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5.6 Multistep Methods

Up until now, we've dealt with one-step methods, i.e. only require points t_i to obtain approximation at t_{i+1}.

Today: use information from >1 previous mesh point to obtain w_{i+1}.

Def. A m-step multistep method for solving IVP $\begin{cases} y' = f(t,y) & a \le t \le b \\ y(a) = \alpha \end{cases}$

has a difference equation of the form

$$
\begin{aligned}
w_{i+1} = a_{m-1}w_i + a_{m-2}w_{i-1} + \dots + a_0 w_{i+1-m} \\
+ h[b_m f(t_{i-1}, w_{i+1}) + b_{m-1} f(t_i, w_i) \\
+ \dots + b_0 f(t_{i+1-m}, w_{i+1-m})]
\end{aligned}
$$

with starting values

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad \dots \quad w_{m-1} = \alpha_{m-1}$$

If $b_m = 0$, the method is explicit.
If $b_m \neq 0$, the method is implicit.

Rmk. Implicit methods generally better, but need to solve eqn at each time step.

Fundamental Theorem of Calculus

$$\int_a^b y'(s)ds = y(b) - y(a)$$

Now on the interval $[t_i, t_{i+1}]$, we have

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} y'(t)dt = y_i + \int_{t_i}^{t_{i+1}} f(t, y(t))dt$$

Since $y(t)$ is not available, we approximate $f(t, y(t))$ by the Lagrange interpolating polynomial $p(t)$ using available nodes.

The difference equation thus becomes

$$w_{i+1} = w_i + \int_{t_i}^{t_{i+1}} p(t)dt.$$

AB Explicit methods : Adams-Bashforth explicit m-step method considers available m nodes, i.e. $(t_{i-j}, f(t_{i-j}, w_{i-j}))$ for $j = 0, 1, \dots m-1$.

$$P_{m-1}(t) = \sum_{j=0}^{m-1} \underbrace{\left(\prod_{\substack{\ell=0 \\ \ell \neq j}}^{m-1} \frac{t_i - t_{i-\ell}}{t_{i-j} - t_{i-\ell}} \right)}_{\text{Lagrange poly. centered at } t_j} f(t_{i-j}, w_{i-j})$$

which yields

$$w_{i+1} = w_i + h \sum_{j=0}^{m-1} b_j f(t_{i-j}, w_{i-j})$$

$$b_j = \frac{1}{h} \int_{t_i}^{t_{i+1}} \left(\prod_{\substack{\ell=0 \\ \ell \neq j}}^{m-1} \frac{t_i - t_{i-\ell}}{t_{i-j} - t_{i-\ell}} \right) dt$$

AM Implicit methods : Adams-Moulton implicit m-step method considers $m+1$ nodes, i.e. (t_{i-j}, w_{i-j}) for $j = -1, 0, 1, \dots m-1$.

Ex. For $m=2$, the AB-2 method is

$$w_{i+1} = w_i + h[b_1 f(t_i, w_i) + b_0 f(t_{i-1}, w_{i-1})]$$

Now we compute the coefficients b_0, b_1:

$$b_0 = \frac{1}{h} \int_{t_i}^{t_{i+1}} \frac{t - t_i}{t_{i-1} - t_i} dt = \frac{-1}{h^2} \cdot \frac{1}{2} (t - t_i)^2 \Big|_{t_i}^{t_{i+1}} = \frac{-1}{2}$$

$$b_1 = \frac{1}{h} \int_{t_i}^{t_{i+1}} \frac{t - t_{i-1}}{t_i - t_{i-1}} dt = \frac{3}{2}$$

Therefore, the method is:

$$w_0 = \alpha, \quad w_1 = \alpha_1$$

$$w_{i+1} = w_i + \frac{h}{2} [3 f(t_i, w_i) - f(t_{i-1}, w_{i-1})]$$

$$i = 0, 1, \dots N-1$$

Rmk. 1. The 1-step AB method is Euler's method.

2. To initialize $w_0, w_1, \dots w_{m-1}$, we can use 1-step methods (typically of the same order as that of the m-step method)

3. $a_{m-1} = 1, \quad a_0 = a_1 = \dots = a_{m-2} = 0$

Thm. (Order of Adams-Bashforth Methods)
-Moulton

The Adams-Bashforth explicit m-step multistep method has local truncation error $O(h^m)$ and is therefore of order m.
Similarly, the Adams-Moulton m-step method has local truncation error $O(h^{m+1})$ and is therefore of order m+1.

Rmk. Proof by Lagrange interpolation error.

Predictor-Corrector Methods

Recall that for implicit methods, must solve an equation to obtain w_{i+1}, e.g. Newton or Secant. (complicated)

Idea: Replace w_{i+1} on RHS with w_{i+1}^{p} computed using an explicit AB method:

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots \quad w_{m-1} = \alpha_{m-1}$$ (initialized w/ one-step method)

$$w_{i+1} = w_i + h\Big[b_0 f(t_{i-m+1}, w_{i-m+1}) + \ldots + b_{m-1} f(t_i, w_i) + b_m f(t_{i+1}, w_{i+1}^{p})\Big)$$

These are called $\overset{w_{i+1}^{p}}{\text{predictor}}$-$\overset{w_{i+1}^{c}}{\text{corrector}}$ methods.

Ex. Use the 2-step Predictor-Corrector AM and AB Methods to find w_2 given $w_1 = \alpha_1$ and the IVP

$$\begin{cases} w'(t) = f(t, y) & a \le t \le b \\ y(a) = \alpha \end{cases}$$

Sol. Apply 2-step AB method to get predictor w_2^p.

$$w_2^p = w_1 + \frac{h}{2}[3f(t_1, w_1) - f(t_0, w_0)]$$

Then apply 2-step AM method to correct w_2^p.

$$w_2 = w_1 + \frac{h}{12}[5f(t_2, w_2^p) + 8f(t_1, w_1) - f(t_0, w_0)]$$

Rmk. Other multistep methods can be obtained using integration of interpolating polynomials over intervals of the form $[t_j, w_{j+1}]$ for $j \le i-1$

5.10 Analysis of General Multistep Methods

A general linear multistep method for approximating the sol. of IVP is defined by the difference equation

$$\begin{cases} w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots \quad w_{m-1} = \alpha_{m-1} \\ w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \ldots + a_0 w_{i-m+1} + \\ \quad \underbrace{h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \ldots + b_0 f(t_{i-m+1}, w_{i-m+1})]}_{F(t_i, h, w_{i+1}, w_i, \ldots w_{i-m+1})} \end{cases}$$

Its local truncation error is given by:

$$\tau_{i+1}(h) = \frac{y_{i+1} - a_{m-1}y_i - \dots - a_0 y_{i-m+1}}{h} - F(t_i, h, w_{i+1}, \dots)$$

$$i = m-1, m, \dots N-1$$

Def. f satisfies Lipschitz condition w.r.t. $\{w_j\}$ if there exists constant $L>0$ such that for every pair of sequences $\{v_j\}_{j=0}^{N}$ and $\{\tilde{v}_j\}_{j=0}^{N}$ and for $i = m-1, m, \dots, N-1$, we have

$$|F(t_i, h, v_{i+1}, \dots v_{i+1-m}) - F(t_i, h, \tilde{v}_{i+1}, \dots \tilde{v}_{i+1-m})| \le L \sum_{j=0}^{m} |v_{i+1-j} - \tilde{v}_{i+1-j}|$$

Rmk. AB and AM this condition if f is Lipschitz continuous.

Def. A multi-step method is convergent if

$$\lim_{h\to 0} \max_{0 \le i \le N} |w_i - y(t_i)| = 0$$

and consistent if

$$\lim_{h\to 0} |\tau_i(h)| = 0 \quad \forall\ i = m, m+1, \dots N$$

AND $$\lim_{h\to 0} |\alpha_i - y(t_i)| = 0 \quad \forall\ i = 1, 2, \dots m-1$$

Rmk. 1. Definitions are similar to that of one-step methods, but also require errors in starting values $\{\alpha_i\}$ to approach 0 as $h \to 0$

2. Know: m-step AB and AM are of order m and m+1, respectively.

$\Rightarrow$ They are consistent!

Stability of Multistep Methods

Consider m-step method:

$$\begin{cases} w_0 = \alpha, \quad w_1 = \alpha_1, \quad \ldots \quad w_{m-1} = \alpha_{m-1} \\ w_{i+1} = a_{m-1} w_i + \ldots + a_0 w_{i+1-m} \\ \qquad\qquad + hF(t_i, h, w_{i+1}, w_i, \ldots, w_{i+1-m}) \end{cases}$$

Its associated characteristic polynomial is given by:

(*) $$p(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \ldots - a_1\lambda - a_0$$

Def. (Root condition)

A multistep method satisfies the root condition if the roots λ_i of the associated characteristic polynomial $p(\lambda)$ satisfy:

- $|\lambda_i| \leq 1$ for any $i = 1, \ldots m$
- all roots with magnitude 1 are simple roots, i.e. if $p(\lambda_i) = 0$, then $p'(\lambda_i) \neq 0$.

Def. A multistep method with the associated characteristic polynomial $p(\lambda)$ is called:

1. **Strongly stable** if it satisfies the root condition and $p(\lambda)$ has $\lambda = 1$ as the only root with magnitude one.
2. **Weakly stable** if it satisfies the root condition and $p(\lambda)$ has more than one root with magnitude one.
3. **Unstable** if it does not satisfy the root condition.

Rmk. Stability of a m-step method easily verified using root condition.

Thm. 5.24 (Connection between convergence, consistency, and stability of multistep methods)

A multistep method is stable iff it satisfies the root condition.
Moreover, it is convergent iff it it is consistent and stable.

Rmk. AB/AM are convergent $\Leftrightarrow$ they satisfy the root condition.