Q.1 $\int \frac{1}{1+x^2} dx$

A) Let $I = \int \frac{1}{1+x^2} dx$

Let $x = \tan\theta$ or $\theta = \arctan(x)$

Differentiating x w.r.t θ

$\frac{dx}{d\theta} = \sec^2(\theta)$ or $dx = \sec^2(\theta) d\theta$

$\therefore I = \int \frac{1}{1+x^2} dx = \int \frac{1}{1+\tan^2\theta} \sec^2\theta d\theta$

$= \int \frac{1}{\sec^2\theta} \sec^2\theta d\theta$

$= \int d\theta = \theta$. Because $\frac{d}{d\theta}(\theta) = 1$

Since $\theta = \arctan(x)$, $\boxed{I = \arctan(x) + C}$

Evaluation of $1 + \tan^2\theta$:

$$
\begin{aligned}
&1 + \tan^2\theta \\
&= 1 + \frac{\sin^2\theta}{\cos^2\theta} \\
&= \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\
&= \frac{1}{\cos^2\theta} && \textbf{since } \sin^2\theta + \cos^2\theta = 1 \\
&= \sec^2\theta && \textbf{similarly, } 1 - \sec^2\theta = \tan^2\theta
\end{aligned}
$$

Differentiation of $\tan\theta$:

$$
\begin{aligned}
&\frac{d}{d\theta}(\tan\theta) \\
&= \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) \\
&= \frac{\cos\theta\cos\theta - \sin\theta(-\sin\theta)}{\cos^2\theta} && \textbf{Quotient Rule: } \frac{d}{d\theta}\frac{f}{g} = \frac{g.f' - f.g'}{g^2} \\
& && \textbf{also } (\cos\theta)' = -\sin\theta \textbf{ and } (\sin\theta)' = \cos\theta \\
&= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\
&= \frac{1}{\cos^2\theta} && \textbf{remember: } \sin^2(x) + \cos^2(x) = 1 \\
&= \sec^2\theta
\end{aligned}
$$