2.4 Convergence Order

Def. Suppose $p_n \to p$ as $n \to p$, $p_n \neq p$ for all n.

If λ, α exist with $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|^\alpha} = \lambda$

then $\{p_n\}$ converges to p with order α and asymptotic error constant λ.

Rmk. 1) Similar definition: $|p_{n+1}-p| \le \lambda |p_n - p|^\alpha$ for large n

2) α reflects convergence speed more than λ.

larger $\alpha \to$ faster convergence

3) if $\cdot\ \alpha = 1$ $(\lambda < 1)$ $\to$ $\{p_n\}$ converges linearly

$\cdot\ \alpha = 2$ $\to$ $\{p_n\}$ converges quadratically

4) different from $O(n^{-p})$ where $|p_n - p| \le k n^{-p}$

Ex. Assume $p_n \to 0$ as $n \to \infty$ with $\lim_{n\to\infty} \frac{|p_{n+1}|}{|p_n|} = \frac{1}{2}$

and $q_n \to 0$ as $n \to \infty$ with $\lim_{n\to\infty} \frac{|q_{n+1}|}{|q_n|^2} = \frac{1}{2}$

Then for all n:

$|p_{n+1}| \approx \frac{1}{2}|p_n| \approx \dots \approx (\frac{1}{2})^{n+1} |p_0|$

$|q_{n+1}| \approx \frac{1}{2}|q_n|^2 \approx \dots \approx (\frac{1}{2})^{2^{n+1}-1} \cdot |q_0|^{2^{n+1}}$

If $p_0 = q_0 = 1$, we can see that $|q_{n+1}| \ll |p_{n+1}|$

$\Rightarrow \{q_n\}$ converges much faster than $\{p_n\}$

Thm. Convergence Order of Bisection

Let $f \in C([a,b])$, $f(a)\cdot f(b) < 0$, $a_1 = a$, $b_1 = b$.

Then the sequence $\{p_n\}$ generated by Bisection converges linearly to the root of $f(x)$.

Pf. Let p be the root of $f(x)$ with $p \in [a,b]$.

Recall that $|p_n - p| \le |b-a| \cdot (\frac{1}{2})^n$ for Bisection.

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} \frac{(\frac{1}{2})^{n+1}}{(\frac{1}{2})^n} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \frac{1}{2}$$

$\Rightarrow$ linear convergence.

Convergence Order of Fixed Point Iteration

Thm 1. Let 1) $g(x) \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$

2) $g(x) \in C'[a,b] \leq k < 1$ for all $x \in (a,b)$

If $g'(p) \neq 0$, then for all $p_0 \neq p$ and $p_0 \in [a,b]$ the sequence $\{p_n\}$ generated by $p_n = g(p_{n-1})$ converges linearly to the unique fixed point $p \in [a,b]$

Pf. By Fixed Point Thm., we know $p_n \to p$ as $n \to \infty$

By MVT, $p_{n+1} - p = g(p_n) - g(p)$

$= g'(\xi_n)(p_n - p)$ by MVT

ξ_n between p_n and p

Since $p_n \to p$ as $n \to \infty$, we have $\xi_n \to p$ as $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lim_{n\to\infty} |g'(\xi)| = |g'(p)| < 1$$

Thus, if $g'(p) \neq 0$, then p_n approaches p linearly with asymptotic error constant $g'(p)$.

Note: Higher order of convergence for fixed point methods can be achieved under additional assumptions.

Thm 2. Let p be a solution of $g(x)$.

Assume $\cdot\, g'(p) = 0$, and

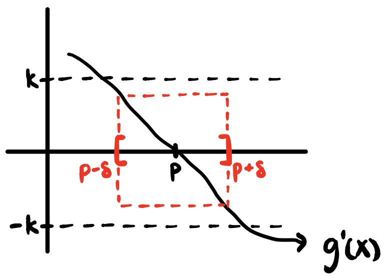
$\cdot\, g''$ continuous with $|g''(x)| < M$ on an open interval I containing p

Then there exists $\delta > 0$ such that for any $p_0 \in [p-\delta, p+\delta]$, the sequence $p_n = g(p_{n-1})$ converges at least quadratically to p, i.e. $|p_{n+1} - p| < \frac{M}{2} |p_n - p|^2$ for all $n \geq n_0$

Pf. ① Show that fixed point iteration converges under given assumptions.

By continuity of $g'(x)$, we can choose $k \in [0, 1]$ and $\delta > 0$ such that:

$$\left.\begin{array}{l} \text{(i) } [p-\delta, p+\delta] \subseteq I, \text{ and} \\ \text{(ii) } g'(p) = \lim_{x \to p} g'(x) = 0 < k \end{array}\right\} \Rightarrow |g'(x)| \leq k \text{ for } x \in [p-\delta, p+\delta]$$



given k, we can find δ st $x \in [p-\delta, p+\delta] \Rightarrow |g'(x)| \leq k$

know: $|g'(x)| \leq k < 1$ on $[p-\delta, p+\delta]$

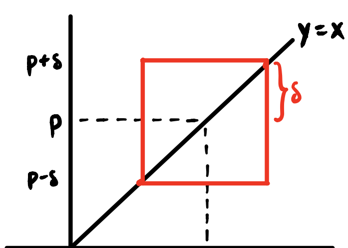
Next, need to show g maps to itself on $[p-\delta, p+\delta]$.

Let $x \in [p-\delta, p+\delta]$.

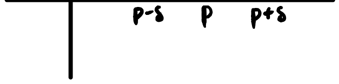
$$\begin{aligned} |g(x) - p| &= |g(x) - g'(x)| \\ &= |g(\xi)| \cdot |x-p| \quad \text{by MVT} \\ &\leq k|x-p| \\ &< |x-p| \quad (*) \end{aligned}$$

$x \in [p-\delta, p+\delta] \Rightarrow |x-p| < \delta$

$\Rightarrow |g(x) - p| < \delta$ by $(*)$



Thus, g maps $[p-\delta, p+\delta]$ into $[p-\delta, p+\delta]$.



Thus, $g(x) \in [p-\delta, p+\delta]$ for all $x \in [p-\delta, p+\delta]$.

$\Rightarrow$ By fixed point thm., g converges to unique sol.