Rohit Wason
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Math 501 Homework (§2.3)

Problem 9. Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is bounded and $sup(A \cup B) = sup\{sup\ A, sup\ B\}$.

Solution. For any $z \in A \cup B$ there are 2 cases: Since A is upper bounded by $sup\ A$ for every $v < sup\ A$, there exists $a' \in A$ such that $v < a'$. It is also the case that $a' \in A \cup B$, so for cases that sup A is an upper bound for $A \cup B$, $sup\ A \cup B = sup\ A$. Similarly for cases that sup B is an upper bound for $A \cup B$, $sup\ A \cup B = sup\ B$.

For all $z \in A \cup B$ two *non exclusive* cases exist:

1. $z \in A$:
   Since A is bounded by $sup\ A$ and $inf\ A$, $inf\ A \leq z \leq sup\ A$. In other words, $A \cup B$ is **bounded** and $sup\ A \cup B = sup\ A$ (from above).

2. $z \in B$:
   Since B is bounded by $sup\ B$ and $inf\ B$, $inf\ B \leq z \leq sup\ B$. In other words, $A \cup B$ is **bounded** and $sup\ A \cup B = sup\ B$ (from above).

In any case, $A \cup B$ is **bounded** and has a supremum as $sup\ A$ or $sup\ B$. Since $sup\ A, sup\ B$ could be distinct, we'll need to pick the greater of the two:

$$sup\ A \cup B = sup\ \{sup\ A, sup\ B\}$$

□