# A Computation Procedure for Reconsideration-Proof Equilibrium

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January 13, 2015

**Abstract**

Reconsideration-proof equilibrium is a refinement proposed by **?** of subgame perfect equilibrium that applies to infinite horizon settings in which time inconsistency is important. A procedure for computing such equilibria is provided. The procedure is applicable under an assumption about time-separability of the utility function. The class of problems that satisfy the assumption includes four of the five examples in **?**.

## 1 Introduction

Reconsideration-proof equilibrium is a refinement proposed by **?** of subgame perfect equilibrium that applies to infinite horizon settings in which time inconsistency is important. To be reconsideration-proof, a strategy must satisfy three properties. First, it must be subgame-perfect. Second, it must have the same continuation value at any histories. Let us call such strategies weakly reconsideration-proof. Accordingly, a continuation value is attached to each weakly reconsideration-proof strategy. Lastly, the strategy must have the highest continuation value among all the weakly reconsideratin-proof strategies.

It is not easy to find reconsideration-proof strategies. Though **?** provides a fixed-point characterization of weakly reconsideratin-proof strategies, it does not tell us much about how to find them.

This paper provides a computation procedure which is applicable under an assumption about time-separability of the utility function. The class of problems that satisfy the assumption includes four of the five examples in **?**.

# 2 Environment

Time is discrete, and indexed by $t$. There is an infinitely lived player.

In each period $t$, there are two subperiods. In the first subperiod, a state variable $z_t \in Z$ is determined, where $Z$ is a set of states. In the second subperiod, knowing the determined value of the state variable $z_t$, the player chooses an action $x_t$ from a set of actions $X$.

It is natural to think that the state variable $z_t$ is determined by a passive player. The passive player formulates an expectation $\{\hat{x}_s\}_{s\geq t}$ on the future actions taken by the infinitely lived player, $\{x_s\}_{s\geq t}$. The state variable $z_t$ is determined by

$$z_t = \xi\left(\{\hat{x}_s\}_{s\geq t}\right),$$

where $\xi$ is a mapping from $X^\infty$ to $Z$.

The utility of the infinitely lived player in period $t$ is given by

$$U(z_t, \{x_s\}_{s\geq t}),$$

where $U$ is a mapping from $X^\infty \times Z$ to $\mathbb{R}$.

A history is an element of $\bigcup_{t\geq 0} X^t$, where $X^0 = \{\emptyset\}$. An element $h^t \in X^t$ is called a history up to $t$.

A strategy $\sigma$ is a mapping from $\bigcup_{t\geq 0} X^t$ to $X$. A path of a strategy $\sigma$ is an element $\{x_t^\sigma\}_{t\geq 0}$ of $X^\infty$ defined by

$$x_0^\sigma = \sigma(\emptyset), \text{ and}$$
$$\text{for each } t \geq 1,\ x_t^\sigma = \sigma\left(\{x_s^\sigma\}_{0\leq s\leq t-1}\right).$$

Similarly, a path of a strategy $\sigma$ after a history $h^{t-1}$ is an element $\{x_s^{\sigma|h^{t-1}}\}_{s\geq t}$ of $X^\infty$ defined by

$$x_t^{\sigma|h^{t-1}} = \sigma(h^{t-1}), \text{ and}$$
$$\text{for each } s \geq t+1,\ x_s^{\sigma|h^{t-1}} = \sigma\left(\left\{h^{t-1}, \left\{x_\tau^{\sigma|h^{t-1}}\right\}_{t\leq\tau\leq s-1}\right\}\right).$$

# 3 Solution Concepts and Characterization

## 3.1 Solution Concepts

**Definition 1.** A strategy $\sigma$ is *subgame perfect* if for any $t$, for any history $h^{t-1}$ up to $t-1$, for any $x \in X$,

$$U(z_t, \{x_s^{\sigma|h^{t-1}}\}_{s\geq t}) \geq U\left(z_t, \left\{x, \{x_s^{\sigma|\{h^{t-1},x\}}\}_{s\geq t+1}\right\}\right),$$

where $z_t = \xi\left(\{x_s^{\sigma|h^{t-1}}\}_{s\geq t}\right)$.

**Definition 2.** A subgame perfect strategy $\sigma$ is *weakly reconsideratin-proof* if it has the same continuation value at any histories.

**Definition 3.** A weakly reconsideratin-proof subgame perfect strategy $\sigma$ is *reconsideration-proof* if it has the highest value among all the weakly reconsideratin-proof subgame perfect strategies.

## 3.2 Characterization

To state the characterization of weakly reconsideratin-proof subgame perfect strategies in **?**, define

$$\begin{aligned} D(V) &= \{\{x_t\}_{t\geq 0} | \text{for each } t \geq 0, U(\xi(\{x_s\}_{s\geq t}), \{x_s\}_{s\geq t}) = V\}, \text{ and} \\ Z(V) &= \{\xi(\{x_t\}_{t\geq 0}) | \{x_t\}_{t\geq 0} \in D(V)\}. \end{aligned}$$

**?** gave the following characterization of weakly reconsideratin-proof subgame perfect strategies.

**Proposition 1.** *[**?**] There exists a weakly reconsideratin-proof subgame perfect strategy with value $V$ if and only if there exists a subset $D^* \subseteq D(V)$ and a subset $Z^* \subseteq Z(V)$ such that*

1. *$Z^* = \xi(D^*)$.*
2. *For all $x \in X$ and all $z \in Z^*$, there exists $d$ in $D^*$ such that $U(z, x, d) \leq V$.*

The characterization does not tell much about how to find such fixed points. In the next section, I specify a subclass of problems and suggest a computation procedure.

# 4 Class of Problems and Computation Procedure

## 4.1 Class of Problems

I suggest a procedure for problems that satisfy the following assumption.

**Assumption 1.** *There exist a constant $\beta$ and a function $u : X^2 \to \mathbb{R}$ such that $U(z_t, \{x_s\}_{s\geq t}) = (1-\beta)\sum_{s=t}^{\infty} \beta^{s-t} u(x_s, x_{s+1})$.*

Using the assumption, I can simplify the characterization of weakly reconsideratin-proof subgame perfect strategies. The simplified characterization will lead to a procedure. To state it, define

$$X_1(V) = \{x \in X | \text{there exists } x' \in X \text{ such that } u(x, x') = V\},$$

$$\text{for each } k,\ X_{k+1}(V) = \{x \in X | \text{there exists } x' \in X_k(V) \text{ such that } u(x, x') = V\},\ \text{and}$$

$$X(V) = \bigcap_{k=1}^{\infty} X_k(V).$$

The next lemma is useful.

**Lemma 1.** *For any $x \in X(V)$, there exists $x' \in X(V)$ such that $u(x, x') = V$.*

**Lemma 2.** *$x \in X(V)$ if and only if there exists $\{x_t\}_{t\geq 0} \in D(V)$ and $t \geq 0$ such that $x_0 = x$.*

*Proof.* Suppose there exists $\{x_t\}_{t\geq 0} \in D(V)$ and $t \geq 0$ such that $x_0 = x$. Then, by definition of $D(V)$, for any $t \geq 0$, $u(x_t, x_{t+1}) = V$. This implies that for any $k \geq 1$, $x \in X_k(V)$. Therefore, $x \in X(V)$.

Suppose $x \in X(V)$. Set $x_0 = x$. By Lemma 1, I can inductively construct a sequence $\{x_t\}_{t\geq 0}$ such that for any $t \geq 0$, $u(x_t, x_{t+1}) = V$. Clearly, $\{x_t\}_{t\geq 0} \in D(V)$. □

Now, the simplified characterization is stated as a proposition.

**Proposition 2.** *There exists a weakly reconsideratin-proof subgame perfect strategy with value $V$ if and only if for all $x \in X$, there exists $x'$ in $X(V)$ such that $u(x, x') \leq V$.*

*Proof.* By Proposition 1, it is sufficient to show the equivalence between the following two statements.

1. there exists a subset $D^* \subseteq D(V)$ and a subset $Z^* \subseteq Z(V)$ such that

   (a) $Z^* = \xi(D^*)$.

   (b) For all $x \in X$ and all $z \in Z^*$, there exists $d$ in $D^*$ such that $U(z, x, d) \leq V$.

2. for all $x \in X$, there exists $x'$ in $X(V)$ such that $u(x, x') \leq V$.

First, since $U$ doesn't depend on $z$ by Assumption 1, the first statement is equivalent to

$$\text{for all } x \in X, \text{ there exists } d \in D(V) \text{ such that } U(x, d) \leq V.$$

Next, using the time separability of $U$, it is equivalent to

$$\text{for all } x \in X, \text{ there exists } \{x_t\}_{t \geq 0} \in D(V)$$
$$\text{such that } (1 - \beta)u(x, x_0) + \beta U(\{x_t\}_{t \geq 1}) \leq V.$$

Since for any $\{x_t\}_{t \geq 0} \in D(V)$, $U(\{x_t\}_{t \geq 1}) = V$, it is equivalent to

$$\text{for all } x \in X, \text{ there exists } \{x_t\}_{t \geq 0} \in D(V) \text{ such that } u(x, x_0) \leq V.$$

By Lemma 2, it is equivalent to the second statement. □

## 4.2 Computation Procedure

The computation procedure for reconsideration-proof equilibrium is the following.

1. Compute $X(V)$ for each $V$, by calculating $\{X_k(V)\}_{k \geq 1}$.

2. Find the largest $V$ such that for all $x \in X$, there exists $x'$ in $X(V)$ such that $u(x, x') \leq V$.

Proposition 2 assures that weakly reconsideratin-proof subgame perfect strategies achieve the largest $V$ are reconsideration-proof. In the next section, I apply the procedure for the examples in ?.

# 5 Examples

**Example 1.** (Example 1 in **?**) Let $u(x, x') = x - x'$ and $X = [0, 1]$. Then,

$$\begin{aligned} X_1(V) &= \{x \in [0,1] | \text{there exists } x' \in [0,1] \text{ such that } x - x' = V\} \\ &= \{x \in [0,1] | x - V \in [0,1]\} \\ &= [0,1] \cap [V, 1+V], \end{aligned}$$

$$\begin{aligned} X_k(V) &= \{x \in [0,1] | \text{there exists } x' \in X_{k-1}(V) \text{ such that } x - x' = V\} \\ &= \{x \in [0,1] | x - V \in X_{k-1}(V)\} \\ &= [0,1] \cap \bigcap_{k=1} [kV, 1+kV], \end{aligned}$$

and

$$X(V) = \begin{cases} \emptyset & \text{for } V \neq 0, \\ [0,1] & \text{for } V = 0. \end{cases}$$

Since if $V = 0$, for any $x \in X$, $u(x, x) = 0 = V$, the value of reconsideration-proof equilibrium is $V = 0$. Any strategies that satisfy for each $t \geq 1$, $\sigma(h^{t-1}) = h^{t-1}_{t-1}$ is reconsideration-proof.

**Example 2.** (Example 2 and 3 in **?**) Let $u(x, x') = \sqrt{xx'}$ and $X = [0, 1]$. Note that $V \in [0, 1]$. Then,

$$\begin{aligned} X_1(V) &= \{x \in [0,1] | \text{there exists } x' \in [0,1] \text{ such that } \sqrt{xx'} = V\} \\ &= \{x \in [0,1] | \sqrt{x} \in [V, \infty)\} \\ &= [V^2, 1], \end{aligned}$$

$$
\begin{aligned}
X_k(V) &= \{x \in [0,1] | \text{there exists } x' \in X_{k-1}(V) \text{ such that } \sqrt{xx'} = V\} \\
&= \{x \in [0,1] | x - V \in X_{k-1}(V)\} \\
&= [V^2, 1],
\end{aligned}
$$

and

$$
X(V) = [V^2, 1].
$$

Since if $V = 1$, for any $x \in X$, $u(x,1) = \sqrt{x} \leq 1 = V$, the value of reconsideration-proof equilibrium is $V = 1$. The strategy that satisfy for each $t \geq 0$, $\sigma(h^{t-1}) = 1$ is reconsideration-proof.

(Example 4 in ?) Let $u(x,x') = x - 2x'$ and $X = [0,1]$. Then,

$$
\begin{aligned}
X_1(V) &= \{x \in [0,1] | \text{there exists } x' \in [0,1] \text{ such that } x - 2x' = V\} \\
&= \{x \in [0,1] | (x - V)/2 \in [0,1]\} \\
&= [0,1] \cap [V, 2 + V],
\end{aligned}
$$

$$
\begin{aligned}
X_k(V) &= \{x \in [0,1] | \text{there exists } x' \in X_{k-1}(V) \text{ such that } x - 2x' = V\} \\
&= \{x \in [0,1] | (x - V)/2 \in X_{k-1}(V)\} \\
&= [0,1] \cap \bigcap_{k=1} [(2^k - 1)V, 2^k + (2^k - 1)V],
\end{aligned}
$$

and

$$
X(V) = \begin{cases} \emptyset & \text{for } V > 0 \text{ or } V < -1, \\ [0,1] & \text{for } -1 \leq V \leq 0. \end{cases}
$$

Since for any $V \in [-1, 0]$, for any $x \in X$, $u(x,1) \leq V$, the value of reconsideration-proof equilibrium is $V = 0$. Any strategies that satisfy for each $t \geq 1$, $\sigma(h^{t-1}) = h^{t-1}_{t-1}/2$ is reconsideration-proof.

**Example 3.** (The example in Section 5 in **?**) Let $u(x,x') = y(x/2 + (1-x')/(1+r))$, and $X = [0,1]$. For a simple exposition, set $y = 1$. Then,

$$
\begin{aligned}
& X_1(V) \\
= \;& \{x \in [0,1] | \text{there exists } x' \in [0,1] \text{ such that } x/2 + (1-x')/(1+r) = V\} \\
= \;& \{x \in [0,1] | (x/2 - V)(1+r) + 1 \in [0,1]\} \\
= \;& [0,1] \cap [2(V - 1/(1+r)), 2V],
\end{aligned}
$$

$$
\begin{aligned}
& X_k(V) \\
= \;& \{x \in [0,1] | \text{there exists } x' \in X_{k-1}(V) \text{ such that } x/2 + (1-x')/(1+r) = V\} \\
= \;& \{x \in [0,1] | (x/2 - V)/(1+r) + 1 \in X_{k-1}(V)\} \\
= \;& [0,1] \cap \bigcap_{k=1} \left[ \left( \left(\tfrac{2}{1+r}\right)^k - 1 \right) 2\frac{V(1+r)-1}{1-r}, \left(\tfrac{2}{1+r}\right)^k \left( 1 + 2\frac{V(1+r)-1}{1-r} \right) - 2\frac{V(1+r)-1}{1-r} \right],
\end{aligned}
$$

and

$$
X(V) = \begin{cases} \emptyset & \text{for } V > \frac{1}{1+r} \text{ or } V < \frac{1}{2}, \\ [0,1] & \text{for } \frac{1}{2} \leq V \leq \frac{1}{1+r}. \end{cases}
$$

For any $V \in [\frac{1}{2}, \frac{1}{1+r}]$, for any $x \in X$, $u(x,1) = x/2 \leq V$. Therefore, $V = \frac{1}{1+r}$ is the value of reconsideration-proof equilibrium.

# References