1. Q — Find the general solution of the recurrence equation $g_{n+2} = 3g_{n+1} - 2g_n$.

A — The recurrence can be written as a homogenous equation, $g_{n+2} - 3g_{n+1} + 2g_n = 0$. On summating:

$$\sum_{n=0}^{\infty} g_{n+2} - 3\sum_{n=0}^{\infty} g_{n+1} + 2\sum_{n=0}^{\infty} g_n = 0 \tag{1}$$

Let $f(x)$ represent the generating function for the sequence $\{r_n : n \geq 0\}$:

$$f(x) = \sum_{n=0}^{\infty} g_n x^n = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots$$

Multiplying (1) by $x^n$:

$$\sum_{n=0}^{\infty} g_{n+2} x^n - 3\sum_{n=0}^{\infty} g_{n+1} x^n + 2\sum_{n=0}^{\infty} g_n x^n = 0 \tag{2}$$

We observe that:

$$\begin{aligned}
\sum_{n=0}^{\infty} g_{n+2} x^n &= g_2 + g_3 x + g_4 x^2 + g_5 x^3 + \cdots \\
x^2 \sum_{n=0}^{\infty} g_{n+2} x^n &= g_2 x^2 + g_3 x^3 + g_4 x^4 + g_5 x^5 + \cdots \\
&= f(x) - g_0 - g_1 x \\
\therefore \sum_{n=0}^{\infty} g_{n+2} x^n &= \frac{f(x) - g_0 - g_1 x}{x^2}
\end{aligned}$$

Also that:

$$\begin{aligned}
\sum_{n=0}^{\infty} g_{n+1} x^n &= g_1 + g_2 x + g_3 x^2 + g_4 x^3 + \cdots \\
x \sum_{n=0}^{\infty} g_{n+1} x^n &= g_1 x + g_2 x^2 + g_3 x^3 + g_4 x^4 + \cdots \\
&= f(x) - g_0 \\
\therefore \sum_{n=0}^{\infty} g_{n+1} x^n &= \frac{f(x) - g_0}{x}
\end{aligned}$$

Therefore (2) can be re-written as:

$$\frac{f(x) - g_0 - g_1 x}{x^2} - 3\frac{f(x) - g_0}{x} + 2f(x) = 0 \tag{3}$$

Simplifying:

$$\frac{f(x) - g_0 - g_1 x - 3(f(x)x^2 - g_0 x^2) + 2f(x)x^2}{x^2} = 0 \tag{4}$$

$$\begin{aligned}
f(x)(1 - x^2) - g_0 - g_1 x + 3g_0 x^2 &= 0 \\
f(x)(1 - x^2) &= g_0 + g_1 x - 3g_0 x^2 \\
f(x) &= \frac{g_0 + g_1 x - 3g_0 x^2}{1 - x^2} \\
&= \frac{g_0 + g_1 x - 3g_0 x^2}{(1 + x)(1 - x)} \\
&= \frac{A}{1 + x} + \frac{B}{1 - x} \\
&= \frac{A(1 - x) + B(1 + x)}{1 - x^2} \\
&= \frac{(A + B) + x(B - A)}{1 - x^2} \\
\frac{g_0 + g_1 x - 3g_0 x^2}{1 - x^2} &= \frac{(A + B) + x(B - A)}{1 - x^2} \\
A + B &= g_0 \\
B - A &= g_1 \\
2B &= g_0 + g_1, B = \frac{g_0 + g_1}{2} \\
\text{and, } 2A &= g_0 - g_1, A = \frac{g_0 - g_1}{2} \\
f(x) &= \frac{g_0 - g_1}{2(1 + x)} + \frac{g_0 + g_1}{2(1 - x)}
\end{aligned}$$

Using the geometric series expansion of $\frac{1}{1-x}$ and $\frac{1}{1+x}$, we conclude that $f(x) = \frac{g_0 - g_1}{2} \sum (-1)^n x^n + \frac{g_0 + g_1}{2} \sum x^n$. In other words

$$\boxed{r_n = \frac{g_0 - g_1}{2}(-1)^n + \frac{g_0 + g_1}{2}}$$