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Dipartimento  
di Ingegneria Gestionale,  
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# Progettazione di un MPC per il controllo di un Continuous-flow Stirred-Tank Reactor

Progetto del corso Controllo Avanzato Multivariabile  
a.a. 2022-2023

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# Outline

- **Introduzione**
- **Modellazione matematica**
- **Formulazione del problema di controllo MPC**
- **Implementazione e simulazioni**



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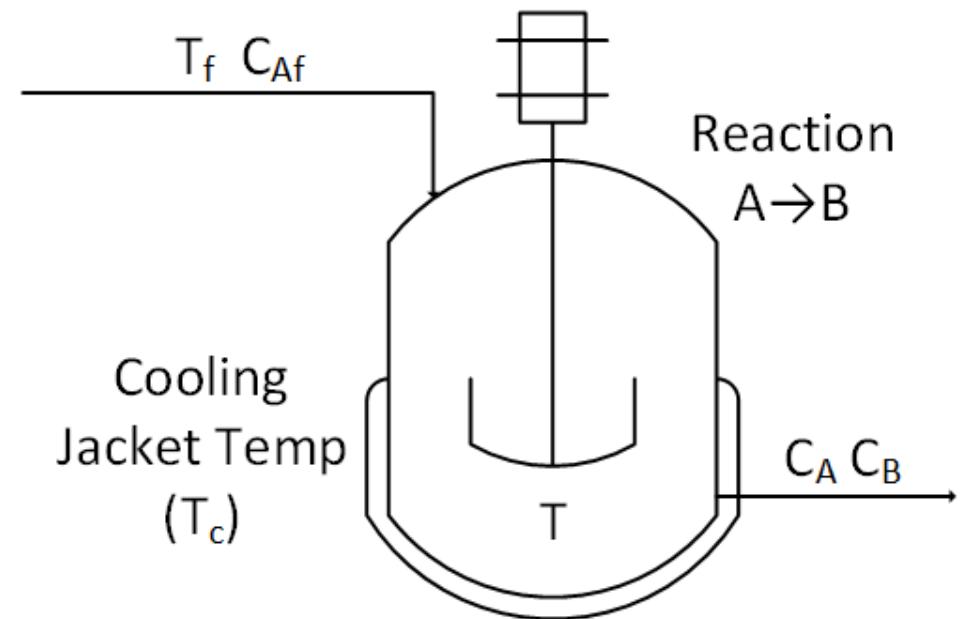


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# Descrizione del sistema

- Un reattore chimico CSTR (Continuous-flow Stirred-Tank Reactor), noto anche come "reattore continuo a serbatoio agitato", è un tipo ideale di reattore continuo che consiste in un serbatoio alimentato da un flusso costante di materia.
- Il reattore è dotato di un sistema di agitazione, grazie al quale si garantisce di avere una composizione uniforme in tutto il serbatoio.



# Descrizione del problema (1)

- Dato un CSTR in cui sta avvenendo una reazione  $A \rightarrow B$ , si vuole controllare la concentrazione del reagente A ( $C_A$ ) usando la tecnica Model Predictive Control (MPC).
- $C_A$  può essere controllata tramite la temperatura del refrigerante ( $T_r$ ).
- Bisogna tenere in considerazione che  $T_r$  deve essere compresa tra 280 [K] e 310 [K].
- $C_A$  deve essere compresa tra 0,38 [mol/l] e 0,954 [mol/l].



# Descrizione del problema (2)

- La condizione iniziale è:
  - $C_A = 0.853 \text{ [mol/l]}$ , concentrazione del reagente A
  - $T = 296.986 \text{ [K]}$ , la temperatura nel reattore
  - $T_C = 292 \text{ [K]}$ , la temperatura del liquido di raffreddamento nel rivestimento.
- Si vuole controllare il sistema all'equilibrio
  - $\bar{C}_A = 0.5054 \text{ [mol/l]}$
  - $\bar{T} = 315.5491 \text{ [K]}$
  - $\bar{T}_C = 308 \text{ [K]}$



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# Modello non lineare

La dinamica del sistema è descritta dalle seguenti equazioni differenziali:

$$\begin{cases} \dot{C}_A(t) = \frac{q_0}{V} \cdot (C_{Af} - C_A(t)) - k_0 \cdot \exp\left(-\frac{E}{R \cdot T(t)}\right) \cdot C_A(t) \\ \dot{T}(t) = \frac{q_0}{V} \cdot (T_f - T(t)) + \frac{(-\Delta H_r) \cdot k_0}{\rho \cdot C_p} \cdot \exp\left(-\frac{E}{R \cdot T(t)}\right) \cdot C_A(t) + \frac{U A}{V \cdot \rho \cdot C_p} \cdot (T_c(t) - T(t)) \\ \dot{T}_c(t) = \frac{T_r(t) - T_c(t)}{\tau_c} \end{cases}$$

Le equazioni sono non lineari.

Per utilizzare la tecnica di controllo *MPC lineare*, è necessario linearizzare il sistema nel punto di equilibrio in cui si vuole controllare il sistema.



# Equilibrio

Si vuole controllare il sistema nel punto di equilibrio:

$$\bar{x} = \begin{cases} \bar{C}_A & = 0.5054 \\ \bar{T} & = 315.5491 \\ \bar{T}_c & = 308 \end{cases}$$

L'ingresso di equilibrio  $\bar{T}_r$  si ottiene dalla terza equazione:

$$0 = \frac{\bar{T}_r - \bar{T}_c}{\tau_c} = \frac{\bar{T}_r - 308}{\tau_c}$$

Da cui:

$$\bar{u} = \bar{T}_r = 308$$



# Sistema tangente (1)

$$\begin{cases} \dot{C}_A(t) &= \frac{q_0}{V} \cdot (C_{Af} - C_A(t)) - k_0 \cdot \exp\left(-\frac{E}{R \cdot T(t)}\right) \cdot C_A(t) \\ \dot{T}(t) &= \frac{q_0}{V} \cdot (T_f - T(t)) + \frac{(-\Delta H_r) \cdot k_0}{\rho \cdot C_p} \cdot \exp\left(-\frac{E}{R \cdot T(t)}\right) \cdot C_A(t) + \frac{U A}{V \cdot \rho \cdot C_p} \cdot (T_c(t) - T(t)) \\ \dot{T}_c(t) &= \frac{T_r(t) - T_c(t)}{\tau_c} \\ y(t) &= C_A(t) \end{cases}$$

$$\begin{cases} \delta \dot{x}(t) &\simeq f_x(\bar{x}, \bar{u}) \delta x(t) + f_u(\bar{x}, \bar{u}) \delta u(t) \\ \delta y(t) &\simeq g_x(\bar{x}, \bar{u}) \delta x(t) + g_u(\bar{x}, \bar{u}) \delta u(t) \end{cases}$$

$$f_x(\bar{x}, \bar{u}) = A = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} & \frac{\delta f_1}{\delta x_3} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} & \frac{\delta f_2}{\delta x_3} \\ \frac{\delta f_3}{\delta x_1} & \frac{\delta f_3}{\delta x_2} & \frac{\delta f_3}{\delta x_3} \end{bmatrix}_{\substack{x=\bar{x} \\ u=\bar{u}}}$$

$$f_u(\bar{x}, \bar{u}) = B = \begin{bmatrix} \frac{\delta f_1}{\delta u} \\ \frac{\delta f_2}{\delta u} \\ \frac{\delta f_3}{\delta u} \end{bmatrix}_{\substack{x=\bar{x} \\ u=\bar{u}}} \quad g_x(\bar{x}, \bar{u}) = C = \left[ \frac{\delta g}{\delta x_1} \frac{\delta g}{\delta x_2} \frac{\delta g}{\delta x_3} \right]_{\substack{x=\bar{x} \\ u=\bar{u}}} \quad g_u(\bar{x}, \bar{u}) = D = \left. \frac{\delta g}{\delta u} \right|_{\substack{x=\bar{x} \\ u=\bar{u}}}$$



# Sistema tangente (2)

$$A = \begin{bmatrix} -k_0 \cdot \exp\left(\frac{-E}{R \cdot \bar{T}}\right) - \frac{q_0}{V} & -\frac{\bar{C}_A \cdot E \cdot k_0}{R} \exp\left(\frac{-E}{R \cdot \bar{T}}\right) \cdot \frac{1}{\bar{T}^2} & 0 \\ \frac{-\Delta H_r \cdot k_0}{\rho C_p} \cdot \exp\left(\frac{-E}{R \cdot \bar{T}}\right) & -\frac{q_0}{V} + \left( \frac{\bar{C}_A \cdot (-\Delta H_r) \cdot k_0}{\rho \cdot C_p} \exp\left(-\frac{E}{R \cdot \bar{T}}\right) \cdot \frac{E}{R \cdot \bar{T}^2} \right) - \frac{U A}{V \cdot \rho \cdot C_p} & \frac{U A}{V \cdot \rho \cdot C_p} \\ 0 & 0 & -\frac{1}{\tau_c} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau_c} \end{bmatrix} \quad C = [1 \ 0 \ 0] \quad D = 0$$

- Si discretizza il sistema con  $T_s = 1$  [s]
- Autovalori del sistema discretizzato:  $\lambda_1 = 0.8257, \lambda_2 = 0.4514, \lambda_3 = 0.5134$



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# Problema di ottimizzazione

$$\min_{\mathbf{u}} \quad J(x(k), u(.)) = \sum_{j=0} \|x(j)\|_Q^2 + \|u(j)\|_R^2 + \|x(N)\|_S^2$$

$$\text{s.t.} \quad x(0) = x(k)$$

$$x(j+1) = Ax(j) + Bu(j)$$

$$x(j) \in \mathcal{X}, u(j) \in \mathcal{U}$$

$$x(N) \in \mathbb{X}_f$$

con  $Q = Q' \geq 0, S = S' \geq 0, R = R' > 0$ .

$$\text{dove } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} C_A \\ T \\ T_c \end{bmatrix}, \text{ e } u = T_r$$

$x \in \mathcal{X}$  dato da:

$$0.38 - \bar{C}_A < x_1 < 0.95 - \bar{C}_A$$
$$-0.1254[mol/l] < x_1 < 0.446[mol/l]$$

$u \in \mathcal{U}$  dato da:

$$280 - \bar{T}_r < u < 310 - \bar{T}_r$$
$$-28[K] < u < 2[K]$$



# Problema di controllo

Il problema di controllo consiste nel risolvere il precedente problema di ottimizzazione ad ogni istante  $k$  ed applicare solo la prima azione di controllo trovata come soluzione (*receding horizon*).



# Outline

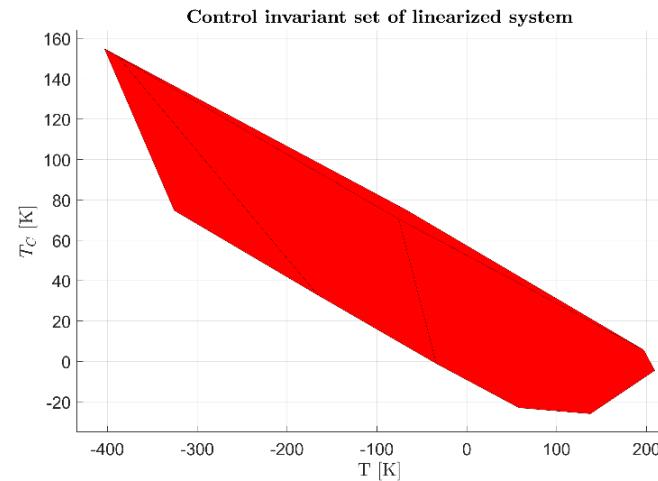
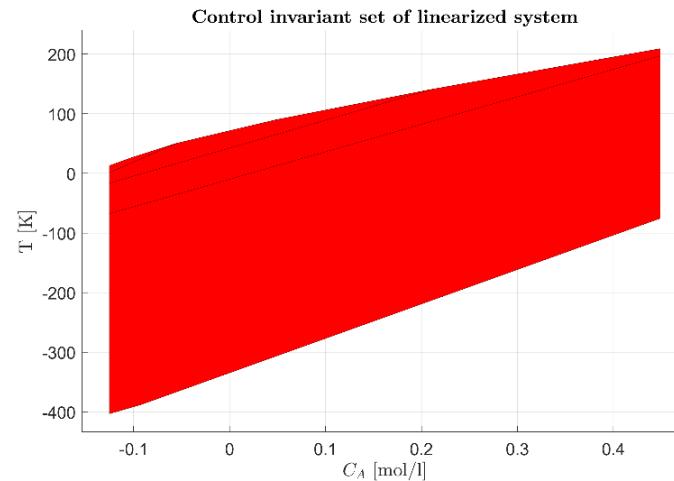
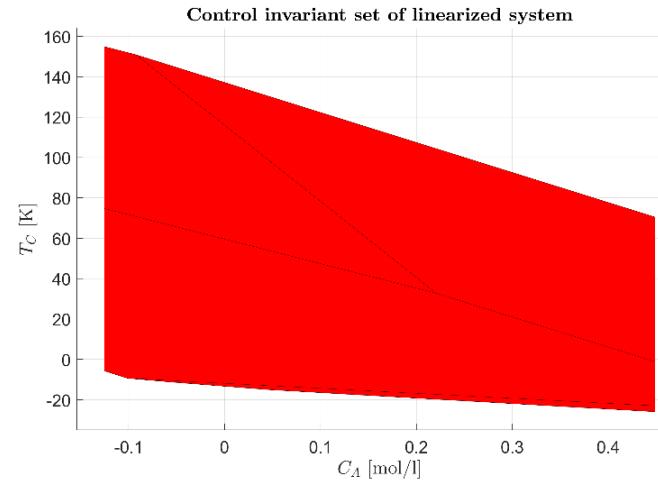
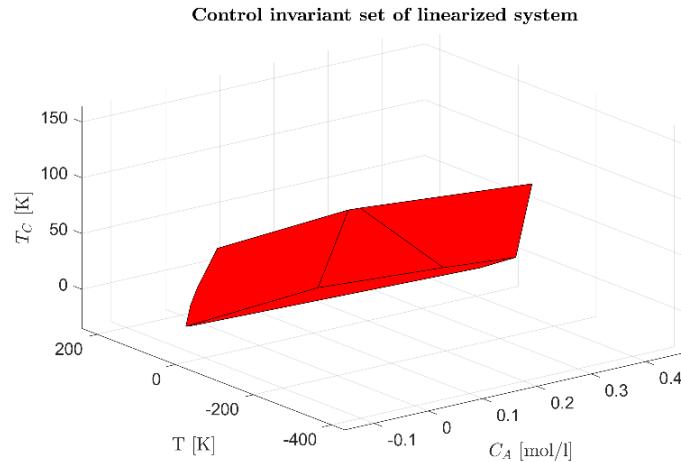
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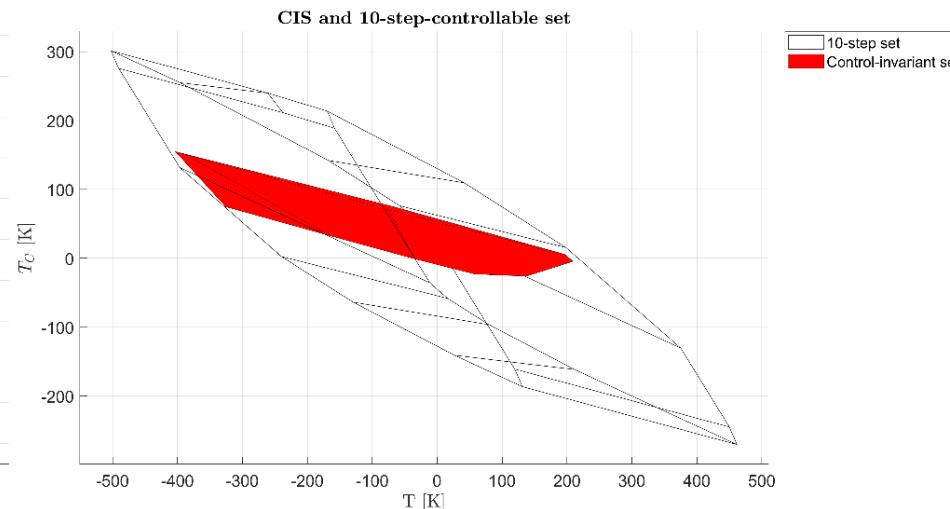
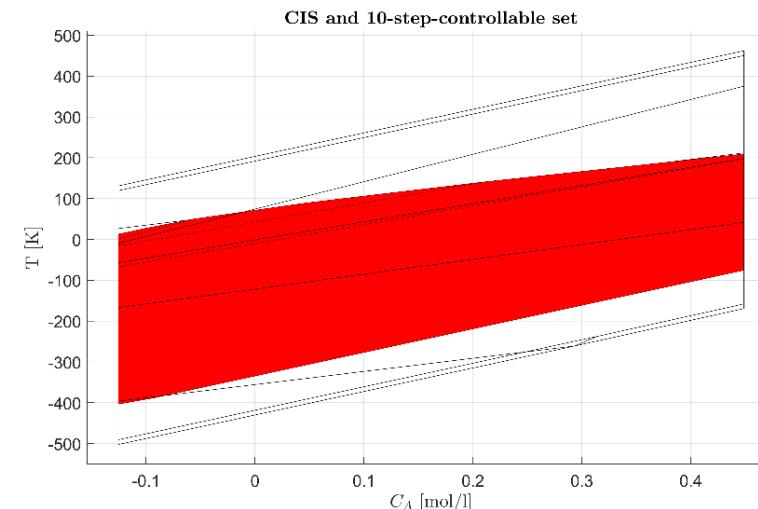
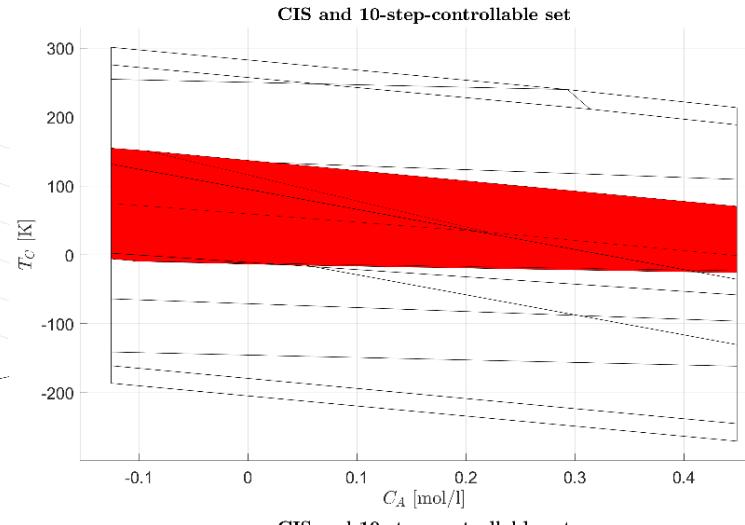
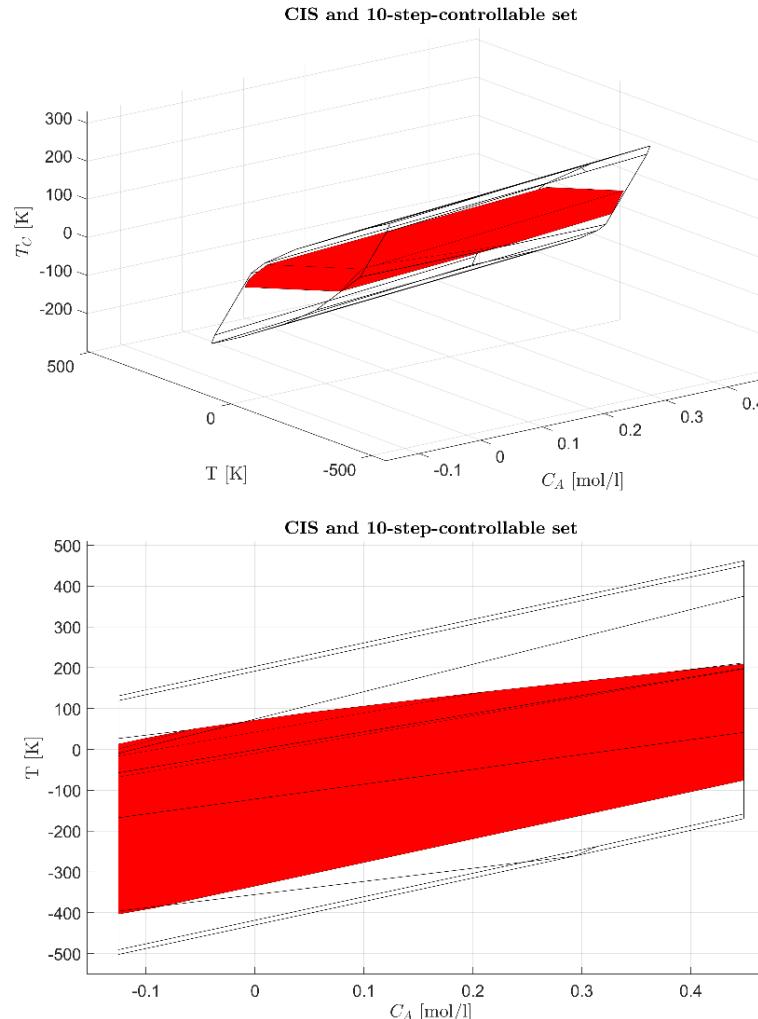
# Simulazione 1: Control invariant set



- $Q = \text{eye}(3)$
- $R = 1$
- $N = 10$
- $T_s = 1$  [s]
- Vincolo terminale  
di diseguaglianza



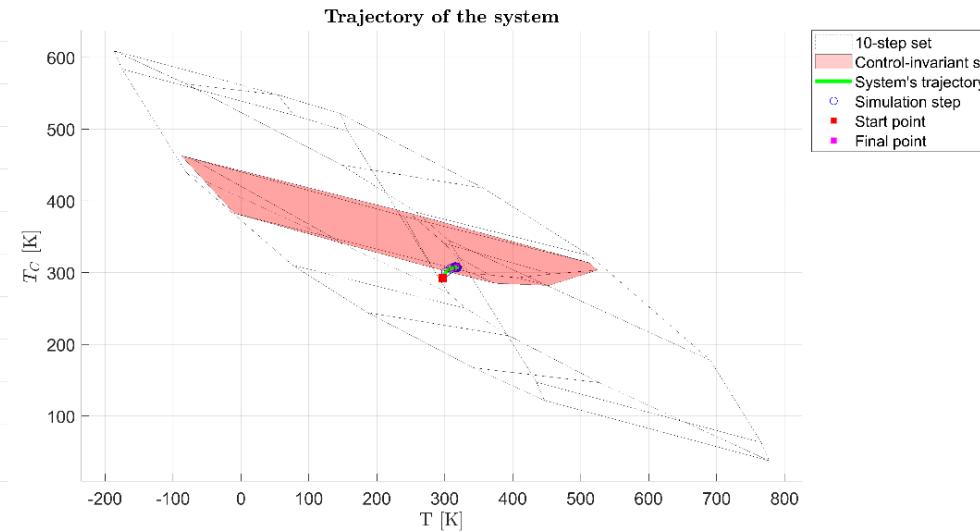
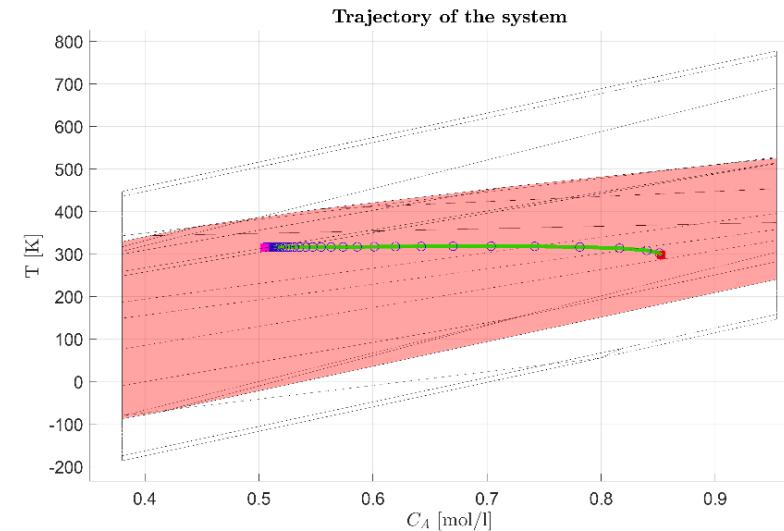
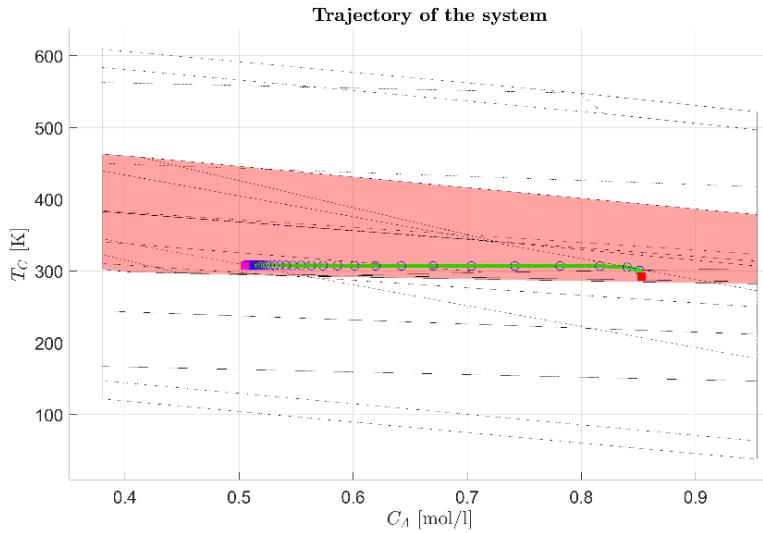
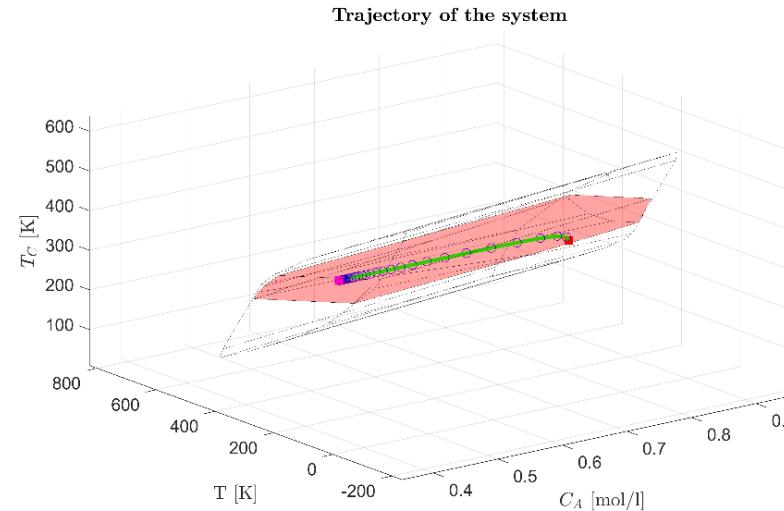
# Simulazione 1: N-Step controllable set



- $Q = \text{eye}(3)$
- $R = 1$
- $N = 10$
- $T_s = 1$  [s]
- Volume CIS:  
10704 [mol\*K<sup>2</sup>/l]
- Volume 10-step CS:  
47466 [mol\*K<sup>2</sup>/l]
- Vincolo terminale di disuguaglianza



# Simulazione 1: Simulazione con MPC

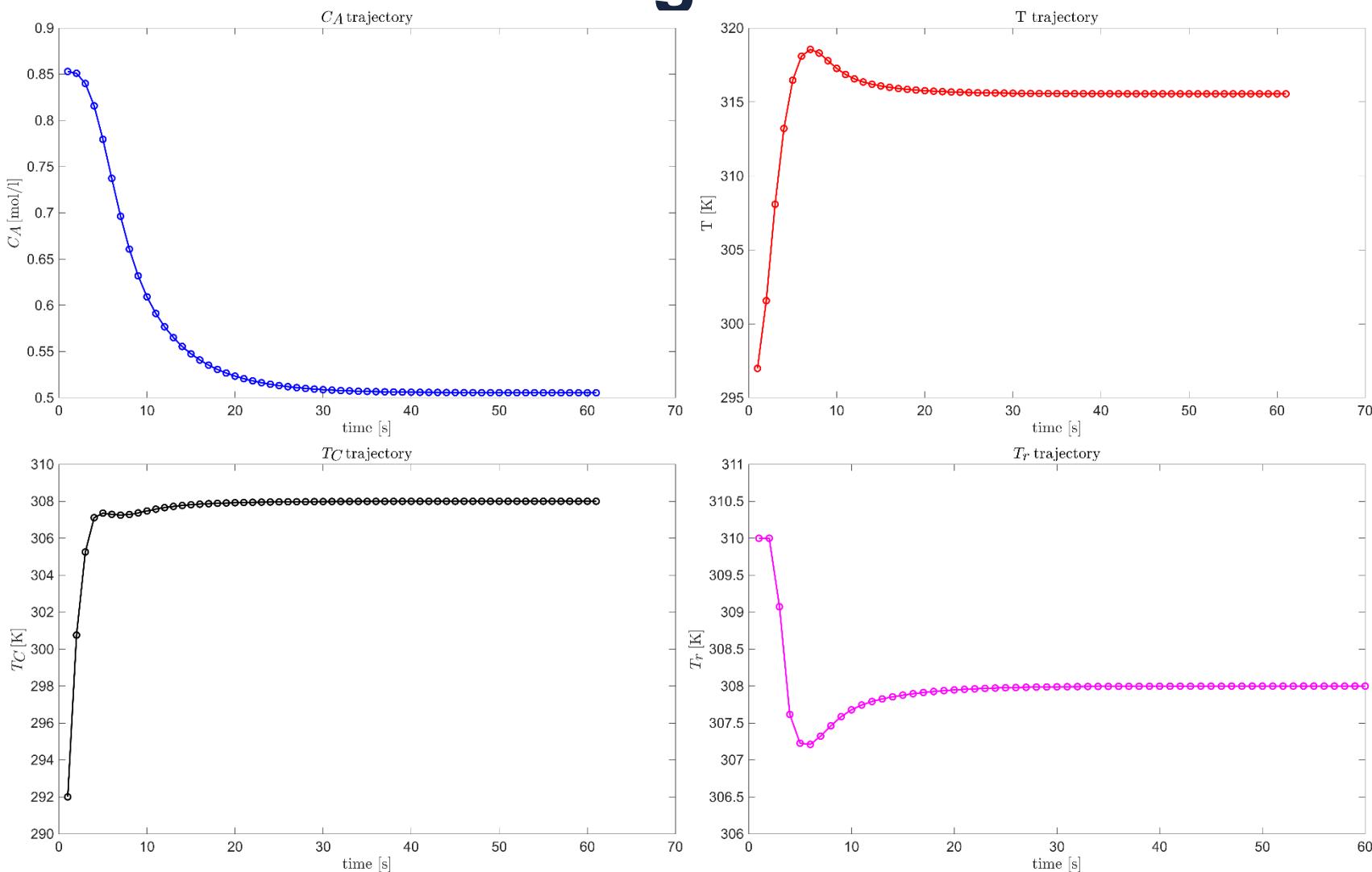


- $Q = \text{eye}(3)$
- $R = 1$
- $N = 10$
- $T_s = 1$  [s]
- Vincolo terminale di diseguaglianza

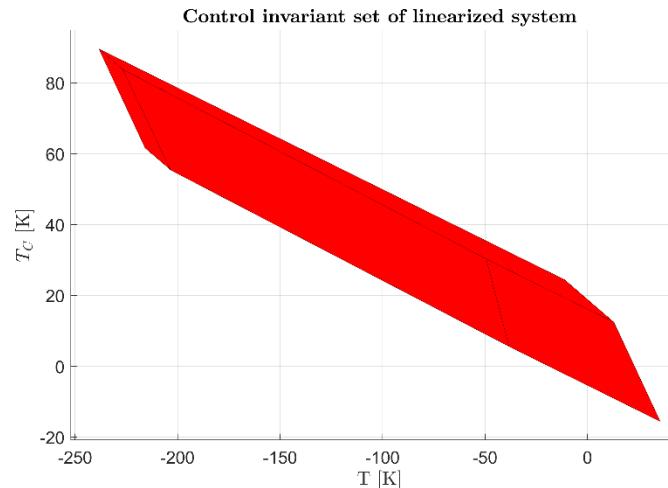
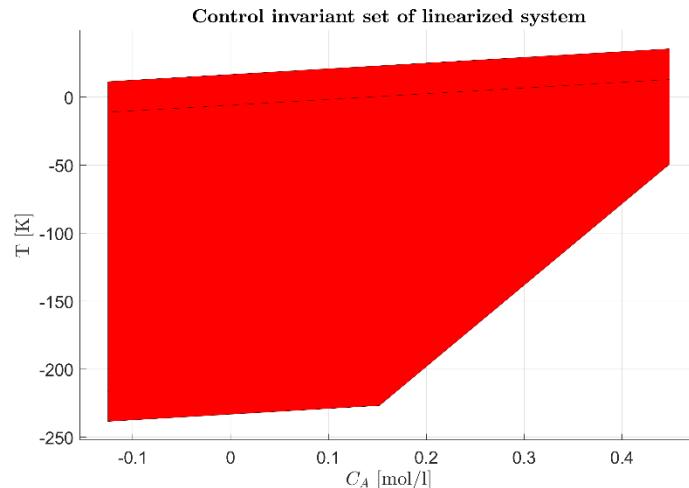
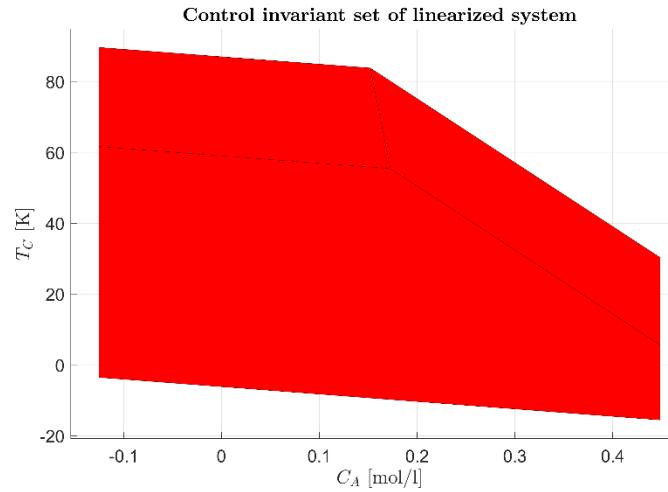
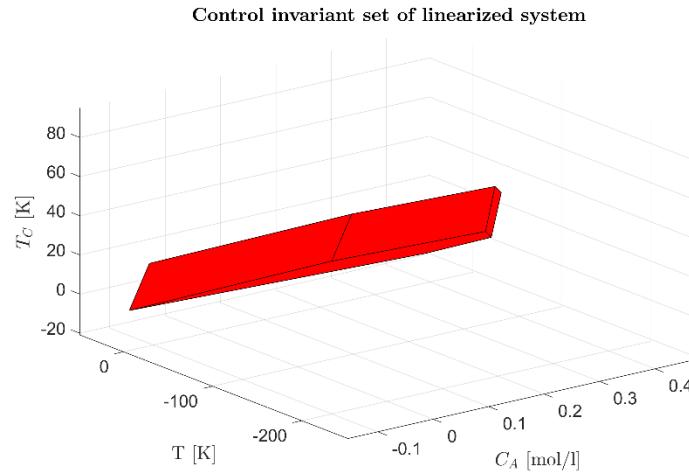


# Simulazione 1: Traiettoria dei singoli stati e ingresso

- $Q = \text{eye}(3)$
- $R = 1$
- $N = 10$
- $T_s = 1$  [s]
- Vincolo terminale di diseguaglianza



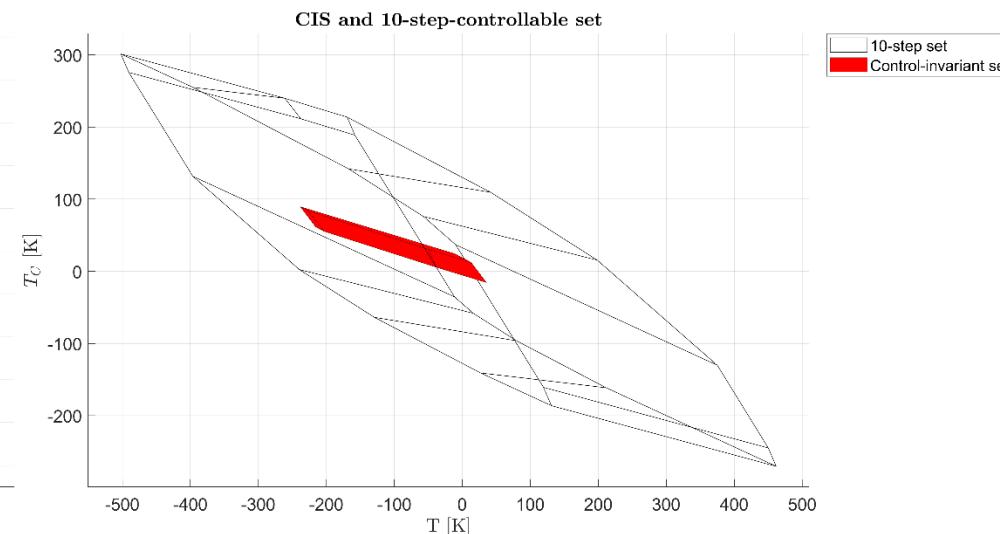
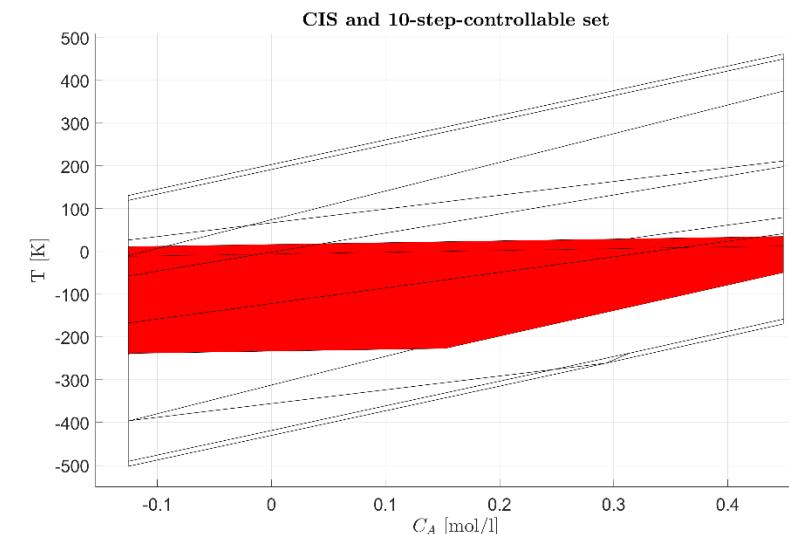
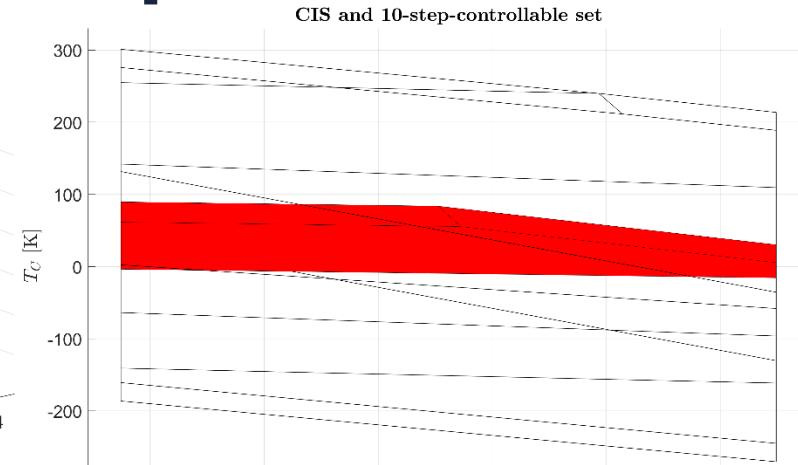
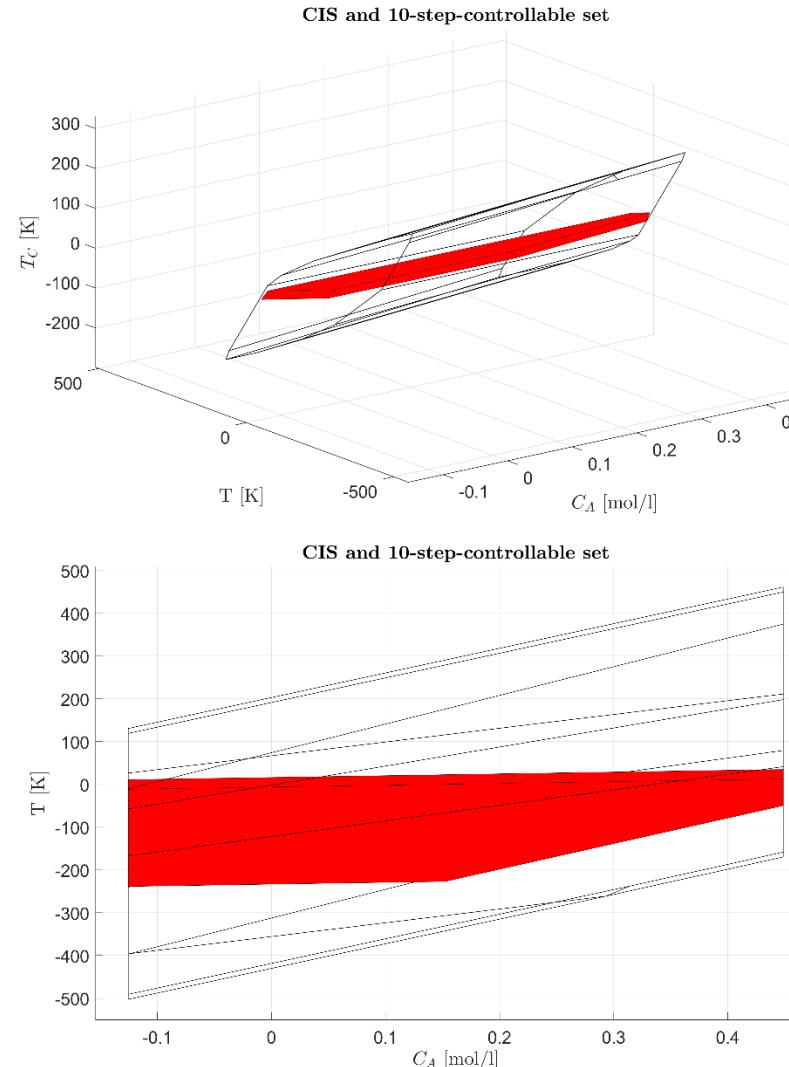
# Simulazione 2: Control invariant set



- $Q = 10000 * \text{eye}(3)$
- $R = 1$
- $N = 10$
- $T_s = 1$  [s]
- Vincolo terminale di disuguaglianza



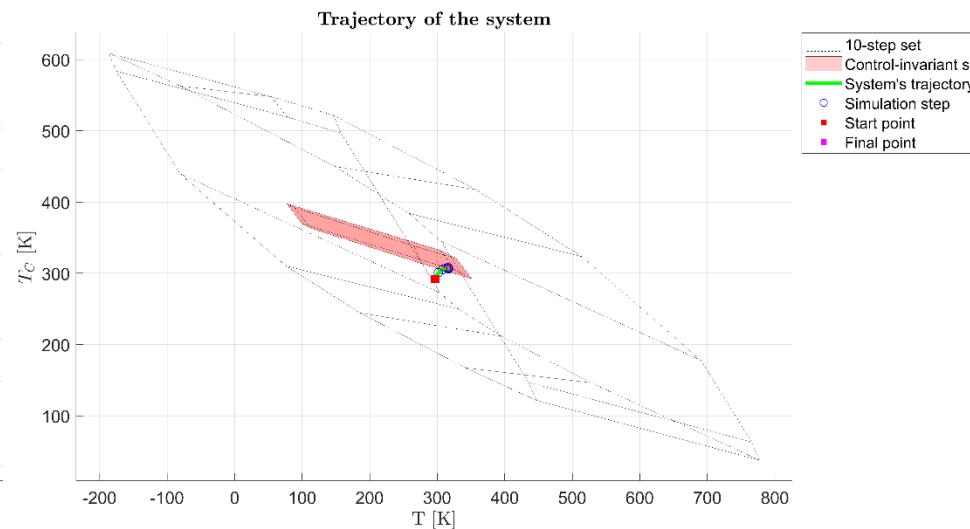
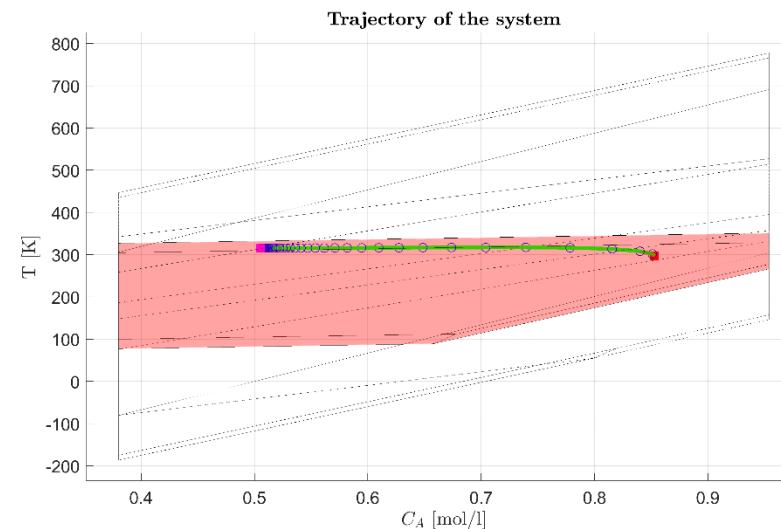
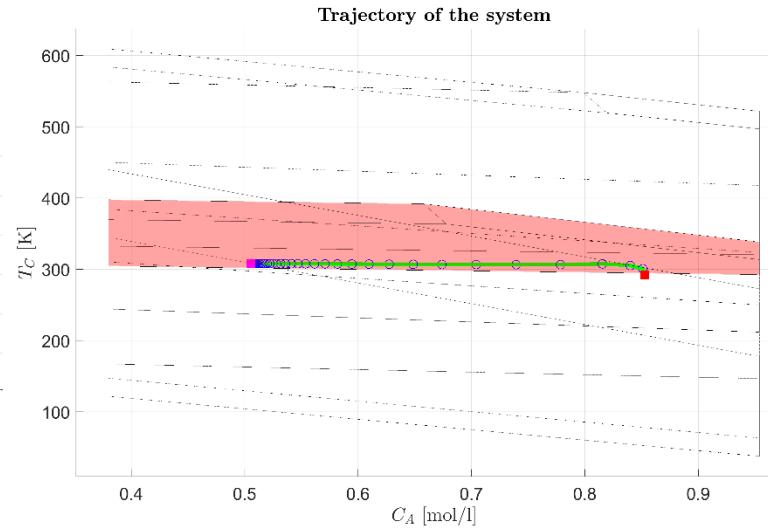
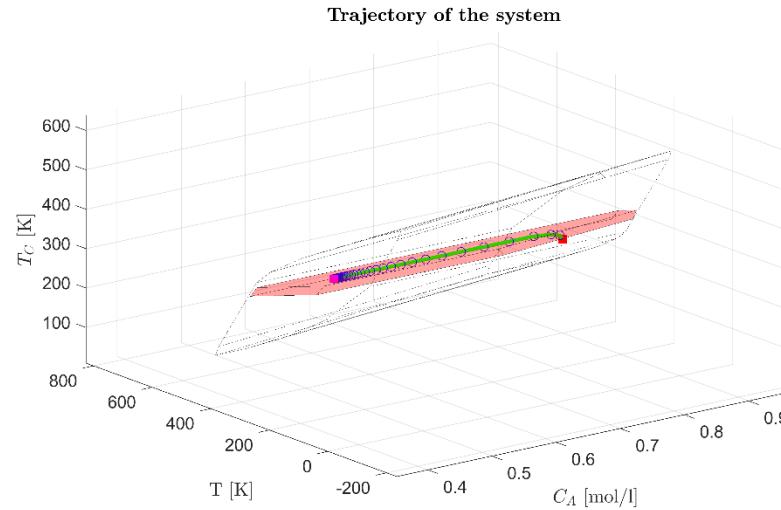
# Simulazione 2: N-Step controllable set



- $Q = 10000 * \text{eye}(3)$
- $R = 1$
- $N = 10$
- Volume CIS:  
2317 [mol\*K<sup>2</sup>/l]
- Volume 10-step CS:  
47466 [mol\*K<sup>2</sup>/l]
- $T_s = 1$  [s]
- Vincolo terminale di disuguaglianza



# Simulazione 2: Simulazione con MPC

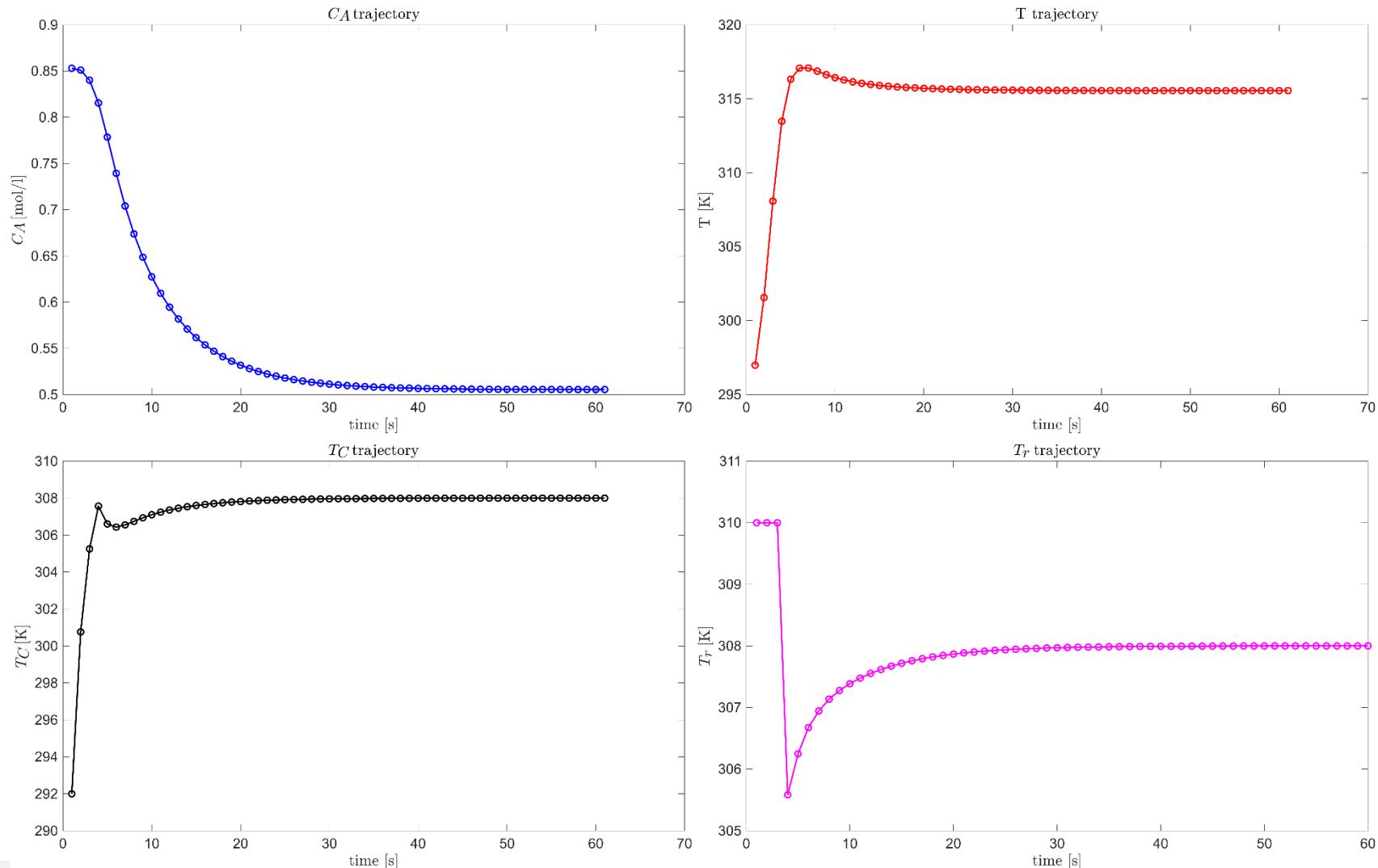


- $Q = 10000 * \text{eye}(3)$
- $R = 1$
- $N = 10$
- $T_s = 1$  [s]
- Vincolo terminale di disuguaglianza

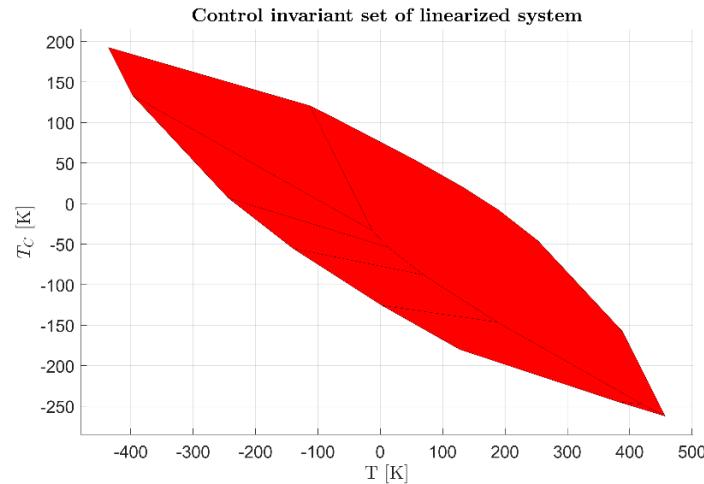
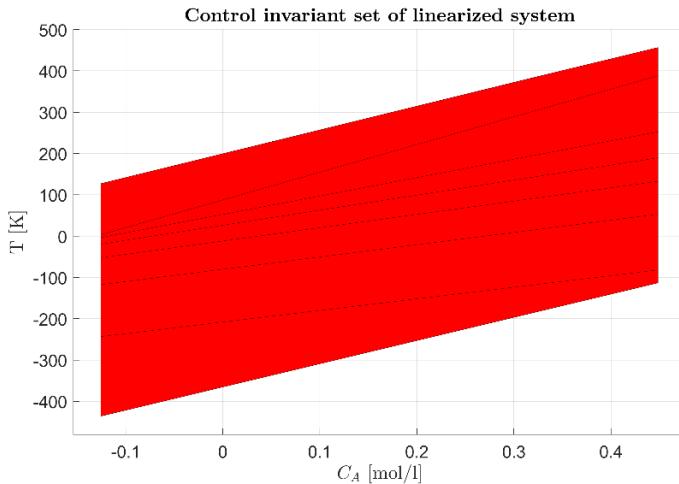
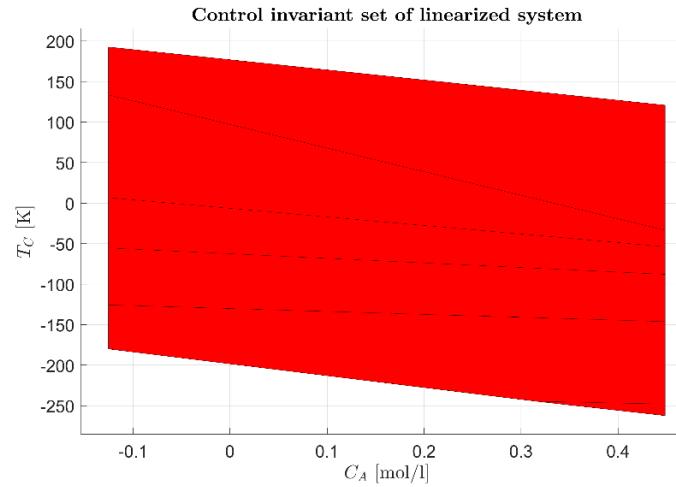
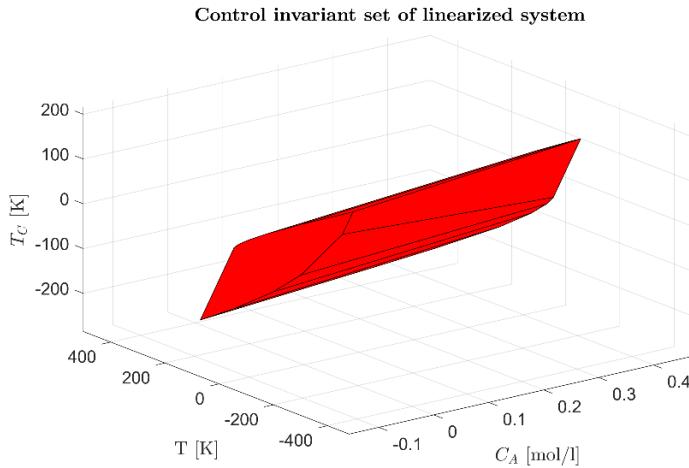


# Simulazione 2: Traiettoria dei singoli stati e ingresso

- $Q = 10000 * \text{eye}(3)$
- $R = 1$
- $N = 10$
- $T_s = 1$  [s]
- Vincolo terminale di diseguaglianza



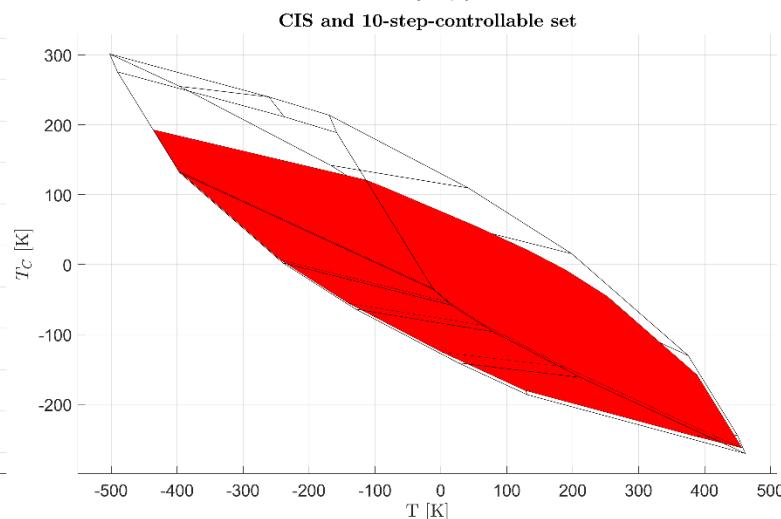
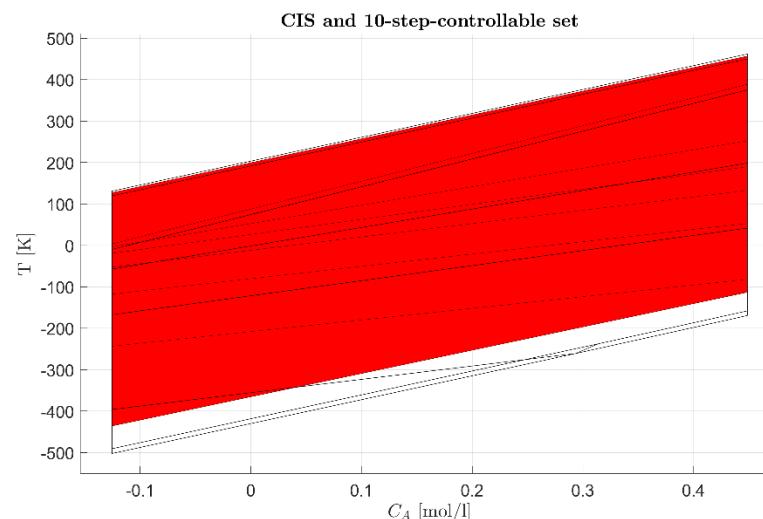
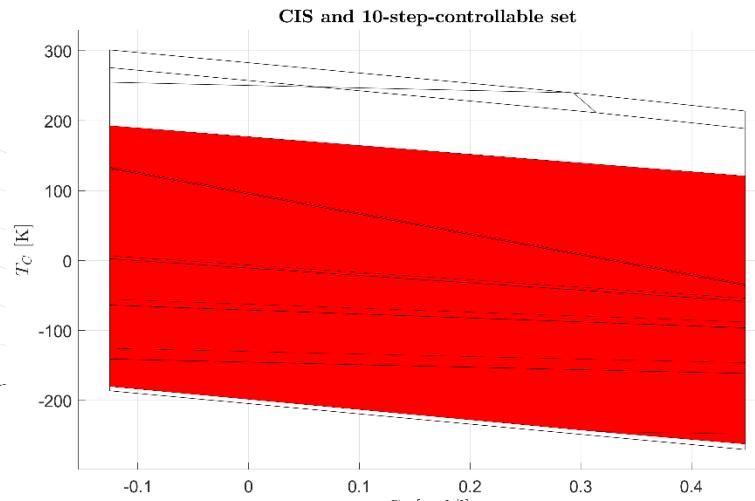
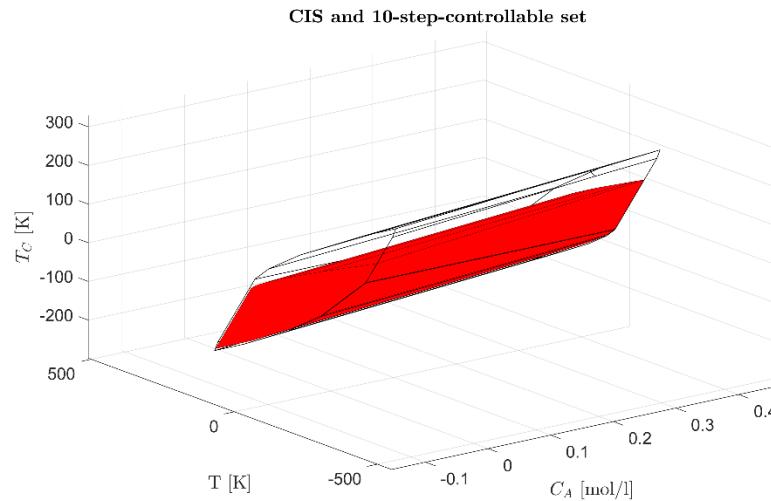
# Simulazione 3: Control invariant set



- $Q = \text{eye}(3)$
- $R = 10000$
- $N = 10$
- $T_s = 1 \text{ [s]}$
- Vincolo terminale di diseguaglianza

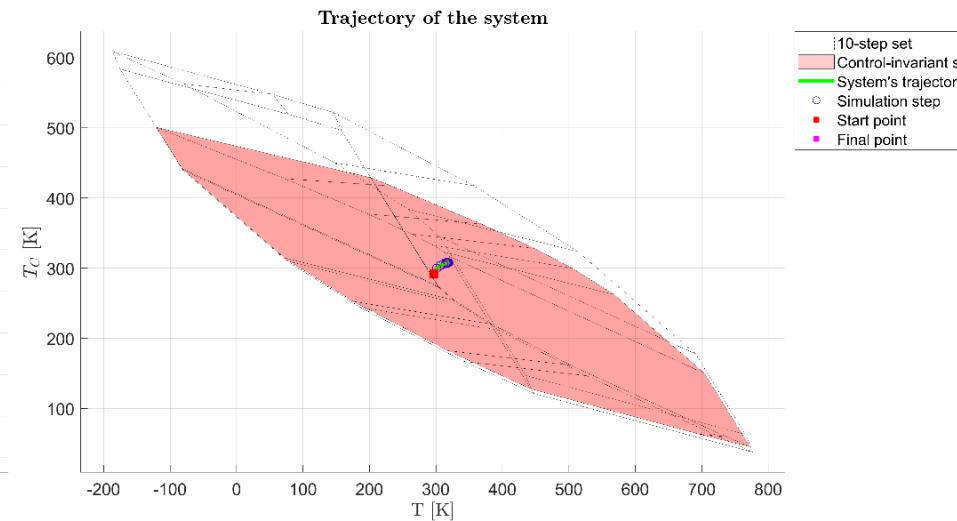
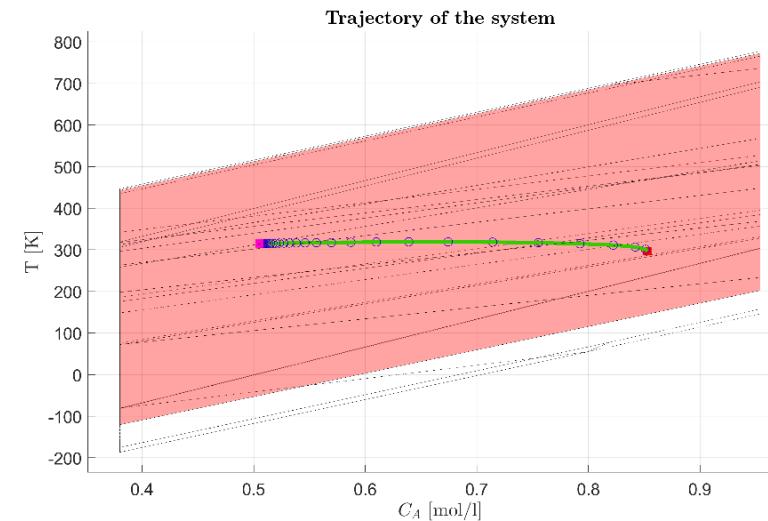
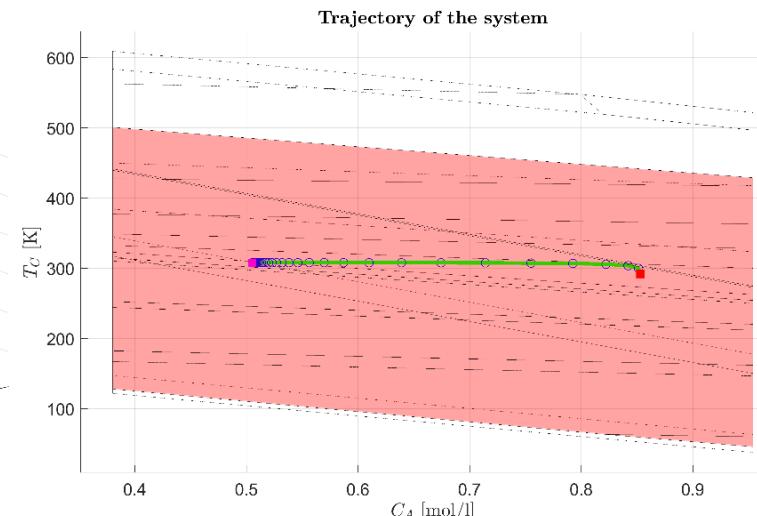
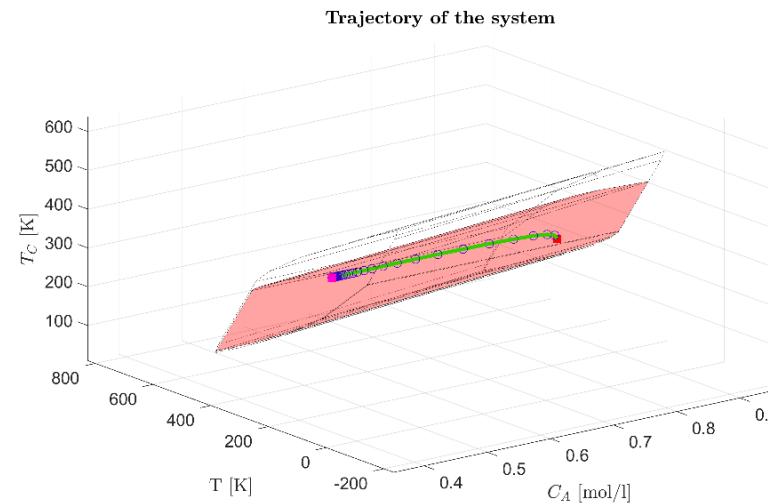


# Simulazione 3: N-Step controllable set



- $Q = \text{eye}(3)$
- $R = 10000$
- $N = 10$
- $T_s = 1$  [s]
- Volume CIS:  
 $32742$  [mol\*K<sup>2</sup>/l]
- Volume 10-step CS:  
 $47466$  [mol\*K<sup>2</sup>/l]
- Vincolo terminale di disuguaglianza

# Simulazione 3: Simulazione con MPC

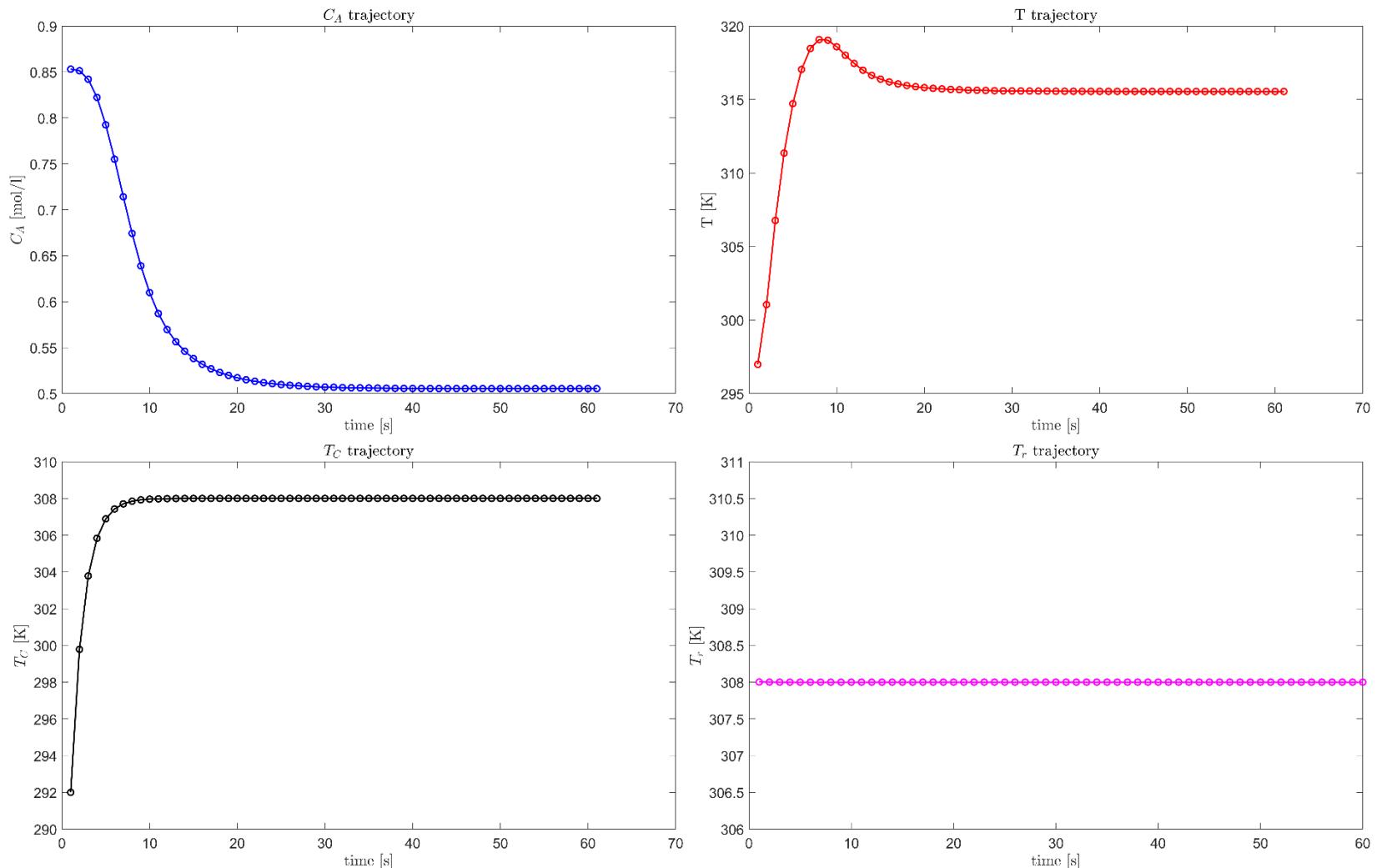


- $Q = \text{eye}(3)$
- $R = 10000$
- $N = 10$
- $T_s = 1$  [s]
- Vincolo terminale di disuguaglianza

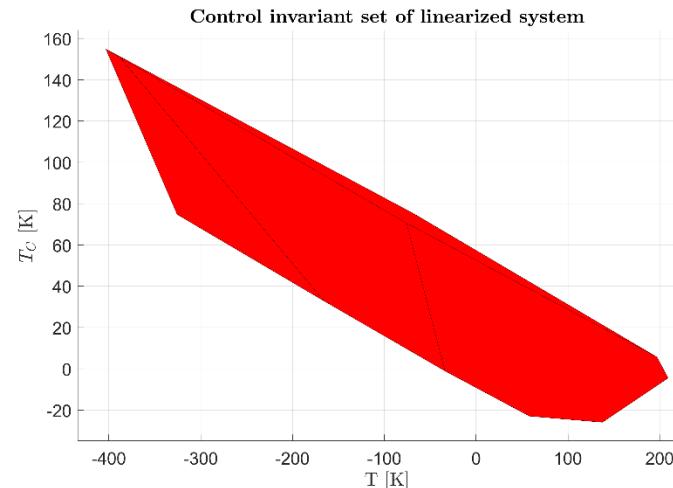
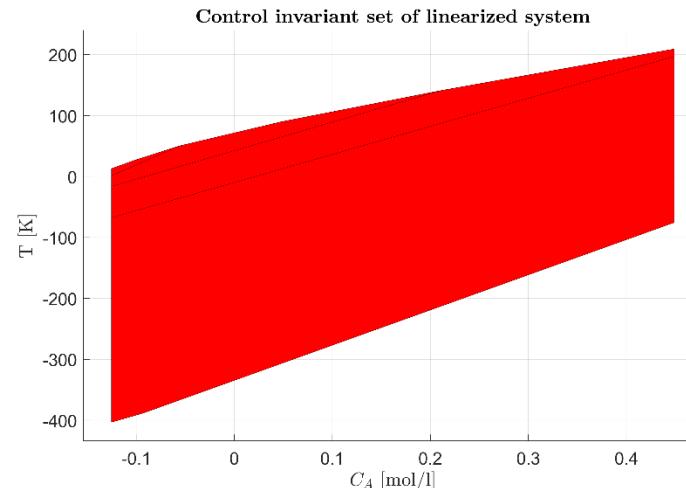
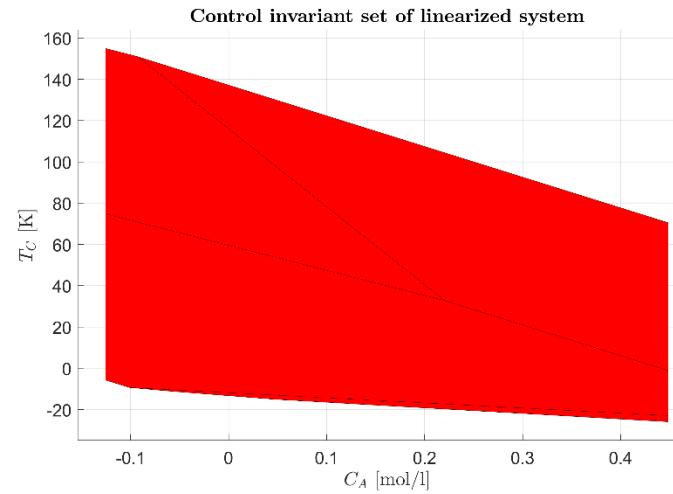
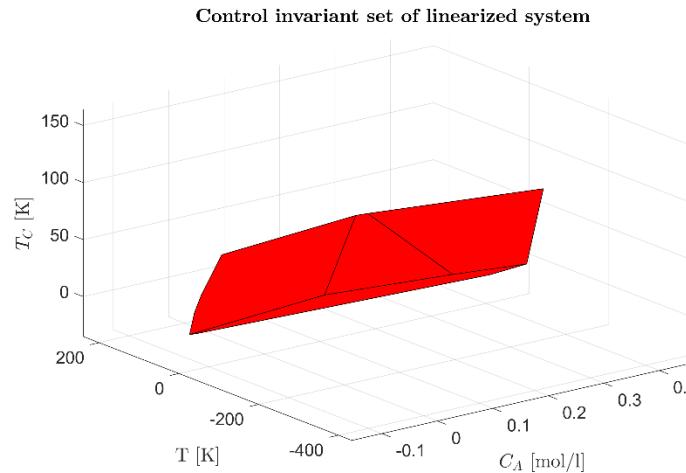


# Simulazione 3: Traiettoria dei singoli stati e ingresso

- $Q = \text{eye}(3)$
- $R = 10000$
- $N = 10$
- $T_s = 1$  [s]
- Vincolo terminale di diseguaglianza



# Simulazione 4: Control invariant set



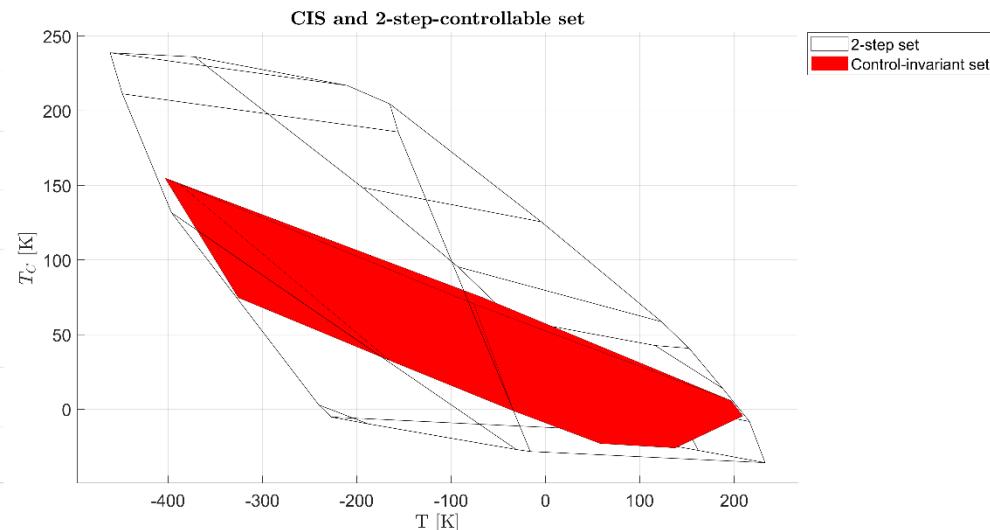
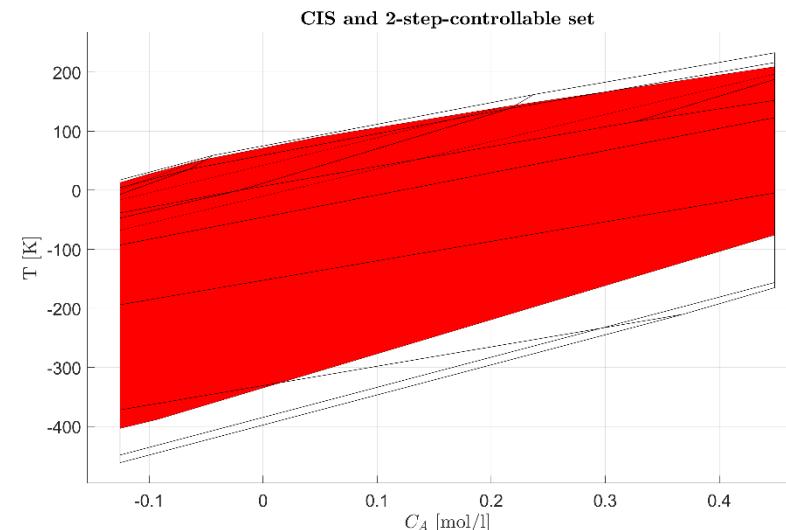
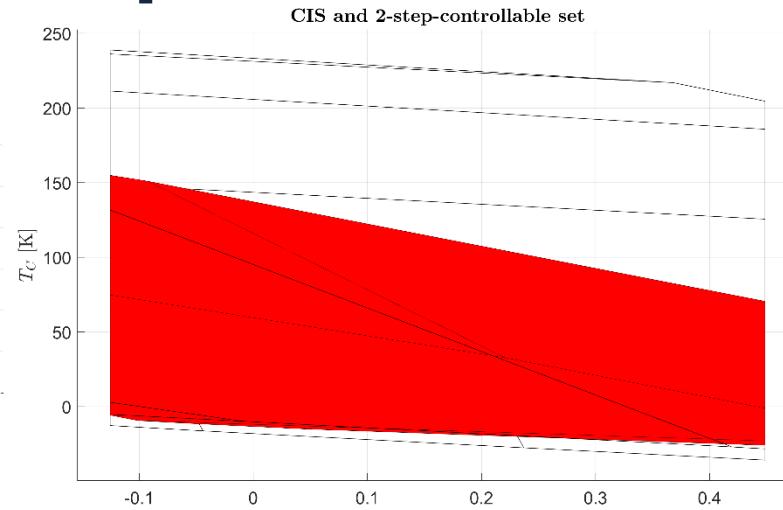
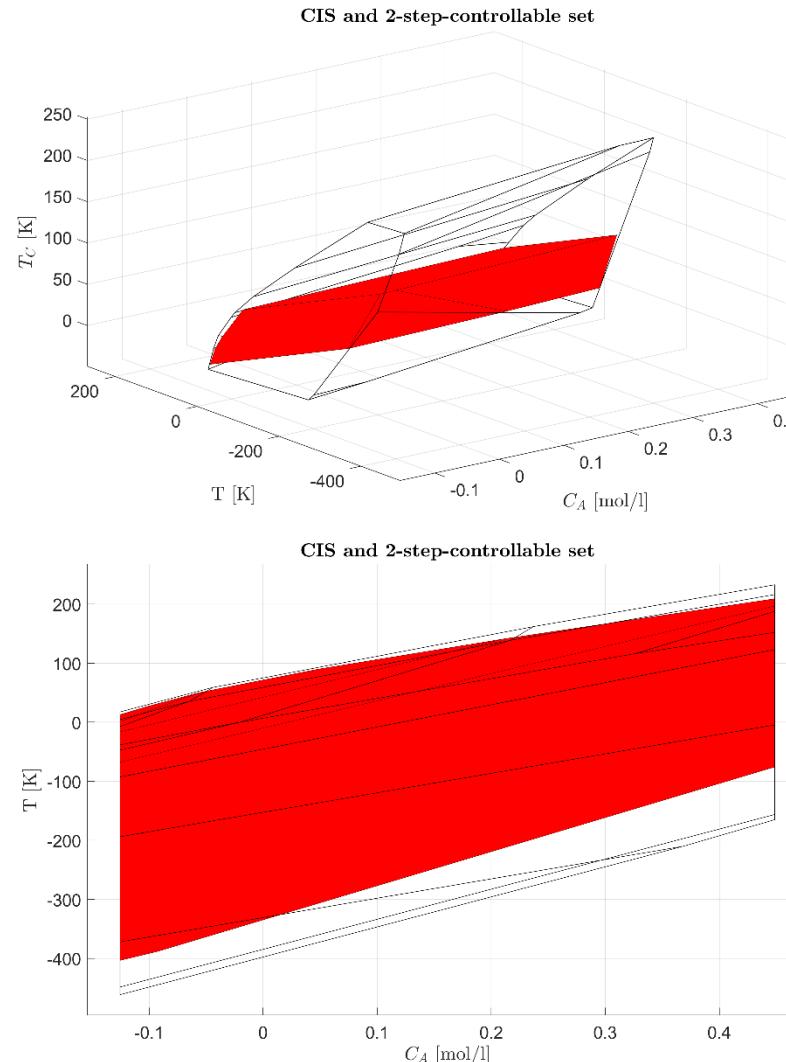
- $Q = 1$
- $R = 1$
- $N = 2$
- $T_s = 1 \text{ [s]}$
- Vincolo terminale di diseguaglianza



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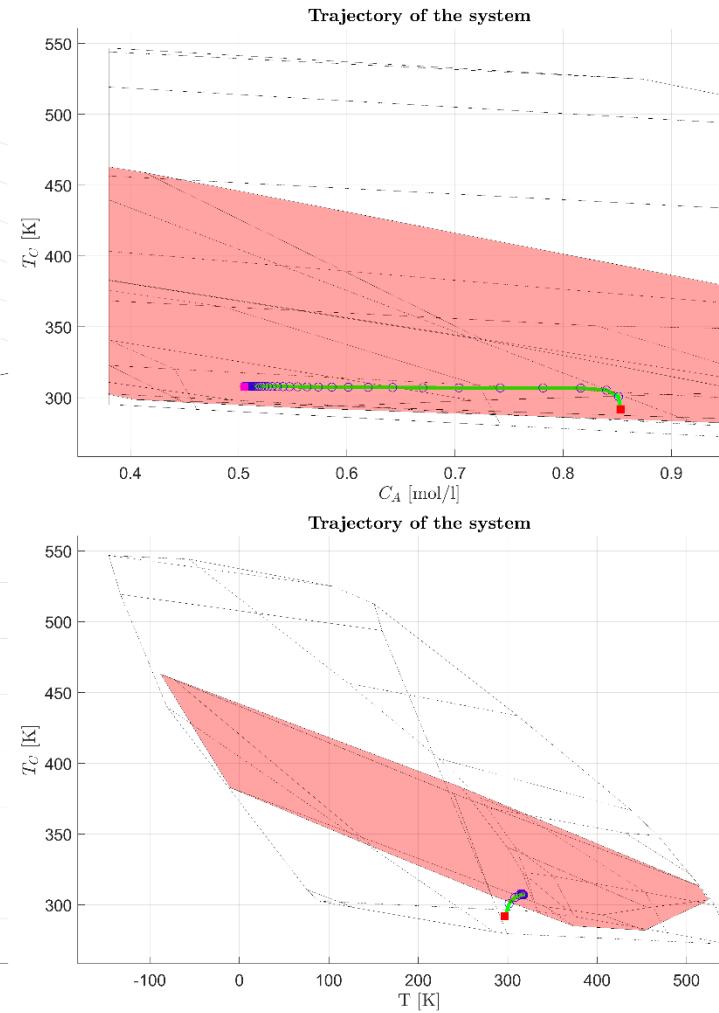
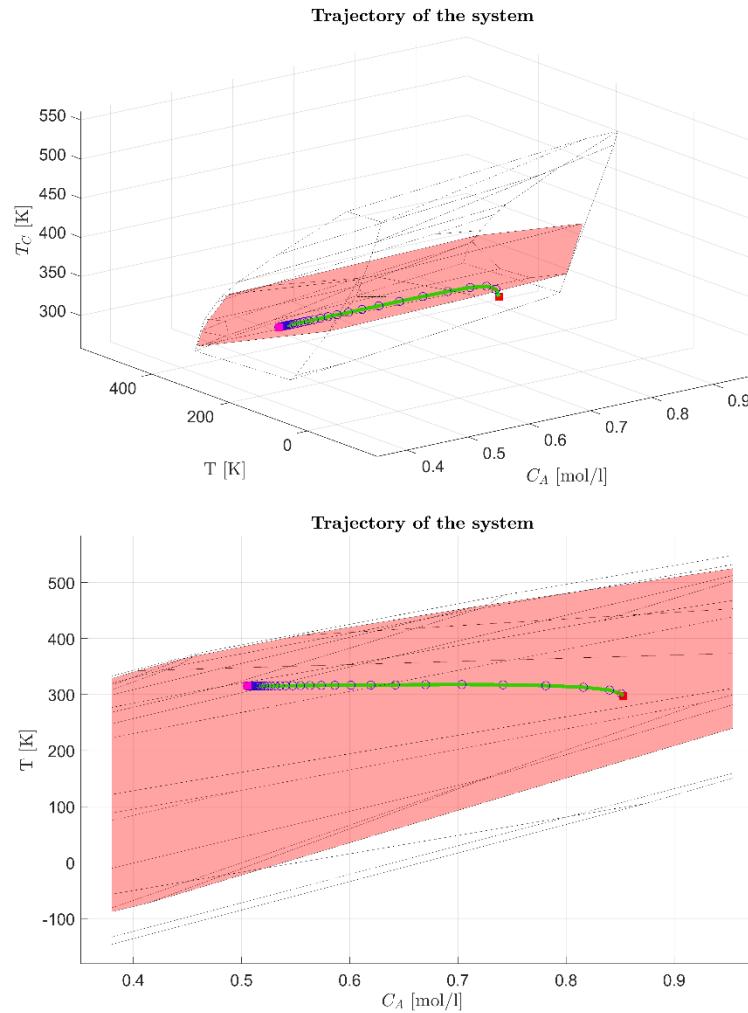
# Simulazione 4: N-Step controllable set



- $Q = 1$
- $R = 1$
- $N = 2$
- $T_s = 1 \text{ [s]}$
- Volume CIS:  $10704 \text{ [mol}^*\text{K}^2/\text{l]}$
- Volume 10-step CS:  $18386 \text{ [mol}^*\text{K}^2/\text{l]}$
- Vincolo terminale di disuguaglianza



# Simulazione 4: Simulazione con MPC

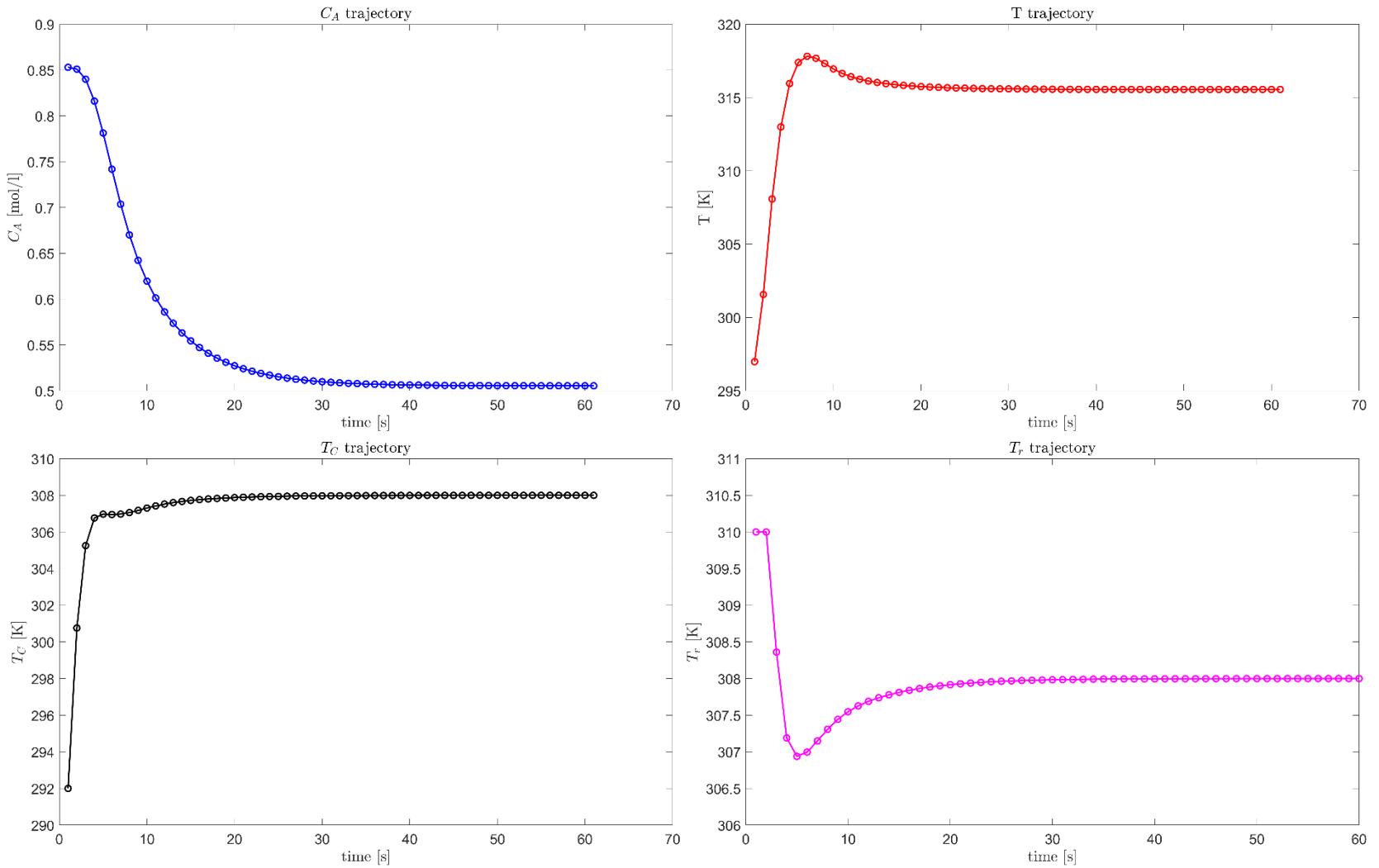


- $Q = 1$
- $R = 1$
- $N = 2$
- $T_s = 1$  [s]
- Vincolo terminale di diseguaglianza

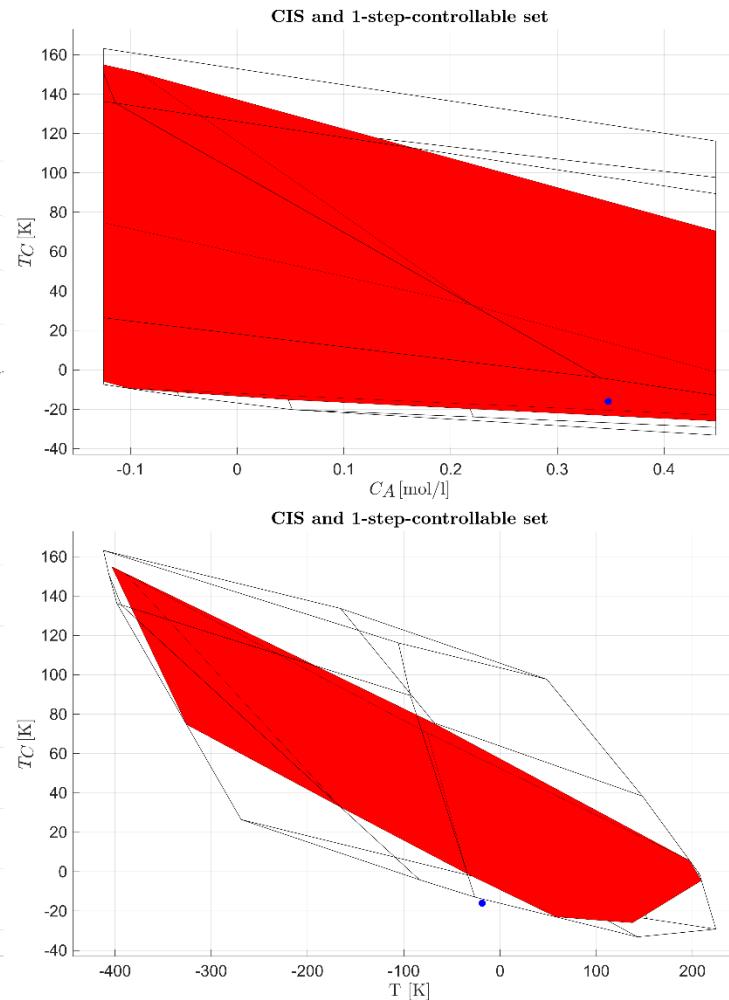
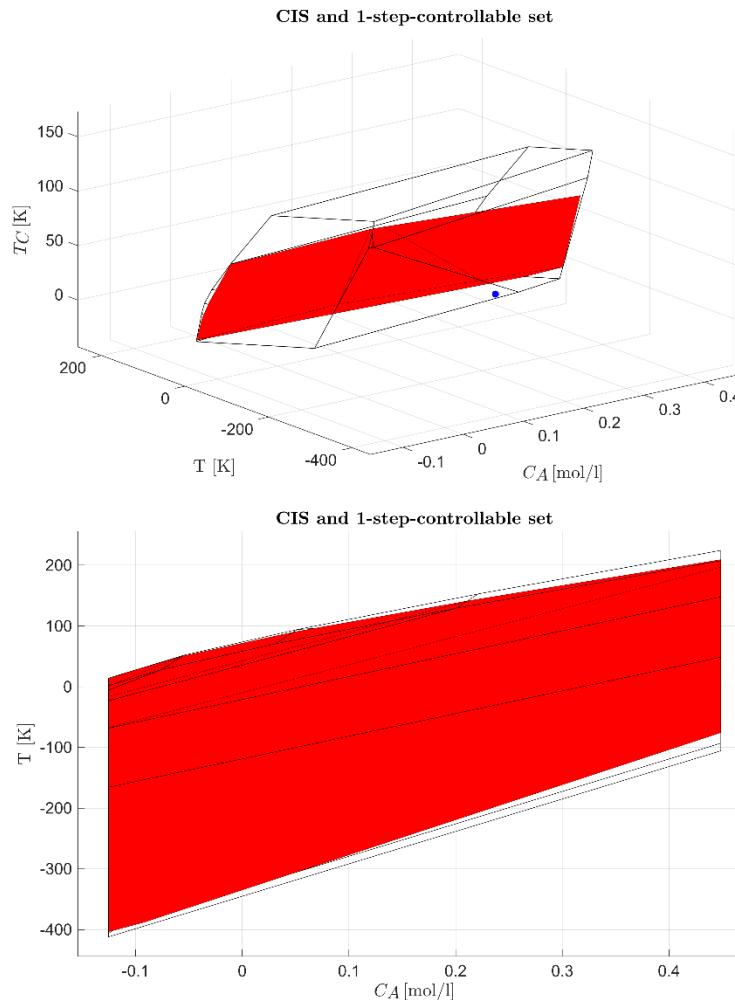


# Simulazione 4: Traiettoria dei singoli stati e ingresso

- $Q = 1$
- $R = 1$
- $N = 2$
- $T_s = 1$  [s]
- Vincolo terminale di diseguaglianza



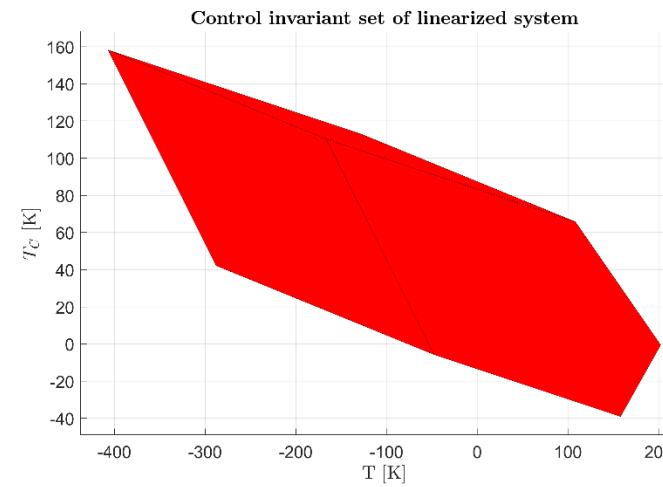
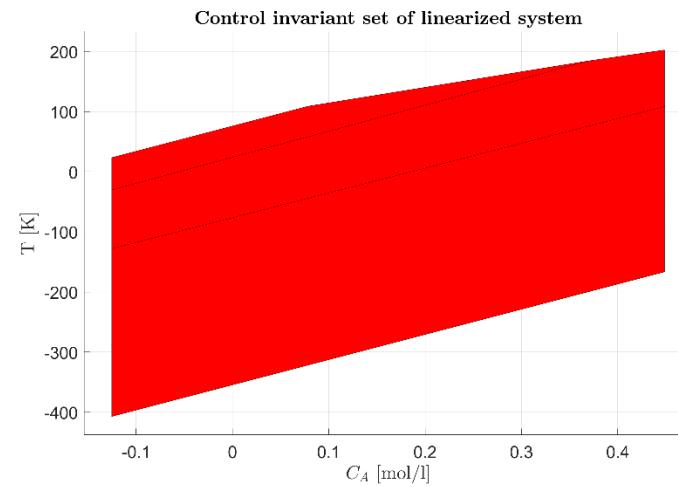
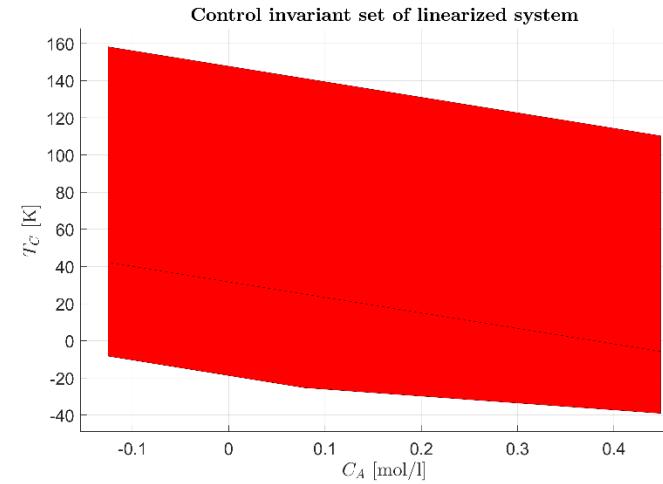
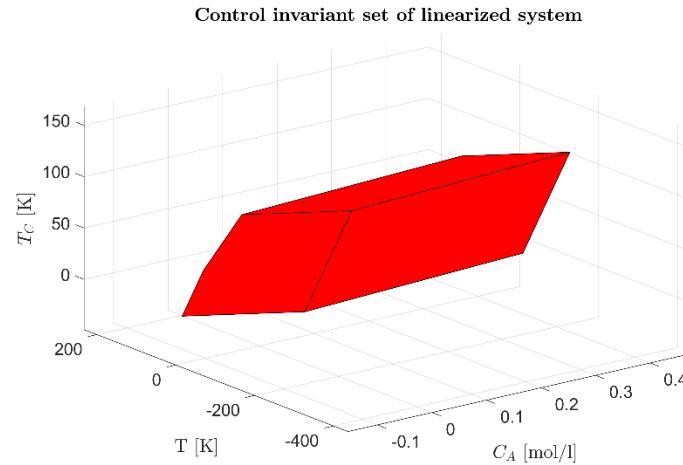
# Simulazione 5: Control invariant set



- $Q = 1$
- $R = 1$
- $N = 1$
- $T_s = 1 \text{ [s]}$
- Vincolo terminale di diseguaglianza
- Problema infeasible



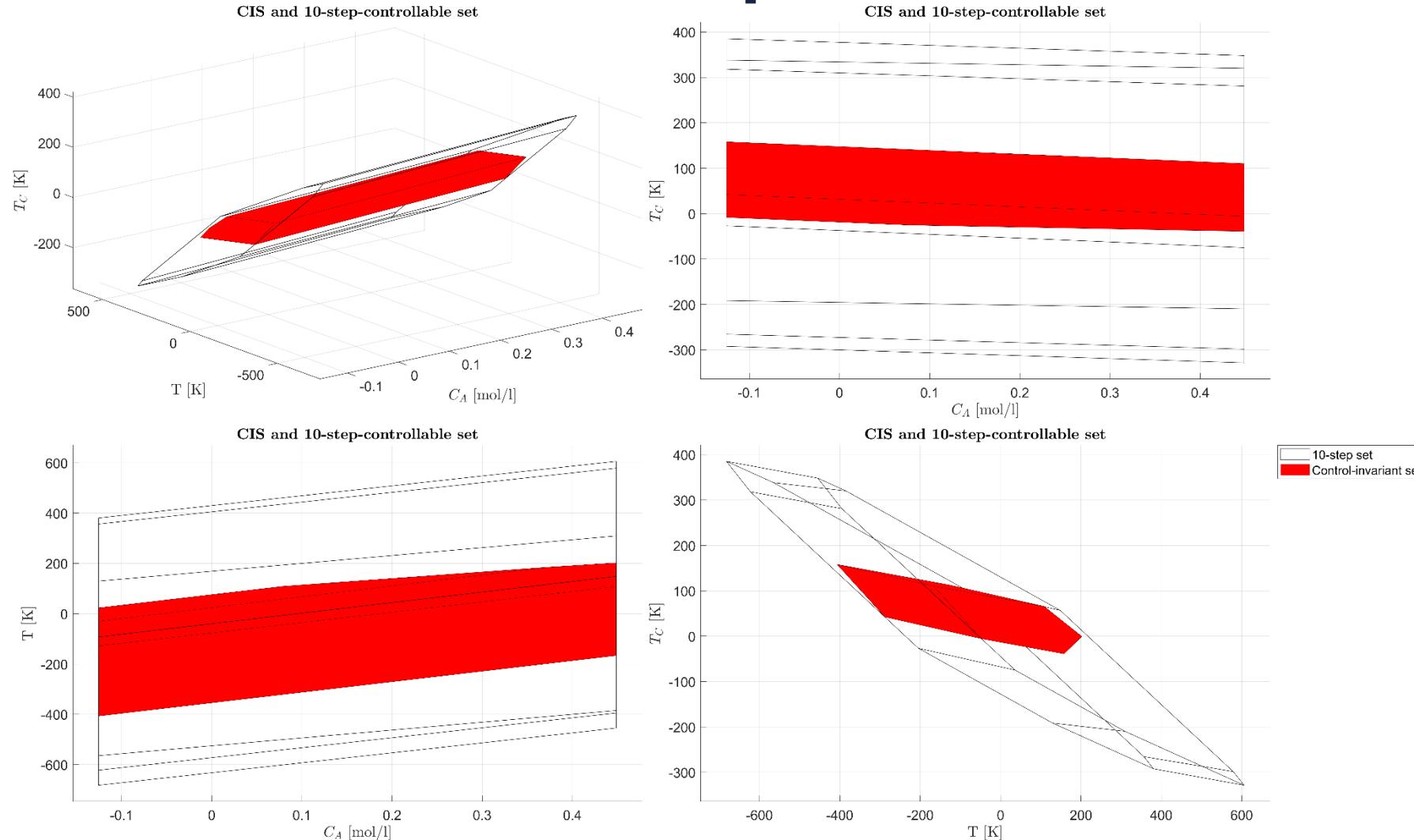
# Simulazione 6: Control invariant set



- $Q = 1$
- $R = 1$
- $N = 10$
- $T_s = 2 \text{ [s]}$
- Volume CIS:  
 $16423 \text{ [mol}^*\text{K}^2/\text{l]}$
- Volume 10-step CS:  
 $68460 \text{ [mol}^*\text{K}^2/\text{l]}$
- Vincolo terminale di disuguaglianza



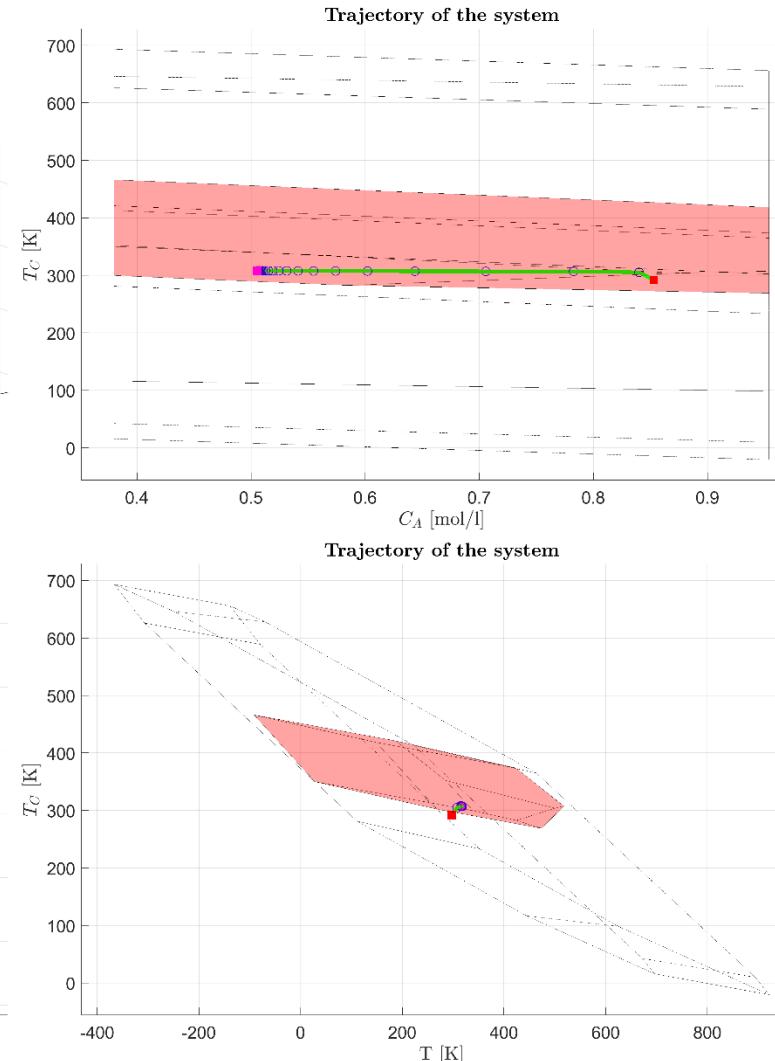
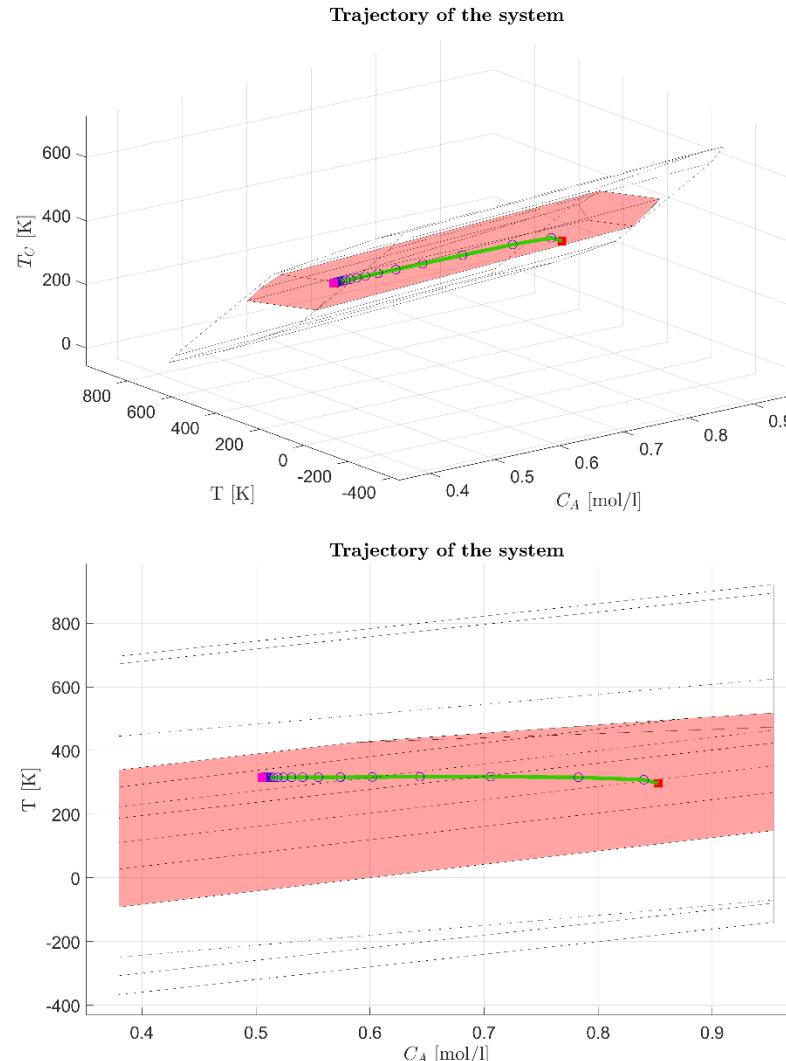
# Simulazione 6: N-Step controllable set



- $Q = 1$
- $R = 1$
- $N = 10$
- $T_s = 2$  [s]
- Vincolo terminale  
di diseguaglianza



# Simulazione 6: Simulazione con MPC

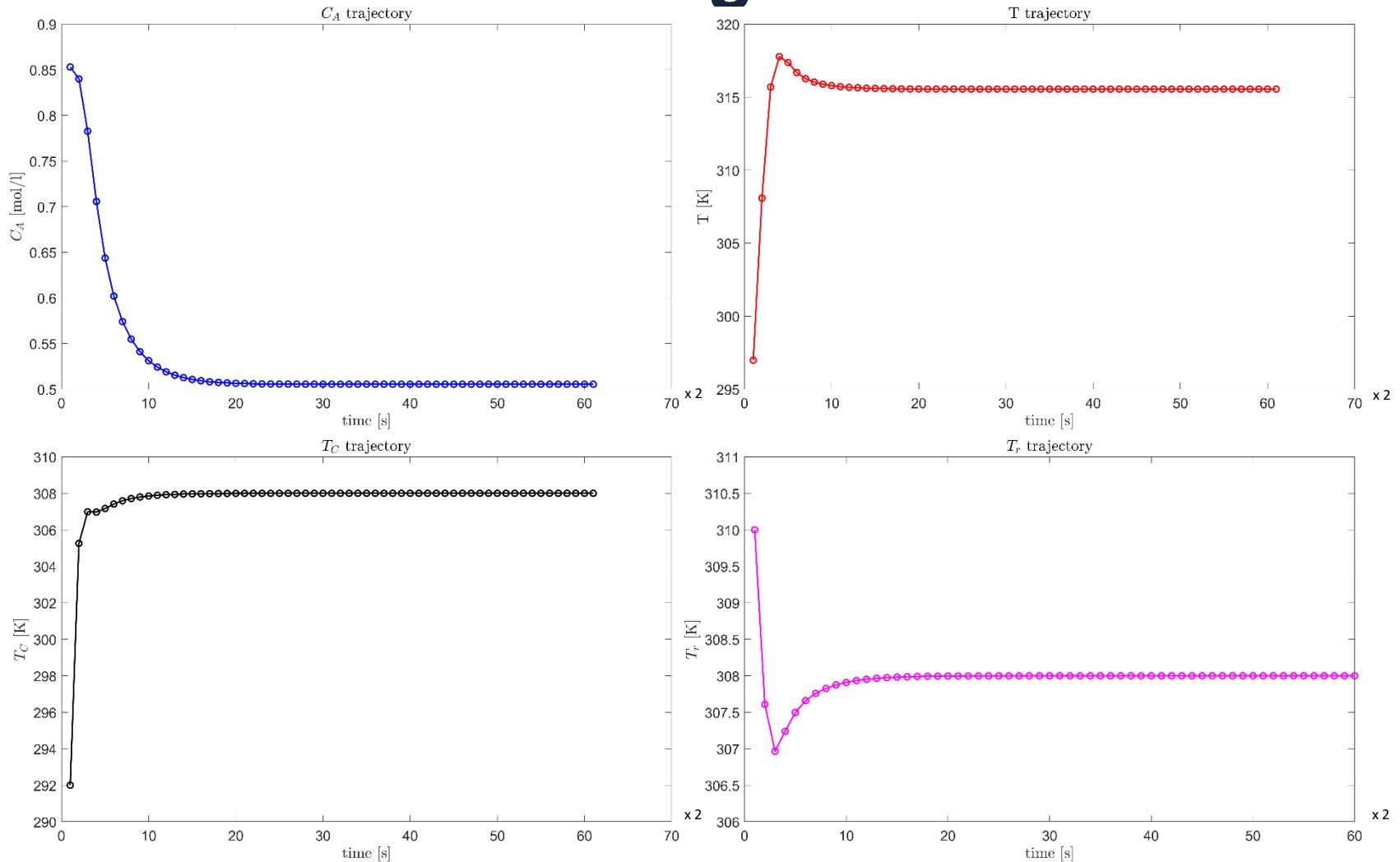


- $Q = 1$
- $R = 1$
- $N = 10$
- $T_s = 2 \text{ [s]}$
- Vincolo terminale di diseguaglianza

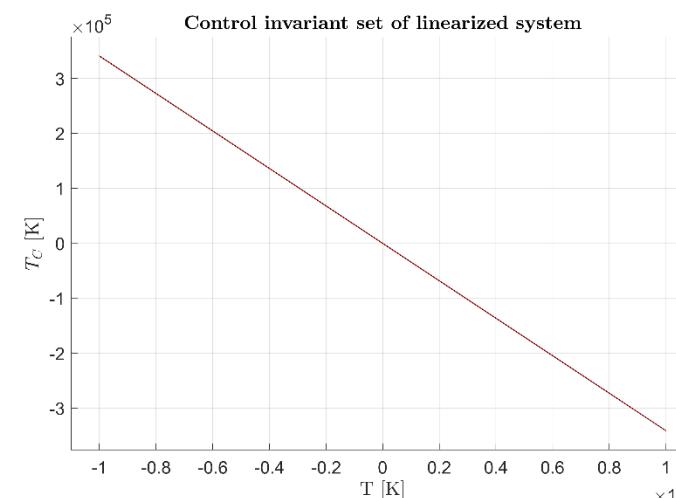
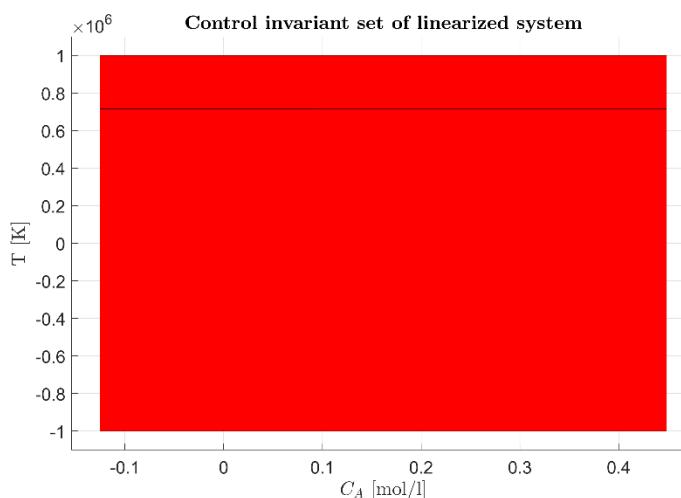
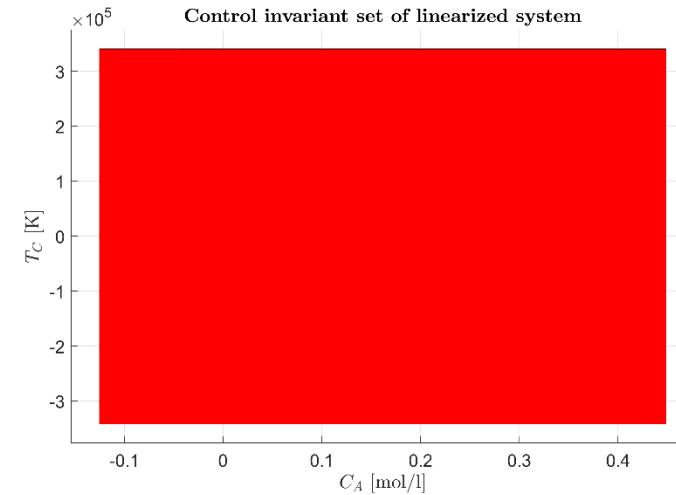
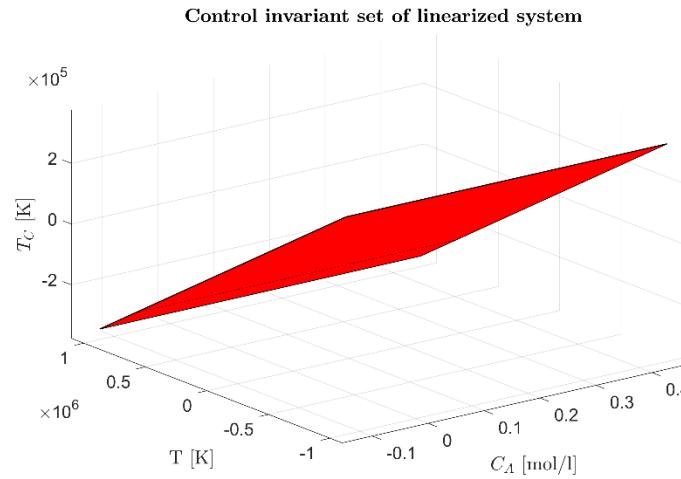


# Simulazione 6: Traiettoria dei singoli stati e ingresso

- $Q = 1$
- $R = 1$
- $N = 10$
- $T_s = 2$  [s]
- Vincolo terminale di diseguaglianza



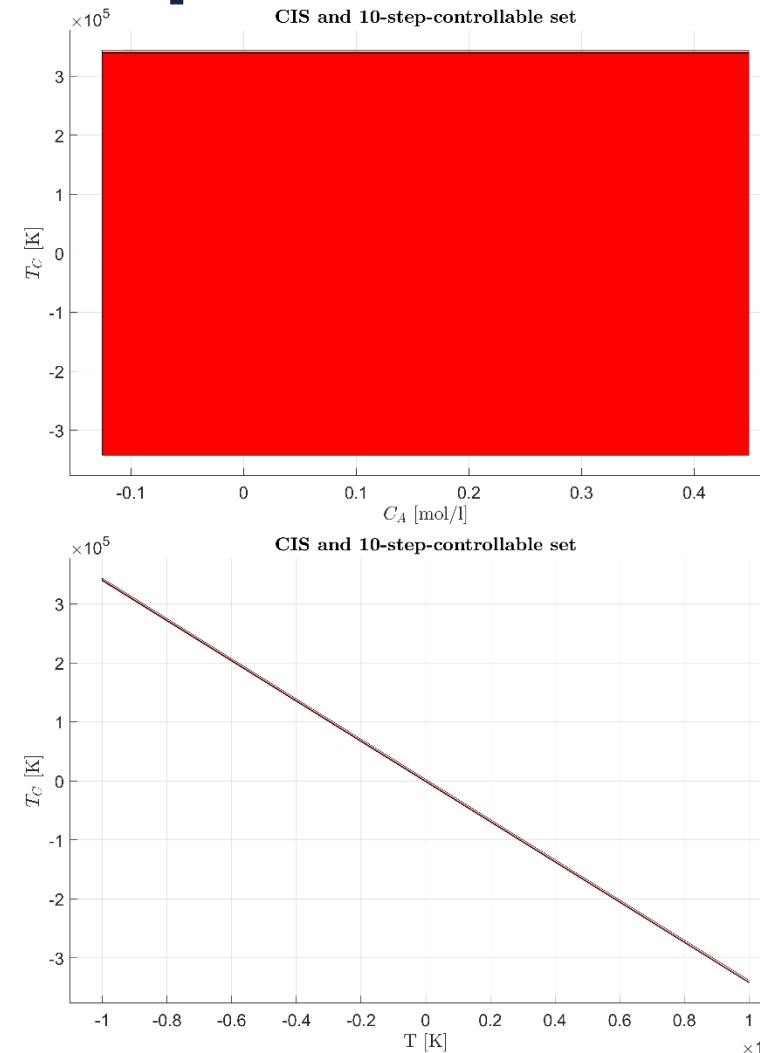
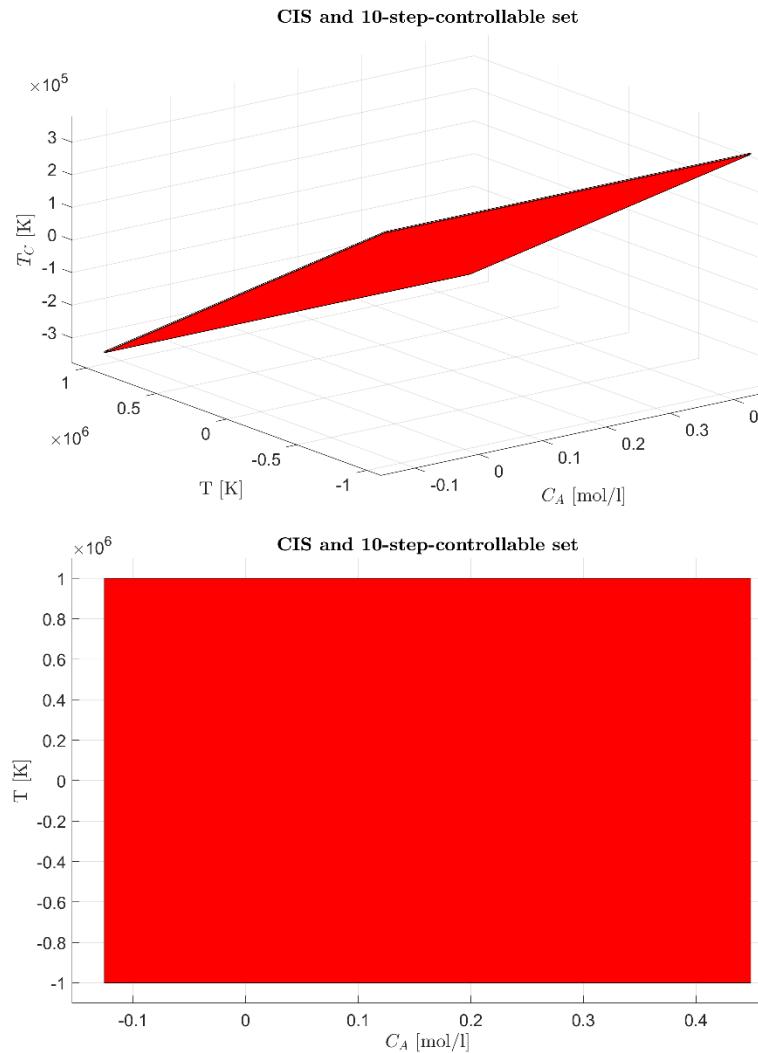
# Simulazione 7: Control invariant set



- $Q = 1$
- $R = 1$
- $N = 10$
- $T_s = 20 [\text{s}]$
- **Vincolo terminale di disuguaglianza**



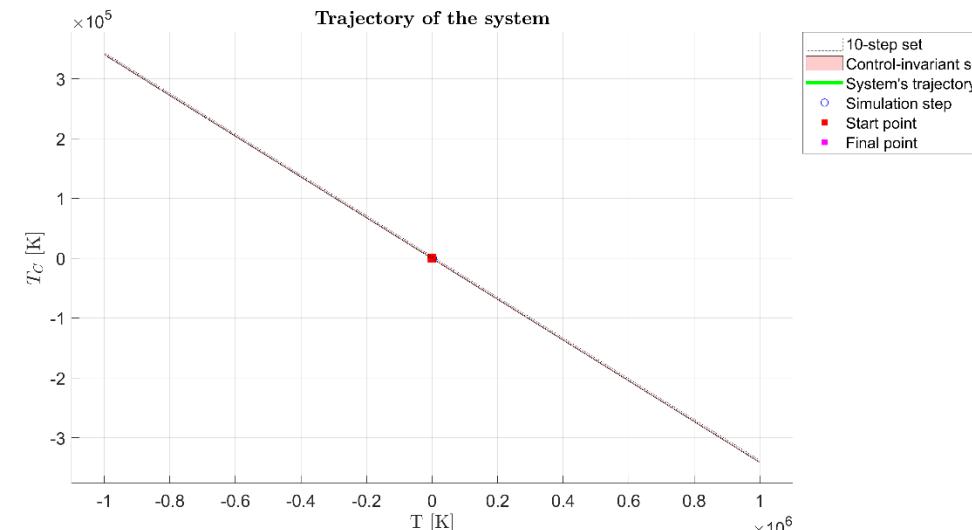
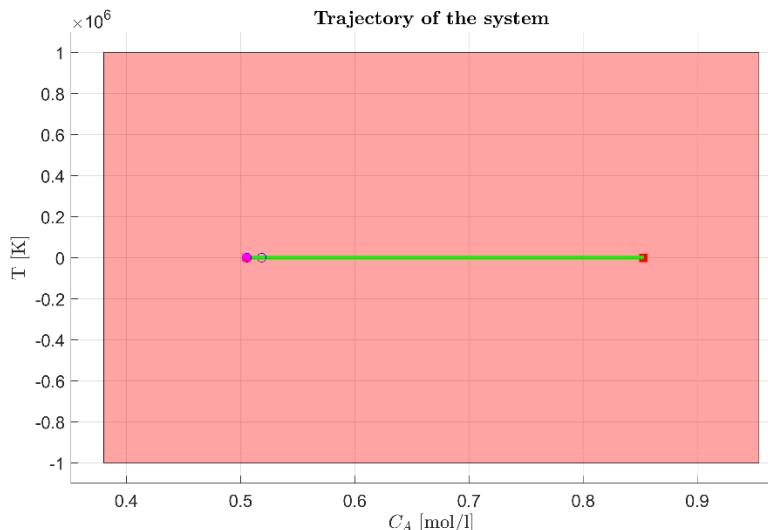
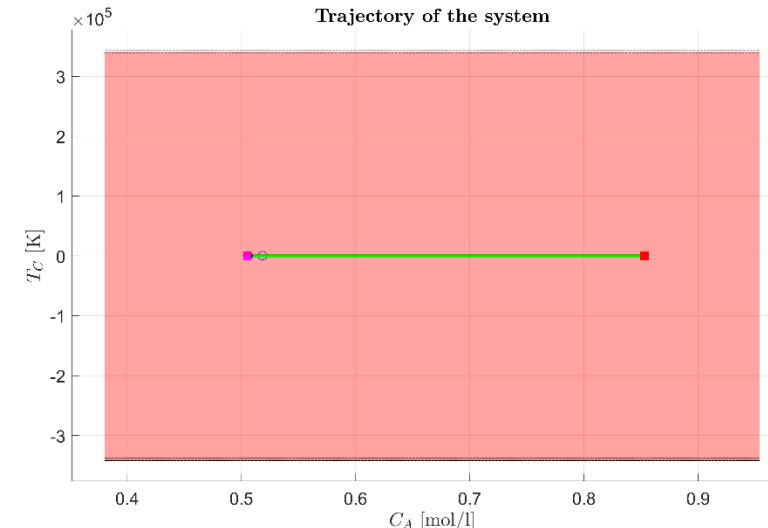
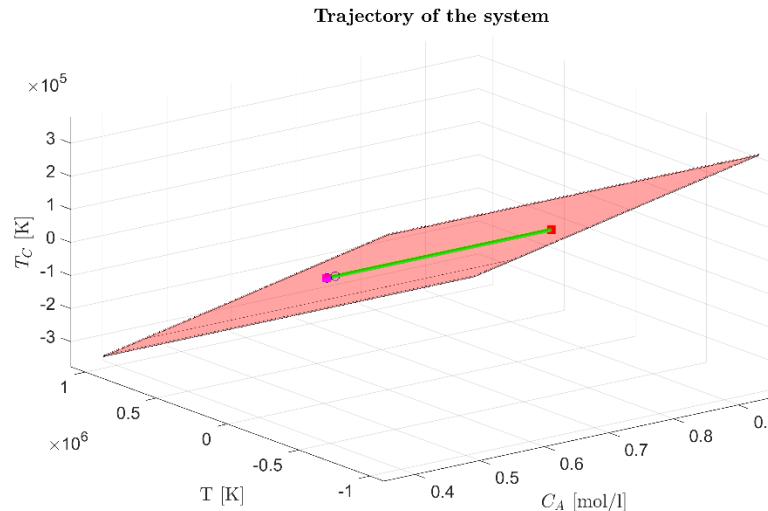
# Simulazione 7: N-Step controllable set



- $Q = 1$
- $R = 1$
- $N = 10$
- $T_s = 20 [\text{s}]$
- Volume CIS:  
 $1,8 \cdot 10^9 [\text{mol} \cdot \text{K}^2/\text{l}]$
- Volume 10-step CS:  
 $5 \cdot 10^9 [\text{mol} \cdot \text{K}^2/\text{l}]$
- Vincolo terminale di disuguaglianza



# Simulazione 7: Simulazione con MPC



$$Q = 1$$

$$R = 1$$

$$N = 10$$

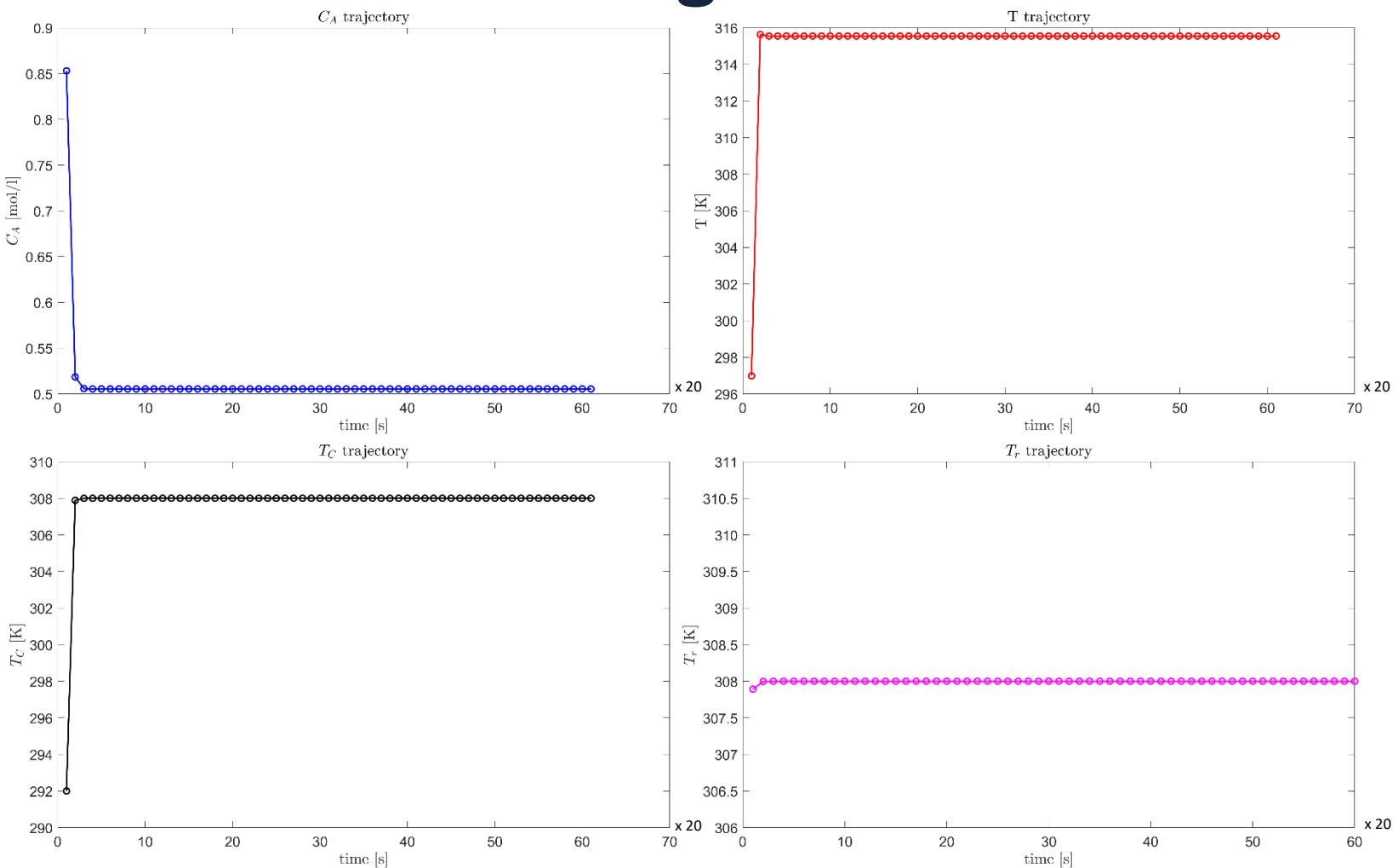
$$T_s = 20 \text{ [s]}$$

Vincolo terminale  
di diseguaglianza

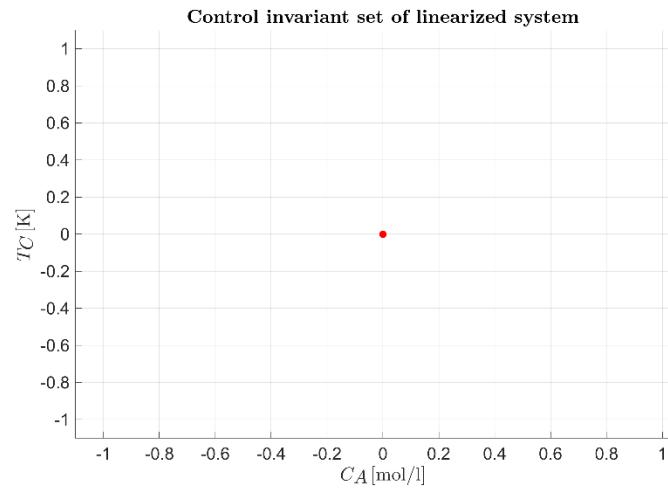
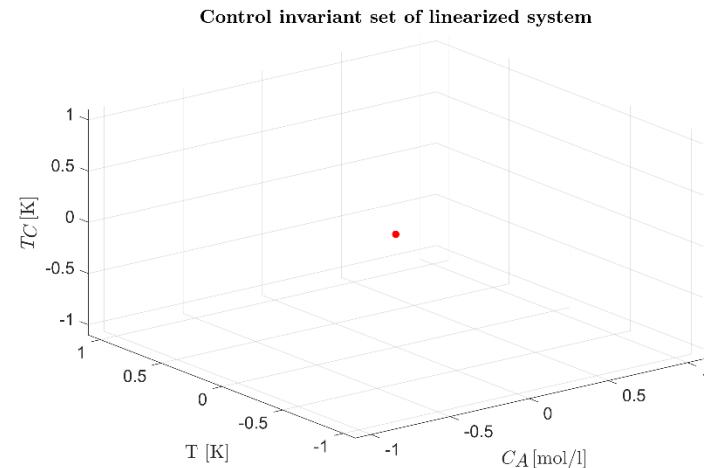


# Simulazione 7: Traiettoria dei singoli stati e ingresso

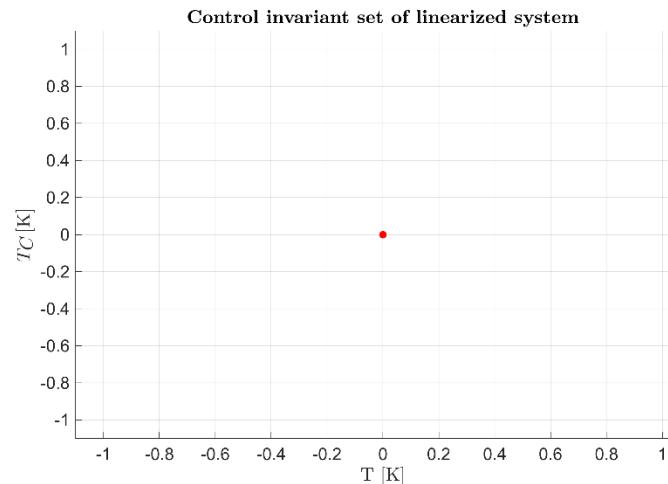
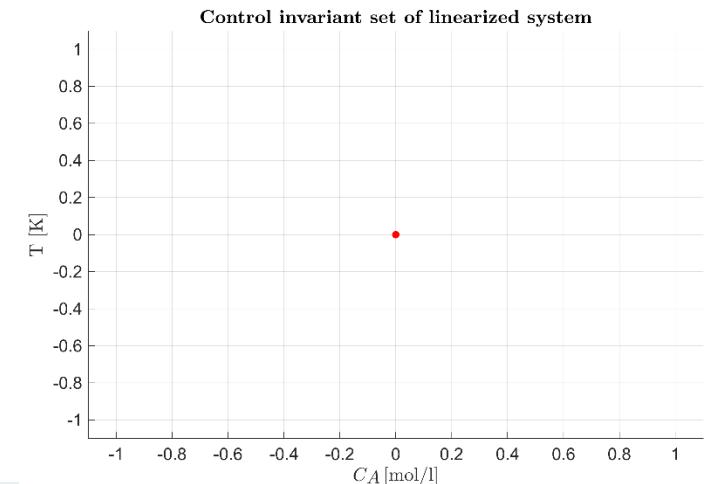
- $Q = 1$
- $R = 1$
- $N = 10$
- $T_s = 20$  [s]
- Vincolo terminale di diseguaglianza



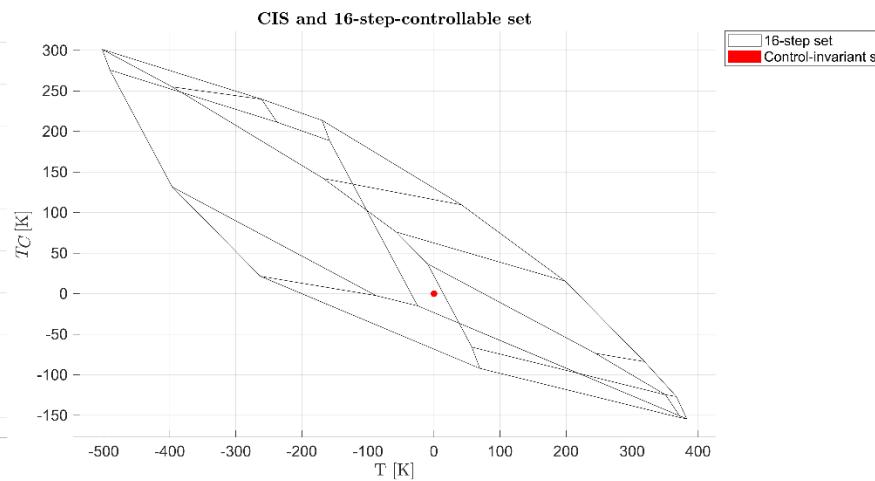
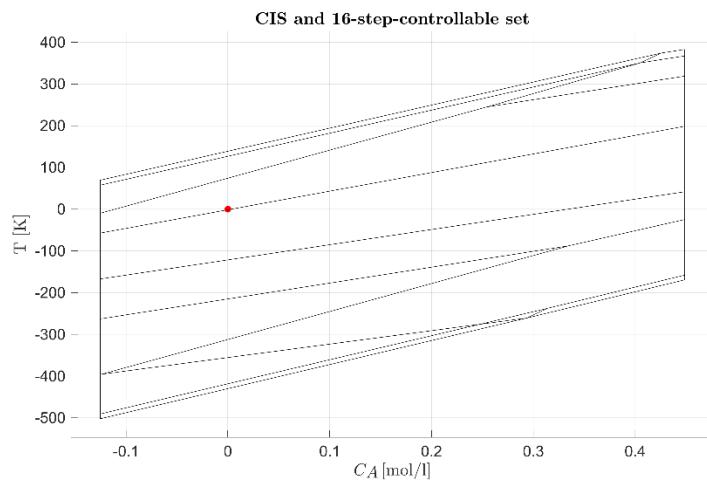
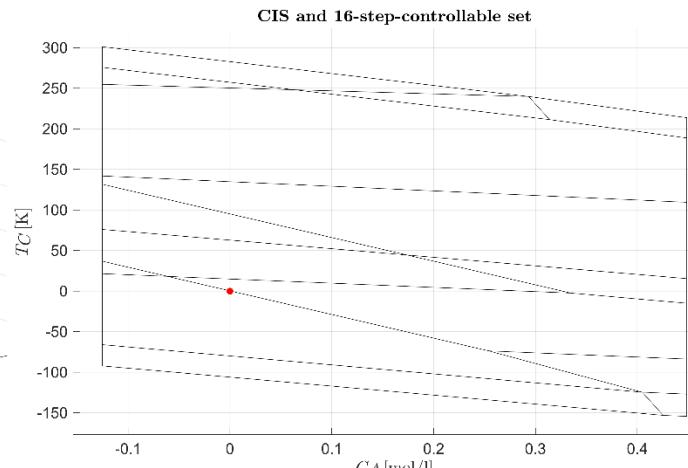
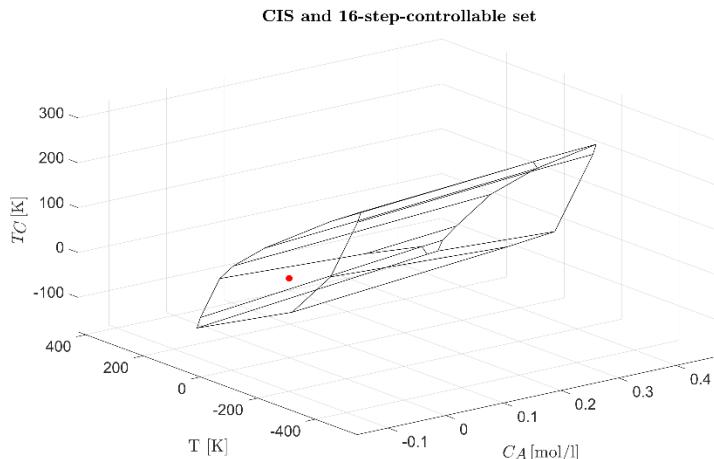
# Simulazione 8: Control invariant set



- $Q = 1$
- $R = 1$
- $N = 16$
- $T_s = 1 \text{ [s]}$
- Vincolo terminale  
di uguaglianza



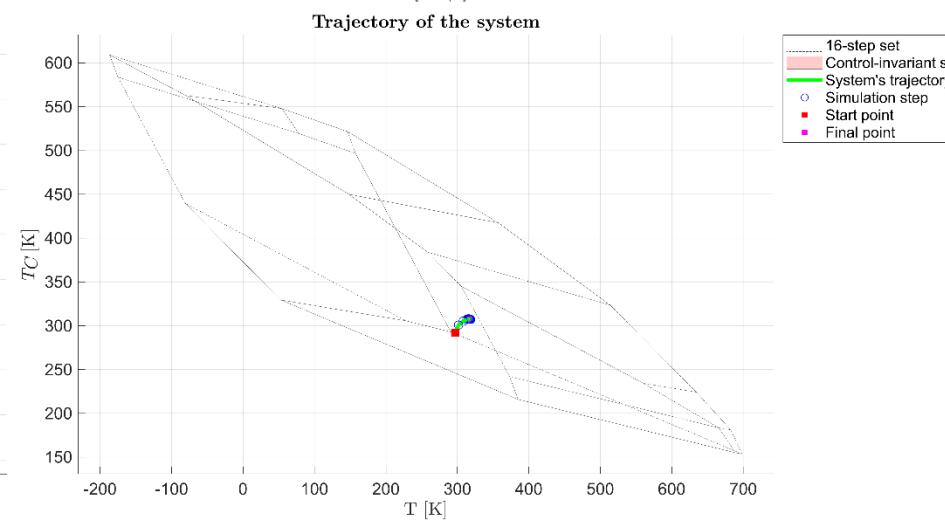
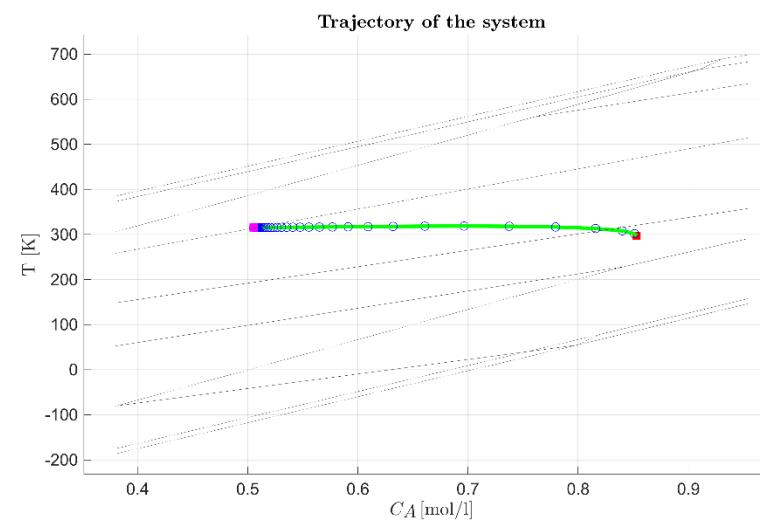
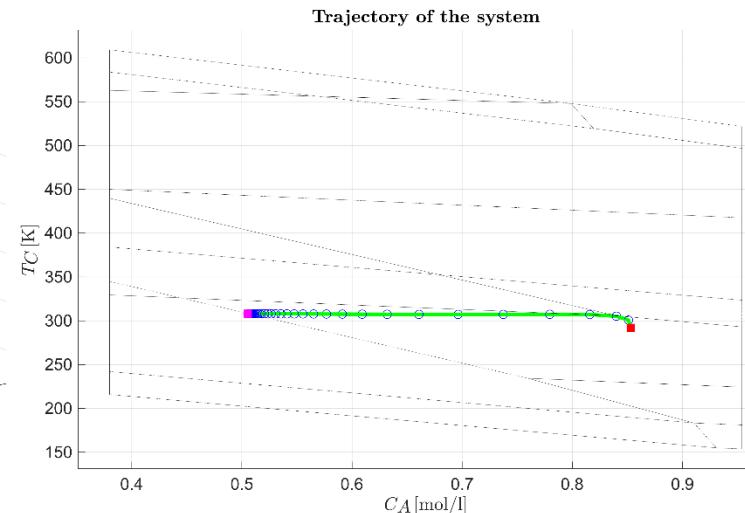
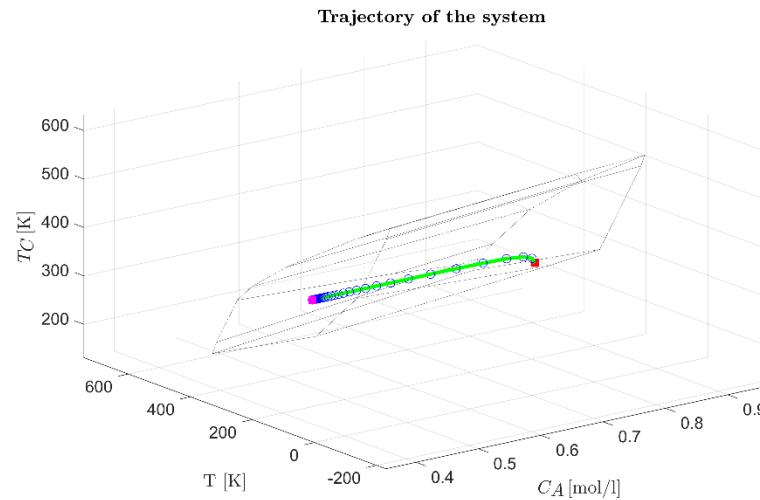
# Simulazione 8: N-Step controllable set



- $Q = 1$
- $R = 1$
- $N = 16$
- $T_s = 1 \text{ [s]}$
- Volume CIS:  
 $0 \text{ [mol}^*\text{K}^2/\text{l}]$
- Volume 10-step CS:  
 $35055 \text{ [mol}^*\text{K}^2/\text{l}]$
- Vincolo terminale di uguaglianza



# Simulazione 8: Simulazione con MPC

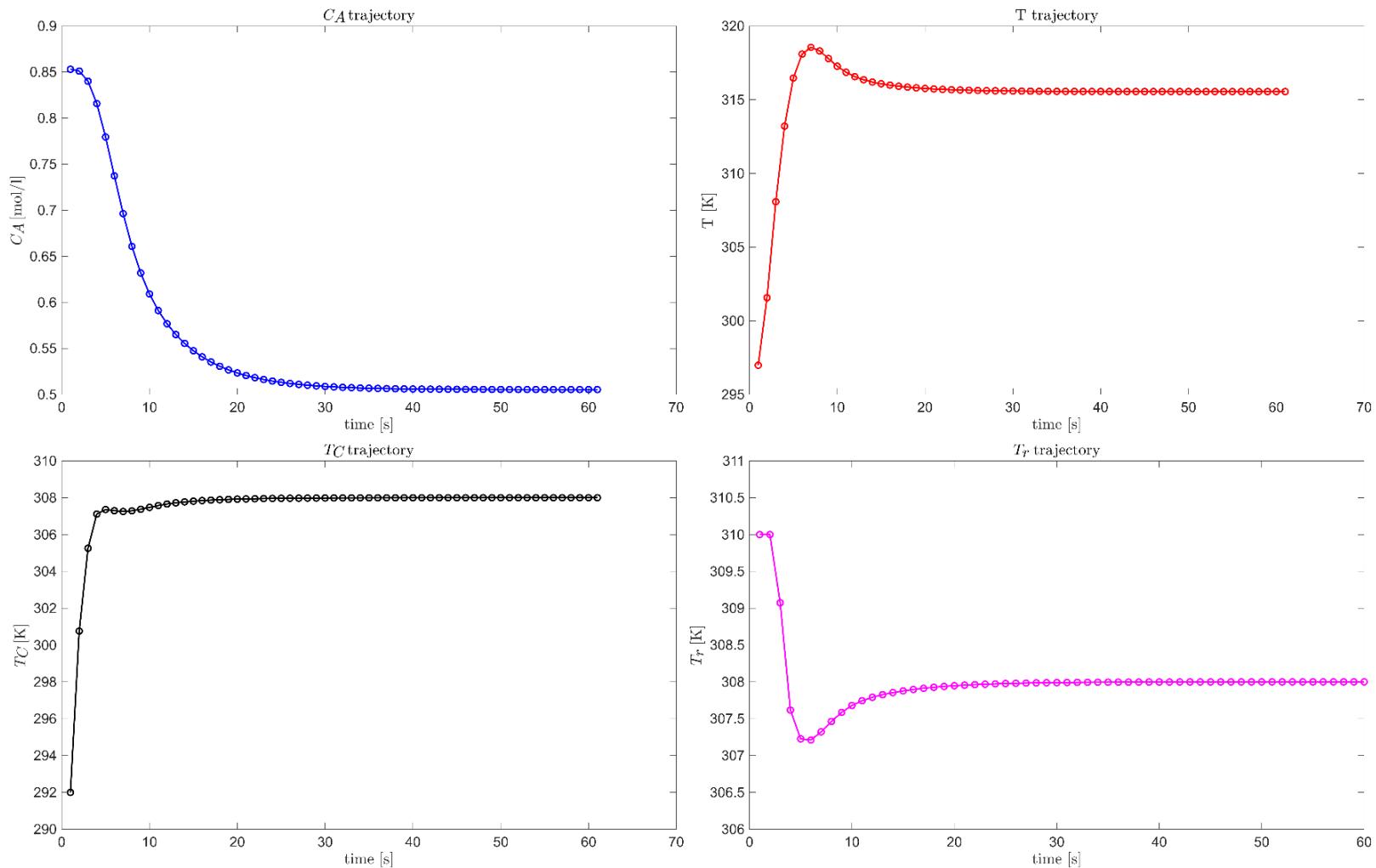


- $Q = 1$
- $R = 1$
- $N = 16$
- $T_s = 1 \text{ [s]}$
- Vincolo terminale di uguaglianza



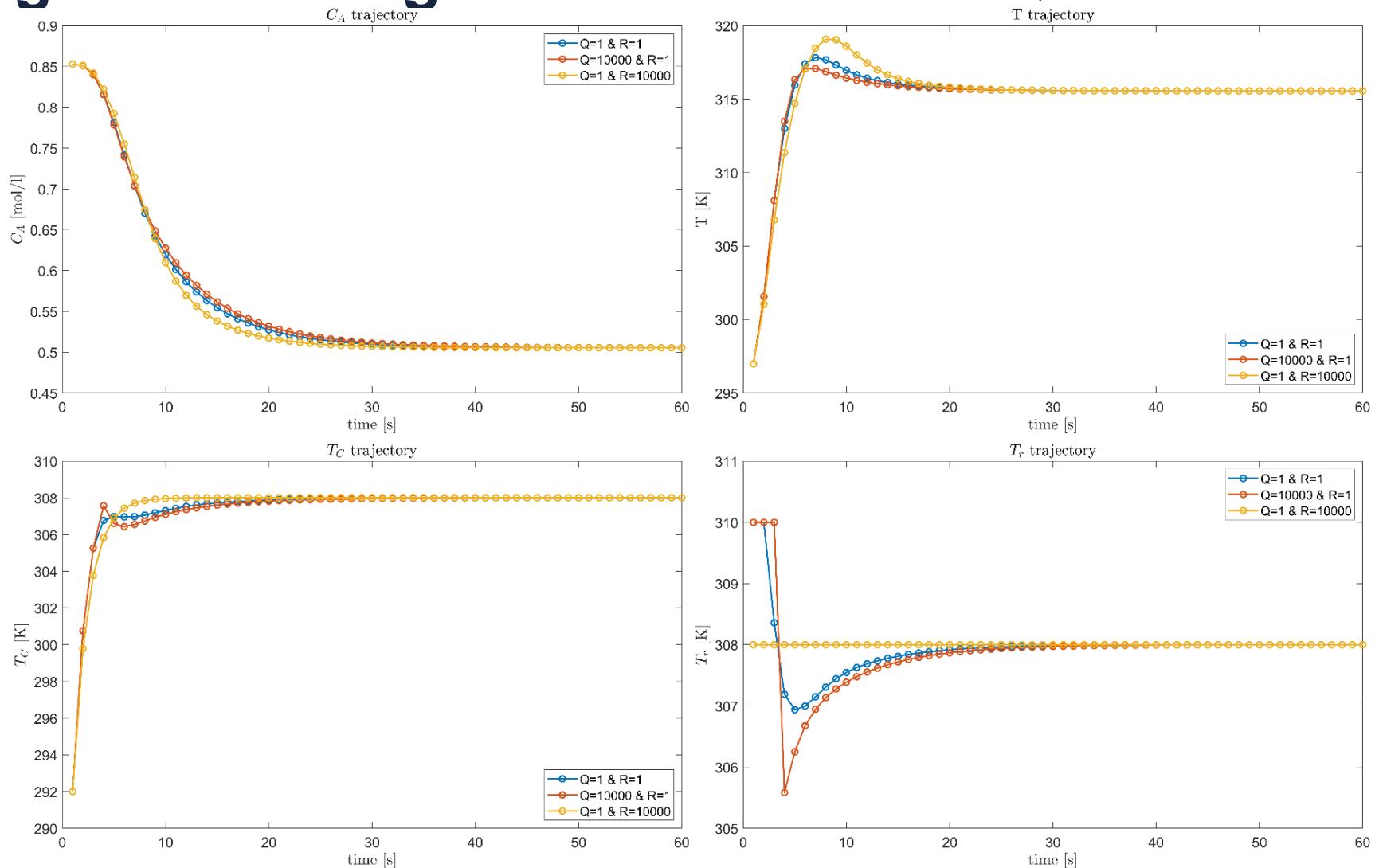
# Simulazione 8: Traiettoria dei singoli stati e ingresso

- $Q = 1$
- $R = 1$
- $N = 16$
- $T_s = 1 \text{ [s]}$
- Vincolo terminale di uguaglianza



# Traiettoria dei singoli stati e ingresso al variare di Q e R

- $N = 10$
- $T_s = 1 \text{ [s]}$
- Vincolo terminale di diseguaglianza





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