# Divisibility by 9

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**Problem.** *How to show that a number is divisible by* 9*?*

**Solution.** *Let's look at division. When a number* $a$ *is divided by another number* $x$ *and leaves* $b$ *as remainder, we can write*

$$a = n \cdot x + b \tag{1}$$

*where* $n$ *is called the "quotient". But if we just focus on the remainder, we call this relationship* ***modular equivalence*** *and write it as* $a \equiv b \pmod{x}$*. Now consider another such statement,*

$$c = m \cdot x + d \tag{2}$$

*or in terms of* ***modular equivalence****,* $c \equiv d \pmod{x}$

*On adding both sides of* (1) *and* (4)*, we get* $(a + c) = (n + m) \cdot x + (b + d)$*. It looks like when the number* $(a + c)$ *is divided by* $x$*, it leaves* $(b + d)$ *as the remainder. This property can be written as*

$$(a + c) \equiv (b + d) \pmod{x} \tag{3}$$

*If we multiplied both sides of* (1) *by a constant number,* $k$ *we would see* $ak = nk \cdot x + bk$ *and again, focussing just on the remainder:*

$$ak \equiv bk \pmod{x} \tag{4}$$

*We shall use* (3) *and* (4) *as* ***properties of modular equivalence****.*

*Now let's consider all the non-negative powers of* 10 *with respect to divisibility by* 9*. We will see that*

$$\begin{aligned} 1 &\equiv 1 \pmod{9} \\ 10 &\equiv 1 \pmod{9} \\ 100 &\equiv 1 \pmod{9} \\ 10^n &\equiv 1 \pmod{9} \end{aligned}$$

*Multiplying all these modular equivalences by arbitrary non-negative constants $a_0, a_1, \ldots a_n$ and using* ***property*** (4) *above:*

$$\begin{aligned} 1 \cdot a_0 &\equiv a_0 \pmod 9 \\ 10 \cdot a_1 &\equiv a_1 \pmod 9 \\ 100 \cdot a_2 &\equiv a_2 \pmod 9 \\ 10^n \cdot a_n &\equiv a_n \pmod 9 \end{aligned}$$

*Adding all these equivalences using* ***property*** (3)*:*

$$1 \cdot a_0 + 10 \cdot a_1 + 100 \cdot a_2 + \cdots + 10^n \cdot a_n \equiv (a_0 + a_1 + a_2 + \cdots + a_n) \pmod 9$$

*Well! the left hand side of the above equivalence is nothing but a representation of a* $(n+1)$*-digit number (e.g.,* $76932 = 1 \cdot 2 + 10 \cdot 3 + 100 \cdot 9 + 1000 \cdot 6 + 10000 \cdot 7$*) and the right hand side is just the sum of its digits!*

*Hence the equivalence statement above shows that* ***when dividing a number by*** 9 ***its remainder is the same as that when dividing the sum of its digits****.*

*E.g.,* 76932 *is divisible by* 9 *since the sum of its digits,* 27 *is (and that is because so is the sum of its digits!)*