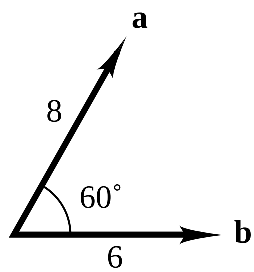
Learning Guide 3 : Work and Energy

<u>Vector Warm-Ups</u> — The Scalar Product

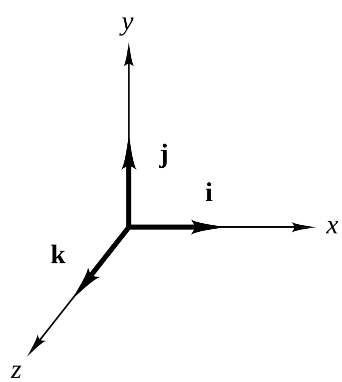
1. What is $\mathbf{a} \cdot \mathbf{b}$? **Key 25**



There is another method of calculating scalar products that is occasionally more convenient than the formula $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$. You will discover this method in parts (2) and (3) and then use it in part (4).

2. $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the unit vectors in the coordinate system drawn in the diagram. Fill in the multiplication table:

$\cdot$	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$
$\hat{\imath}$			
$\hat{\jmath}$			
$\hat{k}$			



3. If you need help turn to Helping Question <u>**1**</u>. **Key 6**

4. In a certain fixed coordinate system, $\mathbf{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$ and $\mathbf{b} = b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k}$. In terms of the components a_x, b_x, a_y, b_y, a_z, and b_z, what is $\mathbf{a} \cdot \mathbf{b}$? Stuck? Turn to Helping Question <u>**2**</u>. **Key 18**

5. Suppose the vectors in part (1) are now given in terms of their components in a rectangular coordinate system. What is $\mathbf{a} \cdot \mathbf{b}$? **Key 32**

Problem I

A stone is released from a height h at time $t = 0$ and falls straight down under the influence of gravity. As you know, its height y at time t is given by $y(t) = h - gt^2/2$.

1. How much work W has the force of gravity done on the stone from the time of its release to time t? If you need help, use Helping Question **3**. **Key 17**

   As you also know, the stone's speed v at time t is given by $v = -gt$.

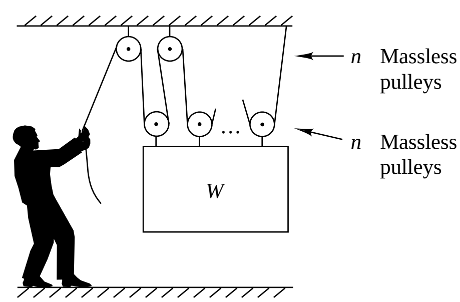
2. From the definition of kinetic energy, what is the kinetic energy K_0 at the start, and what is the kinetic energy K at time t? **Key 31**

3. Verify the work-energy theorem $W = \Delta K$ for this system. **Key 7**

4. From the definition of instantaneous power $P = \mathrm{d}W/\mathrm{d}t$, what is the power delivered by gravity to the stone at time t? From the formula $P = \mathbf{F} \cdot \mathbf{v}$, what is the power delivered by gravity to the stone at time t? Do your two answers agree? **Key 38**

Problem II

A construction worker wants to lift a block of weight $W = 1000$ lb a height $h = 5$ ft off the ground. He attempts to contrive a way to avoid doing 5000 ft·lb of work on the block. One idea he thinks up is to employ $2n$ pulleys, arranged as shown in the sketch. Assume that the tension is the same everywhere in the rope and that friction and the rope's deviation from the vertical can be neglected.



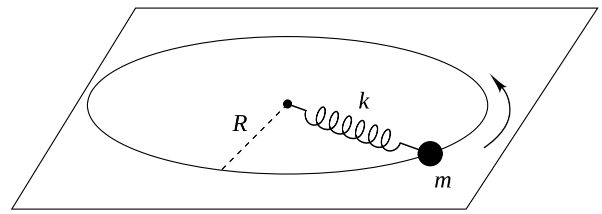
1. Using a force (free-body) diagram, determine the magnitude of the force F the worker needs to exert to lift the block in terms of W and n. Assume that the initial acceleration of the block is negligible and that once in motion the block moves with constant speed. Use Helping Questions **4** and **5** if you need to. **Key 23**

2. Through what distance d does the worker need to pull the rope in order to lift the block h ft? Use Helping Question **6**, if necessary. **Key 5**

3. Does the contraption decrease the amount of work he has to do? **Key 8**

Problem III

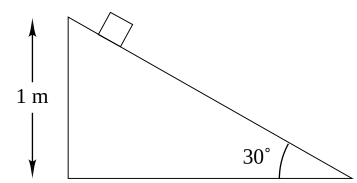
A mass m is attached to a massless spring (with an unstretched length of l_0, and spring constant k) and moves in a circular path of radius R. Assume at first that there is no friction between the mass and the horizontal surface.



1. Find the ratio of the potential energy in the spring to the kinetic energy of the mass in terms of known quantities — i.e., (m, l_0, k, R, g). After a good try, use Helping Question **7**. **Key 1**
2. Can the spring's potential energy equal or exceed the mass's kinetic energy? **Key 26**
3. Now, suppose a demon suddenly "turns on friction" between the mass and the surface. Find the distance d that the mass moves before it stops. Assume that the coefficients of static and kinetic friction are both very small and are a constant $= \mu$. Also assume that the final length of the spring is l_0. (Actually, it will be somewhat longer that l_0, but if μ is small, this can be ignored.) Stuck? Look at Helping Questions **8** and **9**. **Key 30**

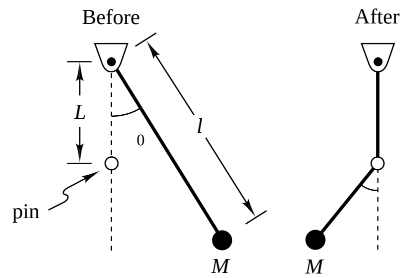
Problem IV

A block slides down a 1-m-high ramp, which is tilted at 30°. The coefficient of kinetic friction between the block and the ramp is $\mu_k = 0.4$. What is the block's speed at the bottom of the ramp? Stuck? Use Helping Question **10** and then Helping Question **11**, if you need to. **Key 40**



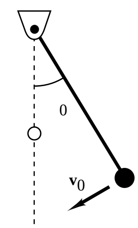
Problem V

A simple pendulum consisting of a mass M at the end of a string of length l is released from rest at an angle θ_0. A pin is located a distance $L < l$ directly below the pivot point.



1. After the string hits the pin, what is the maximum angle α that the pendulum makes with respect to the vertical? Stuck? Use Helping Question **<u>12</u>** and then Helping Question **<u>13</u>**, if you need to. **Key 19**

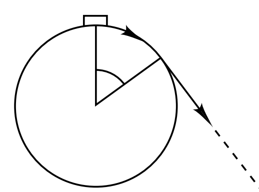
2. If, instead, the mass is given an initial velocity $\mathbf{v}_0$ as sketched here, what is the maximum angle α after the string hits the pin? **Key 12**.



3. How does the answer to part (2) change if $\mathbf{v}_0$ is in the direction opposite to that shown in the figure? **Key 41**

Problem VI

A small mass m starts from rest and slides from the top of a fixed sphere of radius r.



1. If the sphere is frictionless at what angle θ from the vertical does the mass leave the surface? If you need a hint, use Helping Question **<u>14</u>**. **Key 36**

2. Suppose there is a finite friction of $\mu_s = 0.1$ between the mass and the sphere. What is the minimum angle $\theta_{\min}$ at which the mass will start to slide along the sphere? **Key 46**

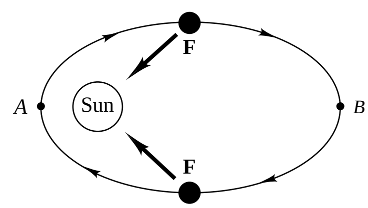
3. The mass is now placed just past this minimum angle and released. The coefficient of kinetic friction μ_k is small but non zero. Does the mass fly off at a larger or a smaller θ than was found in part (1)? Assume that in both cases θ is defined with respect to the top of the sphere. **Key 14**

Problem VII

The power of an automobile engine is usually measured in horsepower (hp) instead of watts (W). One horsepower equals 746 W. A typical automobile engine can sustain an output of 100 hp for a long period of time. Find some clever way to estimate how much sustained power *you* can put out, say, for half an hour. Need an idea? Try Helping Question **15**.
Key 9

Problem VIII

The gravitational force exerted on a planet by the sun is attractive, so the planet's potential energy is greater the farther from the sun it is. The planet moves in slightly elliptical orbit, as indicated in the diagram.



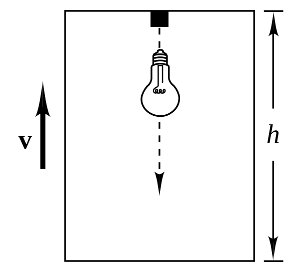
1. Is the planet moving faster at point A or point B? If you can't figure this out, turn to Helping Question **16**. **Key 10**

2. During what part of the orbit is the sun doing positive work on the planet? Negative work? See Helping Question **17**, if you're stuck here. **Key 3**

3. Check that when the sun is doing positive work the power $\mathbf{F} \cdot \mathbf{v}$ is positive; see Helping Question **18**. **Key 15**

Problem IX

The work done by a force depends on the reference frame from which the system is observed. The kinetic energy of a particle also depends on the reference frame. However, in all inertial reference frames the work-energy theorem $W = \Delta K$ holds. In this problem, you will verify these three statements for a sample system viewed from two different reference frames.



An elevator of height h moves upward at constant velocity v. Vibrations cause the elevator's light bulb to fall from its fixture at time $t = 0$. The bulb hits the floor at time t later. Call the mass of the bulb m.

1. One observer views the bulb from the elevator. In this frame, what is the initial kinetic energy K_0? The final kinetic energy K? The work W done by gravity? Since you are trying to verify the work-energy theorem, do not use any energy principles you have learned: calculate the work done from the definition $W = Fd$. **Key 11**

2. A second observer views the bulb from the ground. In this frame, what is the initial kinetic energy K_0? The final kinetic energy K? The work W done by gravity? Again, calculate the work done from the definition $W = Fd$. **Key 20**

3. Verify that the ground observer gets the same work-energy equation as the elevator observer. (The work-energy equation means $W = \Delta K$ expressed in terms of m, v, t, g, and h.) **Key 44**

Helping Questions

1. What is the length of a unit vector? What is the angle between a unit vector and itself? Between two different unit vectors that point along different coordinate axes? Now you know enough to use the formula for the scalar product. **Key 39**

2. How can you use your results from part (2)? **Key 35**

3. The equation $y = h - gt^2/2$ is valid in the coordinate system that has the positive direction of the y-axis pointing upward. In this coordinate system what is the force $\mathbf{F}$? What is the displacement $\mathbf{d}$ in time t? **Key 45**

4. Draw a force (free-body) diagram to determine the tension in the rope. **Key 43**

5. How is the rope tension related to force F? **Key 33**

6. This is a geometry question, not a physics question. The block finishes h ft closer to the ceiling; how much rope must be pulled past the top pulley on the left? **Key 28**

7. How can you use what you know about circular motion to get the kinetic energy of the mass? (Get rid of v^2!) **Key 37**

8. The only thing you know about the distance traveled is the frictional energy dissipated as the mass slows down and stops. How does this help? **Key 34**

9. What is the total energy of the system when the friction comes on? **Key 16**

10. How does the change in the sum of the kinetic energy plus the potential energy relate to the mechanical energy lost to friction? **Key 4**

11. What is the force of friction? Then what is the mechanical energy lost to friction? **Key 22**

12. What quantity is conserved throughout the motion? **Key 2**

13. What are the initial kinetic and potential energies, K_{i} and U_{i}? What are the final kinetic and potential energies K_{f} and U_{f}? Take the potential energy equal to zero at the lowest point of the motion. **Key 27**

14. Use conservation of energy to determine the speed v of the mass at angle θ. What is the normal force when the block just leaves the sphere? Your answer will give you a second equation relating v and θ. **Key 29**

15. How many stairs can you climb in half an hour? **Key 42**

16. Is the *total* energy of the system conserved? When is *kinetic* energy the greatest?
Key 24

17. When the potential energy of the system is decreasing, is the sun doing positive or negative work on the planet? **Key 21**

18. What is the sign of $\mathbf{F} \cdot \mathbf{v}$ if the angle between $\mathbf{F}$ and $\mathbf{v}$ is less than $90°$? Greater than $90°$? **Key 13**

Notes: Conserved Quantities

Now that you've finished Learning Guides 2 and 3, you're able to solve mechanics problems in two quite different ways — by $\mathbf{F} = m\mathbf{a}$ and by energy conservation. As you progress in your study of mechanics, you'll develop an intuition for what types of problems are best solved by $\mathbf{F} = m\mathbf{a}$ and what types of problems should be solved by energy conservation. But while you're learning, a good rule to work by is: *always try to solve the problem first by energy conservation.* If the problem can be solved by both $\mathbf{F} = m\mathbf{a}$ and energy conservation, *it's almost always easier to solve it by energy conservation.* Let's look back at some examples.

Problem III, where a mass attached to a spring slows down by friction, illustrates the point well. It's easy to solve this problem by energy methods once you get the hang of it. In principle, it's also possible to solve it using $\mathbf{F} = m\mathbf{a}$; but to do this, you would first need to know an equation describing the path of the mass. Thus, $\mathbf{F} = m\mathbf{a}$ is the hard way to solve the problem. Problem IV, in which a block slides down a ramp, is a less clear-cut case. The helping questions suggest a combination of attacks, using the work-energy theorem together with $\mathbf{F} = m\mathbf{a}$ to get the mechanical energy lost to friction; this, however, is not really much easier than using only $\mathbf{F} = m\mathbf{a}$. Problem V, with the pendulum and the peg, is similar to problem III: solvable by conservation of energy, but practically impossible by $\mathbf{F} = m\mathbf{a}$. Problem VI, where a block is sliding off a sphere, needs a combination of attacks: one equation from from $\mathbf{F} = m\mathbf{a}$ and one from energy conservation.

Soon you will learn about conservation of linear momentum, as well as conservation of angular momentum. Since linear and angular momenta are both vector quantities with three components each, you will have seven conserved quantities in all – quite a powerful arsenal. *You will be able to solve a wide range of physical problems by conservation laws alone. Some of the physical systems that you will study will be so complex that it won't be possible to identify all the forces; but you will still be able to get important results by using conservation laws alone.*

ANSWER KEY

1.
$$\frac{\text{PE (spring)}}{\text{KE (Mass)}} = \frac{R - l_0}{R}$$

2. Total energy

3. Bottom half; Top half

4. The change in mechanical energy and the mechanical energy lost to friction are equal. Both quantities are negative for this and all problems involving friction.

5. $d = 2nh$

6.

$\cdot$	$\hat{i}$	$\hat{j}$	$\hat{k}$
$\hat{i}$	1	0	0
$\hat{j}$	0	1	0
$\hat{k}$	0	0	1

7.
$$W = \Delta K, \quad \frac{mg^2t^2}{2} = \frac{mg^2t^2}{2} - 0$$

8. No, since $Fd = Wh$ for all choices of n. We know from the work-energy theorem that it must be this way.

9. Around 1/6 horsepower, if your mass is 60 kg.

10. Point A

11. $K_0 = 0$; $K = \frac{1}{2}m(gt)^2$; $W = mgh$.

12.
$$\alpha = \cos^{-1}\left[\left(\frac{1}{l - L}\right)\left(l\cos\theta_0 - L - \frac{v_0^2}{2g}\right)\right]$$

13. $+$, $-$

14. Larger

15. It checks: when the planet is going from B to A the angles between $\mathbf{F}$ and $\mathbf{v}$ is *less* than 90°, so the power delivered is positive.

16.
$$E_{\text{tot}} = \frac{kR}{2}(R - l_0) + \frac{1}{2}k(R - l_0)^2$$

17. $mg^2t^2/2$

18. $a_xb_x + a_yb_y + a_zb_z$

19.
$$\alpha = \cos^{-1}\left(\frac{l\cos\theta_0 - L}{l - L}\right)$$

20. $K_0 = \frac{1}{2}mv^2$; $K = \frac{1}{2}m(-v + gt)^2$ $W = mg(h - vt)$

21. Positive

22.
$$F_k = \mu_k N = \mu_k mg\cos\theta$$
$$W_k = \mathbf{F}_k \cdot \mathbf{d}_k = -\mu_k mgh(\cos\theta/\sin\theta)$$

23. $F = W/2n$

24. Yes; when the potential energy is the least

25. 24

26. No. l_0 cannot be zero for a real spring.

27.

$$K_\mathrm{i} = 0;\ U_\mathrm{i} = mgl(1-\cos\theta_0) K_\mathrm{f} = 0;\ U_\mathrm{f} = mg(l-L)(1-\cos\alpha)$$

28. $2nh$

29. From energy conservation:

$$\frac{1}{2}mv^2 = mgr(1-\cos\theta).$$

From the normal force equaling zero at breakaway:

$$mg\cos\theta = \frac{mv^2}{r}.$$

30.

$$\text{Distance} = \frac{k}{\mu mg}(R-l_0)\left(R-\frac{l_0}{2}\right)$$

31. $K_0 = 0; \quad K = mg^2t^2/2$

32. 24 again

33. They are equal.

34. Energy lost to friction = μmgd = initial total energy.

35. Multiply out $(a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}) \cdot (b_x\hat{\imath} + b_y\hat{\jmath} + b_z\hat{k})$. You will get nine terms, and for each one of them you can use an entry from your multiplication table.

36. $\theta = \cos^{-1}(2/3) \simeq 48.2°$, independent of r and g.

37.

$$\text{Force} = k(R-l_0) = \frac{mv^2}{R},$$

so

$$\frac{1}{2}mv^2 = \frac{1}{2}kR(R-l_0).$$

38.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = mg^2t,$$

which agrees with

$$\mathbf{F}\cdot\mathbf{v} = (-mg\hat{\jmath})\cdot(-gt\hat{\jmath}) = mg^2t.$$

39. $1; 0°; 90°$; so $\hat{\imath}\cdot\hat{\imath} = (1)(1)\cos 0° = 1$, etc.

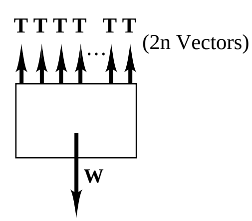
40.

$$v = \sqrt{2gh\,(1-\mu_k\cot\theta)} = 2.5 \text{ m/s}$$

41. The answer doesn't change.

42. Maybe 1 step per second, or 1800 steps. 5 steps is about 1 m. So you can lift your weight through 360 m in 1800 s. What's your power output?

43.



44. The ground observer gets an extra term, minus $mgvt$, on each side of his equation.

45. $-mg$ and $-gt^2/2$
(note both minus signs)

46. $\theta_{\min} = \tan^{-1}\mu = 5.7^\circ$