notes about group theory

September 11, 2020

Important concept: an abstract vector space
An abstract vector space is

1. a set V(whose elements are just called vectors)
2. a set of scalar C
3. operations of addition and scalar multiplication that satisfy the axioms

   axioms:

   (a) commutativity ($v + w = w + v$ for all v, w in V)
   (b) addition associativity ($v + (w + x) = (v + w) + x$ for all v, w, x in V)
   (c) zero vector (there exists a vector 0 in V such that $v + 0 = v$ for all v in V)dd
   (d) anti vector (for all v in V there is a vector $-v$ such that $v+(-v) = 0$)
   (e) vector distributivity ($c(v + w) = cv + cw$ for all v and w in V and scalars c)
   (f) scalar 1 ($1 \cdot v = v$ for all v in V)
   (g) scalar distributivity ($(c_1 + c_2)v = C_1 v + c_2 v$ for all scalars c_1, c_2 and vectors v)
   (h) scalar multiplication associativity ($(c_1 c_2)v = c_1(c_2 v)$ for all scalars c_1, c_2 and vectors v)

some examples: Usually need to check three axioms: the addition closure, zero vector and anti vector.

1. $\mathbb{R}^n$ with a scalar $\mathbb{R}$, with usual addition and scalar multiplication
2. $\mathbb{C}^n$ with a scalar $\mathbb{C}$, with usual addition and scalar multiplication
3. $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$, $n \times n$ matrices, respectively with a scalar $\mathbb{R}$, a scalar $\mathbb{C}$, with the usual addition and scalar multiplication
4. $H_n(\mathbb{C})$, $n \times n$ Hermitian matrices, with a scalar $\mathbb{R}$, with the usual addition and scalar multiplication notice that $H_n(\mathbb{C})$ is actually a subset of $M_n(\mathbb{C})$, just need to check addition closure and zero vector

5. square-integrable complex-valued functions on an interval, $L^2([a,b])$, with a scalar $\mathbb{C}$, with usual addition and scalar multiplication
   the addition closure is a little tricky.

6. square-integrable real-valued functions on an interval, with a scalar $\mathbb{R}$, with the usual addition and scalar multiplication.

7. all complex-coefficient polynomial functions on $\mathbb{R}^3$ of fixed degree l, $P_l(\mathbb{R}^3)$, with a scalar $\mathbb{C}$, with usual addition and scalar multiplication

8. harmonic degree l polynomial, $\mathcal{H}_l(\mathbb{R}^3)$, with a scalar $\mathbb{C}$, with the usual addition and scalar multiplication

9. spherical harmonic polynomials $\tilde{\mathcal{H}}_l$, with a scalar $mathbbC$, with the usual addition and scalar multiplication.

   harmonic degree l polynomial consists of the function which satifies Laplacian equation.

10. $S_n(\mathbb{R})$, symmetric matrices, with a scalar $\mathbb{R}$, with the usual addition and scalar multiplication.

11. $A_n(\mathbb{R})$, antisymmetric matrices, with a scalar $\mathbb{R}$, with the usual addition and scalar multiplication.

    another space: $GL(n,\mathbb{R})$ is not a vector space with a scalar $\mathbb{R}$, it consists of invertible $n \times n$ matrices. because of no zero vectot

some verification of the examples
one function in the set $P_l(\mathbb{R}^3)$, $f(x,y,z) = \sum_{\substack{i,j,k \\ i+j+k=l}} c_{ij} x^i y^j z^k$ if we put the function into spherical coordinates, we're sure that it will be the form of: $f(x,y,z) = r^l\, Y(\theta,\phi)$, where the $Y(\theta,\phi)$ can be any function

but if you use the $\mathcal{H}_l \mathbb{R}^3$ restriction, you will see the corresponding $Y(\theta,\phi)$ has to be a spherical harmonic degree l polynomial.

Thus the familiar sphercal harmonics $Y^l_m(\theta,\phi)$ are just the restriction of particular harmonic polynomial to the unit sphere, which means cutting the r variable the harmonic polynomial is a spherical harmonic polynomial.

Important concept: a basis of a vector space

In physics, there are times when the basis of a space is infinite. Under this cricumstances, the linear combination is considered as an arbitarily large but finite number of terms' combination.

1. $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$, the standard basis are their basis at the same time, they have the same dimension; the symmetric and antisymmetric matrices also form a basis for them, We like to call the symmetric set $S_n(\mathbb{R})$ and the other $A_n(\mathbb{R})$

2. $H_2(\mathbb{C})$, a set of basis $\mathcal{B}$: $\{I, \sigma_x, \sigma_y, \sigma_z\}$

3. $\tilde{\mathcal{H}}_l(\mathbb{R}^3)$, a set of basis $\mathcal{B}$: $\{Y_l^m\}_{-l\leq m\leq l}$, another basis: cartesian basis $\{x, y, z\}$. I don't know the relationship between the two, the dimension is differen ? Anyway, it will be answered afterwards.

Important concept: Components Given $v \in V$ and a basis $\mathcal{B} = \{e_i\}_{i=1,2,\cdots,n}$ for V, we decompose the v into $v = \sum_{i=1}^{n} v^i e_i$, the corresponding coefficients are called *components of v with respect to $\mathcal{B}$* , thus we represent v by the column vector, denoted $[v]_{\mathcal{B}}$ as, $\cdots$

One must keep in mind that: vectors exist indepently of any chosen basis, and their expressions as row or columns vectors depend very much on the choice of the basis.

some example:

1. different bases for $\mathbb{C}^2$ The vector space for a spin $^1/_2$ particle is identifiable with $\mathbb{C}^2$ and the angular momentum operators are given by $L_i = \frac{1}{2}\sigma_i$ The difference between a vector and its component representation is not just pedantic: it can be of real physical importance. An eigenvector of L_z is one-term vector in one basis, and two-term vector in another basis, which means the probability for measuring different observable value.

2. Fourier series of a function.

Important concept: Linear operator

Linear operator is the special situation for linear transformation. There is an important theorem for linear transformation: rank-nullity theorem: $\dim\ker(T)+\dim\mathrm{im}(T) = n$, which is the dimension of vector space V, and T is the linear transformation that from V to W.

This theorem can imply the equivalence between the onto and one-to-one for linear operator.

A linear operator is a function that is based on a vector space. Whenever you see a linear operator, find the corresponding vector space first.

A vector space has an important artribute: all of its linear operators forms a vector space, with a scalar $\mathbb{R}$ or $\mathbb{C}$, with the usual addition and scalar multiplication, denoted as $\mathcal{L}(V)$

Linear operators can take vectors into vectors, and also can take linear operator into linear operator.

some example:

1. Given $A, B \in \mathcal{L}(V)$, we can define a linear operator $ad_A \in \mathcal{L}(\mathcal{L}(V))$acting on B by $ad_A(B) \equiv [A, B]$, this action of A on $\mathcal{L}(V)$ is called the adjoint action or adjoint representation.

   The Heisenberg picture emphasized $\mathcal{L}(v)$ rather than V and interprets the Hamiltonian as an operator in the adjoint representation. For any observable A,
$$\frac{dA}{dt} = iad_H(A)$$

One important property of a linear operator T is whether or not it is invertible. A linear operator is invertible if and only if the only vector it sends to 0 is the zero vector.

For oridinary map, whether it is invertible depends on whether the map is both one-to-one (function value $\implies$ beginning value)and onto ($\forall$ there exists). But, linear operator is more special, the condition is more specific.

One important property: suppose V is finite-dimensional, and $T \in \mathcal{L}(V)$, then T is one-to-one is equivalent to T being onto. So T is one-to-one is equivalent to T being both one-to-one and onto, then equivalent to T is invertible, then equivalent to $T(v) = 0 \implies v = 0$

2. linear operator L_z in the basis of vector space $\mathcal{H}_1(\mathbb{R}^3)$. We already know that there are two familiar sets of basis: one is $\{rY^l_m\}_{-1\leq m\leq 1} = \{\frac{1}{\sqrt{2}}(x+iy), z, \frac{1}{\sqrt{2}}(x-iy)\}$, the other is $\{x, y, z\}$

How the matrices relate to the linear operator: **A linear operator is not the same thing as a matrix, it also exists independently, the identification can only be made once a basis is chosen.** Linear operator on finite-dimensional space can relate to finite-dimensional matrix, while linear operator on infinite-dimensional space can relate to infinite-dimensional matrix.

The action of the linear operator T is determined by its action on the basis vectors:

$$T(v) = T(\sum_{i=1}^{n} v^i e_i) = \sum_{i=1}^{n} v^i T(e_i)$$

which can be decomposed again into the basis combination $= \sum\limits_{i,j=1}^{n} v^i T^j_i e_j$. Thus we know that both $[v]_{\mathcal{B}}$ and $[T(v)]_{\mathcal{B}}$are column vector which are represented by the basis. Because the special form of $[T(v)]_{\mathcal{B}}$ is suspiciously like a result of matrix multiplication,

$$[T(v)]_{\mathcal{B}} = \begin{pmatrix} \sum\limits_{i=1}^{n} v^i T^1_i \\ \sum\limits_{i=1}^{n} v^i T^2_i \\ \vdots \\ \sum\limits_{i=1}^{n} v^i T^n_i \end{pmatrix} = \begin{pmatrix} T^1_1 & T^1_2 & \cdots & T^1_n \\ T^2_1 & T^2_2 & \cdots & T^2_n \\ \vdots & & & \\ T^n_1 & T^n_2 & \cdots & T^n_n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = [T]_{\mathcal{B}}[v]_{\mathcal{B}}$$

we *define* the matrix relationship of linear operator in the basis $\mathcal{B}$, by the equation $[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}}$, from which we can know the concrete result of a linear operator in a basis.

So this is the relationship between matrices and linear operators, if you want to calculate the terms of the corresponding matrix in basis $\mathcal{B}$, use the equations: $T(e_i) = \sum\limits_{j=1}^{n} T^j_i e_j, i = 1, 2, \cdots n$

The definition of the product or composition of two linear operator is: $(AB)(v) = A(B(v))$

Important concept: Dual space A daul vector is an object that eats a vector and spits out a number.

Precise definition: Give a vector space V with a scalar C, a dual vector(or linear functional) on V is a C-valued linear function on V, where linear means: $f(cv + w) = cf(v) + f(w)$. On the left side, there is only calculation in space V, on the right side, there are calculations in space C.

A dual vector is actually a linear function, it has to be based on a vector space, it and its counterparts form the dual space of the vector space. Dual space of vector space V is denoted V^*.

Dual space has an important attribute: V^*, with the scalar C with the usual addition and scalar multiplication is a complex vector space. Also another attribute: it is entirely determined by its value on basis (so does a linear operator, but a linear operator eats a vector and spits another vector).

For a vector space, whose basis is $\mathcal{B} = e_i$, the corresponding dual space basis is defined as: $\mathcal{B}' = e^i$, where $e^i(v) = v^i$. To show why, these dual vector can be a basis for dual space: the definition can implies that $e^i(e_j)) = \delta^i_j$, so $f = \sum\limits_{i=1}^{n} f_i e^i$ in the finite case.

some example:

1. General example of a dual space: let $\mathcal{B} = \{e_i\}$ be a basis for vector space V, we define a set of daul vectors: $e^i(v) = v^i$, whose functions are to pick out the ith component of any vector. The definition of the dual vectors is equal to: $e^i(e_j) = \delta^i_j$. You can see this set is actually a basis for dual space V^*, denoted $\mathcal{B}^*$
2. dual space of $\mathbb{R}^n$, dual space basis: $\mathcal{B}^* = \{e^i\}$, where $e^i(e_j) = \delta^i_j$
3. dual space of $\mathbb{C}^n$, dual space basis: same with the above
4. dual space of $M_n(\mathbb{R})$, dual space basis: $\mathcal{B}^* = \{f^{ij}\}$, where $f^{ij}(A) = A_{ij}$
5. dual space of $M_n(\mathbb{C})$, dual space basis: the same with the above.

some verification: for any dual vector f:

$$
\begin{aligned}
f(v) &= f\left(\sum_{i=1}^{n} v^i e_i\right) \\
&= \sum_{i=1}^{n} v^i f(e_i)
\end{aligned}
$$

$$
e^i(e_j) = \delta^i_j \implies f = \sum_{i=1}^{n} f(e_i) e^i
$$

which is a decomposition based on the $\{f(e_i)\}$, and can be view as $f(v) = [f]^T_{\mathcal{B}^*} \cdot [v]_{\mathcal{B}} =$

Important concept: Non-degenerate Hermitian form A non-degenerate Hermitian form on a vector space V is a C-value function which eats an ordered pair of vectors and spits a scalar, and having the following properties:

1. linearity in the second argument $((v|w_1 + cw_2) = (v|w_1) + c(v|w_2))$
2. Hermiticity $(v|w) = \overline{(w|v)}$complex conjugation
3. non-degeneracyfor each $v \neq 0 \in V$, there exists $w \in V$ such that $(v|w) \neq 0$

and following potencial properties:

1. conjugate-linear in the first argument $(cv_1 + v_2|w) = \bar{c}(v_1|w) + (v_2|w)$
2. for real vector space, symmetric(in this case, the function is called metric)

A metric is a non-degenerate form which on a real vector space, its definition still contains the above three, only but the second one turns into symmetric.

An inner product on a vector space V is a C-valued function which eats an ordered pair of vectors and spits a scalar, and having the following properties:

1. linearity in the second argument
2. Hermiticity
3. positive-definiteness $(v|v) > 0$ for all $v \in V, v \neq 0$

and the following properties:

1. linearity in both arguments.
2. non-degeneracy, because the positive-definiteness implies the non-degeneracy.

One very important use of non-degenerate Hermitian forms is to define preferred sets of bases known as orthonormal bases, which is basis $\mathcal{B} = \{e_i\}$, $(e_i|e_j) = \pm\delta_{ij}$. And if the form is even an inner product, the condition can be $(e_i|e_j) = \delta_{ij}$

some examples of inner product or non-degenerate Hermitian form:

1. inner product: the dot product on $\mathbb{R}^n$, or Euclidean metric on $\mathbb{R}^n$, $(v|w) = \sum_{i=1}^{n} v^i w^i$
2. inner product: Hermitian scalar product on $\mathbb{C}^n$, $(v|w) = \sum_{i=1}^{n} \bar{v}^i w^i$
3. inner product: half trace product on $M_n(\mathbb{C})$, $(A|B) = \frac{1}{2}Tr(A^\dagger B)$
4. only non-denegerate Hermitian form, not an inner product: The Minkowski metric on 4-D spacetime $(\mathbb{R}^4)$, $\eta(v_1, v_2) = x_1x_2 + y_1y_2 + z_1z_2 - t_1t_2$

the Minkowski metric can be written in components as a matrix, just like a linear operator. If we define the components $\eta_{ij} = \eta(e_i, e_j)$, where $e_i \in \mathcal{B}$ standard basis. So the matrix is

$$[\eta]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

then the Minkowski metric can be: $\eta(v_1, v_2) = [v_2]^T_{\mathcal{B}}[\eta]_{\mathcal{B}}[v_1]_{\mathcal{B}}$. Also, the symmetric matrix for the standard basis means all the matrix for any basis are symmetric.

5. inner product, the Hermitian scalar product on $L^2([-a,a])$, $(f|g) \equiv \frac{1}{2}\int_{-a}^{a} \bar{f} g dx$

$$\begin{cases} orthonormalset\{e_i\} \subset \mathcal{H} \\ \forall i\ (e_i|f) = 0 \implies f = 0 \end{cases}$$

$$\xLeftrightarrow{\text{in finite-dimensional case}}$$

$$\begin{cases} span\{e_i\} = \mathcal{H} \\ (e_i|e_j) = \delta_{ij} \end{cases}$$

bu the infinite case can be tricky, need to talk about the completeness in real analysis.

For $L^2([-a,a])$ for which $\left\{e^{i\frac{n\pi x}{a}}\right\}_{n\in\mathbb{Z}}$

How to think about infinite-dimensional Hilbert spaces and their bases: a basis for a Hilbert space is an infinite set whose infinite linear combinations, together with some suitable convergence condition, form the entire vector space.

some verification:

1. the definition of a Hilbert space basis is equivalent to our original definition of a basis for a finite-dimensional inner product space V.

   So you need to see $(e_i|f) = 0 \quad \forall i \implies f = 0$, and $(e_i|e_j) = \delta_{ij}$

The connection between dual vectors and non-degenerate Hermitian forms:

Given a non-degenerate Hermitian form $(\cdot|\cdot)$ on a finite-dimensional vector space V, we can associate to any $v \in V$ a dual vector $\tilde{v} \in V^*$, defined by $\tilde{v}(w) = (v|w)$, denoted $\tilde{v}() = L(v) = (v|\cdot)$When we first introduced the dual space, it is independent from what it is gonna eat. Now using non-degenerate Hermitian form, we construct dual vector from the normal vector

And the original dual space is the set of the linear functions.

The general way to construct the dual space is to use the definition: set of the linear functional and $e^i(v) = v^i$, relationship between space and dual space is obscure. then there are two concrete ways to find them:

1. find a set of basis $\mathcal{B} = \{e_i\}$, then use $e^i(e_j) = \delta^i_j$ to find the corresponding dual basis, then span the dual space

2. use non-degenerate form to build a map, relationship between space and dual space is obvious. then use metric dual definition: $\{L(e_i)\}$

   So the corresponding dual basis in the two concrete way can be different: The daul basis vector e^i is defined relative to the whole basis$\{e_i\}$, while the metric dual $L(e_i)$ only dependes on e_i and the form, does not care about other basis vectors.

The important map: through the non-degenerate Hermitian form, we establish a map between vector to one kind of dual vector. The dual vector that is established by the non-degenerate Hermitian form is called metric dual of v. So on the above we can denote the dual vector as $L(v)$. The map L is conjugate-linear. This map is one-to-one (which can be proved by non-degenerate Hermitian form's non-degenerate property, $L(v) = L(w) \implies v = w$) and onto ($\forall \tilde{v} \in V^*$, there exists $v \in V$, such that $L(v) = \tilde{v}$). I really don't know how to prove this, I hope I will get it.

When Given a basis $\{e_i\}$, under what circumstances do we have $e^i = L(e_i)$, when $(e_i|e_j) = \delta_{ij}$

some example:

1. bras are really dual vectors.

2. consider $\mathbb{R}^4$ with the Minkowski metric, let $\mathcal{B} = \{e_\mu\}_{\mu=1,2,3,4}$ and $\mathcal{B}' = \{e^\mu\}_{\mu=1,2,3,4}$ be the standard basis and dual basis for $\mathbb{R}^4$,

   let $v = \sum_{\mu=1}^{4} v^\mu e_\mu \in \mathbb{R}^4$, then the corresponding dual vector $\tilde{v} = \sum_{\mu=1}^{4} \tilde{v}_\mu e^\mu$, where the components are: $\tilde{v}_\mu = \tilde{v}(e_\mu) = (v|e_\mu) = \sum_\lambda v^\lambda (e_\lambda|e_\mu) = \sum_\lambda v^\lambda \eta_{\lambda\mu}$,

   thus $\tilde{v} = \sum_{\mu=1}^{4} \tilde{v}_\mu e^\mu$ actually equals to $[\tilde{v}]_{\mathcal{B}'} = [\eta]_{\mathcal{B}}[v]_{\mathcal{B}}$

   $v = \sum_{\mu=1}^{4} v^\mu e_\mu$we call the number v^μ contravariant components for the vector, at the same time, we call the number $\tilde{v}_\mu = \tilde{v}(e_\mu) = (v|e_\mu) = \sum_\lambda v^\lambda (e_\lambda|e_\mu) = \sum_\lambda v^\lambda \eta_{\lambda\mu}$, thus $\tilde{v} = \sum_{\mu=1}^{4} \tilde{v}_\mu e^\mu$ covariant components of the vector.

   If the vector is dual vector, the situation is reversed.(the covariant components are its actual components, and the contravariant components are the components of its dual vector.)

   In the $\mathbb{R}^3$ with the Euclidean metric, $[\tilde{v}]_{\mathcal{B}'} = [\eta]_{\mathcal{B}}[v]_{\mathcal{B}}$ stays right, and the $[\eta]_{\mathcal{B}} = I$, so the covariant and contravariant components of any vector in an orthonormal basis are identical. However, when in $\mathbb{R}^4$, the $[\eta]_{\mathcal{B}} \neq I$

3. all of the above is considered under finite-dimensional case, $L^2([-a,a])$ is infinite-dimensional. In this case, map L is still one-to-one, but it is not onto anymore. For example, the dual of Dirac Delta functional $\delta \in L^2([-a,a])$ is not in $L^2([-a,a])$ any more.

some problem:

1. double dual space:
$$J : V \to (V^*)^*$$
$$v \mapsto J_v$$
where J_v acts on $f \in V^*$ by $J_v(f) = f(v)$.

   The function J_v which acts on $f \in V^*$ is linear, because $\forall f, g \in V^*, J_v(f + cg) = (f + cg)(v) = f(v) + cg(v) = J_v(f) + cJ_v(g)$

   The map J is linear, because $\forall v, w \in V, J(v + cw)(f) = f(v + cw) = f(v) + cf(w) = (J(v) + cJ(w))(f)$

   The map J is linear, because:$\forall f \in V^*, J(cv + w)f = f(cv + w) = cf(c) + f(w) = cJ(v) + J(w)$. and the map is one-to-one, and also onto.

   Also, J_v, the function that related to vector v, is a linear function. Notice this linear function's definition dose not involve the non-degenerate Hermitian form.

   there is a property: $\{e_i\}$ is one basis for V, the corresponding dual basis is $\{e^i\}$, then $\{J_{e^i}\}$ is the dual basis for $\{e^i\}$. That means we consider the linear function e^i as a vector and create its corresponding dual basis, which is actually the original basis. $J_{e^i}(e^j) = e^j(e^i) = \delta^{ij}$

2. the transpose of A, a linear operator on V^*, where A is a linear operator on V. A^T is a linear operator that is constructed from the linear operator A. $A^T(f)(v) = f(Av)$, where $v \in V$, $f \in V^*$. The linear operator is on V^*, vector v really is not the point.

   If $\mathcal{B}$ is a basis for V, and $\mathcal{B}^*$ is the corresponding basis, then $[A^T]_{\mathcal{B}^*} = [A]^T_{\mathcal{B}}$.

   the transpose of a matrix has meaning: it is the matrix representation of the transpose of the linear operator represented by the original matrix

$$A^T(\tilde{v}) = \sum_{\lambda,\mu} \tilde{v}_\lambda (A^T)^\lambda_\mu e^\mu \implies$$

$$\begin{aligned}
[A^T(\tilde{v})]_{\mathcal{B}^*} &= \begin{pmatrix} \sum_\lambda \tilde{v}_\lambda (A^T)^\lambda_1 & \sum_\lambda \tilde{v}_\lambda (A^T)^\lambda_2 & \cdots & \sum_\lambda \tilde{v}_\lambda (A^T)^\lambda_n \end{pmatrix} \\
&= \begin{pmatrix} \tilde{v}_1 & \tilde{v}_2 & \cdots & \tilde{v}_n \end{pmatrix} \begin{pmatrix} (A^T)^1_1 & (A^T)^1_2 & \cdots & (A^T)^1_n \\ (A^T)^2_1 & (A^T)^2_2 & \cdots & (A^T)^2_n \\ \vdots & & & \\ (A^T)^n_1 & (A^T)^n_2 & \cdots & (A^T)^n_n \end{pmatrix} \\
&= [\tilde{v}]_{\mathcal{B}^*} [A^T]_{\mathcal{B}^*}
\end{aligned}$$

$$A(v) = \sum_{i,j} v^i T^j_i e_j \implies$$

$$[A(v)]_{\mathcal{B}} = \begin{pmatrix} \sum_{i=1}^{n} v^i A_i^1 \\ \sum_{i=1}^{n} v^i A_i^2 \\ \vdots \\ \sum_{i=1}^{n} v^i A_i^n \end{pmatrix} = \begin{pmatrix} A_1^1 & A_2^1 & \cdots & A_n^1 \\ A_1^2 & A_2^2 & \cdots & A_n^2 \\ \vdots & & & \\ A_1^n & A_2^n & \cdots & A_n^n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = [A]_{\mathcal{B}}[v]_{\mathcal{B}}$$

It is easy to see that for all $v \in V$, we have

$$[\tilde{v}]_{\mathcal{B}^*}[A(v)]_{\mathcal{B}} = [A^T(v)]_{\mathcal{B}^*}[v]_{\mathcal{B}}$$

So we have that $[A^T]_{\mathcal{B}^*} = [A]_{\mathcal{B}}^T$

3. the Hermitian adjoint of A, a linear operator on V, where A is a linear operator on V. $A^\dagger$ is a linear operator that is constructed from the linear linear operator A.

$$A^\dagger = L^{-1} \circ A^T \circ L$$

$$A^\dagger v = (L^{-1} \circ A^T \circ L)v = L^{-1} \circ A^T \tilde{v} = L^{-1}(\langle v| \circ A)$$
$$(A^\dagger v|w) = [L \circ L^{-1}(\langle v| \circ A)]w = (\langle v| \circ A)w = (v|Aw)$$

Under this definition, the linear operator has a property, which based on the V vector space has a non-degenerate Herimitian form: $(A^\dagger v|w) = (v|Aw)$

Also, if $\mathcal{B} = \{e_i\}$ is a basis of V, then $[A^\dagger]_{\mathcal{B}} = [A]_{\mathcal{B}}^\dagger$

$$A(v) = \sum_{i,j} v^i T_i^j e_j \implies$$

$$[A(v)]_{\mathcal{B}} = \begin{pmatrix} \sum_{i=1}^{n} v^i A_i^1 \\ \sum_{i=1}^{n} v^i A_i^2 \\ \vdots \\ \sum_{i=1}^{n} v^i A_i^n \end{pmatrix} = \begin{pmatrix} A_1^1 & A_2^1 & \cdots & A_n^1 \\ A_1^2 & A_2^2 & \cdots & A_n^2 \\ \vdots & & & \\ A_1^n & A_2^n & \cdots & A_n^n \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{pmatrix} = [A]_{\mathcal{B}}[v]_{\mathcal{B}}$$

$$
\begin{aligned}
[A^\dagger(v)]_{\mathcal{B}} &= L^{-1} \circ A^T \circ L(v) = L^{-1} \circ A^T \circ L(\sum_\lambda v^\lambda e_\lambda) \\
&= \sum_\lambda (v^\lambda)^* L^{-1} \circ A^T \circ L(e_\lambda) \\
&= \sum_\lambda (v^\lambda)^* L^{-1} \circ L \circ A e_\lambda \\
&= \sum_\lambda (v^\lambda)^* A e_\lambda \\
&= \sum_\lambda (v^\lambda)^* \begin{pmatrix} A^1_\lambda \\ A^2_\lambda \\ \vdots \\ A^n_\lambda \end{pmatrix} \\
&= \begin{pmatrix} (v^1)^* A^1_1 + (v^2)^* A^1_2 + \cdots (v^n)^* A^1_n \\ (v^1)^* A^2_1 + (v^2)^* A^2_2 + \cdots (v^n)^* A^2_n \\ \vdots \\ (v^1)^* A^n_1 + (v^2)^* A^n_2 + \cdots (v^n)^* A^n_n \end{pmatrix} \\
&= [\tilde{v}]_{\mathcal{B}^*} [A]_{\mathcal{B}} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
e^j A^\dagger(e_i) &= e^j L^{-1} \circ A^T \circ L(v) \\
&= e^j L^{-1} \circ A^T \circ e^i
\end{aligned}
$$

4. Let g be a non-degenerate bilinear form on a vector space V. $\{e_i\}$ to be the basis not necessarily orthonormal

   In an orthonormal basis the matrix of a Hermitian operator is a Hermitian matrix. Is this necessarily true in a non-orthonormal basis? This may let think about matrix representaion does not necessarily based on an orthonormal basis.