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Math 501 Homework (§3.5 Cauchy Sequences)

Problem 1. Let the sequence (x_n) be defined recursively as $x_1 = 1$, and $x_{n+1} = 1 + 1/x_n$ for $n > 1$. Prove that (x_n) converges.

Solution. By Triangle Inequality and definition of the sequence

$$\begin{aligned}
|x_{n+1} - x_n| &\leq |1 + \frac{1}{x_n}| - |x_n| \\
&= \frac{x_n + 1}{x_n} - x_n \text{ (since } x_n > 0, \forall n \in \mathbb{N}) \\
&= \frac{x_n + 1}{1 + \frac{1}{x_{n-1}}} - x_n \\
&= \frac{(x_n + 1)x_{n-1}}{x_{n-1} + 1} - x_n \\
&= \frac{x_n x_{n-1} + x_{n-1}}{x_{n-1} + 1} - x_n \\
&= \frac{x_n x_{n-1} + x_{n-1} - x_n x_{n-1} - x_n}{x_{n-1} + 1} \\
&= \frac{x_{n-1} - x_n}{x_{n-1} + 1} \\
&\leq \frac{|x_{n-1} - x_n|}{x_{n-1} + 1} \text{ (by Triangle Inequality)}
\end{aligned}$$

Moreover, we can show by Induction that $1 \leq x_n \leq 2$. Or

$$\begin{aligned}
2 \leq x_n + 1 \leq 3 \\
\implies \frac{1}{x_n + 1} \leq \frac{1}{2}
\end{aligned}$$

Using the two inequalities above, we have that the ratio

$$\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|} \leq \frac{1}{2}$$

I.e., (x_n) is a *contractive* sequence, hence a Cauchy Sequence, hence convergent. $\square$