Euler's solution to the Basel Problem

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Problem. *Find the infinite sum* $S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$.

Solution. *To solve this problem Euler first shows a general way to factorize a* ***general polynomial of infinite-degree*** *(one that has infinitely many terms in a variable, with powers of the variables raised to* ∞*). But first let's take a* ***quadratic*** *(a polynimial in degree 2) as an example and try to factorize it. Suppose the quadratic* $P(x)$ *satisfies* $P(0) = 1$ *and has for solutions,* $x = a$ *and* $x = b$*. It is easy to see that* $P(x)$ *can be factorized as:*

$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})$

Similarly, a ***third-degree polynomial****,* $P(x)$ *with* $P(0) = 1$ *and solutions* $x = a, x = b$ *and* $x = c$ *can be factorized as:*

$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})(1 - \frac{x}{c})$

Now he chooses a ***general polynomial of infinite-degree****,* $P(x)$ *with* $P(0) = 1$ *and solutions* $x = a, x = b, x = c \dots$ *and extends the same factorization as:*

$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})(1 - \frac{x}{c}) \dots$

Keeping that factorization technique in mind, he then considers an infinite-degree polynomial:

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \dots \tag{1}$$

To solve for x*, a common technique is to equate* $P(x)$ *to zero. Therefore:*

$$\begin{aligned}
P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \ldots &= 0 \\
\frac{x(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \ldots)}{x} &= 0 \quad \textit{(multiplying both numerator and denominator by } x\textit{)} \\
\frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots}{x} &= 0 \\
\therefore P(x) = \frac{\sin(x)}{x} &= 0 \quad \textit{(using Newton's expansion of sine)}
\end{aligned}$$

Newton's expansion of sine *states that:* $sin(x) = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots}{x}$.

To solve for x*, we consider all cases where* $sin(x) = 0$ *i.e.,* $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$

Omitting $x = 0$ *as that will render the fraction* $\frac{sin(x)}{x}$ *undefined, we obtain that* $x = \pm\pi, \pm 2\pi, \pm 3\pi \ldots$ *are solutions to the infinite-degree polynomial,* $P(x)$*.*

Further, since $P(0) = 1 - 0 + 0 - 0 + \cdots = 1$*, it falls in the category of the* ***general polynomials of infinite-degree*** *above. Therefore the polynomial can be factorized as follows:*

$$\begin{aligned}
P(x) &= (1 - \frac{x}{\pi})(1 - \frac{x}{-\pi})(1 - \frac{x}{2\pi})(1 - \frac{x}{-2\pi})(1 - \frac{x}{3\pi})(1 - \frac{x}{-3\pi}) \ldots \\
&= (1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi}) \ldots \\
&= [1 - \frac{x^2}{\pi^2}][1 - \frac{x^2}{4\pi^2}][1 - \frac{x^2}{9\pi^2}] \ldots \quad \textit{(by multilying the terms in pairs)}
\end{aligned}$$

Multilying the terms out for the first two degrees of x*:*

$$\begin{aligned}
P(x) &= 1 + x^2[-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} \ldots] \quad + \textit{ terms with higher degrees of } x \\
&= 1 - x^2[\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} \ldots] \quad + \textit{ terms with higher degrees of } x
\end{aligned}$$

Equating to the definition of $P(x)$ *from (1) above,*

$$
\begin{aligned}
P(x) &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \dots \\
&= 1 - x^2[\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} \dots] \qquad \textit{+ terms with higher degrees of } x
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{3!} &= \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} \dots \qquad \textit{(by equating the coefficients of } x^2\textit{)} \\
\frac{1}{3!} &= \frac{1}{\pi^2}(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \dots)
\end{aligned}
$$

Therefore the original sum,

$$
\boxed{S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \dots = \frac{\pi^2}{6}}
$$