Lenny Wu

**Ex.** Show that the $4^{th}$ order Milne's method given by

$$w_{i+1} = w_{i-3} + \frac{4h}{3}\left[2f(t_i, w_i) - f(t_{i-1}, w_{i-1}) + 2f(t_{i-2}, w_{i-2})\right]$$

is only weakly stable.

**Pf.** The characteristic polynomial of this method is:

$$p(\lambda) = \lambda^4 - 1 = (\lambda+1)(\lambda-1)(\lambda+i)(\lambda-i) = 0$$

$$\rightarrow \lambda = \pm i, \pm 1$$

Since the root condition is satisfied, the method is stable. Additionally, there are multiple roots with magnitude 1 so the method is only weakly stable. $\square$

# 5.11 Stiff Differential Equations

Consider the IVP

$$\begin{cases} y'(t) = -15y & 0 \le t \le 0.5 \\ y(0) = 1 \end{cases}$$

The exact solution is $y(t) = e^{-15t}$ with $y(t) \to 0$ as $t \to \infty$.

- Euler's Method with $h = \frac{1}{4}$ yields

$$w_{i+1} = w_i - 15hw_i$$
$$= (1 - 15h)w_i$$
$$\to w_i = (-2.75)^i, \quad i = 0, 1, \dots N$$

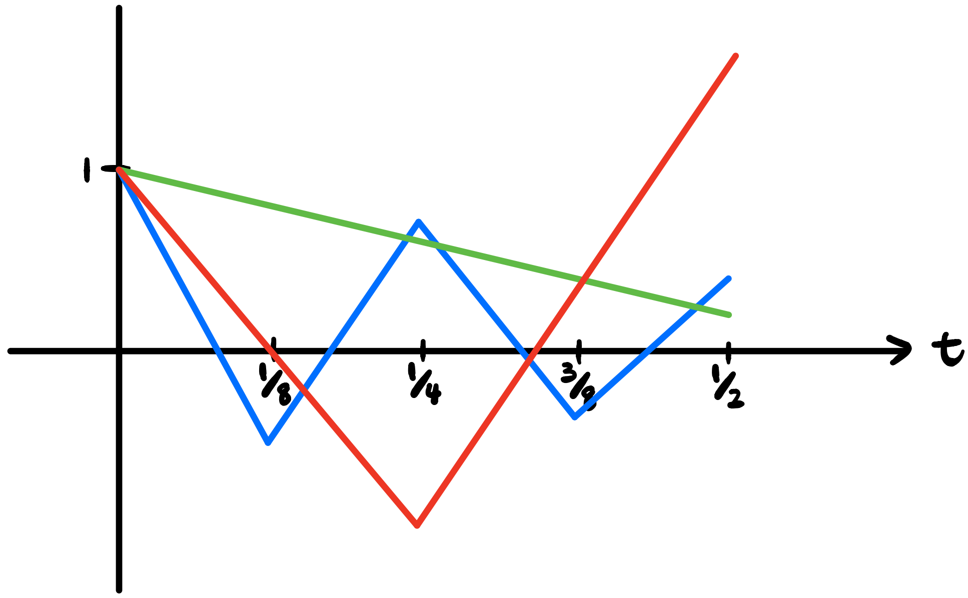
which oscillates wildly and grows quickly.

- Euler's Method with $h = \frac{1}{8}$ yields

$$w_i = (-0.875)^i$$

which oscillates and approaches 0.

- AM-2 with $h = \frac{1}{8}$ produces a solution that decreases monotonically to 0, just as the exact solution does.




**Def.** **Stiff differential equations** are DEs for which some numerical methods are unstable, unless the step size is extremely small.

1. The exact solution has a fast decaying term, of the form $e^{-ct}$ where $c>0$ is large (so that $\frac{d^n}{dt^n}[e^{-ct}]$ decays slower than $e^{-ct}$)
2. Different components of the IVP system evolve on different time scales.

   e.g. $y' = 9y + 5\cos(t)$, $y(0) = \frac{4}{3}$

The fast decaying part of the solution is the transcient solution, and the rest is the steady-state solution.

more important

The approximation properties of a certain numerical method applied to stiff DE can be predicted by examining the error applying the method to the test IVP

$$\begin{cases} y' = \lambda y & t \geq 0,\ \lambda < 0 \\ y(0) = \alpha \end{cases}$$

The exact solution is $y(t) = \alpha e^{\lambda t}$, which has the transcient solution $e^{\lambda t}$ and steady-state solution 0.

Goal: Find when the numerical solutions of a numerical method correctly approximate the solution of the test IVP, i.e. it tends to 0 as $t \to \infty$ when $\lambda < 0$.

Rmk. Want to find a stable $h_0$ for numerical methods s.t. $h < h_0 \Rightarrow$ method stable.

**Def.** The region of absolute stability $R$ of a method is the set

$$R = \{h\cdot\lambda \in \mathbb{C} \mid w_i \to 0 \text{ as } i \to \infty\}$$

where $\{w_i\}$ is generated by applying the method to the IVP.

**Def.** A method is:

- A-stable if its region of abs. stability contains the entire left half plane.
- L-stable if it is A-stable AND at each fixed $t_i$, the numerical solution $w_i$ satisfies $w_i \to 0$ as $Re(\lambda) \to -\infty$.

(Re($\lambda$) always $< 0$)

## Region of Absolute Stability (RAS) for one-step methods

**Ex.** Show that the RAS of Euler's method is

$$R = \{h\cdot\lambda \mid |1+h\lambda| < 1\}$$

which implies $h < \frac{2}{|\lambda|}$

Pf. Apply Euler's method with step size h to the test IVP $y'=\lambda y$, $Re(\lambda)<0$:

$$w_{i+1}=w_i+h\lambda w_i=(1+h\lambda)w_i$$

$$\Rightarrow \quad w_i=(1+h\lambda)^i\cdot\alpha$$

Since the exact sol. is $y(t)=\alpha e^{\lambda t}$, error at $t=i\cdot h$ is:

$$|y(t_i)-w_i|=|e^{ih\lambda}-(1+h\lambda)^i|\cdot|\alpha|$$

Since $Re(\lambda)\to 0$, we know $e^{(h\lambda)j}\to 0$ as $j\to 0$.

To ensure $|y(t_i)-w_i|\to 0$ as $t_i\to\infty$, we need $w_i\to 0$.

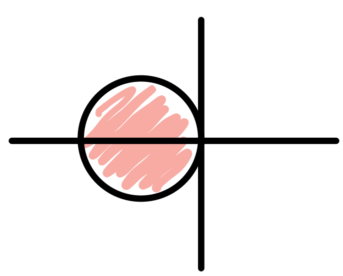
It follows that

$$|1+h\lambda|<1$$

$$\text{i.e.,} \quad -2<h\lambda<0$$

$$h<\frac{2}{|\lambda|}$$

In particular, the RAS of Euler's method is a disk centered at $(-1,0)$ with radius 1 in the complex plane.



**Q.** How about for general one-step methods?

**Thm.** A general one-step method

$$w_{i+1} = w_i + h\phi(t_i, w_i, h),$$

which can be rewritten as $w_{i+1} = Q(h,\lambda) w_i$

has the RAS

$$R = \{ h\lambda \in \mathbb{C} \mid |Q(h,\lambda)| < 1 \}$$

Here, the function $Q(h,\lambda)$ is called the stability function.

**Rmk.** The main step is to derive/compute stability function $Q$.

**Ex.** $n^{th}$ order Taylor method has RAS

$$\left| 1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \dots + \frac{1}{n!}h^n\lambda^n \right| < 1$$

## Region of Absolute Stability (RAS) for multistep methods

Consider a multistep method defined by

$$\begin{aligned} w_{i+1} = {} & a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i-m+1} \\ & + h\big[ b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \dots \\ & + b_0 f(t_{i-m+1}, w_{i-m+1}) \big] \end{aligned}$$

Applying an m-step multistep method to the test IVP

$$w_i = a_{m-1} w_i + \ldots + a_0 w_{i-m+1}$$
$$+ h\lambda [b_m w_{i+1} + b_{m-1} w_i + \ldots + b_0 w_{i-m+1}]$$
$$\text{for } i = m-1, \ldots, N-1$$

Its corresponding stability / characteristic polynomial $Q(z, h\lambda)$ is

$$Q(z, h\lambda) = (1 + h\lambda b_m) z^m - (a_{m-1} + h\lambda b_{m-1}) z^{m-1}$$
$$- \ldots - (a_0 + h\lambda b_0)$$

Here, the numerical solution $w_i$ can be expressed as a linear combination of $\beta_k^i$ with $k = 1, 2, \ldots m$, where $\beta_k$ are the roots of $Q(z, h\lambda)$ for fixed $\lambda$.

To ensure $|w_i| \to 0$, the multistep method has RAS

$$R = \{ h\lambda \in \mathbb{C} \mid |\beta_k| < 1 \text{ for all zeroes of } Q(z, h\lambda) \}$$

Rmk. Stability region = the set of $h\lambda$ s.t. multistep method satisfies root condition for IVP.

**Ex.** Show that the Implicit Trapezoid Method given by
$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \frac{h}{2}[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)] \end{cases}$$
is A-stable but not L-stable.

**Pf.** The stability polynomial for this method is:
$$Q(z,h,\lambda) = (1 - \tfrac{h\lambda}{2})z - (1 + \tfrac{h\lambda}{2})z = 0$$
$$\rightarrow z = \frac{2+h\lambda}{2-h\lambda}$$

Thus the RAS is
$$R = \left\{ h\lambda \in \mathbb{C} \;\middle|\; \left|\frac{2+h\lambda}{2-h\lambda}\right| < 1 \right\}$$

To show A-stability, want $Re(h\lambda) < 0$.

Let $h\lambda = a + ib$.
$$\left|\frac{2+h\lambda}{2-h\lambda}\right| < 1 \iff |2+a+ib| < |2-a-ib|$$
$$\iff (2+a)^2 + (b)^2 < (2-a)^2 + (b)^2$$
$$\iff \underbrace{a}_{Re(h\lambda)} < 0$$

which implies that the RAS for this method is exactly the left half plane

$\Rightarrow$ method is A-stable.

Recall: L-stable if A-stable AND

$w_i \to 0$ as $Re(\lambda) \to -\infty$.

The numerical solutions have the form

$$w_{i+1} = \frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda} w_i$$

$$\Rightarrow w_i = \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^i \alpha$$

$$\to \alpha \text{ as } Re(\lambda) \to -\infty$$

Thus the method is not L-stable. $\blacksquare$