

# Divisibility by 9

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**Problem.** *How to show that a number is divisible by 9?*

**Solution.** *Let's look at division. When a number  $a$  is divided by another number  $x$  and leaves  $b$  as remainder, we can write*

$$a = n \cdot x + b \quad (1)$$

*where  $n$  is called the "quotient". But if we just focus on the remainder, we call this relationship **modular equivalence** and write it as  $a \equiv b \pmod{x}$ . Now consider another such statement,*

$$c = m \cdot x + d \quad (2)$$

*or in terms of **modular equivalence**,  $c \equiv d \pmod{x}$*

*On adding both sides of (1) and (2), we get  $(a + c) = (n + m) \cdot x + (b + d)$ . It looks like when the number  $(a + c)$  is divided by  $x$ , it leaves  $(b + d)$  as the remainder. This property can be written as*

$$(a + c) \equiv (b + d) \pmod{x} \quad (3)$$

*If we multiplied both sides of (1) by a constant number,  $k$  we would see  $ak = nk \cdot x + bk$  and again, focussing just on the remainder:*

$$ak \equiv bk \pmod{x} \quad (4)$$

*We shall use (3) and (4) as **properties of modular equivalence**.*

*Now let's consider all the non-negative powers of 10 with respect to divisibility by 9. We will see that*

$$\begin{aligned} 1 &\equiv 1 \pmod{9} \\ 10 &\equiv 1 \pmod{9} \\ 100 &\equiv 1 \pmod{9} \\ 10^n &\equiv 1 \pmod{9} \end{aligned}$$

Multiplying all these modular equivalences by arbitrary non-negative constants  $a_0, a_1, \dots, a_n$  and using **property** (4) above:

$$\begin{aligned} 1 \cdot a_0 &\equiv a_0 \pmod{9} \\ 10 \cdot a_1 &\equiv a_1 \pmod{9} \\ 100 \cdot a_2 &\equiv a_2 \pmod{9} \\ 10^n \cdot a_n &\equiv a_n \pmod{9} \end{aligned}$$

Adding all these equivalences using **property** (3):

$$1 \cdot a_0 + 10 \cdot a_1 + 100 \cdot a_2 + \dots + 10^n \cdot a_n \equiv (a_0 + a_1 + a_2 + \dots + a_n) \pmod{9}$$

Well! the left hand side of the above equivalence is nothing but a representation of a  $(n+1)$ -digit number (e.g.,  $76932 = 1 \cdot 2 + 10 \cdot 3 + 100 \cdot 9 + 1000 \cdot 6 + 10000 \cdot 7$ ) and the right hand side is just the sum of its digits!

Hence the equivalence statement above shows that **when dividing a number by 9 its remainder is the same as that when dividing the sum of its digits.**

E.g., 76932 is divisible by 9 since the sum of its digits, 27 is (and that is because so is the sum of its digits!)