note about group theory 2

September 20, 2019

1 tensor

Important concept: Tensor A tensor of type (r, s) on a vector space V is a C-valued function T on

$$\underbrace{V \times \cdots \times V}_{r \text{ times}} \times \underbrace{V^* \times \cdots \times V^*}_{s \text{ times}}$$

, which is linear in each argument (multilinearity).

The dual vectors (linear function) are $(1, 0)$ tensors

The vectors are viewed as $(0, 1)$ tensors, don't mistake the two, think about it. The dual vector eats something from V and spits a scalar; while, the vector eats something from V^* and spit a scalar.

linear operators can be viewed as $(1, 1)$ tensors.

Scalars are viewed as $(0, 0)$ tensor.

The set of all tensors of type (r, s) on a vector space V, denoted $\mathcal{T}_s^r(V)$, with a scalar of $\mathbb{C}$, with the usual addition and scalar multiplication, is a vector space.

Now think about components of a tensor. To see this, let $\mathcal{B} = \{e_i\}$ be a basis for V, and the corresponding dual basis $\mathcal{B} = \{e^i\}$. Denote the i th component of the vector v_p as v_p^i, the j th component of dual vector f_q as f_{qj}. Then use the multilinearity, we have:

$$\begin{aligned} T(v_1 \dots v_r, f_1 \dots f_s) &= \\ &= \sum v_1^{i_1} \cdots v_r^{i_r} f_{1j_1} \cdots f_{sj_s} T(e_{i_1} \dots e_{i_r}, e^{j_1} \dots e^{j_s}) \end{aligned}$$

so $T_{i_1 \dots i_r}{}^{j_1 \dots j_s} = T(e_{i_1} \dots e_{i_r}, e^{j_1} \dots e^{j_s})$, are called the components of T in the basis.

Raise and lower the indexes by precomposing with the map L or L^{-1}, where L refers to the metric dual map: $L(v) = (v|\cdot)$,when we have a non-degenerate bilinear form on V. A non-degenerate Hermitian form is linear on the second argument and conjugate linear on the first argument, when scalar is R, the non-degenerate Hermitian form is a non-degenerate bilinear form.

Remember when you doing this, the rank $r + s$ won't change, you lower the first, the second will be raised as the same time. See an example: T is of type$(1,1)$, define another tensor based on T: $\tilde{T}(v,w) = T(v, L(w))$, now $\tilde{T}$ is of type $(2,0)$. The components' symbol is quite tricky, you see the i staff is the subscript and j staff is the supscript, which is diffrent from the components of vector and dual vector. So with $\tilde{T}$ being of type$(2,0)$, its components is denoted $\tilde{T}_{ij}$

Why the symbol is tricky? It is all about the Einstein Summation. For a $(2,0)$ tensor, it acts on $V \times V$. It will be expanded as $T = T_{ij}e^i \otimes e^j$. So that it can use the Einstein Summation.

Actually, for a linear operator $A_i^j = A(e_i, e^j)$, so the symbol is consistent with the tensor components. After all, linear operator is just a kind of tensor.

For example, for a (2, 0) tensor, it acts on $V \times V$, so $T_{ij} = T(e_i, e_j)$ and $T = T_{ij}e^i \otimes e^j$

some examples:

1. components of tensor. Given an operator H, now we consider it as a $(1,1)$ tensor, the components of the tensor are: $H_i^j = H(e_i, e^j) = \langle j|H|i\rangle$, which is also referred as the matrix elements under some representation.

2. Levi-Civita tensor ϵ on $\mathbb{R}^3$ defined by

$$\epsilon(u,v,w) = (u \times v) \cdot w. \qquad u, v, w \in \mathbb{R}^3$$

   It is the oriented volume of a parallelepiped spanned by u, v, w

   Also Levi-Civita symbol $\bar{\epsilon}_{ijk}$ is defined below:

$$\bar{\epsilon}_{ijk} = \begin{cases} +1 & \text{if i, j, k is permutation of } (1,2,3) \\ -1 & \text{if i, j, k is permutation of } (3,2,1) \\ 0 & \text{otherwise} \end{cases}$$

   It is easy to see the Levi-Civita symbol is actually the components of Levi-Civita tensor.

3. The moment of inertia tensor denoted $\mathcal{I}$, is a $2,0$ tensor on $\mathbb{R}^3$. And we know that $\frac{1}{2}\mathcal{I}(\omega,\omega) = KE$ When we raise an index on $\mathcal{I}$ and define it to be the $(1,1)$ tensor (a linear operator): $\mathbf{L} = \mathcal{I}\omega$. Well, usually a $(1,1)$ tensor should be like $\langle\omega|\mathbf{L} = \langle\omega|\mathcal{I}|\omega\rangle = KE$. But for a linear operator, $\mathbf{L} = \mathcal{I}\omega$ is the usual form.

   I have a problem here. $\frac{1}{2}\mathcal{I}\omega^2 = KE$, how can this $(2,0)$ tensor be a tensor? There should be some non-linear calculation, right?

   But if you look it in another way. First, define $(1,1)$ tensor $T(\omega,\tilde{\omega}) = \langle\omega|\mathcal{I}|\omega\rangle$, then you raise and lower its indices using L, which is possible because $\omega \in \mathbb{R}^3$, $\tilde{T}(\omega,\omega) = T(\omega, L(\omega))$. It seems you can build a $(2,0)$ tensor.

   However, two problem:

   (a) can $\mathcal{I}(\omega,\omega)$, the function which actually only takes one vector, be called a $(2,0)$ tensor?

(b) why is there a $\frac{1}{2}$ difference in the two equation ?

4. Multipole moments: scalar potential $\Phi(\mathbf{r})$ of a charge distribution $\rho(\mathbf{r}')$ localized around the origin in $\mathbb{R}^3$ can be expand in a Taylor series as:

$$\Phi(\mathbf{r}) = \frac{1}{4\pi}\left[\frac{Q_0}{r} + \frac{Q_1(\mathbf{r})}{r^3} + \frac{1}{2!}\frac{Q_2(\mathbf{r},\mathbf{r})}{r^5} + \frac{1}{3!}\frac{Q_3(\mathbf{r},\mathbf{r},\mathbf{r})}{r^7} + \cdots\right])$$

We solve the possion equation and find that:

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')\mathrm{d}^3\mathbf{r}', \text{with some factor}$$

Then we expand the $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ in series. For example, in Legendre polymonials.

$$\Phi(\mathbf{r}) = -\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_V \rho(\mathbf{r}')(r')^l P_l(\cos\theta)\mathrm{d}^3\mathbf{r}'$$

Now, we can consider each term respectively: $l=1$ term, $P_1(x) = x$

$$\begin{aligned} & -\frac{1}{r^2}\int_V \rho(\mathbf{r}')(r')^2\cos\theta\mathrm{d}^3\mathbf{r}' \\ =& -\frac{1}{r^2}\int_V \rho(\mathbf{r}')(r')^2\left(\frac{r_i' r_i}{r' r}\right)\mathrm{d}^3\mathbf{r}' \quad \text{drag out the components of} \quad \mathbf{r}' \\ =& \frac{r_i}{r^3}\int_V \rho(\mathbf{r}')r_i'\mathrm{d}^3\mathbf{r}' \quad \text{remember the Einstein summention} \end{aligned}$$

then $l=2$ term, $P_2(x) = \frac{1}{2}(3x^2-1)$,
final is $\frac{r_i r_j}{r^5}\int_V \rho(\mathbf{r}')\frac{1}{2}(3r_i' r_j' - (r')^2)\delta_{ij}\mathrm{d}^3\mathbf{r}'$.

A very important thing is that: function like $\frac{r_i r_j}{r^5}$, can still be multilinear. Check it out. This may help you with above internia tensor problem. That is why it got a $\frac{1}{2}$ difference, because the $(2,0)$ tensor is defined just like this function.

5. Euclidean metric on $\mathbb{R}^n$is a $(2,0)$ tensor.

6. Minkowski metric on $\mathbb{R}^4$ is a $(2,0)$ tensor. These two can be called metric tensors.

7. a metric defined on V, $g_i^j = \delta_i^j$. I don't know the meaning of this.

2 transformation between bases

1. Derive the usual transformation laws that historically were taken as the definition of a tensor. Suppose we have a vector space V and two bases for V, $\mathcal{B}$ and $\mathcal{B}'$.

$$\begin{cases} e_{i'} = A^{j}_{i'} e_j, (\sum j \text{ Einstein Summmation}) \\ e_i = A^{j'}_{i} e_{j'} \end{cases} \implies e_i = A^{j'}_{i} e_{j'} = A^{j'}_{i} A^{k}_{j'} e_k$$

$$\implies e_i = A^{j'}_{i} e_{j'} = \sum_{j',k} A^{j'}_{i} A^{k}_{j'} e_k \implies \sum_{j'} A^{j'}_{i} A^{k}_{j'} = \delta^k_i \implies A^{j'}_{i} A^{k}_{j'} = \delta^k_i$$

$$\begin{cases} e^{i'} = A^{i'}_{j} e^j \\ e^i = A^{i}_{j'} e^{j'} \end{cases} \implies e^i = A^{i}_{j'} A^{j'}_{k} e^k \text{ This is no use, it is the same with the first one}$$

$$\begin{cases} e_{i'} = A^{j}_{i'} e_j, (\sum j \text{ Einstein Summmation}) \\ e_i = A^{j'}_{i} e_{j'} \end{cases} \implies e_{i'} = A^{j}_{i'} A^{k'}_{j} e_{k'} \implies A^{j}_{i'} A^{k'}_{j} = \delta^{k'}_{i'}$$

Notice that $A^j_{i'}$is not anyone's components, because their indices refer to different bases. A real component's indicies should refer to the same basis.

$$e_{i'} = A^{j}_{i'} e_j \implies v^{i'} = A^{i'}_{j} v^j$$

$$\implies [v]_{\mathcal{B}'} = \begin{pmatrix} A^{1'}_{1} & A^{1'}_{2} & \cdots & A^{1'}_{n} \\ A^{2'}_{1} & A^{2'}_{2} & \cdots & A^{2'}_{n} \\ \vdots & & & \\ A^{n'}_{1} & A^{n'}_{2} & \cdots & A^{n'}_{n} \end{pmatrix} [v]_{\mathcal{B}} \iff [v]_{\mathcal{B}'} = A[v]_{\mathcal{B}}$$

$$e_i = A^{j'}_{i} e_{j'} \implies v^i = A^{i}_{j'} v^{j'}$$

$$\implies [v]_{\mathcal{B}} = \begin{pmatrix} A^{1}_{1'} & A^{1}_{2'} & \cdots & A^{1}_{n'} \\ A^{2}_{1'} & A^{2}_{2'} & \cdots & A^{2}_{n'} \\ \vdots & & & \\ A^{n}_{1'} & A^{n}_{2'} & \cdots & A^{n}_{n'} \end{pmatrix} [v]_{\mathcal{B}'} \iff [v]_{\mathcal{B}} = A^{-1}[v]_{\mathcal{B}'}$$

$A^j_{i'}$is not anyone's components, here this matrix exists beyond a tensor product. This is actually not true. We say this, because we automatically associate the dual space basis with the vector space basis. But if we view A as an ordinary tensor, the two bases is not necessarily associated.

So for the tensor basis change:

$$\begin{aligned}
T_{i'_1 \ldots i'_r}{}^{j'_1 \ldots j'_s} &= T(e_{i'_1} \ldots e_{i'_r}, e^{j'_1} \ldots e^{j'_s}) \\
&= T(A^{k_1}_{i'_1} e_{k_1} \ldots A^{k_r}_{i'_r} e_{k_r}, A^{j'_1}_{l_1} e^{l_1} \ldots A^{j'_s}_{l_s} e^{l_s}) \\
&= A^{k_1}_{i'_1} \cdots A^{k_r}_{i'_r} A^{j'_1}_{l_1} \cdots A^{j'_s}_{l_s} T_{k_1 \ldots k_r}{}^{l_1 \ldots l_s} \quad \text{remember to sum}
\end{aligned}$$

This is the standard tensor transformation law, which is taken as the definition of a tensor in much of the physics literature. Here we derived this definition as a consequence of our definition of a tensor as a multilinear function on V and V^*

2. Back to the change of the basis: $e_{i'} = A^j_{i'} e_j$, whereas the components of a vector change like: $v^{i'} = A^{i'}_j v^j$. You see the change is opposite, and that is why the components of a vector are called contravariant, where contra means opposite.

$$e_{i'} = A^j_{i'} e_j$$

$$v^{i'} = A^{i'}_j v^j \implies [v]_{\mathcal{B}'} = \begin{pmatrix} A^{1'}_1 & A^{1'}_2 & \cdots & A^{1'}_n \\ A^{2'}_1 & A^{2'}_2 & \cdots & A^{2'}_n \\ \vdots & & & \\ A^{n'}_1 & A^{n'}_2 & \cdots & A^{n'}_n \end{pmatrix} [v]_{\mathcal{B}} = A[v]_{\mathcal{B}}$$

The components of v transform with the $A^{i'}_j$, $v^{i'} = A^{i'}_j v^j$

whereas the basis vector transform with the $A^j_{i'}$, $e_{i'} = A^j_{i'} e_j$

Similiarly, $e^{i'} = A^{i'}_j e^j$, whereas the components of a dual vector change like: $f_{i'} = A^j_{i'} f_j$. You see the change is opposite. Compared with basis change $e_{i'} = A^j_{i'} e_j$, you see the components of a dual vector change just like the basis, so the components of a dual vector are called covariant, where co means together, the same.

3.

$$v^{i'} = A^{i'}_j v^j \implies [v]_{\mathcal{B}'} = \begin{pmatrix} A^{1'}_1 & A^{1'}_2 & \cdots & A^{1'}_n \\ A^{2'}_1 & A^{2'}_2 & \cdots & A^{2'}_n \\ \vdots & & & \\ A^{n'}_1 & A^{n'}_2 & \cdots & A^{n'}_n \end{pmatrix} [v]_{\mathcal{B}} = A[v]_{\mathcal{B}}$$

$$f_{i'} = A^j_{i'} f_j \implies [f]_{\mathcal{B}'} = {A^{-1}}^T [f]_{\mathcal{B}}$$

Prove that $f(v) = [f]^T_{\mathcal{B}}[v]_{\mathcal{B}} = [f]^T_{\mathcal{B}'}[v]_{\mathcal{B}'}$ We know the definition of f, that $f(v) = [f]^T_{\mathcal{B}}[v]_{\mathcal{B}}$

$$\begin{cases} f(v) = [f]^T_{\mathcal{B}}[v]_{\mathcal{B}} \\ [v]_{\mathcal{B}'} = A[v]_{\mathcal{B}} \\ [f]_{\mathcal{B}'} = {A^{-1}}^T[f]_{\mathcal{B}} \end{cases} \implies f(v) = [f]^T_{\mathcal{B}}[v]_{\mathcal{B}} = [f]^T_{\mathcal{B}'}[v]_{\mathcal{B}'}$$

4. if you have an inner product $(\cdot|\cdot)$ on a real vector space V and an orthonormal basis $\{e_i\}_{i=1,2,3\cdots n}$, then the components of vectors and the corresponding dual vectors are identical

   a vector $v = v^i e_i$, then its corresponding dual vector is $\tilde{v} = L(v) = {v^i}^* L(e_i) = \sum_i v^i e^i$, which means $\tilde{v}_i = v^i$.

   For a real inner product space, the change between one orthonormal basis to another orthonormal basis will be represented by an orthogonal matrix.

   For a complex inner product space, the change between one orthonormal basis to another orthonormal basis will be represented by an unitary matrix.

   But we also have

$$\begin{cases} [v]_{\mathcal{B}'} = A[v]_{\mathcal{B}} \\ [f]_{\mathcal{B}'} = {A^{-1}}^T[f]_{\mathcal{B}} \end{cases}$$

   These equations seems to tell us they should not be identical.

   But owing to

$$\begin{cases} (e_i|e_j) = \delta \\ (e_{i'}|e_{j'}) = \delta \\ e_{i'} = A^j_{i'} e_j \\ e_{j'} = A^i_{j'} e_i \end{cases} \implies$$

$$(e_{i'}|e_{j'}) = A^j_{i'} A^i_{j'} (e_j|e_i) = \sum_{j=1}^{n} A^j_{i'} A^j_{j'} = \delta_{i'j'}$$

$$\implies {A^{-1}}^T A^{-1} = I$$

5. the change of basis for linear operator

   For a linear operator T on V, because of the famous tensor basis change

$$\begin{aligned} {T_{i'_1 \dots i'_r}}^{j'_1 \dots j'_s} &= T(e_{i'_1} \dots e_{i'_r}, e^{j'_1} \dots e^{j'_s}) \\ &= T(A^{k_1}_{i'_1} e_{k_1} \dots A^{k_r}_{i'_r} e_{k_r}, A^{j'_1}_{l_1} e^{l_1} \dots A^{j'_s}_{l_s} e^{l_s}) \\ &= A^{k_1}_{i'_1} \cdots A^{k_r}_{i'_r} A^{j'_1}_{l_1} \cdots A^{j'_s}_{l_s} {T_{k_1 \dots k_r}}^{l_1 \dots l_s} \quad \text{remember to sum} \end{aligned}$$

${T_{i'}}^{j'} = A^i_{i'} A^{j'}_j {T_i}^j$, in matrix form reads $[T]_{\mathcal{B}'} = A[T]_{\mathcal{B}} A^{-1}$, this is the similarity transformation of the matrices.

Linear operator transformation is similarity transformation.

6. **Trace of a linear operator**

   Now we introduce the concept of trace into linear operator: Given a linear operator $T \in \mathcal{L}(V)$ and a basis $\mathcal{B}$ for V, then $Tr(T) = Tr([T]_{\mathcal{B}})$. However, you can prove that the trace of a linear operator does not depend on the choice of the basis. Under the similarity transform the trace of a matrix won't change.

   After the trace of linear operator, let us see another issue about linear operator. $f(Tv) = [f]^T[T][v]$ is invariant under a change of basis. You see the linear operator will have similarity transformation, and vector will $[V]_{\mathcal{B}'} = A[v]_{\mathcal{B}}$, the dual vector will $[f]_{\mathcal{B}'} = {A^{-1}}^T[f]_{\mathcal{B}}$. So it is obvious.

7. the transformation of tensor, $(2,0)$ tensor still from the formalism ${T_{i'_1 \dots i'_r}}^{j'_1 \dots j'_s} = A^{k_1}_{i'_1} \cdots A^{k_r}_{i'_r} A^{j'_1}_{l_1} A^{j'_s}_{l_s} {T_{k_1 \dots k_r}}^{l_1 \dots l_r}$

   Then we have $g_{i'j'} = A^i_{i'} A^j_{j'} g_{ij}$, which is also: $[g]_{\mathcal{B}'} = {A^{-1}}^T[g]_{\mathcal{B}} A^{-1}$.

   Now if g is an inner product and the bases are both orthonormal, then we have $[g]_{\mathcal{B}'} = [g]_{\mathcal{B}} = I$, then $I = {A^{-1}}^T A^{-1}$, which means A is orthogonal.

   Here we say g is an inner product, which is a special kind of Non-degenerate Hermitian form, which is a C-value function, which is a $(2,0)$ type tensor. But here, a linear operator, also a $(1,1)$ tensor, can make the same impact as the $(2,0)$ tensor.

   The Minkowski metric η are not dealing with an inner product, when considering orthonormal bases, one of the four possible matrices are ,

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

   Consider the transformation of η, we have:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = {A^{-1}}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} A^{-1}$$

   which can also be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = A^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} A$$

Matrices satisfying the second equation are known as Lorenz-transformations. what is the point of Lorenz-transformation, where would it be used?

3 the active transformation and passive transformation

Define, a new tensor (the transformation between two bases), $U(e_{i'}) = e_i$, $\forall i$
The components of this tensor: in the basis $\mathcal{B}$, and the dual space basis is the associated one

$$\begin{aligned} U_i^j &= U(e_i, e^j) \\ &= e^j(Ue_i) \\ &= e^j(U(A_i^{k'} e_{k'})) \\ &= A_i^{k'} e^j(U(e_{k'})) \\ &= A_i^{k'} e^j e_k \\ &= A_i^{j'} \end{aligned}$$

This result can be written as

$$[e_i]_{\mathcal{B}} = [U]_{\mathcal{B}}[e_{i'}]_{\mathcal{B}} = A[e_{i'}]_{\mathcal{B}}$$

While, in the former context, we have gotten

$$[v]_{\mathcal{B}'} = A[v]_{\mathcal{B}}$$

$$[e_i]_{\mathcal{B}} = [U]_{\mathcal{B}}[e_{i'}]_{\mathcal{B}} = A[e_{i'}]_{\mathcal{B}}$$

is **active transformation**
And

$$[v]_{\mathcal{B}'} = A[v]_{\mathcal{B}}$$

is **passive transformation**.

1. active transformation has the eye on the bases, while passive transformation has the eye on the components.
2. In the formula of active transformation, the components of the basis vectors are the results under the same basis. While in the formula of passive transformation, the components of the vectors are the results under different bases.
3. they are using the same matrix A, which can be a concrete image of the tensor U, but can also be viewed as a independent object.

4. they will actually lead to the same calculation results. But they have different meanings.

   Active transformation: one considers the coordinate system fixed and interprets the matrix A as taking the physical vector e_i into a new vector $e_{i'}$, where the components of both are expressed in the same coordinate system.

   Passive transformation: the vector v does not change, only the coordinate system in which it is expressed is changed. The matrix A takes the old components of v and gives back the components of the same vector but in a new coordinate.

Chances are that people don't point out whether the matrix is for active transformation or passive transformation. We will have to rely on the context how the pale equation should be interpreted.

some example:

Active and passive transformaitons and Schrodinger and heisenberg pictures.

In Schrodinger picture, the state kets, basis are fixed, it is an active point, the operators change one state ket to another different state ket. While in Heisenberg picture, vectors do not change, but the basis changes, operators are time-dependent, their eigenvectors(which are also the basis) are time-dependent.

$$
\begin{aligned}
\langle \hat{x}(t) \rangle &= \langle \phi | (U^\dagger \hat{x} U) | \phi \rangle \\
&= (\langle \phi | U^\dagger) \hat{x} (U | \phi \rangle)
\end{aligned}
$$

4 tensor product

Important concept: The tensor product

tensor product of two finite-dimensional vector space

Given two finite-dimensional vector space V and W, their tensor product $V \otimes W$ is the set of all C-valued bilinear functions on $V^* \times W^*$. Their functions form a vector space.

The bilinear:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

$$c(v \otimes w) = (cv) \otimes w = v \otimes (cw) \quad c \in \mathbb{C}$$

Every element of this set is a C-valued bilinear function, which is defined by its action on a pair $(h, g) \in V^* \times W^*$

tensor product of two vectors, more generally tensor product of two tensors
Tensor product between two vectors $(v \otimes w)(h, g) = v(h)w(g)$.

Any pair of dual vectors can satisfy the above bilinear properties, so the defintion is OK
Easy to prove that $e_i \otimes f_j$, $i = 1, 2 \cdots n$, $j = 1, 2, \cdots m$ is a basis for $V \otimes W$
For an arbitrary $T \in V \otimes W$, using bilinearity, $T(h, g) = h_i g_j T(e^i, f^j) = h_i g_j T(e^i, f^j) = h_i g_j T^{ij}$. At the same time, $(T^{ij} e_i \otimes f_j)(e^k, e^l) = T^{ij} e_i(e^k) f_j(f^l) = T^{ij} \delta_i^k \delta_j^l = T^{kl}$. Also it is easy to verify that $\{e_i \otimes f_j\}$ is linear independent. so the $\{e_i \otimes f_j\}$ is the basis for space $V \otimes W$.

tensor product's properties

1. bilinearity
2. commute with taking duals $(V \otimes W)^* = V^* \otimes W^*$
3. associative, $(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$

One of the important type of tensor products is the ones of the form

$$\underbrace{V^* \otimes \ldots \otimes V^*}_{r \text{ times}} \otimes \underbrace{V \otimes \ldots \otimes V}_{s \text{ times}}$$

so it is the set of multilinear functions on

$$\underbrace{V \times \ldots \times V}_{r \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{s \text{ times}}$$

Which is the definition of tensor.

This space is of particular interest because it is actually identical to $\mathcal{T}_s^r$. It has the basis $\mathcal{B}_s^r = \{e^{i_1} \otimes \ldots \otimes e^{i_r} \otimes e_{j_1} \otimes \ldots \otimes e_{j_s}\}$, which is also the basis for $\mathcal{T}_s^r$

$$T = T_{i_1 \ldots i_r}{}^{j_1 \ldots j_s} e^{i_1} \otimes \ldots \otimes e^{i_r} \otimes \ldots \otimes e_{j_s}$$

A little calculation can show the relation:

$$\begin{aligned}
&(T_{k_1 \ldots k_r}{}^{l_1 \ldots l_s} e^{k_1} \otimes \ldots \otimes e^{k_r} \otimes e_{l_1} \otimes \ldots \otimes e_{l_s})(e_{i_1} \ldots e_{i_r}, e^{j_1} \ldots e^{j_s}) \\
&= T_{k_1 \ldots k_r}{}^{l_1 \ldots l_s} e^{k_1}(e_{i_1}) \cdots e^{k_r}(e_{i_r}) e_{l_1}(e^{j_1}) e_{l_s}(e^{j_s}) \\
&= T_{k_1 \ldots k_r}{}^{l_1 \ldots l_s} \delta_{l_1}^{k_1} \cdots \delta_{i_r}^{k_r} \delta_{l_1}^{j_1} \cdots \delta_{l_s}^{j_s} \\
&= T_{i_1 \ldots i_r}{}^{j_1 \ldots j_s}
\end{aligned}$$

Now we have two way to think about the components of the tensor: either as the values of the tensor on sets of basis vectors or as the expasion coefficients in the givenm function basis.

Let T_1 and T_2 be tensors of type (r_1, s_1) and $(r_2.s_2)$, respectively, on a vector space V. Then $T_1 \otimes T_2$ is an $r_1 + r_2, s_1 + s_2$ tensor. The tensor product of two tensors is again a tensor. Now you know why it is called 'tensor product'. Tensor product is more like an operation, you apply the operation on tensors, the result is still a tensor, but that does not mean the operation is exactly a tensor. A number plus a number equals to a number, but 'plus ' is not a number.

When we talk about the tensor product of two vector space, it actually concerns about the tensor on the vector space

5 tensor contraction

Another important operation on tensors is contraction, which is the generalization of the trace functional to tensors of arbitrary rank: Given $T = T_{i_1 \dots i_r}{}^{j_1 \dots j_s} e^{j_1} \otimes \dots \otimes e^{i_r} \otimes e_{j_1} \otimes \dots \otimes e_{j_s} \in \mathcal{T}_s^r(V)$.

remember here, we are talking about a very special kind of tensor, $\mathcal{T}_s^r(V)$. They are the functions that is on

$$\underbrace{V \times \ldots \times V}_{r \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{s \text{ times}}$$

Don't take any amazing conclusion granted.

We define a contraction of T to be any $(r-1, s-1)$ tensor resulting from feeding e^i into one of the arguments, e_i into another and then summing over i as implied by the summation convention.But you have to feed e_i and e^i into the arguments at the same time. (one is covariant, the other is contravariant) The original positions of variants have been occupied by the constant vector and dual vector.

$$\tilde{T}(v_1 \dots v_{r-1}, f_1 \dots f_{s-1}) \equiv \sum_i T(v_1 \dots v_{r-1}, e_i, f_1 \dots f_{s-1}, e^i)$$

This is the generalization of trace operation.

$$\mathrm{Tr}_{\mathrm{B}}(\hat{O}) = \sum_m \langle m|\hat{O}|m\rangle_{\mathrm{B}}$$

The $\langle m|$ and $|m\rangle_{\mathrm{B}}$ are the e_i and e^i that you are feeding to the original tensor. You have to feed a hard pair of basis vector.

the components of a contracted tensor

We can see the relationship between the components:

$$
\begin{aligned}
\tilde{T}_{i_1\ldots i_{r-1}}{}^{j_1\ldots j_{s-1}} &= \tilde{T}(e_{i_1}\ldots e_{i_{r-1}}, e^{j_1}\ldots e^{j_{s-1}}) \\
&= T(e_{i_1}\ldots e_{i_{r-1}}, e_i, e^{j_1}\ldots e^{j_{s-1}}, e^i) \\
&= (T_{k_1\ldots k_r}{}^{l_1\ldots l_s} e^{k_1}\otimes\ldots e^{k_r}\otimes e_{l_1}\otimes\ldots e_{l_s})(e_{i_1}\ldots e_i, e^{j_1}\ldots e^i) \\
&= T_{i_1\ldots i_{r_1}k_r}{}^{j_1\ldots j_{s-1}l_s}\delta_i^{k_r}\delta_{l_s}^i \\
&= \sum_i T_{i_1\ldots i_{r-1}i}{}^{j_1\ldots j_{s-1}i}
\end{aligned}
$$

Then we get the relation between the components:

$$\tilde{T}_{i_1\ldots i_{r-1}}{}^{j_1\ldots j_{s-1}} = \sum_l T_{i_1\ldots i_{r-1}l}{}^{j_1\ldots j_{s-1}l}$$

Although, usually, the sum thing should be neglected under summation convetion.

the choice of basis does not affect the contraction

Also, the choice of basis also does not affect the contraction. We can prove that $T(v_1\ldots v_{r-1}, e_i, f_1\ldots f_{s-1}, e^i) = T(v_1\ldots v_{r-1}, e_{i'}, f_1\ldots f_{s-1}, e^{i'})$

How to prove this:

$$T(v_1\ldots v_{r-1}, e_{i'}, f_1\ldots f_{s-1}, e^{i'}) = T(v_1\ldots v_{r-1}, A^i_{i'}e_i, f_1\ldots f_{s-1}, A^{i'}_i e^i)$$

After that, using the multilinear to expand it, remove the zero terms, it is the final result. So the contraction for tensor is well-defined.

More generalization

Similar contractions can be performed on any two arguments of T provided one argument is covariant and the other is contravariant. Not just vector and its corresponding dual vector

If we are working on vector space equipped with a metric g, then we can use the metric to raise and lower the indice, so we contract on any pair of indices, even if they are both covariants or contravariant.

The indices(positions) can be random, but the constant vector has to be associated. Or the whole contraction won't have proper meaning.

some examples:

1. two linear operators A and B, then their tensor product $A\otimes B \in \mathcal{T}_2^2$.

$$(A\otimes B)^{jl}_{ik} = A^j_i B^l_k$$

If we contract the first and the last indices,

$$(A \otimes B)(e_i, v_1, w_1, e^i)$$

Using the relation between the components:

$$\tilde{T}_{i_1 \ldots i_{r-1}}{}^{j_1 \cdots j_{s-1}} = \sum_l T_{i_1 \ldots i_{r-1} l}{}^{j_1 \ldots j_{s-1} l}$$

We get that

$$(A \otimes B)_k^l = A_l^i B_k^l = AB$$

If we contract the second and third indices. We get that

$$(A \otimes B)_i^l = A_i^j B_j^l = BA$$

Here I want to talk about the symbol problem:

$$A_i^j = A(e_j, e^i)$$

So this number should be at the ith row and jth column of the matrix representation of A

2. a $(2,0)$ tensor like the Minkowski metric can be written as $\eta = \eta_{\mu\nu} e^\mu \otimes e^\nu$. More generally, a tensor product like $f \otimes g = f_i g_j e^i \otimes e^j \in \mathcal{T}_0^2$. Notice that the symbol for tensor type set can be confusing, $\mathcal{T}_s^r$, where r times V, and s times V^*, the tensor function takes r vectors and s dual vectors.

3. $V^* \otimes V$ type tensor product, which means it is the function on $V \times V^*$ which is the same as $\mathcal{T}_1^1$, the space of linear operators. Given $T_i^j e^i \otimes e_j \in V^* \otimes V$, define the linear operator by $T(v) = T_i^j e^i(v) e_j = v^i T_i^j e_j$. This kind of tensor product can also be referred as direct product or outer product. Because inner product and outer product both take a dual vector and a vector. But inner product yields a scalar and outer product yields a linear operator.

4. Maxwell stress tensor

   In considering the conservation of total momentum (mechanical plus electromagnetic) in classical electrodynamics, one encounters the symmetric rank 2 Maxwell Stress Tensor

$$T_0^2 = \mathbf{E} \otimes \mathbf{E} + \mathbf{B} \otimes \mathbf{B} - \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2) I$$