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Math 501 Homework (§3.2)

Problem 15. To prove: If $0 < a < b, z_n := (a^n + b^n)^{1/n} \to b$

Solution. We first claim that $(2^{1/n})$ converges to 1. We can prove this by observing that

$$\begin{aligned}
2^{1/n} &= 1 + d_n,\ d_n > 0 \\
2 &= (1 + d_n)^n \geq 1 + nd_n \text{(by Bernoulli's Inequality)} \\
\implies 2 - 1 &\geq nd_n \\
d_n &\leq 1/n \\
\implies |2^{1/n} - 1| = d_n &\leq 1/n
\end{aligned}$$

Since $1/n \to 0$, by theorem 3.1.10, $2^{1/n} \to 1$. Now since

$$\begin{gathered}
0 < a < b \\
0 < a^n < b^n \\
b^n < a^n + b^n < 2b^n \\
(b^n)^{1/n} < (a^n + b^n)^{1/n} < (2b^n)^{1/n} \\
b < z_n < 2^{1/n} b \\
b \leq z_n \leq 2^{1/n} b \text{ (by laxing the inequality)}
\end{gathered}$$

Since we know that $(2^{1/n}) \to 1$ and $(b) \to b$, using Squeeze Theorem, we have that $z_n \to b$. □