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Math 501 Homework (§5.3 Continuous Functions on Intervals)

Problem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective function. Suppose that both f and f^{-1} are continuous on $\mathbb{R}$. Prove that the image of an open bounded interval is open bounded.

Solution. Let $x_1, x_2 \in \mathbb{R}$. If $f^{-1}(x_1) = f^{-1}(x_2)$ we can see that $f(f^{-1}(x_1)) = f(f^{-1}(x_2))$, i.e. $x_1 = x_2$ (since f is bijective). In other words, f^{-1} **is injective**.

Now let $I := (x_1, x_2)$ be open-bounded. The case where $x_1 = x_2$ is trivial since it implies $I = \emptyset$, the empty set, whose image $f(\emptyset) = \emptyset$, which is **open bounded** and we are done.

When $x_1 \neq x_2$, we define a closed-bounded interval, $I' := [x_1, x_2]$. Since f is continuous everywhere, according to **Theorem 5.3.9** the image $f(I')$ is a closed bounded interval, say $[m, M]$.

Since f^{-1} is injective, $[n, M] = [f(x_1), f(x_2)]$ is the image of I'. Dropping the bounds, (n, M) is the image of I, which is **open bounded**. $\square$