notes about group theory 4

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In physics we are often interested in how a particular object behaves under a particular set of transformation. In quantum mechanics, one is often interested in the spin of the ket (which specifies how it transform under rotations), or its behavior under the time-reversal or space inversion transformations. This knowledge leads to 'selection rules' which greatly simplify evaluation of matrix elements.

All physical observable can be considered as 'infinitesimal generators' of particular transformations: for example, the angular momentum operators 'generate' rotations and the momentum operator 'generate' translation.

We begin with the discussion of 'finite' transformations, which share some common properties: one, the performance of two successive transformations is always equivalent to a third transformation; second, every transformation has an inverse which undos it. These transformation actually forms a group, that is one of the most important region of group.

1 Important concept: Group

A group is a set G together with a 'multiplication' operations, that satisfies the follwoing axioms:

1. closure
2. associativity
3. existence of the identity, which can be proved to be unique
4. existence of inverse,which can be proved to be unique

Notice there is no commutativity!

If we add the commutativity, we have commutative or abelian group:

1. commutativity
2. associativity
3. closure
4. existence of the identity

5. existence of inverse

Remember there are right inverse and left inverse. Also cancelation laws are avaliable in groups, right and left.

Another concept is additive group, they are a special kind of group.

some examples:

1. $\mathbb{R}$ together with 'multiplication' being the regular addition, is a group.

   And because its 'multiplication' is somehow like 'addition' (the identity is zero element, the inverse if minus), it is also called an additive group.(no new axioms are added)

2. $\mathbb{R}$ $\{0\}$ together with the regular multiplication, is a group.

3. $\mathbb{C}$ together with the regular addtion, is a group

4. $\mathbb{C}$ $\{0\}$ together with the regular multiplication is a group.

5. let's review the axioms of vector space:

   (a) commutativity
   (b) addition associativity, which can be viewed as the associativity in axioms of group
   (c) zero vector, existence of identity
   (d) anti vector, existence of inverse
   (e) vector distributivity
   (f) identical vector
   (g) scalar distributivity
   (h) scalar multiplication associativity

   For memorizing, two associativity, two distributivity, one commmuativity, two existence, one scalar.

   Maybe the vector space is why there is a kind of group being called additive group. While viewsing the vector spaces as additive groups means we ignore the crucial feature of scalar multiplications,.

6. the general linear group of a vector space V, denoted $GL(V)$, is defined to be the subset of $\mathcal{L}(V)$ consisting of all invertible linear operators on V I think $GL(V)$ should equal to $\mathcal{L}(V)$. Why did you say they are not and $GL(V) \subset \mathcal{L}(V)$.

   You thought so because you viewed $\mathcal{L}(V)$ as a group with addition as the mulitplication. This statement only tells you that $GL(V)$ is a subset not a subgroup of $\mathcal{L}(V)$.

   $GL(V)$ is not a subgroup of $L(V)$. $GL(V)$ is a group with composition as the muliplication while $\mathcal{L}(V)$ is a group with addition as the mulplication.

7. Let V have scalar field C and dimension n, then pick a basis for V, then we can have the corresponding matrix for any $T \in GL(V)$, so these corresponding matrices form a group, which is denoted $GL(n, C)$. Similarly, we get $GL(n, R)$, the real general linear group in n dimension.

   These groups do not occur explicitly very often, but their subgroups do, especially when V is equipped with a non-degenerate Hermitian form. And the set of isometries are the subgroups, which is denoted $\mathrm{Isom}(V)$.

   $\mathrm{Isom}(V)$ consist of the operators which preserve $(\cdot|\cdot)$ in the sense that $(Tv|Tw) = (v|w)$. These preserving operators are always invertible, and $\mathrm{Isom}(V) \subset GL(V)$, so $\mathrm{Isom}(V)$ is a subgroup of $GL(V)$.

   For checking the subgroup, you need to check the closure under multiplication, the existence of identity, the existence of inverse.

   The existence of inverse $(T^{-1}w|T^{-1}w) = (TT^{-1}v|TT^{-1}w) = (v|w)$.

   $\mathrm{Isom}(V)$ is very important, because the matrix representations of the operators in it are the orthogonal matrices, unitary matrices, and the Lorentz transformations, depending on whether V is real or complex and whether $(\cdot|\cdot)$ is definite-positive. These three groups will apear next

   (a) the orthogonal group $O(n)$. Let V be an $n-$dimensional real space, with inner product. The isometries of V can be thought of as operators which preserve $(\cdot|\cdot)$, then preserve length and angles.

   Let see the persevation from the view of components:

$$
\begin{aligned}
(v|w) &= (Tv|Tw) \\
[v]^T[w] &= \delta_{ij}(Tv)^i(Tw)^j \\
&= \delta_{ij}T^i_k v^k T^j_l W^l \\
&= \sum_i v^k T^i_k T^i_l w^l \\
&= [v]^T[T]^T[T][w]
\end{aligned}
$$

   So we have $[T]^T[T] = I$ We can check that $O(n)$ is a group, check the closure under multiplication, the existence of identity, the existence of inverse.

   (b) the Lorentz group $O(n-1, 1)$. Let V be a real vector space, with a Minkowski metric (a non-degenerate Hermitian form and n can be greater to 4) Still from a view of components:

$$
\begin{aligned}
[v]^T[\eta][w] &= \eta(v, w) \\
&= \eta(Tv, Tw) \\
&= [v]^T[T]^T[\eta][T][w]
\end{aligned}
$$

So we get that $[T]^T[\eta][T] = [\eta]$, in components $T^\rho_\mu T^\sigma_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}$. If the basis is orthonormal, then

$$\eta = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{pmatrix}$$

The Lorentz transformations lie at the heart of special relativity.

You can prove that $O(n-1,1)$ form a group, check the closure under multiplication, the existence of identity, the existence of inverse.

(c) unitary group $U(n)$. Let V be a complex space, with an inner product. If T is an isometry, then $(v|w) = (Tv|Tw)$, where $(Tv|Tw) = (T^\dagger Tv|w)$ because of V.

Recall the definition of adjoint. First, construct a linear operator A^Ton V^* based the linear operator A on V, then use the map$L$ which is constructed based on the inner product to construct a linear operator $A^\dagger = L^{-1} \circ A^T \circ L$ on V.

A unitary operator is represented by a unitary matrix in an orthonormal basis, this is not necessarily true in non-orthonormal basis. Always be careful for $[T^\dagger] = [T]^\dagger$. Operators and matrices are not equivalent. Always think about the meaning of the symbols!!!

Notice we have defined the orthogonal matrices, unitary matrices and Lorentz transformations in two ways. It is important to dig in the relation between two diffent yet equivalent ways. In one way, we consider the matrices as those matrices which impliment a basis change(on vector spaces with real inner product, Hermitian inner product and Minkowski metrics). In another way, we consider them as representations in an orthonormal basis for operators which preserve a non-degenerate Hermitian form. An active transformation change the vectors from an old one to a new one, while the passive transformations just show the relationship between the different components. Then we can see that the first way is passive transformation, while the second way is active transformation.

8. the special unitary and orthogonal groups $SU(n)$ and $SO(n)$. They are defined as those matrices in $U(n)$ and $O(n)$ that have determinant equal to 1. Checking the closure under multiplication, the existence of identity, the existence of inverse can show that they are subgroup of $U(n)$ and $O(n)$. They are very important in different fields.

   $SO(n)$ is the group of rotations in n dimensions.

   $SU(2$ is crucial in the theory of angular momentum in quantum mechanics.

   $SU(3)$ is fundamental in particle physics, especially in the mathematical description of quarks.

(a) **The special orthogonal group in two dimension:** $SO(2)$, it is an abelian group. A general form of an element of $SO(2)$ is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

It represents a counterclockwise rotation.

(b) **Special orthogonal group in three dimensions:** $SO(3)$.

Define a rotation first: A rotation in n dimensions is any linear operator R which can be obtained continuously from the identity and takes orthonormal bases to orthonormal bases.(there exists a continuous map $\gamma : [0,1] \to GL(n,\mathbb{R})$ such that $\gamma(0) = I$, and $\gamma(1) = R$). This definition is equivalent to $R \in SO(n)$. Although I don't know how to prove.

The rotation has two general form: one is asscociated with Euler angles, the second one is about abitrary axis.

For the Euler angles, any rotation can be achieved by rotating the given axes by an angle ϕ around the original $z-$axis, then by the angle θ around the new $x-$axis, and finally by an angle ψ around the new $z-$axis.
Take the passive point of view,

$$\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

For the rotation by an arbitrary angle θ about an axis $\hat{n}$.

$$\begin{pmatrix} n_x^2(1-\cos\theta)+\cos\theta & n_xn_y(1-\cos\theta)-n_z\sin\theta & n_xn_z(1-\cos\theta)+n_y\sin\theta \\ n_yn_x(1-\cos\theta)+n_z\sin\theta & n_y^2(1-\cos\theta)+\cos\theta & n_yn_z(1-\cos\theta)-n_x\sin\theta \\ n_zn_x(1-\cos\theta)-n_y\sin\theta & n_zn_y(1-\cos\theta)+n_x\sin\theta & n_z^2(1-\cos\theta)+\cos\theta \end{pmatrix}$$

(c) **Orthogonal group in 3 dimensions** ($O(3)$).

The difference between $SO(3)$ and $O(3)$. In $O(3)$, we also have matrices whose determinats are -1.

Rotations whose determinants are -1 are called **improper rotations**.

Rotations whose determinants are 1 are called **proper rotations**.

Any improper rotation can be written as the product of a proper rotation and the inversion transformation. (Be thought as a proper rotation followed by an inversion transformation.)

If rotation R is an improper rotation, $-R$ is a proper rotation.

One can not continuously go from matrices with $|R| = 1$ to the matrices with $|R| = -1$. The two parts are disconnected. One has to multiply the inversion transformation to go between these two parts. So improper rotations are beyond the definition of rotation above.

(d) **For special unitary group in two complex dimensions:** $SU(2)$, all 2×2 complex matrices A which satisfy $|A| = 1$ and $A^{\dagger}A = I$

There is a close relationship between $SU(2)$ and $SO(3)$. two complex dimensions and three real dimensions.

The generic element of $SU(2)$:

$$\begin{pmatrix} \alpha & \beta \\ -\beta^{*} & \alpha^{*} \end{pmatrix} \quad \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1$$

And there is also another generic form of elements in $SU(2)$.

$$\begin{pmatrix} e^{\frac{i(\psi+\phi)}{2}} \cos\frac{\theta}{2} & ie^{\frac{i(\psi-\phi)}{2}} \sin\frac{\theta}{2} \\ ie^{-\frac{i(\psi-\phi)}{2}} \sin\frac{\theta}{2} & e^{-\frac{i(\psi+\phi)}{2}} \cos\frac{\theta}{2} \end{pmatrix} \psi, \phi, \theta \text{ are Euler angles.}$$

(e) **Lorentz group**

i. **The restricted Lorentz group:** $SO(3,1)_o$. It is defined to be the set of all $A \in O(3,1)$ which satisfy $|A| = 1$ and $A_{44} > 1$.

The restricted Lorentz group $SO(3,1)$ is a subgroup of the extendedn Lorentz group $O(3,1)$.

$O(3,1)$ (extended Lorentz group) can be interpreted as the set of all transformations between inertial reference frames.

$SO(3,1)_o$ (restricted Lorentz group) can be interpreted as those changes of reference frame to those which preserve the orientation of time and space, which is the additional conditions $|A| = 1$ and $A_{44} > 1$ do. The condition $A_{44} > 0$ means that A does not reverse the direction of time, together with $|A| > 0$ implies that A does not reverse the orientation of the space axes.

The most familiar such transformation is probably

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta\gamma \\ 0 & 0 & -\beta\gamma & \gamma \end{pmatrix} \quad -1 < \beta < 1, \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \cosh u$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh u & -\sinh u \\ 0 & 0 & -\sinh u & \cosh u \end{pmatrix} \quad u \in \mathbb{R}, \quad \tanh u = \beta$$

$$S\beta = \beta'$$

$$S^{-1}LS = L'$$

$$L = \begin{pmatrix} \frac{\beta_x^2(\gamma-1)}{\beta^2} + 1 & \frac{\beta_x\beta_y(\gamma-1)}{\beta^2} + 1 & \frac{\beta_x\beta_z(\gamma-1)}{\beta^2} + 1 & -\beta_x\gamma \\ \frac{\beta_y\beta_x(\gamma-1)}{\beta^2} + 1 & \frac{\beta_y^2(\gamma-1)}{\beta^2} + 1 & \frac{\beta_y\beta_z(\gamma-1)}{\beta^2} + 1 & -\beta_y\gamma \\ \frac{\beta_z\beta_x(\gamma-1)}{\beta^2} + 1 & \frac{\beta_z\beta_y(\gamma-1)}{\beta^2} + 1 & \frac{\beta_z^2(\gamma-1)}{\beta^2} + 1 & -\beta_z\gamma \\ -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \frac{u_x^2(\cosh u-1)}{u^2} + 1 & \frac{u_xu_y(\cosh u-1)}{u^2} + 1 & \frac{u_xu_z(\cosh u-1)}{u^2} + 1 & -\frac{u_x}{u}\sinh u \\ \frac{u_yu_x(\cosh u-1)}{u^2} + 1 & \frac{u_y^2(\cosh u-1)}{u^2} + 1 & \frac{u_yu_z(\cosh u-1)}{u^2} + 1 & -\frac{u_y}{u}\sinh u \\ \frac{u_zu_x(\cosh u-1)}{u^2} + 1 & \frac{u_zu_y(\cosh u-1)}{u^2} + 1 & \frac{u_z^2(\cosh u-1)}{u^2} + 1 & -\frac{u_z}{u}\sinh u \\ -\frac{u_x}{u}\sinh u & -\frac{u_y}{u}\sinh u & -\frac{u_z}{u}\sinh u & \cosh u \end{pmatrix}$$

$$\vec{\beta} = \frac{\tanh u}{u}\vec{u}, \quad u = |\vec{u}|$$

This is interpreted passively as a coordinate transformation to a new reference frame that is unrotateed relative to the old frame but is moving uniformly along the $z-$axis with relative velocity β. Such a transformation is often referred to as a boost(a veocity moving).

Any elements from $SO(3,1)_o$ can be decomposed as $A = LR'$, where L is a boost along any direction and R' is a spacial rotation.

$$R' = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} R \in SO(3)$$

ii. **The extended (improper) Lorentz group:** $O(3,1)$.

$O(3,1)$ can be interpreted as the set of all transformations between inertial reference frames without any restrictions. That is we have to add two transformation into this group and make it a new group. We add two transformation: parity(spatial inversion), and time reverse

$$P = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

All the matrices with determinants being 1 must be disconnected by those matrices with determinants being -1, also those which reverse the the orientation of the space must be disconnected with those do not, and those which reverse the direction of time must be disconnected with those do not.

$$\begin{matrix} I & P \\ T & PT \end{matrix}$$

(f) **Special linear group in two complex dimensions:** $SL(2, \mathbb{C})$. This set is defined to be set of all 2×2 complex matrices A with $|A| = 1$.

$SL(2, \mathbb{C})$ bears the same relationship to $SO(3,1)_o$, as $SU(2)$ bears the relationship to $SO(3)$.

Any Elements in $SL(2, \mathbb{C})$ can be decomposed as $A = \tilde{L}\tilde{R}$, where $\tilde{L}$ is a boost with a certain form and $\tilde{R} \in SU(2)$,

$$\tilde{L} = \begin{pmatrix} \cosh\frac{u}{2} + \frac{u_z}{u}\sinh\frac{u}{2} & -\frac{1}{u}(u_x - iu_y)\sinh\frac{u}{2} \\ -\frac{1}{u}(u_x + iu_y)\sinh\frac{u}{2} & \cosh\frac{u}{2} - \frac{u_z}{u}\sinh\frac{u}{2} \end{pmatrix}$$

Any elements from $SO(3,1)_o$ can be decomposed as $A = LR'$, where L is a boost and R' is a special rotation.

$$R' = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} R \in SO(3)$$

9. The group with two elements: $\mathbb{Z}_2$. The set $\mathbb{Z}_2 = \{+1, -1\} \subset \mathbb{Z}$ with the usually multiplication. It is a group and even an abelian group

10. The symmetic group on n letters, denoted S_n. It is also known as the permutation group.It is a set of maps, and the 'multiplication' is the compositon of maps. It is defined to be the set of all one-to-one and onto maps of the set $\{1, 2 \dots n\}$ to itself.

    Any permutation σ is specified by the n numbers $\sigma(i)$, and can be notated as:

    $$\begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$$

    A cyclic permutation is given by $1 \to 2, 2 \to 3, 3 \to 1$. And the copmposition of permutaions can be tricky sometimes, it is the same as the composition of maps. But just remember you have to start from the most right one and then go to the left ones. Don't mistake the order.

    If we have a vector space V consider its $n-$fold tensor product $\mathcal{T}_n^0(V)$, then S_n acts on product by $\sigma(v_1 \otimes v_2 \otimes \dots \otimes v_n) = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(n)}$.

    In the case of n identical particals in quantum mechanics, where the total Hilbert space is naively the $n-$fold tensor product $\mathcal{T}_n^0(\mathcal{H})$ of the single-particle Hilbert space $\mathcal{H}$

2 Important concept: Homomorphism and Isomorphism

We claimed that there is a special relationship between $SU(2)$ and $SO(3)$, as well as between $SL(2, \mathbb{C})$ and $SO(3, 1)_o$. Now we are going to dig in the possible relationship between groups.

Homomorphism: Given two group G and H, a Homomorphism from G to H is a map: $\Phi : G \to H$, such that $\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2) \quad \forall g_1, g_2 \in G$.

1. If $\Phi : G \to H$ is a homomorphism, let e be the identity in G and e' the identity in H. Then $\Phi(e) = e'$ and $\Phi(g^{-1}) = \Phi(g)^{-1}, \forall g \in G$. That means homomorphism can preserves the identity and inverse.
2. A homomorphism should be thought of as a map from one group to another which preserves the multiplicative structure.
3. Note thatΦ is not necessarily one-to-one or onto.

Isomorphism If it is onto then Φ is said to be a homomorphism onto H. If it is onto and one-to-one then we say Φ is an isomorphism.

If a Φ is an isomorphism, then we regard G and H as 'the same group', just with different labels for the elements. While two group G and H are isomorphic we write $G \simeq H$.

To quantify how far a homomorphism is from being one-to-one, we define the kernel of Φ to be the set $K \equiv \{g \in G | \Phi(g) = e'\}$, where e' is the identity in H.

1. If Φ is one-to-one, then $K = \{e\}$.
2. If Φ is not one-to-one, the size of K tells us how far it is from being so. Why? Here is the reason.

   If Φ is not one-to-one, there exists $g_1, g_2 \in G$ such that $\Phi(g_1) = \Phi(g_2) = h \in H$.

   Due to the homomorphism, $\Phi(g_1 g_2^{-1}) = \Phi(g_1)\Phi(g_2)^{-1} = hh^{-1} = e$. Thus $g_1 g_2^{-1} = k \in K$.

   For any $g \in G$, $\forall k \in K$, $\Phi(kg) = \Phi(k)\Phi(g) = e'\Phi(g) = \Phi(g)$.

   Define set $Kg \equiv \{kg | k \in K\}$, the this set Kg consists of those elements of G which get sent to $\Phi(g)$. Thus the size of the set K tells us how far Φ is from being one-to-one.

   At the same time, kernel K is a subgroup of group G(K is also a group).

some examples:

1. An example of a group isomorphism:

   (a) let V be the $n-$dimensional vector space over some scalar C, and a basis $\mathcal{B}$.

   (b) We have the corresponding matrix for any $T \in GL(V)$, denoted $GL(n, V)$.

   (c) This is a map: $GL(V) \to GL(n, C)$, $T \mapsto [T]_{\mathcal{B}}$. This map is one-to-one and onto.

   (d) And we know that $[TU] = [T][U]$, so the map is homomorphism.

   (e) Then this map is an isomorphism, $GL(V) \simeq GL(n, C)$.

   (a) If V has a non-degenerate Hermitian form, and V is real and the form is positive-definite(inner product), the map is $\mathrm{Isom}(V) \simeq O(n)$.

   (b) If V has a non-degenerate Hermitian form, and V is complex and the form is positive-definite, the map is $\mathrm{Isom}(V) \simeq U(n)$.

(c) If V has a non-degenerate Hermitian form, and V is real and the form is a Minkoswki metric, the map us $\mathrm{Isom}(V) \simeq O(n-1,1)$.

2. Linear maps as homomorphisms

   A linear map (a linear transformation) from a vector space $V$to a vector space W is map: $\Phi : V \to W$ that satisfies the usual linearity condition. $\Phi(cv_1 + v_2) = c\Phi(v_1) + \Phi(v_2)$

   But if we see the linearity condition from another view: $\Phi(v_1 + v_2) = \Phi(v_1) + \Phi(v_2)$. Then Φ preserves the multiplicative structure. Also, we've already see the vector space as additive group. Now we have a homomorphism between V and W. linear map is necessarily a homomorphism. (But it is not necessarily onto or one-to-one.)

   And if Φ is one-to-one and onto, we have an isomorphism between V and W, and in particular we refer to it as a vector space isomorphism.

   $$\text{A linear map is an isomorphism} \iff \begin{cases} \dim V = \dim W \\ \phi(v) = 0 \implies v = 0 \end{cases}$$

   The kernel of map Φ is a subspace of V, also known as null space of map Φ. The dimension of K is known as the mullity of map Φ.

   The range of map ϕ is a subspace of W. The dimension of range is known as the rank of map Φ

3. the two groups that are connected by homomorphism do not necessarily have the same 'multiplication'

   the map exp:$\mathbb{R} \to \mathbb{R}^* \quad x \mapsto e^x$, where $\mathbb{R}$ is an additive group of real numbers, and $\mathbb{R}^*$ is the multiplicative group of nonzero real numbers. The map is an isomorphism.

4. the map exp: $\mathbb{C} \to \mathbb{C}^*$, this is a homomorphism, but is not an isomorphism.

5. the map det: $GL(n, C) \to C^* \quad A \mapsto detA$. It is homomorphism for both $C = \mathbb{R}$ and $C = \mathbb{C}$

6. $U(1)$ is the group of 1×1 unitary matrices. It is the set of complex numbers z with $|z| = 1$, and that $U(1)$ is isomorphic to $SO(2)$.

7. We present the relationship in terms of a group homomorphism $\rho : SU(2) \to SO(3)$

   First of all, we consider the vector space of all 2×2 traceless anti-Hermitian matrices, denoted as $\mathfrak{su}(2)$. Now I don't know precisely what does this mean.
   Anti-Hermitian matrices: $A^\dagger = -A$. Properties: The eigenvalues of them are purely imaginary or zero.

An arbitrary element $X \in \mathfrak{su}(2)$ can be written as (traceless anti-Hermitian matrices)

$$X = \frac{1}{2}\begin{pmatrix} -iz & -y - ix \\ y - ix & iz \end{pmatrix}$$

We take $S_x = -\frac{i}{2}\sigma_x, S_y, S_z$ as basis vectors. Then under this basis $\mathcal{B} = \{S_x, S_y, S_z\}$:

$$[X]_{\mathcal{B}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Easy to see that $\det sX = \frac{1}{4}(x^2 + y^2 + z^2) = \frac{1}{4}\left\|[X]\right\|^2$.

(a) **ρ_A is a linear operator.**

$A \in SU(2)$ acts on $X \in \mathfrak{su}(2)$ by the map $X \mapsto AXA^\dagger$.

This map is linear, $X_1 + cX_2 \mapsto A(X_1 + cX_2)A^\dagger = AX_1A^\dagger + cAXA^\dagger$.

$\mathrm{Tr}(AXA^\dagger) = 0$ and $(AXA^\dagger)^\dagger = -AXA^\dagger$. So this linear map is a linear operator on $\mathfrak{su}(2)$. We call it

$$\begin{aligned} \rho_A : \mathfrak{su}(2) &\to \mathfrak{su}(2) \\ X &\mapsto AXA^\dagger \end{aligned}$$

$\forall A \in SU(2)$, there exists such linear operator ρ_A.

(b) **the linear matrix corresponding to ρ_A preserve the norm of $[X]_{\mathcal{B}}$.**

Because this map is a linear operator, $\forall \rho_A$, there exists a matrix $\rho(A)$ such that $\rho(A)[X]_{\mathcal{B}} = [AXA^\dagger]_{\mathcal{B}}$. This matrix has the property of preserving the norm of $[X]_{\mathcal{B}}$.

$$||\rho(A)[X]||^2 = ||[AXA^\dagger]||^2 = 4\det(AXA^\dagger) = 4\det X = ||[X]_{\mathcal{B}}||^2$$

(c) **A map from $A \in SU(2)$ to matrix $\rho(A)$.**

$$\begin{aligned} \rho : SU(2) &\to O(3)(SO(3)) \\ A &\mapsto \rho(A) \end{aligned}$$

This map ρ is a homomorphism, $\rho(AB) = \rho(A)\rho(B)$. The kernel of this homomorphism is $K = \{I, -I\}$. This map is onto (cover) and two-to-one(double) from $SU(2)$ to $SO(3)$. So we call $SU(2)$ as the doule cover of $SO(3)$.

This construction of this map relies on $\mathfrak{su}(2)$, but $\mathfrak{su}(2)$ does not appear in its definition.

8. there is a similar relationship between $SL(2,\mathbb{C})$ and $SO(3,1)_o$.

   Instead of $\mathfrak{su}(2)$, we consider $H_2(\mathcal{C})$. This four-dimensional vector space has a basis $\mathcal{B} = \{\sigma_x, \sigma_y, \sigma_z, I\}$.

$$X = \frac{1}{2}\begin{pmatrix} -iz & -y-ix \\ y-ix & iz \end{pmatrix} X \in \mathfrak{su}(2)$$

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} X \in H_2(\mathbb{C})$$

$$\det X = \frac{1}{4}(x^2+y^2+z^2) = \frac{1}{4}||[X]_{\mathcal{B}}||^2 \quad \text{Euclidean metric on } \mathbb{R}^3$$

$$\det X = t^2 - x^2 - y^2 - z^2 = -\eta([X]_{\mathcal{B}}, [X]_{\mathcal{B}}) \quad \text{Minkowski metric on } \mathbb{R}^4$$

   The matrix $\rho(A)$ preserves the norm of $[X]_{\mathcal{B}}$ in the Minkowski metric on $\mathbb{R}^4$.

$$\rho : SL(2,\mathbb{C}) \to O(3,1)(SO(3,1)_o)$$
$$A \mapsto \rho(A)$$

   This map ρ is again a homomorphism. It is onto and two-to-one. The kernel is also $K = \{I, -I\}$

9. **$\mathbb{Z}_2 = \{+1, -1\} \subset \mathbb{Z}$ and set $\{I, P\} \subset O(3,1)$.**

   Set $\{I, P\} \subset O(3,1)$ and set $\{+1, -1\}$ are both abelian two-element group with $P^2 = I$ and $(-1)^2 = 1$.

$$\Phi : \{I, P\} \to \mathbb{Z}_2$$
$$\Phi(I) = 1$$
$$\Phi(P) = -1$$

   We can also consider set $\{I, T\}$. Since $T^2 = I$, we actually have $\mathbb{Z}_2 \simeq \{I, p\} \simeq \{I, T\}$.

   All two-element groups are isomorphic. There are many groups in physics that are isomorhpic to $\mathbb{Z}_2$

10. **S_n, $\mathbb{Z}_2$ and the sgn**

    S_n: the set of all one-to-one and onto maps of the set $\{1, 2, \cdots n\}$ to itself.

    Define a **transposition** in S_n to be any permutation σ which switches two numbers and leaves all the others alone.

    Any permutations can be written as the product of transpositions (non-uniquely and have the same oddness or evennes of the number of transpositions

Define a map

$$\begin{aligned}
&\mathrm{sgn}: S_n \to \mathbb{Z}_2 \\
&\mathrm{sgn}(\sigma) = \begin{cases} +1 & \sigma \text{ consists of an even number of transpositions} \\ -1 & \sigma \text{ consists of an odd number of transpositions} \end{cases}
\end{aligned} \tag{1}$$

This map is a homomorphism, since $\mathrm{sgn}(\sigma_1\sigma_2) = \mathrm{sgn}(\sigma_1)\mathrm{sgn}(\sigma_2)$.

S_n are used in many places in physics, so the map sgn can really help us.

(a) wedge product of r daul vectors

$$f_1 \wedge \cdots \wedge f_r = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) f_{\sigma(1)} \otimes \cdots f_{\sigma(r)}$$

(b) ϵ tensor, the nonzero components of $\epsilon \equiv e^1 \wedge \cdots \wedge e^r \in \Lambda^n \mathbb{R}^n$ are now $\epsilon_{\sigma(1),\sigma(2),\cdots,\sigma(n)} = \mathrm{sgn}(\sigma)$.

(c) the definition of the determinant

$$\begin{aligned}
|A| &= \sum_{i_1 \ldots i_n} \epsilon_{i_1 \ldots i_n} A_{i_1 1} \cdots A_{i_n n} \\
&= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}
\end{aligned}$$

(d) symmetrization postulate: for' Systems which ontain identical particles, particles of the same mass, charge and spin'. If the particles are all bosons, the states of this kind of system will only be $S^n(\mathcal{H})$, if the particles are all fermions, the states of this kind of system will only be in $\Lambda^n \mathcal{H}$.

Symmetrization postulate: any state of an $n-$particle system is either invariant under a permutation ofthe particles or change sign depending on whether the permutation is even or odd.

To conclude this postulate, we need to see the action of σ on an element of $S^n(\mathcal{H})$ and $\Lambda^n \mathcal{H}$.

i. For elements in $\mathcal{T}_n(V)$, $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in \mathcal{T}_n^0(V)$.

$$\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

ii. For $T = T^{i_1 \cdots i_n} e_{i_1} \otimes \cdots \otimes \cdots e_{i_n} \in S^n(V)$.

$$\begin{aligned}
\sigma(T) &= T^{i_1 \cdots i_n} e_{\sigma(i_1)} \otimes \cdots \otimes \cdots e_{\sigma(i_n)} \quad \text{constant is not affected} \\
&= T^{\sigma^{-1}(j_1) \cdots \sigma^{-1}(j_n)} e_{j_1} \otimes \cdots \otimes e_{j_n} \\
&= T^{j_1 \cdots j_n} e_{j_1} \otimes \cdots \otimes e_{j_n} \\
&\quad \text{the components of a symmetric tensor } T^{\sigma^{-1}(j_1) \cdots \sigma^{-1}(j_n)} = T^{j_1 \cdots j_n} e_{j_1} \\
&= T
\end{aligned}$$

iii. For $T = T^{i_1 \cdots i_n} e_{i_1} \otimes \cdots \otimes \cdots e_{i_n} \in \Lambda^n V$.

$$\begin{aligned}
\sigma(T) &= T^{i_1 \cdots i_n} e_{\sigma(i_1)} \otimes \cdots \otimes \cdots e_{\sigma(i_n)} \quad \text{constant is not affected} \\
&= T^{\sigma^{-1}(j_1) \cdots \sigma^{-1}(j_n)} e_{j_1} \otimes \cdots \otimes e_{j_n} \\
&= \mathrm{sgn}(\sigma) T^{j_1 \cdots j_n} e_{j_1} \otimes \cdots \otimes e_{j_n} \\
&\quad \text{the components of an antisymmetric tensor } T^{\sigma^{-1}(j_1) \cdots \sigma^{-1}(j_n)} = \mathrm{sgn}(\sigma) T^{j_1 \cdots j_n} e_{j_1} \\
&= \mathrm{sgn}(\sigma) T
\end{aligned}$$