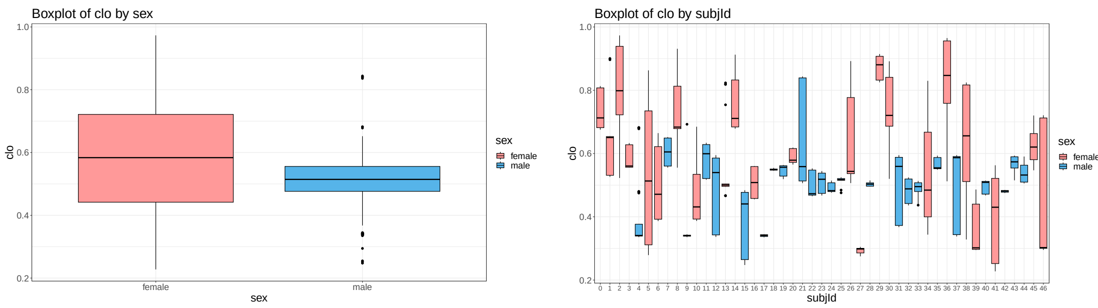
Part 1: Clothing insulation level

1. Data presentation

The data consists of the variable "clo" which is continuous and represents the level of clothing insulation worn by the subject. "t0ut" is also continuous and refers to the outdoor temperature at the time of measurement. "tIn0p" is another continuous variable, representing the indoor operating temperature. "sex" is a factor variable and indicates the sex of the subject. "subjId" is also a factor variable, used to identify the subject. "time" is a continuous variable that represents the time of measurement within the day. "day" is a factor variable that indicates the day within the subject, while "time2" is an integer variable that represents the measurement number within the day. Lastly, "subDay" is a factor variable used as a unique identifier for the day and subject.

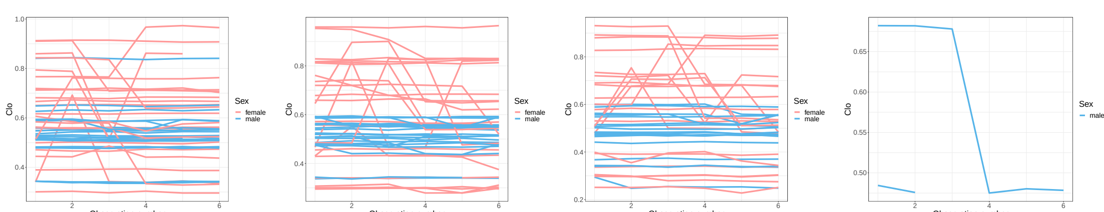


(a) Boxplot of the clo variable grouped by sex, where females are colored red and males are colored blue.

(b) Boxplot of the clo variable for all days (and within each days) grouped by subjId, where females are colored red and males are colored blue.

Figure 1: Boxplots of clo grouped by sex (left) and boxplots of clo grouped by subjId (right).

If we take a look at figure 1a then we see a significant difference in the distribution of clo when grouped by sex. This indicates a clear difference between men and women, when considering the clothing insulation level. Specifically, it seems that women has a wider range and variation, and also a higher mean level of clo. If we now consider figure 1b, then the clo variable is grouped by each individual subject. It is seen that there is a clear variation when inspecting each individual subject. Some has a wider range of clo, whereas some other subjects, e.g. subject 27 has somewhat constant clothing insulation level trough out the days (and within each day). Thus it may be beneficial to account for variation in each subject.



(a) Day 1 (b) Day 2 (c) Day 3 (d) Day 4

Figure 2: Clothing insulation level within each day, where females are colored red and males are colored blue.

When considering figure 2 the most interesting thing is that clo for women seem to change a lot more in contrast to men. Men has a lot more stable clothing insulation level within each day. However, some women does also have a steady clothing insulation level through out the day, so there is definitely a difference between each individual subject.

2. A mixed effect model with subjId as a random effect

As indicated in the previous section, we see that there is a clear depiction between each subject, thus we include subjId as a random effect. Furthermore, we saw in figure 2 that the change in clo is different for each subject, thus we also include a random slope for tOut and tInOp, respectively. This is done as we expect that they have an impact on the clo variable. The initial model we fit is thus given by

$$\begin{aligned} clo_i =& \mu + \beta_1 \cdot day_i + \beta_2 \cdot time_i + \beta_3 \cdot sex_i + \beta_4 \cdot tInOp_i + \beta_5 \cdot tOut_i \\ &+ \beta_6 \cdot sex_i : tOut_i + \beta_7 \cdot sex_i : tInOp_i + \alpha(subjId_i) + z_1(subjId_i) \cdot tInOp_i \\ &+ z_2(subjId_i) \cdot tOut_i + \epsilon_i, \end{aligned} \tag{1}$$

where we have

$$\begin{aligned} \alpha(subjId_i) &\sim \mathcal{N}(0, \sigma_\alpha^2) \\ z_1(subjId_i) &\sim \mathcal{N}(0, \sigma_{z_1}^2) \\ z_2(subjId_i) &\sim \mathcal{N}(0, \sigma_{z_2}^2) \\ \epsilon_i &\sim \mathcal{N}(0, \sigma_\epsilon^2). \end{aligned}$$

Furthermore, $sex_i : tOut_i$ and $sex_i : tInOp_i$ denotes the interaction between sex and tOut and tInOp, respectively. Note that throughout the model building/development we do not use restricted maximum likehood, REML, but the final model will be presented with parameters estimated using REML.

First, we consider if the random slopes are significant by fitting reduced models where each random slope is removed, respectively. The initial model is then compared to the reduced model by computing the ANOVA table. In other words, we do a Likelihood ratio test. The results is as follow:

Variable	npar	AIC	BIC	logLik	deviance	Chisq	Df	Pr(>Chisq)
mixed_reduced	14	-1463.7	-1398.1	745.87	-1491.7			
mixed	17	-1520.2	-1440.5	777.13	-1554.2	62.508	3	1.711e-13 ***

Table 1: Comparison of a reduced model where the random slope tInOp is removed.

Considering the above table, we see that the random slope is highly significant, and thus we should keep it. Now we consider the model, where we remove the random slope tOut:

Variable	npar	AIC	BIC	logLik	deviance	Chisq	Df	Pr(>Chisq)
mixed_reduced	14	-1428.0	-1362.3	727.99	-1456.0			
mixed	17	-1520.2	-1440.5	777.13	-1554.2	98.265	3	< 2.2e-16 ***

Table 2: Comparison of a reduced model where the random slope tOut is removed.

Again, the random slope is highly significant and thus should not be removed. Now we will reduce the model further and consider the fixed effects. The procedure we follow is we compute the type III ANOVA table and remove the most insignificant term to then refit a model (prioritizing higher order terms first). This will be repeated until all terms are significant. The first iteration yields:

Variable	Chisq	Df	Pr(>Chisq)
(Intercept)	61.3109	1	4.874e-15
day	45.1517	3	8.591e-10
time	6.8887	1	0.008674
sex	3.7684	1	0.052229
tInOp	0.3636	1	0.546487
tOut	9.6735	1	0.001869
sex:tOut	0.5406	1	0.462173
sex:tInOp	0.3334	1	0.563635

Table 3: Type III ANOVA table.

Considering the higher order terms, the most insignificant term is sex:tInOp, which then will be removed and the reduced model is fitted. Repeating this until all terms are significant yields the following final model:

$$\begin{aligned} clo_i = \mu &+ \beta_1 \cdot day_i + \beta_2 \cdot time_i + \beta_3 \cdot sex_i + \beta_5 \cdot tOut_i + \beta_6 \cdot sex_i : tOut \\ &+ \alpha(subjId_i) + z_1(subjId_i) \cdot tInOp_i + z_2(subjId_i) \cdot tOut_i + \epsilon_i, \end{aligned} \tag{2}$$

where, as before

$$\begin{aligned} &\alpha(subjId_i) \sim \mathcal{N}(0, \sigma_\alpha^2) \\ &z_1(subjId_i) \sim \mathcal{N}(0, \sigma_{z_1}^2) \\ &z_2(subjId_i) \sim \mathcal{N}(0, \sigma_{z_2}^2) \\ &\epsilon_i \sim \mathcal{N}(0, \sigma_\epsilon^2). \end{aligned}$$

	DF	AIC	BIC
model (2)	15	-1523.8	-1453.5

Table 4: Akaike's- and Bayesian information criteria for the model.

The estimated fixed effect parameters are given table 5 and the variance estimate of the random effects are given in table 6.

Variable	Estimate	Std. Error	2.5%	97.5%
(Intercept)	0.943096	0.062092	0.8212	1.0649
day2	-0.021038	0.007991	-0.03672	-0.005352
day3	-0.046513	0.008916	-0.06401	-0.02901
day4	0.162002	0.044243	0.07516	0.2489
time	0.004648	0.001759	0.001195	0.008101
sexmale	-0.243136	0.084703	-0.4094	-0.07687
tOut	-0.016313	0.003002	-0.02221	-0.01042
tOut:sexmale	0.007871	0.003907	0.0002019	0.015547

Table 5: Estimated fixed effect parameters of the model in equation (2).

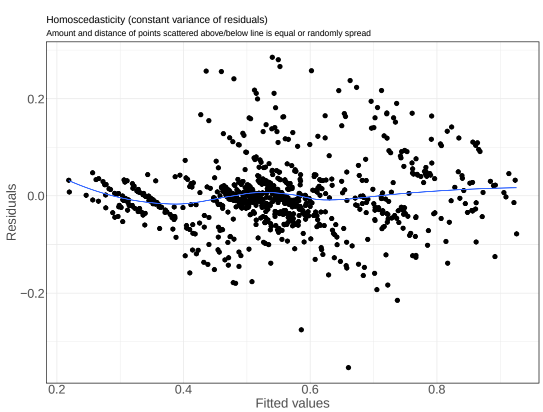
Groups	Name	Variance	Std.Dev.
subjId	(Intercept)	0.1994081	0.44655
	tInOp	0.0008184	0.02861
	tOut	0.0004480	0.02117
Residual		0.0057947	0.07612

Table 6: Estimated random effect variance of the model in equation (2).

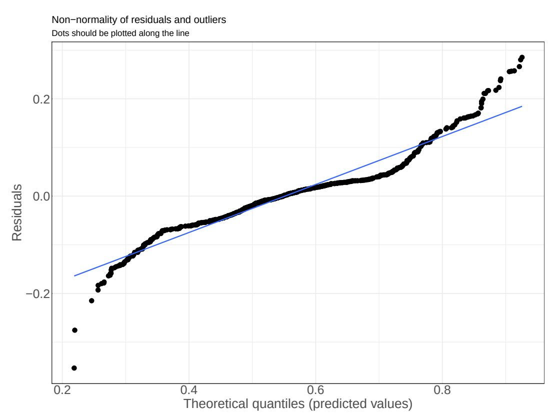
Considering table 5 we see that day2 and day3 are slightly negative, hence contributing to a lower clothing insulation level. On the other hand, day4 is positive and has a significantly larger magnitude indicating that it influences clo more. So, if day4 is one, then we would expect clo to increase by 0.162002, on average, while every other variable is kept constant. What is also worth noting, is that if sexmale is one, then when all other variables are kept constant, then we expect clo to decrease by -0.243136 on average. This to be expected from figure 1a and 2a-2d, where it is seen that males in general has a lower clothing insulation level. Furthermore, tOut is estimated to be -0.016313, thus if the outside temperature is increased by one unit, then we expect clo to decrease by -0.016313 on average which is also to be expected.

From table 6 we see that the standard deviation for the Intercept grouped by subjId is $\sigma_\alpha = 0.44655$. In comparison to the random slopes which are $\sigma_{z_1} = 0.02861$ and $\sigma_{z_2} = 0.02117$ we see that it is significantly larger. This may indicate, that there is a clear difference between each individual subject. The residual standard deviation is $\sigma_\epsilon = 0.07612$ and is the amount of unexplained variation within each subjId.

Based on the derived model, we want to do residual diagnostics. In figure 3a and 3b we have plotted the residuals, and a QQ-plot, respectively. It is clearly seen that the residuals are not homoscedastic and does not follow the theoretical quantiles of the normal distribution.



(a) Residuals of model (2)



(b) QQ-plot of the residuals of model (2)

If we consider the density of the residuals in figure 4, we see that the distribution of the residuals has a lot higher peak and is more slim. Also, it seems to have heavier tails which is also depicted in the QQ-plot in 3b.

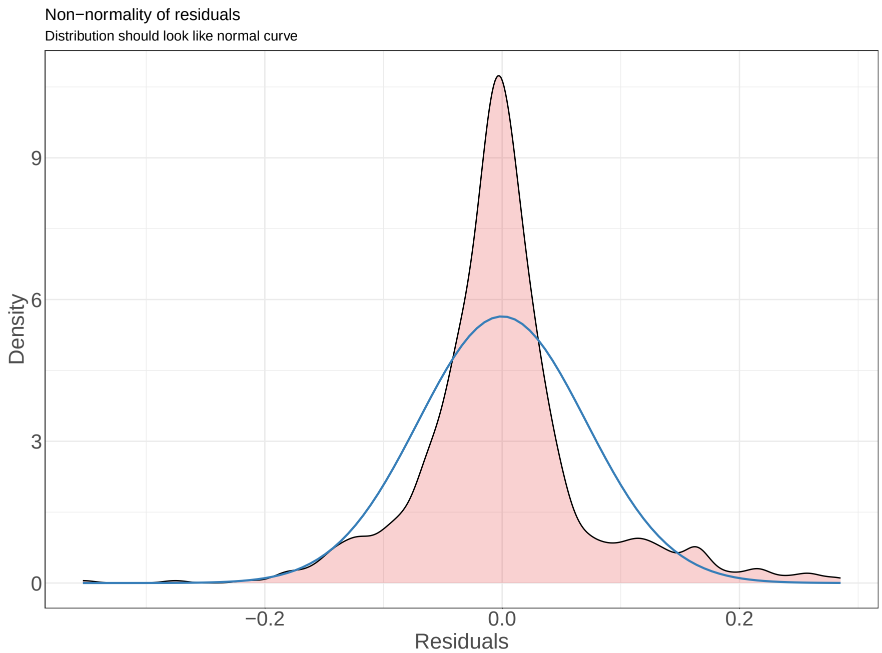


Figure 4: Density of the residuals for model (2) as well as the theoretical normal distribution

Figure 5 shows a QQ-plot of the random effects. Again we see that they lack normality.

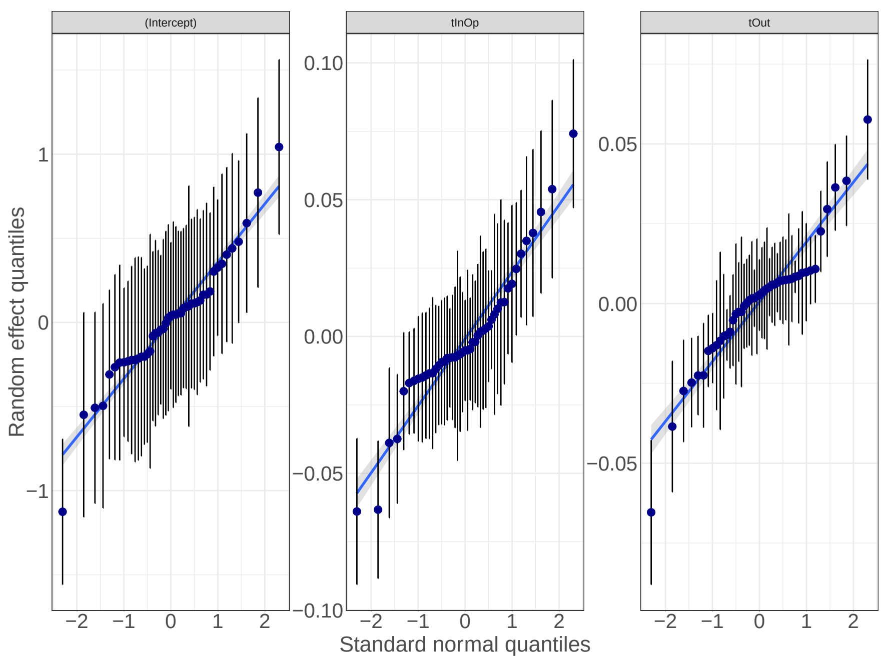


Figure 5: Normal QQ-plots for the random effects of model (2)

Based on the analysis of the residuals, we in general see that they are violated, and thus the model may be inaccurate.

3. A mixed effect model that include subjId and day (nested to subjId) as random effects

We now consider a mixed effect model with subjId and day (nested to subjId) as a random effect. We consider the same initial model as in equation (1), but where we substitute the random effects with subjId and day (nested to subjId). However, as stated in the problem description, we can use subDay to approximate this, thus we end with

$$\begin{aligned} clo_i =& \mu + \beta_1 \cdot day_i + \beta_2 \cdot time_i + \beta_3 \cdot sex_i + \beta_4 \cdot tInOp_i + \beta_5 \cdot tOut_i \\ & + \beta_6 \cdot sex_i : tOut_i + \beta_7 \cdot sex_i : tInOp_i + \alpha(subDay_i) + z_1(subDay_i) \cdot tInOp_i \\ & + z_2(subDay_i) \cdot tOut_i + \epsilon_i, \end{aligned} \tag{3}$$

where we have

$$\begin{aligned} \alpha(subDay_i) &\sim \mathcal{N}(0, \sigma^2_\alpha) \\ z_1(subDay_i) &\sim \mathcal{N}(0, \sigma^2_{z_1}) \\ z_2(subDay_i) &\sim \mathcal{N}(0, \sigma^2_{z_2}) \\ \epsilon_i &\sim \mathcal{N}(0, \sigma^2_\epsilon). \end{aligned}$$

To consider if the random effect structure is appropriate, we formulate nested sub-models and use ANOVA tables, following the exact similar approach as in the previous section. The nested models we consider are where the random slope for tInOp and tOut are removed, respectively. They are called m3a and m3b, respectively. The results of the tests are shown in table (8).

Table 7: Comparison of Models

Model	npar	AIC	BIC	logLik	deviance	Chisq	Df	Pr(¿Chisq)
m3a	14	-1891.8	-1826.2	959.91	-1919.8			
m3	17	-1921.8	-1842.1	977.89	-1955.8	35.944	3	7.695e-08 ***
m3b	14	-1890.8	-1825.1	959.38	-1918.8			
m3	17	-1921.8	-1842.1	977.89	-1955.8	37.008	3	4.583e-08 ***

Table 8: Multiple Likelihood ratio tests

When considering the table, it becomes clear that both terms should be included. Now we proceed to reduce the fixed effect terms by the usual method. The final model is

$$\begin{aligned} clo_i =& \mu + \beta_1 \cdot sex_i + \beta_2 \cdot tInOp_i + \beta_3 \cdot sex_i : tInOp_i + \alpha(subDay_i) + z_1(subDay_i) \cdot tInOp_i \\ & + z_2(subDay_i) \cdot tOut_i + \epsilon_i, \end{aligned} \tag{4}$$

where we have

$$\begin{aligned} \alpha(subDay_i) &\sim \mathcal{N}(0, \sigma^2_\alpha) \\ z_1(subDay_i) &\sim \mathcal{N}(0, \sigma^2_{z_1}) \\ z_2(subDay_i) &\sim \mathcal{N}(0, \sigma^2_{z_2}) \\ \epsilon_i &\sim \mathcal{N}(0, \sigma^2_\epsilon). \end{aligned}$$

	DF	AIC	BIC
model (4)	11	-1928.374	-1876.802

Table 9: Akaike's- and Bayesian information criteria of the model.

Comparing the AIC and BIC of table 9 with the values of table 4, then we a significant decrease in both the AIC and BIC, hence indicating that the model with subjId and day (nested to subjId) as random effects explains the variation of the data better than the model in equation (2).

The estimated fixed effects are given in the table below:

Variable	Estimate	Std. Error	2.5%	97.5%
(Intercept)	1.003310	0.062623	0.8803859	1.1262341
sexmale	-0.397658	0.090520	-0.5753418	-0.2199742
tInOp	-0.015111	0.002127	-0.01928614	-0.01093586
sexmale:tInOp	0.011537	0.003074	0.005502976	0.017571024

Table 10: Estimated coefficient values for the mixed effects model.

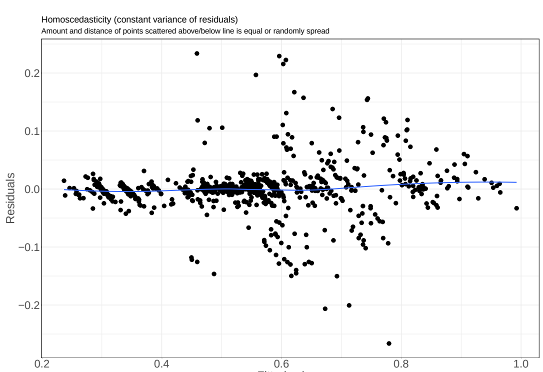
The random effects are depicted in the following table:

Groups	Name	Variance	Std.Dev.
subDay	(Intercept)	0.1414396	0.37608
	tInOp	0.0004330	0.02081
	tOut	0.0001872	0.01368
Residual		0.0024311	0.04931

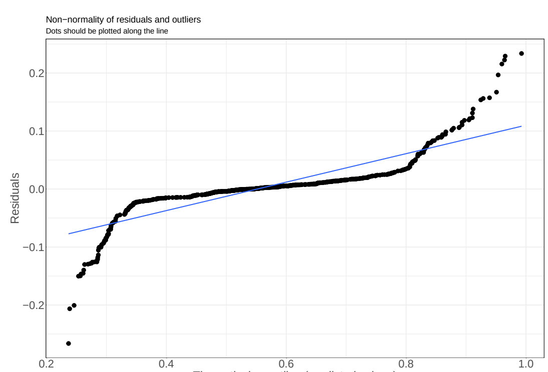
Table 11: Random effect estimates for the linear mixed effects model.

We notice that the variation of the intercept is significantly greater than the other variations. Furthermore, we see that the residual variance is about 2-3 times smaller than the residual variance of model (2) as seen in table 6 which is to be expected, since the variation in the data is better described by this model.

Considering the residuals of the model, we notice that the residuals does not have constant variance and does not seem to be normally distributed as seen in figure 6.



(a) Residuals of model (4)



(b) QQ-plot of the residuals of model (4)

Figure 6

Also when considering the distribution in figure 7, it is more clear that the residuals are not normally distributed, and hence violates this assumption.

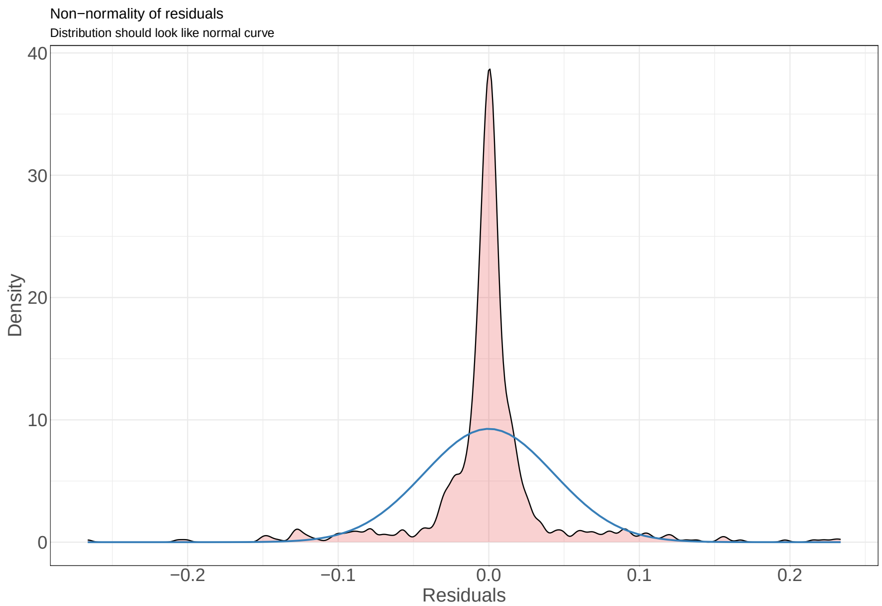


Figure 7: Density of the residuals for model (4) as well as the theoretical normal distribution

Finally, when considering the random effects, they again violate the underlying assumption of normality.

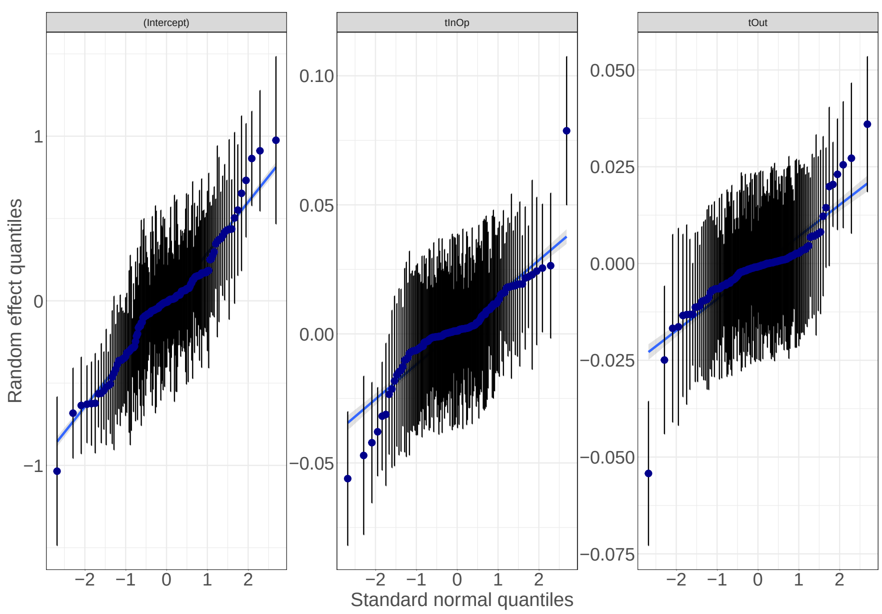


Figure 8: Normal QQ-plots for the random effects of model (4)

Based on this, we remain to conclude that the underlying model assumptions are violated.

4. A mixed effects model with within day auto-correlation

The initial model we consider is based on within day auto-correlation and only random intercepts. We consider two different auto-correlation structures; the auto-regressive process where the present value is based on the preceding value, i.e. AR(1) and the exponential. So, we include a random intercept for subDay and within day auto-correlation within each day.

We reduced the fixed effects of each model by the usual method. This resulted in two different fixed effect structures; for the AR(1) model the terms sex, $tOut$, $tInOp$ and the interaction $sex : tInOp$ are included. On the other hand, the model with the exponential auto-correlation has the terms sex, $tInOp$ and the interaction $sex : tInOp$ included.

Considering table 12 below of the AIC and BIC of the two respective models, we see that both the AIC and BIC of the exponential model is lower. Thus this model describes the variation in the data better than the AR(1) model. In figure 9 we have plotted a variogram for the exponential model which shows that the predicted correlation follows the estimated correlation at different distances quite well. Also, note that the variability increases gradually as distance, or time difference, increases.

	DF	AIC	BIC
AR(1)	8	-1993.055	-1955.548
Exponential	7	-2011.892	-1979.074

Table 12: Akaike's- and Bayesian information criteria for the two models.

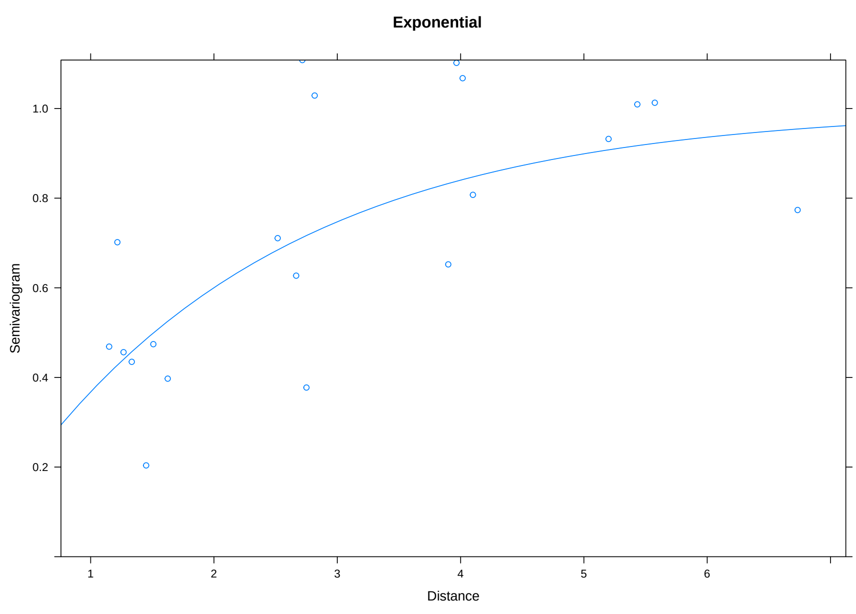


Figure 9: Variogram of the model with the exponential auto-correlation structure

The final model is given by

$$\mathbf{clo} \sim \mathcal{N}(\boldsymbol{\eta}, \mathbf{V}) \tag{5}$$

where

$$\eta_i = \mu + \beta_1(sex_i) + \beta_2(tInOp_i) + \beta_3(sex_i, tInOp_i) \tag{6}$$

and

$$V_{i_1,i_2} = \begin{cases} 0 & \text{if } subDay_{i_1} \neq subDay_{i_2} \\ \nu^2 + \tau^2 \cdot e^{\frac{-|time_{i_2} - time_{i_1}|}{\rho}} & \text{if } subDay_{i_1} = subDay_{i_2} \text{ and } i_1 \neq i_2 \\ \nu^2 + \tau^2 + \sigma^2 & \text{if } i_1 = i_2. \end{cases} \tag{7}$$

Here ν^2 denotes the intercept variation, τ^2 denotes within group variation and ρ is the range.

The fixed effect estimates using REML are given in the table below:

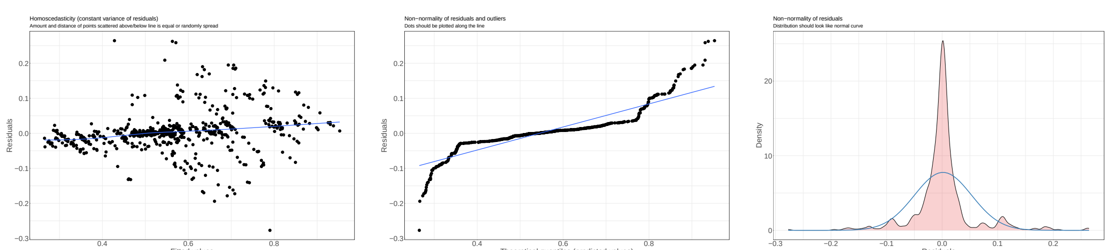
	Value	Std. Error	2.5%	97.5%
(Intercept)	0.91830492	0.06793215	0.784917576	1.051692266
sexmale	-0.35899755	0.09790068	-0.552628043	-0.165367055
tInOp	-0.01220461	0.00245271	-0.017020595	-0.007388616
sexmale:tInOp	0.01024081	0.00354933	0.003271562	0.017210059

Table 13: Fixed effect estimates and 95%-confidence intervals for the mixed effects model.

Group	Variable	Std. Dev. estimate	2.5%	97.5%
	(Intercept)	0.1343275	0.1176429	0.1533784
Residual		0.06435611	0.05779940	0.07165660
Range		2.151807	1.590876	2.910518

Table 14: Standard deviation estimates and 95%-confidence intervals of the random effects in the mixed effects model.

If we consider the residuals of model (5) in figures 10, it is shown that the residuals variance is not constant, in contrast to the assumption. Furthermore, the residuals do not follow the normal distribution, hence the underlying assumption is violated.



(a) Residuals of model (5) plotted versus fitted values.

(b) QQ-plot of the residuals of model (5)

(c) The density of the residuals of model (5)

Figure 10: Residual diagnostics of model (5)

5. Interpretation of model (5)

The final model (5) considers a random intercept for subDay and within day auto-correlation. The exponential auto-correlation model has a lower AIC and BIC values in comparison to the

AR(1) model as seen in table 15. This indicate that it better describes the variation in the data. Furthermore, we see that the model has a significantly smaller AIC and BIC in comparison to all the other models. We notice a big improvement when going from model (2) to model (4) which indeed shows that including a random effect for each day is beneficial. In addition, when incorporating within day auto-correlation shows to be beneficial as well, where exponential structure was best.

Model	# DF	AIC	BIC
model (2)	15	-1523.8	-1453.5
model (4)	11	-1928.374	-1876.802
AR(1)	8	-1993.055	-1955.548
Exponential	7	-2011.892	-1979.074

Table 15: The models we have considered throughout part A.

The fixed effect estimates (Table 13) and random effect standard deviation estimates (Table 14) show the influence of various factors and the associated uncertainty in the mixed-effects model. We notice that the model includes the indoor operating temperature, tInOp, thus we can consider the fixed effects in figure 11.

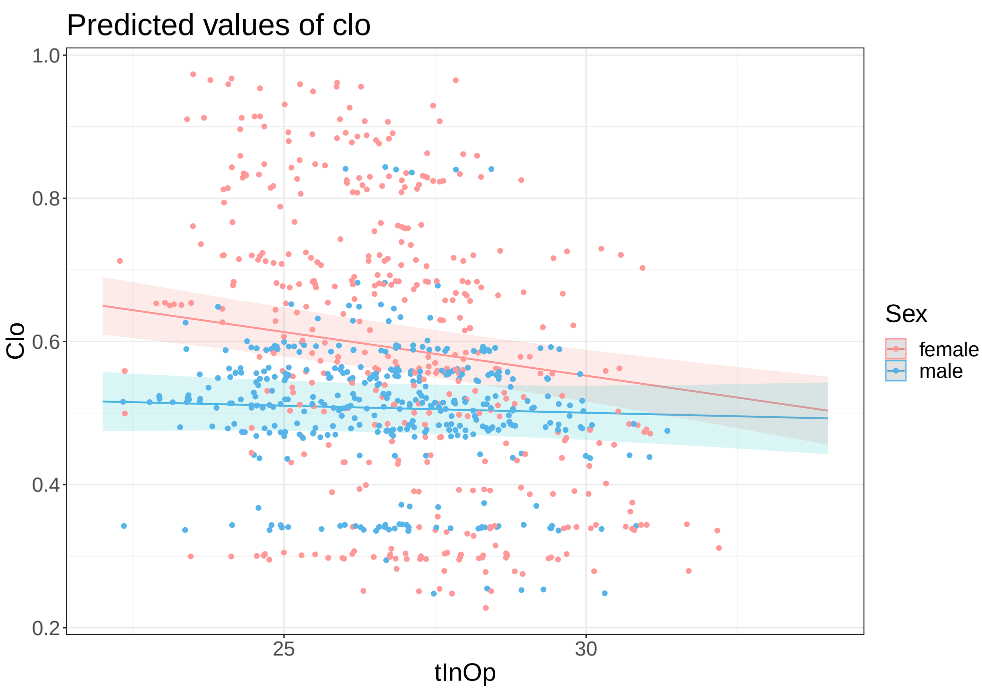
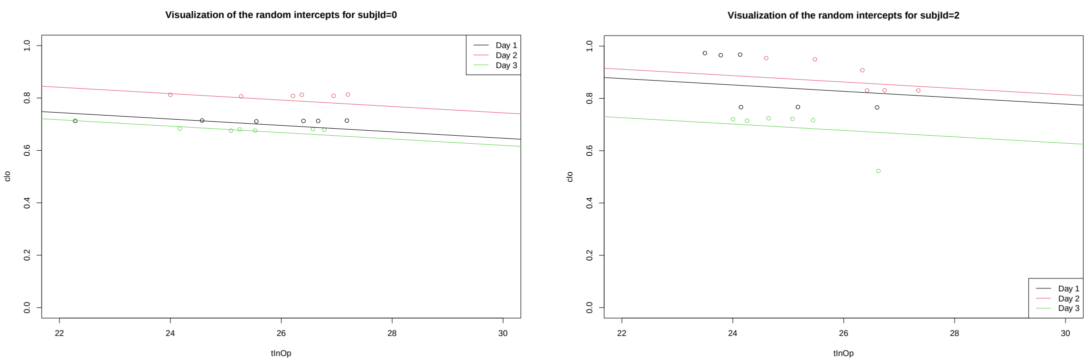


Figure 11: Fixed effect terms plotted on top of the data

We see from figure 11 that the slope for both genders are negative, but females are much more subject to changes in the indoor operating temperature. Furthermore, we notice that level of clothing insulation is lower for the fixed effect for males in comparison to females. Also, the variation seem to be significantly lower for males, which we also saw in figure 1a. In figure 12 we can consider the random effects for subjId 0 and subjId 2. It can be seen that the random effects seem to capture the trend of the data for subjId 0 in figure 12a. However, if we consider subjId 2, then this person seem to change the clothing insulation level throughout the given day, which makes it more difficult to model as can be seen in figure 12b.



(a) Random intercepts for subjId 0 (b) Random intercepts for subjId 2

Figure 12: Random effects

This suggests that there might be room for improving the model to better capture the underlying structure and relationships in the data. Further exploration and adjustments to the model, such as considering non-linear relationships or more complex interaction terms, could potentially enhance the model fit and better represent the data. For instance, if we include a random slope for each day, could possibly lead to a better fit. However, one should notice that the complexity greatly increases, and the optimization problem gets significantly more difficult the more complex structure we consider.

6. Conclusion

In conclusion, the findings in Part A indicate that the mixed effect model, which includes subDay as random effects and inclusion of exponential auto-correlation, provides a better explanation of the variation in the data compared to the considered models.

The estimated fixed effects reveal noteworthy relationships between the variables, such as a negative relationship between sex (male) and the outcome meaning that if the sex of the subject is male, then the expectation of clo is lower. In addition, the parameter of tInOp is also slightly negative, thus if the indoor operating temperature is increased, then we expect the clothing insulation to decrease which is to be expected. The random effects suggest that there is greater variation in the intercept compared to other variations. Additionally, the variation of the intercept in table 14 is significantly larger in comparison to the residual standard deviation. This indicates that there is a significant amount of variation between the groups.

However, it should be noted that the assumptions of constant variance and normally distributed residuals are not met in this model (or in any of the other), as evidenced by the residual plots and density figures. The random effects also do not adhere to the assumption of normality. These violations might have implications for the robustness of the model, and further investigation or refinement may be necessary to address these issues.

Hierarchical models: Random variance

1. Model with subjId as a random intercept

We are considering the model

$$clo_{i,j,k} = \mu + \beta_1(sex_i) + u_i + \epsilon_{i,j,k},$$

where

$$\begin{aligned} u_i &\sim \mathcal{N}(0, \sigma_u^2) \\ \epsilon_{i,j,k} &\sim \mathcal{N}(0, \sigma^2). \end{aligned}$$

Furthermore, $clo_{i,j,k}$ denotes the clothing insulation level for subject i on day j and k refers to the observation number within day i.
This can be rewritten, cf. page 180 in the text book:

$$\mathbf{clo}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{U}_i + \boldsymbol{\epsilon}_i,$$

with

$$\begin{aligned} \mathbf{X}_i &= \mathbf{1}_{N_i \times 2} \\ \boldsymbol{\beta} &= [\mu, \beta_1]^T \\ \mathbf{U}_i &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Psi}_i) \\ \mathbf{Z}_i &= \mathbf{1}_{N_i} \\ \boldsymbol{\epsilon}_i &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_i) \\ \boldsymbol{\Sigma} &= \sigma^2\mathbf{I}_{N_i \times N_i} \\ \boldsymbol{\Psi} &= \sigma_u^2\mathbf{I}_{N_i \times N_i}, \end{aligned}$$

where $\mathbf{1}_{N_i}$ is a column of ones of dimension N_i and $\mathbf{I}_{N_i \times N_i}$ is an identity matrix with dimensions $N_i \times N_i$.
From page 180 in the text book, it follows that the marginal distribution of $\mathbf{clo}_i$ is a multivariate normal distribution, i.e. $\mathbf{clo}_i \sim \mathcal{N}(\mathbf{X}_i\boldsymbol{\beta}, \mathbf{V}_i)$ where

$$E[\mathbf{clo}_i] = E[\mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{U}_i + \boldsymbol{\epsilon}_i] = E[\mathbf{X}_i\boldsymbol{\beta}] + E[\mathbf{Z}_i\mathbf{U}_i] + D[\boldsymbol{\epsilon}_i] = \mathbf{X}_i\boldsymbol{\beta},$$

as the other terms has mean zero. Furthermore we deduce that

$$\begin{aligned} D[\mathbf{clo}_i] = D[\mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{U}_i + \boldsymbol{\epsilon}_i] &= D[\mathbf{X}_i\boldsymbol{\beta}] + D[\mathbf{Z}_i\mathbf{U}_i] + D[\boldsymbol{\epsilon}_i] \\ &= D[\mathbf{Z}_i\mathbf{U}_i] + D[\boldsymbol{\epsilon}_i] \\ &= \mathbf{Z}_i D[\mathbf{U}_i]\mathbf{Z}_i^T + D[\boldsymbol{\epsilon}_i] \\ &= \boldsymbol{\Sigma}_i + \mathbf{Z}_i\boldsymbol{\Psi}_i\mathbf{Z}_i^T = \boldsymbol{\Sigma}_i + \boldsymbol{\Psi}_i = \mathbf{V}_i, \end{aligned}$$

by using the fact that the terms are independent, the fixed effects have zero variance and $\mathbf{Z}_i = \mathbf{1}_{N_i}$ is constant. Given the observations $\mathbf{clo}$ we write the likelihood function as

$$L(\boldsymbol{\theta}; \mathbf{clo}) = \prod_{i=0}^{46} \mathcal{N}(\mathbf{clo}_i; \mathbf{X}_i\boldsymbol{\beta}, \mathbf{V}_i),$$

which implies that the log-likelihood function is given by

$$\ell(\boldsymbol{\theta}; \mathbf{clo}) = \sum_{i=0}^{46} \log \mathcal{N}(\mathbf{clo}_i; \mathbf{X}_i\boldsymbol{\beta}, \mathbf{V}_i) \tag{8}$$

which we will optimize. Here $\boldsymbol{\theta}$ denotes the unknown parameters, $\boldsymbol{\theta} = [\mu,\ \beta_1,\ \sigma^2,\ \sigma_u^2]$. Now that we have expressed the log-likelihood function in equation (8), we can proceed with its optimization. In R, we can use the nlminb function to maximize the log-likelihood function, which requires an initial set of parameter values for $\boldsymbol{\theta}$ and a function that computes the negative log-likelihood given the data and parameter values. The results can be seen in table 16.

Log-likelihood	$\hat{\mu}$	$\hat{\beta_1}$	$\hat{\sigma}^2$	$\hat{\sigma_u}^2$
643.7889	0.591758737	-0.083222822	0.009768104	0.013608124

Table 16: Result of optimizing the log-likelihood in equation (8)

The exact same parameters can be seen when purely using R.

2. Model with subjId and the interaction subjId:day as random intercepts

The model we consider is given by

$$clo_{i,j,k} = \mu + \beta_1(sex_i) + u_i + v_{i,j} + \epsilon_{i,j,k},$$

where

$$\begin{aligned} u_i &\sim \mathcal{N}(0, \sigma_u^2) \\ v_{i,j} &\sim \mathcal{N}(0, \sigma_v^2) \\ \epsilon_{i,j,k} &\sim \mathcal{N}(0, \sigma^2). \end{aligned}$$

Rewriting in accordance to the previous problem, we get

$$\mathbf{clo}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{Z}_i[\mathbf{u}_i, \mathbf{v}_i]^T + \boldsymbol{\epsilon}_i,$$

with

$$\begin{aligned} \mathbf{X}_i &= \mathbf{1}_{N_i \times 2} \\ \boldsymbol{\beta} &= [\mu, \beta_1]^T \\ \mathbf{u}_i &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Psi}_i) \\ \mathbf{v}_i &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Phi}_i) \\ \mathbf{Z}_i &= \mathbf{1}_{N_i \times 2} \\ \boldsymbol{\epsilon}_i &\sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_i) \\ \boldsymbol{\Sigma} &= \sigma^2 \mathbf{I}_{N_i \times N_i} \\ \boldsymbol{\Psi} &= \sigma_u^2 \mathbf{I}_{N_i \times N_i} \\ \boldsymbol{\Phi} &= \sigma_v^2 \mathbf{I}_{N_i \times N_i}. \end{aligned}$$

Again, we want to find the marginal distribution of $\mathbf{clo}_i$, thus we calculate the expectation- and dispersion of $\mathbf{clo}_i$ by the same principles as in the previous problem. We obtain

$$\begin{aligned} E[\mathbf{clo}_i] &= \mathbf{X}_i \boldsymbol{\beta} \\ D[\mathbf{clo}_i] &= \boldsymbol{\Sigma}_i + \boldsymbol{\Psi}_i + \boldsymbol{\Phi}_i. \end{aligned}$$

Again, we can construct the log-likelihood function like previously

$$L(\boldsymbol{\theta}; \mathbf{clo}) = \prod_{i=0}^{46} \mathcal{N}(\mathbf{clo}_i; \mathbf{X}_i \boldsymbol{\beta}, \mathbf{V}_i),$$

which implies that the log-likelihood function is given by

$$\ell(\boldsymbol{\theta}; \mathbf{clo}) = \sum_{i=0}^{46} \log \mathcal{N}(\mathbf{clo}_i; \mathbf{X}_i \boldsymbol{\beta}, \mathbf{V}_i) \tag{9}$$

which we will use to optimize. Here $\boldsymbol{\theta}$ denotes the unknown parameters, $\boldsymbol{\theta} = [\mu,\ \beta_1,\ \sigma^2,\ \sigma_u^2,\ \sigma_v^2]$. Using nlminb to maximize the log likelihood (9) yields

Log-likelihood	$\hat{\mu}$	$\hat{\beta_1}$	$\hat{\sigma}^2$	$\hat{\sigma_u}^2$	$\hat{\sigma_v}^2$
943.1312	0.592420208	-0.084390703	0.003132228	0.010805213	0.009466804

Table 17: Result of optimizing the log-likelihood in equation (9)

The exact same parameters can be seen when using R.

3. Model with subjId and the interaction subjId:day as random intercepts and random variances

Now we will consider the model

$$\begin{aligned} clo_{i,j,k} &= \mu + \beta_1(sex_i) + u_i + v_{i,j} + \epsilon_{i,j,k} \\ u_i &\sim \mathcal{N}(0, \sigma_u^2 \alpha(sex_i)) \\ v_{i,j} &\sim \mathcal{N}(0, \sigma_v^2 \alpha(sex_i)) \\ \epsilon_{i,j,k} &\sim \mathcal{N}(0, \sigma^2 \alpha(sex_i)) \end{aligned}$$

Rewriting this model in matrix-vector form, we get

$$\mathbf{clo}_i = \mathbf{X}_i \boldsymbol{\beta} + \mathbf{Z}_i [\mathbf{u}_i, \mathbf{v}_i]^T + \boldsymbol{\epsilon}_i,$$

where

$$\begin{aligned} &\mathbf{X}_i = \mathbf{1}_{N_i \times 2} \\ &\boldsymbol{\beta} = [\mu, \beta_1]^T \\ &\mathbf{u}_i \sim \mathcal{N}(\mathbf{0}, \alpha(sex_i)\boldsymbol{\Psi}_i) \\ &\mathbf{v}_i \sim \mathcal{N}(\mathbf{0}, \alpha(sex_i)\boldsymbol{\Phi}_i) \\ &\mathbf{Z}_i = \mathbf{1}_{N_i \times 2} \\ &\boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \alpha(sex_i)\boldsymbol{\Sigma}_i) \\ &\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}_{N_i \times N_i} \\ &\boldsymbol{\Psi} = \sigma_u^2 \mathbf{I}_{N_i \times N_i} \\ &\boldsymbol{\Phi} = \sigma_v^2 \mathbf{I}_{N_i \times N_i}. \end{aligned}$$

Again, we want to consider the covariance matrix of $\mathbf{clo_i}$. As $\alpha(sex_i)$ is simply a parameter, then we obtain

$$\mathbf{V}_i = \alpha(sex_i)(\boldsymbol{\Sigma}_i + \boldsymbol{\Psi}_i + \boldsymbol{\Phi}_i).$$

The log-likelihood function is given as previously:

$$\ell(\boldsymbol{\theta}; \mathbf{clo}) = \sum_{i=0}^{46} \log \mathcal{N}(\mathbf{clo}_i; \mathbf{X}_i \boldsymbol{\beta}, \mathbf{V}_i) \tag{10}$$

where $\boldsymbol{\theta}$ denotes the new set of unknown parameters, $\boldsymbol{\theta} = [\mu,\ \beta_1,\ \sigma^2,\ \sigma_u^2,\ \sigma_v^2,\ \alpha_{male},\ \alpha_{female}]$. The results of maximizing the log-likelihood are given in table 18 below.

Log-likelihood	$\hat{\mu}$	$\hat{\beta_1}$	$\hat{\sigma}^2$	$\hat{\sigma_u}^2$	$\hat{\sigma_v}^2$	$\hat{\alpha}_{male}$	$\hat{\alpha}_{female}$
1214.93	0.59259184	-0.08487950	0.01140940	0.05481891	0.12522613	0.02721263	0.44501121

Table 18: Result of optimizing the log-likelihood in equation (10)

Considering table 18 we see that the fixed effects stay approximately the same, whereas the variances changed, especially $\hat{\sigma}_v^2$. It is interesting than $\hat{\alpha}_{female}$ is much greater than $\hat{\alpha}_{male}$, which is to be expected based on the exploratory analysis and part A. This means that the scaling of the variances for females are greater in comparison to males.

4. Theorem 6.7

We are considering

$$\begin{aligned} Y_i|\gamma_i &\sim \mathcal{N}\left(\mu, \frac{\sigma^2}{\gamma_i}\right) \\ \gamma_i &\sim G(1, \phi), \end{aligned}$$

so γ_i has mean 1 and variance $\frac{1}{\phi}$. With this knowledge, we can derive the parameters of the usual Gamma distribution

$$\gamma_i \sim G(\alpha, \beta),$$

which means that

$$\begin{aligned} & E[\gamma_i] = 1 = \frac{\alpha}{\beta} \\ & Var[\gamma_i] = \frac{1}{\phi} = \frac{\alpha}{\beta^2} \\ & \Updownarrow \\ & \alpha = \beta = \phi. \end{aligned}$$

So, in case of γ, the probability density function is given as

$$f_\gamma(\gamma; \phi) = \frac{\phi^\phi}{\Gamma(\phi)} \gamma_i^{\phi-1} e^{-\phi\gamma_i}.$$

Furthermore, the conditional distribution of Y_i given γ_i is given by

$$f_{Y_i|\gamma_i}(y; \mu, \sigma, \phi) = \frac{1}{\sqrt{\frac{\sigma^2}{\gamma_i}}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sqrt{\frac{\sigma^2}{\gamma_i}}}\right)^2}.$$

In order to find the marginal distribution of Y_i, we write the expectation as follows

$$f_{Y_i}(y; \mu, \sigma, \phi) = E[f_{Y_i|\gamma_i}(y; \mu, \sigma, \gamma)] = \int_\gamma f_{Y_i|\gamma_i}(y; \mu, \sigma, \gamma) f_{\gamma_i}(\gamma; \phi)\ d\gamma. \tag{11}$$

First we consider

$$f_{Y_i|\gamma_i}(y;\mu,\sigma,\gamma)f_{\gamma_i}(\gamma;\phi) = \frac{1}{\sqrt{\frac{\sigma^2}{\gamma_i}}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sqrt{\frac{\sigma^2}{\gamma_i}}}\right)^2} \cdot \frac{\phi^\phi}{\Gamma(\phi)}\gamma_i^{\phi-1}e^{-\phi\gamma_i} \tag{12}$$

$$= \frac{1}{\sqrt{\frac{\sigma^2}{\gamma_i}}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sqrt{\frac{\sigma^2}{\gamma_i}}}\right)^2-\phi\gamma_i} \cdot \frac{\phi^\phi}{\Gamma(\phi)}\gamma_i^{\phi-1} \tag{13}$$

$$= \frac{\sqrt{\gamma_i}}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sqrt{\frac{\sigma^2}{\gamma_i}}}\right)^2-\phi\gamma_i} \cdot \frac{\phi^\phi}{\Gamma(\phi)}\gamma_i^{\phi-1} \tag{14}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sqrt{\frac{\sigma^2}{\gamma_i}}}\right)^2-\phi\gamma_i} \cdot \frac{\phi^\phi}{\Gamma(\phi)}\gamma_i^{\phi-1/2}. \tag{15}$$

In step (13) we simply combine the exponentials. Next, we expand $\sqrt{\frac{\sigma^2}{\gamma_i}}$ in step (14). At last, we multiply $\sqrt{\gamma_i}\gamma_i^{\phi-1} = \gamma_i^{\phi-1/2}$ to obtain (15).
Now we insert into the integral of equation (11)

$$f_{Y_i}(y;\mu,\sigma,\phi) = \int_\gamma f_{Y_i|\gamma_i}(y;\mu,\sigma,\gamma)f_{\gamma_i}(\gamma;\phi)\; d\gamma$$

$$= \int_\gamma \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sqrt{\frac{\sigma^2}{\gamma_i}}}\right)^2-\phi\gamma_i} \cdot \frac{\phi^\phi}{\Gamma(\phi)}\gamma_i^{\phi-1/2}\; d\gamma \tag{16}$$

$$= \int_\gamma \frac{\phi^\phi}{\sigma\sqrt{2\pi}\Gamma(\phi)} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sqrt{\frac{\sigma^2}{\gamma_i}}}\right)^2-\phi\gamma_i} \cdot \gamma_i^{\phi-1/2}\; d\gamma \tag{17}$$

$$= \frac{\phi^\phi}{\sigma\sqrt{2\pi}\Gamma(\phi)} \int_\gamma e^{-\frac{1}{2}\left(\frac{y-\mu}{\sqrt{\frac{\sigma^2}{\gamma_i}}}\right)^2-\phi\gamma_i} \cdot \gamma_i^{\phi-1/2}\; d\gamma \tag{18}$$

$$= \frac{\phi^\phi}{\sigma\sqrt{2\pi}\Gamma(\phi)} \int_\gamma \gamma_i^{\phi-1/2} e^{-\gamma_i\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2}}\; d\gamma, \tag{19}$$

by expanding the exponential and rearranging. If we rewrite

$$\gamma_i^{\phi-1/2} = \gamma_i^{(\gamma_i+1/2)-1},$$

then (19) becomes

$$f_{Y_i}(y;\mu,\sigma,\phi) = \frac{\phi^\phi}{\sigma\sqrt{2\pi}\Gamma(\phi)} \int_\gamma \gamma_i^{(\phi+1/2)-1} e^{-\gamma_i\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2}}\; d\gamma,$$

and we notice that the integrand is the kernel in the probability density function of the Gamma distribution, $G\left(\phi+1/2, \left(\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2}\right)^{-1}\right)$. As the integral of a density function by definition is one, we find by adjusting the normalization constant that

$$f_{Y_i}(y;\mu,\sigma,\phi) = \frac{\phi^\phi}{\sigma\sqrt{2\pi}\Gamma(\phi)}\Gamma(\phi+1/2)\cdot\left(\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2}\right)^{-(\phi+1/2)}.$$

By rearranging we obtain

$$\begin{aligned}
f_{Y_i}(y;\mu,\sigma,\phi) &= \frac{1}{\sigma\sqrt{2\pi}}\phi^{\phi}\frac{\Gamma(\phi+1/2)}{\Gamma(\phi)}\cdot\left(\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2}\right)^{-(\phi+1/2)} \\
&= \frac{1}{\sigma\sqrt{2\pi}}\cdot\frac{\sqrt{\phi}}{\sqrt{\phi}}\cdot\phi^{\phi}\frac{\Gamma(\phi+1/2)}{\Gamma(\phi)}\cdot\left(\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2}\right)^{-(\phi+1/2)} \\
&= \frac{1}{\sigma\sqrt{2\pi}\sqrt{\phi}}\cdot\phi^{\phi+1/2}\cdot\frac{\Gamma(\phi+1/2)}{\Gamma(\phi)}\cdot\left(\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2}\right)^{-(\phi+1/2)}
\end{aligned}$$

which is equivalent with

$$f_{Y_i}(y;\mu,\sigma,\phi) = \frac{1}{\sigma\sqrt{2\phi\pi}}\cdot\left(\frac{1}{\phi}\right)^{-(\phi+1/2)}\cdot\frac{\Gamma(\phi+1/2)}{\Gamma(\phi)}\cdot\left(\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2}\right)^{-(\phi+1/2)}. \tag{20}$$

By simplifying (20) we obtain

$$f_{Y_i}(y;\mu,\sigma,\phi) = \frac{1}{\sigma\sqrt{2\phi\pi}}\cdot\frac{\Gamma(\phi+1/2)}{\Gamma(\phi)}\cdot\left(\frac{(y-\mu)^2+2\phi\sigma^2}{2\sigma^2\cdot\phi}\right)^{-(\phi+1/2)}, \tag{21}$$

where we can split the last fraction and simplify further to end with

$$\begin{aligned}
f_{Y_i}(y;\mu,\sigma,\phi) &= \frac{1}{\sigma\sqrt{2\phi\pi}}\cdot\frac{\Gamma(\phi+1/2)}{\Gamma(\phi)}\cdot\left(\frac{(y-\mu)^2}{2\sigma^2\cdot\phi}+1\right)^{-(\phi+1/2)} && (22)\\
&= \frac{1}{\sigma\sqrt{2\phi\pi}}\cdot\frac{\Gamma(\phi+1/2)}{\Gamma(\phi)}\cdot\left(\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2\phi}+1\right)^{-(\phi+1/2)} && (23)
\end{aligned}$$

which we recognize as a student's t-distribution times a constant, so we conclude

$$f_{Y_i} \sim \frac{1}{\sigma}f_0\left(\frac{y-\mu}{\sigma};2\phi\right).$$

5. Find marginal distribution of clo_i

We are now considering the model

$$\begin{aligned}
clo_{i,j,k|u_i,v_i,\gamma_i} &\sim \mathcal{N}(\mu+\beta(sex_i)+u_i+v_{i,j},\sigma^2\alpha(sex_i)/\gamma_i) \\
u_i|\gamma_i &\sim \mathcal{N}(0,\sigma_u^2\alpha(sex_i)/\gamma_i) \\
v_{i,j}|\gamma_i &\sim \mathcal{N}(0,\sigma_v^2\alpha(sex_i)/\gamma_i) \\
\gamma_i &\sim G(1,\phi).
\end{aligned}$$

As we have seen in the previous questions, we can model this as a multivariate normal distribution with mean

$$E[clo_{i,j,k|\gamma_i}] = \mathbf{X}[\mu,\beta]^T$$

and covariance

$$D[clo_{i,j,k|\gamma_i}] = \mathbf{V}_i = \frac{\alpha(sex_i)}{\gamma_i}(\mathbf{\Sigma}_i+\mathbf{\Psi}_i+\mathbf{\Phi}_i).$$

The marginal distribution of $clo_{i,jk}$ is given by the expectation

$$f_{\mathbf{clo}_i} = E[f_{\mathbf{clo}_i|\gamma_i}] = \int_\gamma f_{\mathbf{clo}_i|\gamma_i}f_{\gamma_i}\,d\gamma \tag{24}$$

If we let N_i denote the dimension of $\mathbf{V}_i$, then when inserting the multivariate normal distribution and the Gamma distribution in case of $f_{\mathbf{clo}_i|\gamma_i}$ and f_{γ_i}, respectively, yields

$$f_{\mathbf{clo}_i} = \int_\gamma \frac{1}{\sqrt{(2\pi)^{N_i}\left(\frac{1}{\gamma_i}\right)^{N_i}|\mathbf{V}_i|}} \exp\left(-\frac{1}{2}\gamma_i(\mathbf{X}_i-\boldsymbol{\mu}_i)^T \boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)\right)\frac{\phi^\phi}{\Gamma(\phi)}\gamma_i^{\phi-1}\exp\left(-\phi\gamma_i\right)\, d\gamma \tag{25}$$

$$= \frac{1}{\sqrt{(2\pi)^{N_i}|\mathbf{V}_i|}}\frac{\phi^\phi}{\Gamma(\phi)}\int_\gamma \gamma_i^{\frac{N_i}{2}}\exp\left(-\frac{1}{2}\gamma_i(\mathbf{X}_i-\boldsymbol{\mu}_i)^T \boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)\right)\gamma_i^{\phi-1}\exp\left(-\phi\gamma_i\right)\, d\gamma, \tag{26}$$

by moving the constants outside the integral. Rearranging and simplifying equation (26) results in

$$f_{\mathbf{clo}_i} = \frac{1}{\sqrt{(2\pi)^{N_i}|\mathbf{V}_i|}}\frac{\phi^\phi}{\Gamma(\phi)}\int_\gamma \gamma_i^{\frac{N_i}{2}}\gamma_i^{\phi-1}\exp\left(-\gamma_i\left(\frac{1}{2}(\mathbf{X}_i-\boldsymbol{\mu}_i)^T \boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)+\phi\right)\right) d\gamma. \tag{27}$$

To proceed further, we use the same trick as in problem 4:

$$\gamma_i^{\frac{N_i}{2}}\gamma_i^{\phi-1} = \gamma_i^{\frac{N_i+2\phi}{2}-1},$$

which thus results in

$$f_{\mathbf{clo}_i} = \frac{1}{\sqrt{(2\pi)^{N_i}|\mathbf{V}_i|}}\frac{\phi^\phi}{\Gamma(\phi)}\int_\gamma \gamma_i^{\frac{N_i+2\phi}{2}-1}\exp\left(-\gamma_i\left(\frac{1}{2}(\mathbf{X}_i-\boldsymbol{\mu}_i)^T \boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)+\phi\right)\right) d\gamma. \tag{28}$$

As in problem 4, we notice that the integrand is the kernel of the probability density function of the Gamma distribution, $G\left(\frac{N_i+2\phi}{2},\ \left(\frac{1}{2}(\mathbf{X}_i-\boldsymbol{\mu}_i)^T\boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)+\phi\right)^{-1}\right)$, so by adjusting the normalization constant we obtain

$$f_{\mathbf{clo}_i} = \frac{1}{\sqrt{(2\pi)^{N_i}|\mathbf{V}_i|}}\frac{\phi^\phi}{\Gamma(\phi)}\Gamma\left(\frac{N_i+2\phi}{2}\right)\cdot\left(\frac{1}{2}(\mathbf{X}_i-\boldsymbol{\mu}_i)^T \boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)+\phi\right)^{-\frac{N_i+2\phi}{2}} \tag{29}$$

$$= \frac{1}{\sqrt{(2\pi)^{N_i}|\mathbf{V}_i|}}\frac{\Gamma\left(\frac{N_i+2\phi}{2}\right)}{\Gamma(\phi)}\phi^\phi\cdot\left(\frac{1}{2}(\mathbf{X}_i-\boldsymbol{\mu}_i)^T \boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)+\phi\right)^{-\frac{N_i+2\phi}{2}} \tag{30}$$

$$= \frac{1}{\sqrt{(2\pi)^{N_i}|\mathbf{V}_i|}}\frac{\Gamma\left(\frac{N_i+2\phi}{2}\right)}{\Gamma(\phi)}\phi^\phi\cdot\phi^{-\frac{N_i+2\phi}{2}}\left(\frac{1}{2\phi}(\mathbf{X}_i-\boldsymbol{\mu}_i)^T \boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)+1\right)^{-\frac{N_i+2\phi}{2}}, \tag{31}$$

now we notice that

$$\phi^\phi\cdot\phi^{-\frac{N_i+2\phi}{2}} = \phi^{-N_i/2} = \sqrt{\left(\frac{1}{\phi}\right)^{N_i}},$$

in thus we can simplify equation (31)

$$f_{\mathbf{clo}_i} = \frac{1}{\sqrt{(2\phi\pi)^{N_i}|\mathbf{V}_i|}}\frac{\Gamma\left(\frac{N_i+2\phi}{2}\right)}{\Gamma(\phi)}\left(\frac{1}{2\phi}(\mathbf{X}_i-\boldsymbol{\mu}_i)^T \boldsymbol{V}_i^{-1}(\mathbf{X}_i-\boldsymbol{\mu}_i)+1\right)^{-\frac{N_i+2\phi}{2}}, \tag{32}$$

which is the exact formulation of a multivariate t-distribution. The likelihood function is set up as in previous problems:

$$L(\boldsymbol{\theta}; \mathbf{clo}) = \prod_{i=0}^{46} f_{\mathbf{clo}}(\mathbf{clo}_i|\boldsymbol{\theta})$$

which implies that the log-likelihood is given by

$$\ell(\boldsymbol{\theta}; \mathbf{clo}) = \sum_{i=0}^{46} \log f_{\mathbf{clo}}(\mathbf{clo}_i|\boldsymbol{\theta}) \tag{33}$$

Optimizing the log-likelihood yielded the result given in table 19 below:

Log-likelihood	$\hat{\mu}$	$\hat{\beta}_1$	$\hat{\sigma}^2$	$\hat{\sigma_u}^2$	$\hat{\sigma_v}^2$
1672.93	0.580815992	-0.078536736	0.006381866	0.056257240	0.082303709

$\hat{\alpha}_{male}$	$\hat{\alpha}_{female}$	γ_i
0.025234688	0.080456259	0.094952756

Table 19: Result of optimizing the log-likelihood in equation (33)

Considering the above table, we notice a big increase in the log-likelihood in comparison to the results of optimizing equation (10) or the results in table 18. The estimates; $\hat{\mu}$ and $\hat{\beta}_1$ stays somewhat the same, but we see a significant decrease in the residual variance $\hat{\sigma}^2$. Furthermore, we see that the scaling of variances with respect to sex is decreased by a lot for females, but still shows to be greater than for men which is to be expected.

6. TMB model

A statistical model is considered where an objective function is defined based on a set of input data and parameters. The model is composed of several components, including a mean value, random effects, and noise. The objective function is then optimized to estimate the model parameters using a Laplace approximation. The resulting parameter estimates are then used to report summary statistics and calculate standard errors.

To implement TMB, a C++ code has been done, but when plugging it to R, an error has appeared that has been unable to be solved, regarding that the error is given by a variable that is not in the C++ file nor the R file.

7. Comparison of section 5 and 6 models

As no results have been obtained from question 6, it is not possible to compare them.