1. Q — For each generating function below, give a closed form for the n^{th} term of its associated sequence.

A — Selected answers:

a)

$$(1+x)^{10} = \Sigma_{n=0}^{\infty} \binom{10}{n} x^n \qquad \text{(Newton's Binomial Theorem)}$$

Therefore coefficient of the n^{th} term:

$$\boxed{a_n = \binom{10}{n}}$$

b)

$$\begin{aligned}
\frac{1}{1-x^4} &= \frac{1}{(1-x)(1+x)(1+x^2)} \\
&= \frac{A}{(1-x)} + \frac{B}{(1+x)} + \frac{Cx+D}{(1+x^2)} \\
&= \frac{A(1+x)(1+x^2) + B(1-x)(1+x^2) + (Cx+D)(1-x^2)}{1-x^4} \quad \text{(for all } x\text{)}
\end{aligned}$$

for $x = 1, 4A = 1 \therefore \boxed{A = \frac{1}{4}}$

for $x = -1, 4B = 1 \therefore \boxed{B = \frac{1}{4}}$

substituting:

$(1+x)(1+x^2) + (1-x)(1+x^2) + 4(Cx+D)(1-x^2) = 4$

$x^3(-4C) + x^2(2-4D) + x(4C) + 4D = 2$

$\therefore \boxed{C = 0}, \boxed{D = \frac{1}{2}}$

$$\frac{1}{1-x^4} = \frac{1}{4(1-x)} + \frac{1}{4(1+x)} + \frac{1}{2(1+x^2)}$$

We know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \tag{1}$$

Substituting $-x$ for x,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \tag{2}$$

Differentiating (2) w.r.t x,

$$\frac{d}{dx}(\frac{1}{1+x}) = \frac{-1}{(1+x)^2}$$
$$= -1 + 2x - 3x^2 + \dots$$

$$\therefore \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \dots \tag{3}$$

$$\begin{aligned}
\frac{1}{1-x^4} &= \frac{1}{4}(1+x+x^2+x^3+\dots) + \frac{1}{4}(1-x+x^2-x^3+\dots) + \frac{1}{2}(1-2x+3x^2-\dots) \\
&= \frac{1}{4}(2+2x^2+2x^4+\dots) + \frac{1}{2}(1-2x+3x^2-\dots) \\
&= \frac{1}{2}(1+x^2+x^4+\dots) + \frac{1}{2}(1-2x+3x^2-\dots) \\
&= \frac{1}{2}(2-2x+4x^2-4x^3+6x^4-\dots) \\
&= 1-x+2x^2-2x^3+3x^4-\dots \\
&= \Sigma_{n=0}^{\infty}(-1)^n(n+\frac{1+(-1)^n}{2})x^n \\
&= \Sigma_{n=0}^{\infty}(-1)^n\frac{2n+1+(-1)^n}{2}x^n
\end{aligned}$$

Therefore coefficient of the n^{th} term:

$$\boxed{a_n = (-1)^n\frac{2n+1+(-1)^n}{2}}$$

e)

$$\begin{aligned}
&\frac{1+x^2-x^4}{1-x} \\
&= \frac{1}{1-x} + \frac{x^2}{1-x} - \frac{x^4}{1-x} \\
&= (1+x+x^2+x^3+\dots) + x^2(1+x+x^2+x^3+\dots) - x^4(1+x+x^2+x^3+\dots) \\
&= \Sigma^{\infty}x^n + \Sigma^{\infty}x^{n+2} - \Sigma^{\infty}x^{n+4}
\end{aligned}$$

Therefore coefficient of the n^{th} term:

$$\boxed{a_n = ?}$$