notes about group theory 5

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1 Importan concept: Lie group

Matrix groups that are paramtrizble by a certain number of real parameters, Lie groups, named after Norwegian mathematician Sophus Lie.

The continuous nnature means that we can study **group elements that are infinitely close to the identity**. In physics, they are called **infinitesimal transformations** or **generators** of the group. In math, they are called the **Lie algebra** of the group.

Almost all observables can be built out of elements of certain Lie algebras. Many obejcts and structures can be understood in terms of Lie algebras.

We define a matrix Lie group to be a subgroup $G \subset GL(n, \mathbb{C})$ which is closed. By 'closed', we mean that: for any sequence of matrices $A_n \in G$ which converges to a limit matrix A, either $A \in G$ or $A \notin GL(n, \mathbb{C})$. This says that a limit of matrices in G must either itself be in G, or otherwise be noninvertible. This definition is technical and does not provide much insight into what a Lie group really is.

1. the orthogonal group $O(n)$, whose defintion is $R^T R = I$. Let us consider the function from $GL(, \mathbb{R})$ to itself defined by $f(A) = A^T A$. Each entry of matrix $f(A)$ is a continuous function of the entries of A. So we can say that f is continuous. Now consider a sequence $R_i \in O(n)$that converges to some limit matrix R.

$$\begin{aligned} f(R) &= f\left(\lim_{i\to\infty} R_i\right) \\ &= \lim_{i\to\infty} f(R_i) \quad \text{f is continuous} \\ &= \lim_{i\to\infty} I \\ &= I \end{aligned}$$

   $f(R) = I$, that means $R \in O(n)$, according to the definition of Lie group, $O(n)$ is a Lie group.

2. The Lorentz group $O(n-1, 1)$

3. The unitary group $U(n)$
4. the special unitary group $SU(n)$
5. the special orthogonal groups $SO(n)$

   These matrices are similarly defined by continuous functions

The essence of Lie group

1. Think of Lie groups as groups which can be parametrized in terms of a certain number of real variables. This number is known as the dimension of the Lie group.
2. The dimension of Lie group is also usual (vector space) dimension of its corresponding Lie algebra.
3. Lie group is parametrizable, we can think of it as a kind of multi-dimensional space that also has a group structure and has a distinguished point, identity.
4. There are Lie groups out there which are not matrix Lie group, that is, which can not be described as a subset of $GL(n, \mathbb{C})$ for some n.

1. $\mathcal{H}(\mathbb{R}^3)$ and $\tilde{\mathcal{H}}_l$, the harmonic polynomials and the spherical harmonics.
2. $P_l(\mathbb{R}^3)$ consists of all complex-coefficient polynomial functions on $\mathbb{R}^3$ of fixed degree l.
3. $\mathcal{H}(\mathbb{R}^3)$ consists of all complex-coefficient polynomial functions on $\mathbb{R}^3$ of fixed degree l which satisfy $\Delta f = 0$.
4. $\tilde{\mathcal{H}}_l$ consists of all complex-coefficient polynomial functions on $\mathbb{R}^3$ of fixed degree l which satisfy $\Delta_{S^2} Y(\theta, \phi) = -l(l+1)Y(\theta, \phi)$.

When $l = 1$, $\mathcal{H}_1(\mathbb{R}^3) = P_1(\mathbb{R}^3)$. The bases are respectively $\{x, y, z\}$ and $\{\frac{x+iy}{2}, z, \frac{x-iy}{2}\text{or}\frac{1}{\sqrt{2}}re^{i\phi}\sin\theta, r\cos\theta, \frac{1}{\sqrt{2}}re^{-i\phi}\sin\theta\}$.

We want to know what the matrix corresponding to operator $L_z = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$ is like under two different bases. (to remember L_z, the subscript will form a sequence of z, y, x)

Using $[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}}$. At first, I want to use T_i^j, then I realized that a form is required.
The first basis is called cartesian basis, $\{x, y, z\}$.
Operator $L_z = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$, thus

$$L_z(x) = iy, L_z(y) = -ix, L_z(z) = 0$$

These are actually three set of linear equations, in total nine equations.

$$[L_z]_{\{x,y,z\}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$$

$$[L_z]_{\{x,y,z\}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \\ 0 \end{pmatrix}$$

$$[L_z]_{\{x,y,z\}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which lead us to

$$[L_z]_{\{x,y,z\}} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus we have

$$L_z = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \quad [L_z]_{\{rY_m^l\}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad [L_z]_{\{x,y,z\}} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

With some constant factor, it is determined by the overall normalization of the polynomials.

When $l = 2$,

$$\frac{1}{2}(x+iy)^2, z(x+iy), \frac{1}{\sqrt{2}}(x^2+y^2-z^2), z(x-iy), \frac{1}{2}(x-iy)^2$$

The study of the transformations 'close to ' identity will lead us to Lie algebra.

An abitrary rotation about the $z-$axis looks like

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we take $\epsilon \ll 1$, and use Maclaurin expansion, we have

$$R_z(\epsilon) \approx \begin{pmatrix} 1 & -\epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I + \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = I + \epsilon L_z$$

We should be able to describe a finite rotation through an angle θ as an n-fold iteration of smaller rotations through $\frac{\theta}{n}$

$$R_z(\theta) = [R_Z(^{\theta}/_{n})]^n \approx \left(I + \frac{\theta L_z}{n}\right)^n$$

$$R_z(\theta) = \lim_{n\to\infty} [R_Z(^{\theta}/_{n})]^n \approx \lim_{n\to\infty} \left(I + \frac{\theta L_z}{n}\right)^n = e^{\theta L_z}$$

The set $\{R_z(\theta) = e^{\theta L_z} | \theta \in \mathbb{R}\}$ is a group. $R_z(\theta_1)R_z(\theta_2) = R_z(\theta_1 + \theta_2)$ checks the closure. Then we can see that $\{R_z(\theta) = e^{\theta L_z} | \theta \in \mathbb{R}\}$ is a subgroup of $SO(3)$.

Now we construct a map: $R_z : \mathbb{R} \to SO(3)$. This map is a homomorhpism, one-to-one but not onto.

Continuous homomorphism from the additive group $\mathbb{R}$ to a matrix Lie group G is known as one-parameter subgroup.

1. the set of rotations in $\mathbb{R}^3$ about any particular axis is a one-parameter subgroup, where the parameter can be taken to be the rotation angle.
2. the set of all boosts in a particular direction is a one-parameter subgroup, where the parameter can be taken to be the absolute value u of the rapidity.
3. the set of translations along a particular direction in both momentum and position space are one-parameter subgroups as well.

2 First taste of generator

1. on one hand, using a matrix X to generate a one-parameter subgroup.

   If we have a matrix X such that $e^{tX} \in G \quad \forall t \in \mathbb{R}$, then the map $exp : \mathbb{R} \to G \quad t \mapsto e^{tX}$. Then the image of the map is a one-paramter subgroup. This is the general form of the R_z map.

2. on the other hand, for every one-parameter subgroup, a generator can be found.

   If we have a one-paramter subgroup, and the corresponding map is $\gamma : \mathbb{R} \to G$. First of all, we can claim that $\gamma(0) = I$ (homomorphism preserves the identity). Then we define the constant $\frac{\mathrm{d}\gamma(0)}{\mathrm{d}t}$ to be matrix X(the t is actually $t \in \mathbb{R}$, now $\gamma(r) =$ some matrix). And we have

$$\begin{aligned}\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} &= \lim_{h\to\infty} \frac{\gamma(t+h)-\gamma(t)}{h} \\ &= \lim_{h\to\infty} \frac{\gamma(t)\gamma(h)-\gamma(t)}{h} \\ &= \lim_{t\to\infty} \frac{\gamma(h)-\gamma(0)}{h}\gamma(t) \\ &= X\gamma(t)\end{aligned}$$

This equation has the unique solution $\gamma(t) = e^{tX}$. Thus, every one-parameter subgroups is of the form $exp : \mathbb{R} \to G \quad t \mapsto e^{tX}$. We have a one-to-one correspondence between one-parameter subgroups and matrices X such that $e^{tX} \in G \quad \forall t \in \mathbb{R}$. The matrix X is said to 'generate' the corrrsponding one-parameter subgroup.

3 Importntant concept: Lis algebras

Given a matrix Lie group $G \subset GL(n, \mathbb{C})$, we define the Lie group's Lie algebra $\mathfrak{g}$ to be the set of all matrices $X \in M_n(\mathbb{C})$ such that $e^{tX} \in G \quad \forall t \in \mathbb{R}$. (the set of those matrices that can generate a one-paremeter subgroup). But in physics, the Lie algebra of a matrix Lie group G is $\mathfrak{g}_{\text{physics}} = \{X \in GL(n, \mathbb{C}) | e^{itX} \in G \quad \forall t \in \mathbb{R}\}$. some examples:

1. consider Lie group $GL(n, \mathbb{C})$ and its Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. Consider the definition of Lie algebra: the set of all matrices $X \in M_n(\mathbb{C})$ such that $e^{tX} \in G \quad \forall t \in \mathbb{R}$. We know for any elements in $M_n(\mathbb{C})$, $e^{tX} \in GL(n, \mathbb{C})$, because e^{-tX} is its inverse. So every elements in $M_n(\mathbb{C})$ can be in $\mathfrak{gl}(n, \mathbb{C})$. $\mathfrak{gl}(n, \mathbb{C}) = M_n(\mathbb{C})$.

2. $\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$.

3. consider Lie group $O(3)$ and its Lie algebra $\mathfrak{o}(3)$. From the definition of Lie algebra, for any $X \in \mathfrak{o}(3)$, we have $e^{tX} \in O(3) \quad \forall t \in \mathbb{R}$, which means that $\left(e^{tX}\right)^T e^{tX} = e^{t(X^T)}e^{tX} = I \quad \forall t$. Differentiating this with the respect to t yields that $X^T e^{tX^T} e^{tX} + e^{tX^T} X e^{tX} = 0$. When $t = 0$, we have $X^T + X = 0$. Thus any $X \in \mathfrak{o}(3)$ must be antisymmetric. We can also check that for any real antisymmetric X, matrix e^{tX} will be in $O(3), \quad \forall t \in \mathbb{R}$.(see whether $(e^{tX})^T e^{tX} = I$). Thus $\mathfrak{o}(3)$ is the set of all real antisymmetric matrices.

Proposition: Let $\mathfrak{g}$ be the Lie algebra of a matrix Lie group G. Then $\mathfrak{g}$ is a real vector space, is closed under commutators (it means the commutators are still in the space), and all elements of $\mathfrak{g}$ obey the Jacobi identity.

Prove that $\mathfrak{g}$ is a real vector space. To prove this, we need $\mathfrak{g}$ is closed under real scalar multiplication and $\mathfrak{g}$ is closed under the addition.

1. Closed under real scalar multiplication, if $X \in \mathfrak{g}$, then you can get a fractor from t, thus λX will also be in $\mathfrak{g}$.

2. Closed under addition, this needs Lie product Formula

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m$$

   This formula should be thought of as expressing the addition operation in $\mathfrak{g}$ in terms of the product operation in G.

   Now we want to show that if X, Y is in $\mathfrak{g}$, then $X+Y \in \mathfrak{g}$. You have to check $e^{t(X+Y)}$,But just like the case of closure under scalar multiplication, you only need to check e^{X+Y}. Lie Product Formula tells us that e^{X+Y} is the limit matrix of a sequence $A_m = (e^{\frac{X}{m}} e^{\frac{Y}{m}})^m$, from the definition of Lie group, it is easy to know that $e^{X+Y} \in G$. Thus, $\mathfrak{g}$ is closed under the addition. So $\mathfrak{g}$ is a vector space.

Prove that $\mathfrak{g}$ is closed under commutators.
First of all, $e^{AXA^{-1}} = Ae^X A^{-1}$
Now we want to show that $\mathfrak{g}$ is closed under commucators. If $X \in \mathfrak{g}$ and $A \in G$, then $e^{tAXA^{-1}} = Ae^{tX}A^{-1} \in G \quad \forall t \in \mathbb{R}$, so $AXA^{-1} \in \mathfrak{g}$. Now we have

$$\begin{cases} X \in \mathfrak{g} \\ A \in G \end{cases} \implies AXA^{-1} \in \mathfrak{g}$$

Now let $Y \in \mathfrak{g}$, then $e^{tY} \in G$, let $X \in \mathfrak{g}$, then use the above property, we have $e^{tY}Xe^{-tY} \in \mathfrak{g}$. Compute the derivative of this expression at $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tY} X e^{-tY}\bigg|_{t=0} = YX - XY$$

But we also have

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tY} X e^{-tY}\bigg|_{t=0} = \lim_{h\to 0} \frac{e^{hY}Xe^{-hY} - X}{h}$$

The right side will always be in $\mathfrak{g}$, so we get the commutator is also in $\mathfrak{g}$.

The algebraic structure of the commutator on $\mathfrak{g}$ is closely related to the algebraic structure of the product on G. This is most clearly manifested in the Bake-Campbell-Hausdorff formula, where XY are sufficiently small expresses $e^X e^Y$ as a single exponential. (The size of a matrix $X \in M_n(\mathbb{C})$ is usually expressed by the Hilber-Schmidt norm, defined as $||X|| = \sum_{i,j=1}^{n} |X_{ij}|^2$.)

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\cdots}$$

The Lie product formula expresses Lie algebra addition in terms of group multiplication.

$$e^{X+Y} = \lim_{m\to\infty}\left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m$$

The BCH formula expresses group multiplication in terms of the commutator on the Lie algebra.

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]+\cdots}$$

Some examples:

1. The Lie group of $O(n)$ and $U(n)$: $\mathfrak{o}(n)$ and $\mathfrak{u}(n)$.

   (a) From the definition of Lie algebra, for any $X \in \mathfrak{o}(n)$, we have $e^{tX} \in O(n) \quad \forall t \in \mathbb{R}$, which means that $(e^{tX})^T e^{tX} = e^{t(x^T)}e^{tX} = I, \quad \forall t$. Differentiating this with the respect to t yields that $X^T e^{tX^T} e^{tX} + e^{tX^T} X e^{tX} = 0$. When $t = 0$, we get that $X^T + X = 0$. Thus for any $X \in \mathfrak{o}(n)$, X is antisymmetric.

   (b) Next, for any antisymmetric matrices X, it is easy to see that $(e^{tX})^T e^{tX} = I$, thus $e^{tX} \in O(n)$, thus $X \in \mathfrak{o}(n)$. So $\mathfrak{o}(n)$ is the set of all real, antisymmetric $n\times n$ matrices. We know that real, antisymmetric matrices set are $A_n(\mathbb{R})$, and its dimension is $\frac{n(n-1)}{2}$. So $\mathfrak{o}(n) = A_n(\mathbb{R})$.

2. The Lie group of $U(n)$: $\mathfrak{u}(n)$

   For any $X \in \mathfrak{u}(n)$, we have $e^{tX} \in U(n), \forall t \in \mathbb{R}$, which means that $(e^{tX})^\dagger e^{tX} = I, \forall t \in \mathbb{R}$. Differentiating this with the respect to t yields that $X^\dagger e^{tX^\dagger} e^{tX} + e^{tX^\dagger} X e^{tX} = 0$. When $t = 0$, we get that $X^\dagger + X = 0$. Thus for any $X \in \mathfrak{u}(n)$, X is anti-Hermitian.

   $\mathfrak{u}(n)$ is the set of all anti-Hermitian $n \times n$ matrices.

   (a) We know that any Hermitian matrices multiplied by i will be anti-Hermitian.

   (b) You can use $E_{jj} 1 \le j \le n, (n$ matrices), $E_{ij} + E_{ji}, i < j, 1 \le i < j \le n, (\frac{n^2-n}{2}$ matrices), $i(E_{ij} - E_{ji}), 1 \le i < j \le n, (\frac{n^2-n}{2}$ matrices) as a basis for $H_n(\mathbb{C})$. ($H_n(\mathbb{C})$ is a complex Hermitian matrix vector space, this space is a real vector space. The elements in the space are complex-entry, but the scalar of this space is $\mathbb{R}$. $H_n(\mathbb{C})$ can not form a vector space with respect to $\mathbb{C}$).

   So just multiply the basis for Hermitian matrices, we can get the basis for anti-Hermitian matrices.

   Also note that $\mathfrak{o}(n) \subset \mathfrak{u}(n)$.

3. The Lie algebra of $O(n-1,1)$: $\mathfrak{o}(n-1,1)$.

   Let $X \in \mathfrak{o}(n-1,1)$, by the definition of a Lie algebra, we have $e^{tX^T}[\eta]e^{tX} = [\eta] \quad \forall t$. And always remember that $e^{tX^T} = (e^{tX})^T$. Differentiating with respect to t and evaluating at $t=0$ yields $X^T[\eta] + [\eta]X = 0$. Which is also:

$$\begin{pmatrix} X'^T & -\vec{b} \\ \vec{a} & -X_{nn} \end{pmatrix} + \begin{pmatrix} X' & \vec{a} \\ -\vec{b} & -X_{nn} \end{pmatrix} = 0$$

   So that the general form of X can be written as

$$X = \begin{pmatrix} X'^T & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} X' \in \mathfrak{o}(n-1), \mathbf{a} \in \mathbb{R}^{n-1}$$

   Think of X' as generating the rotation in $n-1$ spatial dimension. And think of $\vec{a}$ generating a boost along the direction it points in $\mathbb{R}^{n-1}$.

4. the Lie algebras of $SO(n)$ and $SU(n)$. The Lie algebra of $SO(n)$ is the set of all matrices $X \in M_n(\mathbb{C})$ such that $e^{tX} \in SO(n), \forall t \in \mathbb{R}$. If we want to check whether X belongs to $SO(n)$, we have to know how to evaluate $\det e^{tX}$.

$$\det e^X = e^{\mathrm{Tr} X}$$

   Prove this proporsition is true when X is diagonalizable.
   X is diagonalizable, so there exists $A \in GL(n,\mathbb{C})$ such that

$$AXA^{-1} = \mathrm{Diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

$$\begin{aligned} \det e^X &= \det(Ae^X A^{-1}) \\ &= \det(e^{AXA^{-1}}) \\ &= \det e^{\mathrm{Diag}(\lambda_1,\cdots,\lambda_n)} \\ &= e^{\mathrm{TrX}} \end{aligned}$$

   Determinant measures how an operator (or matrix) changes volumes. The trace of generator X(also an operator or matrix) gives the rate at which volumes change under the action of the corresponding one-parameter subgroup e^{tX}.

$$\det e^X = e^{\mathrm{Tr} X}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \det e^{tX}\right|_{t=0} = \mathrm{Tr} X$$

So to ensure $\det e^{tX} = 1$, try to make sure $TrX = 0$.

For $\mathfrak{o}(n)$, the condition is already satisfied, $\mathfrak{so}(n) = \mathfrak{o}(n)$. And both of them will be denoted as $\mathfrak{so}(n)$.

For $\mathfrak{su}(n)$, it can be described as the set of traceless, anti-Hermitian $n \times n$ matrices. The tracelessness condition provides one additional constrait beyond the anti-Hermiticity, so $\dim\mathfrak{su}(n) = \dim\mathfrak{u}(n) - 1 = n^2 - 1$.

The basis for all the traceless matrices

$$\{E_{ij}|i \neq j\} \cup \{E_{ii} - E_{i+1,i+1}|1 \leq i < n\} \quad (n^2 - n) + (n - 1)$$

A nice basis for $\mathfrak{su}(n)$ is

$$\{E_{ii} - E_{i+1,i+1}|1 \leq i < n\} \cup \{E_{ij} + e_{ji}|1 \leq i < j \leq n\} \cup \{i(E_{ij} - E_{ji})|1 \leq i < j \leq n\}$$

5. The Lie algebra of $SO(n-1,1)_o$: $\mathfrak{so}(n-1,1) = \mathfrak{o}(n-1,1)$.

   The relationship between $SO(n-1,1)_o$ and $O(n-1,1)$ is the same as that between $SO(n)$ and $O(n)$. We have proved that the Lie algebras of $SO(n)$ and $O(n)$ are the identical.

We can also think of this in the view of the meaning of Lie algebra. Lie algebra is the set of generators X, which are in one-to-one correspondence with 'infinitesimal' transformation $I + \epsilon X$. So, in some sense, we can say that Lie algebra 'contains' the transformations in the Lie group that are close to the identity. We know that the Lie group $O(n)$ is actually disconnected into two parts, one of them is $SO(n)$ and identity transformation is in this part.

No wonder the Lie algebra of $SO(n)$ and $O(n)$ are the identical one. So do $SO(n-1,1)_o$ and $O(n-1,1)$, because we know that $O(n-1,1)$ is disconnected into four parts, and the part that contains I is $SO(n-1,1)_o$.

Now Let's put some physics meaning for the Lie group and Lie algebra some examples:

1. $\mathfrak{so}(2) = \mathfrak{o}(2)$ consists of all antisymmetric 2×2 matrices. We can take

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

   as a basis. You can see that

$$e^{\theta X} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

   X is the 'generator' that generates couterclockwise rotations in the $x - y$ plane.

Now from another point of view, consider the vector field induced by X on $\mathbb{R}^2$, defined as follows: for every $\mathbf{r} \in \mathbb{R}^2$.

$$X^\sharp(\mathbf{r}) = X\mathbf{r} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

so we have: $\nabla X^\sharp = 0$ The flow lines are circles.

2. For any arbitrary element $X \in \mathfrak{so}(3)$ as

$$X = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} = xL_x + yL_y + zL_z$$

We can easily get that $[L_i, L_j] = \sum_{k=1}^{3} \epsilon_{ijk} L_k$ Note that the L_x is not equivalent to physics L_x, remember we said, there is a difference between math Lie group and physics Lie group.

For any $v = (v_x, v_y, v_z) \in \mathbb{R}^3$,

$$Xv = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = (x, y, z) \times (v_x, v_y, v_z) = [X] \times v$$

so that if v lies along the axis $[X]$, then $Xv = 0$. Then we have $e^{tX}[X] = [X]$, because most power series will have $X[X] = 0$.

If we rewrite X:

$$X = \begin{pmatrix} 0 & -n_z & n_x \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}$$

where $\hat{\mathbf{n}} = (n_x, n_y, n_z)$is a unit vector.

$$e^{\theta X} = I + \theta X + \frac{1}{2}(\theta X)^2 + \dots$$

Finally,

$$e^{\theta X} = \begin{pmatrix} n_x^2(1-\cos\theta) + \cos\theta & n_x n_y(1-\cos\theta) - n_z \sin\theta & n_x n_z(1-\cos\theta) + n_y \sin\theta \\ n_y n_x(1-\cos\theta) + n_z \sin\theta & n_y^2(1-\cos\theta) + \cos\theta & n_y n_z(1-\cos\theta) - n_x \sin\theta \\ n_z n_x(1-\cos\theta) - n_y \sin\theta & n_z n_y(1-\cos\theta) + n_x \sin\theta & n_z^2(1-\cos\theta) + \cos\theta \end{pmatrix}$$

If $A(t)$ is a time-dependent orthogonal matrix representing the rotation of a rigid body, then the associated angular velocit bivector in the space frame was

$$[\tilde{\omega}] = \frac{\mathrm{d}A}{\mathrm{d}t} A^{-1}$$

Evaluate it by $A(t) = e^{tX}$, we get that $[\tilde{\omega}] = X$. The angular velocity bivector is just the generator of the rotation. Applying the map J to both side, we get $[\omega] = [X]$, so the pseudovector ω is just the rotation generator expressed in coordinates $[X]$.

Also, the commutator is given by the usual cross product on $\mathbb{R}^3$, because

$$[[X,Y]]_{\mathcal{B}} = [X]_{\mathcal{B}} \times [Y]_{\mathcal{B}}$$

3. $\mathfrak{su}(2)$. It is the set of all 2×2 traceless anti-Hermitian matrices. We take a basis

$$S_x \equiv \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad S_y \equiv \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad S_z \equiv \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

We can easy to check that $[S_i, S_j] = \sum_{k=1}^{3} \epsilon_{ijk} S_k$. It is the same with $\mathfrak{so}(3)$. Also, $Y = xS_x + yS_y + zS_z \in \mathfrak{su}(2)$ is going to be inerpreted as a rotation. It is easy to check that

$$e^{\theta n^i S_i} = \begin{pmatrix} \cos\frac{\theta}{2} - in_z \sin\frac{\theta}{2} & (-in_x - n_y)\sin\frac{\theta}{2} \\ (-in_x + n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z \sin\frac{\theta}{2} \end{pmatrix} = \cos\frac{\theta}{2} I + 2\sin\frac{\theta}{2} n^i S_i$$

Now we can prove that it is a rotation just like we do with the above, Actually, I don't know how to do it, it is not three-dimension, how to prove that the product is going to be zero, when they are lined up.

4. $\mathfrak{so}(3,1)$, we know that for any $X \in \mathfrak{so}(3,1)$,it can be written as

$$X = \begin{pmatrix} X' & \mathbf{a} \\ \mathbf{a} & 0 \end{pmatrix} \text{ where } X' \in \mathfrak{so}(3), \text{ and } \mathbf{a} \in \mathbb{R}^3$$

Now we borrow the L_i from $\mathfrak{so}(3)$, that are

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And embed them into $\mathfrak{so}(3,1)$, then we get $\tilde{L_i}$. Then define new generators for the additional dimension, make them all symmetric K_i. Then we get the commutation relationship:

$$[\tilde{L_i}, \tilde{L_j}] = \sum_{k=1}^{3} \epsilon_{ijk} \tilde{L_k} \quad [\tilde{L_i}, K_j] = \sum_{k=1}^{3} \epsilon_{ijk} \tilde{K_k} \quad [K_i, K_j] = -\sum_{k=1}^{3} \epsilon_{ijk} \tilde{L_k}$$

5. $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$. $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ is the Lie algebra of $SL(2,\mathbb{C})$, it is viewed as a real vector space, so it have the $\mathbb{R}$ subsccript. Since $SL(2,\mathbb{C})$ is the set of all 2×2 complex matrices with the unit determinant, $\mathfrak{sl}(2,\mathbb{C}_{\mathbb{R}})$is the set of all traceless 2×2 complex matrices, using the formula about the determinant and trace.

While actually $\mathfrak{sl}(2,\mathbb{C})$ also is a complex vector space, and it is usually studies from that point of view. But to show the homomorphism from $SL(2,\mathbb{C})$ to $SO(3,1)_o$, we consider it as a real vector space then it will have same structure with $\mathfrak{so}(3,1)$.