(1) Find the explicit formula for a_n, where $a_n = 2^n - 4a_{n-1} - 4a_{n-2}$, with $a_0 = 1$ and $a_1 = 2$.

(Solution)

If $a(n)$ be considered a generating function whose solution represents the n^{th} value (a_n), this recursion can be expressed as a non-homogeneous advancement operator equation on a as $(A^2 + 4A + 4)a = 2^n$ with $a(0) = 1$ and $a(1) = 2$. Or

$$(A + 2)^2 a = 2^n \tag{1}$$

Let's first consider the homogeneous part:

$$(A + 2)^2 a = 0 \tag{2}$$

Since (2) is a polynomial of degree 2, it must have a solution of the form of 2-parameters. One particular solution appears from the equality $(A + 2)a = 0$ or $Aa = -2a$. This is an advancement operator that multiplies the value of the function by -2 at each succession. Hence the particular solution is $a_1 = (-2)^n$. Let's try the other particular solution to be $a_2 = n(-2)^n$:

$$\begin{aligned}
(A+2)^2 a_2 &= (A+2)(A+2)a_2 \\
&= (A+2)(A+2)n(-2)^n \\
&= (A+2)\left[(n+1)(-2)^{n+1} + 2n(-2)^n\right] \\
&= (n+2)(-2)^{n+2} + 2(n+1)(-2)^{n+1} + 2(n+1)(-2)^{n+1} + 4n(-2)^n \\
&= (n+2)(-2)^{n+2} + 4(n+1)(-2)^{n+1} + 4n(-2)^n \\
&= (n+2)(-2)^{n+2} - 2(n+1)(-2)^{n+2} + 4n(-2)^n \\
&= -n(-2)^{n+2} + 4n(-2)^n \\
&= -4n(-2)^n + 4n(-2)^n \\
&= 0
\end{aligned}$$

Hence the homogeneous part of the solution to (1), $a_h = c_1(-2)^n + c_2 n(-2)^n$. Now for the non-homogeneous part, let's try a few particular solutions:

guess #1:

$$\begin{aligned}
a_{nh} &= c_3 2^n. \text{ Plugging it into (1):} \\
c_3 2^n &= 2^n - 4c_3 2^{n-1} - 4c_3 2^{n-2} \\
&= 2^n - 2c_3 2^n - c_3 2^n \\
&= 2^n - 3c_3 2^n \\
c_3 &= \frac{1}{4}
\end{aligned}$$

The explicit formula for a_n is given by combining the homogeneous and the non-homogeneous parts:

$$
\begin{aligned}
a_n &= c_1(-2)^n + c_2 n(-2)^n + \frac{2^n}{4} \\
1 &= c_1 + 0 + \frac{1}{4}, 2 = c_1(-2) + c_2(1)(-2) + \frac{2}{4}, \text{ using initial conditions.} \\
c_1 &= \frac{3}{4}, c_2 = -\frac{3}{2} \\
a_n &= (-2)^n(\frac{3}{4} - \frac{3n}{2}) + \frac{2^n}{4} \\
&= \boxed{(-2)^n(\frac{6-3n}{4}) + \frac{2^n}{4}}
\end{aligned}
$$

(2) If the letters of the word COMBINATORICS are arranged in a sequence, how many arrangements will have two consecutive letters the same?

(Solution)

Available letter combinations: CC, OO, M, B, II, N, A, T, R, S. The possible ways to arrage these letter combinations are 10!

(3) How many positive integers smaller than one million have digits that sum to 25?

(Solution)

The numbers smaller than one million can be written as $a_1a_2a_3a_4a_5a_6$ with $0 \le a_i \le 9$. The possibilities of each a_i can be represented by the generating function:

$A(x) = 1+x^2+x^3\cdots+x^9$. Since all a_i are stringed together, the possibilities of their sums is $A(x)\cdot A(x)\cdot A(x)\cdot A(x)\cdot A(x)\cdot A(x)$

$$
\begin{aligned}
&= (1+x^2+x^3\cdots+x^9)^6 \\
&= (\frac{1-x^{10}}{1-x})^6 \\
&= \frac{1}{(1-x)^6} - \frac{x^{16}}{(1-x)^6}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{1-x} &= 1+x+x^2+x^3+\dots \\
\frac{d}{dx}\frac{1}{1-x} = \frac{-1}{(1-x)^2} &= 1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+\dots \\
\frac{d}{dx}\frac{-1}{(1-x)^2} = \frac{2}{(1-x)^3} &= 2+6x+12x^2+20x^3+30x^4+42x^5+56x^6+72x^7+\dots \\
\frac{d}{dx}\frac{2}{(1-x)^3} = \frac{-6}{(1-x)^4} &= 6+24x+60x^2+120x^3+210x^4+336x^5+504x^6+\dots \\
\frac{d}{dx}\frac{-6}{(1-x)^4} = \frac{24}{(1-x)^5} &= 24+120x+360x^2+840x^3+1680x^4+3024x^5+\dots \\
\frac{d}{dx}\frac{24}{(1-x)^5} = \frac{-120}{(1-x)^6} &= 120+720x+2520x^2+6720x^3+15120x^4+\dots \\
\frac{1}{(1-x)^6} &= -(1+6x+21x^2+56x^3+126x^4+\dots)
\end{aligned}
$$

(7) How many ways can we distribute n different objects to 4 different boxes if at least one object is distributed to box 1, and an odd number of objects is distributed to box 4.

(Solution)

The exponential generating functions for finding the ways n different objects can be distributed among 4 boxes can be written as:

$a_2, a_3 = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, no restrictions. This is also the infinity expansion of e^x.

$a_1 = \sum_{n=1}^{\infty} \frac{x^n}{n!}$, since at least one object is required. This is equivalent to $e^x - 1$.

$a_4 = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$, only odd # of objects.

$$\begin{aligned} 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &= e^x \\ -1 + x - \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &= -e^{-x} \\ a_4 = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots &= \frac{e^x - e^{-x}}{2} \end{aligned}$$

The # of ways for the desired distribution, N is the coefficient of the $\frac{x^n}{n!}$ term in the expansion of the product of these 4 terms:

$$\begin{aligned} S(n) &= e^{2x} \cdot (e^x - 1) \cdot \frac{e^x - e^{-x}}{2} \\ &= (e^{3x} - 1) \cdot \frac{e^x - e^{-x}}{2} \\ &= \frac{e^{4x} - e^{2x} - e^x + e^{-x}}{2} \\ &= \frac{1}{2}\left[\sum \frac{(4x)^n}{n!} - \sum \frac{(2x)^n}{n!} - \sum \frac{(x)^n}{n!} + \sum \frac{(-x)^n}{n!}\right] \\ &= \frac{1}{2}\left[\sum \frac{(4)^n (x)^n}{n!} - \sum \frac{(2)^n (x)^n}{n!} - \sum \frac{(x)^n}{n!} + \sum \frac{(-1)^n (x)^n}{n!}\right] \end{aligned}$$

Therefore $[a_n]S(n) = \boxed{\frac{1}{2}\left[(4)^n - (2)^n - 1 + (-1)^n\right]}$

(8) How many binary sequences of length n are there (consisting 0's and 1's) with no consecutive 1's, except in the rightmost two positions? Explicit formula is required.

(Solution)

We see by observation that the number of desired sequences of length $1, a_1 = 2$, the sequences $\{0\}, \{1\}$ and $a_2 = 4$, the sequences $\{00\}, \{01\}, \{10\}, \{11\}$. Beyond this, a sequence (length > 2) may start with a 0 and using all valid sequences of length $(n-1)$, i.e., a_{n-1}. Alternatively, the sequence may start with a 11 (which would be invalid since length > 2 and this will indicate that there are consecutive 1's that are not in the end). The only other possibility is when the sequence starts with a 10 and uses the valid sequences of length $(n-2)$, i.e. a_{n-2}. We see that a recurrence relation appears

$$a_n = a_{n-1} + a_{n-2} \text{ (for } n > 2 \text{, with } a_1 = 2, a_2 = 4)$$

$$a_{n+2} = a_{n+1} + a_n \text{ (for } n \geq 0)$$

Let $f(x)$ be the a generating function for the sequence $\{a_n : n > 2\}$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

Multiplying the original recurrence relation by x^n and summating

$$
\begin{aligned}
\sum_{n=0}^{\infty} a_{n+2}x^n &= \sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n \\
\frac{1}{x^2}\sum_{n=0}^{\infty} a_{n+2}x^{n+2} &= \frac{1}{x}\sum_{n=0}^{\infty} a_{n+1}x^{n+1} + \sum_{n=0}^{\infty} a_n x^n \\
\frac{1}{x^2}(f(x) - a_0 - a_1 x) &= \frac{1}{x}(f(x) - a_0) + f(x) \\
\frac{1}{x^2}(f(x) - 1 - 2x) &= \frac{1}{x}(f(x) - 1) + f(x) \\
f(x) - 1 - 2x &= xf(x) - x + x^2 f(x) \\
f(x)(1 - x - x^2) &= 1 + x \\
f(x) &= \frac{1+x}{1-x-x^2} \\
f(x) &= \frac{1+x}{(x-r_+)(x-r_-)} \\
\text{where } r_\pm &= \frac{-1 \mp \sqrt{5}}{2}, (r_+ + r_-) = -1 \\
f(x) &= \frac{A}{x-r_+} + \frac{B}{x-r_-} \\
&= \frac{A(x-r_-) + B(x-r_+)}{(x-r_+)(x-r_-)} \\
1 + x &= (A+B)x - (Ar_- + Br_+) \\
A &= -(r_+ + 1), B = 1 - r_- \\
f(x) &= \frac{-(r_+ + 1)}{x - r_+} + \frac{1 - r_-}{x - r_-} \\
&= \frac{r_+ + 1}{r_+ - x} + \frac{r_- - 1}{r_- - x} \\
&= (1 + \frac{1}{r_+})\frac{1}{1 - \frac{x}{r_+}} + (1 - \frac{1}{r_-})\frac{1}{1 - \frac{x}{r_-}} \\
&= (1 + \frac{1}{r_+})\sum(\frac{1}{r_+})^n x^n + (1 - \frac{1}{r_-})\sum(\frac{1}{r_-})^n x^n
\end{aligned}
$$

The coefficient of the n^{th} term,

$$\boxed{a_n = (1 + \frac{1}{r_+})(\frac{1}{r_+})^n + (1 - \frac{1}{r_-})(\frac{1}{r_-})^n}$$

where $r_\pm = \frac{-1 \mp \sqrt{5}}{2}$.