Macalester - Math 312 November 30, 2023

# Midterm Exam II Formulas

**Quadratic Formula** : If we have $ax^2 + bx + c \;=\; 0,$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

**Euler's Formula** Let $z = a + bi$ be a complex number given in rectangular form, then

$$e^z \;=\; e^{a+bi} \;=\; e^a(\cos(b) + i\sin(b))$$

two other related formulas:

(i) if $z = a + bi$, then $r = \sqrt{a^2 + b^2}$, and $\tan(\theta) = \frac{y}{x}$, so $z = re^{i\theta}$

(ii) for any angle $\theta$, $e^{i\theta} = \cos\theta + i\sin\theta$

**Numerical Methods for Systems** In each of the following we are attempting to find an approximate numerical solution $(\hat{x}(t), \hat{y}(t))$ to the (potentially) non-autonomous system

$$\begin{aligned}\frac{dx}{dt} &= F(x, y, t)\\ \frac{dy}{dt} &= G(x, y, t)\end{aligned}$$

with the initial conditions $x(0) = x_0$ and $y(0) = y_0$

where we let $k = \Delta t,$ be the step size, so $t_{i+1} = t_i + k$, and we denote $\hat{y}(t_i)$ by $\hat{y}_i$, and $\hat{x}(t_i)$ by $\hat{x}_i$

**Euler's Method** for any positive integer $i$,

$$\begin{aligned}\hat{x}_i &= \hat{x}_{i-1} + kF(\hat{x}_{i-1}, \hat{y}_{i-1}, t_{i-1})\\ \hat{y}_i &= \hat{y}_{i-1} + kG(\hat{x}_{i-1}, \hat{y}_{i-1}, t_{i-1})\end{aligned}$$

**Heun's Method** for any positive integer $i$,

$$x_{temp} = \hat{x}_{i-1} + kF(\hat{x}_{i-1}, \hat{y}_{i-1}, t_{i-1}) \quad \text{and} \quad y_{temp} = \hat{y}_{i-1} + kG(\hat{x}_{i-1}, \hat{y}_{i-1}, t_{i-1})$$

$$\begin{aligned}\hat{x}_i &= \hat{x}_{i-1} + \frac{k}{2}\Big(F(\hat{x}_{i-1}, \hat{y}_{i-1}, t_{i-1}) + F(\hat{x}_{temp}, \hat{y}_{temp}, t_i)\Big)\\ \hat{y}_i &= \hat{y}_{i-1} + \frac{k}{2}\Big(G(\hat{x}_{i-1}, \hat{y}_{i-1}, t_{i-1}) + G(\hat{x}_{temp}, \hat{y}_{temp}, t_i)\Big)\end{aligned}$$

**Eigenvalues and Eigenvectors** Given a square matrix $A$, its eigenvalues satisfy $\det(A - \lambda I) = 0$

and if $\lambda$ is an eigenvalue for $A$ (real or complex), then an eigenvector $\mathbf{v}$ for $\lambda$ satisfies

$$A\mathbf{v} = \lambda\mathbf{v}$$

**Sinusoidal Forcing Formulas** The following sum to product identity (actually difference to product) is used below:

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Recall the general solution of the DE $y'' + k^2 y = \cos(\omega t)$ has the form

$$y(t) = C_1\cos(kt) + C_2\sin(kt) + \frac{1}{k^2-\omega^2}\cos(\omega t)$$

So, if we choose for our initial conditions $y(0) = 0$ and $y'(0) = 0$, we get

$$y(t) = \frac{1}{k^2-\omega^2}(\cos(\omega t) - \cos(kt))$$

and if will denote $a = \frac{1}{k^2-\omega^2}$, and we will write $y(t) = a(\cos(\omega t) - \cos(kt))$

and using the sum to product identity referenced above we get $y(t) = -2a\sin\left(\frac{\omega+k}{2}t\right)\sin\left(\frac{\omega-k}{2}t\right)$