RELATIVISTIC QUANTUM MECHANICS

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Abstract. We consider, first, how probability densities behave under the Lorentz transformations, and then examine the requirements that would be necessary for there to exist an energy-momentum representation of the gaussian wave packet. A Lorentz invariant Schrödinger equation is proposed from which a relativistic probability density current is obtained as a 4-vector. The method of separation of variables for a time independent potential is applied to the proposed relativistic Schrödinger equation, and we discover that this results in a time independent equation identical to Schrödinger's. We establish that the separation constant is real. We proceed to treat classic one dimensional, and three dimensional problems of quantum mechanics within the context of Special Relativity, and examine angular momentum and spin. Following this we derive the relativistic equation from first principles, and propose a function space for relativistic quantum mechanics. We examine the two particle relativistic Schrodinger equation. We review the tensor formulation electrodynamics within the context of Special Relativity, with the aim of obtaining equations for quatum electrodynamics. Lastly we compare Schrödinger's original equation to the Fokker-Planck equation.

Contents

1. A Note to the Reader

This paper is a work in progress, and may essentially become a book. The sections on Fokker-Planck may be dropped at some point as may the section on the energy-momentum representation of a gaussian, since it was just an exploration. The section on the derivation of a relativistic two particle equation is incomplete, and has been left off with some guesswork. The sections on the barrier and finite square well, spin, and the hydrogen atom haven't been started, and the section on relativistic formulation of electromagnetism is incomplete, and may be moved to an appendix as it is only a review. The section on orbital angular momentum is also incomplete. I'm sure this draft contains many typos, gramatical, and spelling errors which I hope to correct as this paper progresses. If the reader has any comments, they may be directed to strangerland@gmail.com.

2. The Gaussian Distribution Moving at Constant Velocity

Suppose an observer moving at constant velocity, v, along the x-axis, has defined a normal distribution with infinitesimal probability given by

$$g\left(x'\right) dx' = \frac{1}{\sqrt{2\pi}\Delta x'} \exp\left[-\frac{1}{2}\left(\frac{x'}{\Delta x'}\right)^2\right] dx' \tag{2.1}$$

To the stationary observer the standard deviation would appear Lorentz contracted according to the relation

$$\Delta x' = \gamma \Delta x \tag{2.2}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2.3}$$

and

$$\beta = \frac{v}{c} \tag{2.4}$$

Further the Lorentz Transformations give us

$$x' = \gamma (x - vt) \tag{2.5}$$

Now the height of the gaussian does not change due to the motion and the stationary observer sees the probability

$$\begin{aligned} f(x)\,dx &= \frac{1}{\sqrt{2\pi}\gamma\Delta x} \exp\left[-\frac{1}{2}\left(\frac{x-vt}{\Delta x}\right)^2\right] \left|\frac{\partial x'}{\partial x}\right| dx \\ &= \frac{1}{\sqrt{2\pi}\gamma\Delta x} \exp\left[-\frac{1}{2}\left(\frac{x-vt}{\Delta x}\right)^2\right] \gamma dx \\ &= \frac{1}{\sqrt{2\pi}\Delta x} \exp\left[-\frac{1}{2}\left(\frac{x-vt}{\Delta x}\right)^2\right] dx \end{aligned} \tag{2.6}$$

Before we leave this section let us note that we can consider the density to be a conditional denstity given time t.

$$f(x|t) = \frac{1}{\sqrt{2\pi}\Delta x} \exp\left[-\frac{1}{2}\left(\frac{x-vt}{\Delta x}\right)^2\right] \tag{2.7}$$

3. A Second Form of the Gaussian Distribution

Consider the conditional density

$$f(x|t) = \sqrt{\frac{2}{\pi}}\Delta k \exp\left[-2(\Delta k x - \Delta\omega t)^2\right] \tag{3.1}$$

This is a gaussian moving to the right with velocity

$$v = \frac{\Delta\omega}{\Delta k} \tag{3.2}$$

if v is positive, and is normalized with respect to the stationary observer. The variance of this distribution is

$$\operatorname{var}(X) = \frac{1}{4(\Delta k)^2} \tag{3.3}$$

and the conditional expectation

$$\langle x \rangle = vt \tag{3.4}$$

so that

$$\frac{d\langle x \rangle}{dt} = v \tag{3.5}$$

Now let us write

$$
\begin{aligned}
f\left(x|t\right)dx &= \sqrt{\frac{2}{\pi}}\Delta k \exp\left[-2\left(\Delta k x - \frac{\Delta\omega}{c}ct\right)^2\right]dx \\
&= \sqrt{\frac{2}{\pi}}\Delta k_1 \exp\left[-2\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2\right]dx^1 \qquad (3.6)
\end{aligned}
$$

Now in the frame moving along with our gaussian we have the contravariant transformations

$$x'^1 = \gamma\left(x^1 - \beta x^0\right) \qquad (3.7)$$

$$x'^0 = \gamma\left(x^0 - \beta x^1\right) \qquad (3.8)$$

and the covariant Lorentz transformations

$$\Delta k'_1 = \gamma\left(\Delta k_1 + \beta \Delta k_0\right) \qquad (3.9)$$

$$\Delta k'_0 = \gamma\left(\Delta k_0 + \beta \Delta k_1\right) \qquad (3.10)$$

where

$$\beta = -\frac{\Delta k_0}{\Delta k_1} \qquad (3.11)$$

so that we find

$$\Delta k'_1 = \gamma^{-1}\Delta k_1 \qquad (3.12)$$

$$\Delta k'_0 = 0 \qquad (3.13)$$

which yields the velocity of the gaussian in the primed frame

$$-\frac{\Delta k'_0}{\Delta k'_1} = 0$$

as expected. Let us transform our gaussian to the primed frame

$$
\begin{aligned}
f\left(x^1|x^0\right)dx^1 &= \sqrt{\frac{2}{\pi}}\Delta k_1 \exp\left[-2\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2\right]dx^1 \\
&= \sqrt{\frac{2}{\pi}}\gamma^{-1}\Delta k_1 \exp\left[-2\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2\right]\gamma dx^1 \\
&= \sqrt{\frac{2}{\pi}}\gamma^{-1}\Delta k_1 \exp\left[-2\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2\right]\left|\frac{\partial x'}{\partial x}\right|dx^1 \\
&= \sqrt{\frac{2}{\pi}}\Delta k'_1 \exp\left[-2\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2\right]dx'^1 \\
&= \sqrt{\frac{2}{\pi}}\Delta k'_1 \exp\left[-2\left(\Delta k'_1 x'^1 + \Delta k'_0 x'^0\right)^2\right]dx'^1 \\
&= \sqrt{\frac{2}{\pi}}\Delta k'_1 \exp\left[-2\left(\Delta k'_1 x'^1\right)^2\right]dx'^1 \\
&= f\left(x'^1|x'^0\right)dx'^1 \qquad (3.14)
\end{aligned}
$$

and all is still copasetic with the universe. We see that the variance of the transformed conditional density is

$$\operatorname{var}\left(X'^1\right) = \frac{1}{4\left(\Delta k'_1\right)^2} \qquad (3.15)$$

hence

$$\operatorname{var}(X') = \gamma^2 \operatorname{var}(X) \tag{3.16}$$

and

$$\operatorname{var}\left(X'^1\right)\left(\Delta k_1'\right)^2 = \operatorname{var}\left(X^1\right)\left(\Delta k_1\right)^2 = \frac{1}{4} \tag{3.17}$$

However we are hesitant to write

$$\left|\Delta x^1\right| = +\sqrt{\operatorname{var}\left(X^1\right)} \tag{3.18}$$

if we intend that $\left(\Delta x^0, \Delta x^1\right)$ be the components of a Lorentz 2-vector as we find that if

$$\frac{\Delta x^1}{\Delta x^0} = \beta \tag{3.19}$$

then

$$\Delta x'^1 = \gamma\left(\Delta x^1 - \beta \Delta x^0\right) = 0 \tag{3.20}$$

and

$$\begin{aligned}
\Delta x'^0 &= \gamma\left(\Delta x^0 - \beta \Delta x^1\right) \\
&= \gamma \Delta x^0 \left(1 - \beta \frac{\Delta x^1}{\Delta x^0}\right) \\
&= \gamma \Delta x^0 \left(1 - \beta^2\right) \\
&= \gamma^{-1} \Delta x^0 \qquad (3.21)
\end{aligned}$$

which doesn't bode well for Heisenberg's Uncertainty Principle when written in the form

$$\left(\Delta x^1\right)^2 \left(\Delta k_1\right)^2 \geq \frac{1}{4} \tag{3.22}$$

Of course equations (3.11) and (3.19) give us the invariant product

$$\Delta x^0 \Delta k_0 + \Delta x^1 \Delta k_1 = 0 \tag{3.23}$$

which in the primed frame becomes

$$\begin{aligned}
0 &= \Delta x'^0 \Delta k_0' + \Delta x'^1 \Delta k_1' \\
&= \left(\gamma^{-1} \Delta x^0\right) \cdot 0 + 0 \cdot \left(\gamma^{-1} \Delta k_1\right) \qquad (3.24)
\end{aligned}$$

and we see that it is certainly not the case that

$$\left|\Delta x'^1\right| \left|\Delta k_1'\right| \geq \frac{1}{2}$$

Thus we are motivated to rewrite the uncertainty relation in this case given by equation (3.22) as

$$\operatorname{var}\left(x'^1\right) \operatorname{var}\left(-i\frac{\partial}{\partial x'^1}\right) = \operatorname{var}\left(x^1\right) \operatorname{var}\left(-i\frac{\partial}{\partial x^1}\right) = \frac{1}{4} \tag{3.25}$$

with

$$f\left(x^1 | x^0\right) = \Psi^* \Psi \tag{3.26}$$

where perhaps

$$\Psi = \sqrt{\Delta k_1 \sqrt{\frac{2}{\pi}}} \exp\left[i\left(a_0 x^0 + a_1 x^1\right)\right] \exp\left[-\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2\right] \tag{3.27}$$

Actually it is easy to show from the definition of the variance

$$\mathrm{var}\,(x) \quad = \quad \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2 \tag{3.28}$$

by using the contravariant transformation on x, and keeping in mind that we are dealing with densities conditional on t so that $\langle t \rangle = t$, that

$$\gamma^2 \mathrm{var}\,(x) \quad = \quad \mathrm{var}\,(x') \tag{3.29}$$

we also have

$$\begin{aligned} \Delta k' &= \gamma \left(\Delta k - \beta \frac{\Delta \omega}{c} \right) && (3.30) \\ &= \gamma \Delta k \left(1 - \beta \frac{1}{c} \frac{\Delta \omega}{\Delta k} \right) && (3.31) \\ &= \gamma \Delta k \left(1 - \beta^2 \right) && (3.32) \\ &= \gamma^{-1} \Delta k && (3.33) \end{aligned}$$

and that is the *covariant* Lorentz transformation with $\Delta k_0 = -\frac{\Delta \omega}{c}$, hence the uncertainty principle survives the Lorentz transformation

$$\mathrm{var}\,(x')\,\mathrm{var}\,(k') = \mathrm{var}\,(x)\,\mathrm{var}\,(x) \tag{3.34}$$

Its worth reiterating the point that we can have $(c\Delta t, \Delta x)$ a contravariant vector, we can have $(-\Delta\omega/c, \Delta k)$ a covariant vector, we can have

$$\frac{\Delta x}{\Delta t} = \frac{\Delta \omega}{\Delta k} = v \tag{3.35}$$

and we can havevar $(k) = \Delta k$, but then we *cannot* have $\mathrm{var}\,(x) = \Delta x$.

With a little more generality let us assume that f is non-negative such that

$$\int_{-\infty}^{\infty} f\,(\alpha)\, d\alpha = C \tag{3.36}$$

then it can be shown that for $\alpha = k_0 x^0 + k_1 x^1$ we also have

$$\int_{-\infty}^{\infty} f\left(k_0 x^0 + k_1 x^1\right) |k_1|\, dx^1 = C \tag{3.37}$$

Now $f\left(k_0 x^0 + k_1 x^1\right)$ is moving to the "left" with velocity $v = -k_0/k_1$. In the frame moving along with this function we have

$$k_0' x'^0 + k_1' x'^1 = k_0 x^0 + k_1 x^1 \tag{3.38}$$

where $k_0' = 0$ so that $k_1 = \gamma\,(k_1' - \beta k_0') = \gamma k_1'$, and we also have $\gamma dx^1 = dx'^1$. Therefore

$$\begin{aligned} f\left(k_0 x^0 + k_1 x^1\right) |k_1|\, dx^1 &= f\left(k_0 x^0 + k_1 x^1\right) \gamma^{-1} |k_1|\, \gamma dx^1 \\ &= f\left(k_0 x^0 + k_1 x^1\right) |k_1'|\, dx'^1 \\ &= f\left(k_0' x'^0 + k_1' x'^1\right) |k_1'|\, dx'^1 && (3.39) \end{aligned}$$

and of course this holds for any two arbitrary frames, since the frame which sees velocity, v was arbitrary to begin with, and all are equal to the frame moving with the density. Thus for the infinitesimal probabilities

$$\frac{|k_1|}{C} f\left(k_0 x^0 + k_1 x^1\right) dx^1 = \frac{|k_1'|}{C} f\left(k_0' x'^0 + k_1' x'^1\right) dx^1 \tag{3.40}$$

Further if

$$\langle\alpha\rangle = \frac{1}{C}\int_{-\infty}^{\infty}\alpha f(\alpha)\,d\alpha \tag{3.41}$$

$$\left\langle\alpha^2\right\rangle = \frac{1}{C}\int_{-\infty}^{\infty}\alpha^2 f(\alpha)\,d\alpha \tag{3.42}$$

both exist, then so to do the expected values, $\left\langle x^1\right\rangle$ and $\left\langle\left(x^1\right)^2\right\rangle$ hence var $\left(x^1\right)$, where then equation (3.29) holds.

4. Lorentz Transformations of Probability Densities in General

How do we generalize to 2 or 3 spatial dimensions without working to hard? Let f be a Lorentz inavariant real non-negative function of one vector, x^μ, and any number of dual or covectors. The function f may also possibly depend on x_μ, but we will not make this dependence explicit. Say for dependence on one covector, k_μ

$$f\left(k_\mu;x^\mu\right)=f\left(k'_\mu;x'^\mu\right) \tag{4.1}$$

Now we have with $dV=dx^1dx^2dx^3$

$$\int\int\int_{-\infty}^{\infty} f\left(k_\mu;x^\mu\right)dV = C\left(k_\mu\right)<\infty \tag{4.2}$$

$$\int\int\int_{-\infty}^{\infty} f\left(k'_\mu;x'^\mu\right)dV' = C\left(k'_\mu\right)<\infty \tag{4.3}$$

Suppose we desire that

$$\frac{1}{C\left(k_\mu\right)}f\left(k_\mu;x^\mu\right)dV=\frac{1}{C\left(k'_\mu\right)}f\left(k'_\mu;x'^\mu\right)dV' \tag{4.4}$$

since $\gamma dV=dV'$, we must then have

$$\gamma C\left(k_\mu\right)=C\left(k'_\mu\right) \tag{4.5}$$

In fact let D be a list of dual vectors, and let us write

$$\int\int\int_{-\infty}^{\infty} f\left(D;x^\mu\right)dV=C(D) \tag{4.6}$$

then

$$\int\int\int_{-\infty}^{\infty} f\left(D;x^\mu\right)\gamma dV=\gamma C(D) \tag{4.7}$$

so that

$$\int\int\int_{-\infty}^{\infty} f\left(D;x^\mu\right)dV'=\gamma C(D) \tag{4.8}$$

but since

$$f\left(D';x'^\mu\right)=f\left(D;x^\mu\right) \tag{4.9}$$

we have

$$\int\int\int_{-\infty}^{\infty} f\left(D;x^\mu\right)dV'=\int\int\int_{-\infty}^{\infty} f\left(D';x'^\mu\right)dV'=C\left(D'\right) \tag{4.10}$$

therfore

$$C\left(D'\right)=\gamma C\left(D\right) \tag{4.11}$$

Example 1. Let's look at the Multivariate Normal Distribution. For 3 random variables, $\mathrm{x}=\left(x^{1},x^{2},x^{3}\right)$ the probability density is given in matrix form by

$$f\left(\mathrm{x}\right)=\frac{1}{\left(2\pi\right)^{\frac{3}{2}}\left|\Sigma\right|^{\frac{1}{2}}}\exp\left[-\frac{1}{2}\left(\mathrm{x}-\boldsymbol{\mu}\right)^{\mathrm{T}}\Sigma^{-1}\left(\mathrm{x}-\boldsymbol{\mu}\right)\right] \tag{4.12}$$

where suppose

$$\Sigma=\mathrm{A}\mathrm{A}^{\mathrm{T}} \tag{4.13}$$

for non-singular 3×3 matrix, A with entries $a^{i\cdot}_{\cdot j}$ where latin indices, $i,j,k,l,m,n=1,2,3$ per the equations

$$x^{i}=a^{i\cdot}_{\cdot j}z^{j}+\beta^{i}x^{0} \tag{4.14}$$

with

$$\left\langle z^{i}\right\rangle = 0 \tag{4.15}$$

$$\left\langle z^{i}z^{j}\right\rangle = \delta^{ij} \tag{4.16}$$

then

$$\mu^{i}=\left\langle x^{i}\right\rangle=\beta^{i}x^{0} \tag{4.17}$$

and

$$\mathrm{cov}\left(x^{i},x^{j}\right)=\left\langle x^{i}x^{j}\right\rangle-\left\langle x^{i}\right\rangle\left\langle x^{j}\right\rangle=a^{i\cdot}_{\cdot k}a^{j\cdot}_{\cdot k}=\frac{\partial x^{i}}{\partial z^{k}}\frac{\partial x^{j}}{\partial z^{k}} \tag{4.18}$$

Now let

$$2\mathrm{B}=\mathrm{A}^{-1} \tag{4.19}$$

so that

$$f\left(\mathrm{x}\right)=\sqrt{\left(\frac{2}{\pi}\right)^{3}}\left|\mathrm{B}^{\mathrm{T}}\mathrm{B}\right|^{\frac{1}{2}}\exp\left[-2\left(\mathrm{x}-\boldsymbol{\beta}x^{0}\right)^{\mathrm{T}}\mathrm{B}^{\mathrm{T}}\mathrm{B}\left(\mathrm{x}-\boldsymbol{\beta}x^{0}\right)\right] \tag{4.20}$$

now let us write following matrix equation with tensor notation

$$\mathrm{B}\left(\mathrm{x}-\boldsymbol{\beta}x^{0}\right)=b^{i\cdot}_{\cdot j}\left(x^{j}-\beta^{j}x^{0}\right) \tag{4.21}$$

now for greek indice $\mu=1,2,3$ let

$$k^{i\cdot}_{\cdot\mu}=b^{i\cdot}_{\cdot\mu} \tag{4.22}$$

and

$$k^{i\cdot}_{\cdot 0}=-b^{i\cdot}_{\cdot j}\beta^{j} \tag{4.23}$$

Mind however the convention that greek indices $\mu,\nu,\rho,\sigma,\tau,\ldots=0,1,2,3$, and that we have not verified that our newly defined $k^{i\cdot}_{\cdot\mu}$ for $i=1,2,3$ and $\mu=0,1,2,3$ transform covariantly with respect to index μ. None the less we can write

$$f\left(k^{1\cdot}_{\cdot\mu},k^{2\cdot}_{\cdot\nu},k^{3\cdot}_{\cdot\rho};x^{\sigma}\right)=\sqrt{\left(\frac{2}{\pi}\right)^{3}}\left|k^{i\cdot}_{\cdot m}k^{i\cdot}_{\cdot n}\right|^{\frac{1}{2}}\exp\left(-2k^{i\cdot}_{\cdot\mu}k^{i\cdot}_{\cdot\nu}x^{\mu}x^{\nu}\right) \tag{4.24}$$

In matrix form the Covariant Lorentz Transformations without rotations of the spatial coordinates is given by

$$\begin{bmatrix} r'_0 \\ \mathrm{r}' \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\boldsymbol{\beta}^{\mathrm{T}} \\ \gamma\boldsymbol{\beta} & \mathrm{I} + (\gamma-1)\,\frac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{\|\boldsymbol{\beta}\|^2} \end{bmatrix} \begin{bmatrix} r_0 \\ \mathrm{r} \end{bmatrix} \tag{4.25}$$

where

$$\|\boldsymbol{\beta}\|^2 = \left(\beta^1\right)^2 + \left(\beta^2\right)^2 + \left(\beta^3\right)^2 \tag{4.26}$$

and

$$\gamma = \frac{1}{\sqrt{1-\|\boldsymbol{\beta}\|^2}} \tag{4.27}$$

If we apply the covariant Lorentz transformation to $k^{i\cdot}_{\cdot 0}$ we find

$$\begin{aligned} k'^{i\cdot}_{\cdot 0} &= \gamma\left(k^{i\cdot}_{\cdot 0} + \beta^j k^{i\cdot}_{\cdot j}\right) \\ &= \gamma\left(-\beta^j k^{i\cdot}_{\cdot j} + \beta^j k^{i\cdot}_{\cdot j}\right) \\ &= 0 \end{aligned} \tag{4.28}$$

which implies no time dependence in the frame moving with the density as expected. For the inverse transformation

$$\begin{aligned} k^{i\cdot}_{\cdot 0} &= \gamma\left(k'^{i\cdot}_{\cdot 0} - \beta^j k'^{i\cdot}_{\cdot j}\right) \\ &= -\gamma\beta^j k'^{i\cdot}_{\cdot j} \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} k'^{i\cdot}_{\cdot j} &= k^{i\cdot}_{\cdot j} + \gamma\beta^j k^{i\cdot}_{\cdot 0} + (\gamma-1)\,\frac{\beta^j}{\|\boldsymbol{\beta}\|^2}\beta^l k^{i\cdot}_{\cdot l} \\ &= k^{i\cdot}_{\cdot j} + \gamma\beta^j k^{i\cdot}_{\cdot 0} + (1-\gamma)\,\frac{\beta^j}{\|\boldsymbol{\beta}\|^2} k^{i\cdot}_{\cdot 0} \end{aligned} \tag{4.30}$$

so that

$$\begin{aligned} k^{i\cdot}_{\cdot 0} &= -\gamma\beta^j\left(k^{i\cdot}_{\cdot j} + \gamma\beta^j k^{i\cdot}_{\cdot 0} + (1-\gamma)\,\frac{\beta^j}{\|\boldsymbol{\beta}\|^2} k^{i\cdot}_{\cdot 0}\right) \\ &= \gamma k^{i\cdot}_{\cdot 0} - \gamma^2\|\boldsymbol{\beta}\|^2 k^{i\cdot}_{\cdot 0} - \gamma(1-\gamma)\, k^{i\cdot}_{\cdot 0} \\ &= \gamma k^{i\cdot}_{\cdot 0} - \gamma^2\|\boldsymbol{\beta}\|^2 k^{i\cdot}_{\cdot 0} - \gamma k^{i\cdot}_{\cdot 0} + \gamma^2 k^{i\cdot}_{\cdot 0} \\ &= -\gamma^2\|\boldsymbol{\beta}\|^2 k^{i\cdot}_{\cdot 0} + \gamma^2 k^{i\cdot}_{\cdot 0} \\ &= \gamma^2\left(1-\|\boldsymbol{\beta}\|^2\right) k^{i\cdot}_{\cdot 0} \\ &= k^{i\cdot}_{\cdot 0} \end{aligned} \tag{4.31}$$

Nicely from equations (4.22) and (4.23) we should note that

$$\beta^j = \frac{\partial k^{i\cdot}_{\cdot 0}}{\partial k^{i\cdot}_{\cdot j}} \text{ without summation} \tag{4.32}$$

We see from equation (4.24) that the integral

$$\sqrt{\left(\frac{2}{\pi}\right)^3}\int\int\int_{-\infty}^{\infty}\exp\left(-2k^{i\cdot}_{\cdot\mu}k^{i\cdot}_{\cdot\nu}x^{\mu}x^{\nu}\right)dV=\left|k^{i\cdot}_{\cdot m}k^{i\cdot}_{\cdot n}\right|^{-\frac{1}{2}} \tag{4.33}$$

and from equation (4.11) we have

$$\left|k'^{i\cdot}_{\cdot m}k'^{i\cdot}_{\cdot n}\right|^{-\frac{1}{2}}=\gamma\left|k^{i\cdot}_{\cdot m}k^{i\cdot}_{\cdot n}\right|^{-\frac{1}{2}} \tag{4.34}$$

Hence
(4.35)

$$\sqrt{\left(\frac{2}{\pi}\right)^3}\left|k^{i\cdot}_{\cdot m}k^{i\cdot}_{\cdot n}\right|^{\frac{1}{2}}\exp\left(-2k^{i\cdot}_{\cdot\mu}k^{i\cdot}_{\cdot\nu}x^{\mu}x^{\nu}\right)dV=\sqrt{\left(\frac{2}{\pi}\right)^3}\left|k'^{i\cdot}_{\cdot m}k'^{i\cdot}_{\cdot n}\right|^{\frac{1}{2}}\exp\left(-2k'^{i\cdot}_{\cdot\mu}k'^{i\cdot}_{\cdot\nu}x'^{\mu}x'^{\nu}\right)dV'$$

for any two frames, since the choice of the unprimed frame was arbitrary.

5. Energy-Momentum Representation of a Gaussian Packet

This section is somewhat superfluous to the remainder of the paper so the reader may skip it, but we wonder what would it take to obtain and energy-momentum representation of the space-time gaussian packet in one spatial dimension? Well it seems that the natural way of defining the Fourier Transform in the setting of space-time is given by the Foward Transform

$$F\left(k_{\sigma}\right)=\frac{1}{\left(2\pi\right)^2}\int\int\int\int\exp\left(-ik_{\mu}x^{\mu}\right)f\left(x^{\sigma}\right)dx^0dx^1dx^2dx^3 \tag{5.1}$$

and Backward Transform

$$f\left(x^{\sigma}\right)=\frac{1}{\left(2\pi\right)^2}\int\int\int\int\exp\left(ik_{\mu}x^{\mu}\right)F\left(k_{\sigma}\right)dk_0dk_1dk_2dk_3 \tag{5.2}$$

Which we note is Lorentz invariant (has the same form) since the Jacobian determinant of the Lorentz Transformation is equal to unity

$$\det\left(\Lambda^{\mu\cdot}_{\cdot\nu}\right)=\left|\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right|=1 \tag{5.3}$$

so that

$$dx'^0dx'^1dx'^2dx'^3=\left|\left|\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right|\right|dx^0dx^1dx^2dx^3 \tag{5.4}$$

Consider the wave function for a gaussian packet

$$\Psi\left(x|t\right)=\sqrt{\Delta k\sqrt{\frac{2}{\pi}}}\exp\left[i\left(k_0x-\omega_0t\right)\right]\exp\left[-\left(\Delta k\left(x-x_0\right)-\Delta\omega\left(t-t_0\right)\right)^2\right] \tag{5.5}$$

where

$$\langle x\rangle=u\left(t-t_0\right)+x_0 \tag{5.6}$$

so that the group velocity of the packet is given by

$$\frac{d\langle x\rangle}{dt}=u=\frac{\Delta\omega}{\Delta k}=\frac{\Delta x}{\Delta t} \tag{5.7}$$

If we consider $\Psi^*\Psi$ a conditional probability density, $f(x|t)$, we can still compute the expectations

$$\langle t\rangle = \int \Psi^* t \Psi dt = t \tag{5.8}$$

and

$$\left\langle t^2\right\rangle = \int \Psi^* t^2 \Psi dt = t^2 \tag{5.9}$$

to find

$$\operatorname{var}(t) = \left\langle t^2\right\rangle - \langle t\rangle^2 = 0$$

We can also compute the conditional expectation for energy

$$\langle E\rangle = \int \Psi^* i\hbar \frac{\partial}{\partial t}\Psi dt \tag{5.10}$$

and

$$\left\langle E^2\right\rangle = -\hbar^2 \int \Psi^* \frac{\partial^2}{\partial t^2}\Psi dt \tag{5.11}$$

but we will not find an expression of the energy time uncertainty principle,

$$\operatorname{var}(t)\operatorname{var}(E) \geq \frac{\hbar^2}{4}$$

despite the fact that the commutator bracket

$$\left[t, i\hbar\frac{\partial}{\partial t}\right] = -i\hbar \tag{5.12}$$

is well defined suggesting

$$(\Delta t)^2 (\Delta E)^2 \geq \frac{1}{4}\left|\left[t, i\hbar\frac{\partial}{\partial t}\right]\right|^2 = \frac{\hbar^2}{4} \tag{5.13}$$

Suppose we want to obtain an energy-momentum represention of $\Psi(x|t)$. We can first obtain the momentum representation by performing the forward fourier transform

$$\frac{1}{\sqrt{2\pi}}\int \Psi(x|t)\exp(-ikx)\,dx = \phi(k,t) \tag{5.14}$$

to obtain

$$\phi(k,t) = \sqrt{\Delta x\sqrt{\frac{2}{\pi}}}\exp\left(-\Delta x^2 k'^2\right)\exp\left[-ik'\left(x_0 - ut_0\right)\right]\exp(-iat) \tag{5.15}$$

where $k' = k - k_0$ and $a = k'u + \omega_0$

We can proceed no further in our quest unless introduce a marginal or prior wave packet for the random variable t into our computations

$$f(t) = \sqrt{\Delta\omega\sqrt{\frac{2}{\pi}}}\exp\left(-\Delta\omega^2\left(t - t_0\right)^2\right) \tag{5.16}$$

We can now compute the backward fourier transform

$$\frac{1}{\sqrt{2\pi}}\int \phi(k,t)\,f(t)\exp(i\omega t)\,dt \tag{5.17}$$

to obtain
(5.18)
$$\Phi\left(\omega|k\right)g\left(k\right)=\sqrt{\Delta t\sqrt{\frac{2}{\pi}}}\exp\left[-i\left(k'x_0-\omega't_0\right)\right]\exp\left[-\left(\Delta xk'-\Delta t\omega'\right)^2\right]\sqrt{\Delta x\sqrt{\frac{2}{\pi}}}\exp\left(-\Delta x^2k'^2\right)$$

which can be rewritten so that

$$\Phi\left(\omega|k\right)=\sqrt{\Delta t\sqrt{\frac{2}{\pi}}}\exp\left[-i\left(kx_0-\omega t_0\right)\right]\exp\left[-\left(\Delta xk'-\Delta t\omega'\right)^2\right] \tag{5.19}$$

where the expected value

$$\langle\omega\rangle=u\left(k-k_0\right)+\omega_0 \tag{5.20}$$

and marginal or prior packet for k is

$$g\left(k\right)=\sqrt{\Delta x\sqrt{\frac{2}{\pi}}}\exp\left[i\left(k_0x_0-\omega_0t_0\right)\right]\exp\left(-\Delta x^2k'^2\right) \tag{5.21}$$

All this coming from

$$\Psi\left(x|t\right)=\sqrt{\Delta k\sqrt{\frac{2}{\pi}}}\exp\left[i\left(k_0x-\omega_0t\right)\right]\exp\left[-\left(\Delta kx'-\Delta\omega t'\right)^2\right] \tag{5.22}$$

where $x'=x-x_0$ and $t'=t-t_0$.

The introduction of a "prior" or marginal wave packet seems a bit ad hoc, but the assumption that a joint wave packet satisfies

$$\Psi\left(x,t\right)=\Psi\left(x|t\right)f\left(t\right) \tag{5.23}$$

does not contradict the familiar law of probability

$$f\left(x|y\right)f_Y\left(y\right)=f\left(x,y\right)=f\left(y|x\right)f_X\left(x\right) \tag{5.24}$$

since if (5.23) holds, then

$$\begin{aligned} f\left(x,t\right) &= \Psi^*\left(x,t\right)\Psi\left(x,t\right) && (5.25)\\ & \left(\Psi^*\left(x|t\right)f^*\left(t\right)\right)\left(\Psi\left(x|t\right)f\left(t\right)\right) && (5.26)\\ &= \Psi^*\left(x|t\right)\Psi\left(x|t\right)f^*\left(t\right)f\left(t\right) && (5.27)\\ &= f\left(x|t\right)f_T\left(t\right) && (5.28)\end{aligned}$$

We do not have, however

$$\int_{-\infty}^{\infty}\Psi\left(x|t\right)f\left(t\right)dx=f\left(t\right)$$

but

$$\int_{-\infty}^{\infty}\Psi\left(x|t\right)f\left(t\right)dx=\alpha\left(t\right)f\left(t\right) \tag{5.29}$$

so that

$$\int_{-\infty}^{\infty}\Psi\left(x|t\right)dx=\alpha\left(t\right) \tag{5.30}$$

Now what is the magnitude, $|\alpha(t)|^2$? We have

$$\begin{aligned}
\alpha^*(t)\,\alpha(t) &= \left(\int_{-\infty}^{\infty}\Psi^*(x|t)\,dx\right)\left(\int_{-\infty}^{\infty}\Psi(u|t)\,du\right)\\
&= \left(\int_{-\infty}^{\infty}\Psi^*(x|t)\,dx\right)\left(\int_{-\infty}^{\infty}\Psi(x+u|t)\,du\right)\\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\Psi^*(x|t)\,\Psi(x+u|t)\,dx\right)du \qquad (5.31)
\end{aligned}$$

Now

$$\int_{-\infty}^{\infty}\Psi^*(x|t)\,\Psi(x+u|t)\,dx = (\Psi^*\star\Psi)(u) \qquad (5.32)$$

Which is the autocovariance of the wave function, which by the Wiener-Khinchin Theorem

$$\mathcal{F}^{-1}\left[|\Phi(k,t)|^2\right](u) = \int_{-\infty}^{\infty}\Psi^*(x|t)\,\Psi(x+u|t)\,dx \qquad (5.33)$$

Perhaps we should note that for $k=0$

$$\begin{aligned}
\int_{-\infty}^{\infty}\Psi(x|t)\,dx &= \int_{-\infty}^{\infty}\Psi(x|t)\exp(-ikx)\,dx &(5.34)\\
&= \mathcal{F}_x[\Psi(x|t)](k) &(5.35)\\
&= \mathcal{F}_x[\Psi(x|t)](0) &(5.36)\\
&= \phi(0,t) &(5.37)
\end{aligned}$$

6. The Potential for a Gaussian Moving with Constant Velocity

What is the potential in Schrödinger's Equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\,\Psi \qquad (6.1)$$

that leads to the gaussian wave packet

$$\Psi(x|t) = \sqrt{\Delta k\sqrt{\frac{2}{\pi}}}\exp[i(ax-bt)]\exp\left[-(\Delta k x' - \Delta\omega t')^2\right] \qquad (6.2)$$

with conditional expectation of the momentum

$$\langle k\rangle = a \qquad (6.3)$$

where

$$\begin{aligned}
x' &= x - x_0\\
t' &= t - t_0
\end{aligned}$$

Let us write

$$\Psi(x|t) = C_{\Delta k}\,f(x,t)\,g(x,t) \qquad (6.4)$$

Then

$$\frac{\partial\Psi}{\partial t} = C_{\Delta k}\left(\frac{\partial f}{\partial t}g + f\frac{\partial g}{\partial t}\right) \qquad (6.5)$$

Now we find

$$\frac{\partial f}{\partial t} = -ibf \tag{6.6}$$

and

$$\frac{\partial g}{\partial t} = 2\Delta\omega\left(\Delta k x' - \Delta\omega t'\right) g \tag{6.7}$$

so that

$$\frac{\partial \Psi}{\partial t} = \left[2\Delta\omega\left(\Delta k x' - \Delta\omega t'\right) - ib\right]\Psi \tag{6.8}$$

$$= Q\Psi \tag{6.9}$$

Next we find

$$\frac{\partial \Psi}{\partial x} = C_{\Delta k}\left(\frac{\partial f}{\partial x} g + f\frac{\partial g}{\partial x}\right) \tag{6.10}$$

where

$$\frac{\partial f}{\partial x} = iaf \tag{6.11}$$

and

$$\frac{\partial g}{\partial x} = -2\Delta k\left(\Delta k x' - \Delta\omega t'\right) g \tag{6.12}$$

so that

$$\frac{\partial \Psi}{\partial x} = \left[ia - 2\Delta k\left(\Delta k x' - \Delta\omega t'\right)\right]\Psi \tag{6.13}$$

$$= P\Psi \tag{6.14}$$

Then

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial P}{\partial x}\Psi + P\frac{\partial \Psi}{\partial x} \tag{6.15}$$

$$= -2\left(\Delta k\right)^2\Psi + P^2\Psi \tag{6.16}$$

$$= \left[P^2 - 2\left(\Delta k\right)^2\right]\Psi \tag{6.17}$$

Plugging back into Schrödinger's Equation we find

$$i\hbar Q\Psi = -\frac{\hbar^2}{2m}\left[P^2 - 2\left(\Delta k\right)^2\right]\Psi + V\Psi \tag{6.18}$$

so that

$$V\left(x,t\right) = i\hbar Q + \frac{\hbar^2}{2m}\left[P^2 - 2\left(\Delta k\right)^2\right] \tag{6.19}$$

which will be imaginary unless we have

$$\hbar\left\langle k\right\rangle = \hbar a = m\frac{\Delta\omega}{\Delta k} \tag{6.20}$$

since with $L = (\Delta k x' - \Delta\omega t')$ (that's for Lorentz) we have

$$\begin{aligned}\frac{\hbar^2}{2m}\left[P^2 - 2(\Delta k)^2\right] &= \frac{\hbar^2}{2m}\left[(ia - 2\Delta k L)^2 - 2(\Delta k)^2\right] \\ &= \frac{\hbar^2}{2m}\left[\left(-a^2 - 4ia\Delta k L + 4(\Delta k)^2 L^2\right) - 2(\Delta k)^2\right] \qquad (6.21) \\ &= \frac{\hbar^2}{2m}\left[-a^2 - 4ia\Delta k L + 2(\Delta k)^2\left(2L^2 - 1\right)\right] \qquad (6.22)\end{aligned}$$

and

$$i\hbar Q = 2i\hbar\Delta\omega L + \hbar b \qquad (6.23)$$

so that we must have the imaginary terms cancel

$$2i\hbar\Delta\omega L = \frac{\hbar^2}{2m}4ia\Delta k L \qquad (6.24)$$

Hence

$$\Delta\omega = \frac{\hbar a \Delta k}{m} \qquad (6.25)$$

Which gives us equation (6.20). In this condition the potential becomes

$$V(x,t) = \hbar b - \frac{\hbar^2 a^2}{2m} + \frac{1}{m}(\hbar\Delta k)^2\left(2L^2 - 1\right) \qquad (6.26)$$

7. Probability Density Current

Dispensing with the derivation for the moment, the probabilty density current, is defined by

$$\begin{aligned}\mathrm{S}(\mathrm{r},t) &= \frac{i\hbar}{2m}\left[(\nabla\Psi^*)\Psi - \Psi^*(\nabla\Psi)\right] \qquad (7.1) \\ &= \frac{\hbar}{m}\Im\left[\Psi^*(\nabla\Psi)\right] \qquad (7.2)\end{aligned}$$

with probaility density

$$\rho = |\Psi|^2 \qquad (7.3)$$

where the continuity equation expression conservation of probabilty density is

$$\frac{\partial\rho}{\partial t} + \nabla\mathrm{S} = 0 \qquad (7.4)$$

It is tempting to see the 4-component vector

$$S^\mu = (c\rho, \mathrm{S})$$

as a 4-vector, where the continuity equation becomes

$$\partial_\mu S^\mu = 0 \qquad (7.5)$$

In tensor form the spatial components of the probabilty density current become

$$\begin{aligned}S^i &= \frac{i\hbar}{2m}\left[\left(\partial^i\Psi^*\right)\Psi - \Psi^*\left(\partial^i\Psi\right)\right] \qquad (7.6) \\ &= \frac{\hbar}{m}\Im\left[\Psi^*\left(\partial^i\Psi\right)\right] \qquad (7.7)\end{aligned}$$

Where the latin indices range from 1 to 3. But in analogy to the current density in the relativistic formulation of electro-magnetic theory we should have

$$S^{\mu} = \rho_0 \eta^{\mu} \tag{7.8}$$

Where ρ_0 is the probability density in the frame which sees the wave packet as stationary (leaving this notion undefined for the moment).

Consider the one dimension case of the gaussian wave packet of Section 6

$$\Psi\left(x|t\right) = \sqrt{\Delta k \sqrt{\frac{2}{\pi}}} \exp\left[i\left(ax - bt\right)\right] \exp\left[-\left(\Delta k x - \Delta\omega t\right)^2\right] \tag{7.9}$$

with

$$\frac{\partial \Psi}{\partial x} = P\Psi \tag{7.10}$$

where

$$P\left(x,t\right) = ia - 2\Delta k\left(\Delta k x - \Delta\omega t\right) \tag{7.11}$$

then

$$\begin{aligned}
S_x &= \frac{\hbar}{m}\Im\left[\Psi^*\left(\frac{\partial\Psi}{\partial x}\right)\right] && (7.12)\\
&= \frac{\hbar}{m}\left|\Psi\right|^2\Im\left(P\right) && (7.13)\\
&= \frac{\hbar a}{m}\left|\Psi\right|^2 && (7.14)\\
&= \frac{\langle p_x\rangle}{m}\left|\Psi\right|^2 && (7.15)\\
& v_x\left|\Psi\right|^2 && (7.16)
\end{aligned}$$

Since from equation (6.20)

$$\begin{aligned}
\langle p_x\rangle &= m\frac{\Delta\omega}{\Delta k}\\
&= mv
\end{aligned}$$

The components of our 2-vector are

$$\begin{aligned}
S^{\mu} &= \left(c,v\right)\left|\Psi\right|^2 && (7.17)\\
&= \gamma\left(c,v\right)\gamma^{-1}\left|\Psi\right|^2 && (7.18)
\end{aligned}$$

But we have

$$\gamma^{-1}\left|\Psi\right|^2 = \left|\Psi_0\right|^2 \tag{7.19}$$

So that

$$\begin{aligned}
S^{\mu} &= \gamma c\left(1,\beta\right)\left|\Psi_0\right|^2 && (7.20)\\
&= \eta^{\mu}\left|\Psi_0\right|^2 && (7.21)
\end{aligned}$$

Example 2. But what is Ψ_0? Let us rewrite (7.9) as

$$\Psi\left(x|t\right)=\sqrt{\Delta k_1\sqrt{\frac{2}{\pi}}}\exp\left[i\left(a_1x^1+a_0x^0\right)\right]\exp\left[-\left(\Delta k_1x^1+\Delta k_0x^0\right)^2\right] \tag{7.22}$$

where

$$x^0 = ct \tag{7.23}$$
$$\Delta k_0 = -\frac{\Delta\omega}{c} \tag{7.24}$$
$$a_0 = -\frac{b}{c} \tag{7.25}$$

so that

$$\beta=\frac{1}{c}\frac{\Delta\omega}{\Delta k}=-\frac{\Delta k_0}{\Delta k_1} \tag{7.26}$$

In the frame moving with the wave packet we have presumably

$$\Psi_0\left(x'^1|x'^0\right)=\sqrt{\Delta k_1'\sqrt{\frac{2}{\pi}}}\exp\left[i\left(a_1'x'^1+a_0'x'^0\right)\right]\exp\left[-\left(\Delta k_1'x'^1+\Delta k_0'x'^0\right)^2\right] \tag{7.27}$$

where

$$\begin{aligned}\Delta k_0' &= \gamma\left(\Delta k_0+\beta\Delta k_1\right)\\ &= \gamma\left(\Delta k_0-\Delta k_0\right)\\ &= 0\end{aligned} \tag{7.28}$$

But now let us substitute the relativistic momentum for the expected value $\langle\hbar k_1\rangle$

$$\langle k_1\rangle = a_1 \tag{7.29}$$
$$= \frac{m}{\hbar}\frac{\Delta x_1}{\Delta\tau} \tag{7.30}$$
$$= \frac{m}{\hbar}\gamma c\beta_1 \tag{7.31}$$

where

$$c^2\left(\Delta\tau\right)^2=c^2\left(\Delta t\right)^2-\left(\Delta x\right)^2 \tag{7.32}$$

then we should have

$$\begin{aligned}\langle k_1'\rangle &= a_1'\\ &= \frac{m}{\hbar}\frac{\Delta x'^1}{\Delta\tau}\\ &= \frac{m}{\hbar}\frac{0}{\Delta\tau}\\ &= 0\end{aligned} \tag{7.33}$$

But we have

$$a_1'=\gamma\left(a_1+\beta a_0\right) \tag{7.34}$$

so that

$$\begin{aligned}
0 &= a_1 + \beta a_0 \\
&= \hbar a_1 + \beta \hbar a_0 \\
&= m\frac{\Delta x^1}{\Delta \tau} + \frac{\Delta x^1}{\Delta x^0}\hbar a_0 \\
&= m\frac{\Delta x^1}{\Delta x^0}\frac{\Delta x^0}{\Delta \tau} + \frac{\Delta x^1}{\Delta x^0}\hbar a_0 \\
&= mc\gamma + \hbar a_0 \qquad (7.35)
\end{aligned}$$

hence

$$\begin{aligned}
\hbar a_0 &= -mc\gamma \qquad (7.36)\\
&= m\gamma c\beta_0 \qquad (7.37)
\end{aligned}$$

where $-\beta_0 = \beta^0 = 1$, and in terms of $b = -ca_0$, then

$$\hbar b = mc^2\gamma \qquad (7.38)$$

Thus our original gaussian packet should be

$$\Psi\left(x^1|x^0\right) = \sqrt{\Delta k_1\sqrt{\frac{2}{\pi}}}\exp\left[i\frac{m}{\hbar}\gamma c\left(\beta_1 x^1 + \beta_0 x^0\right)\right]\exp\left[-\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2\right] \qquad (7.39)$$

and we see that

$$\begin{aligned}
\hbar a_0' &= \gamma\left(\hbar a_0 + \beta\hbar a_1\right) \\
&= \gamma\left(-mc\gamma + \beta m\frac{\Delta x^1}{\Delta\tau}\right) \\
&= \gamma\left(-mc\gamma + \beta m\frac{\Delta x^1}{\Delta x^0}\frac{\Delta x^0}{\Delta\tau}\right) \\
&= -\gamma\left(mc\gamma - m\beta^2\frac{\Delta x^0}{\Delta\tau}\right) \\
&= -\gamma\left(mc\gamma - m\beta^2\frac{c\Delta t}{\Delta\tau}\right) \\
&= -\gamma\left(mc\gamma - mc\beta^2\gamma\right) \\
&= -\gamma^2 mc\left(1-\beta^2\right) \\
&= -mc \qquad (7.40)
\end{aligned}$$

In the stationary frame we have

$$\Psi_0\left(x'^1|x'^0\right) = \sqrt{\Delta k_1'\sqrt{\frac{2}{\pi}}}\exp\left[i\frac{m}{\hbar}c\left(0\cdot x'^1 - 1\cdot x'^0\right)\right]\exp\left[-\left(\Delta k_1' x'^1 + 0\cdot x'^0\right)^2\right] \qquad (7.41)$$

Noting the quantity $\frac{mc}{\hbar}$ is the reciprical of the Compton wavelength.

All of this is very curious, since the expression for the spatial components of probability density current are derived from a non-relativistic equation, Schrödinger's as follows. First we need the identities

$$\nabla \cdot (f\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g \tag{7.42}$$

$$\nabla \cdot (g\nabla f) = \nabla g \cdot \nabla f + g\nabla^2 f \tag{7.43}$$

from which we obtain

$$\nabla \cdot (f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f \tag{7.44}$$

The first two identities are the same really with f and g interchanged, and their derivation is particularly transparent in tensor notation (latin indices) by the ordinary product rule, and sumation convention

$$\partial^i (f\partial_i g) = \partial^i f \partial_i g + f\partial^i \partial_i g \tag{7.45}$$

Now for volume element $d\lambda$, unfortunately we can't use $d\tau$ or dV for confusion with proper time, τ and potential, V.

$$\frac{\partial \rho}{\partial t} d\lambda = \frac{\partial \Psi^* \Psi}{\partial t} d\lambda \tag{7.46}$$

$$= \left(\frac{\partial \Psi^*}{\partial t} \Psi + \Psi^* \frac{\partial \Psi}{\partial t} \right) d\lambda \tag{7.47}$$

Where Ψ satisfies Schrödinger's Equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \Psi \tag{7.48}$$

The conjugate Ψ^* satisfies the conjugate equation

$$-i\hbar \frac{\partial \Psi^*}{\partial t} = \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \Psi^* \tag{7.49}$$

so that

$$\Psi^* \frac{\partial \Psi}{\partial t} = -\frac{i}{\hbar} \Psi^* \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \Psi \tag{7.50}$$

$$\Psi \frac{\partial \Psi^*}{\partial t} = \frac{i}{\hbar} \Psi \left(-\frac{\hbar^2}{2m} \nabla^2 + V \right) \Psi^* \tag{7.51}$$

Summing we have

$$\Psi^* \frac{\partial \Psi}{\partial t} + \Psi \frac{\partial \Psi^*}{\partial t} = \frac{i}{\hbar} \Psi^* \frac{\hbar^2}{2m} \nabla^2 \Psi - \frac{i}{\hbar} \Psi - \frac{\hbar^2}{2m} \nabla^2 \Psi^* \tag{7.52}$$

$$= \frac{i\hbar}{2m} \left(\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^* \right) \tag{7.53}$$

And with $g = \Psi$ and $f = \Psi^*$ in the identity (7.44) we obtain

$$\frac{\partial \rho}{\partial t} d\lambda = \frac{i\hbar}{2m} \nabla \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \, d\lambda \tag{7.54}$$

$$= -\frac{i\hbar}{2m} \nabla \cdot (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) \, d\lambda \tag{7.55}$$

$$= -\nabla \cdot \mathrm{S} \tag{7.56}$$

With

$$\mathrm{S} = \frac{i\hbar}{2m} \left(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi \right) \tag{7.57}$$

8. A Relativistic Schrödinger Equation

Suppose we desired to obtain a relativistic analogue to Schrödinger's equation

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+V\Psi \tag{8.1}$$

our first impulse is to replace the Laplacian operator, ∇^2 with the D'alembertian operator, $\Box^2$ to obtain

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\Box^2\Psi+V\Psi$$

this equation, however, is not Lorentz invariant due to the left hand side, and the whole point of the endeavor is to obtain a Lorentz invariant equation. So what to do with the left hand side? Well suppose we replace

$$i\hbar\tfrac{\partial\Psi}{\partial t}\quad\text{with}\quad i\hbar\tfrac{d\Psi}{d\tau}$$

where $d\tau$ is the differential of the proper time to obtain, now in tensor notation with the Einstein summation convention

$$i\hbar\frac{d\Psi}{d\tau}=-\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi+V\left(x^\mu\right)\Psi \tag{8.2}$$

expanding the left hand side derivative by the product rule

$$\frac{d\Psi}{d\tau}=\frac{\partial\Psi}{\partial x^0}\frac{dx^0}{d\tau}+\frac{\partial\Psi}{\partial x^1}\frac{dx^1}{d\tau}+\frac{\partial\Psi}{\partial x^2}\frac{dx^2}{d\tau}+\frac{\partial\Psi}{\partial x^3}\frac{dx^3}{d\tau} \tag{8.3}$$

we obtain

$$i\hbar\frac{dx^\mu}{d\tau}\frac{\partial\Psi}{\partial x^\mu}=-\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi+V\left(x^\mu\right)\Psi \tag{8.4}$$

But this is not a proper wave equation, since the variables $x^0=ct$, $x^1=x$, $x^2=y$, and $x^3=z$ must be free to vary independently, and the left hand side treats them as paths with proper time, τ as their parameter. That is $t\left(\tau\right)$, $x\left(\tau\right)$, etc. are one dimensional curves, and they cannot be such. None the less we see a form for a Lorentz invariant equation if we replace derivatives of proper time with some proper velocity 4-vector, η^μ so that we obtain

$$i\hbar\eta^\mu\partial_\mu\Psi=-\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi+V\left(x^\mu\right)\Psi \tag{8.5}$$

We derive at this equation by alternate means in Section 10.2. If $\eta^\mu=(c,0,0,0)$, then

$$\eta^\mu i\hbar\partial_\mu\Psi=ci\hbar\frac{\partial\Psi}{\partial x^0}=ci\hbar\frac{\partial\Psi}{\partial ct}=i\hbar\frac{\partial\Psi}{\partial t} \tag{8.6}$$

hence

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi+V\left(x,y,z,ct\right)\Psi \tag{8.7}$$

If we let $c\to\infty$ we reattain Schrödinger's Equation as a classical limit. Whether or not it has any physical validity, equation (8.5) is a perfectly wonderful linear partial differential equation of second order, though with the presence of $i=\sqrt{-1}$, not all of its coefficients are real numbers. If the potential, V is time independent in the form

$$\eta^0 i\hbar\partial_0\Psi=-\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi+V\left(x^\mu\right)\Psi \tag{8.8}$$

we can apply the method of separation of variable to its solution. We will find that in this case we recover, intact, the time independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = E \tag{8.9}$$

We also obtain through alternate means the relativistic equation in Section 10.2.

The momentum operators now represent the relativistic momentum, but the total energy operator, $i\hbar\partial_0$, of Schrodinger's equation, has a new role. Let us write out the operators explicitly, first the covariant operators

$$\begin{aligned} -i\hbar\partial_\mu &= -i\hbar\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) \\ &= -i\hbar\left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \\ &= \left(-\frac{T}{c}, p_x, p_y, p_z\right) \end{aligned} \tag{8.10}$$

where here T represents[1] the combined rest and kinetic energy, and not the total energy. The contravariant operators give us

$$\begin{aligned} -i\hbar\partial^\mu &= -i\hbar\left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \\ &= -i\hbar\left(-\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) && (8.11) \\ &= \left(\frac{T}{c}, p_x, p_y, p_z\right) && (8.12) \end{aligned}$$

Again note that the operator

$$i\hbar c\partial_0 = i\hbar\frac{\partial}{\partial t} \tag{8.13}$$

is no longer the total energy operator, but now represents the combined rest and kinetic energy. With this formulation, the operator has been placed on equal footing with the momentum operators. But what does the the operator

$$-\hbar^2\Box^2 = \hbar^2\left(\partial^0\partial_0 - \partial^i\partial_i\right) \tag{8.14}$$

represent, well it corresponds to the relativistic equation

$$\left(\frac{T}{c}\right)^2 - p^2 = m^2c^2 \tag{8.15}$$

Thus we would have to regard it as c^2 times the square of a rest mass operator, (m), thus

$$-\hbar^2\Box^2 = (m)^2\, c^2 \tag{8.16}$$

[1]Of course using T as such results in the equation, $T = mc^2$.

Example 3. Let us find the potential in the one dimensional case of equation (8.5) given the gaussian wave packet (7.39) of Section 7. So let us plug the wave packet

$$\Psi\left(x^1|x^0\right) = \sqrt{\Delta k_1 \sqrt{\frac{2}{\pi}}} \exp\left[i\frac{m}{\hbar}\gamma c\left(\beta_1 x^1 + \beta_0 x^0\right)\right] \exp\left[-\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2\right] \tag{8.17}$$

into

$$i\hbar\left(\eta^1\frac{\partial\Psi}{\partial x^1} + \eta^0\frac{\partial\Psi}{\partial x^0}\right) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial\left(x^1\right)^2} - \frac{\partial^2\Psi}{\partial\left(x^0\right)^2}\right) + V\Psi \tag{8.18}$$

We find

$$\frac{\partial\Psi}{\partial x^1} = P\Psi \tag{8.19}$$

where

$$P = ia_1 - 2\Delta k_1 L \tag{8.20}$$

$$L = \Delta k_1 x^1 + \Delta k_0 x^0 \tag{8.21}$$

$$a_1 = \frac{m}{\hbar}\gamma c\beta_1 \tag{8.22}$$

Then since

$$\frac{\partial P}{\partial x^1} = -2\left(\Delta k_1\right)^2 \tag{8.23}$$

we have

$$\begin{aligned}\frac{\partial^2\Psi}{\partial\left(x^1\right)^2} &= \frac{\partial P}{\partial x^1}\Psi + P\frac{\partial\Psi}{\partial x^1}\\ &= \left[P^2 - 2\left(\Delta k_1\right)^2\right]\Psi\end{aligned} \tag{8.24}$$

We also find

$$\frac{\partial\Psi}{\partial x^0} = Q\Psi \tag{8.25}$$

where

$$Q = ia_0 - 2\Delta k_0 L \tag{8.26}$$

$$a_0 = \frac{m}{\hbar}\gamma c\beta_0 \tag{8.27}$$

$$\beta_0 = -1 \tag{8.28}$$

With

$$\frac{\partial P}{\partial x^0} = -2\left(\Delta k_0\right)^2 \tag{8.29}$$

we have

$$\begin{aligned}\frac{\partial^2\Psi}{\partial\left(x^0\right)^2} &= \frac{\partial Q}{\partial x^0}\Psi + Q\frac{\partial\Psi}{\partial x^0}\\ &= \left[Q^2 - 2\left(\Delta k_0\right)^2\right]\Psi\end{aligned} \tag{8.30}$$

Now

$$\begin{aligned}
-\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial\left(x^1\right)^2}-\frac{\partial^2\Psi}{\partial\left(x^0\right)^2}\right) &= -\frac{\hbar^2}{2m}\left\{\left[P^2-2\left(\Delta k_1\right)^2\right]-\left[Q^2-2\left(\Delta k_0\right)^2\right]\right\}\Psi \\
&= -\frac{\hbar^2}{2m}\left\{\left[P^2-Q^2\right]-2\left(\left(\Delta k_1\right)^2-\left(\Delta k_0\right)^2\right)\right\}\Psi \\
&= -\frac{\hbar^2}{2m}\left\{\left[P^2-Q^2\right]-2\Delta k_\mu\Delta k^\mu\right\}\Psi \qquad (8.31)
\end{aligned}$$

and

$$P^2 = -\left(a_1\right)^2-4ia_1\Delta k_1 L+4\left(\Delta k_1\right)^2 L^2 \tag{8.32}$$

$$Q^2 = -\left(a_0\right)^2-4ia_0\Delta k_0 L+4\left(\Delta k_0\right)^2 L^2 \tag{8.33}$$

so that

$$\begin{aligned}
P^2-Q^2 &= -\left[\left(a_1\right)^2-\left(a_0\right)^2\right]-4iL\left[a_1\Delta k_1-a_0\Delta k_0\right]+4L^2\left[\left(\Delta k_1\right)^2-\left(\Delta k_0\right)^2\right] \\
&= -a_\mu a^\mu-4iLa_\mu\Delta k^\mu+4L^2\Delta k_\mu\Delta k^\mu \qquad (8.34)
\end{aligned}$$

hence

$$-\frac{\hbar^2}{2m}\left(P^2-Q^2\right)=\frac{\hbar^2}{2m}\left(a_\mu a^\mu+4iLa_\mu\Delta k^\mu-4L^2\Delta k_\mu\Delta k^\mu\right) \tag{8.35}$$

Now for the left hand side of equation (8.18)

$$i\hbar\left(\eta^1\frac{\partial\Psi}{\partial x^1}+\eta^0\frac{\partial\Psi}{\partial x^0}\right)=i\hbar\left(\eta^1 P+\eta^0 Q\right)\Psi \tag{8.36}$$

$$\begin{aligned}
i\hbar\left(\eta^1 P+\eta^0 Q\right) &= -\hbar\left(\eta^1 a_1+2i\eta^1\Delta k_1 L+\eta^0 a_0+2i\eta^0\Delta k_0 L\right) \\
&= -\hbar\left(\eta^\mu a_\mu+2iL\eta^\mu\Delta k_\mu\right) \qquad (8.37)
\end{aligned}$$

In order that the potential be real valued we see from equations (8.35) and (8.37) that the imaginary terms must cancel

$$\begin{aligned}
-2i\hbar L\eta^\mu\Delta k_\mu &= \frac{\hbar^2}{2m}4iLa_\mu\Delta k^\mu \\
&= 2i\frac{\hbar^2}{m}La^\mu\Delta k_\mu \qquad (8.38)
\end{aligned}$$

so that we must have

$$\eta^\mu = -\frac{\hbar}{m}a^\mu \tag{8.39}$$

$$a^\mu = -\frac{m}{\hbar}\eta^\mu \tag{8.40}$$

but

$$a^\mu=\frac{m}{\hbar}\gamma c\beta^\mu \tag{8.41}$$

so that

$$\eta^\mu=-\gamma c\beta^\mu \tag{8.42}$$

which is opposite in sign to the proper velocity of the packet. To find the potential we return to the relativistic Schrödinger

(8.43)
$$-\hbar\left(\eta^{\mu}a_{\mu}+2iL\eta^{\mu}\Delta k_{\mu}\right)\Psi=-\frac{\hbar^2}{2m}\left(-a_{\mu}a^{\mu}-4iLa_{\mu}\Delta k^{\mu}+4L^2\Delta k_{\mu}\Delta k^{\mu}-2\Delta k_{\mu}\Delta k^{\mu}\right)\Psi+V\Psi$$

Which becomes with the cancelation of imaginary terms

$$\begin{aligned}-\hbar\eta^{\mu}a_{\mu}\Psi &= -\frac{\hbar^2}{2m}\left(-a_{\mu}a^{\mu}+4L^2\Delta k_{\mu}\Delta k^{\mu}-2\Delta k_{\mu}\Delta k^{\mu}\right)\Psi+V\Psi\\ &= -\frac{\hbar^2}{2m}\left(-a_{\mu}a^{\mu}+2\Delta k_{\mu}\Delta k^{\mu}\left(2L^2-1\right)\right)\Psi+V\Psi \qquad (8.44)\end{aligned}$$

but

$$\begin{aligned}a_{\mu}a^{\mu} &= \left(-\frac{m}{\hbar}\eta^{\mu}\right)\left(-\frac{m}{\hbar}\eta_{\mu}\right)\\ &= \frac{m^2}{\hbar^2}\eta^{\mu}\eta_{\mu} \qquad (8.45)\\ &= -\frac{m^2c^2}{\hbar^2} \qquad (8.46)\end{aligned}$$

and

$$\begin{aligned}\eta^{\mu}a_{\mu} &= -\frac{m}{\hbar}\eta^{\mu}\eta_{\mu} \qquad (8.47)\\ &= \frac{mc^2}{\hbar} \qquad (8.48)\end{aligned}$$

so that

$$\begin{aligned}-mc^2\Psi &= -\frac{\hbar^2}{2m}\left(\frac{m^2c^2}{\hbar^2}+2\Delta k_{\mu}\Delta k^{\mu}\left(2L^2-1\right)\right)\Psi+V\Psi \qquad (8.49)\\ &= -\frac{mc^2}{2}\Psi-\frac{\hbar^2}{m}\Delta k_{\mu}\Delta k^{\mu}\left(2L^2-1\right)\Psi+V\Psi \qquad (8.50)\end{aligned}$$

Therefore

$$\begin{aligned}V\Psi &= -\frac{mc^2}{2}\Psi+\frac{\hbar^2}{m}\Delta k_{\mu}\Delta k^{\mu}\left(2L^2-1\right)\Psi\\ &= \left[-\frac{1}{2}mc^2+\frac{\hbar^2}{m}\Delta k_{\mu}\Delta k^{\mu}\left(2L^2-1\right)\right]\Psi \qquad (8.51)\end{aligned}$$

Of course we now have to divide both sides by Ψ where it isn't zero to obtain the Lorentz invariant expression for the potential

$$V = -\frac{1}{2}mc^2+\frac{\hbar^2}{m}\Delta k_{\mu}\Delta k^{\mu}\left(2\left(\Delta k_{\mu}x^{\mu}\right)^2-1\right) \qquad (8.52)$$

Recalling that the proper velocity of the gaussian packet was given by

$$-\eta^{\mu}=\gamma c\beta^{\mu} \qquad (8.53)$$

and

$$\beta^1=\frac{1}{c}\frac{\Delta\omega}{\Delta k}=-\frac{\Delta k_0}{\Delta k_1} \qquad (8.54)$$

Now in the frame in which the potential is stationary

$$\left(\eta^0, \eta^1\right) = (-c, 0) \tag{8.55}$$

$$(\Delta k_0, \Delta k_1) = (0, \Delta k_1) \tag{8.56}$$

so that

$$V_R = -\frac{1}{2}mc^2 + \frac{1}{m}\left(\hbar \Delta k_1\right)^2 \left(2\left(\Delta k_1 x^1\right)^2 - 1\right) \tag{8.57}$$

which is time independent. The corresponding wave function is

$$\Psi = \sqrt{\Delta k_1 \sqrt{\frac{2}{\pi}}} \exp\left[-i\frac{m}{\hbar}\left(0 \cdot x^1 + \eta_0 x^0\right)\right] \exp\left[-\left(\Delta k_1 x^1 + 0 \cdot x^0\right)^2\right] \tag{8.58}$$

with $\eta_0 = c$ as solution to

$$i\hbar\eta^0 \partial_0 \Psi = -\frac{\hbar^2}{2m}\partial^\mu \partial_\mu \Psi + V_R \Psi \tag{8.59}$$

or

$$-i\hbar c \partial_0 \Psi = -\frac{\hbar^2}{2m}\partial^\mu \partial_\mu \Psi + V_R \Psi \tag{8.60}$$

Note that here the time dependent phase factor is

$$\chi(t) = \exp\left(-i\frac{mc^2}{\hbar}t\right) \tag{8.61}$$

since $\eta_0 = c$.

Example 4. Now let us compare this to the potential for the gaussian packet calculated from Schrödinger's Equation proper in Section 6 equation (6.26)

$$V_S = \hbar b - \frac{\hbar^2 a^2}{2m} + \frac{1}{m}\left(\hbar \Delta k\right)^2 \left(2L^2 - 1\right) \tag{8.62}$$

where the potential was calculated with the time dependent phase factor of

$$\chi(t) = \exp(-ibt) \tag{8.63}$$

We have

$$a = a_1 = 0 \tag{8.64}$$

$$L = \Delta k_1 x^1 \tag{8.65}$$

and

$$\frac{b}{c} = \frac{mc}{\hbar} \tag{8.66}$$

$$= -a_0 \tag{8.67}$$

so that equation (6.26) becomes

$$V_S = mc^2 + \frac{1}{m}\left(\hbar \Delta k_1\right)^2 \left(2\left(\Delta k_1 x^1\right)^2 - 1\right) \tag{8.68}$$

with corresponding wave function

$$\Psi = \sqrt{\Delta k_1 \sqrt{\frac{2}{\pi}}} \exp\left[i\frac{m}{\hbar}\left(0 \cdot x^1 - cx^0\right)\right] \exp\left[-\left(\Delta k_1 x^1 + 0 \cdot x^0\right)^2\right] \tag{8.69}$$

as solution to (note the latin indices here)

$$i\hbar c\partial_0\Psi = -\frac{\hbar^2}{2m}\partial^i\partial_i\Psi + V_S\Psi \tag{8.70}$$

We see a difference in the sign of the left hand sides the ordinary Schrödinger equation (8.70) and the relativistic version in (8.59). Explicitly equation (8.59) is

$$-i\hbar c\partial_0\Psi = -\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi + V_R\Psi \tag{8.71}$$

8.1. **Relativistic Probability Density Current.** We are in a position now to find a relativistic analogue of the probability density current based upon equation (8.5). Firstly

$$i\hbar\eta^\nu\partial_\nu\Psi^*\Psi = \left(i\hbar\eta^\nu\partial_\nu\Psi^*\right)\Psi + \Psi^*\left(i\hbar\eta^\nu\partial_\nu\Psi\right) \tag{8.72}$$

The wave equation and the conjugate equation are given by

$$\begin{aligned} i\hbar\eta^\nu\partial_\nu\Psi &= -\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi + V\Psi && (8.73)\\ -i\hbar\eta^\nu\partial_\nu\Psi^* &= -\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi^* + V\Psi^* && (8.74)\end{aligned}$$

so that (8.72) becomes

$$\begin{aligned} i\hbar\eta^\nu\partial_\nu\Psi^*\Psi &= \left(\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi^* - V\Psi^*\right)\Psi + \Psi^*\left(-\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi + V\Psi\right)\\ &= \frac{\hbar^2}{2m}\left[\left(\partial^\mu\partial_\mu\Psi^*\right)\Psi - \Psi^*\left(\partial^\mu\partial_\mu\Psi\right)\right] && (8.75)\end{aligned}$$

The relativistic analogue to the divergence product rule (7.42) is given by

$$\begin{aligned} \partial^\mu\left(f\partial_\mu g\right) &= \partial^\mu f\partial_\mu g + f\partial^\mu\partial_\mu g && (8.76)\\ \partial^\mu\left(g\partial_\mu f\right) &= \partial^\mu g\partial_\mu f + g\partial^\mu\partial_\mu f && (8.77)\end{aligned}$$

If we write

$$\begin{aligned} \partial^\mu f\partial_\mu g &= F^\mu G_\mu && (8.78)\\ &= F_\mu G^\mu && (8.79)\\ &= G^\mu F_\mu && (8.80)\\ &= \partial^\mu g\partial_\mu f && (8.81)\end{aligned}$$

hence

$$\partial^\mu\left(f\partial_\mu g - g\partial_\mu f\right) = f\partial^\mu\partial_\mu g - g\partial^\mu\partial_\mu f \tag{8.82}$$

Then with $f = \Psi$ and $g = \Psi^*$ we find

$$\partial^\mu\left(\Psi\partial_\mu\Psi^* - \Psi^*\partial_\mu\Psi\right) = \Psi\partial^\mu\partial_\mu\Psi^* - \Psi^*\partial^\mu\partial_\mu\Psi \tag{8.83}$$

Plugging this expression into (8.75) yields

$$i\hbar\eta^{\nu}\partial_{\nu}\Psi^{*}\Psi = \frac{\hbar^2}{2m}\left[\partial^{\mu}\left(\Psi\partial_{\mu}\Psi^{*} - \Psi^{*}\partial_{\mu}\Psi\right)\right] \tag{8.84}$$

or

$$\eta^{\nu}\partial_{\nu}\Psi^{*}\Psi = -\frac{i\hbar}{2m}\left[\partial^{\mu}\left(\Psi\partial_{\mu}\Psi^{*} - \Psi^{*}\partial_{\mu}\Psi\right)\right] \tag{8.85}$$

$$= -\frac{i\hbar}{2m}2i\partial^{\mu}\Im\left(\Psi\partial_{\mu}\Psi^{*}\right) \tag{8.86}$$

$$= \frac{\hbar}{m}\partial^{\mu}\Im\left(\Psi\partial_{\mu}\Psi^{*}\right) \tag{8.87}$$

$$= \frac{\hbar}{m}\partial_{\mu}\Im\left(\Psi\partial^{\mu}\Psi^{*}\right) \tag{8.88}$$

Changing the dummy index ν to μ on the left hand side and rearranging terms we find

$$\eta^{\mu}\partial_{\mu}\Psi^{*}\Psi - \frac{\hbar}{m}\partial_{\mu}\Im\left(\Psi\partial^{\mu}\Psi^{*}\right) = 0 \tag{8.89}$$

or

$$\partial_{\mu}\left[\eta^{\mu}\Psi^{*}\Psi - \frac{\hbar}{m}\Im\left(\Psi\partial^{\mu}\Psi^{*}\right)\right] = 0 \tag{8.90}$$

Thus we define

$$S^{\mu} = \eta^{\mu}\Psi^{*}\Psi - \frac{\hbar}{m}\Im\left(\Psi\partial^{\mu}\Psi^{*}\right) \tag{8.91}$$

and

$$\partial_{\mu}S^{\mu} = 0 \tag{8.92}$$

Now we can write

$$S^{0} = \eta^{0}\Psi^{*}\Psi - \frac{\hbar}{m}\Im\left(\Psi\partial^{0}\Psi^{*}\right) \tag{8.93}$$

$$S^{i} = \eta^{i}\Psi^{*}\Psi - \frac{\hbar}{m}\Im\left(\Psi\partial^{i}\Psi^{*}\right) \tag{8.94}$$

In the frame where the potential is stationary we have $\eta^{i} = 0$, and we find further on that in that frame, it is natural to define $\eta^{0} = -c$

$$S^{0}_{\text{rest}} = -c\Psi^{*}\Psi - \frac{\hbar}{m}\Im\left(\Psi\partial^{0}\Psi^{*}\right) \tag{8.95}$$

$$S^{i}_{\text{rest}} = -\frac{\hbar}{m}\Im\left(\Psi\partial^{i}\Psi^{*}\right) \tag{8.96}$$

8.2. **Separation of Variables in the Relativistic Schrödinger Equation.** Our proposed relativistic Schrödinger equation is clearly linear, and can be solved by separation of variables in the rest frame with a time independent potential. If you encounter a time independent potential in your frame of reference, then that potential is certainlt stationary in your frame. This should not be taken as a definition of a stationary potential however, since we can entertain the possibility that a time dependent potential should be considered stationary, but we will not propose a definition here, and will concentrate on solving

$$i\hbar\eta^{0}\partial_{0}\Psi = -\frac{\hbar^2}{2m}\partial^{\mu}\partial_{\mu}\Psi + V\Psi \tag{8.97}$$

with time independent potential. With $\Psi(x,t) = \psi(x)\chi(t)$ letting $t = x^0$ and $x = x^1$ so that $c = 1$, then $\eta^0 = -1$. Thus

$$\begin{aligned} -i\hbar\partial_0\psi\chi &= -\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\psi\chi + V\psi\chi(t) \\ &= -\frac{\hbar^2}{2m}\partial^i\partial_i\psi\chi - \frac{\hbar^2}{2m}\partial^0\partial_0\psi\chi + V\psi\chi \end{aligned} \tag{8.98}$$

So that

$$-i\hbar\psi\partial_0\chi = -\chi\frac{\hbar^2}{2m}\partial^i\partial_i\psi - \psi\frac{\hbar^2}{2m}\partial^0\partial_0\chi + V\psi\chi \tag{8.99}$$

Then dividing bothe sides by $\chi\psi$ and rearanging terms we find

$$\frac{1}{\chi}\frac{\hbar^2}{2m}\partial^0\partial_0\chi - i\hbar\frac{1}{\chi}\partial_0\chi = -\frac{1}{\psi}\frac{\hbar^2}{2m}\partial^i\partial_i\psi + V \tag{8.100}$$

or

$$-\frac{1}{\chi}\frac{\hbar^2}{2m}\partial_0\partial_0\chi - i\hbar\frac{1}{\chi}\partial_0\chi = -\frac{1}{\psi}\frac{\hbar^2}{2m}\partial^i\partial_i\psi + V$$

We obtain for separation constant, α and one spatial dimension

$$\frac{\hbar^2}{2m}\frac{d^2\chi}{dt^2} + i\hbar\frac{d\chi}{dt} + \alpha\chi = 0 \tag{8.101}$$

and

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (V-\alpha)\psi = 0 \tag{8.102}$$

which, significantly, is precisely the same form as the time-independent Schrödinger equation. Thus for the same wave function we expect the ordinary potential and energy eigenstate, and the relativistic potential and energy eigenstate to obey

$$V_S - \alpha_S = V_R - \alpha_R \tag{8.103}$$

so that

$$\alpha_R - \alpha_S = V_R - V_S \tag{8.104}$$

If this difference were always the same constant

$$\alpha_R - \alpha_S = K \tag{8.105}$$

then the difference in energy eigenstates, hence the spectra, would be the same in both the relativistic and non-relativistic formulations. For the gaussian packet at rest we find curiously

$$\alpha_R - \alpha_S = -\frac{3}{2}mc^2 \tag{8.106}$$

This seems due to the sign difference in the relativistic Schrödinger equation (8.71) as compared the ordinary equation (8.70). Now if we change the sign of η^0, thus changing the first component of the proper velocity of the packet from γc to $-\gamma c$, we find that the relativistic potential for the gaussian is unchanged. The same is not true for the non-relativistic potential, equation (8.68) becomes

$$V_S = -mc^2 + \frac{1}{m}(\hbar\Delta k_1)^2\left(2\left(\Delta k_1 x^1\right)^2 - 1\right) \tag{8.107}$$

That is we have changed the sign of b in equation (8.62) as noted in Section 7 after equation (7.37). In this case we have

$$\alpha_R - \alpha_S = \frac{1}{2}mc^2 \tag{8.108}$$

The better explanation for the difference is that we solved the time independent equation in two ways

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_R\psi = -\frac{1}{2}mc^2\psi \tag{8.109}$$

and

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_S\psi = mc^2\psi \tag{8.110}$$

We still need to decide if the separation constant , α, is real in the relativistic formulation, and for that we need a suitable definition of the probability density current. For the moment we shall assume that it is real, and concentrate on the time dependent equation which we can rewrite as

$$\frac{d^2\chi}{dt^2} + i\frac{2m}{\hbar}\frac{d\chi}{dt} + \frac{2m\alpha}{\hbar^2}\chi = 0 \tag{8.111}$$

which factors into

$$\left(\frac{d\chi}{dt} - r_1\right)\left(\frac{d\chi}{dt} - r_2\right)\chi = 0 \tag{8.112}$$

with the roots given by

$$r_1, r_2 = \frac{-i\frac{2m}{\hbar} \pm \sqrt{-\frac{4m^2}{\hbar^2} - 4\frac{2m\alpha}{\hbar^2}}}{2} \tag{8.113}$$

where we find for α real and non-negative

$$r_1, r_2 = -\frac{i}{\hbar}\left(m \mp \sqrt{m^2 + 2m\alpha}\right) \tag{8.114}$$

or with

$$M(\alpha) = +\sqrt{m^2 + 2m\alpha} \tag{8.115}$$

Then

$$r_1, r_2 = -i\left(\frac{m \mp M(\alpha)}{\hbar}\right) \tag{8.116}$$

If we write $r_k = -is_k$, then the factors become

$$\left(\frac{d\chi}{dt} + is_1\right)\left(\frac{d\chi}{dt} + is_2\right)\chi = 0 \tag{8.117}$$

and we find the solution

$$\chi(t) = C_1 \exp(-is_1 t) + C_2 \exp(-is_2 t) \tag{8.118}$$

which can be re-expressed as

$$\chi(t) = \exp\left(-i\frac{m}{\hbar}t\right)\left[C_1 \exp\left(i\frac{M(\alpha)}{\hbar}t\right) + C_2 \exp\left(-i\frac{M(\alpha)}{\hbar}t\right)\right] \tag{8.119}$$

The natural assumption to make at this point is

$$|\chi(t)|^2 = 1 \tag{8.120}$$

which will be satisfied if either constant has unit modulus and the other is zero. Further we want $M(\alpha)$ to be real so that

$$\begin{aligned} m^2 + 2m\alpha &\geq 0 && (8.121)\\ 2m\alpha &\geq -m^2 && (8.122)\\ \alpha &\geq -\frac{m}{2} && (8.123)\end{aligned}$$

Let us choose $C_1 = 0$ and $C_2 = 1$, so that

$$\chi(x^0) = \exp\left(-i\frac{m + M(\alpha)}{\hbar}x^0\right) \tag{8.124}$$

in which case the rest + kinetic energy operator $-i\hbar\partial^0 = i\hbar\frac{\partial}{\partial x^0}$ has the follwing action on χ

$$i\hbar\frac{\partial}{\partial x^0}\exp\left(-i\frac{m + M(\alpha)}{\hbar}x^0\right) = [m + M(\alpha)]\exp\left(-i\frac{m + M(\alpha)}{\hbar}x^0\right) \tag{8.125}$$

It follows that

$$\begin{aligned} \langle -i\hbar\partial^0\rangle &= \left\langle \Psi | i\hbar\frac{\partial}{\partial x^0}\Psi\right\rangle \\ &= \langle\Psi|\,[m + M(\alpha)]\,\Psi\rangle \\ &= [m + M(\alpha)]\,\langle\Psi|\Psi\rangle \\ &= m + M(\alpha) && (8.126)\end{aligned}$$

Converting back to $c \neq 1$ we have

$$\left\langle i\hbar\frac{\partial}{\partial ct}\right\rangle = [m + M(\alpha)]\,c \tag{8.127}$$

or

$$\left\langle i\hbar\frac{\partial}{\partial t}\right\rangle = [m + M(\alpha)]\,c^2 \tag{8.128}$$

Example 5. Let us determine the difference $V_R - \alpha_R$ for the gaussian packet at rest given by equation (8.57)

$$V_R = -\frac{1}{2}mc^2 + \frac{1}{m}(\hbar\Delta k_1)^2\left(2\left(\Delta k_1 x^1\right)^2 - 1\right) \tag{8.129}$$

We have when $c \neq 1$

$$cM(\alpha) = +\sqrt{m^2c^2 + m\alpha} \tag{8.130}$$

and

$$M\left(-\frac{1}{2}mc^2\right) = 0 \tag{8.131}$$

Thus

$$\begin{aligned} V_R - \alpha_R &= V_R - \left(-\frac{1}{2}mc^2\right) \\ &= \frac{1}{m}\left(\hbar\Delta k_1\right)^2\left(2\left(\Delta k_1 x^1\right)^2 - 1\right) \end{aligned} \tag{8.132}$$

Further we see that the expected rest and kinetic energy for the gaussian at rest is

$$\begin{aligned} \left\langle i\hbar\frac{\partial}{\partial t}\right\rangle &= \left[m + M\left(-\frac{1}{2}mc^2\right)\right]c^2 \\ &= mc^2 \end{aligned} \tag{8.133}$$

Example 6. Let us express the expected rest and kinetic energy in proportion to the rest energy

$$\frac{\left[m + M\left(\alpha\right)\right]c^2}{mc^2} = 1 + \frac{M\left(\alpha\right)}{m} \tag{8.134}$$

Now

$$\begin{aligned} \frac{M\left(\alpha\right)}{m} &= \frac{1}{mc}\sqrt{m^2c^2 + 2m\alpha} \\ &= \sqrt{1 + \frac{2\alpha}{mc^2}} \end{aligned} \tag{8.135}$$

Therefore

$$\frac{\left[m + M\left(\alpha\right)\right]}{m} = 1 + \sqrt{1 + \frac{2\alpha}{mc^2}} \tag{8.136}$$

Example 7. Let us write the *total energy* α as the sum rest, kinetic, and potential energies and calculate the expectations

$$\alpha = R + T + V \tag{8.137}$$

$$= mc^2 + M\left(\alpha\right)c^2 + V \tag{8.138}$$

Then

$$\left\langle\alpha\right\rangle = \left\langle mc^2\right\rangle + \left\langle M\left(\alpha\right)c^2\right\rangle + \left\langle V\right\rangle \tag{8.139}$$

so that

$$\alpha = mc^2 + M\left(\alpha\right)c^2 + \left\langle V\right\rangle \tag{8.140}$$

For the gaussian packet and $\alpha = -\frac{1}{2}mc^2$ we have

$$\begin{aligned} -\frac{1}{2}mc^2 &= mc^2 + 0\cdot c^2 + \left\langle V\right\rangle \\ &= mc^2 + \left\langle -\frac{1}{2}mc^2 + \frac{1}{m}\left(\hbar\Delta k_1\right)^2\left(2L^2 - 1\right)\right\rangle \end{aligned}$$

where

$$L = \Delta k_1 x^1 + \Delta k_0 x^0 \tag{8.141}$$

so that

$$
\begin{aligned}
-\frac{3}{2}mc^2 &= \left\langle -\frac{1}{2}mc^2 + \frac{1}{m}\left(\hbar\Delta k_1\right)^2\left(2L^2-1\right)\right\rangle \\
&= -\frac{1}{2}mc^2 + \frac{1}{m}\left(\hbar\Delta k_1\right)^2\left(2\left\langle L^2\right\rangle-1\right) \qquad (8.142)
\end{aligned}
$$

Rearranging

$$
-mc^2 = \frac{1}{m}\left(\hbar\Delta k_1\right)^2\left(2\left\langle L^2\right\rangle-1\right) \qquad (8.143)
$$

When the gaussian is at rest

$$
L = \Delta k_1 x^1 \qquad (8.144)
$$

therefore

$$
\begin{aligned}
-m^2c^2 &= \left(\hbar\Delta k_1\right)^2\left[2\left(\Delta k_1\right)^2\left\langle\left(x^1\right)^2\right\rangle-1\right] \qquad (8.145)\\
&= \left(\hbar\Delta k_1\right)^2\left[2\left(\Delta k_1\right)^2 \operatorname{var}\left(x^1\right)-1\right] \qquad (8.146)\\
& \left(\hbar\Delta k_1\right)^2\left[2\left(\Delta k_1\right)^2 \frac{1}{4\left(\Delta k_1\right)^2}-1\right] \qquad (8.147)\\
&= \left(\hbar\Delta k_1\right)^2\left(\frac{1}{2}-1\right) \qquad (8.148)\\
&= -\frac{1}{2}\left(\hbar\Delta k_1\right)^2 \qquad (8.149)
\end{aligned}
$$

or

$$
\left(\hbar\Delta k_1\right)^2 = 2m^2c^2 \qquad (8.150)
$$

We can think of nothing better to write but

$$
\begin{aligned}
\left(\frac{\Delta E}{c}\right)^2 - \left(\Delta p\right)^2 &= \hbar^2\left[\left(\frac{\Delta\omega}{c}\right)^2 - \left(\Delta k\right)^2\right] \qquad (8.151)\\
&= \left(\Delta m\right)^2 c^2 \qquad (8.152)
\end{aligned}
$$

A rest mass variance? Well if so, then with $\gamma^2\left(\Delta k\right)^2 = \left(\Delta k_1\right)^2$ where $\left(\Delta k_1\right)^2 = \operatorname{var}\left(k_{\text{rest}}\right)$ we have

$$
\begin{aligned}
\left(\Delta m\right)^2 c^2 &= \hbar^2\left[\left(\frac{\Delta\omega}{c}\right)^2 - \left(\Delta k\right)^2\right]\\
&= \hbar^2\left[\left(\frac{\Delta\omega}{c}\right)^2 - \gamma^{-2}\left(\Delta k_1\right)^2\right]\\
&= \hbar^2\left(\frac{\Delta\omega}{c}\right)^2 - \gamma^{-2}\hbar^2\left(\Delta k_1\right)^2\\
&= \hbar^2\left(\frac{\Delta\omega}{c}\right)^2 - \gamma^{-2}2m^2c^2 \qquad (8.153)
\end{aligned}
$$

Hence

$$\hbar^2 \left(\frac{\Delta\omega}{c}\right)^2 = \left[(\Delta m)^2 + 2\gamma^{-2}m^2\right] c^2 \tag{8.154}$$

and in the rest frame

$$(\Delta m)^2 = -2m^2 \tag{8.155}$$

Note that although L is a Lorentz invariant expression, the expectation$\left\langle L^2 \right\rangle$ is not

$$\begin{aligned}
\left\langle L^2 \right\rangle &= \left\langle \left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2 \right\rangle \\
&= \left\langle \left(\Delta k_1 x^1\right)^2 + 2\Delta k_1 x^1 \Delta k_0 x^0 + \left(\Delta k_0 x^0\right)^2 \right\rangle \\
&= \left(\Delta k_1\right)^2 \left\langle \left(x^1\right)\right\rangle^2 + 2\Delta k_1 \left\langle x^1 \right\rangle \Delta k_0 x^0 + \left(\Delta k_0 x^0\right)^2 \qquad (8.156)
\end{aligned}$$

which cannot be factored, and $\left\langle \left(x^1\right)\right\rangle^2 \neq \mathrm{var}\left(x^1\right)$, when the gaussian is in motion.

Example 8. Let us determine the zeroth component of the relativistic probability density current for the wave function

$$\Psi = \exp\left[-i\frac{(m + M(\alpha))\,c}{\hbar} x^0\right] \psi\left(\mathrm{r}\right) \tag{8.157}$$

The zeroth component of the probability current was defined in equation (8.95) as

$$\begin{aligned}
S^0_{\text{rest}} &= -c\Psi^*\Psi - \frac{\hbar}{m}\Im\left(\Psi\partial^0\Psi^*\right) && (8.158)\\
&= -c\Psi^*\Psi - \frac{-i\hbar}{2m}\left(\Psi\partial^0\Psi^* - \Psi^*\partial^0\Psi\right) && (8.159)\\
&= -c\Psi^*\Psi - \frac{i\hbar}{2m}\left(\Psi\partial_0\Psi^* - \Psi^*\partial_0\Psi\right) && (8.160)
\end{aligned}$$

We have

$$\Psi^*\partial_0\Psi = -i\frac{(m + M(\alpha))\,c}{\hbar}\Psi^*\Psi \tag{8.161}$$

so that (we show in the next section that the separation constant α is real)

$$\Psi\partial_0\Psi^* - \Psi^*\partial_0\Psi = 2i\frac{(m + M(\alpha))\,c}{\hbar}\Psi^*\Psi \tag{8.162}$$

plugging we have

$$\begin{aligned}
S^0_{\text{rest}} &= -c\Psi^*\Psi - \frac{i\hbar}{2m}\left[2i\frac{(m + M(\alpha))\,c}{\hbar}\Psi^*\Psi\right] \\
&= -c\Psi^*\Psi + \frac{1}{m}\left(m + M(\alpha)\right)c\Psi^*\Psi \\
&= \frac{M(\alpha)}{m}c\Psi^*\Psi \qquad (8.163)
\end{aligned}$$

In the case of the one dimensional gaussian packet where we have $M(\alpha) = 0$ for $\alpha = -\frac{1}{2}mc^2$, then the zeroth component is zero

$$S^0_{\text{rest}} = 0 \tag{8.164}$$

But we note that $\Psi^*\Psi$ is time independent for the gaussian at rest so

$$\partial_0 \Psi^* \Psi = 0 \tag{8.165}$$

so the classical continuity equation reduces to

$$\nabla \cdot \mathrm{S} = 0$$

The spatial components of the relativistic probability current in the rest frame of the gaussian, we note, are precisely the components of the classical quantum probabity density current as given in equation (8.96), so we have no inconsistancy in this case.

Remark 9. If we had chosen $C_1 = 1$ and $C_2 = 0$ we would end up with the expression

$$\left\langle i\hbar \frac{\partial}{\partial t} \right\rangle = [m - M(\alpha)]\, c^2$$

which for large enough α becomes negative, and this doesn't seem suitable for the combined rest+kinetic energy. Now if we chose neither alternative, that is if neither C_1 or C_2 were zero we could express these constants in terms of the initial conditions, $\chi(0)$ and $\dot{\chi}(0)$

$$\begin{bmatrix} \chi(0) \\ \dot{\chi}(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i\frac{m-M(\alpha)}{\hbar} & -i\frac{m+M(\alpha)}{\hbar} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{8.166}$$

with determinant

$$\det = -\frac{2iM(\alpha)}{\hbar} \tag{8.167}$$

so that

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{i\hbar}{2M(\alpha)} \begin{bmatrix} -i\frac{m+M(\alpha)}{\hbar} & -1 \\ i\frac{m-M(\alpha)}{\hbar} & 1 \end{bmatrix} \begin{bmatrix} \chi(0) \\ \dot{\chi}(0) \end{bmatrix} \tag{8.168}$$

or

$$\begin{aligned} C_1 &= -\frac{i\hbar}{2M(\alpha)} \left[i\frac{m+M(\alpha)}{\hbar}\chi(0) + \dot{\chi}(0) \right] \\ &= \frac{1}{2M(\alpha)} \left[(m + M(\alpha))\,\chi(0) - i\hbar\dot{\chi}(0) \right] \end{aligned} \tag{8.169}$$

and

$$\begin{aligned} C_2 &= \frac{i\hbar}{2M(\alpha)} \left[i\frac{m-M(\alpha)}{\hbar}\chi(0) + \dot{\chi}(0) \right] \\ &= \frac{-1}{2M(\alpha)} \left[(m - M(\alpha))\,\chi(0) - i\hbar\dot{\chi}(0) \right] \end{aligned} \tag{8.170}$$

then we have

$$\begin{aligned} |\chi(t)|^2 &= \left[\overline{C_1}\exp(is_1) + \overline{C_2}\exp(is_2)\right] \left[C_1 \exp(-is_1 t) + C_2 \exp(-is_2 t)\right] \\ &= |C_1|^2 + |C_2|^2 + 2\Re\left\{C_1\overline{C_2}\exp\left[-i(s_1 - s_2)\,t\right]\right\} \\ &= |C_1|^2 + |C_2|^2 + 2\Re\left\{C_1\overline{C_2}\exp\left[2iM(\alpha)\,t/\hbar\right]\right\} \end{aligned} \tag{8.171}$$

recalling that

$$s_1 = \frac{1}{\hbar}\left[m - M(\alpha)\right] \tag{8.172}$$

$$s_2 = \frac{1}{\hbar}\left[m + M(\alpha)\right] \tag{8.173}$$

Our requirement that $|\chi(t)|^2 = 1$ means we must have

$$\Re\left\{C_1\overline{C_2}\exp\left[2iM(\alpha)t/\hbar\right]\right\} = 0 \tag{8.174}$$

If we write

$$C_1\overline{C_2} = R\exp(2i\theta) \tag{8.175}$$

Then

$$\begin{aligned}\overline{C_2}C_1\exp\left(2i\frac{M(\alpha)}{\hbar}t\right) &= R\exp(2i\theta)\exp\left(2i\frac{M(\alpha)}{\hbar}t\right)\\ &= R\exp\left(2i\left(\frac{M(\alpha)}{\hbar}+\theta\right)t\right)\end{aligned} \tag{8.176}$$

hence we want

$$\sin\left[2\left(\frac{M(\alpha)}{\hbar}+\theta\right)t\right] = 0 \tag{8.177}$$

which for continuous time will only happen if

$$\frac{M(\alpha)}{\hbar}+\theta = 0 \tag{8.178}$$

So we need to determine θ which is some function of α, thus

$$\begin{aligned}\overline{C_2}C_1 &= \frac{1}{2M(\alpha)}\left[(m+M(\alpha))\chi(0) - i\hbar\dot{\chi}(0)\right]\frac{-1}{2M(\alpha)}\left[(m-M(\alpha))\overline{\chi(0)} + i\hbar\overline{\dot{\chi}(0)}\right]\\ &= \frac{-1}{4M^2(\alpha)}\left\{\left(m^2 - M^2(\alpha)\right)|\chi(0)|^2 + \hbar^2|\dot{\chi}(0)|^2\right.\\ &\quad\left.+i\hbar\left[(m+M(\alpha))\chi(0)\overline{\dot{\chi}(0)} - (m-M(\alpha))\overline{\chi(0)}\dot{\chi}(0)\right]\right\}\end{aligned} \tag{8.179}$$

Now

$$\begin{aligned}&(m+M(\alpha))\chi(0)\overline{\dot{\chi}(0)} - (m-M(\alpha))\overline{\chi(0)}\dot{\chi}(0)\\ &= m\left(\chi(0)\overline{\dot{\chi}(0)} - \overline{\chi(0)}\dot{\chi}(0)\right) + M(\alpha)\left(\chi(0)\overline{\dot{\chi}(0)} + \overline{\chi(0)}\dot{\chi}(0)\right)\\ &= 2im\Im\left(\chi(0)\overline{\dot{\chi}(0)}\right) + 2M(\alpha)\Re\left(\chi(0)\overline{\dot{\chi}(0)}\right)\end{aligned} \tag{8.180}$$

Therefore

$$\begin{aligned}C_1\overline{C_2} &= \frac{-1}{4M^2(\alpha)}\left\{\left(m^2 - M^2(\alpha)\right)|\chi(0)|^2 + \hbar^2|\dot{\chi}(0)|^2\right.\\ &\quad\left.-2\hbar m\Im\left(\chi(0)\overline{\dot{\chi}(0)}\right) + 2i\hbar M(\alpha)\Re\left(\chi(0)\overline{\dot{\chi}(0)}\right)\right\}\end{aligned} \tag{8.181}$$

so that with $|\chi(0)|^2 = 1$

$$2\theta = \arctan\left\{\frac{2\hbar M(\alpha)\Re\left(\chi(0)\overline{\dot{\chi}(0)}\right)}{(m^2 - M^2(\alpha)) + \hbar^2|\dot{\chi}(0)|^2 - 2\hbar m\Im\left(\chi(0)\overline{\dot{\chi}(0)}\right)}\right\} \tag{8.182}$$

This leads to quite the transcendental equation

$$M(\alpha) = -\frac{\hbar}{2}\arctan\left\{\frac{2\hbar M(\alpha)\Re\left(\chi(0)\overline{\dot{\chi}(0)}\right)}{(m^2 - M^2(\alpha))|\chi(0)|^2 + \hbar^2|\dot{\chi}(0)|^2 - 2\hbar m\Im\left(\chi(0)\overline{\dot{\chi}(0)}\right)}\right\} \tag{8.183}$$

where we could apply the identity

$$\arctan(x) + \arctan\left(x^{-1}\right) = \frac{\pi}{2} \tag{8.184}$$

It would appear a tall order to find α which satisfy the above equation, and be eigenvalues of the time-independent Schrödinger equation (), with the exception of

$$M(\alpha) = 0 \tag{8.185}$$

when $\alpha = -\frac{1}{2}m$ which always satifies. If either of C_1, C_2 are zero then equation (8.178) no longer is valid, since equation (8.174) is automatically satisfied. We may still need to consider equation (8.183) if the time-independent Schrödinger equation allows for a continuum of eigenvalues. We note in finding the potential for the gaussian packet $M(\alpha)$ did not appear as would be the case if $\alpha = -\frac{1}{2}m$.

8.3. **The Separation Constant.** To determine whether the separation constant, α is real for the time independent potential we consider the solution to the relativistic equation which takes the form

$$\Psi\left(\mathrm{r}, x^0\right) = \exp\left(ia_0x^0\right)\psi(\mathrm{r}) \tag{8.186}$$

where

$$a_0 = -\frac{1}{\hbar}[m + M(\alpha)]c \tag{8.187}$$

and

$$M(\alpha) = \frac{1}{c}\sqrt{m^2c^2 + 2m\alpha} \tag{8.188}$$

Consider our new definition of the pronbability density current in the rest frame

$$S^0_{\text{rest}} = -c\Psi^*\Psi - \frac{\hbar}{m}\Im\left(\Psi\partial^0\Psi^*\right) \tag{8.189}$$

$$S^i_{\text{rest}} = -\frac{\hbar}{m}\Im\left(\Psi\partial^i\Psi^*\right) \tag{8.190}$$

where the second line above is just the ordinary probability density current found in the standard theory. We also have the continuity equation

$$\partial_0S^0 + \partial_iS^i = 0 \tag{8.191}$$

Let us find the first term of the continuity equation by first finding

$$\partial_0\Psi^*\Psi = \Psi\partial_0\Psi^* + \Psi^*\partial_0\Psi \tag{8.192}$$

We have

$$\partial_0 \Psi\left(\mathrm{r}, x^0\right) = \partial_0 \exp\left(i a_0 x^0\right) \psi\left(\mathrm{r}\right) \tag{8.193}$$

$$= i a_0 \Psi\left(\mathrm{r}, x^0\right) \tag{8.194}$$

so that

$$\partial_0 \Psi^* \Psi = i\left(a_0 - \overline{a_0}\right) \Psi^* \Psi \tag{8.195}$$

Now let us find the second term of S^0

$$\frac{\hbar}{m} \Im\left(\Psi \partial^0 \Psi^*\right) = -\frac{i\hbar}{2m}\left[\Psi \partial^0 \Psi^* - \Psi^* \partial^0 \Psi\right] \tag{8.196}$$

$$= \frac{i\hbar}{2m}\left[\Psi \partial_0 \Psi^* - \Psi^* \partial_0 \Psi\right] \tag{8.197}$$

$$= \frac{i\hbar}{2m}\left[\Psi\left(-i\overline{a_0}\Psi^*\right) - \Psi^*\left(i a_0 \Psi\right)\right] \tag{8.198}$$

$$= \frac{i\hbar}{2m}\left(-i\right)\left(a_0 + \overline{a_0}\right)\Psi^*\Psi \tag{8.199}$$

$$= \frac{\hbar}{2m}\left(a_0 + \overline{a_0}\right)\Psi^*\Psi \tag{8.200}$$

Then

$$\partial_0 \frac{\hbar}{m} \Im\left(\Psi \partial^0 \Psi^*\right) = \partial_0 \frac{\hbar}{2m}\left(a_0 + \overline{a_0}\right)\Psi^*\Psi \tag{8.201}$$

$$= \frac{i\hbar}{2m}\left(a_0 + \overline{a_0}\right)\left(a_0 - \overline{a_0}\right)\Psi^*\Psi \tag{8.202}$$

Thus

$$\partial_0 S^0 = -ci\left(a_0 - \overline{a_0}\right)\Psi^*\Psi - \frac{i\hbar}{2m}\left(a_0 + \overline{a_0}\right)\left(a_0 - \overline{a_0}\right)\Psi^*\Psi \tag{8.203}$$

$$= - i\left(a_0 - \overline{a_0}\right)\left[c + \frac{\hbar}{2m}\left(a_0 + \overline{a_0}\right)\right]\Psi^*\Psi \tag{8.204}$$

Therfore from the continuity equation we have

$$i\left(a_0 - \overline{a_0}\right)\left[c + \frac{\hbar}{2m}\left(a_0 + \overline{a_0}\right)\right]\Psi^*\Psi = \nabla \cdot \mathrm{S} \tag{8.205}$$

Integrating this equation over all space we find with volume element $d\lambda$

$$\left(a_0 - \overline{a_0}\right)\left[c + \frac{\hbar}{2m}\left(a_0 + \overline{a_0}\right)\right] = \int \nabla \cdot \mathrm{S} d\lambda = \int \mathrm{S} \cdot d\sigma \tag{8.206}$$

where the surface integral vanishes

$$\int \mathrm{S} \cdot d\sigma \to 0 \tag{8.207}$$

So that

$$\left(a_0 - \overline{a_0}\right)\left[c + \frac{\hbar}{2m}\left(a_0 + \overline{a_0}\right)\right] = 0 \tag{8.208}$$

which means that one of the two conditions must hold

$$a_0 - \overline{a_0} = 0 \tag{8.209}$$

$$c + \frac{\hbar}{2m}\left(a_0 + \overline{a_0}\right) = 0 \tag{8.210}$$

Case 1. Let us consider that latter first

$$\Re\left(a_0\right) = -\frac{mc}{\hbar} \tag{8.211}$$

but

$$a_0 = -\frac{1}{\hbar}\left[m + M\left(\alpha\right)\right]c \tag{8.212}$$

which allows

$$M\left(\alpha\right) = \frac{1}{c}\sqrt{m^2c^2 + 2m\alpha} \tag{8.213}$$

to be pure imaginary when α is real and less than $-\frac{1}{2}mc^2$, but we are only interested when α is complex

$$\alpha = a + ib \tag{8.214}$$

then

$$M\left(\alpha\right) = \frac{1}{c}\sqrt{m^2c^2 + 2m\left(a+ib\right)} \tag{8.215}$$

$$= \frac{1}{c}\sqrt{\left(m^2c^2 + 2ma\right) + i2mb} \tag{8.216}$$

$$= \frac{1}{c}\sqrt{r}\exp\left(i\theta/2\right) \tag{8.217}$$

$$= \tfrac{1}{c}\sqrt{r}\left[\cos\left(\theta/2\right) + i\sin\left(\theta/2\right)\right] \tag{8.218}$$

Hence we must have

$$\cos\left(\theta/2\right) = 0 \tag{8.219}$$

which only occurs for $\theta = n\pi$, but we have

$$\theta = \arctan\left[\frac{2mb}{\left(m^2c^2 + 2ma\right)}\right] \tag{8.220}$$

and

$$-\frac{\pi}{2} < \arctan\left[\frac{2mb}{\left(m^2c^2 + 2ma\right)}\right] < \frac{\pi}{2} \tag{8.221}$$

these inequalities are strict so that in this case $\theta \neq n\pi$, except when $b = 0$, hence α is real.

Case 2. Suppose now that $a_0 - \overline{a_0} = 0$, hence a_0 is real, then does there exist complex, α, such that $M\left(\alpha\right)$ is real? In this case from (8.218)

$$\sin\left(\theta/2\right) = 0 \tag{8.222}$$

Which only occurs for $\theta = 2n\pi$ and again

$$n\pi = \arctan\left[\frac{2mb}{\left(m^2c^2 + 2ma\right)}\right]$$

only when $b = 0$, hence α must be real in this case also.

We conclude that the separation constant for the time independent relativistic Schrödinger equation must be real, which we now rechristen as the total energy E.

Example 10. We see two distinct contributions to the kinetic energy, if in a frame where the potential energy is time independent we are given

$$\chi\left(t\right)=\exp\left(i\alpha_{0}ct\right) \tag{8.223}$$

where

$$\begin{aligned}\alpha_{0} &= -\frac{mc}{\hbar}\left(1+\sqrt{1+2\xi}\right) &\text{(8.224)}\\ \alpha_{1} &= 0 &\text{(8.225)}\end{aligned}$$

Measuring the total energy in units, ξ. of rest energy, we see the first contribution is

$$\left\langle T\right\rangle = mc^{2}\sqrt{1+2\xi} \tag{8.226}$$

The second cotribution comes when we transform to a frame that sees the packet in motion with velocity, v, where now

$$\hbar\alpha_{0}^{\prime}=\gamma\hbar\alpha_{0} \tag{8.227}$$

so that the combined rest and kinetic energy becomes

$$\gamma mc^{2}+\gamma mc^{2}\sqrt{1+2\xi}=mc^{2}+\left\langle T\right\rangle \tag{8.228}$$

Thus the expected value of the kinetic energy is given by

$$\left\langle T\right\rangle =\left(\gamma-1\right)mc^{2}+\gamma mc^{2}\sqrt{1+2\xi} \tag{8.229}$$

where $\gamma-1>0$ even when $\xi=-\frac{1}{2}$, the zero of the second term in the expression for the expected value.

8.4. **Determining the Proper Velocity of a Potential.** We do not mean to settle the question of what constitutes a stationary potential, or what proper velocity to attribute to a potential in general, but we will examine this question with some restrictions. First let us consider the case of one spatial dimension and suppose we have a potential, $V\left(x,t\right)$, which is a function of position and time and suppose

$$\begin{aligned}f\left(x\right) &= V\left(x,t_{1}\right) &\text{(8.230)}\\ g\left(x\right) &= V\left(x,t_{2}\right) &\text{(8.231)}\end{aligned}$$

Now let us make the significant assumption that the function, g, is simply the function, f , shifted to the right a distance y, thus

$$g\left(x\right)=f\left(x-y\right) \tag{8.232}$$

then the velocity we attribute to the potential is simply

$$v=\frac{y}{t_{2}-t_{1}} \tag{8.233}$$

in other words we are equating

$$V\left(x-\Delta x,t\right)=V\left(x,t+\Delta t\right) \tag{8.234}$$

Now

$$\frac{\partial V}{\partial t}\Delta t = V(x,t+\Delta t) - V(x,t) \tag{8.235}$$

$$-\frac{\partial V}{\partial x}\Delta x = V(x-\Delta x,t) - V(x,t) \tag{8.236}$$

so that

$$\frac{\partial V}{\partial t}\Delta t = -\frac{\partial V}{\partial x}\Delta x \tag{8.237}$$

or

$$\frac{\Delta x}{\Delta t} = -\frac{\frac{\partial V}{\partial t}}{\frac{\partial V}{\partial x}} \tag{8.238}$$

For a function of three spatial dinesions and time we set

$$V(x-\Delta x, y-\Delta y, z-\Delta z, t) = V(x, t+\Delta t) \tag{8.239}$$

or

$$\frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y + \frac{\partial V}{\partial z}\Delta z = -\frac{\partial V}{\partial t}\Delta t \tag{8.240}$$

which becomes

$$\beta^i \partial_i V = -\partial_0 V \tag{8.241}$$

Example 11. For the relativistic Schrödinger equation

$$i\hbar\eta^\sigma \partial_\sigma \Psi = -\frac{\hbar^2}{2m}\partial^\rho \partial_\rho \Psi + V\Psi \tag{8.242}$$

let us determine the potential for the unnormalized wave function

$$\Psi = fg = \exp\left(i\alpha_\mu x^\mu\right)\exp\left(-k^{i\cdot}_{\cdot\mu}k^{i\cdot}_{\cdot\nu}x^\mu x^\nu\right) \tag{8.243}$$

where

$$\alpha^\mu = \frac{mc}{\hbar}\gamma\beta^\mu \tag{8.244}$$

$$k^{i\cdot}_{\cdot j} = b^{i\cdot}_{\cdot j} \tag{8.245}$$

$$k^{i\cdot}_{\cdot 0} = -b^{i\cdot}_{\cdot j}\beta^j \tag{8.246}$$

$$x^i = a^{i\cdot}_{\cdot j}z^j + \beta^i x^0 \tag{8.247}$$

$$\left\langle z^j \right\rangle = 0 \tag{8.248}$$

$$\left\langle z^i z^j \right\rangle = \delta^{ij} \tag{8.249}$$

$$2b^{i\cdot}_{\cdot j}a^{j\cdot}_{\cdot k} = \delta^i_k \tag{8.250}$$

Then

$$\partial_\sigma f = i\alpha_\sigma f \tag{8.251}$$

$$\partial_\sigma g = -2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^\mu \tag{8.252}$$

$$\partial_\sigma \Psi = \left(i\alpha_\sigma - 2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^\mu\right)\Psi \tag{8.253}$$

where we note that

$$\begin{aligned}
\left\langle k^{i\cdot}_{\cdot\mu}x^{\mu}\right\rangle &= \left\langle k^{i\cdot}_{\cdot 0}x^{0}+k^{i\cdot}_{\cdot i}x^{j}\right\rangle \\
&= \left\langle -b^{i\cdot}_{\cdot j}\beta^{j}x^{0}+b^{i\cdot}_{\cdot j}x^{j}\right\rangle \\
&= \left\langle -b^{i\cdot}_{\cdot j}\beta^{j}x^{0}+b^{i\cdot}_{\cdot j}a^{j\cdot}_{\cdot k}z^{k}+b^{i\cdot}_{\cdot j}\beta^{j}x^{0}\right\rangle \\
&= \left\langle b^{i\cdot}_{\cdot j}a^{j\cdot}_{\cdot k}z^{k}\right\rangle \\
&= b^{i\cdot}_{\cdot j}a^{j\cdot}_{\cdot k}\left\langle z^{k}\right\rangle \\
&= 0 \qquad (8.254)
\end{aligned}$$

Now let us determine

$$\begin{aligned}
\partial^{\sigma}\partial_{\sigma}\Psi &= \mu^{\sigma\rho}\partial_{\rho}\partial_{\sigma}\Psi \\
&= \mu^{\sigma\rho}\partial_{\rho}\left(i\alpha_{\sigma}-2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\right)\Psi
\end{aligned}$$

we have

$$\begin{aligned}
\partial_{\rho}\left(i\alpha_{\sigma}-2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\right) &= -2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}\delta^{\mu}_{\rho} \\
&= -2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\rho} \qquad (8.255)
\end{aligned}$$

so that

$$\partial^{\sigma}\partial_{\sigma}\Psi = \mu^{\sigma\rho}\left[\left(i\alpha_{\sigma}-2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\right)\left(i\alpha_{\rho}-2k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu}\right)-2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\rho}\right]\Psi \qquad (8.256)$$

The product of the terms in parentheses becomes

$$\begin{aligned}
&\mu^{\sigma\rho}\left(i\alpha_{\sigma}-2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\right)\left(i\alpha_{\rho}-2k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu}\right) \\
=\ &\mu^{\sigma\rho}\left(-\alpha_{\sigma}\alpha_{\rho}-2i\alpha_{\sigma}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu}-2i\alpha_{\rho}k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}+4k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu}\right) \\
=\ &-\alpha_{\sigma}\alpha^{\sigma}-2i\alpha^{\rho}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu}-2i\alpha^{\sigma}k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}+4k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\mu^{\sigma\rho}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu} \\
=\ &-\alpha_{\sigma}\alpha^{\sigma}-4i\alpha^{\sigma}k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}+4k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\mu^{\sigma\rho}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu} \qquad (8.257)
\end{aligned}$$

Thus the term

$$\begin{aligned}
-\frac{\hbar^{2}}{2m}\partial^{\rho}\partial_{\rho}\Psi &= -\frac{\hbar^{2}}{2m}\left(-\alpha_{\sigma}\alpha^{\sigma}-4i\alpha^{\sigma}k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}+4k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\mu^{\sigma\rho}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu}-2k^{i\cdot}_{\cdot\sigma}\mu^{\sigma\rho}k^{i\cdot}_{\cdot\rho}\right)\Psi \\
&= \frac{\hbar^{2}}{2m}\left(\alpha_{\sigma}\alpha^{\sigma}+4i\alpha^{\sigma}k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}-4k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\mu^{\sigma\rho}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu}+2k^{i\cdot}_{\cdot\sigma}\mu^{\sigma\rho}k^{i\cdot}_{\cdot\rho}\right)\Psi \qquad (8.258)
\end{aligned}$$

Now let us determine the left hand side of equation ()

$$\begin{aligned}
i\hbar\eta^{\tau}\partial_{\tau}\Psi &= i\hbar\eta^{\tau}\left(i\alpha_{\tau}-2k^{k\cdot}_{\cdot\tau}k^{k\cdot}_{\cdot\beta}x^{\beta}\right)\Psi \\
&= -\left(\hbar\eta^{\tau}\alpha_{\tau}+2i\hbar\eta^{\tau}k^{k\cdot}_{\cdot\tau}k^{k\cdot}_{\cdot\beta}x^{\beta}\right)\Psi \qquad (8.259)
\end{aligned}$$

Now to make the imaginary terms go away we need

$$2i\hbar\eta^{\tau}k^{k\cdot}_{\cdot\tau}k^{k\cdot}_{\cdot\beta}x^{\beta}+\frac{\hbar^{2}}{2m}4i\alpha^{\sigma}k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}=0 \qquad (8.260)$$

which will be the case if

$$\eta^{\tau}=-\frac{\hbar}{m}\alpha^{\tau} \qquad (8.261)$$

therefore

$$\eta^{\tau} = -\gamma c \beta^{\tau} \tag{8.262}$$

So we can now write

$$-\hbar \eta^{\tau} \alpha_{\tau} \Psi = \frac{\hbar^2}{2m}\left(\alpha_{\sigma}\alpha^{\sigma} - 4k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}x^{\mu}\mu^{\sigma\rho}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu} + 2k^{i\cdot}_{\cdot\sigma}\mu^{\sigma\rho}k^{i\cdot}_{\cdot\rho}\right)\Psi + V\Psi \tag{8.263}$$

we have on the left

$$-\hbar \eta^{\tau}\alpha_{\tau} = -mc^2 \tag{8.264}$$

and on the right

$$\alpha_{\sigma}\alpha^{\sigma} = -\frac{m^2c^2}{\hbar^2} \tag{8.265}$$

so that

$$\frac{\hbar^2}{2m}\alpha_{\sigma}\alpha^{\sigma} = \frac{1}{2}mc^2 \tag{8.266}$$

and the expression for the potential

$$V = -\frac{1}{2}mc^2 + \frac{\hbar^2}{m}\mu^{\sigma\rho}\left(2k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\mu}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\mu}x^{\nu} - k^{i\cdot}_{\cdot\sigma}k^{i\cdot}_{\cdot\rho}\right) \tag{8.267}$$

Which we may compare to the relativistic potential found for the gaussian in one dimension of Section 8, Example 3, equation (8.52). We have now

$$\partial_{\tau}V = \frac{4\hbar^2}{m}\mu^{\sigma\rho}k^{i\cdot}_{\cdot\tau}k^{i\cdot}_{\cdot\sigma}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu} \tag{8.268}$$

then setting

$$-\partial_0 V = \beta^l \partial_l V \tag{8.269}$$

we find

$$\mu^{\sigma\rho}\left(\beta^l k^{i\cdot}_{\cdot l} + k^{i\cdot}_{\cdot 0}\right)k^{i\cdot}_{\cdot\sigma}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu} = 0 \tag{8.270}$$

and then for arbitrary

$$y^i = \mu^{\sigma\rho}k^{i\cdot}_{\cdot\sigma}k^{j\cdot}_{\cdot\rho}k^{j\cdot}_{\cdot\nu}x^{\nu} \tag{8.271}$$

we need

$$-\beta^l k^{i\cdot}_{\cdot l} = k^{i\cdot}_{\cdot 0} \tag{8.272}$$

in agreement with our definition at the outset (8.246), where we have assumed that the $k^{i\cdot}_{\cdot l}$ form a non-singular matrix, and of course to be acceptable, the β^i need to satisfy the inequality

$$\left(\beta^1\right)^2 + \left(\beta^2\right)^2 + \left(\beta^3\right)^2 \le c^2 \tag{8.273}$$

9. Solutions to Some One Dimensional Systems

9.1. **Delta Normalization of Plane Waves.** Consider the solution

$$\begin{aligned}\Psi &= A\exp\left(ia_\mu x^\mu\right)+B\exp\left(-ia^\mu x^\mu\right) &(9.1)\\ &= A\exp\left[i\left(a_1x^1+a_0x^0\right)\right]+B\exp\left[-i\left(a^1x^1+a^0x^0\right)\right]\\ &= A\exp\left[i\left(a_1x^1+a_0x^0\right)\right]+B\exp\left[-i\left(a_1x^1-a_1x^0\right)\right]\\ &= \exp\left(ia_0x^0\right)\left[A\exp\left(ia_1x^1\right)+B\exp\left(-ia_1x^1\right)\right] &(9.2)\end{aligned}$$

to the relativistic Schrödinger equation in one spatial dimension

$$-c\partial_0\Psi=-\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi+V\Psi \tag{9.3}$$

The potential that makes this true is given by

$$V=\hbar ca_0-\frac{\hbar^2}{2m}a^\mu a_\mu \tag{9.4}$$

which is a *constant* with respect to differentiation, but not Lorentz invariant. Note that the first term of Ψ is Lorentz invariant but not the second term. At least it doesn't appear so, but we shall remark at the end of this section.

Such a plane wave solution cannot be normalized by ordinary the procedure, but we can apply the delta normalization, thus let

$$\begin{aligned}\Psi_a &= \exp\left(ia_0x^0\right)\left[A\exp\left(ia_1x^1\right)+B\exp\left(-ia_1x^1\right)\right] &(9.5)\\ \Psi_b^* &= \exp\left(-ia_0x^0\right)\left[\overline{A}\exp\left(-ib_1x^1\right)+\overline{B}\exp\left(ib_1x^1\right)\right] &(9.6)\end{aligned}$$

Then

$$\Psi_b^*\Psi_a=|A|^2\exp\left[i\left(a_1-b_1\right)x^1\right]+|B|^2\exp\left[-i\left(a_1-b_1\right)x^1\right]+2\Re\left\{A\overline{B}\exp\left[i\left(a_1+b_1\right)x^1\right]\right\} \tag{9.7}$$

We find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_b^*\Psi_a dx=|A|^2\delta\left(a_1-b_1\right)+|B|^2\delta\left(b_1-a_1\right)+A\overline{B}\delta\left(a_1+b_1\right)+\overline{A}B\delta\left[-\left(a_1+b_1\right)\right] \tag{9.8}$$

where we think of the delta function as being an even function, therefore

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi_b^*\Psi_a dx=\left(|A|^2+|B|^2\right)\delta\left(b_1-a_1\right)+2\Re\left(A\overline{B}\right)\delta\left(b_1+a_1\right) \tag{9.9}$$

Now we choose

$$|A|^2+|B|^2=\tfrac{1}{2\pi}\quad\text{and}\quad\Re\left(A\overline{B}\right)=0 \tag{9.10}$$

so that

$$\int_{-\infty}^{\infty}\Psi_b^*\Psi_a dx=\delta\left(b_1-a_1\right) \tag{9.11}$$

We can also use this procedure to calculate a delta expectation

$$\begin{aligned}\langle p_1\rangle &= \int_{-\infty}^{\infty}\Psi_b^*\left(-i\hbar\partial_1\right)\Psi_a dx\\ &= -i\hbar\int_{-\infty}^{\infty}\Psi_b^*\partial_1\Psi_a dx\\ &= -i\hbar\int_{-\infty}^{\infty}\Psi_b^*\left[ia_1 A\exp\left(ia_1x^1\right)-ia_1B\exp\left(-ia_1x\right)\right]dx\\ &= \hbar a_1\int_{-\infty}^{\infty}\Psi_b^*\left[A\exp\left(ia_1x^1\right)-B\exp\left(-ia_1x\right)\right]dx\end{aligned}$$

and we obtain

$$\langle p_1\rangle = \hbar a_1\left[\left(|A|^2-|B|^2\right)\delta\left(a_1-b_1\right)+2i\Im\left(A\overline{B}\right)\delta\left(a_1+b_1\right)\right] \tag{9.12}$$

We see a problem here with the appearance of the imaginary unit in the second term in parentheses, since we can't have both

$$\Re\left(A\overline{B}\right)=0 \quad \text{and} \quad \Im\left(A\overline{B}\right)=0$$

Well if $|a_1-b_1|$ is sufficiently close to zero, then $a_1+b_1\neq 0$ in which case $\delta\left(a_1+b_1\right)=0$, and we can write

$$\int_{-\infty}^{\infty}\Psi_{a+\Delta a}^*\Psi_a dx=\left(|A|^2+|B|^2\right)\delta\left(\Delta a_1\right) \tag{9.13}$$

and

$$\langle p_1\rangle=\hbar a_1\left[\left(|A|^2-|B|^2\right)\delta\left(\Delta a_1\right)\right] \tag{9.14}$$

Remark 12. We noted that Ψ given in (9.1) was not Lorentz invariant, well at least it does not appear so in the form given, but with a little footwork we can recast it in Lorentz invariant form

$$\Psi = A\exp\left(ia_\mu x^\mu\right)+B\exp\left(-ia^\mu x^\mu\right) \tag{9.15}$$

$$ = \exp\left(ia_0x^0\right)\left[A\exp\left(ia_1x^1\right)+B\exp\left(-ia_1x^1\right)\right] \tag{9.16}$$

It we rename a_0 in the time dependent term α_0, then

$$\begin{aligned}\Psi &= \exp\left(i\alpha_0x^0\right)\left[A\exp\left(ia_1x^1\right)+B\exp\left(-ia_1x^1\right)\right]\\ &= \exp\left(i\alpha_\mu x^\mu\right)\left[A\exp\left(ia_\nu x^\nu\right)+B\exp\left(-ia_\nu x^\nu\right)\right]\end{aligned}$$

where $\alpha_1=0$ and now $a_0=0$. But now we need to reexpress the potential found in equation (9.4) as

$$V=\hbar c\alpha_0-\frac{\hbar^2}{2m}\left[\left(a_1\right)^2-\left(\alpha_0\right)^2\right] \tag{9.17}$$

Let us check ourselves

$$\psi\left(x\right)=A\exp\left(ia_1x^1\right)+B\exp\left(-ia_1x^1\right) \tag{9.18}$$

so that

$$\frac{d^2\psi}{dx^2}=-\left(a_1\right)^2\psi \tag{9.19}$$

Plugging into the time independent equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V\psi=E\psi \tag{9.20}$$

We find

$$V = E - \frac{\hbar^2}{2m}(a_1)^2 \tag{9.21}$$

Then with

$$E = \hbar c\alpha_0 + \frac{\hbar^2}{2m}(\alpha_0)^2 \tag{9.22}$$

we obtain (9.17). We will need to use this trick as we proceed further along.

Its a bit of a masochistic execise, but the solution

$$\Psi = \exp(i\alpha_\mu x^\mu)\left[A\exp(ia_\mu x^\mu) + B\exp(-ia_\mu x^\mu)\right] \tag{9.23}$$

to

$$i\hbar\eta^\mu\partial_\mu\Psi = -\frac{\hbar^2}{2m}\partial^\mu\partial_\mu + V\Psi \tag{9.24}$$

requires a potential

$$V = -\left[\hbar\eta^\mu(\alpha_\mu + a_\mu) + \frac{\hbar^2}{2m}(\alpha^\mu\alpha_\mu + a^\mu a_\mu)\right] \tag{9.25}$$

and the condition

$$\alpha_1 a_1 - \alpha_0 a_0 = 0 \tag{9.26}$$

Condition (9.26) is required to make $\widetilde{\Psi}$ vanish when substituting back into equation (9.24). In the rest frame of equation (9.24) we have

$$\eta^0 = -c \tag{9.27}$$
$$\alpha_1 = 0 \tag{9.28}$$
$$a_0 = 0 \tag{9.29}$$

so that equation (9.25) for the potential reduces to the potential found in equation (9.17). Equation (9.25) seems to depend upon η^μ, but this is illusory since the first term, $\hbar\eta^\mu(\alpha_\mu + a_\mu)$ is Lorentz invariant as is the second term, $\frac{\hbar^2}{2m}(\alpha^\mu\alpha_\mu + a^\mu a_\mu)$ so that (9.25) is identically equal to (9.17), the potential seen in the rest frame.

The algebra is made managable by defining

$$f = \exp(i\alpha_\mu x^\mu) \tag{9.30}$$
$$g = \exp(ia_\mu x^\mu) \tag{9.31}$$
$$\Psi = f(Ag + Bg^*) \tag{9.32}$$
$$\widetilde{\Psi} = f(Ag - Bg^*) \tag{9.33}$$

You should find

$$\begin{array}{cc} \partial f = i\alpha f & (\partial)^2 f = -(\alpha)^2 f \\ \partial g = iag & (\partial)^2 g = -(a)^2 g \\ \partial g^* = -iag^* & (\partial)^2 g^* = -(a)^2 g^* \end{array}$$

where the appropriate subscripts have been omitted, and we're only dealing in Lorentz 2-vectors here, that is α and a are the appropriate covariant components (subscripts) so that

$$\partial\Psi = i\left(\alpha\Psi + a\tilde{\Psi}\right) \tag{9.34}$$

$$(\partial)^2\Psi = -\left[(\alpha)^2 + (a)^2\right]\Psi - 2\alpha a\widetilde{\Psi} \tag{9.35}$$

thus for instance

$$\partial_0\partial_0\Psi = -\left[(\alpha_0)^2 + (a_0)^2\right]\Psi - 2\alpha_0 a_0\widetilde{\Psi} \tag{9.36}$$

9.2. **The Step Potential.** Since the time independent Schrodinger equation takes the same form in both the relativistic and non-relativistic treatment we can import the solutions from some of the classic one dimensional problems, such as the step potential, with no modification. The question arises, however, whether the time dependent phase factor should be adjusted for the possible difference between the relativistic energy eigenstate, E_R, the classical quantum energy eigenstate, E_S. We have

$$V_R - E_R = V_S - E_S \tag{9.37}$$

and the time independent equation

$$-\frac{\hbar^2}{2m}\partial^i\partial_i\psi + (V_R - E_R)\,\psi = 0 \tag{9.38}$$

This consideration arose when we found that the potential for the gaussian packet in the relativistic treatment differed from that found from the classical treatment by a fixed constant as was discussed in Section 8.2 equation (8.106). For the step, barrier, square well potentials we can ignore this consideration, and start with the relativistic potential and energy eigenstates.

The step potential is given by

$$V(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ V & \text{for} \quad x \geq 0 \end{cases} \tag{9.39}$$

The solution for region I where $x < 0$ has the form

$$\psi_1(x) = A\exp\left(ia_1x^1\right) + B\exp\left(-ia_1x^1\right) \tag{9.40}$$

where the 1st term of the wave train

$$\Psi_1\left(x^1, x^0\right) = \exp\left(i\alpha_0x^0\right)\left[A\exp\left(ia_1x^1\right) + B\exp\left(-ia_1x^1\right)\right] \tag{9.41}$$

is moving toward the barrier. The second term represents the reflected wave. Now with a stationary potential it is difficult to consider a particle in motion, for the gaussian packet in motion the quadratic potential was in motion along with it at the same velocity. None the less we are to imagine a particle moving toward the barrier from the left, this consideration is used in setting one of the constants to zero in the solution for region II, where $x \geq 0$. The solution for the constant non-zero potential in region II has the form

$$\psi_2(x) = C\exp\left(ib_1x^1\right) + D\exp\left(-ib_1x^1\right) \tag{9.42}$$

Assuming that there is no wave train incident from the right in

$$\Psi_2\left(x^1, x^0\right) = \exp\left(i\alpha_0x^0\right)\left[C\exp\left(ib_1x^1\right) + D\exp\left(-ib_1x^1\right)\right] \tag{9.43}$$

requires that we set $D = 0$, so that

$$\psi_2(x) = C\exp\left(ib_1x^1\right) \tag{9.44}$$

represents the wave transmitted through the barrier. To obtain an acceptable wave function we require that ψ_1 and ψ_2 have the same magnitude and slope where they meet at $x = 0$. thus we impose the boundary conditions

$$\psi_1(0) = \psi_2(0) \tag{9.45}$$

$$\psi_1'(0) = \psi_2'(0) \tag{9.46}$$

Thus we must have

$$C = A + B \tag{9.47}$$

$$C = \frac{a_1}{b_1}(A - B) \tag{9.48}$$

and we find

Case 1. Now let us consider the case $E > V$, then in region I the time independent equation is given by

$$-\frac{\hbar^2}{2m}\partial^1\partial_1\psi_1 - E\psi_1 = 0 \tag{9.49}$$

we have by equation (9.17)

$$\hbar\alpha_0 = \frac{\hbar^2}{2mc}\left[(a_1)^2 - (\alpha_0)^2\right] \tag{9.50}$$

where from equation (8.187)

$$\hbar\alpha_0 = -\left[m + M(E)\right]c \tag{9.51}$$

with the restriction $E \geq -\frac{1}{2}mc^2$, since

$$M(E) = \frac{1}{c}\sqrt{m^2c^2 + 2mE} \tag{9.52}$$

In region II the time independent equation is given by

$$-\frac{\hbar^2}{2m}\partial^1\partial_1\psi_2 + (V - E)\psi_2 = 0 \tag{9.53}$$

where by equation (9.17) we have

$$V = \hbar c\alpha_0 - \frac{\hbar^2}{2m}\left[(b_1)^2 - (\alpha_0)^2\right] \tag{9.54}$$

But wait a minute mustn't we have both

$$\hbar^2(a_1)^2 = 2mE \tag{9.55}$$

$$\hbar^2(b_1)^2 = 2m(E - V) \tag{9.56}$$

in addition to equations (9.50) and (9.51)? Well in region I we find these requirements perfectly consistent. We have

$$2mc\hbar\alpha_0 = (\hbar a_1)^2 - (\hbar\alpha_0)^2 \tag{9.57}$$

$$= 2mE - (\hbar\alpha_0)^2 \tag{9.58}$$

and

$$\begin{aligned}(\hbar\alpha_0)^2 &= \left[m^2 + 2mM(E) + M(E)^2\right]c^2 \\ &= \left[m^2 + 2mM(E) + \frac{m^2c^2 + 2mE}{c^2}\right]c^2 \\ &= m^2c^2 + 2mM(E)c^2 + m^2c^2 + 2mE \\ &= 2m^2c^2 + 2mM(E)c^2 + 2mE\end{aligned}$$

Thus

$$2mc\hbar\alpha_0 = 2mE - \left(2m^2c^2 + 2mM(E)c^2 + 2mE\right) \tag{9.59}$$

$$mc\hbar\alpha_0 = -\left(m^2c^2 + mM(E)c^2\right) \tag{9.60}$$

$$= -mc^2(m + M(E)) \tag{9.61}$$

or

$$\hbar a_0 = -c(m + M(E)) \tag{9.62}$$

which is indentical to equation (9.51). We should check region II now

$$2mc\hbar\alpha_0 - 2mV = (\hbar b_1)^2 - (\hbar\alpha_0)^2 \tag{9.63}$$

$$= 2m(E - V) - (\hbar\alpha_0)^2 \tag{9.64}$$

$$= 2mE - 2mV - \left(2m^2c^2 + 2mM(E)c^2 + 2mE\right) \tag{9.65}$$

$$= -2mc^2(m + M(E)) - 2mV \tag{9.66}$$

and everything checks out. So let us continue. The boundary conditions () and () give us

$$\frac{B}{A} = \frac{a_1 - b_1}{a_1 + b_1} \tag{9.67}$$

$$\frac{C}{A} = \frac{2a_1}{a_1 + b_1} \tag{9.68}$$

The three spatial components of the relativistic probability current in this frame are identical to the ones obtained from the classical Schrodinger equation so that for the incident, reflected and transmitted waves

$$S_{in} = \frac{\hbar a_1}{m}|A|^2 \tag{9.69}$$

$$S_{re} = -\frac{\hbar a_1}{m}|B|^2 \tag{9.70}$$

$$S_{tr} = \frac{\hbar b_1}{m}|C|^2 \tag{9.71}$$

So the the transmission and reflection coefficients are

$$T = \left|\frac{S_{tr}}{S_{in}}\right| = \frac{b_1}{a_1}\left|\frac{C}{A}\right|^2 = \frac{4a_1b_1}{(a_1 + b_1)^2} \tag{9.72}$$

$$R = \left|\frac{S_{re}}{S_{in}}\right| = \left|\frac{B}{A}\right|^2 = \left(\frac{a_1 - b_1}{a_1 + b_1}\right)^2 \tag{9.73}$$

and

$$T + R = 1 \tag{9.74}$$

Case 2. Suppose $E < V$, then we have

$$\begin{aligned} \hbar^2 (b_1)^2 &= 2m(E-V) < 0 \tag{9.75} \\ V &= \hbar c\alpha_0 - \frac{\hbar^2}{2m}\left((b_1)^2 - (\alpha_0)^2\right) \tag{9.76} \end{aligned}$$

Thus let $b_1 = iB_1$ so that

$$\begin{aligned} \hbar^2 (B_1)^2 &= 2m(V-E) > 0 \tag{9.77} \\ V &= \hbar c a_0 + \frac{\hbar^2}{2m}\left[(B_1)^2 + (\alpha_0)^2\right] \tag{9.78} \end{aligned}$$

and the solution in region II is the decaying exponential

$$\psi_2 = C\exp(-B_1 x) \tag{9.79}$$

In this region

$$\frac{B}{A} = \frac{a_1 - iB_1}{a_1 + iB_1} \tag{9.80}$$

so that the reflection coefficient is

$$R = \left|\frac{B}{A}\right|^2 = 1 \tag{9.81}$$

The probability density current

$$\begin{aligned} S_{tr} &= -\frac{h}{m}\Im\left(\psi_2 \partial^1 \psi_2^*\right) \\ &= -\frac{h}{m}\Im\left(\psi_2 \partial^1 \psi_2\right) \\ &= -\frac{h}{m}\Im\left(-B_1 \psi_2 \psi_2\right) \\ &= 0 \tag{9.82} \end{aligned}$$

Thus the transmission coefficient is

$$T = \left|\frac{S_{tr}}{S_{in}}\right| = 0 \tag{9.83}$$

The time independent wave function, ψ is given by piecing together ψ_1 and ψ_2

$$\psi(x) = \begin{cases} \psi_1(x) & \text{for} \quad x < 0 \\ \psi_2(x) & \text{for} \quad x \geq 0 \end{cases} \tag{9.84}$$

and the only real difference comes from the time dependent phase factor

$$\Psi = \exp\left[-\frac{i}{\hbar}\left(mc + \sqrt{m^2c^2 + 2mE}\right)x^0\right]\psi(x) \tag{9.85}$$

where if we were to measure the total energy in units,ξ of rest energy, mc^2, we can write

$$M(E) = m\sqrt{1+2\xi} \tag{9.86}$$

so that

$$\Psi = \exp\left[-i\frac{mc}{\hbar}\left(1 + \sqrt{1+2\xi}\right)x^0\right]\psi(x) \tag{9.87}$$

As for whether we should consider a particle to be in motion toward the barrier from the left in the presence of a stationary potential, we could consider the problem of crashing a gaussian potential into the step

$$V\left(x^1, x^0\right) = \begin{cases} -\frac{1}{2}mc^2 + \frac{1}{m}\left(\hbar\Delta k_1\right)^2\left(2\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2 - 1\right) & \text{for} \quad x < 0 \\ V - \frac{1}{2}mc^2 + \frac{1}{m}\left(\hbar\Delta k_1\right)^2\left(2\left(\Delta k_1 x^1 + \Delta k_0 x^0\right)^2 - 1\right) & \text{for} \quad x \geq 0 \end{cases} \tag{9.88}$$

but there is no frame where this potential is time independent.

9.3. **The Barrier Potential.** stub

9.4. **The Square Well Potential.** stub

9.5. **The Infinite Square Well.** Rather then let the potential walls go to infinity in the previous section let us specify that $\psi = 0$ outside the interval $[0, L]$ and let the potential inside the closed interval of length L be zero, then we need to solve the time indendent equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \tag{9.89}$$

which has the well known solution

$$\psi\left(x\right) = A\sin\left(a_1 x^1\right) + B\cos\left(a_1 x^1\right) \tag{9.90}$$

where

$$E = \frac{\hbar^2\left(a_1\right)^2}{2m} \tag{9.91}$$

And we impose the boundary conditions

$$\psi\left(0\right) = \psi\left(L\right) = 0$$

which requires that $B = 0$ so that the solution becomes

$$\psi\left(x\right) = A\sin\left(a_1 x^1\right) \tag{9.92}$$

but then

$$\sin\left(a_1 L\right) = 0 \tag{9.93}$$

and this can only happen for

$$a_1 L = n\pi \tag{9.94}$$

thus we find the allowed energy states by substituting (9.94) into (9.91) to obtain

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2} \tag{9.95}$$

Note that we cannot have $n = 0$, since then the wave function is zero everywhere in the box which is equivalent to no particle in the box.

The normaliztion constant is obtained by setting

$$\int_{-\infty}^{\infty}\left|\psi\right|^2 dx = 1 \tag{9.96}$$

Thus we must have

$$|A|^2 \int_0^L \sin^2\left(a_1 x^1\right) dx^1 = |A|^2 \frac{L}{2} = 1 \tag{9.97}$$

Hence

$$A = \sqrt{\frac{2}{L}} \tag{9.98}$$

Now let us write out the full relativistic solution to our problem

$$\Psi\left(x^1, x^0\right) = \sqrt{\frac{2}{L}} \exp\left\{-\frac{i}{\hbar}\left[m + M\left(E_n\right)\right] c x^0\right\} \sin\left(\sqrt{\frac{2mE_n}{\hbar^2}} x^1\right) \tag{9.99}$$

where

$$M\left(E_n\right) = \frac{1}{c}\sqrt{m^2c^2 + 2mE_n} \tag{9.100}$$

which simplifies if we measure the total energy in units of rest mass, that is let us set

$$E_n = \xi_n mc^2 \tag{9.101}$$

so that

$$\xi_n = \frac{1}{2}\left(\frac{\hbar\pi n}{mcL}\right)^2 \tag{9.102}$$

then

$$M\left(E_n\right) = m\sqrt{1 + 2\xi_n} \tag{9.103}$$

$$a_1 = \frac{mc}{\hbar}\sqrt{\xi_n} \tag{9.104}$$

Hence

$$\Psi\left(x^1, x^0\right) = \sqrt{\frac{2}{L}} \exp\left\{-i\frac{mc}{\hbar}\left[1 + \sqrt{1 + 2\xi_n}\right] x^0\right\} \sin\left(\frac{n\pi}{L} x^1\right) \tag{9.105}$$

9.6. **The Harmonic Oscillator.**

9.6.1. *The Newtonian Harmonic Oscillator.* The classical harmonic oscillator of Newtonian mechanics can be derived from Newton's Second Law

$$\mathrm{F} = m\mathrm{a} \tag{9.106}$$

In one dimension with $\mathbf{F} = kx$ and $\mathrm{a} = \ddot{x}$ we have

$$m\ddot{x} = -kx \tag{9.107}$$

a second order ordinary linear differential equation which can be reexpressed as a pair of coupled first order equations with $y = \dot{x}$. we obtain

$$\dot{y} = -\frac{k}{m} x \tag{9.108}$$

$$\dot{x} = y \tag{9.109}$$

This can be put in matrix form by

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \tag{9.110}$$

We can make this matrix equation much more symmetric (actually skew-symmetric) if we let

$$\omega^2 = \frac{k}{m} \geq 0 \tag{9.111}$$

and replace $y = \dot{x}$ with $\dot{x}/\omega$ thus

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x}/\omega \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x}/\omega \end{bmatrix} \tag{9.112}$$

Now we digress for a moment, we can multiply both sides of this equation by the mass, m, to obtain

$$\frac{d}{dt}\begin{bmatrix} mx \\ p/\omega \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} mx \\ p/\omega \end{bmatrix} \tag{9.113}$$

where $p = m\dot{x}$ is the momentum, and we note that if we make the speed of light, c, dimensionless, hence measuring time in units of length and making all velocities dimensionless, then Planck's constant,$\hbar$, now has units of mass times length. If we furthe make $\hbar$ dimensionless, mass is now measured in the recipricol of length, which is delightful seeing that general relativity shows that the presence of mass curves space-time, that units of mass and space-time are so related. In the above equation now mx is dimensionless, as is p/ω, though ω retains it units of mass, the recipricol of length.

Before we proceed to the quantum mechanical description, we would like to obtain the the Hamiltonian of this system

$$\mathcal{H} = T + V \tag{9.114}$$

(where T represents only the kinetic energy) as a function of position, x, and momentum, p , but first we need to find the potential energy of this system. We express force as the gradient of a scalar potential, in fact we choose to write

$$\mathrm{F} = -\nabla V \tag{9.115}$$

For our one dimensional harmonic oscillator this becomes from equation (9.107)

$$\frac{dV}{dx} = kx \tag{9.116}$$

Integrating we find

$$V = \frac{1}{2}kx^2 + C \tag{9.117}$$

where we choose for convenience, $C = 0$. To follow Hamilton's procedure we form the Lagrangian in terms of x and $\dot{x}$

$$\mathcal{L} = T - V \tag{9.118}$$

$$= \frac{1}{2}\left(m\dot{x}^2 - kx^2\right) \tag{9.119}$$

And we see that

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = p \tag{9.120}$$

which we will use to reexpress $\dot{x}$ in terms of p, thereby finding

$$\dot{x} = \frac{p}{m} \tag{9.121}$$

so that the kinetic energy becomes

$$T = \frac{p^2}{m} \tag{9.122}$$

Therefore the Hamiltonian is given by

$$\mathcal{H} = T + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{9.123}$$

where we have written $k = m\omega^2$. We obtain from Hamilton's equations of motion

$$\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial p} \tag{9.124}$$

$$\frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial x} \tag{9.125}$$

equations equivalent to (9.113)

$$\frac{dx}{dt} = \frac{p}{m} \tag{9.126}$$

$$\frac{dp}{dt} = -m\omega^2 x \tag{9.127}$$

9.6.2. *The Quantum Mechanical Harmonic Oscillator.* The time independent equation has always been written

$$\mathcal{H}\psi = E\psi \tag{9.128}$$

where $\mathcal{H}$ is now the Hamiltonian operator, which is obtained from the classical Hamiltonian by the correspondence principle which in turn replaces the classical momentum p with the corresponding momentum operator.

$$p_x = -i\hbar\frac{\partial}{\partial x} \tag{9.129}$$

but now things are going to look a bit odd in the relativistic context of equation (8.5). The right hand side total energy eigenvalue now includes the rest energy, in addition to the kinetic and potential, but the Hamiltonian by the correspondence principle seems to represent only the kinetic and potential. We are planning to use the operator correspondence principle as is to obtain the time independent equation for the harmonic oscillator, so let us look closely. We have the following correspondence as noted in Section 8 equation (8.16)

$$\hbar^2\left(\frac{\partial^2}{\partial\left(ct\right)^2} - \frac{\partial^2}{\partial x^2}\right) \sim m^2c^2 \tag{9.130}$$

Thus

$$-\hbar^2\frac{\partial^2}{\partial x^2} \sim m^2c^2 - \hbar^2\frac{\partial^2}{\partial\left(ct\right)^2} \tag{9.131}$$

then dividing both sides by $2m$ we find

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \sim \frac{1}{2}mc^2 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial\left(ct\right)^2} \tag{9.132}$$

but for the combined relativistic rest and kinetic energy we have the operator correspondence

$$i\hbar\frac{\partial}{\partial ct} \sim \frac{mc^2+T}{c} = p^0 \tag{9.133}$$

Note with $x^0 = ct$

$$\left(i\hbar\frac{\partial}{\partial x^0}\right)\left(i\hbar\frac{\partial}{\partial x_0}\right) = \frac{\partial^2}{\partial\left(ct\right)^2} = \left|i\hbar\frac{\partial}{\partial x^0}\right|^2 \tag{9.134}$$

In any case

$$-\hbar^2\frac{\partial^2}{\partial\left(ct\right)^2} \sim \left(\frac{mc^2+T}{c}\right)^2 = \left(mc+\frac{T}{c}\right)^2 \tag{9.135}$$

We have for the relativistic momentum the correspondence

$$-i\hbar\frac{\partial}{\partial x^1} \sim p_1 \tag{9.136}$$

so that

$$\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \quad &\sim \quad \frac{1}{2}mc^2 + \frac{1}{2m}\left(mc+\frac{T}{c}\right)^2 && (9.137)\\
&\sim \quad mc^2 + T + \frac{1}{2m}\left(\frac{T}{c}\right)^2 && (9.138)\\
&\sim \quad mc^2 + T + \frac{1}{2m}\left(\frac{m\dot{x}^2}{2c}\right)^2 && (9.139)\\
&\sim \quad mc^2 + T + \frac{1}{8}m\left(\dot{x}\beta\right)^2 && (9.140)
\end{aligned}$$

and we see that the operator includes the rest and kinetic energies plus a term that is small for low velocity. Therefore if we form the classical Hamiltonian which represents only the kinetic and potential energies, and replace the momentum terms with the new relativistic momentum operator, the time independent Schrödinger equation does make sense in the relativistic context.

We now find for the quantum mechanical harmonic oscillator the time independent equation is given from classical Hamiltonian (9.123) and the correspondence principle

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi \tag{9.141}$$

Let us do a little algebra to obtain

$$\left(\frac{m\hbar}{\omega}\right)^2\frac{d^2\psi}{m^2dx^2} - m^2x^2\psi = -\frac{2mE}{\omega^2}\psi \tag{9.142}$$

Now we let $q = mx$ and

$$-i\frac{m\hbar}{\omega}\frac{d}{dq} = -i\frac{\hbar}{\omega}\frac{d}{dx} = \frac{p_x}{\omega} = p \tag{9.143}$$

Then with

$$\epsilon = \frac{2mE}{\omega^2} \tag{9.144}$$

equation (9.141) becomes

$$\left(p^2+q^2\right)\psi=\epsilon\psi \tag{9.145}$$

where we will use the operator method of Dirac to solve.

The commutator bracket for p and q is

$$[p,q]=-i\frac{m\hbar}{\omega} \tag{9.146}$$

Now we would like to recast equation (9.145) in terms of operators with a commutaor bracket equal to unity so lets us multiply each side of (9.145) by

$$\alpha=\frac{\omega}{m\hbar} \tag{9.147}$$

to obtain

$$\alpha\left(p^2+q^2\right)\psi=\alpha\epsilon\psi \tag{9.148}$$

and let us define

$$a = \sqrt{\frac{\alpha}{2}}\left(q+ip\right) \tag{9.149}$$

$$a^\dagger = \sqrt{\frac{\alpha}{2}}\left(q-ip\right) \tag{9.150}$$

whose commutator bracket is given by

$$\left[a,a^\dagger\right]=aa^\dagger-a^\dagger a=1 \tag{9.151}$$

so that (9.148) becomes

$$\left(aa^\dagger+a^\dagger a\right)\psi=\varepsilon\psi \tag{9.152}$$

with

$$\varepsilon=\alpha\epsilon=\frac{2E}{\hbar\omega} \tag{9.153}$$

Now if c and $\hbar$ are dimensionless, then so are q, p, α, a, $a^\dagger$, and ε, and this is where we will begin the operator method. If we operate on both sides of (9.152) with $a^\dagger$ we find

$$\varepsilon a^\dagger\psi = a^\dagger\left(aa^\dagger+a^\dagger a\right)\psi \tag{9.154}$$

$$= a^\dagger\left(aa^\dagger+aa^\dagger-1\right)\psi \tag{9.155}$$

so that

$$\left(\varepsilon+1\right)a^\dagger\psi = a^\dagger\left(aa^\dagger+aa^\dagger\right)\psi \tag{9.156}$$

$$= \left(a^\dagger a+a^\dagger a\right)a^\dagger\psi \tag{9.157}$$

$$= \left(aa^\dagger-1+a^\dagger a\right)a^\dagger\psi \tag{9.158}$$

and there results

$$\left(\varepsilon+2\right)a^\dagger\psi=\left(aa^\dagger+a^\dagger a\right)a^\dagger\psi \tag{9.159}$$

hence if ψ is an eigenfunction, then so is $a^\dagger\psi$, and this process can be repeated indefinitely so that

$$\left(\varepsilon+2n\right)\left[\left(a^\dagger\right)^n\psi\right]=\left(aa^\dagger+a^\dagger a\right)\left[\left(a^\dagger\right)^n\psi\right] \tag{9.160}$$

The operator, $a^\dagger$ is called the raising operator, since it raises the energy state with each application. Now if we apply the operator, a, to both sides of (9.152) we find

$$\varepsilon a\psi = a\left(aa^\dagger + a^\dagger a\right)\psi \tag{9.161}$$

$$= a\left(1 + a^\dagger a + a^\dagger a\right)\psi \tag{9.162}$$

hence

$$\left(\varepsilon - 1\right) a\psi = \left(aa^\dagger + aa^\dagger\right) a\psi \tag{9.163}$$

$$= \left(aa^\dagger + a^\dagger a + 1\right) a\psi \tag{9.164}$$

and we find

$$\left(\varepsilon - 2\right) a\psi = \left(aa^\dagger + a^\dagger a\right) a\psi \tag{9.165}$$

and in general

$$\left(\varepsilon - 2n\right)\left[a^n\psi\right] = \left(aa^\dagger + a^\dagger a\right)\left[a^n\psi\right] \tag{9.166}$$

thus the operator, a, is called the lowering operator. The lowering operator, however may not be applied indefinitely, since the total energy eigenstate must be greater than zero. To see this first we note that $\varepsilon = \langle\varepsilon\rangle$, and

$$\langle\varepsilon\rangle = \int_{-\infty}^{\infty}\left(\left|\frac{d\psi}{dq}\right|^2 + q^2\left|\psi\right|^2\right) dq \geq 0 \tag{9.167}$$

since the integrand is positive definite. Thus for some eigenfunction, ψ_0, the ground state, we must have

$$a\psi_0 = 0 \tag{9.168}$$

Now we may rewrite the Schrödinger equation (9.152) in either of the equivalent forms

$$aa^\dagger\psi = \frac{1}{2}\left(\varepsilon + 1\right)\psi \tag{9.169}$$

$$a^\dagger a\psi = \frac{1}{2}\left(\varepsilon - 1\right)\psi \tag{9.170}$$

thus

$$a^\dagger a\psi_0 = 0 = \frac{1}{2}\left(\varepsilon - 1\right)\psi_0 \tag{9.171}$$

and we find from (9.153)

$$\varepsilon = \frac{2E_0}{\hbar\omega} = 1 \tag{9.172}$$

or

$$E_0 = \frac{1}{2}\hbar\omega \tag{9.173}$$

which is the ground state energy. To find the ground state wave function we write from the definition (9.149) of the lowering operator a

$$0 = a\psi_0 \tag{9.174}$$

$$= \sqrt{\frac{\omega}{2m\hbar}}\left(q + i\frac{\partial}{\partial q}\right)\psi_0 \tag{9.175}$$

or

$$\frac{d\psi_0}{dq} + q\psi_0 = 0 \tag{9.176}$$

whose solution is

$$\psi_0 = C_0 \exp\left(-\frac{1}{2}q^2\right) \tag{9.177}$$

where we choose $C_0 = \sqrt{\frac{1}{\sqrt{\pi}}}$ to normalize, since

$$\langle\psi_0|\psi_0\rangle = |C_0|^2 \int \exp\left(-q^2\right) dq \tag{9.178}$$

$$= |C_0|^2 \sqrt{\pi} \tag{9.179}$$

We obtain the wave function for the nth by operating on the eigenfunction, $\exp\left(-\frac{1}{2}q^2\right)$ and normalizing thus

$$\psi_n = C_n \left(a^\dagger\right)^n \exp\left(-\frac{1}{2}q^2\right) \tag{9.180}$$

$$= C_n \sqrt{\sqrt{\pi}} \left(a^\dagger\right)^n \psi_0 \tag{9.181}$$

If we define the nth Hermite polynomial as

$$H_n(q) = \exp\left(\frac{1}{2}q^2\right)\left(q - i\frac{\partial}{\partial q}\right)\exp\left(-\frac{1}{2}q^2\right) \tag{9.182}$$

we may write the nth state eigenfunction as

$$\psi_n = C_n \left(\sqrt{\frac{\alpha}{2}}\right)^n \exp\left(-\frac{1}{2}q^2\right) H_n(q) \tag{9.183}$$

$$= N_n \exp\left(-\frac{1}{2}q^2\right) H_n(q) \tag{9.184}$$

where the normalization constant is found to be (see [1] page 197 for derivation)

$$N_n = \left(\frac{1}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} \tag{9.185}$$

Now recalling that $q = mx$, how do we write the nth eigenfunction in relativistic terms? Our only choice seems to let $b_1 = m$ and $b_0 = 0$ so that

$$\psi_n = N_n \exp\left[-\frac{1}{2}\left(b_0x^0 + b_1x^1\right)^2\right] H_n\left(b_0x^0 + b_1x^1\right) \tag{9.186}$$

Now we seem to have misplaced the actual expression for the total energy eigenstate, E_n, which we find from

$$\varepsilon_n = \varepsilon_0 + 2n \tag{9.187}$$

$$= \frac{2E_0}{\hbar\omega} + 2n \tag{9.188}$$

and

$$\varepsilon_n = \frac{2E_n}{\hbar\omega} \tag{9.189}$$

and therefore since $E_0 = \frac{1}{2}\hbar\omega$ we have

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{9.190}$$

From the time dependent equation we have

$$\chi\left(x^0\right) = \exp\left[-i\frac{mc}{\hbar}\left(1 + \sqrt{1 + 2\xi_n}\right)x^0\right] \tag{9.191}$$

where $\xi_n mc^2 = E_n$ or

$$\xi_n = \frac{\hbar\omega}{mc^2}\left(n + \frac{1}{2}\right) \tag{9.192}$$

then with

$$\alpha_0 = -\frac{mc}{\hbar}\left(1 + \sqrt{1 + 2\xi_n}\right) \tag{9.193}$$

$$\alpha_1 = 0 \tag{9.194}$$

we may write

$$\chi\left(x^0\right) = \exp\left[i\left(\alpha_0 x^0 + \alpha_1 x^1\right)\right] \tag{9.195}$$

and the full relativistic wave function becomes

$$\Psi_n = N_n \exp\left[i\left(\alpha_0 x^0 + \alpha_1 x^1\right)\right] \exp\left[-\frac{1}{2}\left(b_0 x^0 + b_1 x^1\right)^2\right] H_n\left(b_0 x^0 + b_1 x^1\right) \tag{9.196}$$

So does this function satisfy the Relativistic Schrödinger equation (8.5)

$$i\hbar\eta^\mu\partial_\mu\Psi = -\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi + V\left(x^\mu\right)$$

with

$$-\eta^\mu = \gamma c\left(1, \beta\right) \tag{9.197}$$

$$\beta = -\frac{b_1}{b_0} \tag{9.198}$$

under a given Lorentz transformation? Mind you the rest mass remains the rest mass, but

$$b_1' = \gamma\left(b_1 + \beta b_0\right) \tag{9.199}$$

$$= \gamma m \tag{9.200}$$

and

$$b_0' = \gamma\left(b_0 + \beta b_1\right) \tag{9.201}$$

$$= \gamma\beta m \tag{9.202}$$

Note also we must express the potential appropriately

$$V\left(x^\mu\right) = \frac{1}{2}k\left(\lambda_0 x^0 + \lambda_1 x^1\right)^2 \tag{9.203}$$

since $m\omega^2 = k$. For the time independent potential we would have

$$\lambda_0 = 0 \tag{9.204}$$

$$\lambda_1 = 1 \tag{9.205}$$

but is this correct?

9.7. **Potentials which are Time Dependent in Every Reference Frame.** At the end of Section 9.2 above we considered crashing the potential for a gaussian into the stationary potential.

Case 1. So let us consider this and suppose we have a potential of the form

$$V_1 = \begin{cases} V(x-vt) & \text{for} \quad x \leq 0 \\ V(x-vt) + V_B & \text{for} \quad x > 0 \end{cases} \tag{9.206}$$

where

$$\Delta k_1 (x - vt) = \Delta k_1 x^1 + \Delta k_0 x^0 \tag{9.207}$$

We suppose that an observer at rest with respect to the barrier at $x = 0$ would prefer to solve the relativistic equation

$$-i\hbar c \partial_0 \Psi_1 = -\frac{\hbar^2}{2m} \partial^\mu \partial_\mu \Psi_1 + V_1 \Psi_1 \tag{9.208}$$

perhaps by numerical integration. For the potential quadratic in $x - vt$, the position, x_V of the minimum is $x_V = vt$ at time t, and the observer sees the minimum coincide with the barrier at $t = 0$. We'll call this observer, observer I.

Case 2. But there is a second point of view to consider, that of an observer at rest with respect to $V(x - vt)$ at $x' = 0$ where the potential he considers is given by

$$V_2 = \begin{cases} V(x') & \text{for} \quad x' \leq -vt' \\ V(x') + V_B & \text{for} \quad x' > -vt' \end{cases} \tag{9.209}$$

Who we assume prefers to solve

$$-i\hbar c \partial'_0 \Psi_2 = -\frac{\hbar^2}{2m} \partial'^\mu \partial'_\mu \Psi_2 + V_2 \Psi_2 \tag{9.210}$$

again by numerical integration. For the second observer the position, x'_B of the barrier at time t' is $x'_B = -vt'$, both observers agree to synchronize their clocks when the barrier and the minimum coincide, so that observer II sees this occur at time, $t' = 0$.

For the gaussian potential we found in Section 8, Example 3, equation (8.52)

$$V(x - vt) = -\frac{1}{2} mc^2 + \frac{1}{m} (\hbar \Delta k_1)^2 \left[2 \left(\Delta k_1 x^1 + \Delta k_0 x^0 \right)^2 - 1 \right] \tag{9.211}$$

$$V(x') = -\frac{1}{2} mc^2 + \frac{1}{m} (\hbar \Delta k'_1)^2 \left[2 \left(\Delta k'_1 x'^1 \right)^2 - 1 \right] \tag{9.212}$$

and we also have

$$-i\hbar c \partial'_0 = -i\hbar c \gamma \left(\partial_0 + \frac{v}{c} \partial_1 \right) \tag{9.213}$$

$$= -\gamma c (1, \beta) \cdot i\hbar (\partial_0, \partial_1) \tag{9.214}$$

which is in agreement with the relativistic Schrödinger equation (8.5) of Section 8.

Another possible approach to this problem, whose merit we haven't considered, is to solve the time independent equation for the potential V_1 given above (9.206) frozen in an instant of time,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V_1 (x, t) \psi = E \psi \tag{9.215}$$

We solve this equation for each and every instant of time, then little ψ evolves in time as do the energy eigenvalues, E. But E here cannot be the total energy which must be a constant, which does not evolve in time, none the less we find the solution

$$\Psi(x,t) = \psi_{E_t}(x) \tag{9.216}$$

which we plug into

$$
\begin{aligned}
-i\hbar c\partial_0\Psi &= -\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi + V_1\Psi && (9.217)\\
&= -\frac{\hbar^2}{2m}\partial^0\partial_0\Psi - \frac{\hbar^2}{2m}\partial^1\partial_1\Psi + V_1\Psi && (9.218)\\
&= -\frac{\hbar^2}{2m}\partial^0\partial_0\Psi + E_t\Psi && (9.219)
\end{aligned}
$$

or

$$\frac{\hbar^2}{2m}\partial_0\partial_0\Psi + i\hbar c\partial_0\Psi + E_t\Psi = 0 \tag{9.220}$$

We conclude this section with an observation about time independent potentials, from the standpoint of a given location on the x-axis at a particular point in time, all other points on the x-axis are separated by a spacelike interval. An observer cannot be at two points in space at the same time, yet we ask the question what is the probability that a particle, π, will be detected between, say $x^1 = a$ and $x^1 = b$, which we give as

$$\Pr(a \leq l(\pi) \leq b) = \int_a^b \psi_E^*\psi_E dx^1 \tag{9.221}$$

where $l(\pi)$ denotes the location of the particle. Given the time independence here, and observer has time to search any finite portion of the x-axis but not the entire axis. It seems more appropriate to associate an observer at each point on the x-axis, a continuum of observers, and ask what is the probability that an observer located between $x^1 = a$, and $x^1 = b$ will detect a particle. Where the probability is time dependent it seems a reasonable conceit to take this point of view.

10. Derivation of Schrödinger's Equation

10.1. **Derivation of the Time Independent Schrödinger Equation.** For a single particle in three spatial dimensions where the Hamiltonian is independent of time, Hamilton's Principal Function, S whose significance we discuss in Appendix, satisfies

$$\mathcal{H}\left(q^1, q^2, q^3, \frac{\partial S}{\partial q^1}, \frac{\partial S}{\partial q^2}, \frac{\partial S}{\partial q^3}\right) + \frac{\partial S}{\partial t} = 0 \tag{10.1}$$

which has an infinite number of solutions (why?) Let us define the time independent characteristic function S^* by the equation

$$S = S^* - Et \tag{10.2}$$

then S^* satisfies

$$\mathcal{H}\left(q^1, q^2, q^3, \frac{\partial S^*}{\partial q^1}, \frac{\partial S^*}{\partial q^2}, \frac{\partial S^*}{\partial q^3}\right) - E = 0 \tag{10.3}$$

For a single particle of mass m free of geometric constraints and moving under a conservative force, we can use the cartesian coordinates as the generalized coordinates the Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2m}\left[(p_x)^2 + (p_y)^2 + (p_z)^2\right] + V(x,y,z) \tag{10.4}$$

and equation (10.3) becomes

$$\frac{1}{2m}\left[\left(\frac{\partial S^*}{\partial x}\right)^2 + \left(\frac{\partial S^*}{\partial y}\right)^2 + \left(\frac{\partial S^*}{\partial z}\right)^2\right] + V(x,y,z) = E \tag{10.5}$$

With the substitution (what is the significance?)

$$S^* = K \ln(\psi) \tag{10.6}$$

so that

$$\frac{\partial S^*}{\partial x} = \frac{\partial}{\partial x} K \ln(\psi) = \frac{K}{\psi}\frac{\partial \psi}{\partial x} \tag{10.7}$$

and similarly for the remaining coordinates. then equation (10.5) is transformed into

$$\frac{K^2}{2m}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2\right] + (V-E)\,\psi^2 = 0 \tag{10.8}$$

Rather than sovle this equation, Schrödinger asks what partial differential equation must ψ satisfy in order that the integrand

$$f(x,y,z,\psi,\psi_x,\psi_y,\psi_z) = \frac{K^2}{2m}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2\right] + (V-E)\,\psi^2 \tag{10.9}$$

minimize

$$I = \int\int\int_{-\infty}^{\infty} f(x,y,z,\psi,\psi_x,\psi_y,\psi_z)\,dxdydz \tag{10.10}$$

where ψ_x is the first partial derivative of ψ with respect to x, etc. The answer is given by the Euler-Lagrange equation

$$\frac{\partial f}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \psi_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \psi_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \psi_z}\right) = 0 \tag{10.11}$$

where we obtain the time independent Schrödinger equation

$$-\frac{K^2}{2m}\nabla^2\psi + V\psi = E\psi \tag{10.12}$$

where we recognize the constant K as the reduced Planck Constant $\hbar$. To obtain the full Schrödinger equation all we can ask is what partial differential equation does the function

$$\Psi(x,y,z,t) = \exp(-i\omega t)\,\psi(x,y,z) \tag{10.13}$$

where

$$E = \hbar\omega \tag{10.14}$$

Thus we multiply both sides of (10.12) by $\exp(-i\omega t)$, since the equation as it stands involves no derivatives with respect to t, and obtain

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = E\Psi \tag{10.15}$$

but

$$\frac{\partial \Psi}{\partial t} = -i\omega\Psi \tag{10.16}$$

$$E = -i^2\hbar\omega \tag{10.17}$$

so that

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial \Psi}{\partial t} \tag{10.18}$$

10.2. **Derivation of the Relativistic Schrödinger Equation.** Suppose instead of asking what partial differential equation does

$$\Psi\left(x,y,z,t\right) = \exp\left(-i\omega t\right)\psi\left(x,y,z\right) \tag{10.19}$$

satisfy given that little ψ satifies the time independent equation as in the previous section, we ask what differential equation does

$$\Psi\left(x,y,z,t\right) = f\left(t\right)\psi\left(x,y,z\right) \tag{10.20}$$

for some undetermined function, $f\left(t\right)$ given the expectation

$$m^2c^2 = -\hbar^2\left\langle\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right\rangle \tag{10.21}$$

$$= \frac{\hbar^2}{c^2}\left\langle\frac{\partial^2}{\partial t^2}\right\rangle - \hbar^2\left\langle\frac{\partial^2}{\partial x^2}\right\rangle \tag{10.22}$$

Since the potential is time independent we are in the rest frame of the problem so that relativistic momentum is identical with the classical momentum and the combined relativistic kinetic and rest energy is close to the classical kinetic energy plus mc^2, let us further stipulate the expectations

$$-i\hbar\left\langle\frac{\partial}{\partial x}\right\rangle = \langle p_x\rangle \tag{10.23}$$

$$i\hbar\left\langle\frac{\partial}{\partial t}\right\rangle = c\left\langle p^0\right\rangle = mc^2 + \langle T\rangle \tag{10.24}$$

We found in Section 9.6.2 dealing with the harmonic oscillator from equations (9.135) and (9.138) the expectations

$$\frac{\hbar^2}{2m}\left\langle\frac{\partial^2}{\partial\left(ct\right)^2}\right\rangle = -\frac{1}{2m}\left\langle\left(mc+\frac{T}{c}\right)^2\right\rangle \tag{10.25}$$

$$-\frac{\hbar^2}{2m}\left\langle\frac{\partial^2}{\partial x^2}\right\rangle = mc^2 + \langle T\rangle + \frac{1}{2m}\left\langle\left(\frac{T}{c}\right)^2\right\rangle \tag{10.26}$$

First let us find

$$\begin{aligned} c\left\langle p^0\right\rangle &= i\hbar\int\Psi^*\frac{\partial}{\partial t}\Psi dx \\ &= i\hbar\int\left(\psi^*\psi\right)\left[f^*\left(t\right)\frac{\partial}{\partial t}f\left(t\right)\right]dx \\ &= i\hbar f^*\frac{\partial f}{\partial t} \end{aligned} \tag{10.27}$$

so that from (10.24) and (10.27) we have

$$mc^2 + \langle T \rangle = i\hbar f^* \frac{\partial f}{\partial t} \tag{10.28}$$

Second let us find

$$\begin{aligned}
\frac{\hbar^2}{2m}\left\langle \frac{\partial^2}{\partial (ct)^2} \right\rangle &= \frac{\hbar^2}{2m}\int \Psi^* \frac{\partial^2}{\partial (ct)^2}\Psi dx \\
&= \frac{\hbar^2}{2mc^2}\int \Psi^* \frac{\partial^2}{\partial t^2}\Psi dx \\
&= \frac{\hbar^2}{2mc^2} f^* \frac{\partial^2 f}{\partial t^2}\int \psi^*\psi dx \\
&= \frac{\hbar^2}{2mc^2} f^* \frac{\partial^2 f}{\partial t^2} \qquad (10.29)
\end{aligned}$$

so that from (10.25)

$$\frac{1}{2}mc^2 + \langle T \rangle + \frac{1}{2m}\left\langle \left(\frac{T}{c}\right)^2 \right\rangle = -\frac{\hbar^2}{2mc^2} f^* \frac{\partial^2 f}{\partial t^2} \tag{10.30}$$

where we have expanded the right hand side of (10.25). Now together with (10.28) we obtain

$$i\hbar f^* \frac{\partial f}{\partial t} - \frac{1}{2}mc^2 + \frac{1}{2m}\left\langle \left(\frac{T}{c}\right)^2 \right\rangle = -\frac{\hbar^2}{2mc^2} f^* \frac{\partial^2 f}{\partial t^2} \tag{10.31}$$

Multiplying both sides by f and rearranging we obtain

$$\frac{\hbar^2}{2mc^2}(ff^*)\frac{\partial^2 f}{\partial t^2} + i\hbar (ff^*)\frac{\partial f}{\partial t} + \left\{ \frac{1}{2m}\left\langle \left(\frac{T}{c}\right)^2 \right\rangle - \frac{1}{2}mc^2 \right\} f = 0 \tag{10.32}$$

where we can assume that f has unit modulus so that

$$\frac{\hbar^2}{2mc^2}\frac{\partial^2 f}{\partial t^2} + i\hbar \frac{\partial f}{\partial t} + \left\{ \frac{1}{2m}\left\langle \left(\frac{T}{c}\right)^2 \right\rangle - \frac{1}{2}mc^2 \right\} f = 0 \tag{10.33}$$

From equation () we have

$$\begin{aligned}
\frac{1}{2m}\left\langle \left(\frac{T}{c}\right)^2 \right\rangle &= -\frac{\hbar^2}{2m}\left\langle \frac{\partial^2}{\partial x^2} \right\rangle - mc^2 - \langle T \rangle \qquad (10.34) \\
&= E - \langle V \rangle - mc^2 - \langle T \rangle \qquad (10.35)
\end{aligned}$$

and we obtain the alternate form

$$\frac{\hbar^2}{2mc^2}\frac{\partial^2 f}{\partial t^2} + i\hbar \frac{\partial f}{\partial t} + \left\{ E - \langle V \rangle - \langle T \rangle - \frac{3}{2}mc^2 \right\} f = 0 \tag{10.36}$$

We found from the separation of variables procedure for our proposed relativistic Schrödinger equation the time independent equation intact, and the time equation (8.101) here we have converted back to $c \neq 1$ (multiply mass and time by c, and divide energy by c)

$$\frac{\hbar^2}{2mc^2}\frac{d^2\chi}{dt^2} + i\hbar \frac{d\chi}{dt} + E\chi = 0 \tag{10.37}$$

We see that (10.33) alternately (10.36) have the same form as (10.37) so rewriting (10.33) as

$$\frac{\hbar^2}{2mc^2}\frac{\partial^2 f}{\partial t^2}+i\hbar\frac{\partial f}{\partial t}+\alpha f=0 \tag{10.38}$$

Given f satisfies this equation and ψ satisfies the time independent equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi=E\psi \tag{10.39}$$

we ask what partial differential equation must $\Psi=f\psi$ satisfy and what must α be? It is tempting to immediately equate

$$\left\langle\left(\frac{T}{c}\right)^2\right\rangle=m^2c^2+2mE \tag{10.40}$$

or equivalently

$$\langle V\rangle+\langle T\rangle=-\frac{3}{2}mc^2 \tag{10.41}$$

We see the relation between the expression for $\left\langle\left(\frac{T}{c}\right)^2\right\rangle$ and equation (8.115) where again we've converted to $c\neq 1$

$$M\left(E\right)=\frac{1}{c}\sqrt{m^2c^2+2mE} \tag{10.42}$$

Where we would obtain

$$\left\langle\left(\frac{T}{c}\right)^2\right\rangle=M^2\left(E\right)c^2 \tag{10.43}$$

Therefore let us suppose that $\alpha\neq E$, then there exists ϵ such that $\epsilon\alpha=E$. Multiplying () by ϵ and dividing by f we obtain

$$-\frac{\epsilon\hbar^2}{2mc^2}\frac{1}{f}\frac{\partial^2 f}{\partial t^2}-i\epsilon\hbar\frac{1}{f}\frac{\partial f}{\partial t}=E=-\frac{\hbar^2}{2m}\frac{1}{\psi}\nabla^2\psi+V \tag{10.44}$$

Multiplying both sides by $\Psi=f\psi$ we obtain

$$-\frac{\epsilon\hbar^2}{2mc^2}\frac{\partial^2\Psi}{\partial t^2}-i\epsilon\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\Psi+V\Psi \tag{10.45}$$

Rearranging terms we have

$$-i\epsilon\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\left(\nabla^2-\frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi+V\Psi \tag{10.46}$$

If we replace the left hand side with $\eta^\mu=(-\epsilon c,0,0,0)$ we have

$$i\hbar\eta^\mu\frac{\partial\Psi}{\partial x^\mu}=-\frac{\hbar^2}{2m}\left(\nabla^2-\frac{\epsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi+V\Psi \tag{10.47}$$

Now the requirement that the equation be Lorentz invariant, and that η^μ be an acceptable proper velocity vector implies that $\epsilon=1$, thus we obtain (8.5)

$$i\hbar\eta^\mu\frac{\partial\Psi}{\partial x^\mu}=-\frac{\hbar^2}{2m}\partial^\nu\partial_\nu\Psi+V\Psi \tag{10.48}$$

in addition to relations (10.40), (10.41) and (10.43) above. We note that in the derivation of the classical Schrödinger equation when we asked, what partial differential equation does $\exp\left(-i\omega t\right)\psi$

satisfy such that $E = \hbar\omega$, we implicitly defined the role of the operator $i\hbar\frac{\partial}{\partial t}$ so that its expectation be the total energy eigenvalue, which of course we already knew from the time independent equation. In defining a new role for this operator, that its expectation be the combined rest plus kinetic energy, we obtain instead the relativistic Schrödinger equation. If this equation has merit, then apparently realtivistic quantum mechanics does not require a relativistic analogue of the hamiltonian, it remains simply the sum of the kinetic and potential energies.

Remark 13. We were tempted in the preceding to write

$$E = R + T + V \tag{10.49}$$

where upon taking the expectation of both sides where $\langle E\rangle = E$ and $\langle R\rangle = mc^2$ one would obtain

$$E = mc^2 + \langle T\rangle + \langle V\rangle \tag{10.50}$$

Then with (10.41) above, one would find the untenable result

$$E = -\frac{1}{2}mc^2 \tag{10.51}$$

Untenable here, since E represents the total energy eigenvalue (not because it contradicts the famous equation where E represents the combined rest and kinetic energy), and cannot be restricted to that single value. Therefore (10.49) is simply an invalid expression where at least V is a random variable, if the spatial position is taken so. All we do have is

$$E = -\frac{\hbar^2}{2m}\left\langle\nabla^2\right\rangle + \langle V\rangle \tag{10.52}$$

$$E = \frac{1}{2m}\left\langle\left(\frac{T}{c}\right)^2\right\rangle - \frac{1}{2}mc^2 \tag{10.53}$$

$$\langle V\rangle + \langle T\rangle = -\frac{3}{2}mc^2 \tag{10.54}$$

11. The Relativistic Schrödinger Equation for a System of Two Particles

We would like to derive a relativistic equation for two particles following the manner of the preceding Section 10.2 where we assumed a time independent potential at the outset. But a time independent potential for two particles in the relativistic context seems curious for in the context of classical mechanics a time independent potential for two particles is expressed

$$V = V_1(\mathrm{r}_1) + V_2(\mathrm{r}_2) + V_{12}(\mathrm{r}_1, \mathrm{r}_2) \tag{11.1}$$

where V_1 represents the first particle's interaction with the outside universe, V_2 represents the second particle's interaction with the outside universe, and V_{12} represents their interaction with each other and where we cannot write $V_{12}(\mathrm{r}_1, \mathrm{r}_2) = V_I(\mathrm{r}_1) + V_{II}(\mathrm{r}_2)$. We say this seems odd since suppose the first particle is located at r_1 at time $t = t_1$ and the second particle is located at r_2 at some other time $t = t_2$ with respect to some observer. Are we to say that there is an interaction between them acting across time? Relativistically the two particle can only influence each other when the interval

$$I = \|\mathrm{r}_1 - \mathrm{r}_2\|^2 - c^2(t_1 - t_2) \tag{11.2}$$

is timelike, $I<0$, or light-like, $I=0$, otherwise when the interval is spacelike, $I>0$, they can have no influence upon each other. Let us now consider a potential of interaction only of the form

$$V_{12}=\begin{cases} V\left(x^{\mu},y^{\nu}\right) & \text{when} \quad \left(x_{\mu}-y_{\mu}\right)\left(x^{\mu}-y^{\mu}\right)\leq 0 \\ 0 & \text{when} \quad \left(x_{\mu}-y_{\mu}\right)\left(x^{\mu}-y^{\mu}\right)>0 \end{cases} \tag{11.3}$$

Such a potential depends on two independent time variables, x^0 and y^0, and the only way for such a potential to be independent of both time variables, is if the covariant coefficients of both vanish in some reference frame. Is this possible? Well in one spatial dimension where $\Delta x^{\mu}=x^{\mu}-y^{\mu}$, consider the expression

$$\Delta x^{\mu}a_{\mu\nu}\Delta x^{\mu}=\Delta x^{1}a_{11}\Delta x^{1}+\Delta x^{1}a_{10}\Delta x^{0}+\Delta x^{0}a_{01}\Delta x^{1}+\Delta x^{0}a_{00}\Delta x^{0}it \tag{11.4}$$

is certainly possible for this expression to be time independent with $a_{11}=1$, and $a_{01}=a_{10}=a_{00}=0$, thus with

$$a_{\mu\nu}=\begin{cases} 1 & \text{for} \quad \mu=\nu\neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{11.5}$$

we may now express the Coulomb potential as

$$V=\begin{cases} \frac{q_1q_2}{\sqrt{\left(x^{\mu}-y^{\mu}\right)a_{\mu\nu}\left(x^{\nu}-y^{\nu}\right)}} & \text{when} \quad \left(x_{\mu}-y_{\mu}\right)\left(x^{\mu}-y^{\mu}\right)\leq 0 \\ 0 & \text{when} \quad \left(x_{\mu}-y_{\mu}\right)\left(x^{\mu}-y^{\mu}\right)>0 \end{cases} \tag{11.6}$$

With the existence of time independent relativistic potentials of the form (11.3) let us take the time independent Schrodinger Equation for two particles as a given

$$-\left(\frac{\hbar^2}{2m}\partial^i\partial_i+\frac{\hbar^2}{2M}\partial^j\partial_j\right)\psi\left(x^i,y^j\right)+V\psi\left(x^i,y^j\right)=E\psi\left(x^i,y^j\right) \tag{11.7}$$

and suppose

$$\Psi\left(x^{\mu},y^{\nu}\right)=\chi\left(x^0,y^0\right)\psi\left(x^i,y^j\right) \tag{11.8}$$

where we will define the role of the operators in hopes of finding the appropriate PDE for big Ψ. But first let us do some educated guesswork to arrive at the equation we find attractive. First let us suppose that χ has unit modulus so that the joint probability density for both particles is time independent

$$\Psi^*\Psi=\psi^*\left(x^i,y^j\right)\psi\left(x^i,y^j\right) \tag{11.9}$$

and hence so are the marginal densities for each particle. Therefore we ascribe no motion to either marginal density and assume its proper velocity is given by let us say $-\eta^{\mu}=(c,0,0,0)$ where we now take for simplicity $c=1$, $x^0=t$, and $y^0=s$. If the equation we wish to obtain is

$$-i\hbar\left(\frac{\partial\Psi}{\partial x^0}+\frac{\partial\Psi}{\partial y^0}\right)=-\frac{\hbar^2}{2m}\square_x^2\Psi-\frac{\hbar^2}{2M}\square_y^2\Psi+V\Psi \tag{11.10}$$

then separation of variables leads to the time independent Schrödinger equation for two particles

$$-\frac{\hbar^2}{2m}\nabla_x^2\psi-\frac{\hbar^2}{2M}\nabla_y^2\psi+V\psi=E\psi \tag{11.11}$$

and the following equation

$$\frac{\hbar^2}{2m}\frac{\partial^2\chi}{\partial t^2}+i\hbar\frac{\partial\chi}{\partial t}+\frac{\hbar^2}{2M}\frac{\partial^2\chi}{\partial s^2}+i\hbar\frac{\partial\chi}{\partial s}+E\chi=0 \tag{11.12}$$

where we now assume $\chi = f(t)\, g(s)$ and apply separation of variables to this equation to obtain for separation constant, α.

$$\frac{\hbar^2}{2m}\frac{\partial^2 f}{\partial t^2} + i\hbar\frac{\partial f}{\partial t} + (E-\alpha) f = 0 \tag{11.13}$$

$$\frac{\hbar^2}{2m}\frac{g}{\partial t^2} + i\hbar\frac{\partial g}{\partial t} + \alpha g = 0 \tag{11.14}$$

Assuming that the separation constant is real we require

$$E - \alpha \geq -\frac{1}{2}mc^2 \tag{11.15}$$

$$\alpha \geq -\frac{1}{2}Mc^2 \tag{11.16}$$

or

$$-\frac{1}{2}Mc^2 \leq \alpha \leq E + \frac{1}{2}mc^2 \tag{11.17}$$

since the solutions are given by

$$f(t) = \exp(ia_0 t) \tag{11.18}$$

$$g(t) = \exp(ib_0 s) \tag{11.19}$$

where

$$-\hbar a_0 = mc + \sqrt{m^2c^2 + 2m(E-\alpha)} \tag{11.20}$$

$$-\hbar b_0 = Mc + \sqrt{M^2c^2 + 2M\alpha} \tag{11.21}$$

Now we have no guidance in choosing a value for the separation constant, and we may as well view the conbined rest and kinetic energy of one of the particles to be simply its rest energy, if an observer were stationary with respect to that particle. Thus let us say $\alpha = -\frac{1}{2}Mc^2$, then

$$-\hbar b_0 = Mc \tag{11.22}$$

$$-\hbar a_0 = mc + \sqrt{m^2c^2 + 2m\left(E + \frac{1}{2}Mc^2\right)} \tag{11.23}$$

where if we write

$$\xi mc^2 = E + \frac{1}{2}Mc^2 \tag{11.24}$$

then

$$-\hbar a_0 = mc\left(1 + \sqrt{1+2\xi}\right) \tag{11.25}$$

12. The Relativistic Schrodinger Equation in 3 Dimensions

12.1. **Separation of Variables Continued.**

In Section we found that the if the potential is time independent, then the relativistic Schrödinger equation

$$i\hbar\eta^\mu\partial_\mu\Psi = -\frac{\hbar^2}{2m}\partial^\mu\partial_\mu\Psi + V(x^\mu)\Psi \tag{12.1}$$

separated into

$$\frac{\hbar^2}{2m}\frac{d^2\chi}{d\left(x^0\right)^2}+i\hbar\frac{d\chi}{d\left(x^0\right)}+E\chi=0 \tag{12.2}$$

with solution

$$\chi\left(x^0\right)=\exp\left(-i\frac{m+M\left(E\right)}{\hbar}x^0\right) \tag{12.3}$$

for

$$M\left(E\right)=\frac{1}{c}\sqrt{m^2c^2+2mE} \tag{12.4}$$

and the time independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\partial^i\partial_i\psi\left(\mathrm{r}\right)+\left(V-E\right)\psi\left(\mathrm{r}\right)=0 \tag{12.5}$$

recalling the convention that latin indices range from 1 to 3, and greek indices range from 0 to 3. In the case that the potential is zero, the solution to the time independent equation are plane waves of the form

$$\psi\left(\mathrm{r}\right)=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\exp\left[i\left(a_1x^1+a_2x^2+a_3x^3\right)\right] \tag{12.6}$$

where

$$\begin{aligned} E &= \frac{\hbar^2\left\|a\right\|^2}{2m} && (12.7)\\ &= \frac{\hbar^2}{2m}\left[\left(a_1\right)^2+\left(a_2\right)^2+\left(a_3\right)^2\right] && (12.8)\\ &= E_1+E_2+E_3 && (12.9)\end{aligned}$$

For a non zero potential provided it can be expressed

$$V\left(x^1,x^2,x^3\right)=V_1\left(x^1\right)+V_2\left(x^2\right)+V_3\left(x^3\right)=V_i\left(x^i\right) \tag{12.10}$$

the time independent equation can be further separated as

$$\left[-\frac{\hbar^2}{2m}\partial^1\partial_1+V_1\left(x^1\right)\right]\psi+\left[-\frac{\hbar^2}{2m}\partial^2\partial_2+V_2\left(x^2\right)\right]\psi+\left[-\frac{\hbar^2}{2m}\partial^3\partial_3+V_3\left(x^3\right)\right]\psi=E\psi \tag{12.11}$$

Expressing the solution as a product

$$\psi=\psi_1\left(x^1\right)\psi_2\left(x^2\right)\psi_3\left(x^3\right) \tag{12.12}$$

we obtain the independent equations

$$\begin{aligned} \left[-\frac{\hbar^2}{2m}\partial^1\partial_1+V_1\left(x^1\right)\right]\psi_1 &= E_1\psi_1 && (12.13)\\ \left[-\frac{\hbar^2}{2m}\partial^2\partial_2+V_2\left(x^2\right)\right]\psi_2 &= E_2\psi_2 && (12.14)\\ \left[-\frac{\hbar^2}{2m}\partial^3\partial_3+V_3\left(x^3\right)\right]\psi_1 &= E_3\psi_3 && (12.15)\end{aligned}$$

which results in

$$E=E_1+E_2+E_3 \tag{12.16}$$

12.2. **Particle in a Box.** We found in Section 9.5 where we discussed the infinite square well the solution

$$\psi\left(x^{1}\right)=\sqrt{\frac{2}{L_{1}}}\sin\left(\frac{n_{1}\pi}{L_{1}}x^{1}\right) \tag{12.17}$$

where we measured the total energy in units of rest mass

$$E_{1}\left(n_{1}\right)=\xi_{1}\left(n_{1}\right)mc^{2} \tag{12.18}$$

and

$$\xi_{1}\left(n_{1}\right)=\frac{1}{2}\left(\frac{\hbar\pi n_{1}}{mcL_{1}}\right)^{2} \tag{12.19}$$

Then for a particle in a box of side lengths $L_1 \times L_2 \times L_3$, the three dimensional solution

$$\psi\left(\mathrm{r}\right)=\sqrt{\frac{2^{3}}{L_{1}L_{2}L_{3}}}\sin\left(a_{.1}^{1.}x^{1}\right)\sin\left(a_{.2}^{2.}x^{2}\right)\sin\left(a_{.3}^{3.}x^{3}\right) \tag{12.20}$$

with

$$a_{.k}^{k.}=\frac{n_{k}\pi}{L_{k}} \tag{12.21}$$

and

$$\begin{aligned}
E &= E_{1}\left(n_{1}\right)+E_{2}\left(n_{2}\right)+E_{3}\left(n_{3}\right) && (12.22)\\
&= \left[\xi_{1}\left(n_{1}\right)+\xi_{2}\left(n_{2}\right)+\xi_{2}\left(n_{2}\right)\right]mc^{2} && (12.23)\\
&= \xi mc^{2} && (12.24)
\end{aligned}$$

The time dependent phase factor was found to be

$$\begin{aligned}
\chi\left(x^{0}\right) &= \exp\left(i\alpha_{0}x^{0}\right) && (12.25)\\
\alpha_{0} &= -\frac{mc}{\hbar}\left(1+\sqrt{1+2\xi}\right) && (12.26)\\
M\left(E\right) &= m\sqrt{1+2\xi} && (12.27)
\end{aligned}$$

So that at rest with respect to the box we find

$$\Psi\left(x^{0},x^{1},x^{2},x^{3}\right)=\sqrt{\frac{2^{3}}{L_{1}L_{2}L_{3}}}\exp\left(i\alpha_{0}x^{0}\right)\sin\left(a_{.1}^{1.}x^{1}\right)\sin\left(a_{.2}^{2.}x^{2}\right)\sin\left(a_{.3}^{3.}x^{3}\right) \tag{12.28}$$

which we can recast as a Lorentz invariant wave function

$$\Psi\left(x^{0},x^{1},x^{2},x^{3}\right)=\sqrt{\frac{2^{3}}{L_{1}L_{2}L_{3}}}\exp\left(i\alpha_{\mu}x^{\mu}\right)\sin\left(a_{.\nu}^{1.}x^{\nu}\right)\sin\left(a_{.\sigma}^{2.}x^{\sigma}\right)\sin\left(a_{.\rho}^{3.}x^{\rho}\right) \tag{12.29}$$

In the frame at rest with the box we have

$$a_{.\mu}^{i.}=\begin{cases}\frac{n_{i}\pi}{L_{i}} & \text{for} \quad i=\mu\\ 0 & \text{for} \quad i\neq\mu\end{cases} \tag{12.30}$$

and

$$\alpha_{\mu}=\begin{cases}-\frac{mc}{\hbar}\left(1+\sqrt{1+2\xi}\right) & \text{for} \quad 0=\mu\\ 0 & \text{for} \quad 0\neq\mu\end{cases} \tag{12.31}$$

Now let us examine the unnormalized density

$$\Psi^*\Psi = \sin^2\left(b^{1}_{\cdot\nu}y^\nu\right)\sin^2\left(b^{2}_{\cdot\sigma}y^\sigma\right)\sin^2\left(b^{3}_{\cdot\rho}y^\rho\right) \tag{12.32}$$

where

$$y^\mu = \Lambda^{\mu}_{\cdot\nu}x^\nu \tag{12.33}$$

$$b^{i}_{\cdot\mu} = \Lambda^{\cdot\nu}_{\mu\cdot}a^{i}_{\cdot\nu} \tag{12.34}$$

We can apply equation (8.241) to determine the relation between the velocity components, β^i, and the dual vectors, $b^{i}_{\cdot\mu}$, where we can write

$$\beta^i\partial_i\Psi^*\Psi = -\partial_0\Psi^*\Psi \tag{12.35}$$

We have

$$\partial_\mu\Psi^*\Psi = A_i\left(x^\nu\right)b^{i}_{\cdot\mu} \tag{12.36}$$

where we will not write out the functions $A_i\left(x^\nu\right)$ so that

$$\beta^j A_i\left(x^\nu\right)b^{i}_{\cdot j} = -A_i\left(x^\nu\right)b^{i}_{\cdot 0} \tag{12.37}$$

which becomes

$$\beta^j b^{i}_{\cdot j} = -b^{i}_{\cdot 0} \tag{12.38}$$

Let us also examine the normalization constant for the case where the box is not at rest relative to some observer

$$\iiint_V \Psi^*\Psi dV = \iiint_V \sin^2\left(b^{1}_{\cdot\nu}y^\nu\right)\sin^2\left(b^{2}_{\cdot\sigma}y^\sigma\right)\sin^2\left(b^{3}_{\cdot\rho}y^\rho\right)dy^1dy^2dy^3 \tag{12.39}$$

$$= \int_0^{n_3\pi}\int_0^{n_2\pi}\int_0^{n_1\pi}\sin^2\left(u^1\right)\sin^2\left(u^2\right)\sin^2\left(u^3\right)\left|\frac{\partial y^i}{\partial u^j}\right|du^1du^2du^3 \tag{12.40}$$

where the Jacobian is given by

$$\left|\frac{\partial y^i}{\partial u^j}\right| = \left|b^{i}_{\cdot j}\right|^{-1} \tag{12.41}$$

$$= \left|\Lambda^{\cdot\nu}_{j\cdot}a^{i}_{\cdot\nu}\right|^{-1} \tag{12.42}$$

$$= \left|\Lambda^{\cdot k}_{j\cdot}a^{i}_{\cdot k}\right|^{-1} \tag{12.43}$$

$$= \left|\Lambda^{\cdot k}_{j\cdot}\right|^{-1}\left|a^{i}_{\cdot k}\right|^{-1} \tag{12.44}$$

$$= \gamma^{-1}\frac{L_1L_2L_3}{n_1n_2n_3\pi^3} \tag{12.45}$$

where we were able to eliminate the index, $\nu = 0$, after the second step above, since $a^{i}_{\cdot 0} = 0$. The integral

$$\int_0^{n_i\pi}\sin^2\left(u^i\right)du^i = \frac{n_i\pi}{2} \tag{12.46}$$

so that

$$\iiint_V \Psi^*\Psi dV = \frac{\gamma^{-1}L_1L_2L_3}{2^3} \tag{12.47}$$

thus the general form of the packet is given by

$$\Psi\left(y^{\tau}\right)=\sqrt{\frac{\gamma 2^{3}}{L_1 L_2 L_3}}\exp\left(ib^{0\cdot}_{\cdot\mu}y^{\mu}\right)\sin\left(b^{1\cdot}_{\cdot\nu}y^{\nu}\right)\sin\left(b^{2\cdot}_{\cdot\sigma}y^{\sigma}\right)\sin\left(b^{3\cdot}_{\cdot\rho}y^{\rho}\right) \tag{12.48}$$

with

$$\beta^{j}b^{i\cdot}_{\cdot j} = -b^{i\cdot}_{\cdot 0} \tag{12.49}$$

$$y^{\mu} = \Lambda^{\mu\cdot}_{\cdot\nu}x^{\nu} \tag{12.50}$$

$$b^{i\cdot}_{\cdot\mu} = \Lambda^{\cdot\nu}_{\mu\cdot}a^{i\cdot}_{\cdot\nu} \tag{12.51}$$

$$b^{0\cdot}_{\cdot\mu} = \Lambda^{\cdot\nu}_{\mu\cdot}\alpha_{\nu} \tag{12.52}$$

12.3. **Spherically Symmetric Potentials.**

12.3.1. *The Coulomb Potential.* Before we express the Laplacian in spherical coordinates let us consider a spherically symmetric potential, the Coulomb potential for a for a system to two charged particles, one of mass M and charge Q at position r_1, and the second of mass, m and charge q, at r_2. Then the potential is given by

$$V=\frac{qQ}{\|\mathrm{r}_2-\mathrm{r}_1\|}=\frac{qQ}{r} \tag{12.53}$$

which is time independent. But let us consider this carefully in the relativistic context, we have the Minkoski force on charge q given by

$$K^{\mu}=q\eta_{\nu}F^{\mu\nu} \tag{12.54}$$

where the spatial components of K^{μ} are given by

$$\mathrm{K} = \gamma q\left[\mathrm{E}+(\mathrm{v}\times\mathrm{B})\right] \tag{12.55}$$

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^{2}}} \tag{12.56}$$

where v is the velocity of the test charge q, and E and B are due to charge Q. If we observe Q at rest, then $\mathrm{B}=0$, and where $\mathrm{r}_1=0$ we find

$$\mathrm{E}=\frac{Q\hat{\mathrm{r}}}{r^{2}} \tag{12.57}$$

hence the force acting on q is given by

$$\mathrm{F}=\frac{\gamma qQ\hat{\mathrm{r}}}{r^{2}} \tag{12.58}$$

Let us now find the potential from

$$\mathrm{F}=-\nabla V \tag{12.59}$$

We have

$$V = -\int_{\infty}^{r} \mathrm{F} \cdot d\mathrm{r} \tag{12.60}$$

$$= -\int_{\infty}^{r} \frac{\gamma q Q}{r^2} \hat{\mathrm{r}} \cdot d\mathrm{r} \tag{12.61}$$

$$= -\int_{\infty}^{r} \frac{\gamma q Q}{r^3} \mathrm{r} \cdot d\mathrm{r} \tag{12.62}$$

but $d(\mathrm{r}\cdot\mathrm{r}) = 2\mathrm{r}\cdot d\mathrm{r}$ and $d\left(r^2\right) = 2rdr$ so that $\mathrm{r}\cdot d\mathrm{r} = rdr$ so that

$$V = -\int_{\infty}^{r} \frac{\gamma q Q}{r^2} dr \tag{12.63}$$

and we obtain

$$V(r) = \frac{\gamma q Q}{r} \tag{12.64}$$

which is also time independent for constant velocity of the test charge q. Now we ask a question, since the quantum mechanical analysis of the hydrogen atom uses expression (12.53) for the coulomb potential without the appearance of γ, is that merely an approximation to the relativistic expression just obtained, or is it exactly the correct expression? We assume the latter, since in the derivation of the two particle Relativistic Schrödinger equation of Section 11 for a time independent potential, we found the joint probabilty density of the two particle system to be independent of time, and hence the marginal densities for both particles, and in particular for the test charge q, to be independent of time. Hence we take the velocity of the test charge, $\mathrm{v} = 0$ in which case $\gamma = 1$. Therefore we take expression (12.53) for the Coulomb potential to be the correct expression in the relativistic context. This may seem like a bit of hand waving, and possibly circular. What we should say is that if expression (12.64) is time independent, then $\gamma = 1$, since the proper velocity of $\Psi^*\Psi$ is given by, $-\eta^{\mu} = (c, 0, 0, 0)$.

12.3.2. *The Time Independent Schrödinger Equation for Two Particles with a Spherically Symmetric Potential.* In Section 11on The Two Particle Relativistic Schrodinger Equation we recovered by separation of variables the Time Independent Schrödinger equation intact

$$-\frac{\hbar^2}{2m}\nabla_x^2\psi - \frac{\hbar^2}{2M}\nabla_y^2\psi + V\psi = E\psi \tag{12.65}$$

We recovered this equation from the relativistic equation (11.10) which involved two independent time variables, where the potential was independent of both these variables. Suppose further the this potential is spherically symmetric, then it is convenient convert the time independent equation to center of mass andrelative coordinates. The position vector for the center of mass is given by

$$M\left\|\mathrm{y} - \mathrm{R}\right\| = m\left\|\mathrm{R} - \mathrm{x}\right\| \tag{12.66}$$

so that

$$\mathrm{R} = \frac{M\mathrm{y} + m\mathrm{x}}{M + m} \tag{12.67}$$

Classically the kinetic energy of the center of mass is given by

$$T = \frac{1}{2}(M + m)\dot{\mathrm{R}}^2 \tag{12.68}$$

and the kinetic energy of both particles about the center of mass is given by

$$T' = \frac{1}{2}M\left\|\dot{\mathrm{y}} - \dot{\mathrm{R}}\right\|^2 + \frac{1}{2}m\left\|\dot{\mathrm{R}} - \dot{\mathrm{x}}\right\|^2$$

Then with equation (12.67) we find

$$\mathrm{y} - \mathrm{R} = \frac{m}{M+m}(\mathrm{y} - \mathrm{x}) = -\frac{m}{M+m}\mathrm{r} \tag{12.69}$$

$$\mathrm{R} - \mathrm{x} = \frac{M}{M+m}(\mathrm{y} - \mathrm{x}) = -\frac{M}{M+m}\mathrm{r} \tag{12.70}$$

so that

$$\|\mathrm{y} - \mathrm{R}\| = \frac{m}{M+m}\|\mathrm{r}\| \tag{12.71}$$

$$\|\mathrm{R} - \mathrm{x}\| = \frac{M}{M+m}\|\mathrm{r}\| \tag{12.72}$$

Now taking the time derivatives of (12.69) and (12.70) and squaring the magnitudes, the kinetic energy about the center of mass becomes

$$T' = \frac{1}{2}\frac{Mm}{M+m}\dot{\mathrm{r}}^2 \tag{12.73}$$

So that the classical Hamiltonian for this system becomes

$$\mathcal{H} = T + T' + V \tag{12.74}$$

$$= \frac{1}{2}(M+m)\dot{\mathrm{R}}^2 + \frac{1}{2}\mu\dot{\mathrm{r}}^2 + V \tag{12.75}$$

$$= \frac{\mathrm{P}^2}{2(M+m)} + \frac{\mathrm{p}^2}{2\mu} + V \tag{12.76}$$

where the reduced mass of the system is given by

$$\mu = \frac{Mm}{M+m} \tag{12.77}$$

Now as discussed in both Sections 10.2 and 9.6.2 we may use the traditional correspondence principle despite the relativistic setting where the roles of the operators have changed to obtain the Hamiltonian for the time independent equation thus we obtain

$$-\frac{\hbar^2}{2(M+m)}\nabla^2_{cm}\psi - \frac{\hbar^2}{2\mu}\nabla^2_r\psi + V(\mathrm{r})\psi = E\psi \tag{12.78}$$

Now this equation is separable with $\psi = \psi_{cm} \cdot \psi_r$ and $E = E_{cm} + E_r$ and we will concentrate on the equation

$$\frac{\hbar^2}{2\mu}\nabla^2_r\psi_r + V(\mathrm{r})\psi_r = E_r\psi_r \tag{12.79}$$

Now since the potential is spherically symmetric it is convenient to change from cartesian coordinates to spherical where the Laplacian is given by

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

where the angle, ϕ is the azimuth, and θ is the zenith. The transformation equations are given by

$$x^1 = r\sin\theta\cos\phi \tag{12.80}$$
$$x^2 = r\sin\theta\sin\phi \tag{12.81}$$
$$x^3 = r\cos\theta \tag{12.82}$$

and

$$r = \sqrt{\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2} \tag{12.83}$$
$$\theta = \arccos\left(\frac{x^3}{r}\right) \tag{12.84}$$
$$\phi = \arctan\left(\frac{x^1}{x^2}\right) \tag{12.85}$$

Now we write the Laplacian in spherical coordinates in terms of a radial operator, $\mathcal{R}$ and an angular operator, $\mathcal{L}^2$

$$\nabla^2 = \mathcal{R} + \frac{1}{r^2}\mathcal{L}^2 \tag{12.86}$$

where

$$\mathcal{R} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \tag{12.87}$$
$$\mathcal{L}^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{12.88}$$

The equation (12.79) for the relative coordinates is again separable with respect to these operators into

$$r^2\mathcal{R}R\left(r\right) + \frac{2\mu r^2}{\hbar^2}\left[E - V\left(r\right)\right]R\left(r\right) = \Lambda R\left(r\right) \tag{12.89}$$

and

$$\mathcal{L}^2 Y\left(\theta,\phi\right) = -\Lambda Y\left(\theta,\phi\right) \tag{12.90}$$

12.4. **Angular Momentum.** The classical definition of the angular momentum vector is

$$\mathrm{L} = \mathrm{r}\times\mathrm{p} \tag{12.91}$$

whose components are

$$L^1 = x^2p^3 - x^3p^2 \tag{12.92}$$
$$L^2 = x^3p^1 - x^1p^3 \tag{12.93}$$
$$L^3 = x^1p^2 - x^2p^3 \tag{12.94}$$

where we replace the momentum components with the corresponding operators so that the commutator relations are given by

$$\left[L^1, L^2\right] = i\hbar L^3 \tag{12.95}$$
$$\left[L^2, L^3\right] = i\hbar L^1 \tag{12.96}$$
$$\left[L^3, L^1\right] = i\hbar L^2 \tag{12.97}$$

which can be remembered with the mnemonic expression

$$\mathrm{L} \times \mathrm{L} = i\hbar \mathrm{L}$$

Now this suggests we place these quantities into a rank 2 contravariant skew-symmetric tensor

$$L^{\mu\nu} = \begin{bmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & L^3 & -L^2 \\ -K^2 & -L^3 & 0 & L^1 \\ -K^3 & L^2 & -L^1 & 0 \end{bmatrix} \tag{12.98}$$

with the new found role of the operator, $-i\hbar\partial^0 = i\hbar\partial_0$ representing the combined rest and kinetic energy (divided by the speed of light, c) the likely candidates for the quantities K_i are

$$K^1 = x^0p^1 - x^1p^0 \tag{12.99}$$
$$K^2 = x^0p^2 - x^2p^0 \tag{12.100}$$
$$K^3 = x^0p^3 - x^3p^0 \tag{12.101}$$

In terms of operators we have

$$L^{\mu\nu} = -i\hbar \left(x^\mu\partial^\nu - x^\nu\partial^\mu\right) \tag{12.102}$$

Now do these transform correctly according to

$$L^{'\mu\nu} = \sum_{\lambda=0}^{3}\sum_{\sigma=0}^{3} \Lambda^{\mu\cdot}_{\cdot\lambda}\Lambda^{\nu\cdot}_{\cdot\sigma}L^{\lambda\sigma} \tag{12.103}$$

where we take without loss of generality

$$\Lambda^\mu_\lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12.104}$$

Granted for any two ordinary contravariant 4-vectors x^μ and y^μ then $T^{\mu\nu} = x^\mu y^\nu - x^\nu y^\mu$ is a skew-symmetric rank 2 contravariant tensor, but when dealing with operators we have to pay attention to commutivity when verifying its tensor character. The complete set of transformation rules are given by (see [5] page 447)

$$\begin{array}{lll} L'^{01} = L^{01}, & L'^{02} = \gamma\left(L^{02} - \beta L^{12}\right), & L'^{03} = \gamma\left(L^{03} + \beta L^{31}\right) \\ L'^{23} = L^{23}, & L'^{31} = \gamma\left(L^{31} + \beta L^{03}\right), & L'^{12} = \gamma\left(L^{12} - \beta L^{02}\right) \end{array} \tag{12.105}$$

Now taking care not to commute operators

$$\begin{aligned} L'^{01} &= x'^0p'^1 - x'^1p'^0 \\ &= \gamma^2\left(x^0 - \beta x^1\right)\left(p^1 - \beta p^0\right) - \gamma^2\left(x^1 - \beta x^0\right)\left(p^0 - \beta p^1\right) \\ &= \gamma^2\left(x^0p^1 - \beta x^0p^0 - \beta x^1p^1 + \beta^2x^1p^0\right) - \gamma^2\left(x^1p^0 - \beta x^1p^1 - \beta x^0p^0 + \beta^2x^0p^1\right) \\ &= \gamma^2\left(x^0p^1 - x^1p^0\right) - \gamma^2\beta^2\left(x^0p^1 - x^1p^0\right) \\ &= \gamma^2\left(1 - \beta^2\right)\left(x^0p^1 - x^1p^0\right) \\ &= x^0p^1 - x^1p^0 = L^{01} \end{aligned} \tag{12.106}$$

$$\begin{aligned} L'^{02} &= x'^0 p'^2 - x'^2 p'^0 \\ &= \gamma\left(x^0 - \beta x^1\right) p^2 - \gamma x^2\left(p^0 - \beta p^1\right) \\ &= \gamma\left[\left(x^0 p^2 - x^2 p^0\right) - \beta\left(x^1 p^2 - x^2 p^1\right)\right] \\ &= \gamma\left(L^{02} - \beta L^{12}\right) \end{aligned} \tag{12.107}$$

$$\begin{aligned} L'^{03} &= x'^0 p'^3 - x'^3 p'^0 \\ &= \gamma\left[\left(x^0 - \beta x^1\right) p^3 - x^3\left(p^0 - \beta p^1\right)\right] \\ &= \gamma\left[\left(x^0 p^3 - x^3 p^0\right) - \beta\left(x^1 p^3 - x^3 p^1\right)\right] \\ &= \gamma\left[\left(x^0 p^3 - x^3 p^0\right) + \beta\left(x^3 p^1 - x^1 p^3\right)\right] \\ &= \gamma\left(L^{03} + \beta L^{31}\right) \end{aligned} \tag{12.108}$$

Now from electromagnetic theory we have the relation

$$F^{\mu\nu} F_{\mu\nu} = 2\left[\mathrm{B}^2 - \left(\frac{\mathrm{E}}{c}\right)^2\right] \tag{12.109}$$

We have a similar relation for the angular momentum tensor operator

$$L^{\mu\nu} L_{\mu\nu} = 2\left(\mathrm{L}^2 - \mathrm{K}^2\right) \tag{12.110}$$

where

$$\mathrm{L}^2 = \left(L^1\right)^2 + \left(L^2\right)^2 + \left(L^3\right)^2 \tag{12.111}$$

$$= \left(L^{23}\right)^2 + \left(L^{31}\right)^2 + \left(L^{12}\right)^2 \tag{12.112}$$

$$\mathrm{K}^2 = \left(K^1\right)^2 + \left(K^2\right)^2 + \left(K^3\right)^2 \tag{12.113}$$

$$= \left(L^{01}\right)^2 + \left(L^{02}\right)^2 + \left(L^{03}\right)^2 \tag{12.114}$$

Using the Levi-Civita symbol we have

$$L^i = \frac{1}{2}\epsilon_{\mu\nu}^{\ \ 0i} L^{\mu\nu} \tag{12.115}$$

$$K^i = L^{0i} \tag{12.116}$$

If we define the dual tensor oprerator

$$K_{\mu\nu} = \frac{1}{2}\epsilon_{\rho\sigma\mu\nu} L^{\rho\sigma} \tag{12.117}$$

then

$$K_{\mu\nu} L^{\mu\nu} = 4\mathrm{K}\cdot\mathrm{L} \tag{12.118}$$

where this dot product does commute as we see

$$\mathrm{K}\cdot\mathrm{L} = L^{01}L^{23} + L^{02}L^{31} + L^{03}L^{12} \tag{12.119}$$

We may rewrite the commutator relation for angular momentum

$$\left[L^i, L^j\right] = i\hbar L^{ij} \tag{12.120}$$

The cartesian components of angular momentum can be expressed in terms of spherical coordinates as

$$L^1 = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \tag{12.121}$$

$$L^2 = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \tag{12.122}$$

$$L^3 = -i\hbar\frac{\partial}{\partial\phi} \tag{12.123}$$

In the previous section we found that the Laplacian in spherical coordinates could be expressed in terms of a radial operator, $\mathcal{R}$, and an angular operator, $\mathcal{L}^2$ where we now find

$$\mathrm{L}^2 = -\hbar^2\mathcal{L}^2 \tag{12.124}$$

so that the angular equation (12.90) of the previous section becomes

$$\mathrm{L}^2 Y(\theta,\phi) = \hbar^2\Lambda Y(\theta,\phi) \tag{12.125}$$

We would like to solve this equation by an operator method as was done with the harmonic ocsillator in Section 9.6.2. Before we proceed there seems no reason we couldn't convert the angular equation back to cartesian coordinates, in any case if we need to perform the Lorentz transformation on the cartesian coordinate system, then we need to apply the rules of transformation to the skew-symmetric angular momentum tensor operator in order to obtain the correct components of the operator, L.

12.5. **Eigenvalues and Eigenfunctions of the Angular Momentum Operators.** To obtaine the eigenvalues of the angular momentum operators L^2 and L^3 let us define the raising and lowering operators

$$L^+ = L^1 + iL^2 \tag{12.126}$$

$$L^- = L^1 - iL^2 \tag{12.127}$$

so that

$$L^+L^- = \left(L^1\right)^2 + \left(L^2\right)^2 - i\left[L^1, L^2\right] \tag{12.128}$$

$$L^-L^+ = \left(L^1\right)^2 + \left(L^2\right)^2 + i\left[L^1, L^2\right] \tag{12.129}$$

$$\left[L^+, L^-\right] = -2i\left[L^1, L^2\right] = 2\hbar L^3 \tag{12.130}$$

Also making note of the relation

$$L^+L^- + L^-L^+ = 2\left[\left(L^1\right)^2 + \left(L^2\right)^2\right] \tag{12.131}$$

we find the expressions

$$\mathrm{L}^2 = \left(L^1\right)^2 + \left(L^2\right)^2 + \left(L^3\right)^2 \tag{12.132}$$

$$= \frac{1}{2}\left(L^+L^- + L^-L^+\right) + \left(L^3\right)^2 \tag{12.133}$$

$$= L^+L^- - \hbar L^3 + \left(L^3\right)^2 \tag{12.134}$$

$$= L^-L^+ + \hbar L^3 + \left(L^3\right)^2 \tag{12.135}$$

To proceed let us establish the commutation relations

$$\left[\mathrm{L}^2, L^i\right] = 0 \tag{12.136}$$

so let us write with the summation convention

$$\begin{aligned}
\left[\mathrm{L}^2, L^i\right] &= L^j L^j L^i - L^i L^j L^j \\
&= L^j \left(L^i L^j - \left[L^j, L^i\right]\right) - \left(L^j L^i - \left[L^i, L^j\right]\right) L^j \\
&= L^j \left[L^i, L^j\right] + \left[L^i, L^j\right] L^j \quad (12.137) \\
&= L^j i\hbar\epsilon_{ijk} L^k + i\hbar\epsilon_{ijk} L^k L^j \quad (12.138) \\
&= i\hbar \left(\epsilon_{ijk} L^j L^k + \epsilon_{ijk} L^k L^j\right) \quad (12.139) \\
&= i\hbar \left(\epsilon_{ijk} L^j L^k - \epsilon_{ikj} L^k L^j\right) \quad (12.140) \\
&= i\hbar \cdot 0 \quad (12.141)
\end{aligned}$$

where ϵ_{ijk} is the Levi-Civita symbol. Let us take $i = 3$, and then since L^2 and L^3 commute they share eigenfunctions so let us write

$$\mathrm{L}^2 Y^{(m)} = \hbar^2 \Lambda Y^{(m)} \tag{12.142}$$

$$L^3 Y^{(m)} = \hbar m Y^{(m)} \tag{12.143}$$

where the number m is to be determined and we do not assume at this point that m is an integer. We need to take care not to confuse exponents, quantum numbers and tensor indices. The first of the previous equations is just equation (12.125) labeled with the eigenvalue, sans the factor $\hbar$, of the projection equation, with L^3 the projection of L onto the z-axis. So let us now operate on the second equation above with the operator, L^+

$$L^+ L^3 Y^{(m)} = \hbar m L^+ Y^{(m)} \tag{12.144}$$

Now we shall also need the commuation relations

$$\left[L^3, L^+\right] = \hbar L^+ \tag{12.145}$$

$$\left[L^3, L^-\right] = -\hbar L^- \tag{12.146}$$

$$\left[\mathrm{L}^2, L^+\right] = \left[\mathrm{L}^2, L^-\right] = 0 \tag{12.147}$$

and with the first of these we find

$$L^+ L^3 Y^{(m)} = \left(L^3 L^+ - \hbar L^+\right) Y^{(m)} \tag{12.148}$$

hence

$$L^3 L^+ Y^{(m)} = \hbar \left(m + 1\right) L^+ Y^{(m)} \tag{12.149}$$

thus if $Y^{(m)}$ is an eigenfuction of the projection equation, then so is $L^+ Y^{(m)}$. With repeated applications we find

$$L^3 \left[\left(L^+\right)^r Y^{(m)}\right] = \hbar \left(m + r\right) \left[\left(L^+\right)^r Y^{(m)}\right] \tag{12.150}$$

Similarly we find

$$L^3 \left[\left(L^-\right)^r Y^{(m)}\right] = \hbar \left(m - r\right) \left[\left(L^-\right)^r Y^{(m)}\right] \tag{12.151}$$

but as in the case of the harmonic oscillator there may be restrictions on the number of times the raising and lowering operators may be applied to the eigenfuction $Y^{(m)}$, which we will need to investigate. But first we note that for an arbitrary eigenfunction, $Y^{(m)}$, then

$$\mathrm{L}^2\left[L^+Y^{(m)}\right]=L^+\mathrm{L}^2Y^{(m)}=L^+\hbar^2\Lambda Y^{(m)}=\hbar^2\Lambda\left[L^+Y^{(m)}\right] \tag{12.152}$$

and similarly

$$\mathrm{L}^2\left[L^-Y^{(m)}\right]=\hbar^2\Lambda\left[L^-Y^{(m)}\right] \tag{12.153}$$

12.6. **The Relativistic Angular Momentum Operators.** In the previous section we see that the operator, L^2 falls naturally out of the time independent Schrödinger equation when we consider a spherically symmetric potential and change to spherical coordinates. Its eigenfunctions are that same as the operator, L^3 though the eigenvalues differ. We would like to consider now the tensor operators $L^{\mu\nu}$, $K^{\mu\nu}$, $L^{\mu\nu}L_{\mu\nu}=2\left(\mathrm{L}^2-\mathrm{K}^2\right)$, and $K_{\mu\nu}L^{\mu\nu}=4\mathrm{K}\cdot\mathrm{L}$. Rather than concern ourselves with how they might be incorporated or result from the time independent Schrödinger equation, let us instead study these operators systematically, let us list their properties, determine their eigenfunctions and eigenvalues, their null spaces, their expectations and variance. Let us list what we know

(1) $L^{\mu\nu}=-i\hbar\left(x^\mu\partial^\nu-x^\nu\partial^\mu\right)$, the angular momentum tensor operator.
(2) $K_{\mu\nu}=\frac{1}{2}\epsilon_{\rho\sigma\mu\nu}L^{\rho\sigma}$, which is the dual tensor operator.
(3) $L^{\mu\nu}L_{\mu\nu}=2\left(\mathrm{L}^2-\mathrm{K}^2\right)$, which is an invariant scalar operator.
(4) $K_{\mu\nu}L^{\mu\nu}=4\mathrm{K}\cdot\mathrm{L}$, which is an invariant pseudoscalar operator.
(5) $\mathrm{K}\cdot\mathrm{L}=\mathrm{L}\cdot\mathrm{K}$, the dot product of the 3-vectors commute.
(6) $\mathrm{L}^2-\mathrm{K}^2=(\mathrm{L}-\mathrm{K})\cdot(\mathrm{L}+\mathrm{K})=(\mathrm{L}+\mathrm{K})\cdot(\mathrm{L}-\mathrm{K})$ which follow from the previous.

For an arbitrary complex variable, $a+ib$ we have $(a+ib)^2=\left(a^2-b^2\right)+2iab$, thus

$$2\left(\mathrm{L}+i\mathrm{K}\right)\cdot\left(\mathrm{L}+i\mathrm{K}\right)=2\left(\mathrm{L}^2-\mathrm{K}^2\right)+4i\mathrm{L}\cdot\mathrm{K} \tag{12.154}$$

and we may also write

$$\frac{1}{2}\left(L^{\mu\nu}+iK^{\mu\nu}\right)\left(L_{\mu\nu}+iK_{\mu\nu}\right)=2\left(\mathrm{L}^2-\mathrm{K}^2\right)+4i\mathrm{L}\cdot\mathrm{K} \tag{12.155}$$

Now following suit in defining the operators L^+ and L^- we define

$$K^+ = K^1+iK^2 \tag{12.156}$$

$$K^- = K^1-iK^2 \tag{12.157}$$

and we find

$$\left[K^+,K^-\right] = -2i\left[K^1,K^2\right] \tag{12.158}$$

$$\left[K^1,K^2\right] = i\hbar L^{12}=i\hbar L^3 \tag{12.159}$$

$$\left[K^2,K^3\right] = i\hbar L^{23}=i\hbar L^1 \tag{12.160}$$

$$\left[K^3,K^1\right] = i\hbar L^{31}=i\hbar L^2 \tag{12.161}$$

where $x^0\partial^0-\partial^0x^0=+1$ noting that this expression involves the contravariant partial with respect to time, hence we have

$$\left[K^+,K^-\right]=2\hbar L^{12}=2\hbar L^3 \tag{12.162}$$

and hopefully we have done the algebra correctly.

In addition to the skew-symmetric tensor operator $L^{\mu\nu}$ which is analogous to the electromagnetic field tensor, we may also define a symmetric tensor operator analogous to the electromagnetic stress-energy tensor thusly

$$H^{\mu\nu} = \begin{bmatrix} \frac{1}{2}\left(\mathrm{K}^2+\mathrm{L}^2\right) & \Sigma^1 & \Sigma^2 & \Sigma^3 \\ \Sigma^1 & -\sigma^{11} & -\sigma^{12} & -\sigma^{13} \\ \Sigma^2 & -\sigma^{21} & -\sigma^{22} & -\sigma^{23} \\ \Sigma^3 & -\sigma^{31} & -\sigma^{32} & -\sigma^{33} \end{bmatrix} \tag{12.163}$$

where

$$\Sigma = \mathrm{K}\times\mathrm{L} \tag{12.164}$$

and

$$\sigma^{ij} = K^iK^j + L^iL^j - \frac{1}{2}\left(\mathrm{K}^2+\mathrm{L}^2\right)\delta^{ij} \tag{12.165}$$

Now with an ordinary cross product we have $\mathrm{A}\times\mathrm{B} = -\mathrm{B}\times\mathrm{A}$ but we must pay special attention to the expression for Σ, since it contains conbinations such as $x\partial_x$ and $\partial_x x$, therefore let us determine $\mathrm{K}\times\mathrm{L}-\mathrm{L}\times\mathrm{K}$, we will examine the coefficient, a of $\hat{\mathrm{i}}$

$$\begin{aligned} K^2L^3 - K^3L^2 &= L^{02}L^{12} - L^{03}L^{31} \\ &= -\hbar^2\left[\left(x^0\partial^2 - x^2\partial^0\right)\left(x^1\partial^2 - x^2\partial^1\right) - \left(x^0\partial^3 - x^3\partial^0\right)\left(x^3\partial^1 - x^1\partial^3\right)\right] \end{aligned} \tag{12.166}$$

we also have

$$\begin{aligned} L^2K^3 - L^3K^2 &= L^{31}L^{03} - L^{12}L^{02} \\ &= -\hbar^2\left[\left(x^3\partial^1 - x^1\partial^3\right)\left(x^0\partial^3 - x^3\partial^0\right) - \left(x^1\partial^2 - x^2\partial^1\right)\left(x^0\partial^2 - x^2\partial^0\right)\right] \end{aligned} \tag{12.167}$$

and after quite a bit of algebra

$$a - a' = -2\hbar^2\left[x^0x^1\left(\partial^2\partial^2 + \partial^3\partial^3\right) + \left(x^2x^2 + x^3x^3\right)\partial^0\partial^1 - \left(x^0\partial^1 - x^1\partial^0\right)\right] \tag{12.168}$$

Well perhaps we should of said

$$K^2L^3 - K^3L^2 - \left(L^2K^3 - L^3K^2\right) = \left[K^2, L^3\right] - \left[K^3, L^2\right] \tag{12.169}$$

so the i, j, k components of $\mathrm{K}\times\mathrm{L}-\mathrm{L}\times\mathrm{K}$ are respectively

$$\mathrm{i}: \;=\; \left[K^2, L^3\right] - \left[K^3, L^2\right] \tag{12.170}$$

$$\mathrm{j}: \;=\; \left[K^3, L^1\right] - \left[K^1, L^3\right] \tag{12.171}$$

$$\mathrm{k}: \;=\; \left[K^1, L^2\right] - \left[K^2, L^1\right] \tag{12.172}$$

As for an analogue to the Minkowski force due to the electromagnetic field we have for the electromagnetic case with charge, q, and current density 4-vector, J_ν

$$K^\mu = q\eta_\nu F^{\mu\nu} \tag{12.173}$$

$$f^\mu = J_\nu F^{\mu\nu} \tag{12.174}$$

here do not confuse, K^μ, the Minkowski force, with our definition of K given above. The electromagnetic stress-energy tensor satisfies

$$T^{\alpha\beta} = -\frac{1}{\mu_0}\left(F^{\alpha\gamma}g_{\gamma\nu}F^{\nu\beta} + \frac{1}{4}g^{\alpha\beta}F_{\gamma\nu}F^{\gamma\nu}\right) \tag{12.175}$$

hence

$$\partial_\beta T^{\alpha\beta} + g_{\gamma\nu}J^\nu F^{\alpha\gamma} = 0 \tag{12.176}$$

so we expect for a particle of mass, m, analogous operators of the form

$$A^{\nu} = m\eta_{\mu}L^{\mu\nu} \tag{12.177}$$

$$b^{\mu} = mS_{\nu}L^{\mu\nu} \tag{12.178}$$

Where here, S^{μ}, is the probability density 4-vector given in Section 8.1 equation (8.91), and not the Poynting vector analog defined above in equation (12.163)

$$mS_{\mu} = m\eta_{\mu}\Psi\Psi^{*} - \hbar\Im\left(\Psi\partial_{\mu}\Psi^{*}\right) \tag{12.179}$$

Recalling that η^{μ} has the opposite sign to the proper velocity we have ascribed to the wave function, and the equation for A^{ν} takes this into account by transposing the indices for $L^{\mu\nu}$ when compared to the equation for the Minkowski force. I can think of nothing more at the moment but to calculate the expectations of these operators.

12.7. **The Hydrogen Atom.** stub

12.8. **Spin.** stub

13. The Function Space for Relativistic Quantum Mechanics

Let us consider the set of functions $f : \mathbb{R}^4 \to \mathbb{C}$ such that

(1) The functions f are functions of one vector in $\mathbb{R}^4$ as just required, and any number of dual vectors.
(2) The functions f are Lorentz invariant with the Minkowski metric defined on $\mathbb{R}^4$. By Lorentz invariant we mean that the contravariant transformation is applied to the vector and the covariant transformation is applied to the dual vectors.
   (a) If f and g are Lorentz invariant then so are $af + bg$, and fg.
(3) The functions f are square integrable with respect to the volume element $dV = dx^1dx^2dx^3$ of the spatial components

$$\int_{-\infty}^{\infty} f^{*}f dx^1dx^2dx^3 < \infty$$

Then the functions, $f^{*}f$ are suitable as unnormalized probability densities. However once normalized, then $f^{*}f$ are no longer required to be Lorentz invariant. Suitable wave functions are then obtained by multiplication with functions, g of constant modulus.

14. Two Ways of Considering Tensors

14.1. **Definitions of a Tensor.** First let us consider an abstract component free definition of tensor. Given a finite dimensional vector space, and its dual space, which has the same dimension, then a tensor of rank $\begin{pmatrix} r \\ s \end{pmatrix}$ is a multilinear (that is linear in each of its arguments), scalar function of r dual vectors, $\boldsymbol{\sigma}, \ldots, \boldsymbol{\rho}$ and s vectors $\mathrm{u}, \ldots, \mathrm{v}$

$$T\left(\boldsymbol{\sigma}, \ldots, \boldsymbol{\rho}, \mathrm{u}, \ldots, \mathrm{v}\right) = T^{\boldsymbol{\sigma}, \ldots, \boldsymbol{\rho}}{}_{\mathrm{u}, \ldots, \mathrm{v}} \tag{14.1}$$

We can calculate T, for any combination of vectors and dual vectors if we know its value for every combination of some basis of the vector space and some basis of the dual space, That is let e_k be some basis of the vector space and $\boldsymbol{\omega}^j$ be a basis of the dual space, then

$$T\left(\boldsymbol{\omega}^i, \ldots, \boldsymbol{\omega}^j, \mathrm{e}_k, \ldots, \mathrm{e}_l\right) = T^{i,\ldots,j}{}_{k,\ldots,l} \tag{14.2}$$

are called the components of the tensor on the chosen basis. In the component definition of a tensor, the components, T^i, of a tensor of rank $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are the contravariant components of a vector, and the components, T_j of a tensor of rank $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, are the covariant components of a dual vector. A tensor of rank $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in both definitions is simply a scalar.

As such the wave function, Ψ, of a quantum mechanical system, is not multilinear in its vector and dual vector arguments, hence it is not a tensor in this sense, but it is a complex scalar, thus a tensor of rank $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Now let us consider our proposed relativistic Schrodinger equation

$$i\hbar\eta^\mu\partial_\mu\Psi = -\frac{\hbar^2}{2m}\partial^\nu\partial_\nu\Psi + V\Psi \tag{14.3}$$

We can clear the mass from the deniminator on the left hand side to obtain

$$i\hbar\eta^\mu\partial_\mu m\Psi = -\frac{1}{2}\hbar^2\partial^\nu\partial_\nu\Psi + Vm\Psi \tag{14.4}$$

and we see that for a rest mass, $m = 0$, then the wave function for such a particle satifies

$$\partial^\nu\partial_\nu\Psi = 0 \tag{14.5}$$

so that Ψ is a wave train with velocity, c, the speed of light. But as we noted in Section 8 equation (8.16) we may rewrite equation as

$$i\hbar\eta^\mu\partial_\mu m\Psi = -\frac{1}{2}\left(m\right)^2 c^2\Psi + Vm\Psi \tag{14.6}$$

where the rest mass operator (m) satisfies

$$-\hbar^2\Box^2 = \left(m\right)^2 c^2 \tag{14.7}$$

Which fills us with ideas, if we could replace the rest mass, with the stress-energy tensor in some fashion...

15. Vector Potential Formulation of Electro-Dynamics in Tensor Form

We are going to start with the cart before the horse, and state that Maxwell's Equations are contained in the following equivalent expressions

$$\partial^\mu\partial_\mu A^\nu = -\mu_0 J^\nu \tag{15.1}$$

$$\Box^2\mathrm{A} = -\mu_0\mathrm{J} \tag{15.2}$$

where A is the 4-vector potential which satisfies the Lorentz Gauge condition

$$\partial_\nu A^\nu = 0 \tag{15.3}$$

and J is the current density 4-vector

$$J^\nu = \rho_0 \eta^\nu \tag{15.4}$$
$$= \left(c\rho, J^1, J^2, J^3\right) \tag{15.5}$$

The electric and magnetic vectors are obtained from 4-vector potential by

$$\mathrm{E} = -\left(\nabla V + \frac{\partial \mathrm{A}}{\partial t}\right) \tag{15.6}$$
$$E^i/c = -\partial^i \left(V/c\right) + \partial^0 A^i \tag{15.7}$$

and

$$\mathrm{B} = \nabla \times \mathrm{A} \tag{15.8}$$

which can only be expressed as such when $\nabla \cdot \mathrm{B} = 0$. We have the components of A, then given by

$$A^\mu = \left(V/c, A^1, A^2, A^3\right) \tag{15.9}$$

The Lorentz gauge condition implies the continuity equation for the current density 4-vector, since

$$\begin{aligned}
\partial_\nu J^\nu &= -\frac{1}{\mu_0}\partial_\nu \partial^\mu \partial_\mu A^\nu \\
&= -\frac{1}{\mu_0}\partial^\mu \partial_\mu \partial_\nu A^\nu \\
&= -\frac{1}{\mu_0}\partial^\mu \partial_\mu 0 \\
&= 0
\end{aligned} \tag{15.10}$$

However the converse is not true for if the continuity equation holds, then

$$\partial_\nu \partial^\mu \partial_\mu A^\nu = -\mu_0 \partial_\nu J^\nu = 0 \tag{15.11}$$

only implies that the scalar quantity, $\partial_\nu A^\nu = U$ satisfies the wave equation, $\Box^2 U = 0$, and U needn't be zero. All we know is that the continuity equation is a neccessary condition for the Lorentz gauge.

If we were to solve the partial differential equation () in a particular situation we need to know at least the initial conditions

$$\mathrm{A}\left(x, y, z, 0\right) \text{ and } \mathrm{A}_{ct}\left(x, y, z, 0\right)$$

The Lorentz gauge condition allows us to determine the partial derivative of A with respect to $x^0 = ct$

$$\mathrm{A}_{ct}\left(x, y, z, 0\right) = -\left[\mathrm{A}_x\left(x, y, z, 0\right) + \mathrm{A}_y\left(x, y, z, 0\right) + \mathrm{A}_z\left(x, y, z, 0\right)\right] \tag{15.12}$$

But if we impose such an initial condition together with the continuity equation for J, do these together imply that the Lorentz gauge will be satisfied for all x^0? I don't know the answer at the moment.

The electro-magnetic field tensor is defined by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{15.13}$$

which we see is skew-symmetric. The components of F are identified with the components of the electric field E, and magnetic field B as follows

$$F^{\mu\nu} = \begin{bmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & B^3 & -B^2 \\ -E^2/c & -B^3 & 0 & B^1 \\ -E^3/c & B^2 & -B^1 & 0 \end{bmatrix} \tag{15.14}$$

Thus

$$E^i/c = F^{0i} \tag{15.15}$$
$$B^1 = F^{23} \tag{15.16}$$
$$B^2 = F^{31} \tag{15.17}$$
$$B^3 = F^{12} \tag{15.18}$$

and we note the mnemonic for the components of B given by the even permutations, $(1,2,3)$ for B^1, $(2,3,1)$ for B^2, $(3,1,2)$ for B^3. The corresponding odd permutations give $(1,3,2)$ for $-B^1$, $(2,1,3)$ for $-B^2$, $(3,2,1)$ for $-B^3$.

Maxwell's equations can be expressed interms of the filed tensor by

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu \tag{15.19}$$

and

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \tag{15.20}$$

The first is easily obtained

$$\partial_\nu F^{\mu\nu} = \partial_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \tag{15.21}$$
$$= \partial_\nu \partial^\mu A^\nu - \partial_\nu \partial^\nu A^\mu \tag{15.22}$$
$$= \partial^\mu \partial_\nu A^\nu - \partial^\nu \partial_\nu A^\mu \tag{15.23}$$
$$= \partial^\mu 0 - \partial^\nu \partial_\nu A^\mu \tag{15.24}$$
$$= -\left(-\mu_0 J^\mu\right) \tag{15.25}$$

The second requires more work,

$$\partial^\lambda F^{\mu\nu} = \partial^\lambda \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \tag{15.26}$$
$$\partial^\mu F^{\nu\lambda} = \partial^\mu \left(\partial^\nu A^\lambda - \partial^\lambda A^\nu\right) \tag{15.27}$$
$$\partial^\nu F^{\lambda\mu} = \partial^\nu \left(\partial^\lambda A^\mu - \partial^\mu A^\lambda\right) \tag{15.28}$$

to obtain

$$\partial^\lambda F^{\mu\nu} = \partial^\lambda \partial^\mu A^\nu - \partial^\lambda \partial^\nu A^\mu \tag{15.29}$$
$$\partial^\mu F^{\nu\lambda} = \partial^\mu \partial^\nu A^\lambda - \partial^\mu \partial^\lambda A^\nu \tag{15.30}$$
$$\partial^\nu F^{\lambda\mu} = \partial^\nu \partial^\lambda A^\mu - \partial^\nu \partial^\mu A^\lambda \tag{15.31}$$

then summing we obtain

$$\begin{aligned} & \partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} \\ = \; & \partial^\lambda \partial^\mu A^\nu - \partial^\lambda \partial^\nu A^\mu + \partial^\mu \partial^\nu A^\lambda - \partial^\mu \partial^\lambda A^\nu + \partial^\nu \partial^\lambda A^\mu - \partial^\nu \partial^\mu A^\lambda \\ = \; & \left(\partial^\lambda \partial^\mu A^\nu - \partial^\mu \partial^\lambda A^\nu\right) - \left(\partial^\lambda \partial^\nu A^\mu - \partial^\nu \partial^\lambda A^\mu\right) + \left(\partial^\mu \partial^\nu A^\lambda - \partial^\nu \partial^\mu A^\lambda\right) \\ = \; & 0 - 0 + 0 \end{aligned}$$

Notice that if any two indices are repeated, and we are not minding the summation convention here, we find

$$\begin{aligned} & \partial^{\lambda} F^{\mu\mu} + \partial^{\mu} F^{\mu\lambda} + \partial^{\mu} F^{\lambda\mu} \\ = \; & 0 + \partial^{\mu} F^{\mu\lambda} - \partial^{\mu} F^{\mu\lambda} \\ = \; & 0 \end{aligned} \tag{15.32}$$

Therefore we are only interested when no two indices are repeated which leads to 4 independent equations

$$\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0 \tag{15.33}$$
$$\partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = 0 \tag{15.34}$$
$$\partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02} = 0 \tag{15.35}$$
$$\partial^0 F^{31} + \partial^3 F^{10} + \partial^1 F^{03} = 0 \tag{15.36}$$

The first is equivalent to $\nabla \cdot \mathrm{B} = 0$, the reamaining 3 equations express, $\nabla \times \mathrm{E}/c = -\frac{1}{c}\frac{\partial \mathrm{B}}{\partial t}$, which is Faraday's Law, as for equation (15.19) we again have 4 independent equations

$$\mu_0 J^0 = \partial_0 F^{00} + \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} \tag{15.37}$$
$$\mu_0 J^1 = \partial_0 F^{10} + \partial_1 F^{11} + \partial_2 F^{12} + \partial_3 F^{13} \tag{15.38}$$
$$\mu_0 J^2 = \partial_0 F^{20} + \partial_1 F^{21} + \partial_2 F^{22} + \partial_3 F^{23} \tag{15.39}$$
$$\mu_0 J^3 = \partial_0 F^{30} + \partial_1 F^{31} + \partial_2 F^{32} + \partial_3 F^{33} \tag{15.40}$$

where the diagonal terms are zero. The first is equivalent to $\nabla \cdot \mathrm{E} = \frac{1}{\epsilon_0}\rho$, which is Gauss's law, and the remaining express Ampere's Law with Maxwell's correction $\nabla \times \mathrm{B} = \mu_0 \mathrm{J} + \frac{1}{c}\frac{\partial \mathrm{E}}{\partial t}$ where J here are the 3 spatial components of the current density 4-vector.

In the absence of charge Maxwell's equations in terms of the 4-vector potential become

$$\partial^{\mu}\partial_{\mu} A^{\nu} = 0 \tag{15.41}$$

then we have both

$$\partial^{\mu}\partial_{\mu}\partial^{\lambda} A^{\nu} = 0 \tag{15.42}$$
$$\partial^{\mu}\partial_{\mu}\partial^{\nu} A^{\lambda} = 0 \tag{15.43}$$

Subtracting the bottom from the top we have

$$\begin{aligned} \partial^{\mu}\partial_{\mu}\left(\partial^{\lambda} A^{\nu} - \partial^{\nu} A^{\lambda}\right) &= \partial^{\mu}\partial_{\mu} F^{\lambda\nu} \\ &= 0 \end{aligned} \tag{15.44}$$

Equivalently

$$\Box^2 \mathrm{F} = 0 \tag{15.45}$$

which is the wave equation for the electro-magnetic radiation in free space.

16. Comparison of Schrödinger's Equation to the Fokker-Planck Equation

The Fokker-Planck Equation also known as the Forward Kolmogorov Equation is given in one spatial dimension by

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\sigma^2(x,t)\,\rho(x,t) - \frac{\partial}{\partial x}\mu(x,t)\,\rho(x,t) \tag{16.1}$$

Let's see if we can place Schrödinger's into the same *form* as Fokker-Planck. In one spatial dimension we have

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x,t)\,\Psi \tag{16.2}$$

then

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{iV(x,t)}{\hbar}\Psi \tag{16.3}$$

so let

$$\sigma^2(x,t) = \frac{i\hbar}{m} \tag{16.4}$$

Then

$$\frac{\partial \Psi}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 \Psi}{\partial x^2} - \frac{iV(x,t)}{\hbar}\Psi \tag{16.5}$$

and let us set

$$\begin{aligned}\frac{iV(x,t)}{h}\Psi &= \frac{\partial}{\partial x}\mu(x,t)\,\Psi(x,t)\\ &= \frac{\partial \mu}{\partial x}\Psi + \mu\frac{\partial \Psi}{\partial x}\end{aligned} \tag{16.6}$$

Then we have

$$\frac{\partial \mu(x,t)}{\partial x} + \left(\frac{1}{\Psi}\frac{\partial \Psi}{\partial x}\right)\mu(x,t) = \frac{iV(x,t)}{\hbar} \tag{16.7}$$

An ordinary inhomogeneous 1st order linear differential equation for $\mu(x,t)$ in x. Given we have $V(x,t)$ and the corresponding solution, $\Psi(x,t)$ of Schrödinger's Equation, we have an imaginary diffusion term given by equation (16.4), and imaginary drift term obtained by solving (16.7), however we need to specify a boundary condition $\mu(0,t)$. Does this imply a Chapman-Kolmogorov Master Equation for Ψ that allows a derivation of the Fokker-Planck form

$$\frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial}{\partial x}\mu(x,t)\,\Psi(x,t) \tag{16.8}$$

Were inclined to rewrite equation (16.7) as

$$\frac{\partial M(x,t)}{\partial x} + \left(\frac{1}{\Psi}\frac{\partial \Psi}{\partial x}\right)M(x,t) = -V(x,t) \tag{16.9}$$

with

$$M(x,t) = i\hbar\mu(x,t) \tag{16.10}$$

The general solution of the ordinary differential equation

$$\frac{dy}{dx} + a(x)\,y = f(x) \tag{16.11}$$

is given by

$$y\left(x\right)=\eta\left(x\right)\left(y\left(0\right)+\int_{0}^{x}\eta\left(s\right)^{-1}f\left(s\right)ds\right)\tag{16.12}$$

With integrating factor

$$\eta\left(x\right)=\exp\left(-\int_{0}^{x}a\left(s\right)ds\right)\tag{16.13}$$

We have

$$\begin{align}
a\left(x\right) &= \frac{1}{\Psi}\frac{\partial\Psi}{\partial x}\tag{16.14}\\
&= \frac{\partial}{\partial x}\ln\left(\Psi\right)\tag{16.15}
\end{align}$$

so that

$$\begin{align}
\int_{0}^{x}a\left(s\right)ds &= \ln\Psi\left(x,t\right)-\ln\Psi\left(0,t\right)\tag{16.16}\\
&= \ln\left(\frac{\Psi\left(x,t\right)}{\Psi\left(0,t\right)}\right)\tag{16.17}
\end{align}$$

so that

$$\eta\left(x\right)=\frac{\Psi\left(0,t\right)}{\Psi\left(x,t\right)}\tag{16.18}$$

Therfore the solution to equation () is given by

$$M\left(x,t\right)=\frac{\Psi\left(0,t\right)}{\Psi\left(x,t\right)}\left(M\left(0,t\right)-\int_{0}^{x}\frac{\Psi\left(u,t\right)}{\Psi\left(0,t\right)}V\left(u,t\right)du\right)\tag{16.19}$$

Or in the more symetrical form

$$\Psi\left(x,t\right)M\left(x,t\right)=\Psi\left(0,t\right)M\left(0,t\right)-\int_{0}^{x}\Psi\left(u,t\right)V\left(u,t\right)du\tag{16.20}$$

where we seem to be free to choose the boundary condition we please, perhaps $M\left(0,t\right)=0$.

For the Gaussian Packet of Section 6 we have

$$\begin{align}
\frac{1}{\Psi}\frac{\partial\Psi}{\partial x} &= P\notag\\
&= ia-2\Delta k\left(\Delta kx'-\Delta\omega t'\right)\tag{16.21}
\end{align}$$

where $x'=x-x_0$ and $t'=t-t_0$. The potential was found to be quadratic in x,

$$V\left(x,t\right)=\hbar b-\frac{\hbar^2a^2}{2m}+\frac{2}{m}\left(\hbar\Delta k\right)^2\left(\Delta kx'-\Delta\omega t'\right)^2\tag{16.22}$$

Let us evaluate the integration factor in this case, first we have

$$a\left(x\right)=P(x,t)\tag{16.23}$$

Now $P\left(x,t\right)$ is linear in x so that

$$\frac{dP}{dx'}=-2\left(\Delta k\right)^2=\frac{dP}{dx}\tag{16.24}$$

so that

$$-\int_0^x P(x,t)\,dx = \frac{1}{2(\Delta k)^2}\int_{P(0,t)}^{P(x,t)} PdP \tag{16.25}$$

$$= \frac{P(x,t)^2 - P(0,t)^2}{4(\Delta k)^2} \tag{16.26}$$

Thus the integration factor

$$\eta(x) = \exp\left(\frac{P(x,t)^2 - P(0,t)^2}{4(\Delta k)^2}\right) \tag{16.27}$$

$$= \exp\left(\frac{P(x,t)^2}{4(\Delta k)^2}\right)\exp\left(-\frac{P(0,t)^2}{4(\Delta k)^2}\right) \tag{16.28}$$

and we see

$$\frac{\Psi(0,t)}{\Psi(x,t)} = \frac{\exp\left(-\frac{P(0,t)^2}{4(\Delta k)^2}\right)}{\exp\left(-\frac{P(x,t)^2}{4(\Delta k)^2}\right)} \tag{16.29}$$

which however does not imply equality of the denominators:

$$\Psi(x,t) \neq \exp\left(-\frac{P(x,t)^2}{4(\Delta k)^2}\right)$$

The solution becomes

$$\exp\left(-\frac{P(x,t)^2}{4(\Delta k)^2}\right)M(x,t) = \exp\left(-\frac{P(0,t)^2}{4(\Delta k)^2}\right)M(0,t) - \int_0^x \exp\left(-\frac{P(u,t)^2}{4(\Delta k)^2}\right)V(u,t)\,du \tag{16.30}$$

17. The Fokker-Planck Equation

Let be the conditional density of a Markov process be given by

$$f(x,t|y,\tau)\,dy = prob\,(x \geqslant X_t \geqslant x+dx|X_\tau = y)$$

The Master or Chapman-Kolmogorov equation is

$$f(x,t|y,\tau) = \int_z f(x,t|z,s)\,f(z,s|y,\tau)\,dz$$

Now let $\rho(x,t) = f(x,t|x_0,0)$ and

$$W(x|x') = f(x,t+\Delta t|x',t')$$

then the master equation states

$$\rho(x,t+\Delta t) = \int W(x|x')\,\rho(x',t)\,dx'$$

Expanding the left hand side

$$\rho(x,t) + \frac{\partial\rho}{\partial t}\Delta t + ... = \int W(x|x')\,\rho(x',t)\,dx'$$

Now let $r = x - x'$ and $dr = -dx'$ that is $r = \Delta x$ is the jump from x' to x in time Δt so that

$$\rho(x,t) + \frac{\partial \rho}{\partial t}\Delta t + ... = -\int_{\infty}^{-\infty} W(x'+r,x')\,\rho(x',t)\,dr = \int_{-\infty}^{\infty} W(x'+r,x')\,\rho(x',t)\,dr$$

The Taylor series for the expansion of a function of x in the neighboorhood of a is given by,

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + ...$$

Now expand the integrand in a Taylor series wrt x' about x

$$W(x'+r,x')\,\rho(x',t) = W(x+r,x)\,\rho(x,t) + (x'-x)\frac{\partial}{\partial x}W(x+r,x)\,\rho(x,t) + \frac{1}{2!}(x'-x)^2\frac{\partial^2}{\partial x^2}W(x+r,x)\,\rho(x,t) + ...$$

so that

$$\int W(x'+r,x')\,\rho(x',t)\,dr = \rho(x,t)\int W(x+r,x)\,dr - \frac{\partial}{\partial x}\rho(x,t)\int rW(x+r,x)\,dr + \frac{\partial^2}{\partial x^2}\rho(x,t)\,\frac{1}{2}\int r^2W(x+r,x)\,dr +$$

Since $\int W(x+r|x)\,dr = 1$ we have

$$\frac{\partial \rho}{\partial t}\Delta t + ... = -\frac{\partial}{\partial x}\rho(x,t)\int rW(x+r,x)\,dr + \frac{\partial^2}{\partial x^2}\rho(x,t)\,\frac{1}{2}\int r^2W(x+r,x)\,dr + ... =$$

Or with $r = \Delta x$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\rho(x,t)\left(\frac{1}{\Delta t}\int_{-\infty}^{\infty}\Delta x W(x+\Delta x|x)\,d\Delta x\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x,t)\left(\frac{1}{\Delta t}\int_{-\infty}^{\infty}\Delta x^2 W(x+\Delta x|x)\,d\Delta x\right) + ...$$

so that

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\rho(x,t)\,\frac{\langle \Delta x\rangle}{\Delta t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x,t)\,\frac{\left\langle (\Delta x)^2\right\rangle}{\Delta t} + ...$$

with conditional expectations$\langle \Delta x\rangle$ and $\left\langle (\Delta x)^2\right\rangle$ functions of $x, t,$ and Δt giving

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\rho(x,t)\lim_{\Delta t\to 0}\frac{a(x,t,\Delta t)}{\Delta t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x,t)\lim_{\Delta t\to 0}\frac{b^2(x,t,\Delta t)}{\Delta t} + ...$$

and in the limit as $\Delta t \to 0$ we obtain the Fokker-Planck Equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}b^2(x,t)\,\rho(x,t) - \frac{\partial}{\partial x}a(x,t)\,\rho(x,t)$$

which is also known as the Forward Kolmogorov Equation.

Example 14. The solution to the Diffusion Equation

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}$$

subject to initial condition

$$f(x,t_0|x_0,t_0) = \delta(x-x_0)$$

and boundary conditions

$$\lim_{x\to\pm\infty} f(x,t|x_0,t_0)=0$$

is

$$f(x,t|x_0,t_0)=\frac{1}{\sqrt{2\pi(t-t_0)}\sigma}\exp\left(-\frac{(x-x_0)^2}{2(t-t_0)\sigma^2}\right)$$

If we took this as the heat equation $\frac{\sigma^2}{2}$ has units $\frac{length^2}{time}$ and we need

$$f(x,t)=\frac{A}{\sqrt{2\pi(t-t_0)}\sigma}\exp\left(-\frac{(x-x_0)^2}{2(t-t_0)\sigma^2}\right)$$

with

$$f(x_0,t_1)\sqrt{2\pi(t_1-t_0)}\sigma=A$$

Example 15. Let us solve the partial differential equation

$$\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial u}{\partial t}$$

by separation of variables so let

$$u(x,t)=X(x)T(t)$$

then

$$X''(x)T(t)=\frac{2}{\sigma^2}X(x)T'(t)$$

so that

$$\frac{X''(x)}{X(x)}=\frac{2}{\sigma^2}\frac{T'(t)}{T(t)}=-k^2$$

then

$$\frac{T'(t)}{T(t)}=-\frac{\sigma^2k^2}{2}$$

so that

$$T(t)=A\exp\left(-t\sigma^2k^2/2\right)$$

Next we have

$$X''(x)+k^2X(x)=0$$

and factoring we find

$$\left(\frac{d}{dx}+ik\right)\left(\frac{d}{dx}-ik\right)X=0$$

and

$$X=C\exp(ikx)+\bar{C}\exp(-ikx)$$

Now combining the two solution we arrive at

$$u(x,t)=\left[C\exp(ikx)+\bar{C}\exp(-ikx)\right]\exp\left(-t\sigma^2k^2/2\right)$$

Therefore for a discrete set of values of k,

$$u(x,t)=\sum_{k=-\infty}^{\infty}C_k\exp\left(-t\sigma^2k^2/2\right)\exp(ikx)$$

with $C_{-k} = \bar{C}_k$ or in the continuous case,

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp\left(-t\sigma^2 k^2/2\right) \exp(ikx)\, dk$$

Example 16. The Fourier Transform,

$$F(k,t) = \int_{-\infty}^{\infty} f(x,t) \exp(-ikx)\, dx$$

of

$$f(x,t) = \frac{1}{\sqrt{2\pi t}\sigma} \exp\left(-\frac{x^2}{2t\sigma^2}\right)$$

is

$$F(k,t) = \exp\left(-\frac{t\sigma^2 k^2}{2}\right)$$

so that

$$f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{t\sigma^2 k^2}{2}\right) \exp(ikx)\, dk$$

Example 17. If in the Fokker-Planck Equation we have $a(x,t) = b_x^2(x,t)$, then the equation reduces to

$$\frac{\partial p}{\partial t} = \frac{1}{2}\left(b^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2 b^2}{\partial x^2}\right) p$$

which bears a certain resemblance to the Schrodinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi$$

but whereas we might suspect that $\psi^*\psi = f$ obeys the master equation

$$f(x,t|x',t') = \int f(x,t|y,s)\, f(y,s|x',t')\, dy$$

so that $\psi^*\psi$ obeys the fokker-planck equation

$$\frac{\partial \psi^*\psi}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2} b^2 \psi^*\psi - \frac{\partial}{\partial x} a\psi^*\psi$$

we obtain no useful differential equation in ψ alone, since

$$\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2} b^2\psi^*\psi - \left(\frac{\partial a}{\partial x}\psi^*\psi + a\left(\frac{\partial \psi^*}{\partial x}\psi + \psi^*\frac{\partial \psi}{\partial x}\right)\right)$$

so that

$$\frac{1}{\psi^*}\frac{\partial \psi^*}{\partial t} + \frac{1}{\psi}\frac{\partial \psi}{\partial t} = \frac{1}{2\psi^*\psi}\frac{\partial^2}{\partial x^2} b^2\psi^*\psi - \left(\frac{\partial a}{\partial x} + a\left(\frac{1}{\psi^*}\frac{\partial \psi^*}{\partial x} + \frac{1}{\psi}\frac{\partial \psi}{\partial x}\right)\right)$$

whence

$$\frac{\partial \ln \psi^*}{\partial t} + \frac{\partial \ln \psi}{\partial t} = \frac{1}{2\psi^*\psi}\frac{\partial^2}{\partial x^2} b^2\psi^*\psi - \left(\frac{\partial a}{\partial x} + a\left(\frac{\partial \ln \psi^*}{\partial t} + \frac{\partial \ln \psi}{\partial t}\right)\right)$$

which simplifies to

$$2\frac{\partial\Re\left(\ln\psi\right)}{\partial t}=\frac{1}{2\psi^{*}\psi}\frac{\partial^{2}}{\partial x^{2}}b^{2}\psi^{*}\psi-\left(\frac{\partial a}{\partial x}+a\left(2\frac{\partial\Re\left(\ln\psi\right)}{\partial x}\right)\right)$$

The natural log of a complex number, $z=x+iy$ is given by

$$\ln\left(z\right)=\frac{1}{2}\ln\left(z^{*}z\right)+i\left(\arctan\left(\frac{y}{x}\right)+2\pi n\right)$$

and we are left with

$$\frac{\partial\left(\ln\psi^{*}\psi\right)}{\partial t}=\frac{1}{2\psi^{*}\psi}\frac{\partial^{2}}{\partial x^{2}}b^{2}\psi^{*}\psi-\left(\frac{\partial a}{\partial x}+a\frac{\partial\left(\ln\psi^{*}\psi\right)}{\partial x}\right)$$

and we still haven't treated the messy term

$$\frac{1}{2\psi^{*}\psi}\frac{\partial^{2}}{\partial x^{2}}b^{2}\psi^{*}\psi$$

Let us return to the master equation for $\psi^{*}\psi$,

$$\psi^{*}\psi\left(x|x'\right)=\int\psi^{*}\psi\left(x|y\right)\psi^{*}\psi\left(y|x'\right)dy$$

then

$$\begin{aligned}\psi^{*}\left(x|x'\right)\psi\left(x|x'\right) &= \int\left\{\psi^{*}\left(x|y\right)\psi^{*}\left(y|x'\right)\right\}\left\{\psi\left(x|y\right)\psi\left(y|x'\right)\right\}dy\\ &= \int\int\left\{\psi^{*}\left(x|y\right)\psi^{*}\left(y|x'\right)\right\}\delta\left(y-z\right)\left\{\psi\left(x|z\right)\psi\left(z|x'\right)\right\}dzdy\end{aligned}$$

Appendix A. The Lorentz Transformations

Quoting directly from wikipedia, for a boost in an arbitrary direction with velocity, v, it is convenient to decompose the spatial vector, r, into components perpendicular and parallel to the velocity

$$\mathrm{r}=\mathrm{r}_{\perp}+\mathrm{r}_{\|} \tag{A.1}$$

so that

$$\mathrm{v}\cdot\mathrm{r}=\mathrm{v}\cdot\mathrm{r}_{\|} \tag{A.2}$$

then

$$x'^{0}=\gamma\left(x^{0}-\boldsymbol{\beta}^{\mathrm{T}}\mathrm{r}\right) \tag{A.3}$$

$$\mathrm{r}'=\mathrm{r}_{\perp}+\gamma\left(\mathrm{r}_{\|}-\boldsymbol{\beta}^{\mathrm{T}}x^{0}\right) \tag{A.4}$$

where $\boldsymbol{\beta} = \left(\beta^1, \beta^2, \beta^3\right)$. The second equation becomes

$$\begin{aligned}
\mathrm{r}' &= \mathrm{r}_\perp + \gamma \mathrm{r}_\| - \gamma \boldsymbol{\beta}^{\mathrm{T}} x^0 \\
&= \mathrm{r} - \mathrm{r}_\| + \gamma \mathrm{r}_\| - \gamma \boldsymbol{\beta}^{\mathrm{T}} x^0 \\
&= \mathrm{r} + (\gamma - 1)\, \mathrm{r}_\| - \gamma \boldsymbol{\beta}^{\mathrm{T}} x^0 \\
&= \mathrm{r} + (\gamma - 1)\, \frac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}} \mathrm{r}_\| - \gamma \boldsymbol{\beta}^{\mathrm{T}} x^0 \\
&= \mathrm{r} + (\gamma - 1)\, \frac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{\|\boldsymbol{\beta}\|^2} \mathrm{r}_\| - \gamma \boldsymbol{\beta}^{\mathrm{T}} x^0 \\
&= \mathrm{r} + (\gamma - 1)\, \frac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{\|\boldsymbol{\beta}\|^2} \mathrm{r} - \gamma \boldsymbol{\beta}^{\mathrm{T}} \qquad \text{(A.5)}
\end{aligned}$$

since $\boldsymbol{\beta}^{\mathrm{T}} \mathrm{r}_\| = \boldsymbol{\beta}^{\mathrm{T}} \mathrm{r}$ therefore

$$\mathrm{r}' = \left(\mathrm{I} + (\gamma - 1)\, \frac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{\|\boldsymbol{\beta}\|^2} \right) \mathrm{r}$$

$$\begin{bmatrix} x'^0 \\ \mathrm{r}' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma \boldsymbol{\beta}^{\mathrm{T}} \\ -\gamma \boldsymbol{\beta} & \mathrm{I} + (\gamma - 1)\, \frac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{\|\boldsymbol{\beta}\|^2} \end{bmatrix} \begin{bmatrix} x^0 \\ \mathrm{r} \end{bmatrix} \qquad \text{(A.6)}$$

for the covariant transformation we have

$$x'_0 = \gamma \left(x_0 + \boldsymbol{\beta}^{\mathrm{T}} \mathrm{r} \right) \qquad \text{(A.7)}$$

$$\mathrm{r}' = \mathrm{r}_\perp + \gamma \left(\mathrm{r}_\| + \boldsymbol{\beta}^{\mathrm{T}} x_0 \right) \qquad \text{(A.8)}$$

hence

$$\begin{bmatrix} x'_0 \\ \mathrm{r}' \end{bmatrix} = \begin{bmatrix} \gamma & \gamma \boldsymbol{\beta}^{\mathrm{T}} \\ \gamma \boldsymbol{\beta} & \mathrm{I} + (\gamma - 1)\, \frac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{\|\boldsymbol{\beta}\|^2} \end{bmatrix} \begin{bmatrix} x_0 \\ \mathrm{r} \end{bmatrix} \qquad \text{(A.9)}$$

which is also the the form of the inverse transformation

$$\begin{bmatrix} x^0 \\ \mathrm{r} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma \boldsymbol{\beta}^{\mathrm{T}} \\ \gamma \boldsymbol{\beta} & \mathrm{I} + (\gamma - 1)\, \frac{\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{T}}}{\|\boldsymbol{\beta}\|^2} \end{bmatrix} \begin{bmatrix} x'^0 \\ \mathrm{r}' \end{bmatrix} \qquad \text{(A.10)}$$

Appendix B. Random Lorentz 2-vectors

Suppose we want random variables, x and t to be components of a Lorentz 2-vector, (t, x). The components transform according to the Lorentz Transformations

$$\begin{gathered} x' = \gamma\,(x - \beta c t) \\ c t' = \gamma\,(c t - \beta x) \end{gathered} \qquad \text{(B.1)}$$

so that the expectations also transform according to

$$\begin{gathered} \langle x' \rangle = \gamma\,(\langle x \rangle - \beta c \,\langle t \rangle) \\ c\,\langle t' \rangle = \gamma\,(c\,\langle t \rangle - \beta\,\langle x \rangle) \end{gathered} \qquad \text{(B.2)}$$

But suppose we would also like the standard deviations Δx and Δt to transform according to the Lorentz transformation

$$\begin{aligned} \Delta x' &= \gamma\left(\Delta x - \beta c \Delta t\right) \\ c\Delta t' &= \gamma\left(c\Delta t - \beta \Delta x\right) \end{aligned} \tag{B.3}$$

and also have

$$(\Delta x)^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2 \tag{B.4}$$

$$(\Delta t)^2 = \left\langle t^2 \right\rangle - \langle t \rangle^2 \tag{B.5}$$

then

$$\begin{aligned} \left(\Delta x'\right)^2 &= \left\langle x'^2 \right\rangle - \left\langle x' \right\rangle^2 \\ &= \gamma^2 \left\langle \left(x^2 - 2\beta c x t + \beta^2 c^2 t^2\right) \right\rangle - \gamma^2 \left(\langle x \rangle^2 - 2\beta c \langle x \rangle \langle t \rangle + \beta^2 c^2 \langle t \rangle^2 \right) \\ &= \gamma^2 \left\{ \left(\left\langle x^2 \right\rangle - \langle x \rangle^2 \right) + \beta^2 c^2 \left(\left\langle t^2 \right\rangle - \langle t \rangle^2 \right) - 2\beta c \left(\langle xt \rangle - \langle x \rangle \langle t \rangle \right) \right\} \\ &= \gamma^2 \left\{ (\Delta x)^2 + \beta^2 c^2 (\Delta t)^2 - 2\beta c \operatorname{cov}(x,t) \right\} \end{aligned} \tag{B.6}$$

but we also want

$$\begin{aligned} \left(\Delta x'\right)^2 &= \gamma^2 \left(\Delta x - \beta c \Delta t\right)^2 \\ &= \gamma^2 \left((\Delta x)^2 - 2\beta c \Delta x \Delta t + \beta^2 c^2 (\Delta t)^2 \right) \end{aligned} \tag{B.7}$$

thus we require that

$$\operatorname{cov}(x,t) = \Delta x \Delta t \tag{B.8}$$

which implies that the correlation between x and t

$$\rho_{xt} = \frac{\sigma_{xt}}{\sigma_x \sigma_t} = \frac{\Delta x \Delta t}{|\Delta x|\,|\Delta t|} = \pm 1 \tag{B.9}$$

so that x is a linear in t

$$x = at + b \tag{B.10}$$

Appendix C. The Cauchy Schwartz Inequality

C.1. **For real inner product spaces.** We have symmetry, $\langle x|y\rangle = \langle y|x\rangle$, linearity in the 2nd variable, and $\langle x|x\rangle \geq 0$ so that

$$\begin{aligned} 0 &\leq \langle tx + y | tx + y \rangle = \langle tx|tx\rangle + \langle tx|y\rangle + \langle y|tx\rangle + \langle y|y\rangle \\ &= t^2 \langle x|x\rangle + 2t \langle x|y\rangle + \langle y|y\rangle \end{aligned} \tag{C.1}$$

We must have $b^2 - 4ac \leq 0$ with $a = \langle x|x\rangle$, $b = 2\langle x|y\rangle$, and $c = \langle y|y\rangle$ so $4\langle x|y\rangle^2 - 4\langle x|x\rangle \langle y|y\rangle \geq 0$ implies

$$\langle x|x\rangle \langle y|y\rangle \geq \langle x|y\rangle^2 \tag{C.2}$$

C.2. **For complex inner product spaces.** We have conjugate symmetry, $\langle x|y\rangle = \langle y|x\rangle^*$, linearity in the 2nd variable which implies conjugate linearity in the 1st, and $\langle x|x\rangle \geq 0$ so that

$$\begin{aligned} 0 &\leq \langle tx+y|tx+y\rangle = \langle tx|tx\rangle + \langle tx|y\rangle + \langle y|tx\rangle + \langle y|y\rangle \\ &= tt^* \langle x|x\rangle + t^* \langle x|y\rangle + t \langle y|x\rangle + \langle y|y\rangle \end{aligned} \tag{C.3}$$

In particular let

$$t = -\frac{\langle x|y\rangle}{\langle x|x\rangle} \tag{C.4}$$

so that

$$\begin{aligned} 0 &\leq tt^* \langle x|x\rangle^2 + t^* \langle x|y\rangle \langle x|x\rangle + t \langle y|x\rangle \langle x|x\rangle + \langle y|y\rangle \langle x|x\rangle \\ &= \frac{\langle x|y\rangle}{\langle x|x\rangle} \frac{\langle y|x\rangle}{\langle x|x\rangle} \langle x|x\rangle^2 - \frac{\langle y|x\rangle}{\langle x|x\rangle} \langle x|y\rangle \langle x|x\rangle - \frac{\langle x|y\rangle}{\langle x|x\rangle} \langle y|x\rangle \langle x|x\rangle + \langle y|y\rangle \langle x|x\rangle \\ &= \langle x|y\rangle \langle y|x\rangle - \langle y|x\rangle \langle x|y\rangle - \langle x|y\rangle \langle y|x\rangle + \langle y|y\rangle \langle x|x\rangle \\ &= -\langle y|x\rangle \langle x|y\rangle + \langle y|y\rangle \langle x|x\rangle \end{aligned} \tag{C.5}$$

which implies

$$\langle y|y\rangle \langle x|x\rangle \geq \langle y|x\rangle \langle x|y\rangle = |\langle y|x\rangle|^2 \tag{C.6}$$

C.3. **Equality in Cauchy Schwartz.** Suppose $\langle x|x\rangle \langle y|y\rangle = |\langle x|y\rangle|^2$. Clearly this can happen if either $x = 0$ or $y = 0$, but also if $x = \lambda y$. But suppose that $x \neq 0$ and $y \neq 0$ and

$$\langle x|x\rangle \langle y|y\rangle = \langle x|y\rangle \langle y|x\rangle \tag{C.7}$$

Then let

$$\langle x|x\rangle \langle y|y\rangle - \langle x|y\rangle \langle y|x\rangle = 0 \tag{C.8}$$

be the determinant of the system

$$\begin{bmatrix} \langle x|x\rangle & \langle y|x\rangle \\ \langle x|y\rangle & \langle y|y\rangle \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{C.9}$$

which has non-trivial solution. So

$$a \langle x|x\rangle + b \langle y|x\rangle = 0 \tag{C.10}$$

$$a \langle x|y\rangle + b \langle y|y\rangle = 0 \tag{C.11}$$

So that

$$\langle a^* x|x\rangle + \langle b^* y|x\rangle = 0 \tag{C.12}$$

$$\langle a^* x|y\rangle + \langle b^* y|y\rangle = 0 \tag{C.13}$$

Hence

$$\langle a^* x + b^* y|x\rangle = 0 \tag{C.14}$$

$$\langle a^* x + b^* y|y\rangle = 0 \tag{C.15}$$

Then

$$a^* \langle a^* x + b^* y|x\rangle = 0 \tag{C.16}$$

$$b^* \langle a^* x + b^* y|y\rangle = 0 \tag{C.17}$$

So that

$$\langle a^*x + b^*y | a^*x\rangle = 0 \tag{C.18}$$

$$\langle a^*x + b^*y | b^*y\rangle = 0 \tag{C.19}$$

Which implies

$$\langle a^*x + b^*y | a^*x + b^*y\rangle = 0 \tag{C.20}$$

If and only if

$$a^*x + b^*y = 0 \tag{C.21}$$

So that

$$x = \lambda y \tag{C.22}$$

Neither $a = 0$ or $b = 0$, since this would imply $x = 0$ or $y = 0$ contrary to assumption.

C.4. **Miscelaneous.**

Theorem 18. *The partial derivative of a complex variable with respect to its conjugate is zero*

$$\frac{\partial z^*}{\partial z} = 0 \tag{C.23}$$

Proof. First we have

$$\begin{aligned} 2dx &= dz + dz^* \\ &= \frac{\partial (z + z^*)}{\partial z} dz + \frac{\partial (z + z^*)}{\partial z^*} dz^* \\ &= \frac{\partial z}{\partial z} dz + \frac{\partial z^*}{\partial z} dz + \frac{\partial z}{\partial z^*} dz^* + \frac{\partial z^*}{\partial z^*} dz^* \\ &= dz + \frac{\partial z^*}{\partial z} dz + \frac{\partial z}{\partial z^*} dz^* + dz^* \\ &= 2dx + \frac{\partial z^*}{\partial z} dz + \frac{\partial z}{\partial z^*} dz^* \end{aligned} \tag{C.24}$$

so that

$$\frac{\partial z^*}{\partial z} dz + \frac{\partial z}{\partial z^*} dz^* = 0 \tag{C.25}$$

Similarly we have

$$\begin{aligned} 2idy &= dz - dz^* \\ &= \frac{\partial (z + z^*)}{\partial z} dz - \frac{\partial (z + z^*)}{\partial z^*} dz^* \\ &= \frac{\partial z}{\partial z} dz + \frac{\partial z^*}{\partial z} dz - \frac{\partial z}{\partial z^*} dz^* - \frac{\partial z^*}{\partial z^*} dz^* \\ &= dz + \frac{\partial z^*}{\partial z} dz - \frac{\partial z}{\partial z^*} dz^* - dz^* \\ &= 2idy + \frac{\partial z^*}{\partial z} dz - \frac{\partial z}{\partial z^*} dz^* \end{aligned} \tag{C.26}$$

so that

$$\frac{\partial z^*}{\partial z} dz - \frac{\partial z}{\partial z^*} dz^* = 0 \tag{C.27}$$

Then together equations (C.25) and (C.27) imply equation (C.23) $\square$

Appendix D. The Formal Derivation of the Uncertainty Principle

For a hermetian operator, A, we have $\langle \Psi | A\Psi \rangle = \langle A\Psi | \Psi \rangle$ let us define $\langle A \rangle = \langle \Psi | A\Psi \rangle$ and

$$(\Delta A)^2 = \left\langle (A - \langle A \rangle)^2 \right\rangle = \left\langle A^2 \right\rangle - \langle A \rangle^2 \tag{D.1}$$

If we define $A' = A - \langle A \rangle$, then $\langle A' \rangle = 0$ and $(\Delta A')^2 = (\Delta A)^2$.

Now let A and B be hermetian such that $\langle A \rangle = 0$ and $\langle B \rangle = 0$. Then we have

$$(\Delta A)^2 (\Delta B)^2 = \left\langle \Psi | A^2 \Psi \right\rangle \left\langle \Psi | B^2 \Psi \right\rangle = \langle A\Psi | A\Psi \rangle \langle B\Psi | B\Psi \rangle \tag{D.2}$$

By the Cauchy Schwartz Inequality we have

$$\langle A\Psi | A\Psi \rangle \langle B\Psi | B\Psi \rangle \geq |\langle A\Psi | B\Psi \rangle|^2 = |\langle \Psi | AB\Psi \rangle|^2 \tag{D.3}$$

Every operator X can be written as the sum of a hermetian and skew-hermetian operator,

$$X = \frac{1}{2}\left(X + X^\dagger\right) + \frac{1}{2}\left(X - X^\dagger\right) \tag{D.4}$$

where $X^\dagger$ is the adjoint of X and satisfies $\langle \Psi | X\Psi \rangle = \left\langle X^\dagger \Psi | \Psi \right\rangle$. For any operators X and Y, we have $(XY)^\dagger = Y^\dagger X^\dagger$ so that

$$AB = \frac{1}{2}\left(AB + B^\dagger A^\dagger\right) + \frac{1}{2}\left(AB - B^\dagger A^\dagger\right) = \frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA) \tag{D.5}$$

and we now have

$$\begin{aligned} |\langle \Psi | AB\Psi \rangle|^2 &= \left| \Psi | \left(\frac{1}{2}(AB + BA) + \frac{1}{2}(AB - BA) \right) \Psi \right|^2 \\ &= \left| \left\langle \Psi | \frac{1}{2}(AB + BA)\Psi \right\rangle + \left\langle \Psi | \frac{1}{2}(AB - BA)\Psi \right\rangle \right|^2 \\ &= \left| \frac{1}{2}\langle \Psi | (AB + BA)\Psi \rangle + \frac{1}{2}\langle \Psi | (AB - BA)\Psi \rangle \right|^2 \end{aligned} \tag{D.6}$$

By conjugate symmetry of inner products, $\langle x, y \rangle = \langle y, x \rangle^*$, and we may write

$$C = \langle \Psi | AB\Psi \rangle = \langle AB\Psi | \Psi \rangle^* = \left\langle \Psi | (AB)^\dagger \Psi \right\rangle^* = \left\langle \Psi | B^\dagger A^\dagger \Psi \right\rangle^* = \langle \Psi | BA\Psi \rangle^* \tag{D.7}$$

so that

$$\Re(C) = \frac{1}{2}\langle \Psi | (AB + BA)\Psi \rangle \tag{D.8}$$

and

$$i\Im(C) = \frac{1}{2}\langle \Psi | (AB - BA)\Psi \rangle \tag{D.9}$$

Now we have

$$|\langle \Psi | AB\Psi \rangle|^2 = |\Re(C) + i\Im(C)|^2 = \Re(C)^2 + \Im(C)^2 \tag{D.10}$$

Putting everything together so far we have

$$(\Delta A)^2 (\Delta B)^2 \geq \left| \frac{1}{2}\langle \Psi | (AB + BA)\Psi \rangle \right|^2 + \left| \frac{1}{2}\langle \Psi | (AB - BA)\Psi \rangle \right|^2 \tag{D.11}$$

where the first term on the right hand side is the correlation function for the operators, and whose whose minimum value is zero. We define the commutator bracket for the operators

$$[A, B] = AB - BA \tag{D.12}$$

where we note that if a and b are scalars, then

$$[A + a, B + b] = [A, B] \tag{D.13}$$

Thus we have

$$(\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} |[A, B]|^2 \langle \Psi|\Psi \rangle^2 = \frac{1}{4} |[A, B]|^2 \tag{D.14}$$

with $\langle \Psi|\Psi \rangle = 1$ for a normalized wave function.

Equality in Cauchy Schwartz implies either of

(1) $A\Psi = 0$
(2) $B\Psi = 0$ or
(3) $A\Psi = \lambda B\Psi$

For the operators, x and k, where $p = \hbar k$, in the coordinate representation k is represented by $-i\frac{\partial}{\partial x}$ and x by itself. We calculate

$$\left[x, -i\frac{\partial}{\partial x}\right] \Psi = i\Psi \tag{D.15}$$

so that

$$(\Delta x)^2 (\Delta k)^2 \geq= \frac{1}{4} \tag{D.16}$$

Appendix E. The Wave Function with Minimum Uncertainty Product

E.1. **Derivation.** In Appendix D we find the uncertainty product

$$\Delta x \Delta k \geq \frac{1}{2}$$

Where $p = \hbar k$ and $k = -i\frac{\partial}{\partial x}$ in the coordinate representation. We see in Appendix C that equality in Cauchy Schwartz implies either

(1) $x\Psi = 0$ which is not identically the case for non-zero Ψ, and $\Psi = 0$ would not be acceptable since we must have $\langle \Psi, \Psi \rangle = 1$.
(2) $k\Psi = 0$ which would only be the case for $\Psi = const$ in which case $\langle \Psi, \Psi \rangle = 0,\ \infty$, or $-\infty$.
(3) $k\Psi = \lambda x\Psi$ and $\langle \Psi, (xk + kx)\, \Psi \rangle = 0$.

where[2] $\langle x \rangle = 0$ and $\langle k \rangle = 0$. We have then

$$-i\frac{\partial \Psi}{\partial x} = \lambda x \Psi$$

so

$$\frac{1}{\Psi} d\Psi = i\lambda x dx$$

[2] In the last we should say $(k - \langle k \rangle)\, \Psi = \lambda\, (x - \langle x \rangle)\, \Psi$ and $\langle \Psi, (x'k' + k'x')\, \Psi \rangle = 0$ with $x' = x - \langle x \rangle$ and $k' = k - \langle k \rangle$. If $\langle x \rangle = x_0$ and $\langle k \rangle = k_0$, then $(\Delta x)^2 = (\Delta x')^2$ and $(\Delta k)^2 = (\Delta k')^2$. Now we have $\langle \Psi|\, (x'k' + k'x')\, \Psi \rangle = 0$ to attain the minimum uncertainty product, but $\langle \Psi|\, (xk + kx)\, \Psi \rangle = 2x_0k_0$ which is immaterial.

and integrating yields

$$\ln \Psi = \frac{i\lambda x^2}{2} + C$$

so that

$$\Psi = C \exp\left(\left(\frac{a+ib}{2}\right) x^2\right)$$

where $i\lambda = a + ib$. Now $\langle \Psi, (xk + kx)\,\Psi\rangle = 0$ only if $b = 0$ so $i\lambda = a$ thus

$$\Psi = C \exp\left(\frac{1}{2} a x^2\right)$$

but can only be normalized for $a < 0$. Further if we want let $\langle k\rangle = k_0$, then we need

$$(k - k_0)\,\Psi = \lambda x \Psi$$

so that

$$\frac{1}{\Psi} d\Psi = (i\lambda x + ik_0)\, dx$$

which integrates to

$$\Psi = C \exp(ik_0 x) \exp\left(\frac{1}{2} a x^2\right)$$

To normalize we need

$$C^2 \int_{-\infty}^{\infty} \exp\left(a x^2\right) dx = 1$$

and we know to make $a = -\frac{1}{2(\Delta x)^2}$ and $C^2 = \frac{1}{\sqrt{2\pi}\Delta x}$ so that

$$\Psi^* \Psi = \frac{1}{\sqrt{2\pi}\Delta x} \exp\left(-\frac{1}{2}\left(\frac{x}{\Delta x}\right)^2\right)$$

so

$$\Psi = \frac{1}{\sqrt{\Delta x \sqrt{2\pi}}} \exp(ik_0 x) \exp\left(-\frac{1}{4}\left(\frac{x}{\Delta x}\right)^2\right) \tag{E.1}$$

If we want also $\langle x\rangle = x_0$, then

$$\Psi = \frac{1}{\sqrt{\Delta x \sqrt{2\pi}}} \exp(ik_0 x) \exp\left(-\frac{1}{4}\left(\frac{x - x_0}{\Delta x}\right)^2\right)$$

which is in fact the solution to $(k - k_0)\,\Psi = \lambda\,(x - x_0)\,\Psi$.

The Fourier Transform of (E.1) for $k_0 = 0$ is found to be

$$\phi(k) = \sqrt{\Delta x \sqrt{\frac{2}{\pi}}} \exp\left(-(\Delta x)^2 k\right)$$

and with $\Delta x \Delta k = \frac{1}{2}$, and $\langle k\rangle = k_0$ we have

$$\phi(k) = \frac{1}{\sqrt{\Delta k \sqrt{2\pi}}} \exp\left(-\frac{1}{4}\left(\frac{k - k_0}{\Delta k}\right)^2\right)$$

Let the constant group velocity $u = \frac{\Delta\omega}{\Delta k} = \frac{\Delta x}{\Delta t}$ (and not $u = \frac{d\omega}{dk} = \frac{dx}{dt}$ which implies functional relationships between x and t and between ω an k which are not present in the variables of the wave function,) and let us multiply equation (E.1) by the unit modulus factor, $\exp(-i\omega_0 t)$ to obtain

$$\Psi = \frac{1}{\sqrt{\Delta x \sqrt{2\pi}}} \exp\left(i\left(k_0 x - \omega_0 t\right)\right) \exp\left(-\frac{1}{4}\left(\frac{x-ut}{\Delta x}\right)^2\right)$$

where $\langle x \rangle = u\,(t - t_0) + x_0$. For any choice of ω_0 we can write $\langle \omega \rangle = u\,(k - k_0) + \omega_0$. The phase velocity is then $\mathrm{w} = \frac{\omega_0}{k_0}$. We can rewrite this elegantly as

$$\Psi = \sqrt{\Delta k \sqrt{\frac{2}{\pi}}} \exp\left(i\left(k_0 x - \omega_0 t\right)\right) \exp\left(-\left(\Delta k x - \Delta \omega t\right)^2\right)$$

Appendix F. Hamilton's Equations of Motion

Newton's second law forn a single particle in three dimensions moving in a potential V

$$\frac{d}{dt} m\left(\dot{x} + \dot{y} + \dot{z}\right) = -\nabla V \tag{F.1}$$

is cast as an extremal problem by forming the Lagrangian

$$\mathcal{L}\left(x, y, z, \dot{x}, \dot{y}, \dot{z}\right) = T - V \tag{F.2}$$

and minimizing the integral

$$I = \int_{t_1}^{t_2} \mathcal{L} dt \tag{F.3}$$

This will be realized if the Lagrangian satisfies the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \tag{F.4}$$

and similarly for y and z. The problem can be posed fro generalized coordinates, q^i, by making the substitutions $q^1 = x$, $q^2 = y$, and $q^3 = z$, and generalized to any number of particles. Thus the Lagrangian is given by

$$\mathcal{L}\left(q^1, \ldots, q^N, \dot{q}^1, \ldots, \dot{q}^N\right) = T - V \tag{F.5}$$

where the equations of motion are given by the N independent equations

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) = 0 \tag{F.6}$$

These are uncoupled second order differential equations involving $\ddot{q}^i = \frac{d}{dt}\dot{q}^i$, and can be converted to a set of first order coupled equations by the method of Hamilton where we proceed to give a brief overview.

We define the generalized momenta as p^i for $i = 1, \ldots, N$

$$\frac{\partial T}{\partial \dot{q}^i} = p^i \tag{F.7}$$

Where the potential does not depend upon the generalized velocity components, $\dot{q}^i$ we have

$$\frac{\partial T}{\partial \dot{q}^i} = \frac{\partial \mathcal{L}}{\partial q^i} = p^i \tag{F.8}$$

and define the Hamiltonian

$$\mathcal{H}\left(q^1,\ldots,q^N,p^1,\ldots,p^N\right)=\sum_{i=1}^{N}p^i\dot{q}^i-\mathcal{L} \tag{F.9}$$

After expressing the generalized velocity components as functions, $\dot{q}=\dot{q}\left(q^1,\ldots,q^N,p^1,\ldots,p^N\right)$ of the genralized coordinates and momenta with the use of the N equations (F.8) above. We now find

$$\frac{\partial\mathcal{H}}{\partial p^j} = \dot{q}^j+\sum_{i=1}^{N}p^i\frac{\partial\dot{q}^i}{\partial p^j}-\sum_{i=1}^{N}\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\frac{\partial\dot{q}^i}{\partial p^j} \tag{F.10}$$

$$= \dot{q}^j+\sum_{i=1}^{N}\left(p^i-\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right)\frac{\partial\dot{q}^i}{\partial p^j} \tag{F.11}$$

$$= \dot{q}^j \tag{F.12}$$

:Let us substitute (F.7) into (F.9) so that

$$\mathcal{H} = \sum_{i=1}^{N}\frac{\partial T}{\partial\dot{q}^i}\dot{q}^i-\mathcal{L} \tag{F.13}$$

$$= \sum_{i=1}^{N}\frac{\partial T}{\partial\dot{q}^i}\dot{q}^i-(T-V) \tag{F.14}$$

Where we have by a beautiful little theorem of Euler

$$2T=\sum_{i=1}^{N}\dot{q}^i\frac{\partial T}{\partial\dot{q}^i} \tag{F.15}$$

so that the Hamiltonial is found to be the sum of the kinetic and potential energies

$$\mathcal{H}=2T-(T-V)=T+V \tag{F.16}$$

We will digress here for a moment to prove that theorem of Euler, firstly a function $F\left(x,\ldots,z,u,\ldots,v\right)$ is said to be homgeneous of degree N in the variables $u,\ldots v$ if

$$F\left(x,\ldots,z,tu,\ldots,tv\right)=t^N F\left(x,\ldots,z,u,\ldots v\right) \tag{F.17}$$

so let

$$G=F\left(x,\ldots,z,tu,\ldots,tv\right) \tag{F.18}$$

then

$$\frac{\partial G}{\partial u}=t^N\frac{\partial F}{\partial u}=\frac{\partial G}{\partial\left(tu\right)}\frac{\partial\left(tu\right)}{\partial u}=t\frac{\partial G}{\partial\left(tu\right)} \tag{F.19}$$

or

$$\frac{\partial G}{\partial\left(tu\right)}=t^{N-1}\frac{\partial F}{\partial u} \tag{F.20}$$

Now

$$\frac{dG}{dt} = \frac{\partial G}{\partial (tu)}\frac{d(tu)}{dt} + \cdots + \frac{\partial G}{\partial (tv)}\frac{d(tv)}{dt} \tag{F.21}$$

$$= u\frac{\partial G}{\partial (tu)} + \cdots + v\frac{\partial G}{\partial (tv)} \tag{F.22}$$

$$= t^{N-1}\left(u\frac{\partial F}{\partial u} + \cdots + v\frac{\partial F}{\partial v}\right) \tag{F.23}$$

but we also have

$$\frac{dG}{dt} = nt^{N-1}F \tag{F.24}$$

and the result follows.

We have now found in (F.12) the N equations

$$\frac{\partial \mathcal{H}}{\partial p^j} - \dot{q}^j = 0 \tag{F.25}$$

and we would like to minimize the integral of the Lagrangian with respect to time, where from (F.9) we have

$$\mathcal{L} = \sum_{i=1}^{N} p^i\dot{q}^i - \mathcal{H} \tag{F.26}$$

Thus we wish to minimize

$$I = \int_{t_1}^{t_2}\left(\sum_{i=1}^{N} p^i\dot{q}^i - \mathcal{H}\right)dt \tag{F.27}$$

subject to the N equations of constraint (F.25). This is accomplished through the method of Lagrange multipliers albeit with undetermined functions, $\mu_i(t)$ by forming the function

$$F = \sum_{i=1}^{N} p^i\dot{q}^i - \mathcal{H} + \sum_{i=1}^{N}\mu_i(t)\left(\frac{\partial \mathcal{H}}{\partial p^i} - \dot{q}^i\right) \tag{F.28}$$

where we minimize the integral

$$J = \int_{t_1}^{t_2} F dt \tag{F.29}$$

This is called an isoperimetric problen in the calculus of variations, whose solution is given by two sets of N Euler-Lagrange equations

$$\frac{\partial F}{\partial p^j} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{p}^j}\right) = 0 \tag{F.30}$$

$$\frac{\partial F}{\partial q^j} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}^j}\right) = 0 \tag{F.31}$$

Now the function F does not depend on $\dot{p}$ and so

$$\frac{\partial F}{\partial \dot{p}^j} = 0 \tag{F.32}$$

is identically satisfied, thus the first set of N Euler-Lagrange equations read

$$\frac{\partial F}{\partial p^j} = 0 \tag{F.33}$$

or evaluating this expression

$$\frac{\partial F}{\partial p^j} = \dot{q}^j - \frac{\partial \mathcal{H}}{\partial p^j} + \sum_{i=1}^{N} \mu_i\left(t\right) \frac{\partial^2 \mathcal{H}}{\partial p^j \partial p^i} \tag{F.34}$$

$$0 = \sum_{i=1}^{N} \mu_i\left(t\right) \frac{\partial^2 \mathcal{H}}{\partial p^j \partial p^i} \tag{F.35}$$

This matrix equation certainly has the solution, $\mu_i\left(t\right) = 0$, and is the only solution if the determinant

$$\left| \frac{\partial^2 \mathcal{H}}{\partial p^j \partial p^i} \right| \neq 0 \tag{F.36}$$

We accept this here, but is a consequence of the assumption that the potential energy depends only on the generalized coordinates and time, but not on the generalized velocity components, and the assumption that the kinetic energy, T is a positive definite bilinear form in the generalized momenta. Well alright let's look at this we have

$$\frac{\partial^2 \mathcal{H}}{\partial p^j \partial p^i} = \frac{\partial^2 T}{\partial p^j \partial p^i} \tag{F.37}$$

where

$$T = p^i a_{ij} p^j \tag{F.38}$$

so that

$$\frac{\partial T}{\partial p^i} = a_{ij} p^j + p^i a_{ii} \tag{F.39}$$

hence

$$\frac{\partial^2 T}{\partial p^j \partial p^i} = a_{ij} + a_{ji} \tag{F.40}$$

now since the matrix A is positive definite it is symmetric, and its determinant is not zero so

$$\frac{\partial^2 T}{\partial p^j \partial p^i} = a_{ij} + a_{ji} = 2a_{ij} \tag{F.41}$$

Hence (F.36) follows.

Let us now with our unique set of Lagrange multipliers, $\mu_i\left(t\right) = 0$ evaulate the second set of N Euler-Lagrange equations

$$\frac{\partial F}{\partial q^j} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}^j}\right) = 0 \tag{F.42}$$

we might leap immediately to

$$-\frac{\partial \mathcal{H}}{\partial q^j} - \frac{dp^j}{dt} = 0 \tag{F.43}$$

but more properly we should find

$$\sum_{i=1}^{N} p^i \frac{\partial \dot{q}^i}{\partial q^j} - \frac{\partial \mathcal{H}}{\partial q^j} - \frac{d}{dt}\left(p^j + \mu_j\right) = 0 \tag{F.44}$$

where $\mu_j = 0$, and when we consider $\dot{q}^i$ to be independent variables in $F,$ despite our secret knowledge that it may considered a function of the generalized momenta and coordinates, we have

$$\frac{\partial \dot{q}}{\partial q^j} = 0 \tag{F.45}$$

So that Hamilton's equations of motion are obtained from (F.25) and (F.43) as

$$\frac{\partial \mathcal{H}}{\partial p^j} = \frac{dq^j}{dt} \tag{F.46}$$

$$\frac{\partial \mathcal{H}}{\partial q^j} = -\frac{dp^j}{dt} \tag{F.47}$$

Now what more could we ask for, we have already left cartesian coordinates for generalized coordinate, but wait perhaps we can perform a canonical transformation on our generalized coordinates to new set where Hamilton's equations are especially easy to integrate. By a canonical transformation we wish to express our generalized coordinates, q^i and p^i in terms of new variables, Q^i and P^i with a new Hamiltonian, K, which is a function of the new generalized coordinates, and possibly time. Let us stipulate first that there be no functional relationship between the q^i and the Q^j, and that these are completely independent of the p^i and P^j. Now let us suppose the identity

$$\mathcal{L} = \sum_{i=1}^{N} p^i \dot{q}^i - H = \sum_{i=1}^{N} P^i \dot{Q}^i - K + \frac{dS}{dt} \tag{F.48}$$

where S is a function of the q^i, the Q^j and time, t. If we expand the time derivative of $S,$ we find

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial S}{\partial q^i}\dot{q}^i + \frac{\partial S}{\partial Q^i}\dot{Q}^i\right) \tag{F.49}$$

Substituting this back into our identity (F.48) and multiplying by the differential, dt, we obtain

$$\mathcal{L}dt = \sum_{i=1}^{N} p^i dq^i - H dt = \sum_{i=1}^{N} P^i dQ^i - K dt + \frac{\partial S}{\partial t}dt + \sum_{i=1}^{N}\left(\frac{\partial S}{\partial q^i}dq^i + \frac{\partial S}{\partial Q^i}dQ^i\right) \tag{F.50}$$

and rearranging term, there follows

$$\sum_{i=1}^{N}\left(p^i - \frac{\partial S}{\partial q^i}\right)dq^i + \left(K - H - \frac{\partial S}{\partial t}\right)dt - \sum_{i=1}^{N}\left(P^i + \frac{\partial S}{\partial Q^i}\right)dQ^i = 0 \tag{F.51}$$

where the quantities in parentheses must all now be identically zero, thus

$$p^i = \frac{\partial S}{\partial q^i} \tag{F.52}$$

$$P^i = -\frac{\partial S}{\partial Q^i} \tag{F.53}$$

$$K = H + \frac{\partial S}{\partial t} \tag{F.54}$$

The first two of these equations allow us to express our old generalized coordinates, q^i, and p^i in terms of the new variables,Q^i and P^i where we now reexpress S and H , and thus K, solely in terms of the new variables. We obtain new equations of motion by minimizing the integral

$$I = \int_{t_1}^{t_2} \mathcal{L} dt = \int_{t_1}^{t_2} \left(\sum_{i=1}^{N} P^i \dot{Q}^i - K + \frac{dS}{dt} \right) dt \tag{F.55}$$

subject to the constraint

$$\dot{Q} = \frac{\partial K}{\partial P} \tag{F.56}$$

Now it just happens to be that case that the necessary and sufficient condition that the Euler-Lagrange equation be identically satisfied by a function G, is that it be the total derivative of some function S. That is

$$\frac{\partial G}{\partial Q^i} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{Q}^i}\right) = 0 \tag{F.57}$$

$$\frac{\partial G}{\partial P^i} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{P}^i}\right) = 0 \tag{F.58}$$

are identically satisfied when

$$G = \frac{d}{dt} S\left(t, Q^1, \ldots, Q^N, P^1, \ldots P^N\right) \tag{F.59}$$

What this means is that if we form the function

$$F = \sum_{i=1}^{N} P^i \dot{Q}^i - K + \sum_{i=1}^{N} \mu_i\left(t\right) \left(\dot{Q}^i - \frac{\partial K}{\partial P^i} \right) \tag{F.60}$$

to minimize the integral

$$I = \int_{t_1}^{t_2} \left(\sum_{i=1}^{N} P^i \dot{Q}^i - K \right) dt \tag{F.61}$$

subject to the constraint (F.56), then the equations of motion obtained from

$$\frac{\partial F}{\partial P^j} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{P}^j}\right) = 0 \tag{F.62}$$

$$\frac{\partial F}{\partial Q^j} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{Q}^j}\right) = 0 \tag{F.63}$$

are given in exactly the same manner as (F.25) through (F.47) by

$$\frac{\partial K}{\partial P^i} = \frac{dQ^i}{dt} \tag{F.64}$$

$$\frac{\partial K}{\partial Q^i} = -\frac{dP^i}{dt} \tag{F.65}$$

and in makes absolutely no difference if we add to the function, F the total derivative of S, since the Euler-Lagrange equations are linear, which is precisely the problem posed in (F.55) above.

What have we gained,however, by this canonical change of variables? Well suppose the new Hamiltonian, K were identically zero, we would then find

$$\frac{dQ^i}{dt} = 0 \tag{F.66}$$

$$\frac{dP^i}{dt} = 0 \tag{F.67}$$

which then implies that the new generalized coordinates and momenta are a set of arbitrary constants independent of time

$$Q^i = a^i \tag{F.68}$$

$$P^i = b^i \tag{F.69}$$

these are to be determined by the initial conditions of the problem say when $q^i(0)$ and $p^i(0)$ are specified. Further from equation (F.54)

$$H\left(q^1, \ldots, q^N, p^1, \ldots, p^N\right) + \frac{\partial S}{\partial t} = 0 \tag{F.70}$$

where we the Hamiltonian is in terms of the old generalized coordinates and momenta, and the function S is considered terms of the old generalized coordinates, q^i and new, Q^i, Now because of equations (F.52) we discover the Hamilton-Jacobi equation

$$H\left(q^1, \ldots, q^N, \frac{\partial S}{\partial q^1}, \ldots, \frac{\partial S}{\partial q^N}\right) + \frac{\partial S}{\partial t} = 0 \tag{F.71}$$

The complete solution to this PDE is expressed in terms of the generalized coordinates, q^i and N arbitrary constants of integration which we identify with new generalized coordinates, Q^i, thus

$$S = S\left(q^1, \ldots, q^N, Q^1, \ldots, Q^N\right) \tag{F.72}$$

Note that if S is a solution to the Hamilton-Jacobi Equation then so is $S+C$ where C is a constant, and we do not identify this constant of integration with any of the new generalized coordinates. The function S is refered to as Hamilton's Principal function

Now suppose that we may write

$$S = S^* - Et \tag{F.73}$$

where here S^* is not to be taken as the complex conjugate, and does not depend explicitly on time, then since

$$\frac{\partial S}{\partial q^i} = \frac{\partial S^*}{\partial q^i} \tag{F.74}$$

$$\frac{\partial S}{\partial q^i} = -E \tag{F.75}$$

the Hamilton-Jacobi equation becomes

$$H\left(q^1, \ldots, q^N, \frac{\partial S^*}{\partial q^1}, \ldots, \frac{\partial S^*}{\partial q^N}\right) = E \tag{F.76}$$

which is called the *reduced* Hamilton-Jacobi equation. The case of a single particle of mass, m, and free of geometric constraints, we make take the ordinary cartesian coordinates as our generalized coordinates, and the reduced Hamilton-Jacobi equation reads in this case

$$\frac{1}{2m}\left[\left(\frac{\partial S^*}{\partial x}\right)^2+\left(\frac{\partial S^*}{\partial y}\right)^2+\left(\frac{\partial S^*}{\partial z}\right)^2\right]+V\left(x,y,z\right)=E \tag{F.77}$$

Appendix G. The Fourier Transform

G.1. **Definition.** The Forward Fourier Transform

$$F\left(\kappa\right)=\int_{-\infty}^{\infty}f\left(x\right)\exp\left(-2\pi i\kappa x\right)dx \tag{G.1}$$

and the Backward Fourier Transform

$$f\left(x\right)=\int_{-\infty}^{\infty}F\left(\kappa\right)\exp\left(2\pi i\kappa x\right)d\kappa$$

Let $k=2\pi\kappa$, then $p=\hbar k=h\kappa=\frac{h}{\lambda}$ where wave length $\lambda=\frac{1}{\kappa}$ and $\hbar=\frac{h}{2\pi}$. We have Parseval's Theorem, $\int\left|F\left(\kappa\right)\right|^2d\kappa=\int\left|f\left(x\right)\right|^2dx$.

For time and frequency we have

$$G\left(\nu\right)=\int_{-\infty}^{\infty}g\left(t\right)\exp\left(-2\pi i\nu t\right)dt$$

and

$$g\left(t\right)=\int_{-\infty}^{\infty}G\left(\nu\right)\exp\left(2\pi i\nu t\right)d\nu$$

Let $\omega=2\pi\nu$, then $E=\hbar\omega=h\nu$, and period $T=\frac{1}{\nu}$. Again Parseval's $\left|G\left(\nu\right)\right|^2=\left|g\left(t\right)\right|^2$.

G.2. **Informal Proof of the Fourier Integral Transform.** For a function $f\left(x\right)$ defined on an open interval, (a,b), the integral,

$$\int_a^b f\left(x\right)dx=\lim_{\Delta x\to 0}\sum_{a<n\Delta x<b}f\left(n\Delta x\right)\Delta x$$

so if

$$f\left(x\right)=\sum_{n=-\infty}^{\infty}c_n\exp\left(in\pi x/l\right)$$

and

$$c_n=\frac{1}{2l}\int_{-l}^{l}f\left(u\right)\exp\left(-in\pi u/l\right)dx$$

then let

$$\Delta\alpha=\frac{\left(n+1\right)\pi}{l}-\frac{n\pi}{l}=\frac{\pi}{l}$$

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \frac{\Delta\alpha}{2\pi} \left(\int\limits_{-\pi/\Delta\alpha}^{\pi/\Delta\alpha} f(u) \exp(-iun\Delta\alpha)\, du \right) \exp(ixn\Delta\alpha) \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int\limits_{-\pi/\Delta\alpha}^{\pi/\Delta\alpha} f(u) \exp(-iun\Delta\alpha)\, du \right) \exp(ixn\Delta\alpha)\, \Delta\alpha \\
&\to \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{\infty} f(u) \exp(-iu\alpha)\, du \right) \exp(ix\alpha)\, d\alpha
\end{aligned}
$$

G.3. **Other Forms of the Fourier Transform.** If we let $k = 2\pi\kappa$ in (G.1) we obtain the transform pair,

Forward

$$F(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx \tag{G.2}$$

and Backward

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(ikx)\, dk$$

but then Parseval's Theorem reads, $\int |F(k)|^2\, dk = 2\pi \int |f(x)|^2\, dx$.

Now if we define a new function, $\sqrt{2\pi}\tilde{F}(k) = F(k)$ we obtain the transform pair commonly used in quantum physics,

$$\tilde{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx)\, dx \tag{G.3}$$

and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(k) \exp(ikx)\, dk$$

returnin Parseval's Theorem to, $\int \left|\tilde{F}(k)\right|^2 dk = \int |f(x)|^2\, dx$.

G.4. **Some Properties of the Fourier Transform.** The convolution of two functions is defined by

$$
\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \\
&= \int_{-\infty}^{\infty} f(x-u)\, g(u)\, du
\end{aligned}
$$

and on a finite range $[0, x]$ then

$$
\begin{aligned}
(f * g)(x) &= \int_{0}^{x} f(u)\, g(x-u)\, du \\
&= \int_{0}^{x} f(x-u)\, g(u)\, du
\end{aligned}
$$

The Fourier transform of a product is given by

$$\mathcal{F}\left[f\left(x\right)g\left(x\right)\right]\left(k\right)=\left(F*G\right)\left(k\right)=\int_{-\infty}^{\infty}F\left(k-s\right)G\left(s\right)ds$$

$$\mathcal{F}\left[\left(f*g\right)\left(x\right)\right]\left(k\right)=\mathcal{F}\left[f\left(x\right)\right]\left(k\right)\mathcal{F}\left[g\left(x\right)\right]\left(k\right)$$

If

$$\mathcal{F}\left[\psi\left(x\right)\right]\left(k\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi\left(x\right)\exp\left(-ikx\right)dx$$

then

$$\begin{aligned}\mathcal{F}^{*}\left[\psi\left(x\right)\right]\left(k\right)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi\left(x\right)\exp\left(ikx\right)dx\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi\left(-u\right)\exp\left(-iku\right)du\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi\left(-x\right)\exp\left(-ikx\right)dx\end{aligned}$$

therefore

$$\mathcal{F}^{*}\left[\psi\left(x\right)\right]\left(k\right)=\mathcal{F}\left[\psi\left(-x\right)\right]\left(k\right)=\mathcal{F}\left[\psi\left(x\right)\right]\left(-k\right)$$

and

$$\mathcal{F}\left[\psi^{*}\left(x\right)\right]\left(k\right)=\mathcal{F}^{*}\left[\psi^{*}\left(-x\right)\right]\left(k\right)=\mathcal{F}^{*}\left[\psi^{*}\left(x\right)\right]\left(-k\right)$$

An easy mistake to make: Let

$$\mathcal{F}\left[\psi\left(x\right)\right]\left(k\right)=\phi\left(k\right)$$

then note that

$$\mathcal{F}\left[\psi^{*}\left(x\right)\right]\left(k\right)\neq\phi^{*}\left(k\right)$$

in fact

$$\phi^{*}\left(k\right)=\mathcal{F}^{*}\left[\psi^{*}\left(x\right)\right]\left(k\right)=\mathcal{F}\left[\psi^{*}\left(-x\right)\right]\left(k\right)=\mathcal{F}\left[\psi^{*}\left(x\right)\right]\left(-k\right)$$

The characteristic function of the density $\psi^{*}\psi$ is given by (Wiener-Khinchin Theorem)

$$\begin{aligned}\mathcal{F}\left[\psi^{*}\left(x\right)\psi\left(x\right)\right]\left(k\right)&=\mathcal{F}\left[\psi^{*}\left(x\right)\right]*\mathcal{F}\left[\psi\left(x\right)\right]\\&=\int_{-\infty}^{\infty}\mathcal{F}\left[\psi^{*}\left(x\right)\right]\left(u\right)\mathcal{F}\left[\psi\left(x\right)\right]\left(k-u\right)du\\&=\int_{-\infty}^{\infty}\mathcal{F}^{*}\left[\psi^{*}\left(x\right)\right]\left(-u\right)\mathcal{F}\left[\psi\left(x\right)\right]\left(k-u\right)du\\&=-\int_{\infty}^{-\infty}\mathcal{F}^{*}\left[\psi^{*}\left(x\right)\right]\left(s\right)\mathcal{F}\left[\psi\left(x\right)\right]\left(k+s\right)ds\\&=\int_{-\infty}^{\infty}\mathcal{F}^{*}\left[\psi^{*}\left(x\right)\right]\left(s\right)\mathcal{F}\left[\psi\left(x\right)\right]\left(k+s\right)ds\\&=\int_{-\infty}^{\infty}\phi^{*}\left(u\right)\phi\left(k+u\right)du\end{aligned}$$

G.5. The Fourier Transform of a Gaussian.

$$
\begin{aligned}
\phi(k) = \; & \tfrac{C}{\sqrt{2\pi}} \int \exp\left(-\tfrac{1}{4}\left(\tfrac{x}{\Delta x}\right)^2\right) \exp(-ikx)\, dx \\
& = \tfrac{C}{\sqrt{2\pi}} \int \exp\left(-\tfrac{1}{4}\left(\tfrac{x}{\Delta x}\right)^2 - ikx\right) dx \\
& = \tfrac{C}{\sqrt{2\pi}} \int \exp\left(-\left(\tfrac{1}{2\Delta x}\right)^2 \left(x^2 - 4ikx(\Delta x)^2 + \left(2k(\Delta x)^2\right)^2 - \left(2k(\Delta x)^2\right)^2\right)\right) dx \\
& = \tfrac{C}{\sqrt{2\pi}} \exp\left(-k^2(\Delta x)^2\right) \int \exp - \left(\tfrac{1}{2\Delta x}\right)^2 \left(x - 2ik(\Delta x)^2\right)^2 dx \\
& = \tfrac{C}{\sqrt{2\pi}} \exp\left(-k^2(\Delta x)^2\right) \int \exp - \tfrac{1}{2} \left(\tfrac{x - 2ik(\Delta x)^2}{\sqrt{2}\Delta x}\right)^2 dx \\
& = \tfrac{C\sqrt{2}\Delta x}{\sqrt{2\pi}} \exp\left(-k^2(\Delta x)^2\right) \int \exp\left(-\tfrac{1}{2}u^2\right) du \\
& = C\sqrt{2}\Delta x \exp\left(-k^2(\Delta x)^2\right)
\end{aligned}
$$

Then with $C^2 = \frac{1}{\sqrt{2\pi}\Delta x}$ we obtain

$$
\phi(k) = \sqrt{\Delta x \sqrt{\frac{2}{\pi}}} \exp\left(-k^2(\Delta x)^2\right)
$$

and if $\Delta x \Delta k = \frac{1}{2}$ we obtain

$$
\phi(k) = \frac{1}{\sqrt{\Delta k \sqrt{2\pi}}} \exp\left(-\frac{1}{4}\left(\frac{k}{\Delta k}\right)^2\right)
$$

Appendix H. Re-Expressing Equations From $c = 1$ to $c \neq 1$

Type	variable		replace by
mass	m	$\rightarrow$	mc
time	t	$\rightarrow$	ct
proper time	τ	$\rightarrow$	$c\tau$
velocity	v	$\rightarrow$	v/c
proper velocity	η^0	$\rightarrow$	η^0/c
rest and kinetic	E	$\rightarrow$	E/c
angular frequency	ω	$\rightarrow$	ω/c
Gravitation	G	$\rightarrow$	G/c^3

Planck Time, Mass, and Length are given by

$$
t_p = \sqrt{\frac{\hbar G}{c^5}},\ l_p = \sqrt{\frac{\hbar G}{c^3}},\ \text{and } m_p = \sqrt{\frac{\hbar c}{G}}
$$

Appendix I. Baye's Theorem

I.1. **Baye's Theorem.** Baye's Theorem states that for a joint probability distribution, $f(x,t)$ then

$$
f(x|t)\, f_T(t) = f(x,t) = f(t|x)\, f_X(x)
$$

The problem here is that if only $f(x|t)$ is given and whereas we may suppose that the true marginal (prior) distribution $f_T(t)$ exists, we can only make a motivated or unmotivated guess at what it might be. This is why Bayesian statistics are considered subjective.

I.2. **Gamma-Poisson Distribution.** Let K be a Poisson random variable with discrete conditional distribution,

$$\Pr(K = k|S = s) = P(k|s) = \frac{s^k e^{-s}}{k!}$$

so s can be thought of as a time variable where the rate of events per unit time is 1. The mean,

$$\mu_{K|S} = s$$

and the variance

$$\sigma^2_{K|S} = s$$

Let the prior distribution of S be a gamma distribution with parameters $\alpha = 1$ and $\beta = 1/b$. Then

$$f_S(s) = be^{-bs}$$

with mean, $\mu_S = \beta = 1/b$ and variance, $\sigma^2_S = \beta^2 = 1/b^2$. The posterior density of $S = s$ given $K = k$ is given by Baye's Theorem:

$$f(s|k) = \frac{P(k|s) f_S(s)}{\int_0^\infty P(k|s) f_S(s)\, ds}$$

and we find

$$f(s|k) = \frac{(b+1)^{k+1} s^k e^{-(b+1)s}}{k!}$$

with mean

$$\mu_{S|K} = (1+k)/(b+1)$$

and variance

$$\sigma^2_{S|K} = (1+k)/(b+1)^2$$

so in the limit as the event rate $b \to 0$ in the prior distribution,

$$f(s|k) = \frac{s^k e^{-s}}{k!} = P(k|s)$$

with mean

$$\mu_{S|K} = (1+k)$$

and variance

$$\sigma^2_{S|K} = (1+k)$$

And if we let

$$s = \lambda t$$

then

$$P(k|t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

with mean $\mu_{K|T} = \lambda t$ and variance $\sigma^2_{K|T} = \lambda t$.

The prior distribution of T has the density,

$$f_T(t) = be^{-b\lambda t}\frac{ds}{dt} = b\lambda e^{-b\lambda t}$$

and

$$g(t|k) = \frac{(b+1)^{k+1}(\lambda t)^k e^{-(b+1)\lambda t}}{k!}\frac{ds}{dt} = \frac{(b+1)^{k+1}\lambda^{k+1}t^k e^{-(b+1)\lambda t}}{k!}$$

with mean

$$\mu_{T|K} = \frac{(1+k)}{\lambda(b+1)}$$

and variance

$$\sigma^2_{S|K} = (1+k)/\lambda^2(b+1)^2$$

Again in the limit as $b \to 0$ we have

$$g(t|k) = \frac{\lambda^{k+1}t^k e^{-\lambda t}}{k!} = \lambda P(k|\lambda t)$$

with mean

$$\mu_{T|K} = (1+k)/\lambda$$

and variance

$$\sigma^2_{T|K} = (1+k)/\lambda^2$$

The waiting time distribution for successive events in a Poisson process is in fact

$$g(t|0) = \lambda e^{-\lambda t}$$

which could be obtained also by

$$D(t) = \Pr(T \leqslant t) = \Pr(K \geqslant 1|T=t) = 1 - \Pr(K=0|T=t) = 1 - \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = 1 - e^{-\lambda t}$$

so that the density

$$g(t) = D'(t) = \lambda e^{-\lambda t}$$

We note that

$$g(t|k) = \frac{P(k|\lambda t)}{\int_0^\infty P(k|\lambda t)\,dt} = \lambda P(k|\lambda t)$$

isn't an example of Baye's theorem, since $f_S(s) = 1$ isn't a probability density.

$$P(k|s) = \frac{f(s|k)P_K(k)}{\sum\limits_{k=0}^{\infty} f(s|k)P_K(k)}$$

$$P_K(k) = \int_0^\infty P(k|s) f_S(s)\,ds = \int_0^\infty \frac{s^k e^{-s}}{k!} b e^{-bs} ds = \frac{b}{(b+1)^{k+1}}$$

for $b > 0$

$$\sum_{k=0}^{\infty} f(s|k)P_K(k) = \sum_{k=0}^{\infty}\frac{(b+1)^{k+1}s^k e^{-(b+1)s}}{k!}\frac{b}{(b+1)^k} = be^{-(b+1)s}\sum_{k=0}^{\infty}\frac{s^k}{k!} = be^{-(b+1)s}e^s = be^{-bs} = f_S(s)$$

I.3. **Normal-Normal Distribution.** The following function can be interpreted in several ways:

$$f\left(x;y\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-y}{\sigma}\right)^{2}\right)$$

It could a normal distribution for a random variable X with mean y, or it could be for a random variable Y with mean x. Suppose we interpret it to be the conditional distribution for X given the mean y. So let

$$f\left(x|y\right)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{x-y}{\sigma}\right)^{2}\right)$$

Suppose the prior (absolute) distribution of Y is also normal:

$$f_{Y}\left(y\right)=\frac{1}{\sigma_{0}\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{y-\mu_{0}}{\sigma_{0}}\right)^{2}\right)$$

then the joint distribution of X and Y is given by

$$f\left(x,y\right)=f\left(x|y\right)f_{Y}\left(y\right)=f\left(y|x\right)f_{X}\left(x\right)=f\left(y|x\right)\int\limits_{-\infty}^{\infty}f\left(x|y\right)f_{Y}\left(y\right)dy$$

So the posterior distribution of Y given $X=x$ is

$$f\left(y|x\right)=\frac{f\left(x|y\right)f_{Y}\left(y\right)}{\int\limits_{-\infty}^{\infty}f\left(x|y\right)f_{Y}\left(y\right)dy}$$

which is a normal distribution with mean

$$\mu_{Y|X}=\frac{\sigma^{2}\mu_{0}+\sigma_{0}^{2}x}{\sigma^{2}+\sigma_{0}^{2}}$$

and variance

$$\sigma_{Y|X}^{2}=\frac{\sigma^{2}\sigma_{0}^{2}}{\sigma^{2}+\sigma_{0}^{2}}$$

which is independent of any observation made of X.

$$\lim_{\sigma_{0}\to\infty}\mu_{Y|X}=x$$

and

$$\lim_{\sigma_{0}\to\infty}\sigma_{Y|X}^{2}=\sigma^{2}$$

$$\lim_{\sigma_{0}\to 0}\mu_{Y|X}=\mu_{0}$$

and

$$\lim_{\sigma_{0}\to 0}\sigma_{Y|X}^{2}=0$$

As an example suppose $\sigma^{2}=1$, μ_{0} and σ_{0}^{2} are unknown so we make an initial guess at their values, $\mu_{0}=0$ and $\sigma_{0}^{2}=1$, then

$$\mu_{1}=\frac{\sigma^{2}\mu_{0}+\sigma_{0}^{2}x_{1}}{\sigma^{2}+\sigma_{0}^{2}}=\frac{x_{1}}{2}$$

and

$$\sigma_1^2 = \frac{\sigma^2 \sigma_0^2}{\sigma^2 + \sigma_0^2} = \frac{1}{2}$$

where in general

$$\sigma_n^2 = \frac{1}{n+1} \to 0$$

Then

$$\mu_2 = \frac{\mu_1 + \sigma_1^2 x_2}{1 + \sigma_1^2} = \frac{\frac{x_1}{2} + \frac{1}{2} x_2}{1 + \frac{1}{2}} = \frac{x_1 + x_2}{3}$$

and in general

$$\mu_n = \frac{1}{n+1} \sum_{i=1}^{n} x_i \to \bar{x}$$

If we update our prior distribution after each single observation of random variable X according to

$$\mu_n = \frac{\sigma^2 \mu_{n-1} + \sigma_{n-1}^2 x_n}{\sigma^2 + \sigma_{n-1}^2}$$

and

$$\sigma_n^2 = \frac{\sigma^2 \sigma_{n-1}^2}{\sigma^2 + \sigma_{n-1}^2}$$

we obtain after n observations of the random variable X:

$$\mu_n = \frac{\sigma^2 \mu_0 + \sigma_0^2 \left(x_1 + ... + x_n\right)}{\sigma^2 + n\sigma_0^2} = \frac{\frac{\sigma^2}{n} \mu_0 + \sigma_0^2 \bar{x}}{\frac{\sigma^2}{n} + \sigma_0^2}$$

and

$$\sigma_n^2 = \frac{\sigma^2 \sigma_0^2}{\sigma^2 + n\sigma_0^2} = \frac{\left(\frac{\sigma^2}{n}\right) \sigma_0^2}{\left(\frac{\sigma^2}{n}\right) + \sigma_0^2}$$

which are precisely the formulas for the posterior distribution obtained from observing the random variable $\bar{X}$ for a sample of size n with the prior distribution assumed to be $X \sim N\left(\mu_0, \sigma_0^2\right)$. And yet our sequence of updated priors has no relation to the true prior distribution. The sequence tends to a delta distribution, $\sigma_n^2 \to 0$, with mean $\bar{x}$.

I.4. **Beta-Binomial Distribution.** The probability of α successes in n trials where the probability of success is x is given by

$$P\left(\alpha\right) = \left(\begin{array}{c} n \\ \alpha \end{array}\right) x^{\alpha} \left(1 - x\right)^{n-\alpha}$$

then

$$E\left(\alpha\right) = \sum_{\alpha=0}^{n} \alpha \left(\begin{array}{c} n \\ \alpha \end{array}\right) x^{\alpha} \left(1 - x\right)^{n-\alpha} = nx$$

and

$$E\left(\alpha^2\right) = \sum_{\alpha=0}^{n} \alpha^2 \left(\begin{array}{c} n \\ \alpha \end{array}\right) x^{\alpha} \left(1 - x\right)^{n-\alpha} = nx\left(nx + \left(1 - x\right)\right)$$

so that

$$E\left(\left(\alpha - nx\right)^2\right) = nx\left(1 - x\right)$$

and if

$$x = \frac{1}{2}$$

then

$$E\left(\left(\alpha - \frac{n}{2}\right)^2\right) = \frac{n}{4}$$

and then

$$\sqrt{E\left((2\alpha - n)^2\right)} = \sqrt{n}$$

which is the root mean square difference between successes and failures in n trials. Now let $n = \alpha+\beta$ so that $P(\alpha) = \left(\begin{array}{c}\alpha+\beta\\ \alpha\end{array}\right) x^{\alpha}(1-x)^{\beta}$ which we will treat as a function of x and normalize $\int_0^1 h(x)\,dx = \left(\begin{array}{c}\alpha+\beta\\ \alpha\end{array}\right)\int_0^1 x^{\alpha}(1-x)^{\beta}\,dx = \left(\begin{array}{c}\alpha+\beta\\ \alpha\end{array}\right) B(\alpha+1,\beta+1)$, then the function $f(x) = \frac{x^{\alpha}(1-x)^{\beta}}{B(\alpha+1,\beta+1)}$ is a continuous density for the random variable X. Now $\left(\begin{array}{c}\alpha+\beta\\ \alpha\end{array}\right) = \frac{(\alpha+\beta)!}{\alpha!\beta!} = \frac{\alpha+\beta}{\alpha\beta}\frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!} = \frac{\alpha+\beta}{\alpha\beta}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{\alpha+\beta}{\alpha\beta}\frac{1}{B(\alpha,\beta)}$

Let $f(\alpha|x) = \left(\begin{array}{c}n\\ \alpha\end{array}\right) x^{\alpha}(1-x)^{n-\alpha}$ be the conditional distribution of α given $X = x$, and let

$$g(x) = \begin{cases} 1 & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Be the prior density of X. The uniform distribution is a special case of the beta distribution: $g(x) = \frac{x^0(1-x)^0}{B(1,1)}$

Then the joint distribution-density is $f(\alpha, x) = \left(\begin{array}{c}n\\ \alpha\end{array}\right) x^{\alpha}(1-x)^{n-\alpha}$ and

$$\int_0^1 \sum_{\alpha=0}^{n} \left(\begin{array}{c}n\\ \alpha\end{array}\right) x^{\alpha}(1-x)^{n-\alpha}\,dx = 1$$

so that $f(x|\alpha) = \frac{x^{\alpha}(1-x)^{n-\alpha}}{B(\alpha+1,n-\alpha+1)}$

Let us consider a sequence of n Bernoulli trials with $\alpha \in \{0,1\}$. We have $f(\alpha|x) = \left(\begin{array}{c}1\\ \alpha\end{array}\right) x^{\alpha}(1-x)^{n-\alpha} = x^{\alpha}(1-x)^{n-\alpha}$

Let $g_0(x) = \frac{x^0(1-x)^0}{B(1,1)}$ be an initial guess at the prior distribution of X which we will update after each observation in the sequence of Bernoulli trials. Then we have

$$g_1(x) = \frac{x^{\alpha_1}(1-x)^{1-\alpha_1}}{B(1+\alpha_1, 1+1-\alpha_1)}$$

then

$$g_2(x) = \frac{x^{\alpha_1+\alpha_2}(1-x)^{2-\alpha_1-\alpha_2}}{B(1+\alpha_1+\alpha_2, 1+2-\alpha_1-\alpha_2)}$$

and in general

$$g_n(x) = \frac{x^{\alpha}(1-x)^{\beta}}{B(1+\alpha, 1+\beta)}$$

with $\alpha = \sum_{i=1}^{n} \alpha_i$ and $\beta = n - \sum_{i=1}^{n} \alpha_i$. We can regard α as the number of steps forward in a discrete random walk, and β as the number of steps backward.

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