

# 第 1 章

## § 5 行列式按行 (列) 展开

例 12

证明范德蒙德 (Vandermonde) 行列式

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

solution: 数学归纳法

当  $n = 1$  或当  $n = 2$  时显然成立

$$\begin{aligned} D_n &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & x_2 - x_1 & \cdots & x_n - x_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_2 - x_1 & x_3 - x_1 & \cdots & x_n - x_1 \\ \vdots & \vdots & \ddots & \vdots \\ x_2^{n-2}(x_2 - x_1) & x_3^{n-2}(x_3 - x_1) & \cdots & x_n^{n-2}(x_n - x_1) \end{vmatrix} \\ &= \left( \prod_{i=2}^n (x_i - x_1) \right) \left( \prod_{2 \leq i < j \leq n} (x_j - x_i) \right) \\ &= \prod_{1 \leq i < j \leq n} (x_j - x_i) \end{aligned}$$

Q.E.D

## 习题

1. 利用对角线法则计算下列三阶行列式:

$$(1) \begin{vmatrix} 2 & 0 & 1 \\ 1 & -4 & -1 \\ -1 & 8 & 3 \end{vmatrix} = -24 + 0 + 8 + 16 - 0 - 4 = -4$$

$$(2) \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

$$(3) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a \\ = (a-b)(b-c)(c-a)$$

$$(4) \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = 3xy(x+y) - x^3 - y^3 - (x+y)^3 \\ = -2x^3 - 2y^3$$

## 2 算逆序对数 (略)

### 3. 写出四阶行列式中含有因子 $a_{11}a_{23}$ 的项

答:  $-a_{11}a_{23}a_{32}a_{44}$  和  $a_{11}a_{23}a_{34}a_{42}$

### 4. 计算下列各行列式:

$$(1) \begin{vmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 0 & 1 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 2 & -10 \\ 1 & 2 & 0 & 2 \\ 10 & 3 & 2 & -14 \\ 0 & 0 & 1 & 0 \end{vmatrix} \\ = (-1)^{3+4} \begin{vmatrix} 4 & -1 & -10 \\ 1 & 2 & 2 \\ 10 & 3 & -14 \end{vmatrix} \\ = \begin{vmatrix} 4 & -1 & 10 \\ 1 & 2 & -2 \\ 10 & 3 & 14 \end{vmatrix} \\ = \begin{vmatrix} 9 & 9 & 10 \\ 0 & 0 & -2 \\ 17 & 17 & 14 \end{vmatrix} \\ = 0$$

$$(2) \begin{vmatrix} 2 & 1 & 4 & 1 \\ 3 & -1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 5 & 0 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 4 & 1 \\ 5 & 0 & 6 & 2 \\ 7 & 0 & 7 & 4 \\ 5 & 0 & 6 & 2 \end{vmatrix} \\ = \begin{vmatrix} 5 & 6 & 2 \\ 7 & 7 & 4 \\ 5 & 6 & 2 \end{vmatrix} \\ = 0$$

$$(3) \begin{vmatrix} -ab & ac & ae \\ bd & -cd & de \\ bf & cf & -ef \end{vmatrix} = adf \begin{vmatrix} -b & c & e \\ b & -c & e \\ b & c & -e \end{vmatrix} \\ = adf(bce + 3bce) \\ = 4abcdef$$

$$(4) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ b+c & a-b & a-c \end{vmatrix} = 0$$

$$(5) \begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix} = \begin{vmatrix} 0 & 1+ab & a & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix} = \begin{vmatrix} 1+ab & a & 0 \\ -1 & c & 1 \\ 0 & -1 & d \end{vmatrix} = (1+ab)cd + 1 + ab + ad = abcd + ab + ad + cd + 1$$

$$(6) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 1 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -3 \\ 2 & -2 & -2 \\ -1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & -4 \\ 0 & -4 & -4 \\ -1 & -1 & -1 \end{vmatrix} = -\begin{vmatrix} 0 & -4 \\ -4 & -4 \end{vmatrix} = 16$$

5. 求解下列方程:

$$(1) \begin{vmatrix} x+1 & 2 & -1 \\ 2 & x+1 & 1 \\ -1 & 1 & x+1 \end{vmatrix} = 0$$

$$(x+1)^3 - 4 = 6(x+1)$$

设  $y = x+1$

$$y^3 - 6y - 4 = 0$$

$$(y+2)(y^2 - 2y - 2) = 0$$

$$y = -2 \text{ or } \sqrt{3} + 1 \text{ or } -\sqrt{3} + 1$$

$$x = -3 \text{ or } \sqrt{3} \text{ or } -\sqrt{3}$$

$$(2) \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & a & b & c \\ x^2 & a^2 & b^2 & c^2 \\ x^3 & a^3 & b^3 & c^3 \end{vmatrix} = 0 \quad (a \neq b \neq c)$$

$$(x - a)(x - b)(x - c)(a - b)(a - c)(b - c) = 0 \quad (Vandermonde 行列式)$$

$$x = a \text{ or } b \text{ or } c$$

6. 证明:

$$(1) \quad \begin{vmatrix} a^2 & ab & b^2 \\ 2a & a+b & 2b \\ 1 & 1 & 1 \end{vmatrix} = (a-b)^3$$

$$\begin{vmatrix} a^2 & ab & b^2 \\ 2a & a+b & 2b \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a(a-b) & ab & b(b-a) \\ a-b & a+b & b-a \\ 0 & 1 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} a(a-b) & b(b-a) \\ a-b & b-a \end{vmatrix}$$

$$= -b(a-b)^2 + a(a-b)^2$$

$$= (a-b)^3$$

$$(2) \quad \begin{vmatrix} ax+by & ay+bz & az+bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix} = (a^3 + b^3) \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$\begin{vmatrix} ax+by & ay+bz & az+bx \\ ay+bz & az+bx & ax+by \\ az+bx & ax+by & ay+bz \end{vmatrix} = a \begin{vmatrix} x & ay+bz & az+bx \\ y & az+bx & ax+by \\ z & ax+by & ay+bz \end{vmatrix} + b \begin{vmatrix} y & ay+bz & az+bx \\ z & az+bx & ax+by \\ x & ax+by & ay+bz \end{vmatrix}$$

$$= a^2 \begin{vmatrix} x & ay+bz & z \\ y & az+bx & x \\ z & ax+by & y \end{vmatrix} + b^2 \begin{vmatrix} y & z & az+bx \\ z & x & ax+by \\ x & y & ay+bz \end{vmatrix}$$

$$= a^3 \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} + b^3 \begin{vmatrix} y & z & x \\ z & x & y \\ x & y & z \end{vmatrix}$$

$$= a^3 \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix} + b^3 \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$= (a^3 + b^3) \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}$$

$$(3) \quad \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = 0$$

$$\begin{aligned}
& \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} \\
& = \begin{vmatrix} a^2 & 2a+1 & 2a+3 & 2a+5 \\ b^2 & 2b+1 & 2b+3 & 2b+5 \\ c^2 & 2c+1 & 2c+3 & 2c+5 \\ d^2 & 2d+1 & 2d+3 & 2d+5 \end{vmatrix} \\
& = \begin{vmatrix} a^2 & 2a+1 & 2 & 2 \\ b^2 & 2b+1 & 2 & 2 \\ c^2 & 2c+1 & 2 & 2 \\ d^2 & 2d+1 & 2 & 2 \end{vmatrix} \\
& = 0
\end{aligned}$$

$$(4) \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)(a+b+c+d)$$

$$\begin{aligned}
& \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & c-a & d-a \\ 0 & b(b-a) & c(c-a) & d(d-a) \\ 0 & b^2(b^2-a^2) & c^2(c^2-a^2) & d^2(d^2-a^2) \end{vmatrix} \\
& = (b-a)(c-a)(d-a) \begin{vmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2(b+a) & c^2(c+a) & d^2(d+a) \end{vmatrix} \\
& = (b-a)(c-a)(d-a)(c-b)(d-b) \begin{vmatrix} 1 & 1 \\ c(a+b+c) & d(a+b+d) \end{vmatrix} \\
& = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)(a+b+c+d)
\end{aligned}$$

$$(5) \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a_0 & a_1 & a_2 & a_3 \end{vmatrix} = a_3x^3 + a_2x^2 + a_1x + a_0$$

$$\begin{aligned}
& \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a_0 & a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ x^2 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a_0 + a_1x & a_1 & a_2 & a_3 \end{vmatrix} \\
& = \begin{vmatrix} x^2 & -1 & 0 \\ 0 & x & -1 \\ a_0 + a_1x & a_2 & a_3 \end{vmatrix} \\
& = \begin{vmatrix} 0 & -1 & 0 \\ x^3 & x & -1 \\ a_0 + a_1x + a_2x^2 & a_2 & a_3 \end{vmatrix} \\
& = \begin{vmatrix} x^3 & -1 \\ a_0 + a_1x + a_2x^2 & a_3 \end{vmatrix} \\
& = a_3x^3 + a_2x^2 + a_1x + a_0
\end{aligned}$$

设  $n$  阶行列式  $D = \det(a_{ij})$ ，把  $D$  上下翻转、或逆时针旋转  $90^\circ$ 、或依副对角线翻转，依次得

$$D_1 = \begin{vmatrix} a_{n1} & \cdots & a_{nn} \\ \vdots & & \vdots \\ a_{11} & \cdots & a_{1n} \end{vmatrix}, D_2 = \begin{vmatrix} a_{1n} & \cdots & a_{nn} \\ \vdots & & \vdots \\ a_{11} & \cdots & a_{n1} \end{vmatrix}, D_3 = \begin{vmatrix} a_{nn} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{11} \end{vmatrix}$$

求证  $D_1 = D_2 = (-1)^{\frac{n(n-1)}{2}} D, D_3 = D$ 。

solution:

对于  $D_1$ ，从下往上交换，可知  $D_1 = (-1)^{\frac{n(n-1)}{2}} D$

对于  $D_2$ ，可知

$$\begin{aligned} D_2 &= (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{vmatrix} \\ &= (-1)^{\frac{n(n-1)}{2}} D \end{aligned}$$

对于  $D_3$ ，可知

$$\begin{aligned} D_3 &= (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} a_{n1} & \cdots & a_{11} \\ \vdots & & \vdots \\ a_{nn} & \cdots & a_{1n} \end{vmatrix} \\ &= (-1)^{\frac{n(n-1)}{2} + \frac{n(n-1)}{2}} \begin{vmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{vmatrix} \\ &= (-1)^{n(n-1)} D \\ &= D \end{aligned}$$