Steffensen's Method

Let $\{\Delta^2\}(p_n) = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$

0) $p_0^{(0)}$	$p_1^{(0)} = g(p_0^{(0)})$	$p_2^{(0)} = g(p_1^{(0)})$
1) $p_0^{(1)} = \{\Delta^2\}(p_0^{(0)}) = p_0^{(0)} - \frac{(p_1^{(0)}-p_0^{(0)})^2}{p_2^{(0)}-2p_1^{(0)}+p_0^{(0)}}$	$p_1^{(1)} = g(p_0^{(1)})$	$p_2^{(1)} = g(p_1^{(1)})$
		g
2) $p_0^{(2)} = \{\Delta^2\}(p_0^{(1)})$	$p_1^{(2)} = g(p_0^{(2)})$	$p_2^{(2)} = g(p_1^{(2)})$
$\vdots$		
n) $p_0^{(n)} = \{\Delta^2\}(p_0^{(n-1)})$	$p_1^{(n)} = g(p_0^{(n)})$	$p_2^{(n)} = g(p_1^{(n)})$

Thm. Steffensen's Method

Suppose $g(x)=x$ has solution p with $g'(p)\neq 1$.
If there exists $\delta>0$ st $g\in C^3[p-\delta, p+\delta]$, then
Steffensen's Method gives quadratic convergence
for $p_0\in[p-\delta, p+\delta]$

2.6 Zeros of Polynomials

Def. A polynomial of degree n has the form
$p(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$

Rmk. - a_i's are coefficients of p
- $p(x)=0$ is a polynomial of degree 0

Thm. Fundamental Theorem of Algebra
If $p(x)$ is a polynomial of degree $n\geq 1$
with $a_i\in\mathbb{C}$, $i=0,1,\dots n$, then $p(x)$ has

at least one complex root.

Cor. 1 If $p(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then there exist unique $x_1, x_2, \dots x_k$ and unique $\underbrace{m_1, m_2, \dots m_k}_{\text{integers}}$ satisfying $\sum_{i=1}^{m} m_i = 0$ st

$$p(x) = a_n (x-x_1)^{m_1} (x-x_2)^{m_2} \cdots (x-x_k)^{m_k}$$

Rmk. Cor. 1 $\Rightarrow$ collection of zeroes of p_n are unique, and if each zero x_i counted as many times as its multiplicity m_i, then p_n has exactly n zeroes.

Cor. 2 Let $P(x)$ and $Q(x)$ be polynomials of degree at most n.
If $x_1, x_2, \dots x_k$ with $k > n$ are distinct numbers such that $P(x_i) = Q(x_i)$, $i = 1, \dots k$ then $P(x) = Q(x)$ for all values of x.

Rmk. To show 2 polynomials of degree at most n are the same, we only need to show that they agree on $n+1$ values.

Pf. sketch of Cor. 2:

$R(x) = P(x) - Q(x)$, $\deg(R) \leq n$; R has $n+1$ roots.

$$\rightarrow R(x) \equiv 0 \Rightarrow P(x) = Q(x)$$

Ex. If $P(x)$ with $\deg(P(x)) = n$, and $P(x_i) = x_i^n$ for $x_1 = 1$, $x_2 = 2$, $\dots$ $x_n = n$, $x_{n+1} = n+1$, then $P(x) = x^n$ (by Cor. 2).