Euler's solution to the Basel Problem

3/4/2020

Problem. *Find the infinite sum* $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$.

Solution. *To solve this problem Euler first factorizes a quadratic,* $P(x)$ *with* $P(0) = 1$ *and solutions* $x = a$ *and* $x = b$ *as:*

$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})$

Similarly, a third-degree polynomial, $P(x)$ *with* $P(0) = 1$ *and solutions* $x = a, x = b$ *and* $x = c$ *can be factorized as:*

$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})(1 - \frac{x}{c})$

In general, an infinite-degree polynomial, $P(x)$ *with* $P(0) = 1$ *and solutions* $x = a, x = b, x = c \ldots$ *can be factorized as:*

$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})(1 - \frac{x}{c}) \ldots$

Then he considers an infinite-degree polynomial:

$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \ldots$

Since $P(0) = 1$*, to find the solutions to* $P(x)$*:*

$0 = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \ldots$ *or* $0 = \frac{x(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \ldots)}{x}$

$\therefore \frac{sin(x)}{x} = 0$*, using Newton's expansion of* $sin(x)$.

To solve for x*, we consider all cases where* $sin(x) = 0$ *i.e.,* $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$

Omitting $x = 0$ *since that will render the fraction* $\frac{sin(x)}{x}$ *undefined, we obtain that* $x = \pm\pi, \pm 2\pi, \pm 3\pi \ldots$ *are solutions to the infinite-degree polynomial,* $P(x)$*. Thus the polynomial can be factorized as follows:*

$P(x) = (1 - \frac{x}{\pi})(1 - \frac{x}{-\pi})(1 - \frac{x}{2\pi})(1 - \frac{x}{-2\pi})(1 - \frac{x}{3\pi})(1 - \frac{x}{-3\pi}) \ldots$

$= (1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi})\ldots$

Multilying the terms in pairs, we obtain:

$P(x) = [1 - \frac{x^2}{(\pi)^2}][1 - \frac{x^2}{(2\pi)^2}][1 - \frac{x^2}{(3\pi)^2}]\ldots$

$= [1 - \frac{x^2}{\pi^2}][1 - \frac{x^2}{4\pi^2}][1 - \frac{x^2}{9\pi^2}]\ldots$

Multilying the terms out for the first two terms::

$P(x) = 1 + x^2[-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2}\ldots]$ *+ higher terms of* x

$= 1 - x^2[\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2}\ldots]$ *+ higher terms of* x

Equating to the definition of $P(x)$ *and comparing coefficients of* x^2*,*

$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2}\ldots$

$\frac{1}{3!} = \frac{1}{\pi^2}(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\ldots)$

Therefore the original sum,

$S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}\ldots = \frac{\pi^2}{6}.$