Roots – rational or not

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Conjecture. *A natural root of a natural number is either an integer or irrational.*

Proof. We begin by observing that if a prime number y divides any power of a natural number x^p, it also divides x because otherwise it would have to be "composed" of other factors of x, which is impossible since y is prime.

Now let $n, k \in \mathbb{N}$ with $k > 1$. We will prove that $n^{1/k}$ is either an integer or an irrational number.

$n = 1$ is trivial. For $n > 1$ let's assume that $n^{1/k}$ is **rational**, i.e.,

$$n^{1/k} = \frac{p}{q}$$

with $gcd(p, q) = 1$.

This implies that $p^k = nq^k$ or n divides p^k.

If n is prime this also implies that n divides p (from the observation made above) or that $p = np'$ for some factor p'. Using this and the fact that $k > 1$, the preceeding equation can be written as $nq^k = n^2 {p'}^2 p^{k-2}$ or $q^k = n{p'}^2 p^{k-2}$, i.e. n divides q^k. But we know this means n divides q. Hence we see that p and q have a common divisor $n(> 1)$. This is a contradiction to our assumption that $gcd(p, q) = 1$.

If n is not prime let $n_a, n_b, n_c \ldots$ be all prime factors of n and $n' = n_a \cdot n_b \cdot n_c \ldots$ be their product. Since n divides p^k so does n' and so do each of $n_a, n_b, n_c \ldots$. Again from the observation above, each of $n_a, n_b, n_c \ldots$ also divide p, and so does n'. In the same way as above we can show that now p and q share a common divisor $n'(> 1)$ which is contrary to the assumption above.

So we see that in all exhaustive cases, $n^{1/k}$ cannot be rational. So it must be either an integer or an irrational number. □