Horner's Method

Newton's Method to find zeros of $p(x)$

$\Rightarrow$ need to evaluate $p(x)$ and $q(x)$ repeatedly

Note p and p' both polynomials

Q: How can we evaluate polynomials efficiently?

Idea nested evaluation / synthetic division

Thm. Horner's Method

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Define $b_n = a_n$

$$b_k = a_k + b_{k+1} x_0, \quad k = n-1, n-2, \dots 1, 0$$

Then $b_0 = P(x_0)$.

Moreover, if $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + b_2 x + b_1$,

then $P(x) = (x - x_0) Q(x) + b_0$.

Pf. $(x - x_0) \cdot Q(x) + b_0$

$= (x - x_0)[b_n x^{n-1} + b_{n-1} x^{n-2} + \dots + b_2 x + b_1] + b_0$

$= [b_n x^n + b_{n-1} x^{n-1} + \dots + b_2 x^2 + b_1 x] - [b_n x_0 x^n + \dots + b_2 x_0 x + b_1 x_0] + b_0$

$= b_n x^n + (b_{n-1} - b_n x_0) x^{n-1} + \dots + (b_1 - b_2 x_0) x + (b_0 - b_1 x_0)$

By hypothesis, $b_n = a_n$, $b_k - b_{k+1} x_0 = a_k$

$= a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$= P(x)$

$\Rightarrow P(x) = (x - x_0) Q(x) + b_0$ and $P(x_0) = b_0$

Rmk. - can use $b_k = a_k + b_{k+1} x_0$, $k = n-1, \dots 1, 0$ $(b_n = a_n)$

to evaluate P at x_0 in nested manner

- for P_n ($\deg \le n$), require at most n multiplications and n additions

Ex. Use Horner's Method to evaluate

$$P(x) = 2x^4 - 3x^2 + 3x - 4 \quad \text{at} \quad x_0 = -2$$

Sol.

$x_0=-2$	$a_4=2$	$a_3=0$	$a_2=-3$	$a_1=3$	$a_0=-4$
		$b_4x_0=-4$	$b_3x_0=8$	$b_2x_0=-10$	$b_1x_0=14$
	$b_4=2$	$b_3=-4$	$b_2=5$	$b_1=-7$	$b_0=10$ $\shortparallel$ $P(x_0)$

Rmk.

1. $P(x) = (((2x+0)-3)+3)x+4$ ← 4 nested mult, 3 add
2. Let $P(x) = (x-x_0)Q(x) + b_0$. Then $P'(x_0) = Q'(x_0)$.
   This means we can obtain $P'(x_0)$ by applying Horner's Method to $Q(x)$.
   (new iterates $c_k = b_k + c_{k+1}x_0$)

   Here, $c_n = b_n$

   $$\begin{array}{ccccc} a_n & \to & b_n & \to & c_n \\ a_{n-1} & \to & b_{n-1} & \to & c_{n-1} \\ & & \vdots & & \\ a_1 & \to & b_1 & \to & c_1 \\ a_0 & \to & b_0 & \to & c_0 \end{array}$$

3. If $f(x) = P(x)$ in Newton's Method, then we can use Horner's Method (aka synthetic division) to compute $f(x_0)$ and $f'(x_0)$ more efficiently.
   From implementation perspective:

   $$[b_0, c_0] = \text{Horner's}(\underline{a}, x_0) \qquad \underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

4. Can be interpreted as a (linear) neural network!

$NN(x_0) = u_n(x_0)$ where $u_{k+1} = \sigma(u_k x_0 + a_k)$

nonlinear activation function

$b_k = a_k + b_{k+1} x_0$ vs. $b_k = \sigma(a_k + b_{k+1} x_0)$

2.6 Deflation

Def. Procedure to find all zeros of polynomials succesively by applying Newton's Method.

1) Choose initial guess $p_0^{(1)}$ and find approximate zero $\hat{x}_1$ of $P_n(x)$ with degree n using Newton's.
2) Find zeros of $Q(x)$ and get $\hat{x}_2$.
   To find $\hat{x}_2$, choose initial guess $p_0^{(k)}$ and apply Newton's to $Q_k(x)$

$$P_n(x) \approx (x-\hat{x}_1)(x-\hat{x}_2)\,Q_2(x) \qquad k=2$$

$$\vdots$$

$$P_n(x) \approx (x-\hat{x}_1)(x-\hat{x}_2)\cdots(x-\hat{x}_{n-2})\,Q_{n-2}(x) \qquad k=n-2$$

   $Q_{n-2}(x)$ quadratic $\rightarrow$ use quadratic formula
3) Obtain refined solutions $x_1, x_2, \dots x_n$ by applying Newton's Method to $p_n(x)$ with initial guesses $\hat{x}_1, \hat{x}_2, \dots \hat{x}_n$, respectively.

Rmk. Inaccuracy increases as k increases in step 2. Reduce error via step 3.