Lesson Plan - Irrationality of $\sqrt{2}$

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Background story (for class)

The set of real numbers, $\mathbb{R}$ is filled with both rational and irrational numbers. While the rationals are very well understood in our vocabulary (think how often we use $1/2$ in our daily usage), the irrationals are still very mysterious. In fact, it is a proven fact that there are more (*way many more*) irrationals than there are rationals.

But the goal of this lesson is not to prove that fact (it takes some higher-level math to do that, actually). We will discuss what baffled the Pythagoreans in ancient Greece when they discovered that the hypotenuse of a "unit-right-triangle" did not measure up to any number they knew. (brief discussion: Pythagorean formula for the hypotenuse, $c^2 = a^2 + b^2$).

Supporting concepts

The following skills are useful to revise while teaching this material:

- "Proof by contradiction",
- Deriving a logical conclusion from known facts,
- State contrapositive, given a logical statement,
- Recognize the divisibility rules for common divisors $(3, 4, 9, 11, \dots)$,
- Understand "greatest common divisor".

Learning Goals

By the end of this lesson the students will be able to

- Understand that irrational numbers are also real,
- Demonstrate, using basic vocabulary, that $\sqrt{2}$ is irrational.

Specific Expectations

- Express a rational number in the p/q form, where $\gcd(p, q) = 1$,
- Recognize a contradiction, while trying to express $\sqrt{2}$ as p/q.

Outline of the Course Content

Unit Title	Contents
Unit-1	Review: Prime Factorization
Unit-2	Review: $\gcd()$ and the fact that for rationals $p/q, \gcd(p, q) = 1$
Unit-3	Build the proof with class

Course Content

Review: Prime Factorization

It's the fundamental fact of arithmetic, that every natural number $(1, 2, 3, \ldots)$ can be written as a product of powers of Primes. Examples:

- $25 = 2^0 \cdot 3^0 \cdot 5^2$
- $112 = 2^4 \cdot 3^0 \cdot 5^0 \cdot 7^1$
- $1 = 2^0 \cdot 3^0 \cdot 5^0 \ldots$
- $13 = 2^0 \cdot 3^0 \cdot 5^0 \cdot 7^0 \cdot 11^0 \cdot 13^1$ [class discussion: How are Primes unique in this context?]
- Class activity: Ask one group of students to think of a number, and another to come up with its Prime factorization.

Review: Greatest Common Divisor ($\mathsf{gcd}()$)

Common divisors (factors), as the name suggests, are just factors of two (or more) numbers that are common. The greatest of them is called the $\mathsf{gcd}()$. Examples:

- $\mathsf{gcd}(14, 7) = 7$
- $\mathsf{gcd}(22, 33) = 11$
- $\mathsf{gcd}(25, 112) = 1$ [introduce: co-primes]
- $\mathsf{gcd}(13, 20) = 1$ [class discussion: How are Primes unique in this context?]

The proof

The instructor divides the class into small groups and alternates these steps between the groups, helping them derive a conclusion from the prior step(s):

- To prove: $\sqrt{2}$ is irrational.
- Suppose the above statement is false.
- This means that $\sqrt{2}$ can be written as p/q where $\mathsf{gcd}(p, q) = 1$:

$$\sqrt{2} = \frac{p}{q}$$

- Squaring both sides, we get:

$$2 = \frac{p^2}{q^2}$$

- This implies that:

$$p^2 = 2q^2$$

- Thus p^2 is even. The only way this would be true is, if p is even (class discussion: Why is this the case?)
- This means that p^2 is actually divisible by 4.
- Hence q^2 and (using similar argument) q must be even.
- But if both p and q are even, $\mathsf{gcd}(p, q) = 2 \neq 1$.
- **This is a contradiction!**
- The conclusion: Our assumption in the first step that $\sqrt{2}$ can be written as $p/q, \mathsf{gcd}(p, q) = 1$, or that $\sqrt{2}$ is rational, must be wrong.
- The required statement is *hence proved.*