Metric Space

A metric space (X, d) is a set X with a function $d: X \times X \to \mathbb{R}^{\geq 0}$ satisfying

(1) $\forall\, x, y \in X \; [d(x,y) = 0 \iff x = y]$ (positivity)

(2) $\forall\, x, y \in X \; [d(x,y) = d(y,x)]$ (symmetry)

(3) $\forall\, x, y, z \in X \quad d(x,z) \leq d(x,y) + d(y,z)$ (Δ inequality)

Ex. $\mathbb{R}$ $\qquad d(x,y) = |x-y|$

$\mathbb{R}^2$ $\qquad d((x_1,x_2),(y_1,y_2))$

$$= \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

$\mathbb{R}^n$ $\qquad d(\bar{x},\bar{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

X any set $\qquad d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

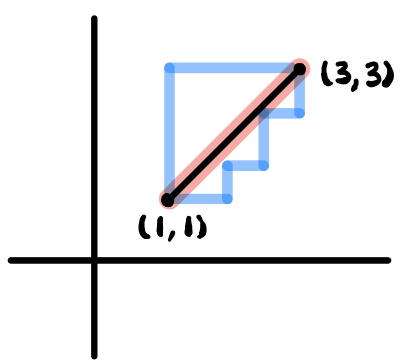
discrete metric

$\mathbb{R}^n$ taxicab metric

$\vec{x} = (x_1, x_2, \ldots x_n)$

$\vec{y} = (y_1, y_2, \ldots y_n)$

$d(\vec{x},\vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$



In Euclidean metric

$$d((1,1),(3,3)) = \sqrt{8}$$

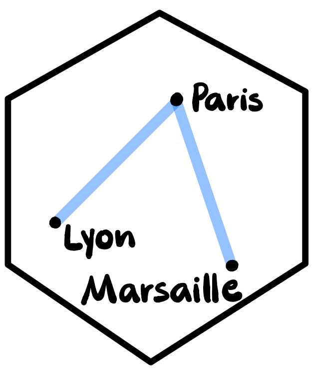
In taxicab metric:

$$d((1,1),(3,3)) = |3-1| + |3-1| = 4$$

Paris metric

Suppose $(a,b) \in \mathbb{R}^2$. Define a metric on $\mathbb{R}^2$ by:

$$d((x_1,x_2),(y_1,y_2)) = d_{Eucl}((x_1,x_2),(a,b)) + d_{Eucl}((a,b),(y_1,y_2))$$



Induced Metric

If (X,d) is a metric space and $Y \subseteq X$ is a subset, we may view Y as a metric space with the "induced metric", i.e. $d|_{Y^2}$.

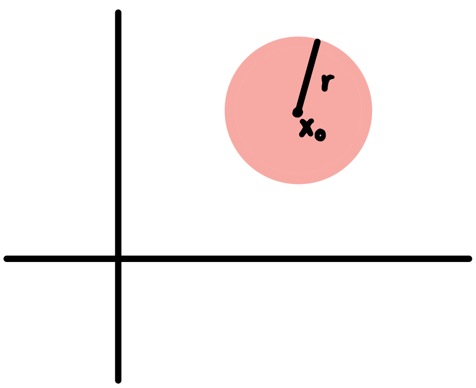
Convergence

Suppose (X, d) is a metric space and $(X_n)_{n\in\mathbb{N}}$ is a sequence of points in X.
Then we say $(X_n)_{n\in\mathbb{N}}$ converges to $x\in X$, or write $\lim_{n\to\infty} x_n = x$, if
if $\forall \varepsilon>0 \ \exists N \ \forall n\geq N \ d(x_n, x) < \varepsilon$.
(equivalent to $\lim_{n\to\infty} d(x_n, x) = 0$)

Balls

Suppose (X, d) is a metric space, $x_0 \in X$, and $r\in\mathbb{R}^{>0}$. We define the ball of points of radius r about x_0 to be the set $B_{(X,d)}(x_0, r) = \{x\in X \mid d(x-x_0) < r\}$.

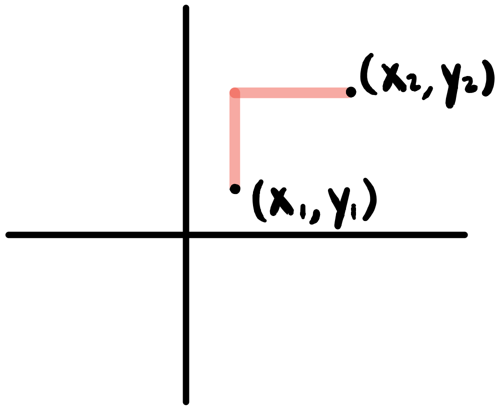
when (X, d) clear from context, write $B(x_0, r)$



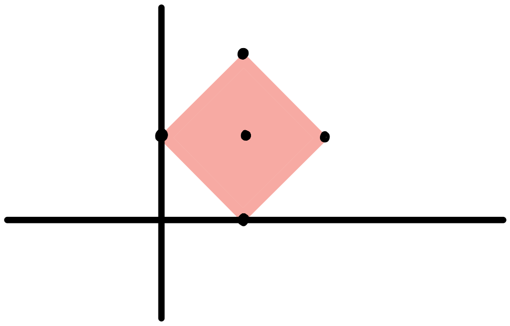
open ball

What about the taxicab metric on $\mathbb{R}^2$?

$$d_{taxi}((x_1,y_1),(x_2,y_2)) = |x_1-x_2| + |y_1-y_2|$$



$$B((1,1),1) = \{(x,y) \mid |x-1|+|y-1|<1\}$$



Discrete metric

X any set

$$d_{discrete}(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y \end{cases}$$

$B(x,1) = x$

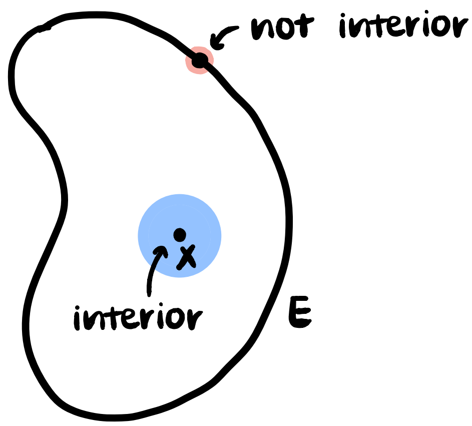
$B(x,2) = X$

3 Kinds of Points

$x \in X$, $E \subseteq X$

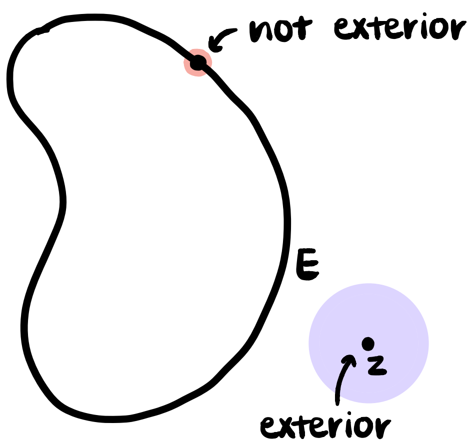
1. Interior point

   We say x is an interior point of E if there is some $r>0$ s.t. $B(x,r) \subseteq E$.



2. Exterior point

   We say $z \in X$ is an exterior point of E if there is some $r>0$ s.t. $B(z,r) \cap E = \phi$.



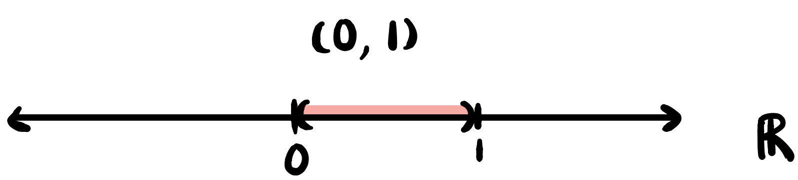
3. Boundary point

We say x is a boundary point of E if x is neither interior nor exterior to E.

If $E \subseteq X$, we define the interior of E, $\mathrm{int}(E)$, as the set of interior points, the exterior of E, $\mathrm{ext}(E)$, as the set of exterior points, and the boundary of E, ∂E, as the set of boundary points.

Def. Suppose $E \subseteq X$. We say $x_0 \in X$ is an adherent point of E if, for every $r > 0$, $B(x_0, r) \cap E \neq \emptyset$.

Ex.



$d(x,y) = |x-y|$

0 is an adherent point of $(0, 1)$.

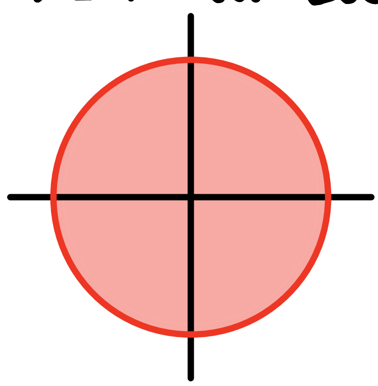
$B(0,r) \cap (0,1) \ni \min(\frac{1}{2}, \frac{r}{2})$

for arbitrary $r > 0$.

$\rightarrow$ 0 is an adherent point.

Set of adherent points of $(0,1)$ is $[0,1]$.

Ex. $E = B((0,0),1) \subseteq \mathbb{R}^2$ w/ Euclidean metric



Thm. Suppose $E \subseteq X$.

The following are equivalent:

(1) x_0 is an adherent point of E.

(2) x_0 is either interior or boundary to E.

(3) There is a sequence $(x_n)_{n \in \mathbb{N}}$ of points in E such that $\lim_{n \to \infty} x_n = x$.

1.1.12 $d_{\ell^2}(x,y) \le d_{\ell^1}(x,y) \le \sqrt{n}\, d_{\ell^2}(x,y)$

Thm. (Uniqueness of Limits)

If (X, d) is a metric space and $x_n \to x$ and $y_n \to y$ then $x = y$.

Pf. Take $\varepsilon > 0$. By def. of $x_n \to n$, there is $N_1 \in \mathbb{N}$ s.t. for $n > N_1$, $d(x_n, x) < \frac{\varepsilon}{2}$.

Similarly, there is N_2 s.t. for $n > N_2$, $d(x_n, y) < \frac{\varepsilon}{2}$.

Then for $N > \max\{N_1, N_2\}$, we have for $n > N$

$$0 \le d(x,y) \le d(x, x_n) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Since ε was arbitrary, $d(x,y) = 0$, so $x = y$. ▫

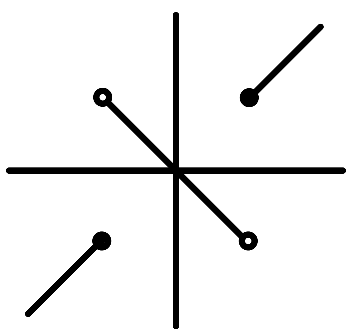
Thm. If (X, d_{disc}) is a discrete metric space and $x_n \to x$ in this space, then there is N s.t. $x_n = x$ for $n > N$.

Pf. Since $x_n \to x$, $\exists$ N s.t. for $n > N$, we have $d_{disc}(x_n, x) < 1$.

Let the base space be $\mathbb{R}$.

$f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } |x| \geq 1 \\ -x & \text{if } |x| < 1 \end{cases}$$



Define a metric on $\mathbb{R}$ by

$$d(x,y) = |f(x) - f(y)|$$

When do two metrics d_1, d_2 on X have the same limiting properties?

Def. We say (X, d_1) and (X, d_2) are topologically equivalent if for every $x \in X$ and $r > 0$, $\exists\ s > 0$ s.t. $B_{d_1}(x, s) \subseteq B_{d_2}(x, r)$ and for every $x \in X$ and $r' > 0$, $\exists\ s' > 0$ s.t. $B_{d_2}(x, s') \subseteq B_{d}(x, r')$

Thm. (X, d_1) and (X, d_2) are topologically equivalent if and only if ($x_n \to x$ in d_1 iff $x_n \to x$ in d_2).

Pf. Suppose these are topologically equivalent, and $x_n \to x$ in d_1. WTS: $x_n \to x$ in d_2.

Consider $B_{d_2}(x, \varepsilon)$ for any $\varepsilon > 0$.

By topological equivalence, $\exists\ \varepsilon'$.

Recall (X, d) where $d: X \times X \to [0, \infty)$

is a metric space if d satisfies

1. $\forall\, x, y \in X: \quad d(x,y) = 0 \iff x = y$
2. $\forall\, x, y \in X: \quad d(x,y) = d(y,x)$
3. $\forall\, x, y, z \in X: \quad d(x,z) \le d(x,y) + d(y,z)$

Ex. $X = \mathbb{R} \qquad d(x,y) := |x-y|$

Ex. $X = \{ F : [0,1] \to \mathbb{R},\ F \text{ is continuous} \}$

$d: X \times X \to [0, \infty)$

$(f,g) \mapsto d(f,g) := \sup |f(x) - g(x)| \quad x \in [0,1]$

in order to conclude that (X, d) is M.S.

we need to verify that d satisfies 1, 2, 3

① $d(f,g) = 0 \iff \sup\limits_{x \in [0,1]} |f(x) - g(x)| = 0$

since

$0 \le f(x) - g(x) \le 0 \qquad \forall\ x \in [0,1]$

$f(x) = g(x) \qquad \forall\ x \in [0,1]$

$\Rightarrow f = g$

② $d(f,g) = \sup\limits_{x \in [0,1]} |f(x) - g(x)| = \sup\limits_{x \in [0,1]} |g(x) - f(x)|$

$= d(g,f)$

③ $f, g, h \in X$

$$
\begin{aligned}
d(f,g) &= \sup_{x\in[0,1]} |f(x)-g(x)| \\
&= \sup_{x\in[0,1]} |f(x)-h(x)+h(x)-g(x)| \\
&\leq \sup_{x\in[0,1]} (|f(x)-h(x)| + |h(x)-g(x)|) \\
&\leq \sup_{x\in[0,1]} |f(x)-h(x)| + \sup_{x\in[0,1]} |h(x)-g(x)| \\
&= d(f,h) + d(h,g)
\end{aligned}
$$

Ex. $X := \{f : [0,1] \to \mathbb{R},\ f \text{ continuous}\}$

$d(f,g) := \int_0^1 |f(x)-g(x)|\,dx$

① $d(f,g) = 0 \iff \int_0^1 |f(x)-g(x)|\,dx = 0$

$L(x) := |f(x)-g(x)| \geq 0$

L is continuous on $[0,1]$

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$$
\begin{aligned}
\int_0^1 L(x)\,dx = 0 &\iff L(x) = 0 \quad \forall\ x\in[0,1] \\
&\iff |f(x)-g(x)| = 0 \quad \forall\ x\in[0,1] \\
&\iff f(x) = g(x) \quad \forall\ x\in[0,1]
\end{aligned}
$$

②
$$
\begin{aligned}
d(f,g) &= \int_0^1 |f(x)-g(x)|\,dx \\
&= \int_0^1 |g(x)-f(x)|\,dx \\
&= d(g,f)
\end{aligned}
$$

③ $f, g, h \in X$

$$d(f,g) = \int_0^1 |f(x) - g(x)|\,dx$$
$$= \int_0^1 |f(x) + h(x) + h(x) - g(x)|\,dx$$
$$\le \int_0^1 |f(x) + h(x)|\,dx + \int_0^1 |h(x) - g(x)|\,dx$$
$$\le d(f,h) + d(h,g)$$

Def. Let (X, d) be a metric space.
A sequence (x_n) in X converges to $x \in X$ iff
$\lim_{n\to\infty} |x_n - x| = 0$

Ex. 1 $X = \mathbb{R}$

$x_n = \frac{1}{n}$, $x = 0$

$d(x,y) := |x-y|$

$d(\frac{1}{n}, 0) = |\frac{1}{n} - 0| = \frac{1}{n}$

$\to 0$ as $n \to \infty$

d_{disc} : discrete metric on $\mathbb{R}$

$$d_{disc}(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

$d_{disc}(\frac{1}{n}, 0) = 1$

$$\lim_{n\to\infty} d_{disc}\left(\tfrac{1}{n}, 0\right) = 1 \neq 0$$

$\Rightarrow$ $\left(\frac{1}{n}\right)$ doesn't converge to 0

as $n \to \infty$ w.r.t. d_{disc}

$\frac{1}{n} \to 0$ in $(\mathbb{R}, d)$

$\frac{1}{n} \not\to 0$ in $(\mathbb{R}, d_{disc})$

Prop. (Uniqueness of limits)

Let (x_n) be a sequence in a M.S. (X, d).

If $x, x' \in X$ and $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = x'$,

then $x = x'$.

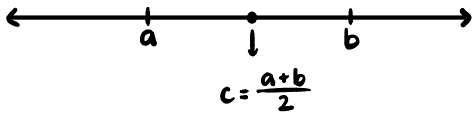
Pf. $\lim_{n\to\infty} x_n = x \rightarrow \lim_{n\to\infty} d(x_n, x) = 0$

$\lim_{n\to\infty} x_n = x' \rightarrow \lim_{n\to\infty} d(x_n, x') = 0$

$$0 \leq d(x, x') \leq \underbrace{d(x, x_n)}_{\downarrow 0} + \underbrace{d(x_n, x')}_{\downarrow 0}$$

$$0 \leq d(x, x') \leq 0$$

$$\Rightarrow d(x, x') = 0$$



$|x-c| \le |b-c| = r$

$|b-c| = |c-a|$

$[a,b] = \{x \in \mathbb{R} : |x-c| \le r\}$

$(X,d) \quad x_0 \in X$

Def. (Open balls)

Let (X,d) be a metric space, $x_0 \in X$ and $r > 0$, the open ball of radius r centered at x_0 is defined as:

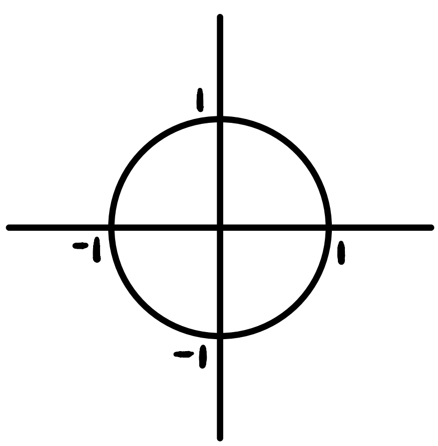
$$B_{(X,d)}(x_0, r) := \{x \in X : d(x, x_0) < r\}$$

Ex. $X = \mathbb{R}^2$

$d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$B_{(\mathbb{R}^2,d_2)}((0,0),1) = \{(x,y) \in \mathbb{R} \mid \sqrt{x^2+y^2} \le 1\}$

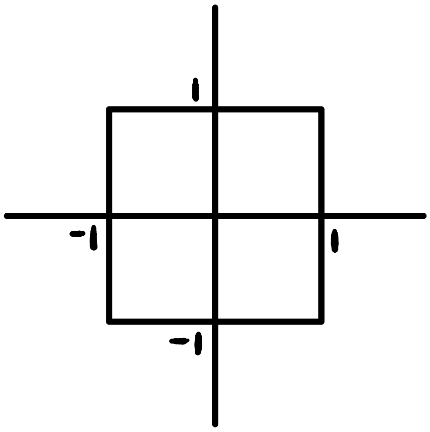
↳ unit ball at (0,0)



Ex. $X = \mathbb{R}^2$

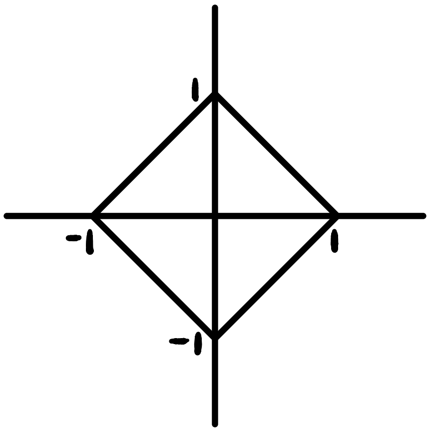
$$d_\infty((x_1, y_1), (x_2, y_2)) := \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

$$B_{(\mathbb{R}^2, d_\infty)}((0,0), 1) = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} \leq 1\}$$



Ex. $X = \mathbb{R}^2$

d_1



Def. (Interior, Exterior, Boundary)

Let (X, d) be a m.s., $E \subseteq X$, $x_0 \in X$

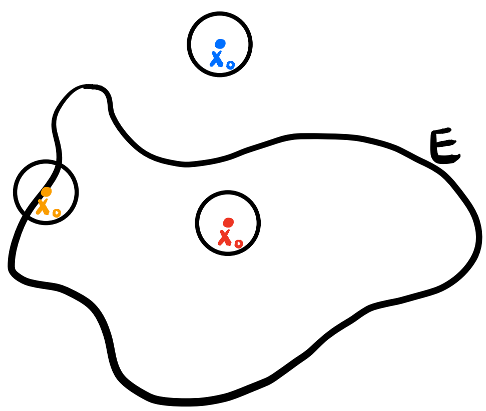
- x_0 is an **interior point** of E if there exists $r>0$ s.t. $B(x_0, r) \subseteq E$.

  $x_0 \in \text{int}(E)$

- x_0 is an **exterior point** of E if it is an interior point of $X \setminus E$.

  $x_0 \in \text{ext}(E)$ $\qquad$ E^c

- x_0 is a boundary point of E if it is neither an interior or exterior point of E.



Rmk. x_0 is an exterior point of E

$\Leftrightarrow \exists\ r>0$ s.t. $B(x_0, r) \cap E = \emptyset$

Def. (open, closed sets)

Let (X,d) be a m.s., $E \subseteq X$.

- E is open if for all $x \in E$, there exists $r > 0$ s.t. $B(x,r) \subseteq E$
- E is closed if its complement $X \setminus E$ is open.

Prop. Every open ball is open.

Pf. Let $B(x,r)$ be an open ball in (X,d).

Suppose $y \in B(x,r)$.

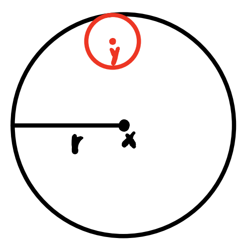
By definition, $d(x,y) < r$.

$$r' := r - d(x,y) > 0$$

We claim that $B(y,r') \subseteq B(x,r)$.

Indeed, $\forall\ z \in B(y,r')$, we have that

$$\begin{aligned} d(z,x) &\le d(z,y) + d(y,x) \\ &< r' + d(y,x) \\ &= r - d(x,y) + d(y,x) \\ &= r \end{aligned}$$



So $B(x,r)$ is open.