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Math 501 Homework (§2.1)

Problem 9. Let $K := \{s + t\sqrt{2} : s, t \in \mathbb{Q}\}$. Show that K satisfies the following:

Solution. a) $x_1, x_2 \in K \implies x_1 + x_2 \in K, x_1 x_2 \in K$:

Let's prove **closure of addition** on $\mathbb{Q}$: if $s, t \in \mathbb{Q}$ there exist $s_n, s_d, t_n, t_d \in \mathbb{R}$ such that $s = s_n/s_d, t = t_n/t_d$. Therefore $s + t$

$$
\begin{aligned}
&= \frac{s_n}{s_d} + \frac{t_n}{t_d} \\
&= \frac{s_n t_d}{s_d t_d} + \frac{t_n s_d}{s_d t_d} \\
&= \frac{1}{s_d t_d}(s_n t_d + t_n s_d) && (s_d t_d, s_n t_d, t_n s_d \in \mathbb{R}; \text{Distributive}) \\
&= \frac{r}{s_d t_d} && (r = s_n t_d + t_n s_d) \\
&\in \mathbb{Q}
\end{aligned}
$$

Similarly for **closure on multiplication** st

$$
\begin{aligned}
&= \frac{s_n}{s_d}\frac{t_n}{t_d} \\
&= \frac{s_n s_d}{t_n t_d} && (\text{closure on multiplication in } \mathbb{R}) \\
&\in \mathbb{Q}
\end{aligned}
$$

We can now write $x_1 + x_2$

$$
\begin{aligned}
&= s_1 + t_1\sqrt{2} + s_2 + t_2\sqrt{2} : s, t \in \mathbb{Q} \\
&= (s_1 + s_2) + (t_1 + t_2)\sqrt{2} && (\mathbb{Q} \subset \mathbb{R}; \text{Associativity of addition}) \\
&= u + v\sqrt{2}, u, v \in \mathbb{Q} && (\text{closure in } \mathbb{Q} \text{ from above}) \\
&\in K
\end{aligned}
$$

Similarly x_1x_2

$$\begin{aligned}
&= (s_1 + t_1\sqrt{2})(s_2 + t_2\sqrt{2}) \\
&= s_1(s_2 + t_2\sqrt{2}) + t_1\sqrt{2}(s_2 + t_2\sqrt{2}) && \text{(Distributive on } \mathbb{R}) \\
&= s_1s_2 + s_1t_2\sqrt{2} + t_1\sqrt{2}(s_2 + t_2\sqrt{2}) && \text{(Distributive on } \mathbb{R}) \\
&= s_1s_2 + (s_1t_2 + t_1s_2 + t_1t_2)\sqrt{2} \\
&= u + v\sqrt{2}, u, v \in \mathbb{Q} && \text{(closures in } \mathbb{Q} \text{ from above)}
\end{aligned}$$

□

Solution. b) If $x \neq 0, x \in K$ then $1/x \in K$:

Let $x = s + t\sqrt{2} : s, t \in \mathbb{Q}$ We can say that $1/x$

$$\begin{aligned}
&= \frac{1}{s + t\sqrt{2}} \\
&= \frac{s - t\sqrt{2}}{(s + t\sqrt{2})(s - t\sqrt{2})} \\
&= \frac{s - t\sqrt{2}}{s^2 - 2t^2} \\
&= (s - t\sqrt{2})\frac{1}{s^2 - 2t^2} \\
&= \frac{s}{s^2 - 2t^2} + \frac{-t\sqrt{2}}{s^2 - 2t^2} && \text{(Distributive on } \mathbb{R}) \\
&= \frac{s}{s^2 - 2t^2} + \frac{-t}{s^2 - 2t^2}\sqrt{2} && \text{(Conjugative on } \mathbb{R}) \\
&= s\frac{1}{s \cdot s - 2 \cdot t \cdot t} + (-t)\frac{1}{s \cdot s - 2 \cdot t \cdot t}\sqrt{2} \\
&= u + v\sqrt{2} : u, v, \in \mathbb{Q} && \text{(Closures on } \mathbb{Q}) \\
&\in R
\end{aligned}$$

□