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Math 501 Homework (§5.1, Continuous Functions)

Problem 14. Given $A := (0, \infty)$ and $k : A \to \mathbb{R}$ defined by

$$\begin{cases} 0 & x \text{ irrational,} \\ n & x \text{ rational of the form m/n}, \gcd(m, n) = 1 \end{cases}$$

Prove that k is unbounded on every open interval in A and that it is not continuous anywhere in A.

Solution. Let $I := (a, b) \subseteq A$ be an open interval. We find any two rationals

$$p = \frac{m_1}{n_1}, q = \frac{m_2}{n_2} \in I$$

such that $\gcd(n_1, n_2) = 1$. This is possible since, according to Density, there are infinitely many rationals in I. We take the average of p, q as

$$t = \frac{m_3}{n_1 n_2}$$

Notice that $k(t) = n_1 n_2$, which is greater than both $k(p) = n_1$ and $k(q) = n_2$. In other words, no matter how large $k(p)$ or $k(q)$, we can find a larger $k(t)$ satisfying $p, q, t \in I$.

Since I is arbitrary, we conclude that k is **unbounded** on every open interval in A – *including* A itself. This means for any $M > 0$ it is possible to find $x_M \in A$ satisfying $|k(x_M)| > M$. In other words, k is **discountinuous everywhere** on A. $\square$