(X, d)

$E \subseteq X$ is open if $x \in E \Leftrightarrow \exists\, r > 0$ s.t. $B(x, r) \subseteq E$

E is closed if $X \setminus E$ is open

a set w/ only one pt

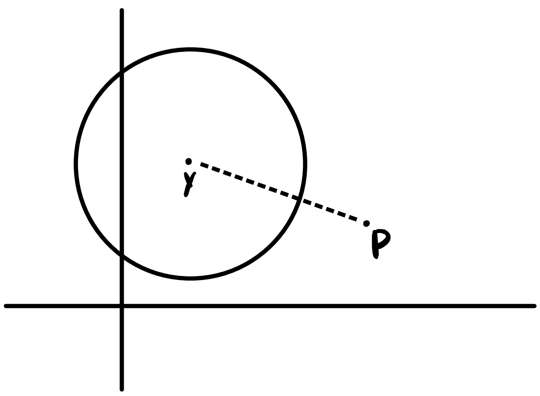
Prop. Every singleton is closed.

Let (X, d) be a m.s. and $P \in X$. Consider the set $S = \{P\}$. Then S is closed.

Pf. By definition, need to show $X \setminus \{P\}$ is open.

If $y \in X \setminus \{P\}$, then $y \neq P$.

Hence $d(P, y) > 0$.



let $r_i = \frac{d(y, P)}{2}$

then $B(y, r) \subseteq X \setminus \{P\}$

(works $\forall\, y \Rightarrow X \setminus \{P\}$ is open)

Prop. Let d be the discrete metric on X.

Then every subset $E \subseteq X$ is open and closed.

Pf. Let $E \subseteq X$, $x \in E$.

$B(x, \frac{1}{2}) = \{x\}$

$\Rightarrow E$ is open.

Since this argument works for all subsets of X, E^c is open.

$\Rightarrow E$ is closed.

In $(\mathbb{R}, d)$:

- (a, b) open, not closed.
- $[a, b]$ closed, not open
- $(a, b]$, $[a, b)$ not open, not closed

Prop. Let (X, d) be a m.s.

① ϕ and X are always open and closed.

② if $\{A_i : i \in I\}$ is any collection of open sets, then $\bigcup_i A_i$ is open.

③ if $\underbrace{A_1, A_2, \dots A_n}_{\text{finite}}$ are open, then $\bigcap_{i=1}^{n} A_i$ is open.

Pf. ① ϕ is open.

If $x \in X$ then $\forall\, r > 0$, $B(x, r) \subseteq X$, then X is open.

Moreover, $X^c = \phi$ is open.

$\Rightarrow X$ is closed.

② Let $x \in \bigcup_{i \in I} A_i \Rightarrow \exists\, i \in I$ s.t. $x \in A_i$, then $\exists\, r > 0$ s.t.

$B(x, r) \subset A_i \subseteq \bigcup_{i \in I} A_i$

$\Rightarrow \bigcup_{i \in I} A_i$ is open.

③ Let $x \in \bigcap_{i=1}^{n} A_i$. Then $x \in A_i$ for all $i \in \{1, 2, \dots n\}$.

Since A_i is open for all $i \in \{1, 2, \dots n\}$,

$\exists$ r_i s.t. $B(x, r_i) \subseteq A_i$.

$r := \min\{r_i : 1 \le i \le n\}$

Since $r \le r_i$ $\forall$ $1 \le i \le n$, $B(x, r) \subset B(x, r_i) \subset A_i$ $\forall$ $1 \le i \le n$.

$\Rightarrow$ $B(x, r) \subset \bigcap_{i=1}^{n} A_i$.

$\Rightarrow$ $\bigcap_{i=1}^{n} A_i$ is open.

Ex. Consider $X = \mathbb{R}$.

$d(x, y) := |x - y|$

$A_n := (-\frac{1}{n}, \frac{1}{n})$

$\bigcap_{i=1}^{\infty} A_i = \{0\}$ is closed and not open.

$(\bigcap_{i \in I} F_i)^c = \bigcup_{i \in I} F_i^c$ is open if F_i^c is open.

$\Updownarrow$

F_i is closed.

Cor. Any $\cap$ of closed sets is closed.

Any finite $\cup$ of closed sets is closed.

Prop. A set E in a metric space is open $\Leftrightarrow$ it is the union of open balls.

Pf. $\Rightarrow$ Using the fact that any open ball is open, and any arbitrary union of open sets is open, we conclude that it is a union of open balls is open

$\Leftarrow$

Let $x \in E$. Since E is open, $\exists$ $r_x > 0$ s.t. $B(x, r_x) \subseteq E$.

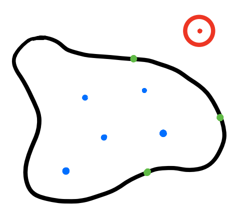
Then $\bigcup_{x \in E} B(x, r_x) \subseteq E = \bigcup_{x \in E} \overset{\in B(x,r_x)}{x} \subseteq \bigcup_{x \in E} B(x, r_x)$

$$\Rightarrow E = \bigcup_{x \in E} B(x, r_x)$$

Def. Let (X, r_X) be a metric space, $E \subseteq X$, $x_0 \in X$.

We say that x_0 is an adherent point of E if $\forall$ $r > 0$: $B(x_0, r) \cap E \neq \phi$.

$E \subseteq X$



: exterior points

, : adherent points

y is an exterior point of E

$\Leftrightarrow$ $\exists$ $r > 0$ s.t. $B(y, r) \subseteq X \setminus E$

$\Leftrightarrow$ $\exists$ $r > 0$ s.t. $B(y, r) \cap E = \phi$

Prop. Let (X, r_X) be a metric space, $E \subseteq X$, $x_0 \in X$.

The following are equivalent:

1. x_0 is an adherent point of E.
2. x_0 is not an exterior point of E.
3. x_0 is an interior point or a boundary point of E.
4. $\exists$ a sequence $(x_n) \in E$ s.t. $\lim_{n\to\infty} x_n = x_0$.

Pf. $1 \Leftrightarrow 2 \Leftrightarrow 3$.

$1 \Rightarrow 4$: Let x_0 be an adherent point of E.

By def. $\forall\, r>0$ we have that $B(x_0, r) \cap E \neq \phi$.

Let $r_n = \frac{1}{n}$, then $\forall\, n \in \mathbb{N}$ we have that $B(x_0, r_n) \cap E \neq \phi$.

$\Rightarrow \exists\, x_n \in B(x_0, r_n) \cap E$

$\Rightarrow x_n \in E$ and $x_n \in B(x_0, r_n)$

$(x_n) \subseteq E$ $\qquad$ $d(x_n, x_0) < r_n$ $\qquad \frac{1}{n}$

$\Downarrow$

$x_n \to x_0$ as $n \to \infty$

$4 \Rightarrow 1$: Let $(x_n) \subseteq E$ s.t. $x_n > x_0$.

$\forall\, r>0$ there exist $N \in \mathbb{N}$ s.t. $d(x_n, x_0) < r$ $\forall\, n \geq N$.

In particular, $d(x_N, x_0) < r$

$\Rightarrow x_N \in B(x_0, r)$

Moreover, $x_N \in E$.

$\Rightarrow B(x_0, r) \cap E \neq \phi$

$\Rightarrow x_0$ is an adherent point of E.

Def. The set of all adherent points of a set E in a metric space (X,d) is called the closure of E, and denoted by $\bar{E}$.

Prop. Let (X,d) be a metric space, $E \subseteq X$.
The following are equivalent:

1. E is closed.
2. $E = \bar{E}$
3. If (x_n) is a convergent sequence in E, then $\lim_{n\to\infty} x_n \in E$.

Pf. $1 \Leftrightarrow 2$.

$$\begin{aligned} E \text{ closed} &\Leftrightarrow X \setminus E \text{ open.} \\ &\Leftrightarrow X \setminus E = \operatorname{int}(X \setminus E) \\ &\Leftrightarrow X \setminus E = \operatorname{ext}(E) \\ &\Leftrightarrow X \setminus \operatorname{ext}(E) = E \end{aligned}$$

(X,d) m.s.

$E \subseteq X$

$$\operatorname{int}(E) \subseteq E \subseteq \bar{E} = \operatorname{int}(E) \cup \underbrace{\partial E}_{\text{boundary}}$$

$\operatorname{int}(E) = E$ $\Updownarrow$ E is open

$E = \bar{E}$ $\Updownarrow$ E is closed

Prop. The closure of any set is closed.

Pf. Let $E \subseteq X$.

(wts: $\bar{E}$ is closed, $\bar{E} = \bar{\bar{E}}$)

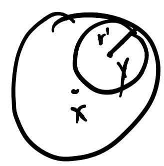
$\subseteq$: We know that $\bar{E} \subseteq \bar{\bar{E}}$.

$\supseteq$: Suppose $x \in \bar{\bar{E}}$.

$\Rightarrow$ x is an adherent point of $\bar{E}$.

$\Rightarrow \forall\ r>0,\ B(x,r) \cap \bar{E} \neq \phi$.

$\Rightarrow \exists\ y \in B(x,r) \cap \bar{E}$.



$y \in B(x,r) \Rightarrow d(x,y) < r$.

Define $r' := r - d(x,y)$.

$B(x,r') \subseteq B(x,r)$.

$y \in \bar{E} \Rightarrow$ y is an adherent point of E.

$\Rightarrow B(y,r') \cap E \neq \phi$

$\Rightarrow \exists\ z \in B(y,r') \subseteq B(x,r)$.

$\Rightarrow z \in B(x,r)$.

$$
\begin{aligned}
d(z,x) &\leq d(z,y) + d(y,x) \\
&< r' + d(x,y) \\
&= r
\end{aligned}
$$

$\Rightarrow z \in B(x,r)$.

$\Rightarrow B(x,r) \cap E \neq \phi \quad \forall\ r>0$

$\Rightarrow x \in \bar{E}$

$\Rightarrow \bar{\bar{E}} \subseteq \bar{E}$

Since $\bar{E} = \bar{\bar{E}}$, $\bar{E}$ is closed.

Relative Notions

Def. (X,d) m.s. and $Y \subseteq X$.

Denote by d_Y the metric of X restricted to $Y \times Y$.

This makes $(Y, d_{Y\times Y})$ a m.s. In this case,

we say $E \subseteq Y$ is relatively open (relatively closed) if

E is open (closed) in $(Y, d_{Y\times Y})$.

Thm. (X,d) m.s., $Y \subseteq X$, $Y \neq \phi$.

Then for every $E \subseteq Y$,

1. E is relatively open $\Leftrightarrow$ $\exists\ A \subseteq X$ open s.t. $E = A \cap Y$.
2. E is relatively closed $\Leftrightarrow$ $\exists\ B \subseteq X$ s.t. $E = B \cap Y$.

Pf. 1. E is open in (y, d_Y) $\Leftrightarrow$ $\exists\ A \subseteq X$ open s.t. $E = A \cap Y$.

Denote $B_X(x,r) = \{z \in X,\ d(x,z) < r\}$

and $B_Y(x,r) = \{z \in Y,\ d(x,z) < r\}$

Note: $B_Y(x,r) = B_X(x,r) \cap Y$.

$\Leftarrow$

If $E = A \cap Y$ where $A \subseteq X$ is open

$x \in E \Rightarrow x \in A$.

Since A is open in X, $\exists\ r_x > 0$ s.t. $B_X(x, r_x) \subset A$.

$$\begin{aligned} B_Y(x, r_x) &= B_X(x, r_x) \cap Y \\ &\subseteq A \cap Y \\ &= E. \end{aligned}$$

$\Rightarrow$ E is open in (y, d_Y)

$\Rightarrow$

If E is open in (Y, d_Y)

$\forall\ x \in E\ \exists\ r_x > 0$ s.t. $B_Y(x, r_x) \subseteq E$.

$E = \bigcup_{x\in E} B_Y(x, r_x)$

Define $A := \bigcup_{x \in E} B_X(x, r_x)$. A is open in X.
(union of open sets)

$$
\begin{aligned}
Y \cap A &= Y \cap \bigcup_{x\in E} B_X(x, r_x) \\
&= \bigcup_{x \in E} Y \cap B_X(x, r_x) \\
&= E.
\end{aligned}
$$

Ex. $X = \mathbb{R}$ $\quad d = |\cdot|$

1. $Y = \mathbb{Q}$

- $E = \underbrace{[0, 1]}_{B} \cap \mathbb{Q} = B \cap Y$ is relatively closed.
  closed in $(\mathbb{Q}, |\cdot|)$

- $E = [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$ is relatively closed and relatively open.

  $\sqrt{2}, \sqrt{3} \notin \mathbb{Q}$

  $\Rightarrow [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q} = (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$
  relatively open

2. $Y = [0, 2)$

   $E = [1, 2) = Y \cap [1, 2]$ is relatively closed.