi = -1, \text{ and } (\sqrt[4]{1})_3 = \cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4} = -i.$$

   Hence, $\sqrt[4]{1} = \left\{1,\ i,\ -1,\ -i\right\}$.

   b. Because $-1 = e^{i\pi}$, we have

$$\sqrt[6]{-1} = (e^{i\pi})^{\frac{1}{6}} = e^{\frac{\pi+2k\pi}{6}} \qquad \text{for } k = 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

   It follows that

$$
\begin{aligned}
(\sqrt[6]{-1})_0 &= e^{\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{1}{2}i \quad (\sqrt[6]{-1})_1 = e^{\frac{\pi}{2}} = i, \\
(\sqrt[6]{-1})_2 &= e^{\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i,\ (\sqrt[6]{-1})_3 = e^{\frac{7\pi}{6}} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i, \\
(\sqrt[6]{-1})_4 &= e^{\frac{3\pi}{2}} = -i \quad \text{and} \quad (\sqrt[6]{-1})_5 = e^{\frac{11\pi}{6}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i.
\end{aligned}
$$

Hence,

$$\sqrt[6]{-1} = \left\{(\sqrt[6]{-1})_0,\ (\sqrt[6]{-1})_1,\ (\sqrt[6]{-1})_2,\ (\sqrt[6]{-1})_3,\ (\sqrt[6]{-1})_4,\ (\sqrt[6]{-1})_5\right\}.$$

c. Because $\sqrt[3]{1-i} = \left(\sqrt{2}e^{-\frac{\pi}{4}}\right)^{\frac{1}{3}} = \sqrt[6]{2}e^{\frac{-\frac{\pi}{4}+2k\pi}{3}}$ for $k = 0,\ 1,\ 2,$

$$
\begin{aligned}
(\sqrt[3]{1-i})_0 &= \sqrt[6]{2}e^{\frac{-\frac{\pi}{4}}{3}} = \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right), \\
(\sqrt[3]{1-i})_1 &= \sqrt[6]{2}e^{\frac{-\frac{\pi}{4}+2\pi}{3}} = \sqrt[6]{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right),\ . \\
(\sqrt[3]{1-i})_2 &= \sqrt[6]{2}e^{\frac{-\frac{\pi}{4}+2\pi}{3}} = \sqrt[6]{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)
\end{aligned}
$$

Hence, $\sqrt[3]{1-i} = \left\{(\sqrt[3]{1-i})_0,\ (\sqrt[3]{1-i})_1,\ (\sqrt[3]{1-i})_2\right\}.$

d. Because $\sqrt[3]{1-i} = \left(e^{-i\frac{\pi}{2}}\right)^{\frac{1}{3}} = \left(e^{\frac{-\frac{\pi}{2}+2k\pi}{3}}\right)$ for $k = 0,\ 1,\ 2,$

$$\begin{aligned}
(\sqrt[3]{-i})_0 &= e^{-i\frac{\pi}{6}} = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - i\frac{1}{2},\\
(\sqrt[3]{-i})_1 &= e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i,\\
(\sqrt[3]{-i})_2 &= e^{i\frac{7\pi}{6}} = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - i\frac{1}{2}.
\end{aligned}$$

Hence, $\sqrt[3]{-i} = \left\{\frac{\sqrt{3}}{2} - i\frac{1}{2},\ i,\ -\frac{\sqrt{3}}{2} - i\frac{1}{2}\right\}.$

e.

$$-1 + \sqrt{3}i = 2\left(\cos(\pi - \frac{\pi}{3}) + i\sin(\pi - \frac{\pi}{3})\right) = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right),$$

$$(-1 + \sqrt{3}i)^{\frac{1}{4}} = \sqrt[4]{2}\left(\cos\frac{\frac{2\pi}{3}+2k\pi}{4} + i\sin\frac{\frac{2\pi}{3}+2k\pi}{4}\right) \text{ for } k = 0,\ 1,\ 2,\ 3.$$

For $k - 0,\ 1,\ 2,\ 3,$ we obtain

$$\left[(-1)+\sqrt{3}i)^{\frac{1}{4}}\right]_0 = \sqrt[4]{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) = \sqrt[4]{2}\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right),$$

$$\left[(-1+\sqrt{3}i)^{\frac{1}{4}}\right]_1 = \sqrt[4]{2}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) = \sqrt[4]{2}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right),$$

$$\left[(-1+\sqrt{3}i)^{\frac{1}{4}}\right]_2 = \sqrt[4]{2}\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right) = \sqrt[4]{2}\left(-\frac{\sqrt{3}}{2}-i\frac{1}{2}\right),$$

$$\left[(-1+\sqrt{3}i)^{\frac{1}{4}}\right]_3 = \sqrt[4]{2}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right) = \sqrt[4]{2}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right).$$

Hence, $(-1+\sqrt{3}i)^{\frac{1}{4}} = \left\{\left[(-1)+\sqrt{3}i)^{\frac{1}{4}}\right]_k : k=0,\ 1,\ 2,\ 3\right\}.$

2. **2.** Because $-8 = 8(\cos\pi + i\sin\pi)$, we have

$$\sqrt[3]{-8} = 2\left[\cos\left(\frac{\pi+2k\pi}{3}\right)+i\sin\left(\frac{\pi+2k\pi}{3}\right)\right] \text{ for } k=0,\ 1,2.$$

Hence,

$$(\sqrt[3]{-8})_0 = 2\left(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 1 + \sqrt{3}i,$$

$$(\sqrt[3]{-8})_1 = 2\left(\cos\left(\frac{\pi+2\pi}{3}\right) + i\sin\left(\frac{\pi+2\pi}{3}\right)\right) = 2(\cos\pi + i\sin\pi) = -2,$$

$$(\sqrt[3]{-8})_2 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i.$$

All solutions of $z^3 = -8$ are $(\sqrt[3]{-8})_k$ for $k = 0,\ 1,\ 2.$

3. **3.**

a. By (10.4.9), the solution of the equation is

$$z = \frac{-2 + \sqrt{4 - 4(1-i)(1+i)}}{2(1-i)} = \frac{-2 + \sqrt{-4}}{2(1-i)} = \frac{-1 + \sqrt{-1}}{(1-i)}.$$

By Example 10.4.1, $\sqrt{-1} = \pm i$. Hence, $z_0 = \frac{-1+i}{1-i} = -1$ and

$$z_1 = -\frac{(1+i)^2}{(1-i)(1+i)} = -\frac{2i}{2} = -i.$$

Hence, $z_0 = -1$ and $z_1 = -i$ are the solutions of

$$(1-i)z^2 + 2z + (1+i) = 0.$$

b. By (10.4.9), the solution of the equation is

$$z = \frac{-2 + \sqrt{4 - 4(1-i)}}{2} = -1 + \sqrt{i}$$

Because $\sqrt{i} = (e^{i\frac{\pi}{2}})^{\frac{1}{2}} = e^{i\frac{(\frac{\pi}{2}+2k\pi)}{2}}$ for $k = 0,\ 1,$

$$z_0 = -1 + e^{i\frac{\pi}{4}} = -1 + \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = -1 + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

and

$$z_1 = -1 + e^{i\frac{5\pi}{4}} = -1 + \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -1 - \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

Hence, z_0 and z_1 are the solutions of $z^2 + 2z + (1-i) = 0$.