A better estimate using Lagrange polynomials

Thm. Suppose $x_0, x_1, \dots x_n$ are distinct numbers in the interval $[a,b]$ and $f \in C^{(n+1)}[a,b]$. Then for each $x \in [a,b]$, there exists $\xi(x) \in [a,b]$ between $x_0, x_1, \dots x_n$ such that

$$f(x) = p(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

where $p(x) = \sum_{k=0}^{n} L_{n,k}(x) \cdot f(x_k)$

Rmk. Lagrange error R_L is similar to Taylor polynomial error:

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1} \quad \text{vs.} \quad \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

but error in R_L is "spread" across the different nodes.

Ex. Given $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$ for $f(x) = \frac{1}{x}$

a) Determine the error form for the Lagrange polynomial $p(x)$.

b) Determine the maximum error when $p(x)$ is used to approximate $f(x)$ for $x \in [2,4]$

Sol. a) $f'(x) = \frac{-1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = \frac{-6}{x^4}$

Error: $R_L(x) = \frac{f'''(\xi(x))}{3!}\underbrace{(x-x_0)(x-x_1)(x-x_2)}_{g(x)}$

$$\approx \frac{-6}{3!}(\xi(x))^{-4}(x-2)(x-2.75)(x-4) \qquad \xi(x) \in (2,4)$$

b) want to find $\max_{x \in (2,4)} |R_2(x)|$

Note: $|\xi(x)^{-4}| \le 2^{-4} = \frac{1}{16}$

Let $g(x) = (x-2)(x-2.75)(x-4)$

$$= x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22$$

To find max values of g on $[2,4]$,

first find critical points:

$$g'(x) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = 0$$

$$x = \frac{7}{3} \longrightarrow g\left(\frac{7}{3}\right) = \frac{25}{108}$$

$$x = \frac{7}{2} \qquad g\left(\frac{7}{2}\right) = \frac{-9}{16}$$

$$\left(\left|\frac{-9}{16}\right| > \left|\frac{25}{108}\right|\right)$$

$\Rightarrow \max\limits_{x \in [2,4]} |g(x)| \le \frac{9}{16}$. Hence max error is

$$|R_1(x)| \le \frac{6}{3!} |\xi(x)^{-4}| |g(x)| \le \left|\frac{1}{16}\right| \left|\frac{9}{16}\right|$$

Note In general, we have to evaluate g at boundary points.

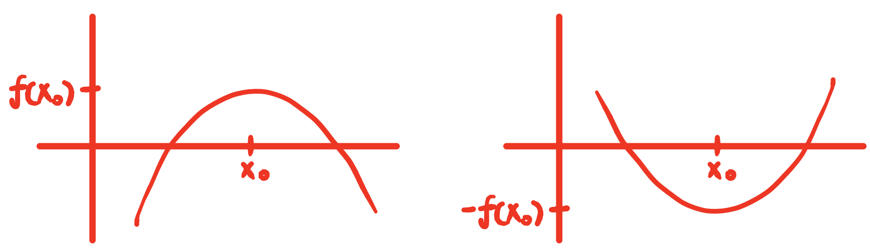
Rmk. 1. Power series form $\equiv$ Lagrange form but with different basis:

Power series basis: $\{x^n, x^{n-1}, \dots x, 1\}$

Lagrange basis: $\{L_{n,n}(x), L_{n,n-1}(x), \dots L_{n,1}(x), L_{n,0}(x)\}$

2. Can use fmin bound (fun, x_1, x_2) to find maximum of a function on (x_1, x_2).

note: $\max\limits_x f(x) = -\min\limits_x f(x)$



3. Adding a new node x_i changes all bases

$L_{n,j}$s to $L_{n+1,j}$s