Chapter 1

Recurrent Problem

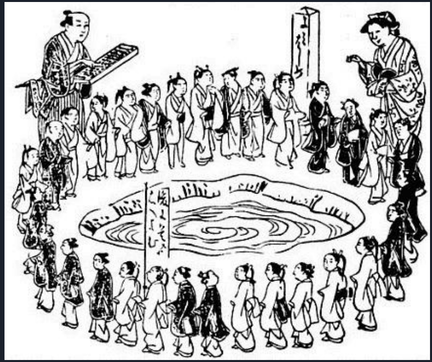
What is Recurrence?

The solution to a problem can be constructed from the solutions to smaller instances of the same problem.

The Josephus Problem

<u>Background:</u>

<u>Flavius Josephus</u> is a Jewish historian living in the 1st century. According to his account, he and his 40 comrade soldiers were trapped in a cave, surrounded by Romans. They chose suicide over capture and decided that they would form a circle and start killing one by skipping every two others. By luck, or maybe by the hand of God, Josephus and another man remained the last and gave up to the Romans.



Problem Definition

Given:

n people numbered 1 to *n* around a circle

Rules:

Eliminate every second remaining person until only one survives.

Problem:

Determine the survivor’s number.

Strategy

- Introduce appropriate notation:

J(n) denotes the survivor's assigned number.

- Look at small cases first:

n	$J(n)$
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5

Insights from Small Cases

- We may start with:
  An even number $2n$ of people
  An odd number $2n + 1$ of people
- First round eliminates all people assigned with even numbers.
- $2n$ is the last person eliminated in the first round.

A General Solution Pattern

Consider the cases of starting number of people separately:

- <u>An even number 2*n* of people:</u>

After 1st round, n people remain: 1, 3, 5, …., 2n-3, 2n-1

How would the survivor of these n people relate to J(n)?

$J(2n) = 2J(n)-1$

- <u>An odd number 2n + 1 of people</u>

After 1st round, n+1 people remain: 1, 3, 5, …, 2n-3, 2n-1, 2n+1

In order to relate to J(n), we need to eliminate one more person.

After the first elimination in 2nd round, *n* people remain:

3, 5, ... , 2n-3, 2n-1, 2n + 1

How would the survivor of these n people relate to J(n)?

$J(2n + 1) = 2J(n) + 1$

Recurrence Relation for J(n)

We end up with the following recurrence relation, split into the cases of the starting number of people being even or odd:

$J(1) = 1$

$J(2n) = 2J(n) - 1$ for $n > 0$

$J(2n + 1) = 2J(n) + 1$ for $n > 0$

That is, the solution to the **original problem** can be constructed from the solution to a **half-size subproblem**.

Question: How many times of unfolding are required for calculating J(100)? >> **6 times**: J(50), J(25), J(12), J(6), J(3), J(1)

However, the above relation only gives us indirect information. Instead, we need a closed-form solution to the above recurrence relation.

Guessing Closed Form for J(n)

n	$J(n)$
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15
16	1

Pattern of the Closed Form for J(n)

n	$J(n)$
1	1
2	1
3	3
4	1
5	3
6	5
7	7
8	1
9	3
10	5
11	7
12	9
13	11
14	13
15	15
16	1

A Hypothesized Closed Form for J(n)

It seems that we can split n into $(log_2 n + 1)$ groups, such that:

- A a group starting with the number 2^g contains 2^g numbers.
- Each starts with a *power of 2* (i.e., $2^0, 2^1, 2^2, 2^3, \ldots$).
- $J(n)$ in the *beginning* of the group is always 1.
- $J(n)$ increases by 2 as we move towards the *end* of the group.

How would you formulate this?

Given any number in the interval of $[1, n]$, we can locate it by:

- identifying *which group* it belongs to;
- measuring *how far* it is from the beginning of the group.

For a given number that belongs to the group starting with 2^g and is h away from the beginning of its belonging group, we have:

$$\boxed{J(2^g + h) = 2h + 1} \qquad \text{for } g \geq 0 \text{ and } 0 \leq h < 2^g$$

Practice Problem

Prove the closed form of $J(n)$ using Induction Method.