If f is integrable on $[0, b]$, $0<b<\infty$, and

f is of exponential order, $|f(x)| \le Me^{cx}$,

then f has a Laplace transform $F(p)$ for $p>c$

and $F(p) \to 0$ as $p \to \infty$.

Rmk. In fact, if $F(p)$ exists, then $F(q)$ exists for all $q>p$

and $F(q) \to 0$ as $q \to \infty$.

$$L[f(x)] = \int_0^\infty f(x) e^{-px}\, dx$$

[relies on integration by parts + dominated convergence]

Consequence: p^n, $\sin(p)$ cannot be the Laplace transform of <u>any</u> function f.

§50 Inverse Laplace Transform

Suppose that f, g are continuous and $L[f] = L[g]$, i.e. $F(p) = G(p)$ for all p (large).

$\rightarrow \int_0^\infty f(x) e^{-px}\, dx = \int_0^\infty g(x) e^{-px}\, dx$

$\rightarrow \int_0^\infty (f(x) - g(x)) e^{-px}\, dx = 0$ for all p (large)

$\rightarrow \int_0^\infty (f(x) - g(x))(ae^{-p_1 x} + be^{-p_2 x})\, dx = 0$ for all p_1, p_2 and $a, b \in \mathbb{R}$

(not obvious) $\rightarrow \int_0^\infty (f(x) - g(x))\, \varphi(x)\, dx = 0$

$\varphi(x)$ continuous, decays sufficiently fast

$\rightarrow f(x) = g(x)$

i.e. a continuous f is uniquely determined by its Laplace transform

$L[f(x)] = F(p)$

$f(x) = L^{-1}[f(x)]$

L^{-1} inverse Laplace transform

Properties: ① L linear $\to$ L^{-1} linear

$$L^{-1}[a\cdot F(p)+b\cdot G(p)]=a\cdot L^{-1}[F(p)]+b\cdot L^{-1}[G(p)]$$

② $L[e^{ax}f(x)]=F(p-a)$

$$\Rightarrow \quad e^{ax}f(x)=L^{-1}[F(p-a)]$$

③ $L[f(x-a)]=e^{-ap}F(p)$

$$\Rightarrow \quad f(x-a)=L^{-1}[e^{-ap}F(p)]$$

Ex. (i) $L^{-1}\left[\frac{1}{p^2}\right]=x$

$$\Leftrightarrow \; L[x]=\frac{1}{p^2}$$

(ii) $L^{-1}\left[\frac{1}{(p-a)^2}\right]=e^{ax}\cdot x$

(iii) Similarly, $L^{-1}\left[\frac{1}{(p-a)^{n+1}}\right]=\frac{1}{n!}e^{ax}x^n$

$\frac{n!}{p^{n+1}}=L[x^n]$ (exercise! by convention: $f(x)=0$ for $x<0$)

$\to$ we can use linearity + partial fractions to compute L^{-1} of rational functions

Thm. **Fundamental Theorem of Algebra**

Every polynomial $q(z)=b_0+b_1z+\ldots+b_nz^n$ with $n\geq 1$, $b_0,\ldots b_n\in\mathbb{C}$ has a root, i.e. there is $w\in\mathbb{C}$ such that $q(w)=0$.

Cor. $q(z)=c(z-z_1)^{k_1}\cdot\ldots\cdot(z-z_m)^{k_m}$

where $z_1,\ldots z_m$ are the distinct roots of q, $c\in\mathbb{C}$. $(n=\sum_{i=1}^{m}k_i)$

Proof: long division

Cor. Partial Fraction Decomposition

Let $k<n$, $a_i, b_i\in\mathbb{C}$, q as above,

$$R(z)=\frac{a_0+a_1z+\ldots+a_kz^k}{b_0+b_1z+\ldots+b_nz^n}$$
$$=\frac{A_{11}}{z-z_1}+\frac{A_{12}}{(z-z_1)^2}+\ldots+\frac{A_{1k_1}}{(z-z_1)^{k_1}}$$

$$+\dots+\frac{A_{m1}}{z-z_m}+\dots+\frac{A_{mk_m}}{(z-z_m)^{k_m}}$$

Rmk. (i) If $q(x)=b_0+\dots+b_nx^n$ has only real coefficients $b_i\in\mathbb{R}$, $q(w)=0$,

then $0=\bar{0}=\overline{q(w)}=q(\bar{w})$ $\left(\bar{\cdot}=\text{complex conjugate}\right)$

i.e. complex roots always appear in pairs w, $\bar{w}$.

(ii) If also $a_0,\dots a_k\in\mathbb{R}$, then we obtain a partial fraction decomposition with real coefficients as follows:

$$\frac{A}{(x-w)^k}+\frac{\bar{A}}{(x-\bar{w})^k}$$

$$=\frac{A(x-\bar{w})^k+\bar{A}(x-w)^k}{((x-w)(x-\bar{w}))^k}\qquad \begin{matrix} w=\alpha+i\beta \\ \bar{w}=\alpha-i\beta \end{matrix}\quad \alpha,\beta\in\mathbb{R}$$

$$=\frac{2\operatorname{Re}(A(x-\alpha+i\beta)^k)}{((x-a)^2+\beta^2)^k}$$

$$=\sum_{\ell\le k}\frac{\gamma_\ell x+\delta_\ell}{((x-a)^2+\beta^2)^\ell}$$

Ex. $R(x)=\frac{36x}{(x+3)^2(x^2+9)}=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{x+3i}+\frac{D}{x-3i}$

$B=R(x)\cdot(x+3)^2-\left(\frac{A}{x-3}-\frac{C}{x+3i}-\frac{D}{x-3i}\right)(x+3)^2$ for all x

$x=-3 \rightarrow B=\frac{36(-3)}{9+9}=-6$

$C=\frac{36(-3i)}{(-3i+3)^2(-3i-3i)}=\dots=\frac{1}{-i}=i$

$D=\dots=-i$ (so $\bar{C}=D$)

$x=0 \rightarrow A=0$

So $R(x)=\frac{-6}{(x+3)^2}+\frac{i}{x+3i}+\frac{-i}{x-3i}$