Roots – rational or not

January 23, 2021

Conjecture. *A real root of a natural number is either an integer or irrational.*

Proof. We begin by observing that if a prime number y divides any power x^p, it also divides x because otherwise it would have to be "composed" of other factors of x, which is impossible for a prime.

Now let $n, k \in \mathbb{N}$ with $k \geq 2$. We will prove that $n^{1/k}$ is either an **integer** or an **irrational number**. The only other possibility for a real root being a **strictly rational** number (of the form $p/q : p, q \in \mathbb{N}$ with $gcd(p, q) = 1$).

$n = 1$ is trivial. For $n > 1$ let's assume that $n^{1/k}$ is **strictly rational**. I.e.,

$$n^{1/k} = \frac{p}{q}$$

This implies that $p^k = nq^k$ or that n **divides** p^k.

If n **is prime** this also implies that n divides p (from the first observation we made) or that $p = np'$ for some factor p'. Using this and the fact that $k \geq 2$, the last equation can be written as $nq^k = n^2 {p'}^2 p^{k-2}$ or $q^k = n{p'}^2 p^{k-2}$, i.e. n divides q^k. But we know this means n divides q. Hence we see that p and q have a common divisor $n (> 1)$. This is a contradiction to the definition of **strictly rational** numbers: $gcd(p, q) = 1$.

If n **is not prime** let $n_a, n_b, n_c \dots$ be all the prime factors of n and $n' = n_a \cdot n_b \cdot n_c \dots$ be their product. Since n divides p^k so does n' and so do each of $n_a, n_b, n_c \dots$. Again, from the observation above each of $n_a, n_b, n_c \dots$ also divide p, and so does n'. In the same way as above we can show that now p and q share a common divisor $n'(> 1)$ which is, again, contrary to the assumption above.

In this way we see that in no case can $n^{1/k}$ be **strictly rational**. So it must be either an integer or an irrational number. $\square$