Theorem (3.1.5)**.** *Let $X = (x_n)$ be a sequence of real numbers, and let $x \in \mathbb{R}$. The following statements are equivalent.*

(a) *X converges to x.*

(b) *For every $\epsilon > 0$, there exists a natural number K such that for all $n > K$, the terms x_n satisfy $|x_n - x| < \epsilon$.*

(c) *For every $\epsilon > 0$, there exists a natural number K such that for all $n > K$, the terms $x - \epsilon < x_n < x + \epsilon$.*

(d) *For every ϵ-neigborhood $V_\epsilon(x)$ or x, there exists a natural number K such that for all $n \geq K$, the terms x_n belong to $V_\epsilon(x)$.*

Theorem (3.1.9)**.** *Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then the m-tail $X_m = \{x_{m+n} : n \in \mathbb{N}\}$ of X converges if and only if X converges. In this case, $\lim X_m = \lim X$.*

Theorem (3.1.10)**.** *Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim (a_n) = 0$ and if for some $C > 0$ and some $m \in \mathbb{N}$ we have $|x_n - x| \leq Ca_n$ for all $n \geq m$, then it follows that $\lim (x_n) = x$.*

Theorem (3.2.2)**.** *A convergent sequence of real numbers is bounded.*

Theorem (3.2.3)**.** *(a)* *Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y, respectively, and let $c \in \mathbb{R}$. Then the sequences $X + Y, X - Y, X\dot{Y}$, and cX converge to $x + y, x - y, xy$ and cx, respectively.*

(b) *If $X = (x_n)$ converges to x and $Z = (z_n)$ is a sequence of nonzero real numbers that converges to $z \neq 0$, then the quotient sequence X/Z converges to x/z.*

Theorem (3.2.4)**.** *If $X = (x_n)$ is a convergent sequence of real numbers and if $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim (x_n) \geq 0$.*

Theorem (3.2.5)**.** *If $X = (x_n)$ and $Y = (y_n)$ are convergent sequences of real numbers and if $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $\lim(x_n) \leq \lim(y_n)$.*

Theorem (3.2.6)**.** *If $X = (x_n)$ is a convergent sequence and if $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq \lim(x_n) \leq b$.*

Theorem (3.2.7)**.** ***Squeeze Theorem*** *Suppose that $X = (x_n), Y = (y_n)$ and $Z = (z_n)$ and sequences of real numbers such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and that $\lim(x_n) = \lim(z_n)$. Then $Y = (y_n)$ is convergent and*

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

Theorem (3.2.9)**.** *If $x = \lim(x_n)$, then $|x| = \lim(|x_n|)$.*

Theorem (3.2.10)**.** *If* $\lim(x_n) = x \geq 0$, *then* $\lim(\sqrt{x_n}) = \sqrt{x}$.

Theorem (3.2.11)**.** *Let* $X = (x_n)$ *be a sequence of positive real numbers such that* $L := \lim(x_{n+1}/x_n)$ *exists. If* $L < 1$, *then* (x_n) *converges and* $\lim(x_n) = 0$.

Theorem (3.3.2)**.** ***Monotone Convergence Theorem*** *A monotone sequence of real numbers is convergent if and only if it is bounded. Further*

(a) If $X = (x_n)$ *is a bounded increasing sequence, then*

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

(b) If $Y = (y_n)$ *is a bounded decreasing sequence, then*

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$$

Theorem (3.4.2)**.** *If a sequence* $X = (x_n)$ *of real numbers converges to a real number* x, *then any subsequence* $X' = (x_{n_k})$ *of* X *also converges to* x.

Theorem (3.4.4)**.** *Let* $X = (x_n)$ *be a sequence of real numbers. Then following are equivalent*

(i) The sequence $X = (x_n)$ *does not converge to* $x \in \mathbb{R}$

(ii) There exists an $\epsilon_0 > 0$ *such that for any* $k \in \mathbb{N}$, *there exists* $n_k \in \mathbb{N}$ *such that* $n_k \geq k$ *and* $|x_{n_k} - x| \geq \epsilon_0$

(iii) There exists an $\epsilon_0 > 0$ *and a subsequence* $X' = (x_{n_k})$ *of* X *such that* $|x_{n_k} - x| \geq \epsilon_0$ *for all* $k \in \mathbb{N}$

Theorem (3.4.5)**.** ***Divergence Criteria*** *If a sequence* $X = (x_n)$ *of real numbers has either of the following prooperties, then* X *is divergent*

(i) X *has two convergent subsequences* $X' = (x_{n_k})$ *and* $X'' = (x_{r_k})$ *whose limits are not equal.*

(ii) X *is unbounded.*

Theorem (3.4.7)**.** ***Monotone Subsequence Theorem*** *If* $X = (x_n)$ *is a sequence of real numbers, then there's a subsequence of* X *that is monotone.*

Theorem (3.4.8)**.** ***Bolzano-Weierstraß Theorem*** *A bounded sequence of real numbers has a convergent subsequence.*

Theorem (3.4.9)**.** *Let* $X = (x_n)$ *be a bounded sequence of real numbers and let* $x \in \mathbb{R}$ *have the property that every convergent subsequence of* X *converges to* x. *Then the sequence* X *converges to* x.

Theorem (3.4.11)**.** ***Limit Superior and Limit Inferior***

If $X = (x_n)$ *is a bounded sequence of real numbers, then the following statements for a real number* x^* *are equivalent* *(a)* $x^* = \limsup(x_n)$.

(b) *If* $\epsilon > 0$, *there are at most a finite number of* $n \in \mathbb{N}$ *such that* $x^* + \epsilon < x_n$, *but an infinite number of* $n \in \mathbb{N}$ *such that* $x^* - \epsilon < x_n$.

(c) *If* $u_m = \sup\{x_n : n \geq m\}$, *then* $x^* = \inf\{u_m : m \in \mathbb{N}\} = \lim(u_m)$.

(d) *If* S *is the set of subsequential limits of* (x_n), *then* $x^* = \sup S$.

Similarly, The following statements are equivalent

(a) $x^* = \liminf(x_n)$.

(b) *If* $\epsilon > 0$, *there are at most a finite number of* $n \in \mathbb{N}$ *such that* $x^* + \epsilon > x_n$, *but an infinite number of* $n \in \mathbb{N}$ *such that* $x^* - \epsilon > x_n$.

(c) *If* $u_m = \inf\{x_n : n \leq m\}$, *then* $x^* = \sup\{u_m : m \in \mathbb{N}\} = \lim(u_m)$.

(d) *If* S *is the set of subsequential limits of* (x_n), *then* $x^* = \inf S$.

Theorem (3.4.12)**.** *A bounded sequence* (x_n) *is convergent if and only if* $\limsup(x_n) = \liminf(x_n)$.

Definition (3.5.1)**.** *A sequence of real numbers,* $X = (x_n)$ *is said to be a* ***Cauchy sequence*** *if for every* $\epsilon > 0$ *there exists as natural number* $H(\epsilon)$ *such that for all natural numbers* $n, m \geq H(\epsilon)$, *the terms* x_n, x_m *satisfy* $|x_n - x_m| < \epsilon$.

Lemma (3.5.3)**.** *If* $X = (x_n)$ *is a convergent sequence of real numbers, then* X *is a Cauchy sequence.*

Lemma (3.5.4)**.** *A Cauchy sequence of real numbers is bounded.*

Theorem (3.5.5)**.** *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Definition (3.5.7)**.** *We say that a sequence of real numbers* $X = (x_n)$ *is* ***contractive*** *if there exists a constant* $C, 0 < C < 1$, *such that*

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$$

Theorem (3.5.8)**.** *Every contractive sequence is a Cauchy sequence, and hence is convergent.*

Corollary (3.5.10)**.** *If* $X = (x_n)$ *is a contractive sequence with constant* $C, 0 < C < 1$, *and if* $x^* := \lim X$ *then*

(i) $|x^* - x_n| \leq \frac{C^{n-1}}{1-C}|x_2 - x_1|$,

(ii) $|x^* - x_n| \leq \frac{C}{1-C}|x_n - x_{n-1}|$.

Definition (3.6.1)**.** *Let* $X = (x_n)$ *be a sequence of real numbers.*

(i) *We say that* (x_n) ***tends to*** $+\infty$, *and write* $\lim(x_n) = +\infty$, *if for every* $\alpha \in \mathbb{R}$ *there exists a natural number* $K(\alpha)$ *such that if* $n > K(\alpha)$, *then* $x_n > \alpha$.

(ii) *We say that* (x_n) ***tends to*** $-\infty$, *and write* $\lim(x_n) = -\infty$, *if for every* $\beta \in \mathbb{R}$ *there exists a natural number* $K(\beta)$ *such that if* $n > K(\beta)$, *then* $x_n < \beta$.

We say that (x_n) *is* ***properly divergent*** *if either* $\lim(x_n) = +\infty$ *or* $\lim(x_n) = -\infty$.

Theorem (3.6.3)**.** *A monotone sequence of real numbers is properly divergent if and only if it is unbounded*

(a) *If* (x_n) *is an unbounded increasing sequence, then* $\lim(x_n) = +\infty$.

(b) *If* (x_n) *is an unbounded decreasing sequence, then* $\lim(x_n) = -\infty$.

Theorem (3.6.4)**.** *Let* (x_n) *and* (y_n) *be two sequence of real numbers and suppose for all* $n \in \mathbb{N}$

$$x_n \leq y_n$$

(a) *If* $\lim(x_n) = +\infty$, *then* $\lim(y_n) = +\infty$.

(b) *If* $\lim(y_n) = -\infty$, *then* $\lim(x_n) = -\infty$.

Theorem (3.6.5)**.** *Let* (x_n) *and* (y_n) *be two sequences of positive real numbers and suppose that for some* $L \in \mathbb{R}, L > 0$ *we have*

$$\lim(x_n/y_n) = L$$

Then $\lim(x_n) = +\infty$ *if and only if* $\lim(y_n) = +\infty$.

Theorem (3.7.3)**.** ***The*** n***th Term Test*** *If the series* $\sum x_n$ *converges, then* $\lim(x_n) = 0$.

Theorem (3.7.4)**.** ***Cauchy Criterion for Series*** *The series* $\sum x_n$ *converges if and only if for every* $\epsilon > 0$ *there exists* $M(\epsilon) \in \mathbb{N}$ *such that if* $m \geq n \geq M(\epsilon)$, *then*

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_n| < \epsilon$$

Theorem (3.7.7)**.** ***Comparison Test*** *Let* $X = (x_n)$ *and* $Y = (y_n)$ *be sequences of real numbers and suppose that for some* $K \in \mathbb{N}$ *we have* $0 \leq x_n \leq y_n$ *for* $n \geq K$.

(a) *Then the convergence of* $\sum y_n$ *implies the convergence of* $\sum x_n$.

(b) *The divergence of* $\sum x_n$ *implies the divergence of* $\sum y_n$.

Theorem (3.7.8)**.** ***Limit Comparison Test*** *Suppose that* $X = (x_n)$ *and* $Y = (y_n)$ *are strictly positive sequences and suppose that the following limit exists in* $\mathbb{R}$

$$r := \lim(\frac{x_n}{y_n})$$

(a) If $r \neq 0$ *then* $\sum x_n$ *is convergent if and only if* $\sum y_n$ *is convergent.*

(b) If $r = 0$ *then* $\sum y_n$ *is convergent if and only if* $\sum x_n$ *is convergent.*