# Proof that Best Distribution of Edge Weights for Maximization in $K_n$ & Divided Cycles Stems from Longest Cycle

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We begin by defining $K_n$ as a complete graph with $n$ vertices. The number of edges in $K_n$ is given by ${}_nC_2$. We denote the edges of $K_n$ as $e_1, e_2, \ldots, e_m$ where $m = {}_nC_2$. The weights assigned to these edges are $w(e_1), w(e_2), \ldots, w(e_m)$, with the assumption that $w(e_1) > w(e_2) > \ldots > w(e_m)$. The total graph value $V(K_n)$ is defined as the sum of the products of the weights of the edges across all spanning trees of $K_n$. According to Cayley's formula, the number of spanning trees in $K_n$ is $n^{n-2}$. Each spanning tree consists of $n-1$ edges. To analyze the contribution of a specific edge $e_i$ to the total graph value, we express this contribution as $C_i = w(e_i) * T_i$ where, $T_i$ is the number of spanning trees that include the edge $e_i$. When an edge $e_i$ is included in a spanning tree, the remaining $n-2$ vertices can be obtained by any spanning tree formed from the remaining $n-1$ vertices. Therefore, the number of spanning trees that include edge $e_i$ is given by $T_i = n^{n-3}$. This leads to the conclusion that the total contribution to the graph value from all edges can be expressed as:

$$V(K_n) = \sum_{i=1}^{m} C_i = \sum_{i=1}^{m} w(e_i) * n^{n-3}$$

To maximize $V(K_n)$, we need to maximize the sum. If the largest weights are assigned to the edges that form the longest cycle in $K_n$, then these edges will appear in a larger number of spanning trees, thus increasing their overall contribution to $V(K_n)$. In $K_n$, the longest cycle is a Hamiltonian cycle that includes all $n$ vertices. If the weights of the edges in the Hamiltonian cycle are the largest, each time a spanning tree is formed by including an edge from this cycle, it will significantly contribute to the total value. Given the multitude of spanning trees possible in $K_n$, the overall contribution from the edges of the longest cycle will dominate the total graph value. Consequently, by placing the largest weights on the edges of the longest cycle in $K_n$, we ensure that these edges contribute maximally to the total graph value since they will appear in a larger number of spanning trees. **Q.E.D.**

We define the graph $G$ as consisting of a cycle $C_n$ with $n$ vertices and an additonal edge $e_{n+1}$ that connects two non-adjacent vertices within the cycle.

The edges of $G$ can be denoted as $e_1, e_2, \ldots, e_n$ for the edges of the cycle, and $e_{n+1}$ for the extra edge. We assign weights to these edges as $w(e_1), w(e_2), \ldots, w(e_n), w(e_{n+1})$, where we assume $w(e_1) \geq w(e_2) \geq \ldots \geq w(e_n) \geq w(e_{n+1})$. The total graph value $V(G)$ s defined as the sum of the products of the weights of the edges across all spanning trees of $G$. A spanning tree consists of $n$ vertices and $n-1$ edges. To analyze the contribution of each edge to the total graph value, consider first the spanning trees that do not include the extra edge. When a spanning tree is formed by removing one edge from the cycle $C_n$, the edge $e_{n+1}$ can be included. The contribution of each spanning tree that includes $e_{n+1}$ can be expressed as $C_{e_{n+1}} = w(e_{n+1}) * T_{e_{n+1}}$, where $T_{e_{n+1}}$ is the number of spanning trees that include the edge $e_{n+1}$. For each edge $e_i$ of the cycle that is removed, we form a spanning tree that includes $e_{n+1}$. Next, we consider the spanning trees that include each edge $e_i$ of the cycle. The contribution from these edges can be expressed as $C_i = w(e_i) * T_i$, where $T_i$ is the number of spanning trees that include edge $e_i$. Each edge can be included in $T_i = n^{n-3}$ spanning trees. The total contribution to the graph value from all edges can then be expressed as:

$$V(G) = \sum_{i=1}^{n} C_i + C_{e_{n+1}} = \sum_{i=1}^{n} w(e_i) * n^{n-3} + w(e_{n+1}) * T_{e_{n+1}}$$

. To maximize $V(G)$, we need to maximize the expression $\sum_{i=1}^{n} w(e_i) + w(e_{n+1})$. The optimal assignment of weights occurs when the largest weights are placed on the edges of the cycle $C_n$ while assigning the smallest weight to the extra edge $e_{n+1}$. This ensures that the edges of the cycle contribute maximally to the total graph value because they will appear in a greater number of spanning trees compared to the extra edge. **Q.E.D.**