Continous functions on metric spaces

Def. Let (X, d_X) and (Y, d_Y) be m.s.,
$f : X \to Y$ a function,
$x_0 \in X$.

f is continuous at x_0 if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$
s.t. $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$

$\forall\ \varepsilon > 0\ \exists\ \delta > 0$ s.t. $d_Y(f(x), f(x_0)) < \varepsilon$
$\forall\ x$ s.t. $d_X(x, x_0) < \delta$

Lemma $F : X \to Y$ is continuous at x_0
$\Leftrightarrow$ for any sequence $(x_n) \subset X$ s.t.
$\lim_{n \to \infty} d_X(x_n, x_0) = 0$, $f(x_n) \xrightarrow{d_Y} f(x_0)$.
$\Leftrightarrow \lim_{x_n \to x_0} f(x_n) = f(x_0)$.

Thm. $f : X \to Y$. T.F.A.E.:

(i) f is continuous

(ii) For every open set $V \subseteq Y$, $f^{-1}(V)$ is open in (X, d_X).

(iii) For every closed set $F \subseteq Y$, $f^{-1}(F)$ is closed in (X, d_X).

$1 \Rightarrow 2$

Pf. Assume $f: X \to Y$ is continuous.

Let $V \subseteq Y$ be open.

Take $x_0 \in f^{-1}(V)$.

$\Rightarrow f(x_0) \in V$.

$\exists$ $r > 0$ s.t. $B_{d_Y}(f(x_0), r) \subseteq V$ because V is open in Y.

Since f is continuous at x_0, $\forall$ $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t. if $\underbrace{d_X(x, x_0) < \delta}_{\Updownarrow \; x \in B_{d_X}(x_0, \delta)}$, then $\underbrace{d_Y(f(x), f(x_0)) < \varepsilon}_{\Updownarrow \; f(x) \in B_{d_Y}(f(x_0), \varepsilon)}$.

If $\varepsilon < r$, then $B_{d_Y}(f(x_0), \varepsilon) \subset B_{d_Y}(f(x_0), r)$.

$\therefore$ for any $x \in B_{d_X}(x_0, \delta)$, we have $f(x) \in B_{d_Y}(f(x_0), \varepsilon) \subset V$.

$\Rightarrow x \in f^{-1}(V)$.

Therefore $B_{d_X}(x_0, \delta) \subset B_{d_Y}(f(x_0), r)$.

$x_0 \in \operatorname{int}(f^{-1}(V))$.

$\Rightarrow f^{-1}(V)$ is open in X.

$2 \Rightarrow 3$

Let F be a closed set in Y.

$\Leftrightarrow F^c$ is open in Y.

$\Rightarrow f^{-1}(F^c)$ is open in X.

wts: $f^{-1}(F^c) = (f^{-1}(F))^c$

$x \in f^{-1}(F^c) \Leftrightarrow f(x) \in F^c$

$\Leftrightarrow f(x) \notin F$

$\Leftrightarrow x \notin f^{-1}(F)$

$\Leftrightarrow x \in f^{-1}(F)^c$

$3 \Rightarrow 1$

Assume f is not continuous.

Suppose $\lim_{n\to\infty} f(x_n) = x$ and $\lim_{n\to\infty} f(x_n) \neq f(x)$ for some $x \in X$ and $(x_n) \subset X$.

Consider $A := \{f(x_n) : x_n \in X\}$, $B := \{x_n, n \in \mathbb{N}\}$

If $f(x_0) \notin \bar{A}$, then $x_0 \notin f^{-1}(\bar{A})$.

$x_0 \in \bar{B}$ and $B \subseteq f^{-1}(A) \subseteq f^{-1}(\bar{A})$.

$\Rightarrow x_0 \in \bar{B} \subseteq \overline{f^{-1}(\bar{A})} = f^{-1}(\bar{A})$

$\Rightarrow x_0 \in f^{-1}(\bar{A})$. $\lightning$

If $f(x_0) \in \bar{A}$, then since $f(x_n)$ does not converge to $f(x_0)$, $\exists$ a subsequence $\{f(x_{n_k})\}$ s.t. $\exists\ \varepsilon_0$ s.t. $d_Y(f(x_0), f(x_{n_k})) \geq \varepsilon_0$.

Consider $A' := \{f(x_{n_k}) : x_{n_k} \in X\}$, $B' := \{x_{n_k}, n_k \in \mathbb{N}\}$

Since $f(x_0) \notin \bar{A'}$, we can proceed as before. $\blacksquare$

$x_0 \notin f^{-1}(\bar{A'})$

Continuity and compactness

Thm. $f: X \to Y$ continuous, $K \subseteq X$ compact
Then $f(K) := \{f(x) : x \in K\}$ is compact in Y.

Pf.① Suppose (y_n) is a sequence in $f(K)$.
$\forall\, n \in \mathbb{N}\ \exists\ x_n \in K$ s.t. $f(x_n) = y_n$.

$\Rightarrow (x_n) \subseteq K$.

Since K is compact, $\exists\ (x_{n_j}) \subseteq (x_n)$ and $x \in K$
s.t. $\lim_{n_j \to \infty} x_{n_j} = x$.
Since f is continuous, $\lim_{n_j \to \infty} f(x_{n_j}) = f(x)$.

$$\underset{\displaystyle \lim_{n_j \to \infty} y_{n_j}}{\underset{\shortparallel}{\lim_{n_j \to \infty} f(x_{n_j})}} = f(x)$$

$\Rightarrow f(K)$ is a compact subset of Y.

Pf.② Suppose V_α, $\alpha \in I$, is an open cover for K.

$\Rightarrow f(K) \subseteq \bigcup_{\alpha \in I} V_\alpha$

are

Since f is continuous and V_α is open $\forall\ \alpha \in I$,
$f^{-1}(V_\alpha)$ is open in X.

$\Rightarrow K \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) \subseteq \bigcup_{\alpha \in I} f^{-1}(V_\alpha)$ (*)

$x \in f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right) \Rightarrow f(x) \in \bigcup_{\alpha \in I} V_\alpha$

$\Rightarrow \exists\ \alpha \in I$ s.t. $f(x) \in V_\alpha$

$\Rightarrow x \in f^{-1}(V_\alpha)$

$\Rightarrow f^{-1}(V_\alpha)$, $\alpha \in I$ is an open cover for K.

Since K is compact, $\exists$ $F \subseteq I$ finite s.t.
$K \subseteq \bigcup_{\alpha \in F} f^{-1}(V_\alpha)$

$\Rightarrow f(K) \subseteq \bigcup_{\alpha \in F} V_\alpha$

$\Rightarrow f(K)$ is compact. $\square$

Def. $f: X \to Y$ is uniformly continuous if $\forall$ $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t. $d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$

Thm. A continuous function on a compact space is uniformly continuous.

Pf. Let $\varepsilon > 0$.

Since $f: X \to X$ is continuous at every $x \in X$, (for each $x \in X$), $\exists$ $\delta(x) > 0$ s.t. $d_X(x, x') < \delta$

$\Rightarrow d_Y(f(x), f(x_j)) < \frac{\varepsilon}{2}$

$X = \bigcup_{x \in X} x \subseteq \bigcup_{x \in X} B(x, \varepsilon)$

This is an open cover for X.

Since X is compact, $\exists$ $x_1, x_2, \dots x_n$ s.t.

$x \in X \subseteq \bigcup_{i=1}^{n} B(x_i, \frac{\delta(x_i)}{2})$

$\exists$ $j \in \{1, \dots, n\}$ s.t. $x \in B(x_j, \frac{\delta(x_j)}{2})$

$d_Y(f(x), f(x_j))$

$$\Rightarrow d_X(x, x_j) < \frac{\delta(x_j)}{2}$$

$$\Rightarrow d_Y(f(x), f(x_j)) < \frac{\varepsilon}{2}$$

Moreover,

$$d_X(x, x_j) \leq \frac{\delta(x_j)}{2} + \frac{\delta(x_j)}{2}$$

$$= \delta(x_j)$$

$$d_Y(f(x), f(x_j)) \leq d_Y(f(x), f(x_j)) + d_Y(f(x_j), f(x'))$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

So f is uniformly continuous. $\square$

Continuity and connectedness

Def. Let (X, d) be a m.s.
We say X is **disconnected** if $\exists$ disjoint, nonempty, open subsets $V, W \subseteq X$ s.t. $X = V \cup W$.

Def. X is **connected** if X is not disconnected.

Obs. X is disconnected

$\Leftrightarrow$ $\exists$ $V \subset X$, $V \neq \phi$, $V \neq X$ s.t. V is open and closed.

$\forall$ $V \subset X$, $X = V \cup V^c$.

$\hookrightarrow \neq \phi$ if $V \neq X$

Thm. Suppose $f: X \to Y$ is continuous and $E \subseteq X$ is connected. Then $f(E)$ is connected.

Pf. Suppose $f(E)$ is disconnected.

$\exists$ V, W s.t. $f(E) = V \cup W$.

V, W are open, $V \neq \phi$, $W \neq \phi$, $V \cap W = \phi$.

Consider the restriction of f to E:

$f_{|E} : E \to V \cup W$ is continuous.

$$E = \underbrace{f_{|E}^{-1}(V)}_{\text{open}} \cup \underbrace{f_{|E}^{-1}(W)}_{\text{open}}$$

disjoint

$\Rightarrow$ E is disconnected. ↯