Rohit Wason
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Math 501 Homework (§5.4 Uniform Continuity)

Problem 1. Let I be closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $\epsilon > 0$ then there exists a Lipschitz function $g_\epsilon : I \to \mathbb{R}$ such that $|f(x) - g_\epsilon(x)| < \epsilon$ for all $x \in I$

Solution. Take g_ϵ, a piecewise linear function on $I \to \mathbb{R}$. By definition, if we divide I in a finite number of disjoint intervals $I_1, I_2, \ldots, I_m$, g_ϵ is a linear function on each interval I_k. Let the slopes of these linear functions be m_k respectively and $M = \max(m_k)$.
Since M is the largest slope of g_ϵ between any two points $x, u \in I, x \neq u$ we have

$$|\frac{g_\epsilon(x) - g_\epsilon(u)}{x - u}| \leq M$$

We thus prove that **all piecewise linear functions** are Lipschitz functions. Combining this result with **Theorem 5.4.13** we prove the existence of a Lipschitz function $g_\epsilon : I \to \mathbb{R}$ such that

$$|f(x) - g_\epsilon(x)| < \epsilon$$

□