2.3 Newton's Method

Idea: Approximate f by a linear function at each iteration

twice cont. diff'able on (a,b)

Suppose $f \in C^2[a,b]$ and $p \in [a,b]$ is a root of f.

Let $p_0 \in (a,b)$ be an approximation to p such that $f'(p) \neq 0$ and $|p-p_0|$ is small.

Then $f(p) = f(p_0) + f'(p_0)(p-p_0) + f''(\xi(p)) \cdot \frac{(p-p_0)^2}{2}$

where $\xi(p)$ is between p_0 and p.

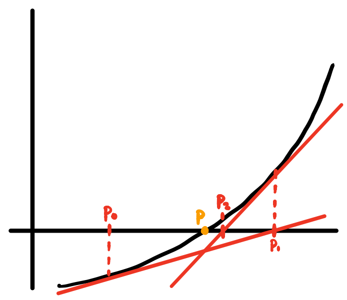
$f(p)=0 \rightarrow 0 \approx f(p_0) + f'(p_0)(p-p_0)$

$\rightarrow p \approx p_0 - \frac{f(p_0)}{f'(p_0)} = p_1$

Newton iteration:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad \text{for all } n \geq 1$$

Geometrically:



If p_0 is not close enough to p, Newton's Method might diverge.

Ex. Let $f(x) = x^2 - 3$. Use Newton's Method to find a root of f with accuracy $\varepsilon = 10^{-8}$, with $p_0 = 1.5$.

Sol. $p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \frac{p_{n-1}^2 - 3}{2p_{n-1}}$

By direct computation, we get $p_3 = 1.7320581$,

$f(p_3) < 10^{-8}$, $|p_3 - p| < 10^{-8}$

Rmk. ① equivalent fixed point iteration to Newton:

$p_n = g(p_{n-1})$ where $g(x) = x - \frac{f(x)}{f'(x)}$, $f'(p) \neq 0$.

Also, $g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2}$

If p is a zero of f, then $g'(p) = 0$!

② two shortcomings of Newton's Method:

1 - need $f'(p_n)$ at each iteration (can be very expensive in high dimensions)

We could approximate f' by $f'(p_{n-1}) = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$ (secant method)

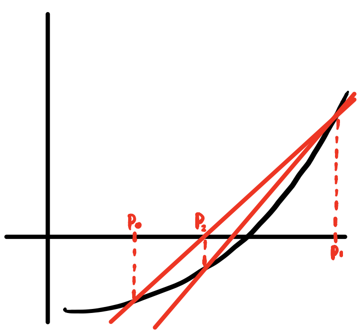
2 - need to start "close" to the root $p_0 \approx p$

Secant Method

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

Rmk. need 2 initial conditions p_0, p_1

Geometrically:



Note Newton's and Secant methods are NOT root-bracketing

(p may not be between p_n and p_{n-1})

Method of False Position (MFP)

Idea: use Secant Method with root-bracketing ($p \in [p_{n-1}, p_n]$)

1) Initialize p_0, p_1 s.t. $f(p_0) \cdot f(p_1) < 0$

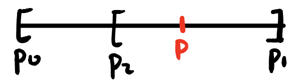
$\Rightarrow p_0 \in [p_0, p_1]$

Let $p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)}$

(p_2 is root of line passing through $(p_0, f(p_0))$, $(p_1, f(p_1))$)

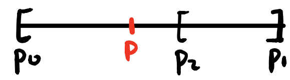
2) While $n \le \text{maxIter}$

- if $\text{sign}(f(p_1)) \cdot \text{sign}(f(p_2)) < 0$, $\rightarrow$ $p \in [p_1, p_2]$



  set $p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)}$

- if $\text{sign}(f(p_1)) \cdot \text{sign}(f(p_2)) > 0$, $\rightarrow$ $p \notin [p_1, p_2]$



  set $p_3 = p_2 - \frac{f(p_2)(p_2 - p_0)}{f(p_2) - f(p_0)}$ (note: $p \in [p_0, p_2]$

In general, $p_n = p_3$, $p_{n-1} = p_2$, $p_{n-2} = p_1$, $p_{n-3} = p_0$.

Note: may require 3 previous points.