1. Q — Solve $a_{n+1} = 2a_n - 2, n \geq 1, a_0 = 5$ using Generating Functions.

A — This is a non-homogeneous recurrence function, $2a_n - a_{n+1} = 2$. Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$ represent the generating function of the desired sequence $\{a_n : n \geq 0\}$. On multiplying both sides of the recurrence by x^n and summating, we obtain

$$2\Sigma a_n x^n - \Sigma a_{n+1} x^n = 2\Sigma x^n \tag{1}$$

Since $\Sigma a_{n+1} x^n$ has a_{n+1} as the coefficient on x_n and is missing a_0, it can be written as $\frac{f(x)-a_0}{x} = \frac{f(x)-5}{x}$, using initial value. Thus (1) is re-written as

$$2f(x) - \frac{f(x)-5}{x} = 2\Sigma x^n = 2\frac{1}{1-x} \tag{2}$$

Simplifying:

$\frac{2xf(x)-f(x)+5}{x} = \frac{2}{1-x}$

$2xf(x) - f(x) + 5 = \frac{2x}{1-x}$

$f(x)(2x-1) = \frac{2x}{1-x} - 5$

$$f(x) = \frac{5-7x}{(1-2x)(1-x)} \tag{3}$$

Using partial factorization, $f(x) = \frac{A}{1-2x} + \frac{B}{1-x} = \frac{A(1-x)+B(1-2x)}{(1-2x)(1-x)}$. I.e., $5 - 7x = A(1-x) + B(1-2x)$ for all x. Using $x = 1$, $B = 2$ and using $x = \frac{1}{2}$, $A = 3$. Thus (3) is re-written as

$$\begin{aligned} f(x) &= \frac{3}{1-2x} + \frac{2}{1-x} \\ &= 3\Sigma 2^n x^n + 2\Sigma x^n \qquad \text{(using Geometric Series)} \end{aligned}$$

We now observe that the coefficient on x^n in $f(x)$ is:

$\boxed{a_n = 3 \cdot 2^n + 2}$

which is the desired solution.