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Math 501 Homework (§3.5)

Problem 15. Approximate the root $0 < r < 1$ of equation $x^3 - 5x + 1 = 0$ within 10^{-4}.

Solution. We first rewrite the equation as $x = \frac{1}{5}(x^3 + 1)$. Then we pick an arbitrary $0 < x_1 < 1$ and define

$$x_{n+1} := \frac{1}{5}(x_n^3 + 1) \tag{15.1}$$

Let's use Induction to show that $0 < x_n < 1$ for all n. With the base clear, and assuming a particular $0 < x_n < 1$, we can see that $1 < x_n^3 + 1 < 2$ and $1/5 < (x_n^3 + 1) < 2/5$. Thus using (15.1), since $0 < x_{n+1} = x_n^3 + 1$

$$0 < x_n < 1 \tag{15.2}$$

Now let's look at the difference in subsequent terms of x_n

$$\begin{aligned}
&\Big|x_{n+2} - x_{n+1}\Big| \\
&= \Big|\frac{1}{5}(x_{n+1}^3 + 1) - \frac{1}{5}(x_n^3 + 1)\Big| = \frac{1}{5}\Big|x_{n+1}^3 - x_n^3\Big| \\
&= \frac{1}{5}\Big|x_{n+1}^2 + x_{n+1}x_n + x_n^2\Big|\Big|x_{n+1} - x_n\Big| \\
&\leq \frac{3}{5}\Big|x_{n+1} - x_n\Big| && \text{(since } 0 < x_n < 1\text{)}
\end{aligned}$$

This implies that $X = (x_n)$ is a **contractive sequence** with $C = \frac{3}{5}$ and that $\lim X = r$ exists. Substituting r in (15.1) we get $r = \frac{1}{5}(r^3 + 1)$ or $r^3 - 5r + 1 = 0$. Hence r is a root of the given equation. Now to approximate the value of r we can

1. Pick $x_1 = 0.5$
2. Calculate $x_2 = 0.225$
3. Calculate $x_3 = 0.202278125$
4. ...

In order to know how many such iterations we need, we use Corollary 3.5.10(i) from the book which states that for a contractive sequence with limit, r and contractive constant, $C = \frac{3}{5}$ here,

$$\begin{aligned} \Big|r - x_n\Big| &\leq \frac{C^{n-1}}{1-C}|x_2 - x_1| \\ &\leq \frac{3^{n-1}}{5^{n-2}}(\frac{0.275}{2}) \\ &\leq \frac{3^{n-1}}{5^{n-2}}(0.1375) \end{aligned}$$

This inequality tells us where to stop, depending on the precision we seek in the value of r. E.g., for the required precision we know that $|r - x_n| \leq 10^{-4}$. This means the following equation applies

$$\begin{aligned} \frac{3^{n-1}}{5^{n-2}} &= 10^{-4} \\ (\frac{3}{5})^{n-1} &= 5 \times 10^{-4} \\ (n-1)\ln\frac{3}{5} &= \ln 5 - 4\ln 10 \\ (n-1) - 0.51082562376 &= -7.60090245954 \\ n &\approx 16 \end{aligned}$$

We can iterate manually, or use a computer program to calculate the desired approximation of r:

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x=0.5; for(n in (2:16)) x=(x^3+1)/5; print(x)
0.2016397
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