The question is posed whether we repaint two six-sided dice such that the distribution of getting the results is still the same, i.e., the number of ways for getting $(2, 3, 4, \ldots)$ does not change. All six sides of the dice must have a non-zero integer. Repetetion is allowed.

We start by representing the outcomes of a dice roll in terms of a **Generating Function**. In a standard dice there is one way to roll a 1, one way to roll a 2, and so on. This fact can be represented as a Generating Function, G_1 that generates the sequence of all possibilities when rolling a dice:

$G_1 = 1 \cdot x^1 + 1 \cdot x^2 + 1 \cdot x^3 + 1 \cdot x^4 + 1 \cdot x^5 + 1 \cdot x^6$ — one way to roll each result. The powers of $x, 1, 2, 3 \ldots$ are a notation to represent the face (result) that is rolled.

According to the similarities between algebra of polynomials and that of the Generating Functions, it follows that the representation of rolling two six-faced dice is:

$$(x + x^2 + x^3 + x^4 + x^5 + x^6) \cdot (x + x^2 + x^3 + x^4 + x^5 + x^6)$$

$$= (x + x^2 + x^3 + x^4 + x^5 + x^6)^2 \tag{1}$$

Hence the present question can be restated as if there is a way to rewrite the expression (1) as a product of two polynomials P and Q, according to the **rules**:

1. The sum of coefficients of the terms in P and Q is 6, each.
2. The exponents of x are all non-zero integers.

It can bee seen that voilating #1 above would result in a dice that is **not** 6-faced, and #2 is necessary to make Generating Functions work.

We start by factorizing (1) as a product of P and Q as follows:

PQ

$$\begin{aligned} &= x^2(1 + x + x^2 + x^3 + x^4 + x^5)^2 && \text{factoring out } x \\ &= x^2(1 + x)^2(1 + x^2 + x^4)^2 && (\text{factoring out } 1 + x \\ & && \text{since } x = -1 \text{ is a root} \\ & && 1 - 1 + 1 - 1 + 1 - 1 = 0, \\ & && \text{and using long-division}). \end{aligned}$$

Now to factorize $(1 + x^2 + x^4)$ we observe that it is a quadratic in x^2, $[1 + (x^2) + (x^2)^2]$. To solve the following for x^2:

$$1 + (x^2) + (x^2)^2 = 0 \tag{2}$$

$x^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$

Solving for complex roots,

$x = \pm\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ are the 4 roots to (2). Lets call them $r_1, \bar{r_1}, r_2, \bar{r_2}$. These are **complex conjugate pairs**. The sums & products of the complex conjugates can be calculated as:

$$\boxed{\begin{aligned} &r_1 + \bar{r_1} \\ &\quad = (\frac{1}{2} + \frac{\sqrt{3}}{2}i) + (\frac{1}{2} - \frac{\sqrt{3}}{2}i) \\ &\quad = 1 \\ &r_1\bar{r_1} \\ &\quad = (\frac{1}{2} + \frac{\sqrt{3}}{2}i)(\frac{1}{2} - \frac{\sqrt{3}}{2}i) \\ &\quad = \frac{1}{4} - \frac{3}{4}(-1) \\ &\quad = \frac{1}{4} + \frac{3}{4} \\ &\quad = 1 \end{aligned}}$$

Similar results can be found for r_2 and $\bar{r_2}$. Therefore (2) can be written as:

$$\begin{aligned} &1 + x^2 + x^4 \\ &\quad = (x - r_1)(x - \bar{r_1})(x - r_2)(x - \bar{r_2}) \\ &\quad = [x^2 - x(r_1 + \bar{r_1}) + r_1\bar{r_1}][x^2 - x(r_2 + \bar{r_2}) + r_2\bar{r_2}] \\ &\quad = (x^2 - x + 1)(x^2 + x + 1) && \text{(using properties of conjugates).} \end{aligned}$$

To continue factorizinbg (1):

$$\begin{aligned} &(P)(Q) \\ &\quad = x^2(1+x)^2(1-x+x^2)^2(1+x+x^2)^2 \quad \text{using (2) above} \end{aligned}$$

We want to break this up as a product of P, Q so they each represent a six-faced dice. The number of coefficients carried by each of these terms are as follows:

$(x)(x)$	$(1+x)(1+x)$	$(1-x+x^2)(1-x+x^2)$	$(1+x+x^2)(1+x+x^2)$
$1+1$	$2+2$	$1+1$	$3+3$

Since we want each of P, Q to contain 6 as a sum of coefficients, we need the terms with sums of 2 and 3 in each P and Q:

P	Q
$[(1+x)(1+x+x^2)\ldots]$	$[(1+x)(1+x+x^2)\ldots]$

On expansion in their current form, both P and Q would generate a **constant term** (one without a power of x) which would amount to a face with no number on it. This is against the rules of the desired dice (see above). To account for this this we must use an x term in each P and Q:

P	Q
$[(x)(1+x)(1+x+x^2)\ldots]$	$[(x)(1+x)(1+x+x^2)\ldots]$

To use the remaining terms we have two choices. We could use one in each P and Q, but this will result in two dice identical to the original ones. The other choice is to use them both in either P or Q:

P	Q
$[(x)(1+x)(1+x+x^2)(1-x+x^2)]$	$[(x)(1+x)(1+x+x^2)]$

Upon expansion, the product PQ looks like:

$(x^8+x^6+x^5+x^4+x^3+x)(x^4+2x^3+2x^2+x)$,

both with non-zero coefficients, and the sum of coefficients being 6.

We now have the desired pair of non-standard dice painted:

$[8,6,5,4,3,1]$ and $[4,3,3,2,2,1]$.

Since this pair is derived from the original pair of standard dice, the distribution of getting the results $2,3,4\ldots$ is the same.