(X,d) $(\mathbb{R}, |\cdot|)$

Prop. $f: X \to \mathbb{R}$ continuous.

(a) If X is compact, f attains its min and max.

(b) If X is connected, $a, b \in X$ and $f(a) \le y \le f(b)$ then $\exists$ $c \in X$ s.t. $f(c) = y$. (IVT)

Pf. (a) $f(X) \subseteq \mathbb{R}$

$y := \sup f(X)$

$\exists$ $(x_n) \subset X$ s.t. $\lim_{n\to\infty} f(x_n) = y$.

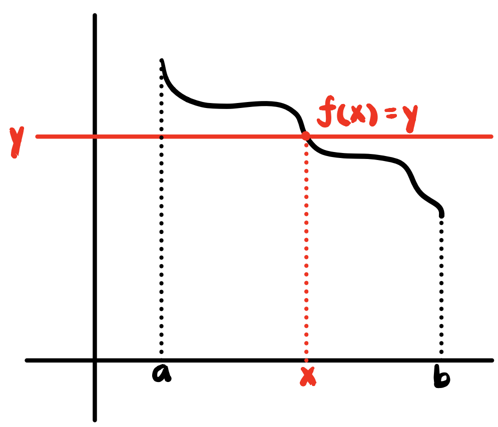
Since X is compact, $\exists$ $(x_{n_k}) \subseteq (x_n)$ and $x \in X$ s.t. $\lim_{n_k\to\infty} x_{n_k} = x$.

f continuous $\Rightarrow \lim_{n_k\to\infty} f(x_{n_k}) = f(x)$,, y

$\Rightarrow f(x) = y = \sup f(X)$

$\Rightarrow f(x) = \max_{z\in X} f(z)$. ⊠

(b)



Assume $\nexists$ $c \in X$ s.t. $f(c) = y$.

Then $\forall$ $x \in X$, $f(x) > y$ or $f(x) < y$.

Define $A := \{x \in X : f(x) < y\} = f^{-1}(-\infty, y)$

$B := \{x \in X : f(x) > y\} = f^{-1}(y, \infty)$

$\Rightarrow a \in A, \quad b \in B$

$\Rightarrow A \neq \emptyset, \quad B \neq \emptyset$

Since f is continuous, the preimage of any open set is open, then A and B are open.

Moreover, $A \cap B \neq \emptyset$.

$$X = f^{-1}(f(X)) = f^{-1}(-\infty, y) \cup f^{-1}(y, \infty)$$
$$= A \cup B$$

$\Rightarrow X$ is disconnected. $\lightning$

$(X, d) \quad a, b \in X$

exercise: a function from a connected m.s. to a discrete m.s. is continuous $\Leftrightarrow$ it is constant.

Thm. Connected components partition X.

i.e. they are disjoint and their union is X.

In particular, every point $x \in X$ is contained in a unique connected component.

Pf. $\forall \ x \in X$, let $C(x) := \bigcup \{C : x \in C \text{ and } C \text{ is connected}\}$

C is connected $\forall \ x \in X$.

Def. We say $E \subseteq X$ is a connected component of X if E is connected and if $E \subsetneq Y \subseteq X$, then Y is discrete.

"E is maximally connected"

If $C(X) \cap C(Y) \neq \emptyset$, then $C(X) \cup C(Y)$ is connected.

Observe that $x \in C(X) \subseteq C(X) \cup C(Y)$

$$\Rightarrow C(X) \subseteq C(X) \cup C(Y) \subseteq C(X)$$

$$\Rightarrow C(X) = C(X) \cup C(Y)$$

Similarly, $C(Y) = C(X) \cup C(Y)$. ∎

Def. Let (X, d_X), (Y, d_Y) be m.s., (f_n) be a sequence of functions $(f_n : X \to Y)$, $f : X \to Y$.
We say **f_n converges to f pointwise** on X if $\forall\ x \in X$ $\lim_{n\to\infty} f_n(x) = f(x)$.

$\left(\forall\ x \in X\ \exists\ N(\varepsilon, x) \in \mathbb{N}\ \text{ s.t. } d_Y(f_n(x), f(x)) < \varepsilon \text{ for all } n > N. \right)$

Ex. $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f : \mathbb{R} \to \mathbb{R}$

$f_n(x) := \frac{x}{n}$ $\quad f(x) = 0$

For fixed $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = x \lim_{n\to\infty} \frac{1}{n}$

$= 0 = f(x)$

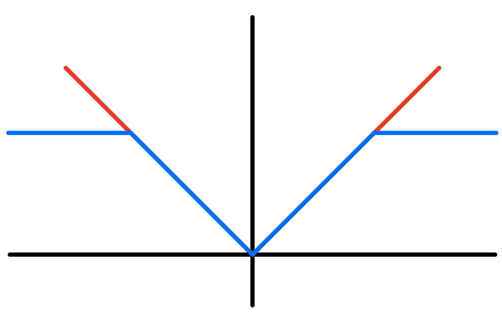
Ex. $f_n : \mathbb{R} \to \mathbb{R}$ $\quad f : \mathbb{R} \to \mathbb{R}$

$x \mapsto \min\{|x|, n\} \leq n$ $\quad x \mapsto |x|$

If x is fixed, then

$\lim_{n\to\infty} \min\{|x|, n\} = |x|$

$\Rightarrow f_n$ converges to f pointwise



- f_n is bounded above $\forall\ n$
- f is not bounded

Ex. $f_n : [0,1] \to \mathbb{R}$ $\quad x \mapsto x^n$

$f : [0,1] \to \mathbb{R}$

$1 \mapsto 1$

$x \mapsto 0 \quad \forall\, x \neq 1$

If x is fixed, then

$$\lim_{n\to\infty} x^n = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ f_n converges to f pointwise (but not uniformly)

- f is not continuous
- f_n is continuous $\forall\, n \in \mathbb{N}$

Ex. $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) := \begin{cases} 2n & \text{if } \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

$f : [0,1] \to \mathbb{R}$

$x \mapsto 0$

$\int_0^1 f(x)\,dx = 0$

$\lim_{n\to\infty} f_n(x) = 0 \qquad \forall\, n > \frac{1}{x}, \quad f_n(x) = 0$

$\Rightarrow$ f_n converges to f pointwise

$$\begin{aligned}
\int_0^1 f_n(x)\,dx &= \overset{0}{\int_0^{\frac{1}{2n}} f_n(x)\,dx} + \int_{\frac{1}{2n}}^{\frac{1}{n}} f_n(x)\,dx + \overset{0}{\int_{\frac{1}{n}}^1 f_n(x)\,dx} \\
&= \int_{\frac{1}{2n}}^{\frac{1}{n}} f_n(x)\,dx = \int_{\frac{1}{2n}}^{\frac{1}{n}} 2n\,dx \\
&= 2n\left(\frac{1}{n} - \frac{1}{2n}\right) \\
&= 1
\end{aligned}$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1 \neq \int_0^1 f(x)\,dx = 0$$

Def. Given f and $(f_n): X \to Y$, we say that f_n converges uniformly to f on X if $\forall\ \varepsilon > 0\ \exists\ N(\varepsilon) \in \mathbb{N}$ s.t. $\forall\ n \geq N$, $\forall\ x \in X$, $d_Y(f_n(x), f(x)) < \varepsilon$.

Rmk. uniform convergence $\underset{\not\Leftarrow}{\Rightarrow}$ pointwise convergence

Thm. If $f_n \to f$ uniformly on X and every f_n is continuous at x_0, then also f is continuous at x_0.

In particular, uniform limit of continuous functions is continuous.

Pf. $\forall\ \varepsilon > 0$, $\exists\ N \in \mathbb{N}$: $\forall\ n \geq N$ $\forall\ x \in X$, $d_Y(f_n(x), f(x)) < 1$.

By assumption, f_n is continuous at x_0

$\Rightarrow \forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon, x_0) > 0$ s.t.

$$d_X(x, x_0) < \delta \text{ implies } d_Y(f(x_0), f(x)) < \varepsilon.$$

$$d_Y(f(x), f(x_0)) \leq \overbrace{d_Y(f(x), f_N(x))}^{<\varepsilon} + \overbrace{d_Y(f_N(x), f_N(x_0))}^{<\varepsilon} + \overbrace{d_Y(f(x_0), f_N(x))}^{<\varepsilon}$$

$$< 3\varepsilon \quad \forall\ x \text{ s.t. } d_X(x, x_0) < \delta$$

So f is continuous.

Thm. If $f_n \to f$ uniformly on X and f_n is f_n is bounded for every $n \in \mathbb{N}$, then f is also bounded.

($f: X \to Y$ is bounded if $f(X)$ is bounded)

Pf. For $\varepsilon = 1$, $\exists\ N \in \mathbb{N}\ \ \forall\ n \geq N\ \ \forall\ x \in X$,
$d_Y(f_n(x), f(x)) < 1$.

Since f_N is bounded, $\exists\ y \in Y$ and $r > 0$ s. t. $f_N(X) \subseteq B(y, r)$.

$$\Rightarrow\ d_Y(f_n(x), y) < 1 \qquad \forall\ x \in X$$

$$\Rightarrow\ d_Y(f(x), y) \leq d_Y(f_n(x), y) + d_Y(f_n(x), f(x)) < 1 + r \qquad \forall\ x \in X$$

$$\Rightarrow\ f(X) \subseteq B(y, 1+r).$$

So f is bounded.

Prop. Suppose $f_n \to f$ uniformly on X, f_n is continuous for all $n \in \mathbb{N}$, $(x_n) \subset X$ and $x \in X$ s.t. $\lim_{n\to\infty} x_n = x$.
Then $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Pf. By previous Thm., f is continuous $\Rightarrow$
$\forall \ \varepsilon > 0 \ \exists \ \delta > 0$ s.t. $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.
Since $\lim_{n\to\infty} x_n = x$, $\exists \ N_1 \in \mathbb{N}$ s.t. $d_X(x, x_n) < \delta \ \forall \ n \geq N_1$.
Combining the two previous observations, we conclude that

$$d_Y(f(x), f(x_n)) < \varepsilon \quad \forall \ n \geq N_1.$$

By uniform convergence, $\exists \ N_2 \in \mathbb{N}$ s.t. $\forall \ n \geq N_2$ $\forall \ w \in X$, $d_Y(f(w), f_n(w)) < \varepsilon$.
Let $N = \max(N_1, N_2)$.

$$d_Y(f(x), f_n(x_n)) \leq \underbrace{d_Y(f(x), f(x_n))}_{<\varepsilon} + \underbrace{d_Y(f(x_n), f_n(x_n))}_{<\varepsilon} < 2\varepsilon$$

So $\lim_{n\to\infty} f_n(x_n) = f(x)$.

(X, d_X) (Y, d_Y) $(f_n): X \to Y$, $f: X \to Y$

We say that (f_n) converges pointwise to f if $\forall\ x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$.

- pointwise limit of bounded functions is not necessarily bounded.
- pointwise limit of continuous functions is not necessarily continuous.

We say that (f_n) converges uniformly to f if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$, $\forall\ x \in X$, $d_Y(f(x), f_n(x)) < \varepsilon$.