(cont.) Convergence Order of Fixed Point Iteration

Thm 2. Let p be a solution of $g(x)$.

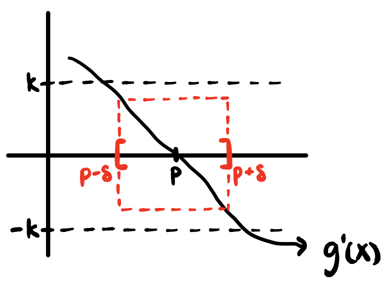
Assume
- $g'(p)=0$, and
- g'' continuous with $|g''(x)|<M$ on an open interval I containing p

Then there exists $\delta>0$ such that for any $p_0\in[p-\delta, p+\delta]$, the sequence $p_n=g(p_{n-1})$ converges at least quadratically to p, i.e. $|p_{n+1}-p|<\frac{M}{2}|p_n-p|^2$ for all $n\geq n_0$

Pf. ① Show that fixed point iteration converges under given assumptions.

By continuity of $g'(x)$, we can choose $k\in[0,1]$ and $\delta>0$ such that:

$$\left.\begin{array}{l}\text{(i) } [p-\delta, p+\delta]\subseteq I, \text{ and} \\ g'(p)=\lim_{x\to p} g'(x)=0<k\end{array}\right\} \Rightarrow |g'(x)|\leq k \text{ for } x\in[p-\delta, p+\delta]$$



given k, we can find δ st $x\in[p-\delta, p+\delta] \Rightarrow |g'(x)|\leq k$

know: $|g'(x)|\leq k<1$ on $[p-\delta, p+\delta]$

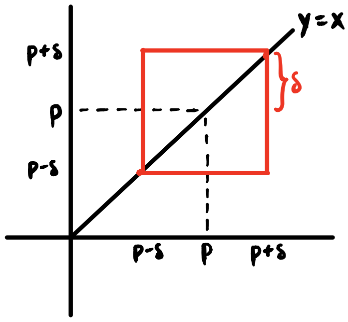
Next, need to show g maps to itself on $[p-\delta, p+\delta]$.

Let $x\in[p-\delta, p+\delta]$.

$$\begin{aligned}|g(x)-p| &= |g(x)-g'(x)| \\ &= |g(\xi)|\cdot|x-p| \quad \text{by MVT} \\ &\leq k|x-p| \\ &< |x-p| \quad (*)\end{aligned}$$

$x \in [p-\delta, p+\delta] \Rightarrow |x-p| < \delta$

$\Rightarrow |g(x)-p| < \delta$ by (*)



Thus, g maps $[p-\delta, p+\delta]$ into $[p-\delta, p+\delta]$.

Thus, $g(x) \in [p-\delta, p+\delta]$ for all $x \in [p-\delta, p+\delta]$.

$\Rightarrow$ By fixed point thm., g converges to unique sol.

② Show quadratic convergence.

Expand $g(x)$ at p for $x \in [p-\delta, p+\delta]$:

$$g(x) = g(p) + \underbrace{g'(p)(x-p)}_{=0 \text{ by assumption}} + \frac{g''(\xi_n)}{2!}(x-p)^2$$

ξ_n betw. x and p

$$\Rightarrow g(x) = g(p) + \frac{g''(\xi_n)}{2}(x-p)^2$$

$$\Rightarrow \frac{|p_{n+1}-p_n|}{|p_n-p|^2} = \frac{|g''(\xi_n)|}{2}$$

by assumption

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1}-p_n|}{|p_n-p|^2} = \lim_{n\to\infty} \frac{|g''(\xi_n)|}{2} = \frac{1}{2}|g''(\underbrace{\lim_{n\to\infty}\xi_n}_{=p})| < \frac{M}{2}$$

$g''(p) \neq 0 \Rightarrow \{p_n\}$ converges quadratically to p

$g''(p) = 0 \Rightarrow \{p_n\}$ converges at higher order (≥ 3) to p.

Rmk. Need p_0 to be sufficiently close to p:

$p_0 \in [p-\delta, p+\delta]$

Convergence Order of Newton's Method

Ex. Let $p \in (a,b)$ be a zero of $f \in C^2([a,b])$.

Construct a fixed point problem $g(x) = x$ associated with root-finding problem, $f(x)=0$, such that

$p_n = g(p_{n-1})$ converges quadratically.

Sol. Set $g(x) = x - \phi(x) \cdot f(x)$

Goal: find $\phi(x)$ to get quadratic convergence.

$g'(x) = 1 - \phi'(x) f(x) - \phi(x) f'(x)$

$\rightarrow g'(p) = 1 - \phi(p) \cdot f'(p)$ (want $g'(p)=0$)

To obtain convergence, want $g'(p)=0$.

$\Rightarrow \phi(p) = \dfrac{1}{f'(p)}$

Choose $\phi(x) = \dfrac{1}{f'(x)} \longrightarrow g(x) = x - \dfrac{f(x)}{f'(x)}$

$\Rightarrow$ Fixed pt iter. is $\underbrace{p_n = g(p_{n-1}) = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})}}_{\text{Newton's Method!}}$

Rmk. ① If $f(p)=0$ and $f'(p) \neq 0$, then for any p_0 sufficiently close to p, (st $g \in C^1[a,b]$, $g(x) \in [a,b]$, $g'(x) \le k$) Newton's Method will converge at least quadratically.

② If $f(p)=0$, then for p_0 close to p, Secant Method converges to p with order $\dfrac{\sqrt{5}+1}{2} \approx 1.618$.