

1. Q — Sophie's coefficients $G(n, k)$ for $n \geq 0$, k arbitrary integer are defined as follows:

$$G(n, k) = \begin{cases} 1, & \text{if } n = k = 0 \\ 0, & \text{if } k < -n \text{ or } k > n \\ G(n-1, k-1) + G(n-1, k) + G(n-1, k+1), & \text{for } n > 0 \text{ and } -n \leq k \leq n \end{cases}$$

Compute the coefficients $G(4, 0)$ and $G(4, 1)$.

A — The recursive definition of G is used to calculate the values as

follows:

$$\mathbf{G(4, 0)} = G(3, -1) + G(3, 0) + G(3, 1) = 5 + 5 + 3 = \mathbf{13}$$

$$\begin{aligned} G(3, -1) &= G(2, -2) + G(2, -1) + G(2, 0) &= 1 + 2 + 2 = 5 \\ G(3, 0) &= G(2, -1) + G(2, 0) + G(2, 1) &= 2 + 2 + 1 = 5 \\ G(3, 1) &= G(2, 0) + G(2, 1) + G(2, 2) &= 2 + 1 + 0 = 3 \end{aligned}$$

$$\begin{aligned} G(2, -2) &= G(1, -3) + G(1, -2) + G(1, -1) &= 1 \\ G(1, -3) & &= 0 \\ G(1, -2) & &= 0 \end{aligned}$$

$$\begin{aligned} G(1, -1) &= G(0, -2) + G(0, -1) + G(0, 0) &= 1 \\ G(0, -2) & &= 0 \\ G(0, -1) & &= 0 \\ G(0, 0) & &= 1 \end{aligned}$$

$$\begin{aligned} G(2, -1) &= G(1, -2) + G(1, -1) + G(1, 0) &= 0 + 1 + 1 = 2 \\ G(1, -2) & &= 0 \\ G(1, -1) & &= 1 \\ G(1, 0) &= G(0, -1) + G(0, 0) + G(0, 1) &= 0 + 1 + 0 = 1 \\ G(0, 1) & &= 0 \end{aligned}$$

$$\begin{aligned} G(2, 0) &= G(1, -1) + G(1, 0) + G(1, 1) &= 1 + 1 + 0 = 2 \\ G(1, 1) &= G(0, 0) + G(0, 1) + G(0, 2) &= 0 + 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} G(2, 1) &= G(1, 0) + G(1, 1) + G(1, 2) &= 1 + 0 + 0 = 1 \\ G(1, 2) & &= 0 \end{aligned}$$

$$\begin{aligned} G(2, 2) &= G(1, 1) + G(1, 2) + G(1, 3) &= 0 + 0 + 0 = 0 \\ G(1, 3) & &= 0 \end{aligned}$$

$$G(4, 1) = G(3, 0) + G(3, 1) + G(3, 2) = 5 + 3 + 1 = \mathbf{9}$$

$$G(3, 2) = G(2, 1) + G(2, 2) + G(2, 3) = 1 + 0 + 0 = 1$$

$$G(2, 3) = 0$$