Chapter 6

Special Numbers

Stirling Numbers

- Stirling Number First Kind
- Stirling Number Second Kind

Stirling Number Second Kind

Table 258 Stirling's triangle for subsets.

n	$\left\{ {n \atop 0} \right\}$	$\left\{ {n \atop 1} \right\}$	$\left\{ {n \atop 2} \right\}$	$\left\{ {n \atop 3} \right\}$	$\left\{ {n \atop 4} \right\}$	$\left\{ {n \atop 5} \right\}$	$\left\{ {n \atop 6} \right\}$	$\left\{ {n \atop 7} \right\}$	$\left\{ {n \atop 8} \right\}$	$\left\{ {n \atop 9} \right\}$
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7 +	6 x3	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

Stirling Number Second Kind

The symbol $\left\{ {n \atop k} \right\}$

stands for the number of ways to partition a set of n things into k nonempty subsets.
For example, there are seven ways to split a four-element set into two parts:

$$\begin{array}{llll} \{1,2,3\} \cup \{4\}, & \{1,2,4\} \cup \{3\}, & \{1,3,4\} \cup \{2\}, & \{2,3,4\} \cup \{1\}, \\ \{1,2\} \cup \{3,4\}, & \{1,3\} \cup \{2,4\}, & \{1,4\} \cup \{2,3\}; & \end{array}$$

Thus $\left\{ {4 \atop 2} \right\} = 7.$

$\left\{ {n \atop k} \right\}$ Can be read as ‘n subset k’ .

The case $k = 0$ is a bit tricky. Things work out best if we agree that there's just one way to partition an empty set into zero nonempty parts; hence ${0 \brace 0} = 1$. But a nonempty set needs at least one part, so ${n \brace 0} = 0$ for $n > 0$.

Let's look at small k. There's just one way to put n elements into a single nonempty set; hence ${n \brace 1} = 1$, for all $n > 0$. On the other hand ${0 \brace 1} = 0$, because a 0-element set is empty.

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Stirling Number Second Kind

What happens when $k = 2$? Certainly $\left\{ {0 \atop 2} \right\} = 0$. If a set of $n > 0$ objects is divided into two nonempty parts, one of those parts contains the last object and some subset of the first $n-1$ objects. There are 2^{n-1} ways to choose the latter subset, since each of the first $n-1$ objects is either in it or out of it; but we mustn't put all of those objects in it, because we want to end up with two nonempty parts. Therefore we subtract 1:

So,

$$\left\{ {n \atop 2} \right\} = 2^{n-1} - 1, \qquad \text{integer } n > 0.$$

Stirling Number Second Kind

Now we can compute $\left\{ {n \atop k} \right\}$ for all.

Given a set of $n > 0$ objects to be partitioned into k nonempty parts, we either put the last object into a class by itself in $\left\{ {n-1 \atop k-1} \right\}$

ways), or we put it together with some nonempty subset of the first n-1 objects. There are $k\left\{ {n-1 \atop k} \right\}$

possibilities in the latter case, because each of the $\left\{ {n-1 \atop k} \right\}$

ways to distribute the first n - 1 objects into k nonempty parts gives k subsets that the nth object can join.

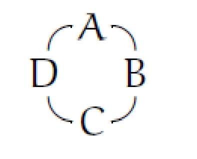
Hence,

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}, \qquad \text{integer } n > 0.$$

Stirling Number First Kind

$\left[{n \atop k}\right]$ counts the number of ways to arrange n objects into k cycles instead of subsets. We verbalize $\left[{n \atop k}\right]$ by saying “n cycle k."

The cycle



can be written more compactly as '[A, B, C,D]', with the understanding that

$$[A, B, C, D] = [B, C, D, A] = [C, D, A, B] = [D, A, B, C];$$

a cycle “wraps around" because its end is joined to its beginning. On the other hand, the cycle [A;,B, C,D] is not the same as [A,B,D,C] or [D, C, B,A].

Stirling Number First Kind

Table 259 Stirling's triangle for cycles.

n	$\left[{n \atop 0}\right]$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$	$\left[{n \atop 6}\right]$	$\left[{n \atop 7}\right]$	$\left[{n \atop 8}\right]$	$\left[{n \atop 9}\right]$
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

Stirling Number First Kind

There are eleven different ways to make two cycles from four elements:

$$\begin{array}{llll} [1,2,3]\,[4]\,, & [1,2,4]\,[3]\,, & [1,3,4]\,[2]\,, & [2,3,4]\,[1]\,, \\ [1,3,2]\,[4]\,, & [1,4,2]\,[3]\,, & [1,4,3]\,[2]\,, & [2,4,3]\,[1]\,, \\ [1,2]\,[3,4]\,, & [1,3]\,[2,4]\,, & [1,4]\,[2,3]\,; & \end{array} \tag{6.4}$$

hence $\left[{4 \atop 2}\right] = 11$.

Stirling Number First Kind

A singleton cycle (that is, a cycle with only one element) is essentially the same as a singleton set (a set with only one element).

Similarly, a 2-cycle is like a 2-set, because we have [A, B] = [B,A] just as [A, B} = {B,A}. But

there are two different 3-cycles, [A, B,C] and [A, C, B].

Notice, for example,

that the eleven cycle pairs in (6.4) can be obtained from the seven set pairs in (6.1) by making two cycles from each of the 3-element sets.

In general, n!\n = (n - 1)! different n-cycles can be made from any n-element set, whenever n > 0.

Therefore we have,

$$\left[{n \atop 1} \right] = (n-1)!\,, \qquad \text{integer } n > 0.$$

Stirling Number First Kind

This is much larger than the value $\left\{ {n \atop 1} \right\} = 1$ for Stirling subset numbers.

In fact, it is easy to see that the cycle numbers must be at least as large as the subset numbers,

$$\left[{n \atop k} \right] \geqslant \left\{ {n \atop k} \right\}, \qquad \text{integers } n, k \geqslant 0,$$

because every partition into non empty subsets leads to at least one arrangement of cycles. Equality holds when all the cycles are necessarily singletons or doubletons, because cycles are equivalent to subsets in such cases. This happens

$$\left[{n \atop n} \right] = \left\{ {n \atop n} \right\}; \qquad \left[{n \atop n-1} \right] = \left\{ {n \atop n-1} \right\}.$$

when k = n and when k = n - 1; hence

$$\left[{n \atop n} \right] = \left\{ {n \atop n} \right\} = 1; \qquad \left[{n \atop n-1} \right] = \left\{ {n \atop n-1} \right\} = \binom{n}{2}.$$

Stirling Number First Kind

We can derive a recurrence for $\begin{bmatrix} n \\ k \end{bmatrix}$ by modifying the argument we used for $\begin{Bmatrix} n \\ k \end{Bmatrix}$

Every arrangement of n objects in k cycles either puts the last object into a cycle by itself $\left(\text{in } \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \text{ ways}\right)$

or inserts that object into one of the $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ cycle arrangements of the first n - 1 objects. In the latter

case, th eads to

$$[A, B, C, D], \qquad [A, B, D, C], \qquad \text{or} \qquad [A, D, B, C]$$

when we insert a new element D, and there are no other possibilities. Summing over all j gives a total of n-1 way ence is therefore

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \qquad \text{integer } n > 0.$$

Recurrences

Recurrences:

$$\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} .$$

$$\left[{n \atop k} \right] = (n-1)\left[{n-1 \atop k} \right] + \left[{n-1 \atop k-1} \right] .$$

Special values:

$$\left\{ {n \atop 0} \right\} = \left[{n \atop 0} \right] = [n=0] .$$

$$\left\{ {n \atop 1} \right\} = [n>0] ; \qquad \left[{n \atop 1} \right] = (n-1)!\,[n>0] .$$

$$\left\{ {n \atop 2} \right\} = (2^{n-1}-1)[n>0] ; \qquad \left[{n \atop 2} \right] = (n-1)!\,H_{n-1}\,[n>0] .$$

$$\left\{ {n \atop n-1} \right\} = \left[{n \atop n-1} \right] = \binom{n}{2} .$$

$$\left\{ {n \atop n} \right\} = \left[{n \atop n} \right] = \binom{n}{n} = 1 .$$

$$\left\{ {n \atop k} \right\} = \left[{n \atop k} \right] = \binom{n}{k} = 0 , \qquad \text{if } k > n.$$

Eulerian Numbers

Another triangle of values pops up now and again, this one due to Euler [104, §13; 110, page 485], and we denote its elements by $\left\langle {n \atop k} \right\rangle$. The angle brackets in this case suggest "less than" and "greater than" signs; $\left\langle {n \atop k} \right\rangle$ is the number of permutations $\pi_1 \pi_2 \ldots \pi_n$ of $\{1, 2, \ldots, n\}$ that have k *ascents*, namely, k places where $\pi_j < \pi_{j+1}$. (Caution: This notation is less standard than our notations $\left[{n \atop k} \right]$, $\left\{ {n \atop k} \right\}$ for Stirling numbers. But we'll see that it makes good sense.)

For example, eleven permutations of $\{1, 2, 3, 4\}$ have two ascents:

$$\begin{array}{llllll} 1324\,, & 1423\,, & 2314\,, & 2413\,, & 3412\,; & \\ 1243\,, & 1342\,, & 2341\,; & 2134\,, & 3124\,, & 4123\,. \end{array}$$

(The first row lists the permutations with $\pi_1 < \pi_2 > \pi_3 < \pi_4$; the second row lists those with $\pi_1 < \pi_2 < \pi_3 > \pi_4$ and $\pi_1 > \pi_2 < \pi_3 < \pi_4$.) Hence $\left\langle {4 \atop 2} \right\rangle = 11$.

Eulerian Numbers

Table 268 Euler's triangle.

n	$\left\langle {n \atop 0} \right\rangle$	$\left\langle {n \atop 1} \right\rangle$	$\left\langle {n \atop 2} \right\rangle$	$\left\langle {n \atop 3} \right\rangle$	$\left\langle {n \atop 4} \right\rangle$	$\left\langle {n \atop 5} \right\rangle$	$\left\langle {n \atop 6} \right\rangle$	$\left\langle {n \atop 7} \right\rangle$	$\left\langle {n \atop 8} \right\rangle$	$\left\langle {n \atop 9} \right\rangle$
0	1									
1	1	0								
2	1	1	0							
3	1	4	1	0						
4	1	11	11	1	0					
5	1	26	66	26	1	0				
6	1	57	302	302	57	1	0			
7	1	120	1191	2416	1191	120	1	0		
8	1	247	4293	15619	15619	4293	247	1	0	
9	1	502	14608	88234	156190	88234	14608	502	1	0

Eulerian Numbers

Table 268 lists the smallest Eulerian numbers; notice that the trademark sequence is 1, 11, 11, 1 this time. There can be at most $n-1$ ascents, when $n > 0$, so we have $\left\langle {n \atop n} \right\rangle = [n=0]$ on the diagonal of the triangle.

Euler's triangle, like Pascal's, is symmetric between left and right. But in this case the symmetry law is slightly different:

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-1-k} \right\rangle, \qquad \text{integer } n > 0; \tag{6.34}$$

The permutation $\pi_1\pi_2\ldots\pi_n$ has $n-1-k$ ascents if and only if its "reflection" $\pi_n\ldots\pi_2\pi_1$ has k ascents.

Let's try to find a recurrence for $\left\langle {n \atop k} \right\rangle$. Each permutation $\rho = \rho_1\ldots\rho_{n-1}$ of $\{1,\ldots,n-1\}$ leads to n permutations of $\{1,2,\ldots,n\}$ if we insert the new element n in all possible ways. Suppose we put n in position j, obtaining the permutation $\pi = \rho_1\ldots\rho_{j-1}\,n\,\rho_j\ldots\rho_{n-1}$. The number of ascents in π is the same as the number in ρ, if $j=1$ or if $\rho_{j-1} < \rho_j$; it's one greater than the number in ρ, if $\rho_{j-1} > \rho_j$ or if $j = n$. Therefore π has k ascents in a total of $(k+1)\left\langle {n-1 \atop k} \right\rangle$ ways from permutations ρ that have k ascents, plus a total of $\big((n-2)-(k-1)+1\big)\left\langle {n-1 \atop k-1} \right\rangle$ ways from permutations ρ that have $k-1$ ascents. The desired recurrence is

$$\left\langle {n \atop k} \right\rangle = (k+1)\left\langle {n-1 \atop k} \right\rangle + (n-k)\left\langle {n-1 \atop k-1} \right\rangle, \qquad \text{integer } n > 0. \tag{6.35}$$

Harmonic Numbers

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}, \quad \text{integer } n \geqslant 0.$$

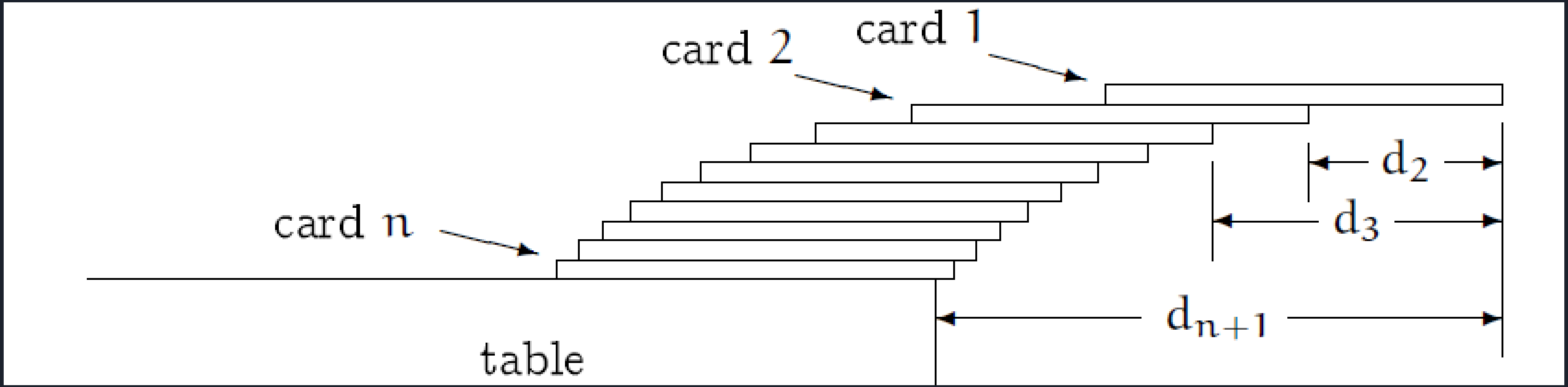
These numbers appear so often in the analysis of algorithms that computer scientists need a special notation for them. We use Hn, the 'H' standing for harmonic," since a tone of wavelength 1/n is called the n th harmonic of a tone whose wavelength is 1.

n	0	1	2	3	4	5	6	7	8	9	10
H_n	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$	$\frac{7129}{2520}$	$\frac{7381}{2520}$

Harmonic Numbers

It can be said that, H_n is never an integer when $n>1$.

Let's go for a card trick.



Harmonic Number

Given

n cards

A table

The problem is to find out the largest possible overhang by stacking the cards up over the table's edge, subject to the laws of gravity

Requirement:

- the edges of the cards to be parallel to the edge of the table
- Each card is 2 units long

Harmonic Number

With one card,

we get maximum overhang when its center of gravity is just above the edge of the table. The center of gravity is in the middle of the card, so we can create half a cardlength, or 1 unit, of overhang.

With two cards,

it's not hard to convince ourselves that we get maximum overhang when the center of gravity of the top card is just above the edge of the second card, and the center of gravity of both cards combined is just above the edge of the table. The joint center of gravity of two cards will be in the middle of their common part, so we are able to achieve an additional half unit of overhang.

So if we generalize, it defines that

The center of gravity of the top k cards lies just above the edge of the k+1st card

(which supports those top k). The table plays the role of the n+1st card.

Harmonic Numbers

Let's go for algebraically

Let,

dk be the distance from the extreme edge of the top card to the corresponding edge of the **kth** card from the top.

Then d1 = 0, and

we want to make dk+1 the center of gravity of the first k cards:

$$d_{k+1} = \frac{(d_1 + 1) + (d_2 + 1) + \cdots + (d_k + 1)}{k}, \quad \text{for } 1 \leqslant k \leqslant n.$$

Harmonic Number

Now multiply this equation by k

$$k d_{k+1} = k + d_1 + \cdots + d_{k-1} + d_k , \qquad k \geqslant 0;$$

Then,

$$(k-1) d_k = k - 1 + d_1 + \cdots + d_{k-1} , \qquad k \geqslant 1.$$

Now subtracting these two equations

$$k d_{k+1} - (k-1) d_k = 1 + d_k , \qquad k \geqslant 1;$$

Hence,

$$d_{k+1} = d_k + 1/k.$$

Harmonic Number

Now, we can say that

The second card will be offset half a unit past the third, which is a third of a unit past the fourth, and so on. The general formula

$$d_{k+1} = H_k$$

If we set k=n then

$$d_{n+1} = H_n$$

The total overhang when n cards are stacked.

Worm on the rubber band

An amazing problem to show harmonic number in another guise

Explanation: A slow but persistent worm, W, starts at one end of a meter-long rubber band and crawls **one centimeter per minute** toward the other end.

At the **end of each minute**, an equally persistent keeper of the band, **K**, whose sole purpose in life is **to frustrate W**, stretches it **one meter**.

Thus

after one minute of crawling, W is 1 centimeter from the start and 99 from the nish; then K stretches it one meter. During the stretching operation W maintains his relative position, 1% from the start and 99% from **the finish;**

so W is now 2 cm from the starting point and 198 cm from the goal. After W crawls for another minute the score is 3 cm traveled and 197 to go;

Worm on the rubber band

Let's write down some formulas.

When K stretches the rubber band, the fraction of it that W has crawled stays the same.

Thus he crawls 1/100th of it the first minute,

1/200th the second,

1/300th the third, and so on.

After **n** minutes the fraction of the band that he's crawled is

$$\frac{1}{100}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) = \frac{H_n}{100}.$$

Practice problem

Definition: A slow but persistent Superworm,SW, starts at one end of a meter-long rubber band and crawls **50 centimeter per minute** toward the other end.

At the **end of each minute**, an equally persistent keeper of the band, **K**, whose sole purpose in life is **to frustrate SW**, stretches it **one meter**.

Now find out where will be the SW on the band after n minutes?

This problem is called SuperWorm on the rubber band.