notes about group theory 3

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Application of the tensor product in quantum mechanics: there are some issues that are quite counterintuitive. We often encounter Hilbert spaces like $L^2([-a,a])$ and $L^2(\mathbb{R})$. These bases are non-denumerably infinite. And some of the basis vector even do not belong to the Hilbert space.

For position operator $\hat{x}$ acts on functions $\phi(x) \in L^2(\mathbb{R})$ by $\hat{x}\phi(x) = x\phi(x)$, its eigenvector $\delta(x-x_0)$ are not in the Hilbert space. In the basis $\{\delta(x-x_0)\}_{x_0\in\mathbb{R}}$, $\phi(x) = \int_{-\infty}^{\infty} \mathrm{d}x'\phi(x')\delta(x-x')$, which in Dirac notation is $|\phi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x'\phi(x')|x'\rangle$.

Because $\rangle x|x'\rangle = \delta(x-x')$.

We have: $\phi(x) = \langle x|\phi\rangle$. The components can be interpreted either as epxansion coefficients, or the value of a given dual vector on the vector.

Momentum representation: we also expand the square-integrable functions using the odd basis $\{e^{ipx}\}_{p\in\mathbb{R}}$.

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p\phi(p)e^{ipx}$$

When we talk about representation, we are considering the components, $\psi(x)$ will not change, x is a symbol in it, but not position representation.

The transformation between two representations, we know that

$$\langle x_0|p\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\delta(x-x_0)e^{ipx} = e^{ipx_0}$$

We can construct a new Hilbert space out of two Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$, we can also construct linear operation on $\mathcal{H}_1 \otimes \mathcal{H}_2$ out of linear operators on $\mathcal{H}_1$ and $\mathcal{H}_2$. Given linear operator A_i on $\mathcal{H}_i$ $i = 1,2$, define a linear operator $A_1 \otimes A_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by

$$(A_1 \otimes A_2)(v \otimes w) = (A_1 v) \otimes (A_2 w)$$

Under this definition, we have $(A \otimes B)(C \otimes D) = AC \otimes BD$ In physics, we usually encounter the situation where one of A_1, A_2 is identity. They are uaually abbreviated, don't confuse yourself.

We can construct a new inner product out of two Hilbert space:

$$((v_1 \otimes v_2)|(w_1 \otimes w_2) = (v_1, w_1)_1 \cdot (v_2, w_2)_2$$

Vector operators are defined to be sets of operators that transform sa three-dimensional vectors under the adjoint action of the total angular momentum operators J_i. That is, a vector operator is a set of operators $\{B_i\}_{i=1,2,3}$ that satisfies:

$$ad_{J_i}(B_j) = [J_i, B_j] = i\sum_{k=1}^{3} \epsilon_{ijk} B_k$$

Adding degrees of freedom is implemented by taking tensor products of the corresponding Hilbert spaces. The total Hilbert space for the system is $L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2s+1}$, we can take a basis $\{|\mathbf{r}\rangle \otimes |m\rangle\}$, where $\mathbf{r} \in \mathbb{R}^3$ and $-s \leq m \leq s$

So an arbitrary ket $|\psi\rangle$ has expansion:

$$|\psi\rangle = \sum_{m=-s}^{s} \int \mathrm{d}^3 r \psi_m(\mathbf{r}) |\mathbf{r}, m\rangle$$

1 Symmetric tensors and antisymmetric tensors

A symmetric $(r, 0)$ tensor is an $(r, 0)$ tensor whose value is unaffected by the interchange of any two of its arguments. All the symmetric $(r, 0)$ tensor form a vector space, denoted $S^r(V^*)$.

A symmetric $(0, r)$ tensor is an $(0, r)$ tensor whose value is unaffected by the interchange of any two of its arguments. All the symmetric $(0, r)$ tensor form a vector space, denoted $S^r(V)$

An antisymmetric (alternating) $(r, 0)$ tensor is one whose values changes sign after transposition of any two of its arguments.

An antisymmetric (alternating) $(0, r)$ tensor is one whose values changes sign after transposition of any two of its arguments.

Attention, we are talking about tensor $(r, 0)$ and $(0, r)$. Concerning tensor $(1, 1)$, we can not say it is a symmetric tensor. We have to use a metric if possible to lower or raise the indices.

We can see the denotion is again confusing.

1. The symmetric $(r, 0)$ tensors are denoted $S^r(V^*)$
2. The symmetric $(0, r)$ tensors are denoted $S^r(V)$.
3. The antisymmetric $(r, 0)$ tensors are denoted $\Lambda^r V^*$
4. The antisymmetric $(0, r)$ tensors are denoted $\Lambda^r V$.

These four symbols are totally different from the original tensor symbol $T^r_s(V)$. They are designed specifically for these two special tensors. Just remember them, and don't get confused. One way to think about them logically is to let $r = 1$. $S^1(V^*) = V^*$ and $S^1(V) = V$. And we define $\Lambda^1 V^* = V^*$ and $\Lambda^1 V = V$.

some examples of symmetric tensors:

1. For space $S^2(\mathbb{R}^{2*})$, consider the set $\{e^1 \otimes e^1, e^2 \otimes e^2, e^1 \otimes e^2 + e^2 \otimes e^1\} \subset S^2(\mathbb{R}^2)^*$. You can check that the set is linearly independent and can be used to expand the vector in space $S^2(\mathbb{R}^{2*})$. Since the Euclidean metric g on $\mathbb{R}^2$ is in the space, we check its representation under this basis. Actually, under this basis, $g = e^1 \otimes e^1 + e^2 \otimes e^2$And the symbol g can appear in many situations. Be sure to recognize it.

2. Other examples can be: the Euclidean metric on $\mathbb{R}^3$, the Minkowski metric on $\mathbb{R}^4$, the momeent of inertia tensor, the Maxwell stress tensor

3. Multipole moments and harmonic polynomials The taylor expansion of the scalar potential have Q_l, which are the symmetric rank l multipole moment tensors. Each symmetric tensor Q_l can be interpreted as a degree l polynomial f_l, by evaluating on l copies of $\mathbf{r} = (x^1, x^2, x^3)$

   $$f_l(\mathbf{r}) \equiv Q_l(\mathbf{r} \ldots \mathbf{r}) = Q_{i_1 \ldots i_r} x^{i_1} \cdots x^{i_r}$$

   There should be sum of every i_k through 1 to 3

   The multipole moments are just fixed polynomials, which is in turn correspond to symmetric tensors.

   The tracelessness of the Q_l, the fact that $\sum_k Q_{i_1 \ldots k \ldots k \ldots i_n} = 0$. Prove of the tracelessness: the scalar potential must obeys the Laplace equation, so every term $\frac{f_l(\mathbf{r})}{r^{2l+1}}$. Then easy to see that f_l must be harmonic polynomials, then Q_l must be traceless. But why, why Q_l have to be traceless when f_l are harmonic polynomials?

4. The polynomial associated to the Euclidean metric tensor $g = \sum_{i=1}^{3} e^i \otimes e^j$

   $$g(\mathbf{r}, \mathbf{r}) = e^1(\mathbf{r})e^1(\mathbf{r}) + \ldots = (x^1)^2 + \ldots$$

5. the symmetric tensor in $s^3(\mathbb{R}^3)$ associated to the polynomial $x^2 y$

   $$Q(\mathbf{r}, \mathbf{r}, \mathbf{r}) = (x^1)^2 x^2 = e^1 \otimes e^1 \otimes e^2$$

Antisymmetric tensors An antisymmetric tensor is also called alternating tensor. An antisymmetric $(r, 0)$ tensor is one whose value changes sign under transposition of any two of its arguments,

$$T(v_1 \ldots v_i \ldots v_j \ldots v_r) = -T(v_1 \ldots v_j \ldots v_i \ldots v_r)$$

Also antisymmetric $(0, r)$ tensors are defined similarly, both sets form vector spaces, denoted $\Lambda^r V^*$ and $\Lambda^r V$. For $r = 1$ we define $\Lambda^1 V^* = V^*$ and $\Lambda^1 V = V$

Properties:

1. $T(v_1 \ldots v_r) = 0$ if $v_i = v_j$ for any $i \neq j$

2. From the above, we get $T(v_1 \ldots v_r) = 0$ if $\{v_1 \ldots v_r\}$ is linearly dependent.

3. Also, we get that:

   if $\dim V = n \implies$ the only tensor in $\Lambda^r V^* \text{and} \Lambda^r V \quad \text{for} \quad r > n \quad$ is the 0 tensor

An important operation on antisymmetric tensors is the wedge product: Given $f, g \in V^*$ we define the wedge product of f and g, denoted $f \wedge g$, to be the antisymmetric $(2,0)$ tensor defined by $f \wedge g = f \otimes g - g \otimes f$

1. It is easy to check that $f \wedge g = -g \wedge f$
2. and also $f \wedge f = 0$
3. Also, $\{e^i \wedge e^j\}_{i<j}$ is a basis of $\Lambda^2 V^*$. Note that $e^i \wedge e^j$ and $e^j \wedge e^i$ are not linearly independent.

   $\{e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_r}\}_{i_1 < i_2 < \cdots i_r}$ is a basis for $\Lambda^r V^*$.
4. r-fold wedge, sum of all the tensor products of the form $f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_r}$ where each term gets a + or - sign depending on whether an odd or an even number of transpositions of the factors are necessary to obtain it from $f_1 \otimes f_2 \otimes \cdots \otimes f_r$.

$$f_1 \wedge f_2 = f_1 \otimes f_2 - f_2 \otimes f_1$$

$$\begin{aligned} f_1 \wedge f_2 \wedge f_3 = & f_1 \otimes f_2 \otimes f_3 + f_2 \otimes f_3 \otimes f_1 + f_3 \otimes f_1 \otimes f_2 \\ & - f_3 \otimes f_2 \otimes f_1 - f_2 \otimes f_1 \otimes f_3 - f_1 \otimes f_3 \otimes f_2 \end{aligned}$$

5. $\{v_1 \ldots v_r\}$ is a linearly dependent set, then $T(v_1 \ldots v_r) = 0$
6. $\{f_1 \ldots f_r\} \subset V^*$ is linear dependent, then $f_1 \wedge \ldots \wedge f_r = 0$.
7. If $\dim V = n$, then any set of more than n vectors must be linearly dependent, then $\Lambda^r V = \Lambda^r V^* = 0$ for $r > n$

Some examples:

1. Systems which contain identical particles, particles of the same mass, charge and spin. For bosons only states in $S^n(\mathcal{H})$ are observed, while for fermions only states in $\Lambda^n \mathcal{H}$. All the particles are either fermions or bosons. This restriction of the total Hilbert space to either $S^n(\mathcal{H})$ or $\Lambda^n \mathcal{H}$ is known as the symmetrization postulate.

   If we have two fermions, we cannot measure the same values for a complete set of quantum numbers for both particles, since then the state would have to include a term of the form $|\psi\rangle|\psi\rangle$ and thus could not belong to $\Lambda^2 \mathcal{H}$

   (a) Consider two identical spin 1/2 fermions fixed in space, so that $\mathcal{H}_{tot} = \Lambda^2 \mathbb{C}^2$. The 2 of $\mathbb{C}^2$ means in one ket there are two numbers, and the 2 of the Λ^2 means the basis is in the form of $f_1 \wedge f_2$. Combining the two, we know that $\mathcal{H}_{tot} = \Lambda^2 \mathbb{C}^2$ is one-dimentsional with the basis vector being $|0,0\rangle = |\frac{1}{2}\rangle|-\frac{1}{2}\rangle - |-\frac{1}{2}\rangle|\frac{1}{2}\rangle$.

(b) But for two distinguishable spin 1/2 fermions, the total space is $\mathcal{H}_{tot} = \mathbb{C}^2 \otimes \mathbb{C}^2$, we have a set of basis which has four basis vector: $|0,0\rangle = |\frac{1}{2}\rangle|-\frac{1}{2}\rangle - |-\frac{1}{2}\rangle|\frac{1}{2}\rangle$, $|1,1\rangle = |\frac{1}{2}\rangle|\frac{1}{2}\rangle$, $|1,0\rangle = |\frac{1}{2}\rangle|-\frac{1}{2}\rangle + |-\frac{1}{2}\rangle|\frac{1}{2}\rangle$, $|1,-1\rangle = |-\frac{1}{2}\rangle|-\frac{1}{2}\rangle$

2. More complete Levi-Civita tensor.

   Consider $\mathbb{R}^n$ with the standard inner product, $\{e_i\}$ is an orthonormal basis for it, consider the tensor $\epsilon \equiv e^1 \wedge \ldots \wedge e^n \in \Lambda^n \mathbb{R}^{n*}$ vector space $\Lambda^n \mathbb{R}^{n*}$ is a one-dimensional space.

$$\epsilon_{i_1 \ldots i_n} = \begin{cases} 0 & if\{i_1 \ldots i_n\} \text{ contains a repeated index} \\ -1 & if\{i_1 \ldots i_n\} \text{is an odd rearrangement or a cyclic permutation of} \{1 \ldots n\} \\ +1 & if\{i_1 \ldots i_n\} \text{is an even rearrangement or an anti-cyclic permutation of} \{1 \ldots n\} \end{cases}$$

   The $\Lambda^n \mathbb{R}^n *$ is one-dimensional, and that ϵ is the basis for it.

   Always remember you have to have inner product.

3. **Determinant**

   (a) Determinant can be the form of cofactor expansion.

   (b) Take an $n \times n$ matrix A and consider its n columns as n column vectors in $\mathbb{R}^n$, construct the ϵ tensor using the standard basis and inner product on $\mathbb{R}^n$, $|A| \equiv \epsilon(A_1 \ldots A_n)$, in components $|A| = \sum_{i_1 \ldots i_n} \epsilon_{i_1 \ldots i_n} A_{i_1 1} \cdots A_{i_n n}$.

   If we view determinant as antisymmetric tensor, lots of its properties can have a simple explanation.

   (a) sign change under interchange of columns

   (b) invariant under addition of rows

   (c) factoring of scalars.

   (d) interpretation as the oriented volume of the skew n-cube obtained by applying A to the standard n-cube.

   More about the oriented volume: for any non-degenerated matrix A, it can send the standard orthonormal basis $\{e_1, \cdots, e_n\}$to a new basis $\{Ae_1, \cdots, Ae_n\}$.

   The skew cube spanned by the new basis: its volume is

$$\begin{aligned} \epsilon(Ae_1, \cdots, Ae_n) &= \epsilon(\sum_{i_1} A_{i_1\,1} e_{i_1}, \cdots, \sum_{i_n} A_{i_n\,n} e_{i_n}) \\ &= \sum_{i_1, i_2, \cdots, i_n} A_{i_1\,1} \cdots A_{i_n\,n} \epsilon(e_1, \cdots, e_n) \\ &= \sum_{i_1, i_2, \cdots, i_n} A_{i_1\,1} \cdots A_{i_n\,n} \epsilon_{i_1, \cdots, i_n} \\ &= \det A \end{aligned}$$

4. **Orientations and ϵ tensor**

   Tensor ϵ is defined based on an orthonormal basis for $\mathbb{R}^n$. So different orthonormal basis can lead to different ϵ tensor.

$$
\begin{aligned}
\epsilon' &= e^{1'} \wedge \cdots e^{n'} \\
&= A^{1'}_{i_1} \cdots A^{n'}_{i_n} e^{i_1} \wedge \cdots e^{i_n} \\
&= \sum_{i_1, i_2, \cdots, i_n} A^{1'}_{i_1} \cdots A^{n'}_{i_n} \epsilon_{i_1 \cdots i_n} e^1 \wedge \cdots e^n \\
&= |A| \epsilon
\end{aligned}
$$

   Any two bases related by any transformation with $|A| > 0$ are said to have the same orientation.

   Any two bases related by any transformation with $|A| < 0$ are said to have the opposite orientation.

   Define an orientation as a maximal set of bases all having the same orientation.

   $\mathbb{R}^n$ has exactly two orientations.

   Tensor ϵ does not depend on a particular choice of orthonormal basis, but it does depend on a metric and a choice of orientation, where the orientation chosen is the one determined by the standard basis.

2 pseudovector or axial vector in $\mathbb{R}^3$

A pseudovector or axial vector is a tensor on $\mathbb{R}^3$ (type unclear) whose components transform like vector under rotation but do not change sign under inversion. (including angular velocity vector ω, the magnetic field vector $\mathbf{B}$, all the cross product of two vectors.)I saw an interesting example: imagine a rotating wheel, now it is going forward, if you see it in a mirror, it would also be going forward. So a pseudovector will be attached with a sign after mirror.

Bivectors: some of the pseudovectors who also happen to be elements of $\Lambda^2\mathbb{R}^3$ (space $\Lambda^2\mathbb{R}^3$ has the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$. Bivectors are necessarily antisymmetric $(0, 2)$ tensor)

If $\alpha \in \Lambda^2\mathbb{R}^3$, then we expand it as $\alpha = \alpha^{23} e_2 \wedge e_3 + \alpha^{31} e_3 \wedge e_1 + \alpha^{12} e_1 \wedge e_2$

$$
\begin{aligned}
\alpha &= \alpha^{23} e_2 \wedge e_3 + \alpha^{31} e_3 \wedge e_1 + \alpha^{12} e_1 \wedge e_2 \\
&= \alpha^{23}(e_2 \otimes e_3 - e_3 \otimes e_2) + \alpha^{31}(e_3 \otimes e_1 - e_1 \otimes e_3) + \alpha^{12}(e_1 \otimes e_2 - e_2 \otimes e_1)
\end{aligned}
$$

$$
\begin{cases}
\alpha(e^1, e^1) = \alpha(e^2, e^2) = \alpha(e^3, e^3) = 0 \\
\alpha(e^1, e^2) = -\alpha(e^2, e^1) = \alpha^{12} \\
\alpha(e^1, e^3) = -\alpha(e^3, e^1) = -\alpha^{31} \\
\alpha(e^2, e^3) = -\alpha(e^3, e^2) = \alpha^{23}
\end{cases}
$$

We define a map J in components as:

$$
\begin{aligned}
J :& \Lambda^2\mathbb{R}^3 \to \mathbb{R}^3 \\
& \alpha^{ij} \mapsto (J(\alpha))^i \equiv \frac{1}{2}\epsilon^i_{jk}\alpha^{jk} \equiv \frac{1}{2}\epsilon_{pjk}\delta^{ip}\alpha^{jk}
\end{aligned}
$$

So that $J(r \wedge p) = r \times p$

$$r \wedge p = (r^1p^2 - r^2p^1)e_1 \wedge e_2 + (r^3p^1 - r^1p^3)e_3 \wedge e_1 + (r^2p^3 - r^3p^2)e_2 \wedge e_3$$

Now we have a lot of concepts: pseudovectors, cross product, bivector. First, cross product is a binary operation, while the other are actually all special tensors.

Now let us see, why the bivectors transform like vectors under rotation, but without a sign change under inversion.

Under two orthonormal bases, bivector is a $(0, 2)$ tensor, the matrix transformation law is

$$[\alpha]_{\mathcal{B}'} = A[\alpha]_{\mathcal{B}}A^T$$

$$J([\alpha]_{\mathcal{B}'}) = J(A[\alpha]_{\mathcal{B}}A^T)$$

While the transformation law for the associated vector $J(\alpha)$ is

$$[J(\alpha)]_{\mathcal{B}'} = A[J(\alpha)]_{\mathcal{B}}$$

map J commutes with arbitrary rotation, so that bivectors and vectors behave the same under rotation.

Proof: map J commutes with any rotation.

$$A[J(\alpha)]_{\mathcal{B}} = J(A[\alpha]_{\mathcal{B}}A^T)$$

which in components is:

$$A^{i'}_j\alpha^j = \frac{1}{2}\epsilon^{i'}_{k'l'}A^{k'}_m A^{l'}_n\alpha^{mn}$$

On the left side sum j, while on the right side sum mn first and then sum $k'l'$

We now compute

$$
\begin{aligned}
\frac{1}{2}\epsilon^{i'}_{k'l'}A^{k'}_m A^{l'}_n\alpha^{mn} &= \frac{1}{2}\epsilon_{p'k'l'}\delta^{i'p'}A^{k'}_m A^{l'}_n\alpha^{mn} \\
&= \frac{1}{2}\sum_q \epsilon_{p'k'l'}A^{i'}_q A^{p'}_q A^{k'}_m A^{l'}_n\alpha^{mn} \text{because } A \text{is an orthogonal matrix, its componens replace} \delta \\
&= \frac{1}{2}\sum_q \epsilon_{qmn}|A|A^{i'}_q\alpha^{mn} \quad \epsilon_{p'k'l'}A^{p'}_q A^{k'}_m A^{l'}_m = |A|\epsilon_{qmn} \\
&= \frac{1}{2}|A|\epsilon^q_{mn}A^{i'}_q\alpha^{mn} \text{resume the Einstein Summation, one in subscript and one in supscript} \\
&= |A|A^{i'}_q\alpha^q \text{the definition of map } J \\
&= A^{i'}_q\alpha^q \quad \text{or} \quad -A^{i'}_q\alpha^q
\end{aligned}
$$

When A is a rotation, $A(J(\alpha)) = J(A(\alpha))$, when A is an inversion $-I$, $J(-\alpha) = -1 \cdot (-I) \cdot J(\alpha)$.

Also, you can see from the equation $J([\alpha]_{\mathcal{B}'}) = J(A[\alpha]_{\mathcal{B}} A^T)$, that the components of a bivector won't change sign under inversion.

3 pseudovectors that are not cross product

Let K and K' be two orthonormal bases for $\mathbb{R}^3$, where K is time-dependent. We refer to K as body frame and K'as space frame. Let A be the time-dependent matrix of the basis transformation taking K' to K, so that $[\mathbf{r}]_{K'} = A[\mathbf{r}]_K$.

From the space frame, we calculate the velocity:

$$\begin{aligned} [\mathbf{v}]_{K'} &= \frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{r}]_{K'} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}A[\mathbf{r}]_K \quad [\mathbf{r}]_K \text{is a constant vector} \\ &= \frac{\mathrm{d}A}{\mathrm{d}t}(A^{-1}[\mathbf{r}]_{K'}) \end{aligned}$$

Now we need to explore the property of $\frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}$.

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(I) = \frac{\mathrm{d}}{\mathrm{d}t}(AA^T) = \frac{\mathrm{d}A}{\mathrm{d}t}A^T + A\frac{\mathrm{d}A^T}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}A^T + \left(\frac{\mathrm{d}A}{\mathrm{d}t}A^T\right)^T$$

We see that $\frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}$ is actually an antisymmetric matrix

We then define an angular velocity bivector

$$[\tilde{\omega}]_{K'} = \frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}$$

Then the angular vector is

$$\omega = J(\tilde{\omega})$$

Any time-dependent rotation matrix A can be associated with an antisymmetric matrix $\frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}$

$$\frac{\mathrm{d}A}{\mathrm{d}t}A^{-1} = \begin{pmatrix} 0 & -\omega^{3'} & \omega^{2'} \\ \omega^{3'} & 0 & -\omega^{1'} \\ -\omega^{2'} & \omega^{1'} & 0 \end{pmatrix}$$

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}[r]_{K'} &= \begin{pmatrix} 0 & -\omega^{3'} & \omega^{2'} \\ \omega^{3'} & 0 & -\omega^{1'} \\ -\omega^{2'} & \omega^{1'} & 0 \end{pmatrix} \begin{pmatrix} x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} \\ &= \begin{pmatrix} \omega^{2'}\omega^{3'} - \omega^{3'}\omega^{2'} \\ \omega^{3'}\omega^{1'} - \omega^{1'}\omega^{3'} \\ \omega^{1'}\omega^{2'} - \omega^{2'}\omega^{1'} \end{pmatrix} \\ &= [\omega \times r]_{K'} \quad K' \text{ is the space frame} \end{aligned}$$

In the body frame

$$[\tilde{\omega}]_K = A^{-1}\frac{\mathrm{d}A}{\mathrm{d}t}$$

Instead in space frame

$$[\tilde{\omega}]_{K'} = \frac{\mathrm{d}A}{\mathrm{d}t}A^{-1}$$

some problems:

1. Explore the properties of orthogonal matrices. Again, this is the set of real invertible matrices A satisfying $A^T = A^{-1}$, denoted $O(n)$.

   $O(n)$ is not a subspace of $M_n(\mathbb{R})$, there is no zero, and it is not closed under addition.

   O(n) is a group, the composition is still orthogonal, the inverse is still orthogonal, the identity is an orthogonal matrix.

   The orthogonal matrices A with $|A| = 1$, the rotations, form a subgroup, denoted $SO(n)$. To check the subgroup, composition, inverse, identity, associativity. However, the matrices with $|A| = -1$ do not form a subgroup, there is no identity.

2. compute the dimension of the space of $(0, r)$ antisymmetric tensors $\Lambda^r V$