Rohit Wason
Spring 2021

Math 501 Homework (§6.1, Derivatives)

Problem 12. If $r > 0$ is rational and $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) := \begin{cases} x^r \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Determine those values of r for which $f'(0)$ exists.

Solution. First consider $x^s, s = m/n \in \mathbb{Q}$.
When $x < 0$, $\lim_{x \to 0^-} x^s = 0$ for n odd, but does not exist for n even (**even roots** of negative numbers aren't real). Therefore

$$\lim_{x \to 0} x^s = \begin{cases} 0, & n \text{ odd} \\ DNE, & n \text{ even.} \end{cases} \tag{12.1}$$

Now for $f'(0)$ to exist, $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$ must exist (definition of Derivative).

For $x \neq 0$, using $-1 \leq \sin(1/x) \leq 1$ we can write $-|x^{r-1}| \leq f(x)/x \leq |x^{r-1}|$, or

$$-|g(x)| \leq \frac{f(x)}{x} \leq |g(x)|, \text{for } g(x) := x^{r-1} \tag{12.2}$$

Case I: $0 < r < 1$.
Let $s = 1 - r > 0$, or $g(x) = x^s, s = m/n \in \mathbb{Q}$. We know from (12.1) that $\lim_{x \to 0} g(x) = 0$ is defined **only when n is odd.**

Case II: $r \geq 1$.
Let $s = r - 1 > 0$, and $g(x) = x^s, s = m/n \in \mathbb{Q}$. As above, $\lim_{x \to 0} g(x) = 0$ is defined **only when n is odd.**

Hence all $r = m/n$, n odd, lead to $\lim_{x \to 0} |g(x)| = -\lim_{x \to 0} |g(x)|$. Using **Squeeze Theorem** and (12.2) this is a sufficient condition for $\lim_{x \to 0} \dfrac{f(x)}{x} = 0$ hence the existence of $f'(0)$. $\square$