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Math 501 Homework (§5.4, Uniform Continuity)

Problem 2. Show that $f(x) := 1/x^2$ is uniformally continuous on $A := [1, \infty)$, but that it is not so on $B := (0, \infty)$.

Solution. Let $x, u \in A$. The ratio $|(f(x) - f(u))/(x - u)|$

$$\begin{aligned} = |\frac{1/x^2 - 1/u^2}{x - u}| &= |\frac{u^2 - x^2}{x^2 u^2 (x - u)}| \\ &= |\frac{x + u}{x^2 u^2}| \\ &= \frac{1}{x}\frac{1}{u^2} + \frac{1}{x^2 u} \end{aligned}$$

Now since $x \geq 1$, $1/x \leq 1$ and $1/x^2 \leq 1$. The same can be established for u. Therefore the above ratio

$$|\frac{f(x) - f(u)}{x - u}| \leq 2,$$

which makes f satisfy the **Lipschitz condition**. This also makes it **uniformally continuous** on A.

In the case of $B := (0, \infty)$, if we fix $\delta > 0$ and set points $x_\delta = \delta/2 \in B$ and $u_\delta = \delta/4 \in B$ we have that $|x_\delta - u_\delta| = \delta/2 < \delta$.
However the quantity $|f(x_\delta) - f(u_\delta)| = |4/\delta^2 - 16/\delta^2| = 12/\delta^2$.
This means there exists $\epsilon_0 = 12/\delta^2$ such that $|f(x_\delta) - f(u_\delta)| \geq \epsilon_0$ which is one of the **Nonuniform Continuity Criteria**. Hence f is not uniformally continuous on B. $\square$