notes about the group theory 7

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Important Concept: A representation of a group It is a vector space V together with a group homomorphsim $\Pi : G \to GL(V)$. Sometimes they are written as a pair (Π, V). We call the space V representation space. If V is a real vector space, we call the representation real representation, same with complex vector space.

We treat the elements of the group G as operators, though intuitively they don't seem like an operator, or you just don't view them as an operator. This is just like when you touch the quantum mechanics, all the variables are set to be operator.

If G is a matrix Lie group and V is finite-dimensional. Then the group homomorphism $\Pi : G \to GL(V)$is continuous, then Π induces a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ by

$$\pi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\pi(e^{tX})\bigg|_{t=0}$$

Any homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ for some V is known as a Lie algebra representation.

Every finite-dimensional representation of a Lie group G induces a representation of the corresponding Lie algebra. The converse is not true.

In many application, the vector space V is equipped with an inner product which preserved by the operators $\Pi(g)$. In other words, $\Pi(g) \in \mathrm{Isom}(V) \quad \forall g \in G$, thus the group homomorphism is actually $\Pi : G \to \mathrm{Isom}(V)$. In this case, we say that Π is a unitary representation. Ok, here is a question. Now that there is a homomorphism from $G \to GL(V)$ and $\mathrm{Isom}(V) \subset GL(V)$, why does the homomorphism from $G \to \mathrm{Isom}(V)$ exist?

Oh, it suddenly hit me that homomorphism is not isomorphism. they have to have the same diemnsion. It is perfectly possible that there is still a homomorphism from G to $\mathrm{Isom}(V)$.

Now consider the Lie algebra homomorphism induced by the group homomorphism. It is $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. For any $X \in \mathfrak{g}$, $\pi(X) = \Pi(e^{tX})$. Thus we have $(v|w) = (\Pi(e^{tX})v|\Pi(e^{tX})w)$.

Employing our standard trick of differentiating with respect to t and evaluating at $t = 0$.

Use the proposition: Let $G \to H$ be a continuous homomorphism, this will induces a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ given by $\phi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(X)\big|_{t=0}$

$$\begin{aligned}(\pi(X)v|w) + (v|\pi(X)w) &= (\pi(X)v|w) + (\pi(X)^\dagger v|w) \\ &= ([\pi(X) + \pi(X)^\dagger]v|w)\end{aligned}$$

So $([\pi(X) + \pi(X)^\dagger]v|w) = 0$.

So the above group representation $\Pi : G \to \mathrm{Isom}(V)$, induces a Lie algebra representation $\pi : \mathfrak{g} \to \mathfrak{isom}(V)$. Now this Lie algebra representation is called a unitary representation.

Some examples:

1. Let our representation space be $V = \mathbb{C}$. We can show that $GL(\mathbb{C}) \simeq GL(1, \mathbb{C}) \simeq \mathbb{C}^*$. So $\Pi : G \to \mathbb{C}^*$ is a group homomorphism, hence a representation.

   If you want to show that $GL(\mathbb{C}) \simeq GL(1, \mathbb{C})$.

   We always know that $GL(V) \to GL(n, C)$ is a homomorphism and even isomorphism.

   We know nothing about G, we just claim that $V = \mathbb{C}$, then we get a group homomorphism.

2. Trivial representation: For any group G and vector space V, define the trivial reprentation of G on V, by the homomorphism $\Pi : \Pi(g) = I, \quad \forall g \in G$.

   Now suppose that G is a Lie group, Π induces a Lie algebra representation. It is $\pi(X) = 0 \quad \forall X \in \mathfrak{g}$.

   From the above two representation, we can summarize something. They don't request anything from G, but one request the V and the other request the homomorphim.

3. Fundatmental representation This one request G to be a matrix Lie group and lets V be $\mathbb{C}^n$. Actually, here $V = \mathbb{C}^n$ is the most important thing.

   : Let G be a matrix Lie group, then $G \subset GL(n, C) = GL(C^n)$ for some n. Interpreting elements in G as operators, and $GL(n, C)$ as matrices, we get a homomorphism, thus a representation. This is called a fundamental or standard representation.

   If $G = O(3)$ or $SO(3)$, then $V = \mathbb{R}^3$, and the fundamental representation is known as vector representation.

   If $G = SU(2)$, then $V = \mathbb{C}^2$, and the fundamental representation is known as spinor representation.

   If $G = SO(3, 1)_o$ or $O(3, 1)$, then $V = \mathbb{R}^4$, and then fundamental representation is known as four-vector representation.

If $G = SL(2,\mathbb{C})$, then $V = \mathbb{C}^2$, and the fundamental representation is known as spinor representation, the vectors in this representation are referred to as left-handed spinors, and are used to describe massless relativistic spin $\frac{1}{2}$ particles.

The fundamental representation of $SO(3)$, $O(3)$ and $SU(2)$ is unitary representation. But the representation of $SO(3,1)_o$, $O(3,1)$ and $SL(2,\mathbb{C})$ are not unitary reprensentation, because these groups preserves the Minkowski metric, which is not an inner product.

Each of these group representations induces a representation of the corresponding Lie algebra which then goes by the same name, and which is also given by interpreting the elements as linear operators.

Since Lie algebras are vector space and a representation is a linear map, we can describe any Lie algebra representation just by figuring out the image of basic vectors under π. examples:

1. The adjoint representaion request the homomorphism, and request the relationship between G and V, which is actually a result from the request of the homomorphism. Note that it does not provide a concrete V.

   The adjoint representation Ad is a map from G to $GL(\mathfrak{g})$, where the operator $Ad_A, A \in G$ is defined by $Ad_A(X) = AXA^{-1}, X \in \mathfrak{g}$.

   In the context of representation theory, the Ad homomorphism is known as adjoint representation $(Ad, \mathfrak{g})$.

   We now consider the corresponding Lie algebra representation $(ad, \mathfrak{g})$, which acts as $\mathrm{ad}_X(Y) = [X, Y], X, Y \in \mathfrak{g}$.

   From the discussion before, we know that Ad_A will preserve a metric on $\mathfrak{g}$ known as Killing form, thus $\mathrm{Ad} : G \to \mathrm{Isom}(\mathfrak{g})$.

   Now put $G = SU(2)$, we get that $Ad : SU(2) \to \mathrm{Isom}(SU(2)) \simeq O(3)$. So, $\mathrm{Ad} : SU(2) \simeq O(3)$

   For $\mathfrak{so}(3)$ with basis $\mathcal{B} = \{L_i\}$