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Math 560 Homework (#6, Test of significance)

Problem 1. To assess the accuracy of a laboratory scale, a standard weight is repeatedly weighed a total of n times. The scale readings are independent and Normally distributed with an unknown mean μ and a known standard deviation $\sigma = 0.01$ grams.

Solution. a) How large does n need to be to guarantee that a two-sided 95% confidence interval for μ has a margin of error no larger than 0.003?

We know that

$$n \geq (\frac{z^* \sigma}{m})^2$$

Also, for $C = 95\%$, $z^* = 1.96$. So we have that

$$n \geq (\frac{1.96 \times 0.01}{0.003})^2$$
$$n \geq 42.69$$

Thus we see that n needs to be **at least** $\boxed{43}$.

(b) What is the smallest confidence level for which a two-sided confidence interval for μ is guaranteed to have margin of error no more than 0.003 based on a sample size $n = 20$?

Again, using the inequality above,

$$20 \geq (\frac{0.01z^*}{0.003})^2 = 11.11z^{*2}$$
$$z^{*2} \leq 1.8$$
$$\therefore z^* \leq 1.341$$

For $z^* < 0$ we are to use the area to the left of this point on the Normal curve. On looking up $z^* = 1.341$ in the *Normal table*, we find P-Value $= 90.99\%$ which is the desired confidence level. $\square$

Problem 2. A population follows a Normal distribution with mean μ and standard deviation $\sigma = 3$. A random sample with replacement of size $n = 10$ is taken from this population to test the hypothesis $H_0 : \mu = 5$.

Solution. (a) If the test of H_0 against the alternative $H_a : \mu < 5$ and we observe the sample mean to be $\bar{x} = 2$, should H_0 be rejected at level $\alpha = .05$?

We are to compare $H_0 : \mu = 5$ vs. $H_a : \mu < 5$. The statistic

$$z = \frac{\bar{x} - \mu_o}{\sigma/\sqrt{n}} = \frac{2 - 5}{3/\sqrt{10}} \approx -3.16$$

Looking up -3.16 in the *Normal table*, we get the P-Value= 0.0008 which is significantly smaller than $\alpha = .05$, so H_0 is to be **rejected**.

(b) If the test of H_0 against the alternative $H_a : \mu > 5$ and we observe the sample mean to be $\bar{x} = 2$, should H_0 be rejected at level $\alpha = .05$?

From (a), since $z = -3.16$ the P-Value in this case will be the area to the *right* of the probability at z, which $= 1 - P(z) = 1 - 0.0008 = 0.9992$ which is significantly larger than $\alpha = .05$, so H_0 is **not to be rejected**.

□

Problem 3. Your friend generated $n = 5$ observations from a Normal distribution using a random number generator in R. Your friend remembers that the standard deviation was $\sigma = 2$ but forgets what was used for the mean $\mu =?$.
Your friend thinks that the value used for μ might have been $\mu_0 = 10$. Is there strong evidence against this hypothesis? To answer this question, perform a test of significance at level $\alpha = .05$. Carefully state the hypotheses, calculate an appropriate test statistic, compute the P-value, and state your conclusion.

Solution. So the hypothesis H_0 states that the mean is $\mu_0 = 10$ We observe, from the sample, that the sample mean, $\bar{x} = 8.32287$.

Hence we are to compute the test statistic, z that represents the likelihood of μ being close to μ_0. This statistic is given by:

$$\begin{aligned} z &= \frac{\bar{x} - \mu_o}{\sigma/\sqrt{n}} \\ &= \frac{8.3228 - 10}{2/\sqrt{5}} \\ &\approx -1.8751 \end{aligned}$$

To find the P-Value for this statistic, we lookup -1.8751 in the *Normal table*, and since this value is to the left of 0, the Normal mean, we're looking for the area to the left of -1.8751.

P-Value ≈ 0.0307. This value is smaller than our confidence-level $\alpha = 0.05$, hence our friend's hypothesis $\mu_0 = 10$ must be **rejected**! $\square$