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Math 560 Homework (#3)

Problem 1. Using the given probability distribution of X.

Solution. (a) $P(2 < X < 3)$

$$
\begin{aligned}
&= \int_2^3 \frac{2}{x^3} dx \\
&= 2\int_2^3 x^{-3} dx \\
&= -\left[x^{-2}\right]_2^3 \\
&= -\left[3^{-2} - 2^{-2}\right] \\
&\approx \boxed{0.12975}
\end{aligned}
$$

(b) $P(X < 2 \vee X > 3)$ can be seen as the probability of all events for $X > 1$, excluding $P(2 < X < 3)$ (from above) and hence $= P(X > 1) - P(2 < X < 3)$. Since $P(X > 1) = 1$, being the probability of all exhaustive cases

$$P(X < 2 \vee X > 3) = 1 - 0.12975 = \boxed{0.87025}$$

(c) $P(X < 2)$

$$
\begin{aligned}
&= \int_{-\infty}^2 \frac{2}{x^3} dx \\
&= \int_{-\infty}^1 \frac{2}{x^3} dx + \int_1^2 \frac{2}{x^3} dx \\
&= 0 - \left[x^{-2}\right]_1^2 \\
&= \left[1 - \frac{1}{\sqrt{2}}\right] \\
&\approx \boxed{0.29289}
\end{aligned}
$$

□

Problem 2. The scores on a particular exam follow a $Normal(74, 8)$ distribution. On "standardizing" this variable we get $Z = \frac{X-\mu}{\sigma}$.

Solution. (a) Therefore the probability that a randomly selected student has an exam score less than 80 $= P(X < 80) = P(Z < \frac{80-74}{8})$
$= P(Z < 0.75) = \boxed{0.7734}$ (using the *Standard Normal Cumulative Probabilities*).

(b) The probability that a randomly selected student has an exam score between 80 and 90 $= P(80 < X < 90) = P(0.75 < Z < \frac{90-74}{8})$ like above. This is $= P(0.75 < Z < 2) = P(Z < 2) - P(Z < 0.75)$.
Using the *Standard Normal Cumulative Probabilities* we see that $P(Z < 0.75) = 0.7734$ and $P(Z < 2) = 0.9772$. Therefore $P(0.75 < Z < 2) = 0.9772 - 0.7734 = \boxed{0.2038}$

(c) To find a value x such that the probability that a randomly selected student has an exam score less than x is 0.90 we reverse lookup the *Standard Normal Cumulative Probabilities* for cummulative probablities close to 0.90 and see that

$P(Z < 1.28) = 0.8997$ and $P(Z < 1.29) = 0.9015$. From the definition, $X = Z\sigma + \mu$. Therefore $1.28(8) + 74 < x < 1.29(8) + 74$. Or

$$84.24 < x < 84.32$$

□

Problem 3. The distributions of nonword- and word-errors are given. The total number of words written are 500. Also, the the correlation between the number of nonword errors and the number of word errors is 0.55. Let X be the number of word-, and Y the number of nonword-errors.

Solution. (a) The mean number of word-errors,

$$\mu_X = \sum_{i=1}^{4} x_i p_i$$

Using the given distribution, we have

$$\mu_X = 0(0.2) + 1(0.4) + 2(0.3) + 3(0.1) = 1.3$$

Similarly,

$$\mu_Y = \sum_{i=1}^{3} y_i p_i$$

since we only have 3 distinct values in the sequence. Again using the given distribution, we have

$$\mu_Y = 0(0.6) + 1(0.3) + 2(0.1) = 0.5$$

The mean of the total number of errors (nonword errors plus word errors) is the sum of both the means: $\mu_{X+Y} = \mu_X + \mu_Y$

$$= 1.3 + 0.5 = \boxed{1.8}$$

(b) To find the standard deviation, we first calculate the variance of each variable:

$$\begin{aligned}\sigma_X^2 &= \sum_{i=1}^{4} (x_i - \mu_X)^2 p_i \\ &= (0 - 1.3)^2(0.2) + (1 - 1.3)^2(0.4) + (2 - 1.3)^2(0.3) + (3 - 1.3)^2(0.1) \\ &= 0.81\end{aligned}$$

Similarly

$$\begin{aligned}\sigma_Y^2 &= \sum_{i=1}^{3} (y_i - \mu_Y)^2 p_i \\ &= (0 - 0.5)^2(0.6) + (1 - 0.6)^2(0.3) + (2 - 0.6)^2(0.1) \\ &= 0.394\end{aligned}$$

Now since X and Y have a correlation $\rho = 0.55$ we have that the combined variance

$$\begin{aligned}\sigma_{X+Y}^2 &= \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y \\ &= 0.81 + 0.394 + (0.55)\sqrt{0.81}\sqrt{0.394} \\ &\boxed{\approx 1.5147}\end{aligned}$$

$\square$

Problem 4. Suppose that 70% of homeowners in a town have a dog (let's call this event, D), 35% have an alarm system (event A), and 20% have both a dog and an alarm system ($A \cap D$).

Solution. (a) According to *de Morgan's Rule*, the probability that a randomly selected homeowner from this town has either a dog or an alarm system

$$\begin{aligned} P(A \cup D) &= P(A) + P(D) - P(A \cap D) \\ &= 0.70 + 0.35 - 0.20 \\ &= \boxed{0.85} \end{aligned}$$

(b) The probability that a randomly selected homeowner from this town has an alarm system given that the homeowner has a dog is given by

$$\begin{aligned} P(A|D) &= \frac{P(A \cap D)}{P(D)} \\ &= \frac{0.20}{0.70} \\ &= 0.28 \end{aligned}$$

$\square$