Ex. $f:[-\pi,\pi)\to\mathbb{R}$ such that

(a) f is bounded

(b) f is piecewise cont.

(c) f has only finitely many min & max

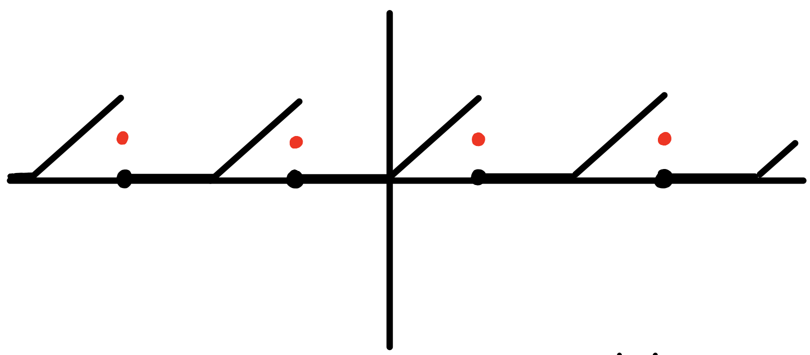
then f has bounded variation.

If in addition, $f:[-\pi,\pi)\to\mathbb{R}$ is integrable,

then $f(x)\sim\frac{1}{2}a_0+\sum_{k=1}^{\infty}[a_k\cos(kx)+b_k\sin(kx)]$

$$=\frac{1}{2}[\tilde{f}(x^+)+\tilde{f}(x^-)].$$

Ex. $f(x)=\begin{cases}0 & -\pi\le x<0\\ x & 0\le x<\pi\end{cases}$



• : pts where F.S. disagrees w/ $\tilde{f}$

$$f(x)=\frac{\pi}{4}-\frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{\cos(2k-1)x}{(2k-1)^2}+\frac{(-1)^{k+1}\sin(kx)}{k}\right)$$

for $x\in(-\pi,\pi)$

and at $x=-\pi,\pi$, the value of Fourier series is $\frac{1}{2}(f(x^+)+f(x^-))=\frac{1}{2}(0+\pi)=\frac{\pi}{2}$

Then: $\frac{\pi}{2}=\frac{\pi}{4}-\frac{2}{\pi}\sum_{k=1}^{\infty}\left(\frac{-1}{(2k-1)^2}+0\right)$

$$\to\frac{\pi^2}{8}=\frac{\pi}{2}\cdot\frac{\pi}{4}+\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

note: $\sum_{k=1}^{\infty}\frac{1}{k^2}=\sum_{k=1}^{\infty}\frac{1}{(2k)^2}+\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$

$$=\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{\pi^2}{8}$$

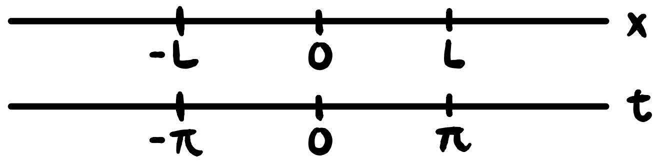
$$\rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{8} \cdot \frac{4}{3} = \frac{\pi^2}{6}$$

Fourier series on arbitrary intervals:

Suppose $f: \mathbb{R} \to \mathbb{R}$ is $2L$-periodic

i.e. $f(x+2L) = f(x)$ for all $x \in \mathbb{R}$.

or $f: [-L, L) \to \mathbb{R}$



$$x = \frac{L}{\pi} \cdot t$$

Then $g(t) = f(\underbrace{\tfrac{L}{\pi} \cdot t}_{=x})$ is 2π-periodic.

Fourier coefficients of $g(t) = f(x)$:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(kt)\, dt$$
$$= \frac{1}{L} \int_{-L}^{L} g\left(\frac{L}{\pi} \cdot x\right) \cos\left(k \frac{\pi}{L} \cdot x\right) dx$$
$$= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi}{L} x\right) dx$$

Similarly, $b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi}{L} x\right) dx$

$$f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{L} x\right) + b_k \sin\left(\frac{k\pi}{L} x\right) \right)$$

Ex. Find the Fourier series of

$$f(x) = \begin{cases} 0 & x \in [-2, 0) \\ 1 & x \in [0, 2) \end{cases}$$

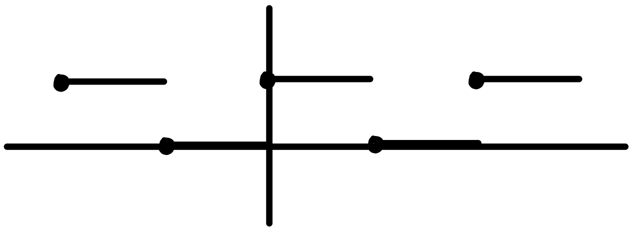
Recall: $h(t) = \begin{cases} 0 & t \in [-\pi, 0) \\ \pi & t \in [0, \pi) \end{cases}$

$$h(t) \sim \frac{\pi}{2} + 2\sum_{k=1}^{\infty} \frac{\sin(2k-1)t}{2k-1}$$

and $f(x) = \frac{1}{\pi} h(\frac{\pi}{2}x)$

Thus $f(x) \sim \frac{1}{\pi}(\text{Fourier series of } h)(\frac{\pi}{2}\cdot x)$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k-1)\frac{\pi}{2}x}{2k-1}$$

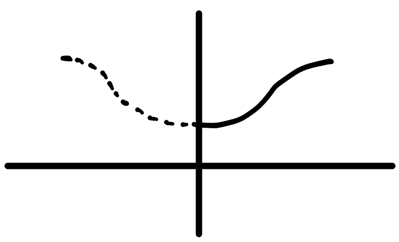


Fourier series of $f = \begin{cases} \tilde{f}(x) & x \neq -4, -2, 0, \dots \\ \frac{1}{2} & x \text{ even} \end{cases}$

Fourier sin/cosine series

Consider $f : [0, \pi] \to \mathbb{R}$. Then

$$\tilde{f}_{even} = \begin{cases} f(x) & x \in [0, \pi] \\ -f(x) & x \in [-\pi, 0) \end{cases}$$



is the extension of f to an even function on $[-\pi, \pi]$

Thus $\tilde{f}_{even}(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)$

where $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}_{even}(x)\cos(kx)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(-x)\cos(kx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\,dx$$

$-x = t,\ -dx = dt:$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(t)\cos(k(-t))(-1)\,dt + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\,dx$$

In particular, if f has bounded variation,

then $f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos(kx)$

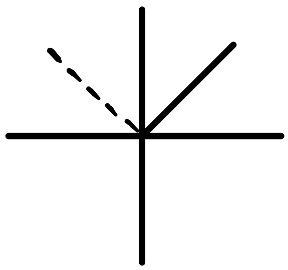
$$= \tfrac{1}{2}(f(x^+) + f(x^-)) \quad \text{for } x \in [0, \pi]$$

i.e. we have an expansion of $f : [0,\pi] \to \mathbb{R}$ into cosine terms;

this is the Fourier cosine series of f.

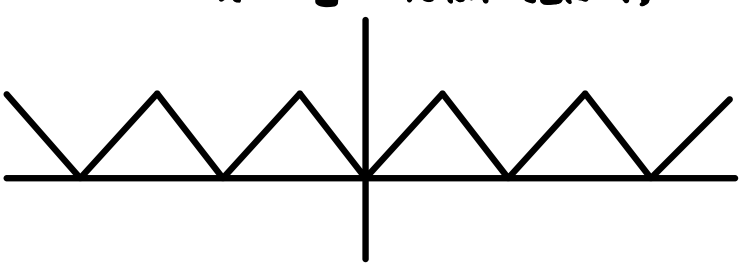
Ex. $f : [0,\pi] \to \mathbb{R}$, $f(x) = x$

$\tilde{f}_{even}(x) = |x|$



$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi f(x)\cos(kx)\,dx \\ &= \frac{2}{\pi}\int_0^\pi x\cos(kx)\,dx \\ &= \frac{2}{\pi}\left[\frac{1}{k}x\sin(kx)\Big|_0^\pi - \frac{1}{k}\int_0^\pi \sin(kx)\,dx\right] \\ &= 0 + \frac{2}{k^2\pi}\cos(kx)\Big|_0^\pi \\ &= \frac{2}{k^2\pi}((-1)^k - 1) \\ &= \begin{cases} 0 & k \text{ even} \\ \frac{-4}{k^2\pi} & k \text{ odd} \end{cases} \end{aligned}$$

$\longrightarrow x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}$ for $x \in [0,\pi]$



extension of $|x|$ to $\mathbb{R}$ which is continuous.

Thus we get convergence on $[0,\pi]$, in particular at $0, \pi$.