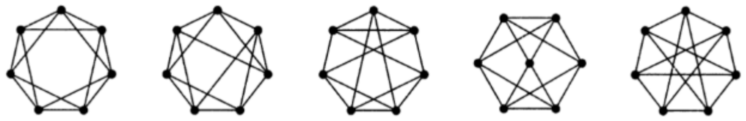
MAT 306 Graph Theory - Problem Set 1

Problem 1

Determine which pairs of graphs below are isomorphic, presenting the proof by testing the smallest possible number of pairs.

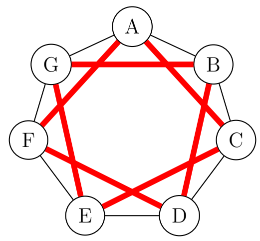


Solution:

To determine if the five graphs G_1, G_2, G_3, G_4, and G_5 are isomorphic, we can compare their vertices, edges, cycle lengths, and degree sequences.

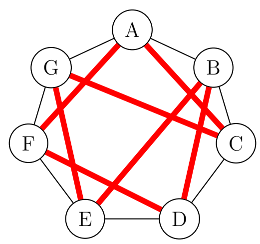
1. All graphs have the same number of vertices (7) and edges (14). The degree sequence for all graphs is $4, 4, 4, 4, 4, 4, 4$. Since the graphs are isomorphic if and only if their complements are isomorphic, we will test their cycle lengths.

2. **Graph G_1** has vertices A, B, C, D, E, F, G with degrees $4, 4, 4, 4, 4, 4, 4$ and a cycle of length **7**.



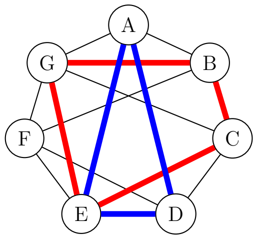
G_1

3. **Graph G_2** also has vertices A, B, C, D, E, F, G with degrees $4, 4, 4, 4, 4, 4, 4$ and a cycle of length **7**.



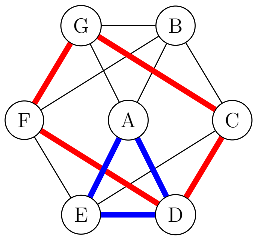
G_2

4. **Graph G_3** has vertices A, B, C, D, E, F, G with degrees $4, 4, 4, 4, 4, 4, 4$ but consists of two component cycles of lengths **3** and **4**.



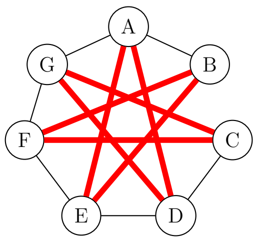
G_3

5. **Graph G_4** is similar to G_3 with vertices A, B, C, D, E, F, G and consists of two component cycles of lengths **3** and **4**.



G_4

6. **Graph G_5** has vertices A, B, C, D, E, F, G with degrees $4, 4, 4, 4, 4, 4, 4$ and a cycle of length **7**.



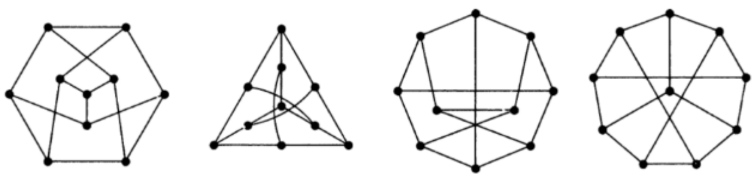
G_5

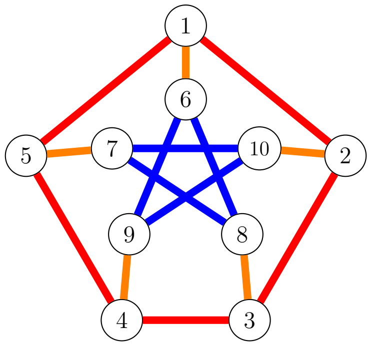
Thus, we conclude that $\mathbf{G_1} \cong \mathbf{G_2} \cong \mathbf{G_5}$ are pairwise isomorphic graphs that are cycles of length 7, while $\mathbf{G_3} \cong \mathbf{G_4}$ are isomorphic to each other but not 7-cycles.

Therefore, $\mathbf{G_1, G_2, G_5 \ncong G_3, G_4.}$

Problem 2

Prove that the graphs below are all drawings of the Petersen graph.





Petersen_Graph

Solution:

The Petersen graph has 10 vertices, 15 edges, and consists of 2 graphs (red, blue) with 1 connector graph (orange). We can use the disjointness definition of adjacency to show that each drawing maintains the same adjacency properties.

1. List the vertices of the Petersen graph:

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

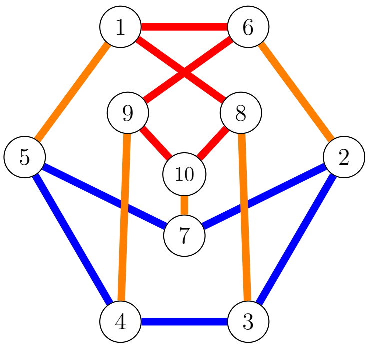
2. Define adjacency based on the edges:

$$\begin{aligned} E = \{&(1,2), (1,5), (1,6), (2,3), (2,10), (3,4), (3,8), \\ &(4,9), (4,5), (5,7), (7,10), (6,8), (6,9), (7,8), (9,10)\} \end{aligned}$$

3. Separate and take two disjoint graphs (e.g. $(v_1, v_2, v_3, v_4, v_5)$ & $(u_1, u_2, u_3, u_4, u_5)$) then add the matching edges of each disjoint graph (e.g. $\{(v_1, u_1), (v_2, u_2), (v_3, u_3), (v_4, u_4), (v_5, u_5)\}$) to make a Petersen graph.

   (a)

   C_1



Graph C_1 has 10 vertices.

$$V(C_1) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Take 2 disjoint graphs in C_1

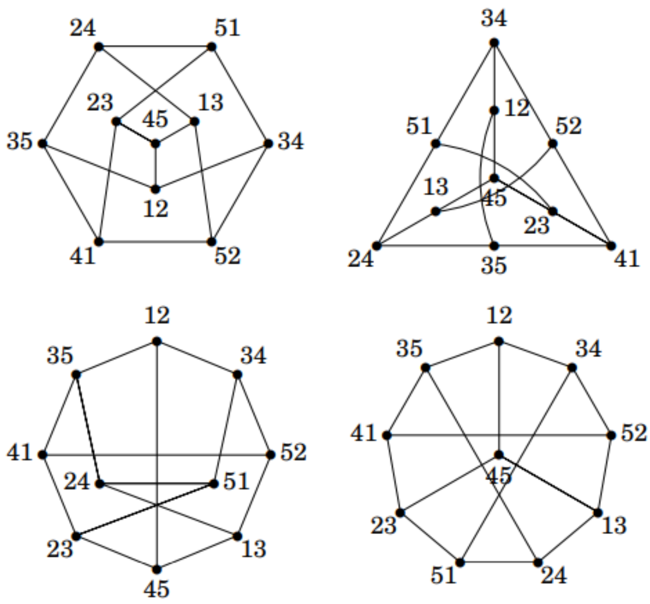
$$\begin{aligned} \color{red}{d_1} &= \{1, 6, 8, 9, 10\} \\ \color{blue}{d_2} &= \{2, 3, 4, 5, 7\} \end{aligned}$$

If we add 5 edges between d_1 & d_2 we get:

$$d_j \quad = \{(1,5),(9,4),(10,7),(8,3),(6,2)\}$$

Hence, the given graph is a **copy** of a **Petersen graph**.

4. Isomorphism is established by finding an adjacency-preserving bijection between vertex sets, which corresponds to relabeling graphs or permuting rows and columns of adjacency matrices to match, demonstrating structural symmetry. The number of isomorphisms from one graph to another is the same as the number of isomorphisms from the graph to itself. Here are the different arrangements of the graph but similar vertices and edges.



Therefore, all of the drawings above are copies of the Petersen Graph.

Problem 3

Prove that a self-complementary graph with n vertices exists if and only if n or $n-1$ is divisible by 4. **Hint:** When n is divisible by 4, generalize the structure of P_4 by splitting the vertices into four groups. For $n \equiv 1 \mod 4$, add one vertex to the graph constructed for $n-1$.

Statement: Prove that a self-complementary graph with n vertices exists if and only if n or $n-1$ is divisible by 4.

Definitions: A **self-complementary graph** is a graph that is isomorphic to its complement. The **complement** of a graph G is a graph $\overline{G}$ on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Proof

1: If $n \equiv 0 \mod 4$

- Structure: Partition the n vertices into 4 groups of $k = \frac{n}{4}$ vertices each.

- Graph Construction: Connect vertices within each group and connect vertices from different groups in a specific manner.

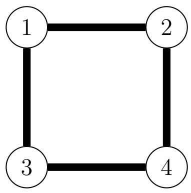
2: If $n \equiv 1 \mod 4$

- Start with a self-complementary graph with $n-1$ vertices.
- Add a new vertex and connect it to exactly half of the existing vertices.

Conclusion

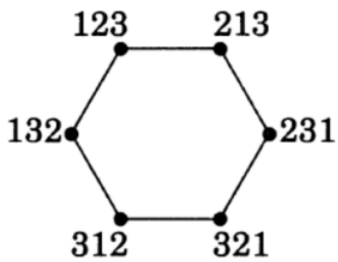
Therefore, we conclude that a self-complementary graph with n vertices exists if and only if n or $n-1$ is divisible by 4.

Graph Illustration



Problem 4

Let G be the graph whose vertices are the permutations of $\{1, 2, 3, \ldots, n\}$, with two permutations $a_1, ..., a_n$ and $b_1, ..., b_n$ adjacent if they differ by interchanging a pair of adjacent entries. Prove that G_n is connected.



G_3

Understanding the Graph Structure

- **Vertices:** The vertices of G_n are all $n!$ permutations of the set $\{1, 2, \ldots, n\}$.
- **Edges:** An edge exists between two vertices if they differ by a single adjacent transposition.

Proving Connectivity

To prove that G_n is connected, we need to show that there is a path between any two permutations in G_n.

1. **Adjacent Transpositions:** Any permutation can be expressed as a product of adjacent transpositions.

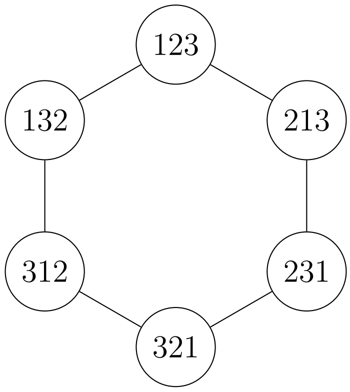
2. **General Case:** For any two permutations σ and τ in S_n, we can find a sequence of adjacent transpositions that transforms σ into τ.

3. **Induction:**

   - **Base case:** For $n = 1$, there is only one permutation, so the graph is trivially connected.
   - **Inductive Step:** Assume that G_k is connected for k. For G_{k+1}, any permutation can be transformed into any other by first fixing the last element and then applying the inductive hypothesis to the first k elements.

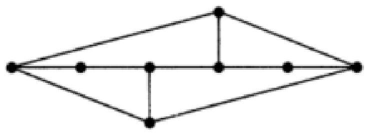
Conclusion

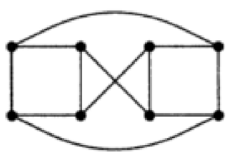
Since we can always find a sequence of adjacent transpositions to connect any two permutations,**Therefore,** we conclude that the graph G_n is **connected**.



Problem 5

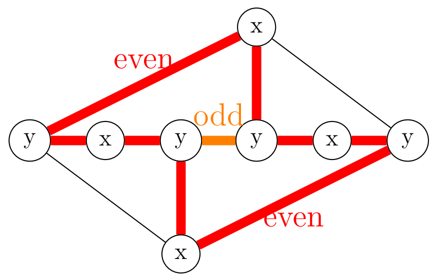
In each graph below, find a bipartite subgraph with the maximum number of edges. Prove that this is the maximum, and determine whether this is the only bipartite subgraph with this many edges.





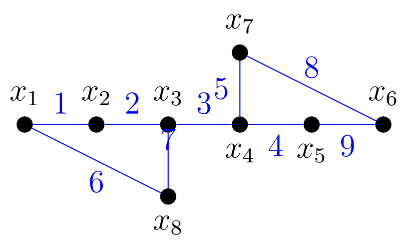
Solution:

1. The graph has a unique largest bipartite subgraph, which is formed by removing the central edge (the edge connecting vertices labeled "x" and "y"). After this deletion, the resulting subgraph is bipartite because the remaining vertices can be divided into two independent sets: one containing all "x" vertices and the other containing all "y" vertices. If removing an edge makes a graph bipartite, that edge must be part of all odd cycles in the graph, as bipartite graphs do not contain odd cycles. In this case, the two highlighted odd cycles share only the central edge, indicating that no other edge is common to all odd cycles. **Then,** the central edge is crucial for maintaining the odd cycle structure of the graph.



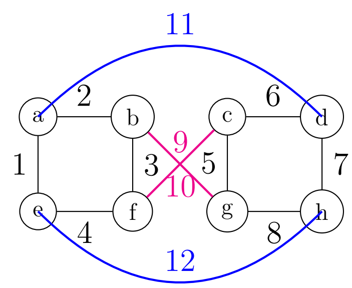
B_1

Therefore, the bipartite graph that contains the maximum number of edges is:



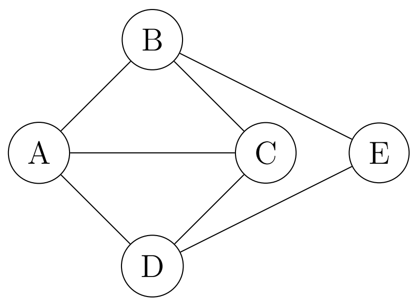
2. In the given graph, the largest bipartite subgraph contains 10 edges and is not unique. For instance, if we delete the edges (11) and (12), we get a bipartite graph with the sets (X = a, f, g, d) and (Y = e, b, c, h). Alternatively, deleting the edges (3) and (10) also results in a bipartite subgraph with the same number of edges, where (X = a, f, c, h) and (Y = e, b, g, d). Although these two subgraphs are isomorphic, they are distinct subgraphs, similar to how the Petersen graph has multiple claws.

   Therefore, we conclude that at least two edges(blue or magenta) must be deleted to make the graph bipartite with maximum number of edges.



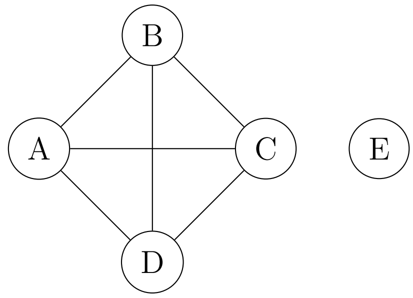
Problem 6

Let G be an n-vertex simple graph, where $n \geq 2$. Determine the maximum possible number of edges in G under each of the following conditions:



Solution:

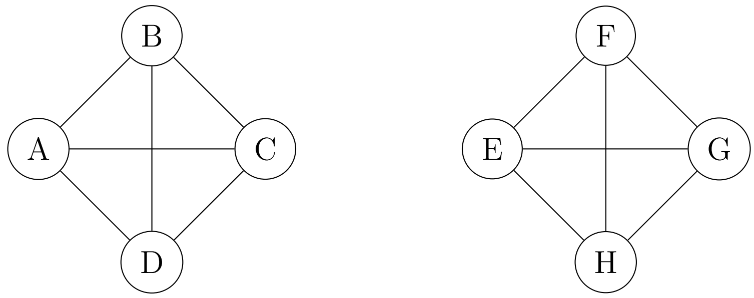
(a) If G has an independent set of size a:



$$\text{Max edges} = \frac{n(n-1)}{2} - \frac{a(a-1)}{2}$$

This formula accounts for the total number of edges in a complete graph minus the edges that would connect the vertices in the independent set.

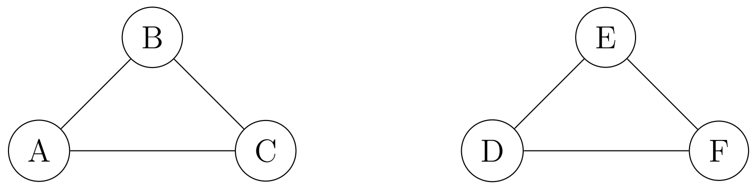
(b) If G has exactly k components:



$$\text{Max edges} = \frac{n(n-1)}{2} - \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$$

where n_i is the number of vertices in the i-th component. This formula subtracts the maximum edges possible in each component from the total edges in a complete graph.

(c) If G is disconnected:



$$\text{Max edges} = \frac{n(n-1)}{2} - (\text{edges that would connect components})$$

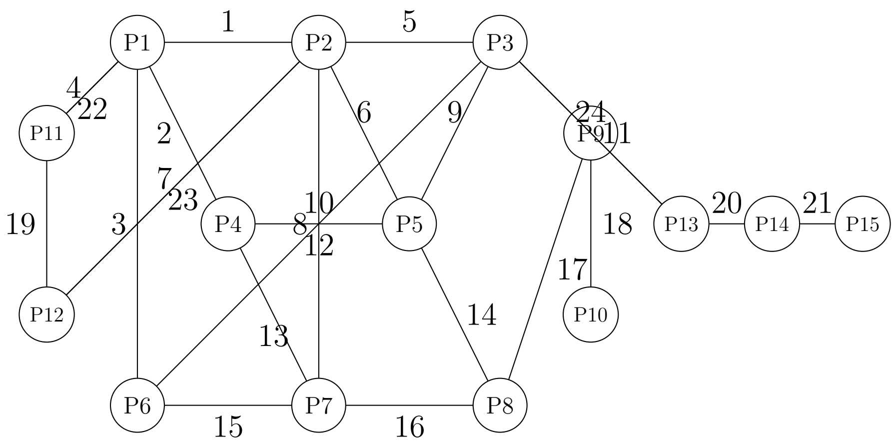
This can be approximated by considering the maximum edges within the components while ensuring at least two components exist.

Therefore, we find that it is varies based on the conditions such as the presence of an independent set, the number of components, and whether the graph is disconnected, connected with specific formulas to calculate the maximum edges in each scenario.

Problem 7

Each game of *bridge* involves two teams of two partners each. Consider a club in which four players cannot play a game if two of them have previously been partners that night. Suppose that 15 members arrive, but one decides to study graph theory. The other 14 people play until each has been a partner with four others. Next, they succeed in playing six more games (12 partnerships), but after that, they cannot find four players containing no pair of previous partners. Prove that if they can convince the graph theorist to play, then at least one more game can be played.

Graph Representation



Let $G = (V, E)$ be an undirected graph where V represents the players and E represents the partnerships. Each player is a vertex, and an edge exists between two vertices if those two players have partnered in a game.

Counting the Partnerships:

- Initially, each of the 14 players is partner with 4 others to formed a total of:

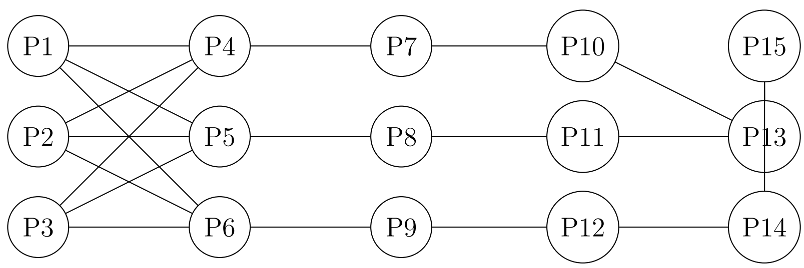
$$\frac{14 \times 4}{2} = 28$$

- After 6 more games, 12 additional partnerships occurred, bringing to the total of:

$$28 + 12 = 40$$

Graph Constraints

The problem states that after these games, they cannot find four players such that no two of them have previously partnered. This means that any subset of four vertices contains at least one edge among them, indicating that the graph does not contain an independent set of size 4.



By Turán's theorem, the maximum number of edges in a triangle-free graph on 14 vertices is 49. Since the complement graph $\overline{G}$ has 51 edges, it must contain a triangle. Adding the 15th player (universal vertex in $\overline{G}$), this triangle combines with the universal vertex to form a complete graph K_4 in $\overline{G}$, corresponding to an independent set of size 4 in G, allowing one more game. Thus, the graph theorist enables another game.

Conclusion

By including the extra player in the team, we can say that it increase the total number of available partnerships, making at least one more game possible. **Therefore,** by letting the graph theorist join in, they will help create fresh team combinations, making it possible to play at least one more game.

Problem 8

Let n be a positive integer. Let d be a list of n nonnegative integers with even sum whose largest entry is less than n and differs from the smallest entry by at most 1. Prove that d is graphic. Hint: Use the Havel-Hakimi Theorem.

Let the degree sequence d have entries k or $k-1$, with the following properties:

- The sum of the degrees is even.
- The maximum degree is less than n (the number of vertices).

According to the Havel-Hakimi algorithm, we can determine if a degree sequence is graphic by performing the following steps:

1. Iteratively remove the largest degree k from the sequence.
2. Reduce the next k degrees by 1.

The new sequence retains the following properties:

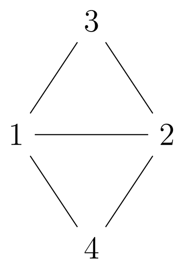
- The entries differ by at most 1.
- The sum of the degrees remains even.
- The maximum degree is less than $n-1$.

We will use induction on n:

- **Base Case:** For $n = 1$, the degree sequence is trivially graphic.
- **Inductive Step:** Assume the statement holds for all sequences of size n. We show it holds for size $n+1$.

Each step maintains the graphicability of the degree sequence, hence we conclude that d is graphic.

Answer: $\boxed{d \text{ is graphic.}}$



Problem 9

Suppose that G is a graph and D is an orientation of G that is strongly connected. Prove that if G has an odd cycle, then D has an odd cycle. Hint: Consider each pair $\{v_i, v_{i+1}\}$ in an odd cycle $(v_1, \dots, v_k)$ of G.

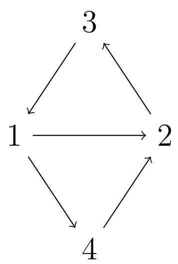
Proof:

1. **Odd Cycle in G:** Let $C = (v_1, v_2, \dots, v_k)$ be an odd cycle in G where k is odd.
2. **Orientation:** Since D is a strong orientation, for every edge (v_i, v_{i+1}) in C, we can assign a direction.
3. **Cycle in D:** If we traverse the cycle C in G, we can maintain the directionality in D. Since C is odd, the traversal will return to the starting vertex v_1 after an odd number of steps, thus forming an odd cycle in D.

Let C be an odd cycle in G. In the orientation D, C becomes a closed directed walk. Decompose this walk into directed cycles. Since C has odd length, the sum of cycle lengths is odd. At least one cycle must be odd (even sum impossible), hence D contains an odd cycle.

Conclusion:

Therefore, D must also contain an odd cycle.



Problem 10

Given an ordering $\sigma = v_1, \dots, v_n$ of the vertices of a tournament, let $f(\sigma)$ be the sum of the lengths of the feedback edges, meaning the sum of $j - i$ over edges $v_j v_i$ such that $j > i$. Prove that every ordering minimizing $f(\sigma)$ places the vertices in non-increasing order of outdegree. Hint: Determine how $f(\sigma)$ changes when consecutive elements of σ are exchanged.

Proof:

1. **Understanding Feedback Edges:** A feedback edge $(v_j v_i)$ contributes $j - i$ to $f(\sigma)$ if $j > i$. The goal is to minimize this sum.
2. **Outdegree Consideration:** Let $d(v)$ denote the outdegree of vertex v. If we place vertices in non-increasing order of outdegree, the vertices with higher outdegree will appear earlier in the ordering.
3. **Effect of Swapping:** Consider two consecutive vertices v_i and v_{i+1} in the ordering. If $d(v_{i+1}) > d(v_i)$, swapping them will reduce the number of feedback edges involving v_{i+1} and increase those involving v_i, leading to a higher contribution to $f(\sigma)$.

Consider an ordering σ minimizing $f(\sigma)$. Suppose u precedes v in σ with outdegree(u) $<$ outdegree(v). Swapping u and v changes $f(\sigma)$ by $a - c + \Delta$, where a (edges from u left), c (edges from v left), and Δ accounts for the $u - v$ edge. Since outdegree(u) $<$ outdegree(v), swapping decreases $f(\sigma)$, contradicting minimality. Thus, non-increasing outdegree order is required.

Conclusion:

Therefore, to minimize $f(\sigma)$, the vertices must be ordered in non-increasing order of outdegree.