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# Math 501 Homework (§3.6 Properly Divergent Sequences)

**Problem 1.** Prove that $x_n = \sqrt{n^2 - n - 1}$ is properly divergent.

**Solution.** Let's first prove, by induction, that ultimately, $x_n \geq n - 1$. The **base step** is clear at $n = 2$. For the induction step, assume the claim is true for a given $n$, and hence

$$\begin{aligned} x_n = \sqrt{n^2 - n - 1} &\geq n - 1 \\ x_n^2 = n^2 - n - 1 &\geq n^2 - 2n + 1 \\ x_n^2 + 2n &\geq n^2 + 1 \end{aligned}$$

Squaring the subsequent term $x_{n+1}^2$ we get

$$\begin{aligned} &= (n+1)^2 - (n+1) - 1 \\ &= n^2 + 2n - n - 1 \\ &= x_n^2 + 2n \\ &\geq n^2 + 1 \text{ (from above)} \end{aligned}$$

In other words $x_{n+1} \geq \sqrt{n^2 + 1}$ which is $> (n+1) - 1$. Therefore $x_n \geq n - 1$ for $n \geq 2$.

By offsetting $n$ and letting $y_n := (n - 1)$ we have that $\lim(y_n) = +\infty$. Since $x_n \geq y_n$, $\lim(x_n) = +\infty$. I.e., $\sqrt{n^2 - n - 1}$ is properly divergent. □