Chapter 1

Recurrent Problem

Lines in the Plane

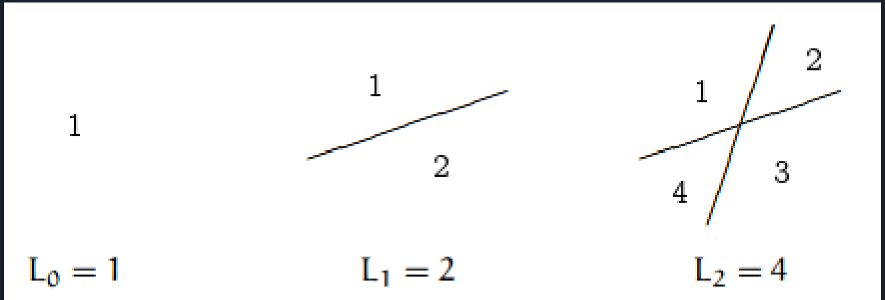
Given: A two-dimensional plane and n straight lines (whose both ends extend infinitely)

Rules: Place the n lines in whatever way you like.

Problem: What is the maximum number L_n of regions defined by the n lines in the plane?

Strategy:
LOOK AT SMALL CASE FIRST

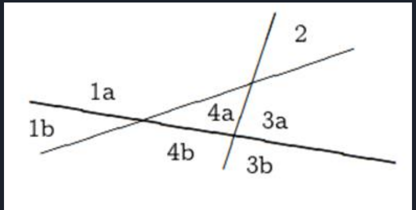
Guess the closed form……………..



You might think it is 2^n??

Let’s see some bigger case.

Determine Recurrence Eqn.



❖ $L_3 = 4 + 3 = 7$

Suffice:

- ❖ The n^{th} line (for $n > 0$) increases the number of regions by k if and only if it splits k of the old regions, and
- ❖ it splits k old regions if and only if it hits the previous lines in $k - 1$ different places.
- ❖ Two lines can intersect in at most one point. Therefore the new line can intersect the $n-1$ old lines in at most $n-1$ different points, and we must have $k \leq n$.

We have established the upper bound

$$L_n \leqslant L_{n-1} + n, \qquad \text{for } n > 0.$$

Determine Recurrence Eqn.

Necessity:

We simply place the n^{th} line in such a way that it's not parallel to any of the others (hence it intersects them all), and such that it doesn't go through any of the existing intersection points (hence it intersects them all in different places). I.e.

$$L_n \geq L_{n-1} + n, \qquad \text{for } n > 0.$$

Hence, the recurrence equation is is,

$$L_0 = 1;$$
$$L_n = L_{n-1} + n, \qquad \text{for } n > 0.$$

Determine the Closed Form

Here,

$$\begin{aligned} L_n &= L_{n-1} + n \\ &= L_{n-2} + (n-1) + n \\ &= L_{n-3} + (n-2) + (n-1) + n \\ &\quad \vdots \\ &= L_0 + 1 + 2 + \cdots + (n-2) + (n-1) + n \\ &= 1 \ + \ S_n\,, \qquad \text{where } S_n = 1 + 2 + 3 + \cdots + (n-1) + n. \end{aligned}$$

Now,

$$\begin{array}{rcccccccccc} S_n &=& 1 &+& 2 &+& 3 &+ \cdots +& (n-1) &+& n \\ +\,S_n &=& n &+& (n-1) &+& (n-2) &+ \cdots +& 2 &+& 1 \\ \hline 2S_n &=& (n+1) &+& (n+1) &+& (n+1) &+ \cdots +& (n+1) &+& (n+1) \end{array}$$

Or,

$$S_n = \frac{n(n+1)}{2}, \qquad \text{for } n \geqslant 0.$$

The Solution

$$L_n = \frac{n(n+1)}{2} + 1, \qquad \text{for } n \geqslant 0.$$

Practice Problem

Prove the closed form of L_n using Induction Method.

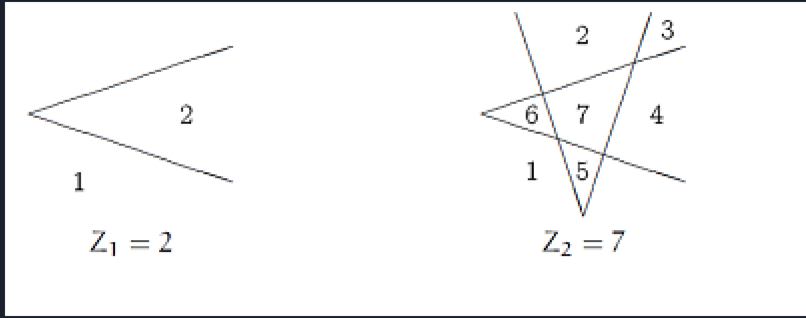
Bent Lines in the Plane

What if we use bent lines instead of straight lines??

Bent Lines in the Plane

Given: Suppose that instead of straight lines we use bent lines, each containing one "zig".

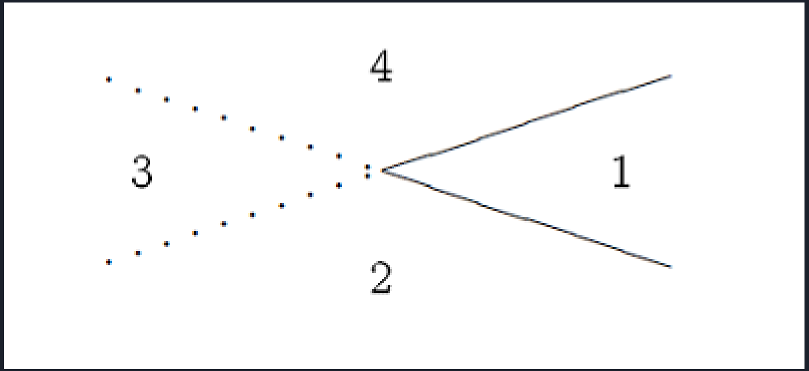
Problem: What is the maximum number Z_n of regions determined by n such bent lines in the plane?



Bent Lines in the Plane

From these small cases, and after a little thought, we realize that a bent line is like two straight lines except that regions merge when the “two" lines don't extend past their intersection point.

Regions 2, 3, and 4, which would be distinct with two lines, become a Single region when there's a bent line; we lose two regions.



So, The Solution

$$
\begin{aligned}
Z_n &= L_{2n} - 2n = 2n(2n+1)/2 + 1 - 2n \\
&= 2n^2 - n + 1, \qquad \text{for } n \geqslant 0.
\end{aligned}
$$

Practice Problem

What is the maximum number regions definable by n zig-zag lines,

each of which consists of two parallel infinite half-lines joined by a straight segment?