Notes about group theory 6

2019/3/6

1 The Lie algebra of classical and quantum physics

1.1 $\mathfrak{o}(2) = \mathfrak{so}(2)$

$\mathfrak{so}(2)$ is one-dimensional. Take as a basis

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then we have

$$e^{\theta X} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

So X does generate the rotations.

1.2 $\mathfrak{so}(3)$

$\mathfrak{so}(3) = \mathfrak{o}(3)$ is the set of all the antisymmetric matricess.

Take as a basis

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

And we have $[L_i, L_j] = \sum_{k=1}^{3} \epsilon_{ijk} L_k$.

For any rotation in $\mathfrak{so}(3)$,

$$X = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} = xL_x + yL_y + zL_z$$

$$Xv = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = (x, y, z) \times (v_x, v_y, v_z) = [X]_{\{L_x, L_y, L_z\}} \times v$$

So we have $X[X]_{\{L_x,L_y,L_z\}} = 0$. Then we have $e^{tX}[X]_{L_x,L_y,L_z} = [X]_{L_x,L_y,L_z}$.

Also, using the basis $\mathcal{B} = \{L_x, L_y, L_z\}$ for $\mathfrak{so}(3)$, we have

$$[[X,Y]]_{\mathcal{B}} = [X]_{\mathcal{B}} \times [Y]_{\mathcal{B}}$$

Which means, in components, the commutator is given by the usual cross product on $\mathbb{R}^3$.

1.3 $\mathfrak{su}(2)$

$\mathfrak{su}(2)$ is the set of all the traceless anti-Hermitian matrices. Take as a basis

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} S_y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S_z = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

And we have $[S_i, s_j] = \sum_{k=1}^{3} \epsilon_{ijk} S_k$.

1.4 $\mathfrak{so}(3,1)$

Take $\{\tilde{L}_i, K_i\}$as a basis

$$\tilde{L}_i \equiv \begin{pmatrix} L_i & \vec{0} \\ \vec{0} & 0 \end{pmatrix}$$

And

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} K_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We have $[\tilde{L}_i, \tilde{L}_j] = \sum_{k=1}^{3} \epsilon_{ijk} \tilde{L}_k$, $[\tilde{L}_i, K_j] = \sum_{k=1}^{3} \epsilon_{ijk} K_k$, $[K_i, K_j] = -\sum_{k=1}^{3} \epsilon_{ijk} \tilde{L}_k$. ($L_i$ are the spatial rotation, and K_i are the boosts along their corresponding axes).

Before, we have

$$L = \begin{pmatrix} \frac{\beta_x^2(\gamma-1)}{\beta^2}+1 & \frac{\beta_x\beta_y(\gamma-1)}{\beta^2}+1 & \frac{\beta_x\beta_z(\gamma-1)}{\beta^2}+1 & -\beta_x\gamma \\ \frac{\beta_y\beta_x(\gamma-1)}{\beta^2}+1 & \frac{\beta_y^2(\gamma-1)}{\beta^2}+1 & \frac{\beta_y\beta_z(\gamma-1)}{\beta^2}+1 & -\beta_y\gamma \\ \frac{\beta_z\beta_x(\gamma-1)}{\beta^2}+1 & \frac{\beta_z\beta_y(\gamma-1)}{\beta^2}+1 & \frac{\beta_z^2(\gamma-1)}{\beta^2}+1 & -\beta_z\gamma \\ -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \frac{u_x^2(\cosh u-1)}{u^2}+1 & \frac{u_xu_y(\cosh u-1)}{u^2}+1 & \frac{u_xu_z(\cosh u-1)}{u^2}+1 & -\frac{u_x}{u}\sinh u \\ \frac{u_yu_x(\cosh u-1)}{u^2}+1 & \frac{u_y^2(\cosh u-1)}{u^2}+1 & \frac{u_yu_z(\cosh u-1)}{u^2}+1 & -\frac{u_y}{u}\sinh u \\ \frac{u_zu_x(\cosh u-1)}{u^2}+1 & \frac{u_zu_y(\cosh u-1)}{u^2}+1 & \frac{u_z^2(\cosh u-1)}{u^2}+1 & -\frac{u_z}{u}\sinh u \\ -\frac{u_x}{u}\sinh u & -\frac{u_y}{u}\sinh u & -\frac{u_z}{u}\sinh u & \cosh u \end{pmatrix}$$

Actually, explicitly summing the exponential power series yields that

$$e^{u^i K_i} = \begin{pmatrix} \frac{u_x^2(\cosh u-1)}{u^2}+1 & \frac{u_x u_y(\cosh u-1)}{u^2}+1 & \frac{u_x u_z(\cosh u-1)}{u^2}+1 & -\frac{u_x}{u}\sinh u \\ \frac{u_y u_x(\cosh u-1)}{u^2}+1 & \frac{u_y^2(\cosh u-1)}{u^2}+1 & \frac{u_y u_z(\cosh u-1)}{u^2}+1 & -\frac{u_y}{u}\sinh u \\ \frac{u_z u_x(\cosh u-1)}{u^2}+1 & \frac{u_z u_y(\cosh u-1)}{u^2}+1 & \frac{u_z^2(\cosh u-1)}{u^2}+1 & -\frac{u_z}{u}\sinh u \\ -\frac{u_x}{u}\sinh u & -\frac{u_y}{u}\sinh u & -\frac{u_z}{u}\sinh u & \cosh u \end{pmatrix}$$

1.5 $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$

$SL(2,\mathbb{C})$ is the set of all complex matrices with unit determinant.

$\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$ is the set of all traceless complex matrices, which is viewed as a real vector space here.

Take $\{S_i, \tilde{K}_i\}$ as a basis for real vector space $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$.

$$S_1 \equiv \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} S_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ o & i \end{pmatrix} \tilde{K}_i = iS_i$$

We also have that $[S_i, S_j] = \sum_{k=1}^{3} \epsilon_{ijk} S_k$, $[S_i, \tilde{K}_j] = \sum_{k=1}^{3} \epsilon_{ijk} \tilde{K}_k$, $[\tilde{K}_i, \tilde{K}_j] = -\sum_{k=1}^{3} \epsilon_{ijk} S_k$

$$e^{u^i \tilde{K}_i} = \begin{pmatrix} \cosh\frac{u}{2} + \frac{u_z}{u}\sinh\frac{u}{2} & -\frac{1}{u}(u_x - iu_y)\sinh\frac{u}{2} \\ -\frac{1}{u}(u_x + iu_y)\sinh\frac{u}{2} & \cosh\frac{u}{2} - \frac{u_z}{u}\sinh\frac{u}{2} \end{pmatrix}$$

$\mathfrak{sl}(2,\mathbb{C})$ denote the set of all traceless complex matrices, viewed as a complex vector space.

Take $\{S_i\}$ as a basis is enough.

2 Lie algebra, different definition

A real abstract Lie algebra is defined to be a real vector space $\mathfrak{g}$ equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ called the Lie bracket which satisfies Antisymmetry and Jacobi identity.

A real vector space equipped with a special bilinear map.

1. $[X, Y] = -[Y, X] \quad \forall X, Y \in \mathfrak{g}$ Antisymmetry
2. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \quad \forall X, Y, Z \in \mathfrak{g},$ Jacobi identiy

The bracket can be the commucator but it can also be something else.

Some examples:

1. $\mathfrak{gl}(V)$: the Lie algebra of linear operators on a vector space

   Let V be a vector space(possibly infinite-dimensional). And $\mathcal{L}(V)$ is the set of all linear operators on V.

   Define the bracket to be $[T, U] = TU - UT, \forall T, U \in \mathcal{L}(V)$.

Now consider $\mathcal{L}(V)$ to be equipped with the bracket (bilinear map), we can see this map satisfies the two requests. So $\mathcal{L}(V)$ equipped with the bracket is a Lie algebra. We denote it by $\mathfrak{gl}(V)$.

Set $\mathcal{L}(V)$ and set $\mathfrak{gl}(V)$ are the same set, but we consider the latter having an additional structure.

2. $\mathfrak{isom}(V)$: the Lie algebra of anti-Hermitian operators

   Now let V be an inner product space. The inner product (non-degenerate Hermitian form) allows us to define an operator's adjoint.

   We then define $\mathfrak{isom}(V) \subset \mathfrak{gl}(V)$ to be the set of all anti-Hermitian operators. Why do we define it as the set of all anti-Hermitian operators. There must be a reason!!!

   Then $\mathfrak{isom}(V)$ is a Lie algebra. (The set of all anti-Hermitian operators, is a real vector space, and equipped with commucator as the Lie bracket).

3. The Possion bracket on phase space. Consider a physical system with a $2n-$dimensional phase space P parameterised by n generalized coordinates q_i and the n conjugate momenta p_i.

   The set of all complex-valued, infinitely differentiable functions on P is a real vector space which we will denote by $\mathcal{C}(P)$. Then we can turn it into a Lie algebra just like what we did to $\mathcal{L}(V)$, this time we use Possion bracket as our Lie bracket. It is easy to check the antisymmetry and Jacobi identity.

   Though $\mathcal{C}(P)$ is in general infinite-dimensional, it often has interesting finite dimensional Lie subalgbras. If $P = \mathbb{R}^6$, which can be seen as the case in cartesian coordinates for $\mathbb{R}^3$.

   Consider the three components of the angular momentum,

   $$J_1 = q_2p_3 - q_3p_2 \quad J_2 = q_3p_1 - q_1p_3 \quad J_3 = q_1p_2 - q_2p_1$$

   They are viewed as functions of p_i and q_i. We check that the functions satisfy:

   $$J_i, J_j = \sum_{k=1}^{3} \epsilon_{ijk} J_k$$

   This is a familiar angular momentum commutation relations in $\mathfrak{so}(3)$. This implies that $\mathfrak{so}(3)$ is a Lie subalgebra of $\mathcal{C}(\mathbb{R}^6)$.

   But what do the functions J_i have to do with generators of rotations? The answer has to do with a general relationship between symmetries and conserved quantities.

   There is something about classical mechanics. I am not familiar with these stuff, I can't understand the whole text

4. If we have a one-dimensional system with position coordinate q and conjugate momentum p, then $p = \mathbb{R}^2$ and $\mathcal{C}(\mathbb{R}^2)$ contains Heisenberg algebra. What operations will p and q associate to?

   One of the most basic representations of p and q as operators is on the vector space $L^2(\mathbb{R})$, where

   $$\hat{q}f(x) = xf(x) \quad \hat{p}f(x) = -\frac{\mathrm{d}f}{\mathrm{d}x}$$

   If we exponentiate $\hat{p}$, then

   $$e^{t\hat{p}}f(x) = f(x-t)$$

   So $\hat{p}$ generates the translations along the $x-$axis. If we work in the momentum representation, we know that $i\hat{q} = -\frac{\mathrm{d}}{\mathrm{d}p}$, thus

   $$e^{it\hat{q}}\Phi(p) = \Phi(p-t)$$

   So $\hat{q}$ is actually a translation in momentum space.

   For this simple example, the canonical transformation is just

   $$Q \equiv \frac{p+iq}{\sqrt{2}} \quad P \equiv \frac{p-iq}{\sqrt{2}i}$$

   From one span to another, span$\{q, p, 1\}$ to span$\{Q, P, 1\}$. $\{q, p\} = 1$, and still $\{Q, P\} = 1$

3 Homomorphism and Isomorphism in Lie algebra

We have experienced the group homomorphism between $SU(2)$ and $SO(3)$, as well as $SL(2,\mathbb{C})$ and $SO(3,1)_o$. It is time to check the relationship between $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$, as well as $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ and $\mathfrak{so}(3,1)$.

Show how the relationships between Lie algebras arise as a consequence of the relationships between the corresponding groups.

A Lie algebra homomorphism from a Lie algebra $\mathfrak{g}$ to a Lie algebra $\mathfrak{h}$ to be a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ that preserves the Lie bracket, in the sense that$[\phi(X), \phi(Y)] = \phi([X,Y]) \quad \forall X, Y \in \mathfrak{g}$. In ordinary homomorphism, we have 'multiplication' as the binary operation, here we have $[,]$ as the binary operation.

If ϕ is a vector space isomorphism, then it is called Lie algebra isomorphism. (If ϕ is a vector space isomorphism, $\mathfrak{g}$ and $\mathfrak{h}$ should have the same dimension). We also write $\mathfrak{g} \simeq \mathfrak{h}$.

One way to prove that two Lie algebras are isomorphic is to use structute constants. We take bases for both of the Lie algebras. The commutation relations take the form

$$[X_i, X_j] = \sum_{k=1}^{n} {c_{ij}}^k X_k$$

$$[Y_i, Y_j] = \sum_{k=1}^{n} {d_{ij}}^k Y_k$$

the numbers ${c_{ij}}^k$ and ${d_{ij}}^k$ are called structure constants. They depend on the choice of the bases. So this can be tricky.

If one can exhibit bases such that ${c_{ij}}^k = {d_{ij}}^k \quad \forall i, j, k$, then it is easy to check that the Lie algebras are isomorphic.

Some examples:

1. $\mathfrak{gl}(V) \simeq \mathfrak{gl}(n, C)$.

   $\mathfrak{gl}(V)$ is the set of all the linear operators. $\mathfrak{gl}(n, \mathbb{C})$ is the set of matrix.

   Let V be an $n-$dimensional vector space over a set of scalars C. If we choose a basis for it, we can establish a map from the operators to the matrix representations. It is easy to see that $\mathfrak{gl}(V) \simeq \mathfrak{gl}(n, C)$.

2. $\mathfrak{isom}(V) \simeq \mathfrak{o}(n)$ and $\mathfrak{isom}(V) \simeq \mathfrak{u}(n)$

   If V is an inner product space, we can restrict this isomorphism to $\mathfrak{isom}(V) \subset \mathfrak{gl}(V)$ to get $\mathfrak{isom}(V) \simeq \mathfrak{o}(n)$, when $C = \mathbb{R}$ and $\mathfrak{isom}(V) \simeq \mathfrak{u}(n)$, when $C = \mathbb{C}$.

3. Let $\mathfrak{g}$ be a Lie algebra. We use the bracket to turn $X \in \mathfrak{g}$ into a linear operator ad_X on $\mathfrak{g}$, by defining $\mathrm{ad}_X(Y) = [X, Y], \quad \forall Y \in \mathfrak{g}$.

   Then we have a map

   $$\begin{aligned} \mathrm{ad} : &\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) && (1) \\ &X \mapsto \mathrm{ad}_X && (2) \end{aligned}$$

   where $\mathfrak{gl}(\mathfrak{g})$, is just the set of linear operators which are on $\mathfrak{g}$. By checking the defintion, $[\phi(X), \phi(Y)] = \phi([X, Y])$, in this case, that is $[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]}$. If you put the defintion of ad_X into it.

   $$\begin{aligned} [\mathrm{ad}_X, \mathrm{ad}_Y](Z) &= (\mathrm{ad}_X \cdot \mathrm{ad}_Y - \mathrm{ad}_Y \cdot \mathrm{ad}_X)(Z) \\ &= \mathrm{ad}_X([Y, Z]) - \mathrm{ad}_Y([X, Z]) \\ &= [X, [Y, Z]] - [Y, [X, Z]] \qquad (3) \end{aligned}$$

   And on the other side,

   $$\mathrm{ad}_{[X,Y]}(Z) = [[X, Y], Z].$$

   So this is actually Jacobi identity for Lie algebra $\mathfrak{g}$.

   Also, note that the map is $\to \mathfrak{gl}(\mathfrak{g})$. Why do you think so?

4 Important Concept: Continuous homomorphism to Lie algebra homomorphism

Let $\Phi : G \to H$ be a continuous homomorphism from a matrix Lie group G to a matrix Lie group H. Then this induces a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$ given by

$$\phi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\Big|_{t=0}$$

proof: Let $\Phi : G \to H$ be such a homomorphism, and let e^{tX} be a one-parameter subgroup in G.(X is a certain matrix)(Actually every elements in $\mathfrak{g}$will form a one-parameter subgroup of G. It is one of the definition of Lie algebra). Then $\{\Phi(e^{tX})\}$ is a one-parameter subgroup in H(need to be verified). So by the definition of Lie algebra, one of the generator:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\Big|_{t=0} \equiv Z$$

This is actually one generator in $\mathfrak{g}$ corresponding to another generator in $\mathfrak{h}$. Thus we define a map ϕ from $\mathfrak{g}$ to $\mathfrak{h}$, and it follows that

$$\phi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\Big|_{t=0} \implies \Phi(e^{tX}) = e^{t\phi(X)}$$

So now we need to check: one, it is a linear map, two, it preserves the Lie bracket. That is $\phi(sX) = s\phi(X)$, $\phi(X+Y) = \phi(X)+\phi(Y)$ and $[\phi(X), \phi(Y)] = \phi([X, Y])$.

1.

$$\begin{aligned}
\phi(sX) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tsX})\Big|_{t=0} \\
&= \frac{\mathrm{d}(st)}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}(st)}\Phi(e^{tsX})\Big|_{t=ts=0} \\
&= s\,\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\Big|_{t=0} \\
&= s\phi(X)
\end{aligned}$$

2.

$$\begin{aligned}
\phi(X+Y) &= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{t(X+Y)})\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\lim_{m\to\infty}(e^{\frac{tX}{m}}e^{\frac{tY}{m}})^m)\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\lim_{m\to\infty}\Phi((e^{\frac{tX}{m}}e^{\frac{tY}{m}})^m)\Big|_{t=0} \quad \text{continuous homomorphism} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\lim_{m\to\infty}(\Phi(e^{\frac{tX}{m}})\Phi(e^{\frac{tY}{m}}))^m\Big|_{t=0} \quad \Phi(e^{tX}) = e^{t\phi(X)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\lim_{m\to\infty}(e^{\frac{t\phi(X)}{m}}e^{\frac{t\phi(Y)}{m}})^m\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}e^{t(\phi(X)+\phi(Y))}\Big|_{t=0}
\end{aligned}$$

3.

$$\begin{aligned}
[\phi(X),\phi(Y)] &= \frac{\mathrm{d}}{\mathrm{d}t}e^{t\phi(X)}\phi(Y)e^{-t\phi(X)}, \quad \frac{\mathrm{d}}{\mathrm{d}t}e^{tY}Xe^{-tY}\Big|_{t=0} = [Y,X] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\phi(Y)\Phi(e^{-tX}), \quad \Phi(e^{tX}) = e^{t\phi(X)} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})(\frac{\mathrm{d}}{\mathrm{d}s}e^{s\phi(Y)})\Phi(e^{-tX}) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX})\Phi(e^{sY})\Phi(e^{-tX}) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{tX}e^{sY}e^{-tX}) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(e^{e^{tY}sYe^{-tX}}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\Phi(e^{se^{tX}Ye^{-tX}}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\phi(e^{tX}Ye^{-tX}), \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t)) = \phi(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)) \\
\text{Don't know this} &= \phi(\frac{\mathrm{d}}{\mathrm{d}t}e^{tX}Ye^{-tX}) \\
&= \phi([X,Y])
\end{aligned}$$

some examples:

1. first, we had a homomorphism $\rho : SU(2) \to SO(3)$, defined by

$$[AXA^{\dagger}]_{\mathcal{B}} = \rho(A)[X]_{\mathcal{B}}, \text{where} X \in \mathfrak{su}(2), A \in SU(2), \rho(A) \in SO(3)$$

The induced Lie algebra homomorphism ϕ:

$$\phi(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\rho(e^{tY})\Big|_{t=0}$$

$$\implies \rho(e^{tY}) = e^{t\phi(Y)}$$

Take S_x as an example

$$\rho(e^{tS_x}) = e^{t\phi(S_x)}$$

Also

$$\rho(e^{tS_x})[X]_{\mathcal{B}} = [e^{tS_x} X e^{tS_x}{}^{\dagger}]_{\mathcal{B}} = [e^{tS_x} X e^{-tS_x}]_{\mathcal{B}}$$

Then,

$$e^{t\phi(S_x)}[X]_{\mathcal{B}} = [e^{tS_x} X e^{-tS_x}]_{\mathcal{B}}$$

I still don't know how to deduce that $\rho(S_i) = L_i$

$$S_x \equiv \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad S_y \equiv \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad S_z \equiv \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
\phi(S_x) &= \frac{\mathrm{d}}{\mathrm{d}t}\rho(e^{tS_x})\Big|_{t=0} \\
\phi(S_x)[X]_{\mathcal{B}} &= \frac{\mathrm{d}}{\mathrm{d}t}\rho(e^{tS_x})\Big|_{t=0}[X]_{\mathcal{B}} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\rho(e^{tS_x})[X]_{\mathcal{B}}\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}[e^{tS_x} X e^{tS_x}{}^{\dagger}]_{\mathcal{B}}\Big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}[e^{tS_x} X e^{-tS_x}]_{\mathcal{B}}\Big|_{t=0} \\
&= [S_x, X]_{\mathcal{B}}
\end{aligned}$$

Then you can calculate the componeents of $\phi(S_x)$.

Now we have a homomorphism $\rho : SU(2) \to SO(3)$. This induces a Lie algebra homomorphism from $\mathfrak{su}(2) \to \mathfrak{so}(3)$, which is

$$\phi(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\rho(e^{tY})\Big|_{t=0}$$

Put the basis $\{S_i\}$ into it, we get $\phi(S_i) = L_i$. Because the basis have the same commutation relations, it also means the structure constants are the same. So this is a Lie algebra isomorphism. $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$.

2. Another Lie algebra isomorphism: $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \simeq \mathfrak{so}(3,1)$, which arises from the homomorphism $\rho : SL(2,\mathbb{C}) \to SO(3,1)_o$. ρ is defined just as the above $[AXA^{\dagger}]_{\mathcal{B}} = \rho(A)[X]_{\mathcal{B}}$, where $A \in SL(2,\mathbb{C})_{\mathbb{R}}, X \in H_2(\mathbb{C}), \mathcal{B} = \{\sigma_i, I\}$. The induced ϕ is still:

$$\phi(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\rho(e^{tY})\bigg|_{t=0}.$$

We will get $\phi(S_i) = \tilde{L_i}$ and $\phi(\tilde{K_i}) = K_i$. So we get another Lie algebra isomorphism.

3. Consider a matrix Lie group G and its Lie algebra $\mathfrak{g}$. We know that for any $A \in G$ and $X \in \mathfrak{g}$, $AXA^{-1} \in \mathfrak{g}(e^{AXA^{-1}})$. To see whether AXA^{-1} belongs to $\mathfrak{g}$, we need to check whether $e^{AXA^{-1}=Ae^XA^{-1}}$ belongs to $\mathfrak{g}$

   So for every $A \in G$, we can construct an corresponding operator Ad_A on $\mathfrak{g}$ by $\mathrm{Ad}_A(X) = AXA^{-1}, X \in \mathfrak{g}$

   (a) We then have a map

   $$\begin{aligned} \mathrm{Ad} : G &\to GL(\mathfrak{g}) \\ A &\mapsto \mathrm{Ad}_A \end{aligned}$$

   (b) which can be verified as a homomorphism from G to $GL(\mathfrak{g})$(It is easy to check that $\mathrm{Ad}_A\mathrm{Ad}_B = \mathrm{Ad}_{AB}$).

   (c) Then this homomorphism induces a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where ϕ should be:

   $$\phi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{e^{tX}}\bigg|_{t=0}$$

   This is an operator on $\mathfrak{g}$, to figure out what it is, you have to make it act on sth.

   $$\begin{aligned} (\phi(X))(Y) &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}_{e^{tX}}(Y))\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(e^{tX}Ye^{-tY})\bigg|_{t=0} \\ &= [X,Y] \end{aligned}$$

(a) This induced Lie algebra homomorphism is actually what we have encountered:

$$\begin{aligned} \mathrm{ad}: \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ X &\mapsto \mathrm{ad}_X \end{aligned}$$

the map constructed by defining $\mathrm{ad}_X(Y) = [X, Y]$, $X, Y \in \mathfrak{g}$.
'ad' is the infinitesimal version of 'Ad', the Lie bracket is the infenitesimal version of the similarity transformation.

(b) 'Ad' is a homomorphism, so it takes e^{tX} to $\mathrm{Ad}_{e^{tX}}$. Since e^{tX} is a one-parameter subgroup of G, $\mathrm{Ad}_{e^{tX}}$ will be a one-parameter subgroup of $GL(\mathfrak{g})$

(c)
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}_{e^{tX}}(Y))\bigg|_{t=0} = [X, Y] \implies \mathrm{Ad}_{e^{tX}} = e^{t\mathrm{ad}_X} \implies$$
$$\mathrm{Ad}_{e^{tX}} Y = e^{t\mathrm{ad}_X} Y \implies$$
$$e^{tX} Y e^{-tX} = Y + t[X, Y] + \frac{t^2}{2}[X, [X, Y]] + \frac{t^3}{3!}[X, [X, [X, Y]]] + \cdots$$

(d) Let $X, H \in M_n(C)$, then we have $[X, H] \iff e^{tX} H e^{-tX} = H, \forall t \in \mathbb{R}$.
The invariance properties of the Hamiltonian can be formulated in terms of conmmutators with the corresponding generators.

(e) The map $\mathrm{Ad}: G \to GL(\mathfrak{g})$ can be more specific, once we know that the operators Ad_X usually preserve a metric on real vector space $\mathfrak{g}$. So we specify it as $\mathrm{Ad}: G \to \mathrm{Isom}(\mathfrak{g})$.

One important fact to take away is that the correspondence between matrix Lie group and Lie algebras is not one-to-one. Two different matrix Lie groups might have the same Lie algebra.

5 Problem

1. $SO(n)$ is the set of all linear operators

$$\begin{cases} \text{which can take orthonormal bases into orthonormal bases} \\ \text{can be obtained continuously from the identity} \end{cases}$$

(a)
$$\begin{cases} \text{which can take orthonormal bases into orthonormal bases} \\ \text{can be obtained continuously from the identity} \end{cases} \implies R \in O(n)$$

Then how to prove $\det R = 1$?

(b) If $R \in SO(n)$, then R must take orthonormal bases into orthonormal bases.

How to prove that R is continuously obtainable from the identity. Using induction.

i. the claim is trivially true for $SO(1)$

ii. suppose that the claim is true for $n-1$. Take $R \in SO(n)$ and show that it can be continuously connected (via orthogonal similarity transformation) to a matrix of the form

$$\begin{pmatrix} 1 & \\ & R' \end{pmatrix} \quad R' \in SO(n-1)$$

2. Euler's theorem that any $R \in SO(3)$ has an eigenvector with eigenvalue 1.

(This means that alll vectors v proportional to this eigenvector are invariant under R, $Rv = v$, so this rotation fixes a line , which is obviously the axis of rotation.)

$$\begin{aligned} \det(R-I) &= \det(R-I)^T \\ &= \det(R^T - I) = \det(R^{-1} - R^{-1}R) = \det(R^{-1})\det(I-R) \\ &= -\det(R-I) \end{aligned}$$

3. The matrix

$$\begin{pmatrix} n_x^2(1-\cos\theta)+\cos\theta & n_xn_y(1-\cos\theta)-n_z\sin\theta & n_xn_z(1-\cos\theta)+n_y\sin\theta \\ n_yn_x(1-\cos\theta)+n_z\sin\theta & n_y^2(1-\cos\theta)+\cos\theta & n_yn_z(1-\cos\theta)-n_x\sin\theta \\ n_zn_x(1-\cos\theta)-n_y\sin\theta & n_zn_y(1-\cos\theta)+n_x\sin\theta & n_z^2(1-\cos\theta)+\cos\theta \end{pmatrix}$$

is just the components form (in the standard basis) of the linear operator.

$$R(\hat{n},\theta) = L(\hat{n}) \otimes \hat{n} + \cos\theta(I - L(\hat{n}) \otimes \hat{n}) + \sin\theta \hat{n}\times$$

$L(\hat{n}) \otimes \hat{n}$ is just the projection onto the axis of rotation.

$\cos\theta(I - L(\hat{n}) \otimes \hat{n}) + \sin\theta\hat{n}\times$ give a counterclockwise rotation by θ in the plane perpendicualr to $\hat{n}$.

4. Prove that $SO(3,1)_o$ is a subgroup of $O(3,1)$. The definition of $SO(3,1)_o$ consists of three conditions $|A| = 1$, $A_{44} > 1$, $[A]^T[\eta][A] = [\eta]$.

(a) show that $I \in SO(3,1)_o$

(b) show that :if $A \in SO(3,1)_o$, then $A^{-1} \in SO(3,1)_o$

i. $|A^{-1}| = 1$

ii. $[A^{-1}]^T[\eta][A^{-1}] = [\eta]$, and then $[A^T]^T[\eta][A^T] = [\eta]$

iii. write out the 44 components from the $[A]^T[\eta][A] = [\eta]$ for A and A^{-1}.

$$\begin{pmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$a_0^2 = 1 + \vec{a}^2, \quad a_0 = A_{44}, \vec{a} = \begin{pmatrix} A_{14} \\ A_{24} \\ A_{34} \end{pmatrix}$$

So we get two equations from A and A^{-1}.

$$a_0^2 = 1 + \vec{a}^2 \quad b_0^2 = 1 + \vec{b}^2$$

Also using the equation $AA^{-1} = I$, which can leads to $a_0 b_0 = 1 - \vec{a} \cdot \vec{b}$.
So we have three equations about a_0, b_0 totally,

$$\begin{cases} a_0^2 = 1 + \vec{a}^2 \\ b_0^2 = 1 + \vec{b}^2 \\ a_0 b_0 = 1 - \vec{a} \cdot \vec{b} \end{cases}$$

So we can finally show that $b_0 > 1$

(c) show that: if $A, B \in SO(3,1)_o$, then $AB \in SO(3,1)_o$

5. Prove that any $A \in SO(3,1)_o$ can be written as a product of a rotation and a boost.

6. find an explicit formula for the map: $\rho : \mathrm{SU}(2) \to O(3)$ and prove that ρ maps $SU(2)$ onto $SO(3)$ and has kernel $\pm I$.

The genetic element of $SU(2)$ looks like

$$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1$$

An arbitraty $X \in \mathfrak{su}(2)$

$$X = \frac{1}{2} \begin{pmatrix} -iz & -y - ix \\ y - ix & iz \end{pmatrix} \quad , x, y, z \in \mathbb{R}$$

So $[AXA^\dagger]_\mathcal{B}$ can be calculated. We can then try compute $\rho(A)$ from $\rho(A)[X]_\mathcal{B} = [AXA^\dagger]_\mathcal{B}$.

Two ways to parametrize $\alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1$.

$$\alpha = e^{i\frac{(\psi+\phi)}{2}} \cos\frac{\theta}{2} \quad \beta = ie^{\frac{\psi-\phi}{2}} \sin\frac{\theta}{2}$$

$$\alpha = \cos\frac{\theta}{2} - in_z \sin\frac{\theta}{2} \quad \beta = (-in_x - n_y) \sin\frac{\theta}{2}$$

7. Let G be a matrix Lie group. Its Lie algebra $\mathfrak{g}$ comes equipped with a symmetric $(2,0)$ tensor known as its Killing Form, denoted K and defined by $K(X,Y) = -\mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$.