Lesson Plan - Infinite Primes

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Introduction

When discussing the Prime numbers, we often assume there are infinitely many of them. However, this fact is not so obvious. When I first posed this question to a friend, their first reaction was 'aren't there infinitely many of everthing?'

Many obvious things around us take some careful explaining and the infinitude of Primes is one of them. Primes have amused humans since early Greeks, and very little is known about the patterns these numbers exhibit. However, the fact that there is no limit to Prime numbers has been known to us at least since 300 B.C. While the earliest proof, due to Euclid, was different, the present proof is used more commonly today.

Learning Goals

By the end of this lesson the students will be able to

- Understand "proof by contradiction",
- Generalize the concept of 'divisibility',
- Demonstrate, using basic vocabulary, that there is no limit to the Prime numbers.

Specific Expectations

- Recognize the divisibility rules for common divisors $(3, 4, 9, 11, \dots)$,
- Derive a logical conclusion from known facts,
- State contrapositive, given a logical statement,
- Recognize a contradiction,

Outline of the Course Content

Unit Title	Contents
Unit-1	Use Truth Tables to discuss logical contradiction.
Unit-2	Discuss "contrapositive"
Unit-3	Present the proof

Course Content

Logical Contradiction

A powerful method of proof that is frequently used in mathematics is **proof by contradiction**. This method is based on he fact that a logical statement, P can either be true or false, but never both. The idea is to prove that the statement P is true by showing that it cannot be false. This is done by assuming that P is false and proving that it leads to a **contradiction** (e.g., "2 is odd" is a logical contradiction).

Table 1

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Class discussion: Does this table show a contradiction?

Contrapositive

The **contrapositive** of the conditional statement $P \implies Q$ is the conditional statement $\neg Q \implies \neg P$. This is achieved since in the first statement P was a *necessary condition* for Q, the negation of Q in the second statement leads us to the negation of P.

Breakout activity: (come up with examples similar to the following):

- If $x = 3$, then $x + 4 = 7$.
- If $x + 4 \neq 7$, then $x \neq 3$.

The proof

The instructor divides the class into small groups and alternates these steps between the groups, helping them derive a conclusion from the prior step(s):

- To prove: There are infinitely many Primes.
- Suppose the above statement is false.
- This implies there are finitely man Prime numbers. Let these be $p_1, p_2, p_3, \ldots, p_n$.
- Now construct a new number, $p = p_1 \times p_2 \times p_3 \times \cdots \times p_n + 1$.
- Observe: p is larger than any of the primes (discuss examples if needed).
- Since $p_1, p_2, p_3, \ldots, p_n$ constitute all primes, p can't be a prime (define: Composite). Thus p is Composite, i.e., is divisible by one of the primes $p_1, p_2, p_3, \ldots, p_n$.
- But when we divide p by either of $p_1, p_2, p_3, \ldots, p_n$, we get 1 as a remainder.
- **This is a contradiction!**
- The conclusion: Our assumption in the first step "there are finitely many Prime numbers", must be wrong.
- The required statement is *hence proved.*