Proof By Induction

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Reading Exercise. *Suppose that in a gossiping village of n people, every resident knows exactly one rumor. Any resident can call another one over the phone, at which point they can share every rumor they know. Show that if n is at least 4, then $2n-4$ phone calls is sufficient to achieve that every resident knows every rumor.*

Proof. By induction on n, the number of people.
Basis: Let C_n denote the number of calls required to achieve the goal. For $n = 4$, the 4 people (p_1, p_2, p_3, p_4) will need to make the following calls (in this order):
$p_{1,2}$ results in $p_1 = (1, 2)$ and $p_2 = (1, 2)$
$p_{3,4}$ results in $p_3 = (3, 4)$ and $p_4 = (3, 4)$
$p_{2,3}$ results in $p_2 = (1, 2, 3, 4)$ and $p_3 = (1, 2, 3, 4)$
$p_{1,4}$ results in $p_1 = (1, 2, 3, 4)$ and $p_4 = (1, 2, 3, 4)$
Therefore, $C_4 = 4$
Induction hypothesis: Assume For n people, the number of calls require to achieve the goal:

$$C_n = 2n - 4 \tag{1}$$

Induction: For $n+1$ people, pick one person, say p_x and have another, say p_y call p_x to learn their rumor. This is one call:

$$C_{1,2} = 1 \tag{2}$$

The remaining n people (excluding p_x) can learn all their rumor and that of p_x's in $2n - 4$ calls, using (1) above:

$$C_n = 2n - 4 \tag{3}$$

Now the only person remaining to be "caught up" is p_x, who can call p_y again. This will be one more call:

$$C_{1,x} = 1 \tag{4}$$

Adding the equations (2), (3) and (4), the total calls for $n + 1$ people:

$$C_{n+1} = 2n - 4 + 1 + 1 = 2n + 2 - 4 = 2(n + 1) - 4 \tag{5}$$

The truth of the proposition fon a specific n implies truth for $n + 1$. Hence the proposition must be true for all $n > 4$. $\square$