Chapter 7

Generating Functions

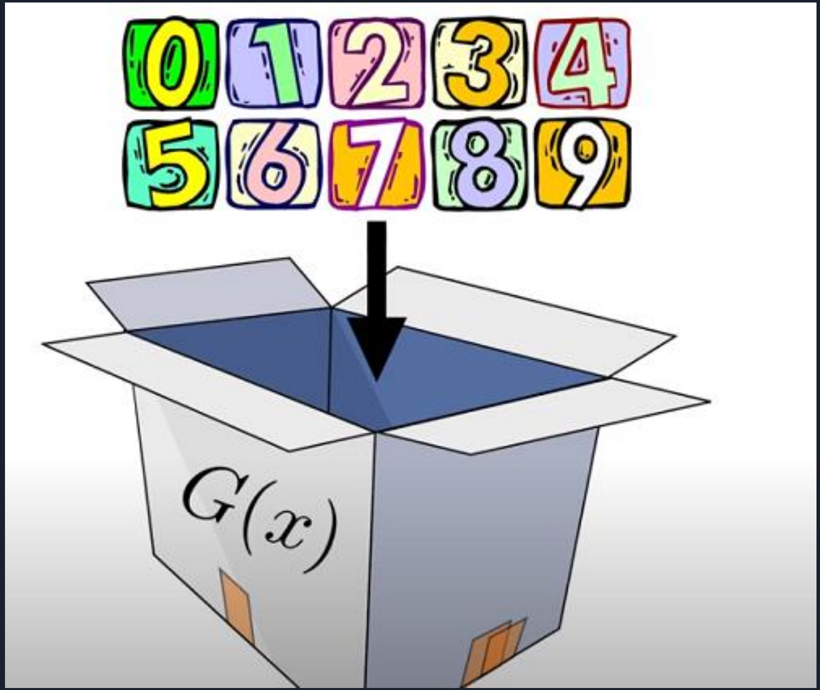
What is Generating Function?

Generating Function:

$$G(x) = \sum_{k=0}^{inf} \mathrm{ak}\, x^{k}$$

What is Generating Function?

Generating Function: A companion trick to record bunch of numbers often probabilities associated with some integers and counts.



Generating Functions

Let’s imagine a dice , it’s a good looking dice but not perfect having the probabilit

Of a number.

Probability of number 1 is P1

Probability of number 1 is P2

Probability of number 1 is P3

Probability of number 1 is P4

Probability of number 1 is P5

Probability of number 1 is P6

Generating Functions

Can you create the function which can contain all the information of these Probabilities?

So that we can calculate all the probability values.

Now, we need a way to isolate these values. The simplest way to do that by using a polynomial.

So,

$$G(x) = p_1x + p_2x^2 + p_3x^3 + p_4x^4 + p_5x^5 + p_6x^6$$

Generating Functions

So,

$$G(x) = p_1x + p_2x^2 + p_3x^3 + p_4x^4 + p_5x^5 + p_6x^6$$

How can we get P1 from this function? Let’s take first derivative

$$G'(x) = p_1 + 2p_2x + 3p_3x^2 + 4p_4x^3 + 5p_5x^4 + 6p_6x^5$$

$$G'(0) = p_1$$

Generating Functions

So,

$$G(x) = p_1x + p_2x^2 + p_3x^3 + p_4x^4 + p_5x^5 + p_6x^6$$

How can we get P1 from this function? Let's take first derivative

$$G'(x) = p_1 + 2p_2x + 3p_3x^2 + 4p_4x^3 + 5p_5x^4 + 6p_6x^5$$

How about P2? We will take another derivative.

$$G''(x) = 2p_2 + 3 \cdot 2p_3x + 4 \cdot 3p_4x^2 + 5 \cdot 4p_5x^3 + 6 \cdot 5p_6x^4$$

$$G''(0) = 2p_2$$

$$\frac{1}{2}G''(0) = p_2$$

Generating Functions

What about Pk?

$$\boxed{p_k \stackrel{?}{=} \frac{1}{k!}\frac{d^k G(x)}{dx^k}\bigg|_{x=0}}$$

Generating Functions

Now Le’ts think again.

If x=1, what will happen to first derivative/

$$G'(x) = p_1 + 2p_2x + 3p_3x^2 + 4p_4x^3 + 5p_5x^4 + 6p_6x^5$$

$$G'(1) = p_1 + 2p_2 + 3p_3 + 4p_4 + 5p_5 + 6p_6$$

Generating Functions

Another cool thing happens if we multiply our generating functions.

G(x) * G(x)

$$G^2(x) = \underbrace{(p_1 x + p_2 x^2 + \dots)}_{G(x)}\underbrace{(p_1 x + p_2 x^2 + \dots)}_{G(x)}$$

Generating Functions

The lowest power of x will be 2 and the highest will be 12

So, do you see any patterns?

Let’s imagine the number of dice is 2.

Now, think the probability having 2.

$$G^2(x) = (\underline{p_1 x} + p_2 x^2 + \ldots)(\underline{p_1 x} + p_2 x^2 + \ldots)$$

$$\underline{p_1 p_1} x^2$$

Probability of having 1 and 1

$$G^2(x) = (p_1 x + p_2 x^2 + \ldots)(p_1 x + p_2 x^2 + \ldots)$$

$$\begin{aligned} & p_1 p_1 x^2 \\ & + (p_1 p_2 + p_2 p_1) x^3 \\ & + (p_1 p_3 + p_2 p_2 + p_3 p_1) x^4 \\ & + \ldots \end{aligned}$$

Generating Functions

Is this a coincidence?

Let's see another case. To have 4 in total, you should have

1, 3

2, 2

3, 1

So total num of ways of having 4 = 3

$$G^2(x) = (p_1x + p_2x^2 + \ldots)(p_1x + p_2x^2 + \ldots)$$

$$\begin{aligned} &p_1p_1x^2 \\ &+(p_1p_2 + p_2p_1)x^3 \\ &+(\underline{p_1p_3 + p_2p_2 + p_3p_1})x^4 \\ &+\ldots \end{aligned}$$

Cases to have a total of four

Generating Functions

If we multiply G(x) with another G(x), then the resulting function will be another generating function.

So from Generating Functions, we can say

- Counting problem is so easy
- To study random graphs without arbitrary distribution

So, Let’s go for another example

Generating Functions

Now , we you want to choose 10 candies from this bunch of red, blu

And green candies.

Somehow, you want to have

Even number of red candies

$$(1 + x^2 + x^4 + x^6 + x^{10})$$

More than 6 blue candies

$$(x^7 + x^8 + x^9 + x^{10})$$

Less than 3 green candies

$$(1 + x + x^2)$$

Generating Functions

Now from this Generated function

You will find the coefficients of X^10

The number of coefficients will be

The ans of choosing 10 candies from

These bunch of candies.

$$(1 + x^2 + x^4 + x^6 + x^{10}) \times (x^7 + x^8 + x^9 + x^{10}) \times (1 + x + x^2)$$

Problem no 1

A girl can choose two items from the basket containing 1 apple, 1 pear, 1 orange, 1 banana and 1 papaya. How many ways she can do this?

Let’s see the solution:

(0 apple + 1 apple) (0 pear + 1 pear) (0 orange + 1 orange) and like so on……………

$(x^0 + x^1) (x^0 + x^1) (x^0 + x^1) (x^0 + x^1) (x^0 + x^1)$

$= (1 + x)^5$

$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

So, she wants to choose 2 items. Now look at the x^2 coefficient which is 10.

The ans is : 10.

Modifying Problem no 1

Suppose there are 2 apples (considered to be identical) instead of 1. Rest of all the items are same. Now how can the two items be chosen?

Solution:

(0 apple + 1 apple + 2 apple)   (0 pear + 1 pear) (0 orange + 1 orange) and like so on................

$= (x^0 + x^1 + x^2) (1 + x)^4$

$=(1+x+x^2) (1+x)^4$

$= (1+x+x^2) (1 + 4x + 6x^2 + 4x^3 + x^4)$

$= (1 + 4x + 6x^2 + 4x^3 + x^4 + x + 4x^2 + 6x^3 + 4x^4 + x^5 + x^2 + 4x^3 + 6x^4 + 4x^5 + x^6$

$= (6 + 4 + 1) x^2$

$=11 x^2$

Ans: 11 ways.

Problem no 2

Suppose you go to a bakery just before it closes. Remaining there are 3 cheese, 2 cherry and 4 raspberry pastries left. How many ways can you choose exactly 7 pastries?

Solution:

$(x^0 + x^1 + x^2 + x^3)(x^0 + x^1 + x^2)(x^0 + x^1 + x^2 + x^3 + x^4)$

Cheese cherry raspberry

$= 1 + 3x + 6x^2 + 9x^3 + 11x^4 + 11x^5 + 9x^6 + 6x^7 + 3x^8 + x^9$

= 6 ways.

So , if we want to know the ways to choose of 5 pastries or 4 pastries. You can give ans from this simple one equation. Isn't it??

Modifying Problem no 2

What if the raspberry pastries come in box of 2 (that means in even numbers)? Other amounts of problem 2 remains same? Now how many ways to choose 7 pastries?

Solution:

$(x^0 + x^1 + x^2 + x^3)\ (x^0 + x^1 + x^2)\ (x^0 + x^2 + x^4)$

Cheese cherry raspberry

$= 1 + 2x + 4x^2 + 5x^3 + 6x^4 + 6x^5 + 5x^6 + 4x^7 + 2x^8 + x^9$

= 4 ways.

Thank You!!