Euler's solution to the Basel Problem

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Problem. *Find the infinite sum* $S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$.

Solution. *To solve this problem Euler first shows a general way to factorize a **general polynomial of infinite-degree** (one that has infinitely many terms in a variable, with powers of the variables raised to ∞). But first let's take a **quadratic** (a polynimial in degree 2) as an example and try to factorize it. Suppose the quadratic $P(x)$ satisfies $P(0) = 1$ and has for solutions, $x = a$ and $x = b$. It is easy to see that $P(x)$ can be factorized as:*

$$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})$$

*Similarly, a **third-degree polynomial**, $P(x)$ with $P(0) = 1$ and solutions $x = a, x = b$ and $x = c$ can be factorized as:*

$$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})(1 - \frac{x}{c})$$

*Now he chooses a **general polynomial of infinite-degree**, $P(x)$ with $P(0) = 1$ and solutions $x = a, x = b, x = c \dots$ and extends the same factorization as:*

$$P(x) = (1 - \frac{x}{a})(1 - \frac{x}{b})(1 - \frac{x}{c}) \dots$$

Keeping that factorization technique in mind, he then considers an infinite-degree polynomial:

$$P(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \dots \tag{1}$$

To solve for x, a common technique is to equate $P(x)$ to zero. Therefore:

$$\begin{aligned}
P(x) &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\dots && = 0 \\
& \frac{x(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}\dots)}{x} && = 0 \quad \textit{(multiplying both numerator and denominator by x)} \\
& \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\dots}{x} && = 0 \\
\therefore P(x) &= \frac{\sin(x)}{x} && = 0 \qquad \textit{(using Newton's expansion of sine)}
\end{aligned}$$

Newton's expansion of sine *states that: $sin(x) = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\dots}{x}$.*

To solve for x, we consider all cases where $sin(x) = 0$ i.e., $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi \dots$

Omitting $x = 0$ as that will render the fraction $\frac{sin(x)}{x}$ undefined, we obtain that $x = \pm\pi, \pm 2\pi, \pm 3\pi \dots$ are solutions to the infinite-degree polynomial, $P(x)$.

Further, since $P(0) = 1 - 0 + 0 - 0 + \dots = 1$, it falls in the category of the ***general polynomials of infinite-degree*** *above. Therefore the polynomial can be factorized as follows:*

$$\begin{aligned}
P(x) &= (1 - \frac{x}{\pi})(1 - \frac{x}{-\pi})(1 - \frac{x}{2\pi})(1 - \frac{x}{-2\pi})(1 - \frac{x}{3\pi})(1 - \frac{x}{-3\pi})\dots \\
&= (1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi})\dots \\
&= [1 - \frac{x^2}{\pi^2}][1 - \frac{x^2}{4\pi^2}][1 - \frac{x^2}{9\pi^2}]\dots && \textit{(by multilying the terms in pairs)}
\end{aligned}$$

Multilying the terms out for the first two degrees of x:

$$\begin{aligned}
P(x) &= 1 + x^2[-\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2}\dots] && + \textit{ terms with higher degrees of } x \\
&= 1 - x^2[\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2}\dots] && + \textit{ terms with higher degrees of } x
\end{aligned}$$

Equating to the definition of $P(x)$ from (1) above,

$$\begin{aligned} P(x) &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \cdots \\ &= 1 - x^2[\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} \cdots] \qquad + \textit{terms with higher degrees of } x \end{aligned}$$

$$\begin{aligned} \frac{1}{3!} &= \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} \cdots \qquad \textit{(by equating the coefficients of } x^2\textit{)} \\ \frac{1}{3!} &= \frac{1}{\pi^2}(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots) \end{aligned}$$

Therefore the original sum,

$$\boxed{S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots = \frac{\pi^2}{6}}$$