Divisibility by 11

6/23/2020

Problem. *How to show that a number is divisible by* 11*?*

Solution. *Let's look at division. When a number a is divided by another number x and leaves b as remainder, we can write*

$$a = n \cdot x + b \tag{1}$$

where n is called the "quotient". But if we just focus on the remainder, we call this relationship ***modular equivalence*** *and write it as $a \equiv b \pmod{x}$. Now consider another such statement,*

$$c = m \cdot x + d \tag{2}$$

or in terms of ***modular equivalence****, $c \equiv d \pmod{x}$*

On adding both sides of (1) and (2), we get $(a + c) = (n + m) \cdot x + (b + d)$. It looks like when the number $(a + c)$ is divided by x, it leaves $(b + d)$ as the remainder. This property can be written as

$$(a + c) \equiv (b + d) \pmod{x} \tag{3}$$

If we multiplied both sides of (1) by a constant number, k we would see $ak = nk \cdot x + bk$ and again, focussing just on the remainder:

$$ak \equiv bk \pmod{x} \tag{4}$$

We shall use (3) and (4) as ***properties of modular equivalence****.*

We see that $10 = 1 \cdot 11 + (-1)$ which, in terms of ***modular equivalence*** *we be written as $10 \equiv -1 \pmod{11}$.*

Now let's consider all the non-negative powers of 10 with respect to divisibility *by* 11*. We will see that*

$$
\begin{aligned}
1 &\equiv 1 \pmod{11} \\
10 &\equiv -1 \pmod{11} \\
100 &\equiv 1 \pmod{11} \\
1000 &\equiv -1 \pmod{11} \\
10000 &\equiv 1 \pmod{11} \\
&\vdots
\end{aligned}
$$

Multiplying all these modular equivalences by arbitrary non-negative constants $a_0, a_1, \ldots a_n$ *and using* ***property*** (4) *above:*

$$
\begin{aligned}
1 \cdot a_0 &\equiv a_0 \pmod 9 \\
10 \cdot a_1 &\equiv -a_1 \pmod 9 \\
100 \cdot a_2 &\equiv a_2 \pmod 9 \\
1000 \cdot a_3 &\equiv -a_3 \pmod 9 \\
10000 \cdot a_4 &\equiv a_4 \pmod 9 \\
&\vdots \\
10^n \cdot a_n &\equiv \pm a_n \pmod 9
\end{aligned}
$$

Adding all these equivalences using ***property*** (3)*:*

$$1 \cdot a_0 + 10 \cdot a_1 + 100 \cdot a_2 + 1000 \cdot a_3 + 10000 \cdot a_4 \cdots + 10^n \cdot a_n \equiv (a_0 - a_1 + a_2 - a_3 + a_4 \cdots + a_n) \pmod{11}$$

Well! the left hand side of the above equivalence is nothing but a representation of a $(n+1)$*-digit number (e.g.,* $76934 = 1 \cdot 4 + 10 \cdot 3 + 100 \cdot 9 + 1000 \cdot 6 + 10000 \cdot 7$*) and the right hand side is the alternating sum and difference of its digits!*

Hence the equivalence statement above shows that ***when dividing a number by*** 11 ***its remainder is the same as that when dividing the alternating difference ans sum of its digits (starting with a difference)****.*

E.g., 76934 *is divisible by* 11 *since the sum of its digits is* 11