LOGIC AND TRUTH TABLES
Lesson 4

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INTRODUCTION

- Propositional calculus (or logic) is the study of the logical relationship between objects called propositions and forms the basis of all mathematical reasoning and all automated reasoning.

Definition

A proposition is a statement that is either true or false, but not both (we usually denote a proposition by letters; p, q, r, s, . . .).

The value of a proposition is called its truth value; denoted by T or1 if it is true and F or 0 if it is false.

Opinions, interrogative and imperative sentences are not propositions.

Truth table:

P
0
1

Examples

Example (Propositions)

- Today is Monday.
- The derivative of sin x is cos x.
- Every even number has at least two factors.

Example (Not Propositions)

- C++ is the best language.
- When is the pretest?
- Do your homework.

Logical Connectives

Connectives are used to create a compound proposition from two or more other propositions.

- Negation (denoted ¬ or !)
- And (denoted ^) or Logical Conjunction
- Or (denoted _) or Logical Disjunction
- Exclusive Or (XOR, denoted)
- Implication (denoted !)
- Biconditional; "if and only if" (denoted $)

Negation

- A proposition can be negated. This is also a proposition. We usually denote the negation of a proposition p by $\neg p$.

Example (Negated Propositions)

- Today is not Monday.
- It is not the case that today is Monday.
- It is not the case that the derivative of sin x is cos x.

Truth table:

P	$\neg p$
0	1
1	0

Logical And

- The logical connective And is true only if both of the propositions are true. It is also referred to as a conjunction

Example (Logical Connective: And)

- It is raining and it is warm.
- $(2 + 3 = 5) \wedge (\sqrt{2} < 2)$
- Cat is dead and Cat is not dead.

Truth table:

p	q	p ^q
0	0	0
0	1	0
1	0	0
1	1	1

Logical Or

- The logical disjunction (or logical or) is true if one or both of the propositions are true.

Example (Logical Connective: Or)

- It is raining or it is the second day of lecture.
- $(2 + 2 = 5) \vee (\sqrt{2} < 2)$
- You may have cake or ice cream

Truth table:

p	q	p vq
0	0	0
0	1	1
1	0	1
1	1	1

Exclusive Or

- The exclusive or of two propositions is true when exactly one of its propositions is true and the other one is false.

Example (Logical Connective: Exclusive Or)

- The circuit is either is on or off.
- Let $ab < 0$, then either $a < 0$ or $b < 0$ but not both.
- You may have cake or ice cream, but not both.

Truth table:

p	q	p ʘq
0	0	0
0	1	1
1	0	1
1	1	0

Implications

- Definition

Let p and q be propositions. The implication $p \rightarrow q$ is the proposition that is false when p is true and q is false and true otherwise.

Here, p is called the “hypothesis” (or “antecedent” or “premise”) and q is called the “conclusion” or “consequence”.

Truth table:

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Implications

The implication $p \rightarrow q$ can be equivalently read as

- if p then q
- p implies q
- if p, q
- p only if q
- q if p
- q when p
- q whenever p
- p is a sufficient condition for q (p is sufficient for q)
- q is a necessary condition for p (q is necessary for p)
- q follows from p

Examples

- If you buy your air ticket in advance, it is cheaper.
- If x is a real number, then $x^2 \geq 0$.
- If it rains, the grass gets wet.
- If the sprinklers operate, the grass gets wet.
- If 2 + 2 = 5 then all unicorns are pink.

Exercise

Which of the following implications is true?

- If -1 is a positive number, then $2 + 2 = 5$.

true: the hypothesis is obviously false, thus no matter what the conclusion, the implication holds.

- If -1 is a positive number, then $2 + 2 = 4$.

true: for the same reason as above

- If $\sin x = 0$ then $x = 0$.

false: x can be any multiple of π ; i.e. if we let $x = 2\pi$ then clearly $\sin x = 0$, but $x \neq 0$. The implication "if $\sin x = 0$ then $x = k\pi$ for some integer k" is true.

Bi conditional

The bi conditional $p \leftrightarrow q$

is the proposition that is true when p and q have the same truth values. It is false otherwise

Note that it is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$

Truth table:

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Examples

$p \leftrightarrow q$ can be equivalently read as

- p if and only if q
- p is necessary and sufficient for q
- if p then q, and conversely

- $x > 0$ if and only if x^2 is positive.
- The alarm goes off iff a burglar breaks in.
- You may have pudding if and only if you eat your meat

Exercise

Which of the following biconditionals is true?

- $x^2 + y^2 = 0$ if and only if $x = 0$ and $y = 0$

true: both implications hold.

- $2 + 2 = 4$ if and only if $\sqrt{2} < 2$

true: for the same reason above.

- $x^2 \geq 0$ if and only if $x \geq 0$.

false: The converse holds. That is, "if $x \geq 0$ then $x^2 \geq 0$ ". However, the implication is false; consider $x = -1$. Then the hypothesis is true, $(-1)^2 = 1^2$ 0 but the conclusion fails

Truth Tables

- Truth Tables are used to show the relationship between the truth values of individual propositions and the compound propositions based on them

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Using Logical Equivalences

- Logical equivalences can be used to construct additional logical equivalences.

Example: Show that $(p \wedge q) \rightarrow q$ is a tautology

$$
\begin{array}{rlll}
((p \wedge q) \rightarrow q) & \iff & \neg(p \wedge q) \vee q & \text{Implication Law} \\
 & \iff & (\neg p \vee \neg q) \vee q & \text{De Morgan's Law (1st)} \\
 & \iff & \neg p \vee (\neg q \vee q) & \text{Associative Law} \\
 & \iff & \neg p \vee 1 & \text{Negation Law} \\
 & \iff & 1 & \text{Domination Law}
\end{array}
$$

Using Logical Equivalences

Example (Exercise 17)[1]: Show that

$$\neg(p \leftrightarrow q) \iff (p \leftrightarrow \neg q)$$

Sometimes it helps to start out with the second proposition.
$(p \leftrightarrow \neg q)$

$$\begin{array}{lll}
\iff & (p \rightarrow \neg q) \wedge (\neg q \rightarrow p) & \text{Equivalence Law} \\
\iff & (\neg p \vee \neg q) \wedge (q \vee p) & \text{Implication Law} \\
\iff & \neg\left(\neg\left((\neg p \vee \neg q) \wedge (q \vee p)\right)\right) & \text{Double Negation} \\
\iff & \neg\left(\neg(\neg p \vee \neg q) \vee \neg(q \vee p)\right) & \text{De Morgan's Law} \\
\iff & \neg\left((p \wedge q) \vee (\neg q \wedge \neg p)\right) & \text{De Morgan's Law} \\
\iff & \neg\left((p \wedge q) \vee (\neg p \wedge \neg q)\right) & \text{Commutative Law} \\
\iff & \neg(p \leftrightarrow q) & \text{Equivalence Law}
\end{array}$$

Using Logical Equivalences

Show that

$$\neg(q \rightarrow p) \vee (p \wedge q) \iff q$$

$\neg(q \rightarrow p) \vee (p \wedge q)$

$$\begin{array}{lll}
\iff & (\neg(\neg q \vee p)) \vee (p \wedge q) & \text{Implication Law} \\
\iff & (q \wedge \neg p) \vee (p \wedge q) & \text{De Morgan's \& Double Negation} \\
\iff & (q \wedge \neg p) \vee (q \wedge p) & \text{Commutative Law} \\
\iff & q \wedge (\neg p \vee p) & \text{Distributive Law} \\
\iff & q \wedge 1 & \text{Identity Law} \\
\iff & q & \text{Identity Law}
\end{array}$$

THANK YOU