PS1, Question 3, part C

Derivation of matrix equations:

A pair of 2 images is related by a pure rotation, i.e. a transformation by rotation matrix R. Then for a pair of corresponding 3D direction vectors, $\mathbf{d}_i$ and $\mathbf{d}_i'$, representing a direction vector in image1 and the rotated version of that direction vector in image2, the following is true:

$$\mathbf{d}_i' = \mathbf{R}\mathbf{d}_i$$

Using the notation d_i^k to mean the kth component of direction vector index *i* in image1, and the prime version $\mathrm{d}_i^{k'}$ to mean the kth component of direction vector index *i* in image2, this can be written:

$$\mathbf{R}\mathbf{d}_i = \begin{pmatrix} \mathbf{r}_1^T\mathbf{d}_i \\ \mathbf{r}_2^T\mathbf{d}_i \\ \mathbf{r}_3^T\mathbf{d}_i \end{pmatrix} = \begin{pmatrix} r_{11}\mathrm{d}_i^x + r_{12}\mathrm{d}_i^y + r_{13}\mathrm{d}_i^z \\ r_{21}\mathrm{d}_i^x + r_{22}\mathrm{d}_i^y + r_{23}\mathrm{d}_i^z \\ r_{31}\mathrm{d}_i^x + r_{32}\mathrm{d}_i^y + r_{33}\mathrm{d}_i^z \end{pmatrix},$$ where in the first equation, $\mathbf{r}_j^T$ denotes the jth row of R,

i.e. the 1x3 vector of row j elements. The term on the right of the second equation uses r_{ij} to denote the (*i*,*j*) element of the rotation matrix, and d_i^x, d_i^y, and d_i^z to represent the x, y, and z components of the direction vector $\mathbf{d}_i$ of the first image. Continuing, we get:

$$\mathbf{R}\mathbf{d}_i = \begin{pmatrix} \mathrm{d}_i^x r_{11} + \mathrm{d}_i^y r_{12} + \mathrm{d}_i^z r_{13} + (0)r_{21} + (0)r_{22} + (0)r_{23} + (0)r_{31} + (0)r_{32} + (0)r_{33} \\ (0)r_{11} + (0)r_{12} + (0)r_{13} + \mathrm{d}_i^x r_{21} + \mathrm{d}_i^y r_{22} + \mathrm{d}_i^z r_{23} + (0)r_{31} + (0)r_{32} + (0)r_{33} \\ (0)r_{11} + (0)r_{12} + (0)r_{13} + (0)r_{21} + (0)r_{22} + (0)r_{23} + \mathrm{d}_i^x r_{31} + \mathrm{d}_i^y r_{32} + \mathrm{d}_i^z r_{33} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{d}_i^x & \mathrm{d}_i^y & \mathrm{d}_i^z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_i^x & \mathrm{d}_i^y & \mathrm{d}_i^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{d}_i^x & \mathrm{d}_i^y & \mathrm{d}_i^z \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{21} \\ r_{22} \\ r_{23} \\ r_{31} \\ r_{32} \\ r_{33} \end{pmatrix}$$

The above applies to a single $\mathbf{d}_i$ to $\mathbf{d}_i'$ correspondence. We can find other direction vectors in the original image, and the rotated versions of those direction vectors, and stack them on top of each other. For example, doing this with 3 such correspondences of direction vectors, we get:

$$\begin{pmatrix} \mathbf{d}_1' \\ \mathbf{d}_2' \\ \mathbf{d}_3' \end{pmatrix} = \begin{pmatrix} & \mathbf{d}_1^T & & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \mathbf{d}_1^T & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & \mathbf{d}_1^T & \\ & \mathbf{d}_2^T & & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \mathbf{d}_2^T & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & \mathbf{d}_2^T & \\ & \mathbf{d}_3^T & & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & & \mathbf{d}_3^T & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & & \mathbf{d}_3^T & \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{21} \\ r_{22} \\ r_{23} \\ r_{31} \\ r_{32} \\ r_{33} \end{pmatrix}$$

Or explicitly writing every element:

$$\begin{pmatrix} d_1^{x'} \\ d_1^{y'} \\ d_1^{z'} \\ d_2^{x'} \\ d_2^{y'} \\ d_2^{z'} \\ d_3^{x'} \\ d_3^{y'} \\ d_3^{z'} \end{pmatrix} = \begin{pmatrix} \mathrm{d}_1^x & \mathrm{d}_1^y & \mathrm{d}_1^z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_1^x & \mathrm{d}_1^y & \mathrm{d}_1^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{d}_1^x & \mathrm{d}_1^y & \mathrm{d}_1^z \\ \mathrm{d}_2^x & \mathrm{d}_2^y & \mathrm{d}_2^z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_2^x & \mathrm{d}_2^y & \mathrm{d}_2^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{d}_2^x & \mathrm{d}_2^y & \mathrm{d}_2^z \\ \mathrm{d}_3^x & \mathrm{d}_3^y & \mathrm{d}_3^z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_3^x & \mathrm{d}_3^y & \mathrm{d}_3^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{d}_3^x & \mathrm{d}_3^y & \mathrm{d}_3^z \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{21} \\ r_{22} \\ r_{23} \\ r_{31} \\ r_{32} \\ r_{33} \end{pmatrix}$$

This can be solved for the 9 elements $r_{ij}$of $\boldsymbol{r}$ by matrix inversion:

$$\begin{pmatrix} r_{11} \\ r_{12} \\ r_{13} \\ r_{21} \\ r_{22} \\ r_{23} \\ r_{31} \\ r_{32} \\ r_{33} \end{pmatrix} = \begin{pmatrix} \mathrm{d}_1^x & \mathrm{d}_1^y & \mathrm{d}_1^z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_1^x & \mathrm{d}_1^y & \mathrm{d}_1^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{d}_1^x & \mathrm{d}_1^y & \mathrm{d}_1^z \\ \mathrm{d}_2^x & \mathrm{d}_2^y & \mathrm{d}_2^z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_2^x & \mathrm{d}_2^y & \mathrm{d}_2^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{d}_2^x & \mathrm{d}_2^y & \mathrm{d}_2^z \\ \mathrm{d}_3^x & \mathrm{d}_3^y & \mathrm{d}_3^z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{d}_3^x & \mathrm{d}_3^y & \mathrm{d}_3^z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{d}_3^x & \mathrm{d}_3^y & \mathrm{d}_3^z \end{pmatrix}^{+} \begin{pmatrix} d_1^{x'} \\ d_1^{y'} \\ d_1^{z'} \\ d_2^{x'} \\ d_2^{y'} \\ d_2^{z'} \\ d_3^{x'} \\ d_3^{y'} \\ d_3^{z'} \end{pmatrix}$$