Proofs for Cycles and Complete Graphs

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Let the two weighted n-cycles be denoted as C_1 and C_2. The Edges of C_1 can be represented as $E_1 = \{e_1, e_2, \dots, e_n\}$ and the edges of C_2 as $E_2 = \{f_1, f_2, \dots, f_n\}$. The weights assigned to the edges of both C_1 and C_2 are derived from the same set $W = \{w_1, w_2 \dots, w_n\}$ and are assigned randomly; *i.e.*, for some weight $w_n = e_a$ and $w_n = f_b$ where $a, b \leq n$. We will define a mapping $m : E_1 \to E_2$ such that $m(e_i) = f_i$ for $i = 1, 2, \dots, n$. To demonstrate that this mapping is a bjection, we must show that it is both injective and surjective. To establish injectivity, suppose there exist indices i and j such that $m(e_i) = m(e_j)$. This implies that $f_i = f_j$. Since the edges f_i and f_j correspond to distinct connections in the cycle C_2, we must have $i = j$. Therefore, the mapping m is injective. Next, to show surjectivity, consider any edge $f_k \in E_2$. There exists an edge $e_k \in E_1$ such that $m(e_k) = f_k$. This mapping ensures that every edge in E_2 is accounted for by at least one edge in E_1, thus satisfying the surjectivity condition. Having established that m is both injective and surjective, we conclude that m is a bijection. This one-to-one correspondence between the edges of C_1 and C_2 is independent of the order in which the weights are assigned to the edges. onsequently, since the edges are matched in a bijective manner, the sum of the products of the weights associated with spanning trees of C_1 and C_2 remains invariant under the assignments of weights. Thus, the total graph values for both cycles are equal:

$$TGV(C_1) = TGV(C_2).$$

Q.E.D.

Let the two weighted n-complete graphs be denoted K_1 and K_2. The edges of K_1 can be represented as $E_1 = \{e_{ij} | 1 \leq i < j \leq n\}$, and the edges of K_2 as $E_2 = \{f_{ij} | 1 \leq i < j \leq n\}$. The weights assigned to the edges of both K_1 and K_2 are derived from the same set $W = \{w_1, w_2, \dots, w_n\}$, where $m = \frac{n(n-1)}{2}$ represents the total number of edges in a complete graph. We will define a mapping $p : E_1 \to E_2$ such that $p(e_{ij}) = f_{ij}$ for each edge in E_1 corresponding to the edge f_{ij} in E_2. To demonstrate that this mapping is a bijection, we must show that it is both injective and surjective. To establish injectivity, suppose there exist indices (i, j) and (k, l) such that $p(e_{ij}) = p(e_{kl})$. This implies that $f_{ij} = f_{kl}$. Since the edges f_{ij} and f_{kl} correspond to distinct pairs of vertices in the complete graph K_2, it must follow that $(i, j) = (k, l)$. Therefore, the mapping p is injective. Next, to show surjectivity, consider any edge $f_{kl} \in E_2$. There exists an edge $e_{kl} \in E_1$ such that $p(e_{kl} = f_{kl})$. This mapping ensures that every edge in E_2 is account for by at least one edge in E_1, thus satisfying the surjectivity condition. Having established that p is both injective and surjective, we conclude that p is a bijection. This one-to-one correspondence between the edges of K_1 and K_2 is independent of the order in which the weights are assigned to the edges. Consequently, since the edges are matched in a bijective manner, the sum of the products of the weights associated with spanning trees of K_1 and K_2 remains invariant under the assignment of weights. Thus, the total graph values for both complete graphs are equal:

$$TGV(K_1) = TGV(K_2).$$

Q.E.D.