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Ch.11 Boundary Value Problems (BVPs) for ODEs

Recall the second order IVP is of the form

$$(1)\quad \begin{cases} y''(x) = f(x, y, y') & a \le t \le b \\ y(a) = \alpha, \quad y(b) = \beta \end{cases}$$

The BVP is called **linear** if

$$f(x, y, y') = p(x)y' + q(x)y + r(x)$$

and nonlinear otherwise.

Note: we have conditions of y on boundary $x=a$ and $x=b$ rather than the initial conditions (one boundary) y, y' on $x=a$.

Thm. 11.1 (Well-posedness of BVP)

Let f in (1) be obtained on $\Omega = [a, b] \times \mathbb{R}^2$.

If (i) f, f_y, $f_{y'}$ are continuous on Ω

(ii) $f_y(x, y, y') > 0$ for all $(x, y, y') \in \Omega$

(iii) $|f_{y'}(x, y, y')| \le M$ for all $(x, y, y') \in \Omega$ for some $M > 0$

then the BVP has a unique solution.

Ex. Show that the BVP

$$y'' + e^{-xy} + \sin y' = 0 \quad \text{for all } \lambda \in [1, 2], \quad y(1) = y(2) = 0$$

has a unique solution.

Sol. $f(x,y,y') = -e^{-xy} - \sin y' \quad \forall\ x \in [1,2]$

and $f_y(x,y,y') = xe^{-xy} > 0$,

$|f_{y'}(x,y,y')| = |-(0)(y')| \le 1$

$\Rightarrow$ BVP has a unique solution by Thm. 11.1

Linear BVPs

Corollary 11.2 (Uniqueness of linear BVPs)

If the linear BVP

$$(2)\ \begin{cases} y''(x) = p(x)y' + q(x)y + r(x), & \forall\ x \in [a,b] \\ y(a) = \alpha,\ y(b) = \beta \end{cases}$$

satisfies (i) $p(x), q(x), r(x)$ continuous on $[a,b]$

(ii) $q(x) > 0$ on $[a,b]$

then the BVP has a unique solution.

Pf. Let $f(x,y,y') = p(x)y' + q(x)y + r(x)$.

Since $p(x), q(x), r(x)$ are continuous on $[a,b]$

$f(x,y,y')$ is continuous on $[a,b]$.

Next, analyze the partial derivatives $f_y, f_{y'}$:

$$f_y = q(x) > 0 \quad \forall\ x \in [a,b]$$

$$|f_{y'}| = |p(x)| \quad \forall\ x \in [a,b]$$

Note that boundedness of $f_{y'}$ is guaranteed by continuity of $p(x)$ on $[a,b]$.

By Thm. 11.1, the BVP has a unique solution.

To solve the BVP, first note that the solution to (2) can be written as

(3) $$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} \cdot y_2(x)$$

where y_1 is the sol. to the IVP

(4) $$\begin{cases} y_1'' = p(x)y_1' + q(x)y_1 + r(x) & \forall\ x \in [a,b] \\ y_1(a) = \alpha, \quad y_1'(a) = 0 \end{cases}$$

and y_2 is the sol. to the IVP

(5) $$\begin{cases} y_2'' = p(x)y_2' + q(x)y_2 + r(x) & \forall\ x \in [a,b] \\ y_2(a) = 0, \quad y_2'(a) = 1 \end{cases}$$

To check this, take 2 derivatives of (3) and check that it satisfies (2), that is,

$$y'(x) = y_1'(x) + \underbrace{\frac{\beta - y_1(b)}{y_2(b)}}_{=k} \cdot y_2'(x)$$

$$\begin{aligned} y''(x) &= y_1''(x) + k y_2''(x) \\ &= p(x)y_1'(x) + q(x)y_1(x) + r(x) + k[p(x)y_2'(x) + q(x)y_2(x)] \\ &= p(x)\big(\underbrace{y_1'(x) + ky_2'(x)}_{=y'}\big) + q(x)\big(\underbrace{y_1(x) + ky_2(x)}_{=y}\big) + r(x) \\ &= p(x)y' + q(x)y + r(x) \end{aligned}$$

and boundary conditions $y(a) = y_1(a) = \alpha_1$

since $y(a) = y_1(a) + \frac{\beta - y_1(b)}{y_2(b)} \cdot \underbrace{y_2(a)}_{0}$

and $y(b) = y_1(b) + k y_2(b)$

$$= y_1(b) + \frac{\beta - y_1(b)}{y_2(b)} \cdot y_2(b)$$

$$= \beta$$

which satisfies (2).

Solving the Linear BVP

To solve the linear BVP, we solve the IVPs (4) and (5) by first transforming each eqn in a system of 1^{st} order IVPs, e.g., letting $u_1 = y$ and $u_2 = y'$, (4) can be written as:

$$\begin{cases} u_1'(x) = u_2 & a \le x \le b \\ u_2'(x) = p(x)u_2 + q(x)u_1 + r(x) \end{cases} \qquad \begin{cases} u_1(a) = \alpha \\ u_2(a) = 0 \end{cases}$$

Similarly, letting $v_1 = y$ and $v_2 = y'$, (5) can be written as

$$\begin{cases} v_1'(x) = v_2 & a \le x \le b \\ v_2'(x) = p(x)v_2 + q(x)v_1 \end{cases} \qquad \begin{cases} v_1(a) = \alpha \\ v_2(a) = 0 \end{cases}$$

Then use any numerical methods to solve system of 1^{st} order IVPs.

Rmk. If y_1 increases rapidly as x increases from a to b, then $\beta \ll y_1(b)$ and $k \approx \frac{-y_1(b)}{y_2(b)}$ will cause loss of significant digits.

Remedy: we can "flow" backwards instead to reduce one-off errors:

$$\begin{cases} y_1'' = p(x)y' + q(x)y + r(x) & a \le x \le b \\ y(b) = \beta, \quad y'(b) = 0 \end{cases}$$

$$\begin{cases} y_2'' = p(x)y' + q(x)y & a \le x \le b \\ y(b) = \beta, \quad y'(b) = 1 \end{cases}$$

and $y(x) = y_1(x) + \left(\frac{\alpha - y_1(a)}{y_2(a)}\right) \cdot y_2(a)$

11.2 Nonlinear BVPs: the shooting method

To solve a nonlinear 2^{nd} order BVP, we consider a sequence of IVPs:

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha, \quad y'(a) = \theta_k \end{cases} \quad \text{slopes} \qquad (1)$$

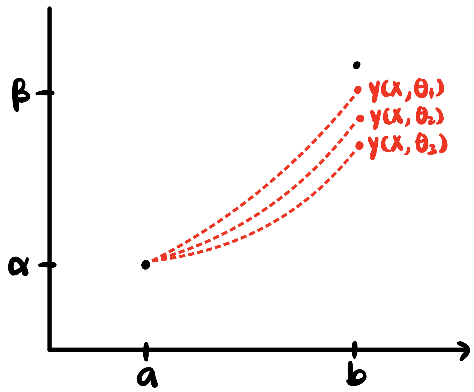
such that the solution $y(x, \theta_k)$ to (1) and $y(x)$ to the BVP satisfy

$$\lim_{k\to\infty} \underbrace{y(b, \theta_k)}_{\text{sol. of IVP}} = \underbrace{y(b)}_{\text{sol. of BVP}} = \beta$$

Rmk. The idea here is to iteratively change the slope θ_k until we get

$$y(b, \theta^*) = y(b)$$

hence "shooting method".



To determine θ_k, we can apply root-finding methods, e.g. Newton's Method or Secant Method, to the equation

$$F(\theta) = y(b, \theta) - \beta = 0$$

(i) Secant Method: solve IVP (1) with θ_k updated by

$$\theta_k = \theta_{k-1} - \frac{\theta_{k-1} - \theta_{k-2}}{F(\theta_{k-1}) - F(\theta_{k-2})} \cdot F(\theta_{k-1})$$

$$= \theta_{k-1} - \frac{(y(b, \theta_{k-1}) - \beta)(\theta_{k-1} - \theta_{k-2})}{y(b, \theta_{k-1}) - y(b, \theta_{k-2})} \qquad k = 2, 3, \dots$$

(ii) Newton's Method:

$$\theta_k = \theta_{k-1} - \frac{F(\theta_{k-1})}{F'(\theta_{k-1})}$$

$$= \theta_{k-1} - \frac{y(b, \theta_{k-1}) - \beta}{\frac{dy}{d\theta}(b, \theta_{k-1})}$$

Rmk. We do not immediately know / have access to $\frac{dy}{d\theta}(b, \theta_{k-1})$.

However, we can compute $\frac{dy}{d\theta}$ by solving another IVP.

We derive $\frac{dy}{d\theta}$ as follows: consider IVP

$$(2) \begin{cases} y''(x,\theta) = f(x, y(x,\theta), y'(x, \theta)), & a \le x \le b \\ y(a,\theta) = \alpha & y'(a,\theta) = \theta \end{cases}$$

Note: here, $y'(x,\theta) = \frac{dy}{dx}$ (prime w.r.t. x)

Taking partial derivative of (2) w.r.t. θ:

$$\begin{aligned} \frac{\partial y''(x,\theta)}{\partial \theta} &= \frac{\partial f}{\partial \theta}(x, y(x,\theta), y'(x, \theta)) \\ &= \cancel{\frac{\partial f}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial \theta}}_{0}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial \theta} \end{aligned}$$

$$\Rightarrow \frac{\partial y''}{\partial \theta} = \frac{\partial f}{\partial y}(x, y(x,\theta), y'(x,\theta)) \cdot \frac{\partial y(x,\theta)}{\partial \theta} + \frac{\partial f}{\partial y'}(x, y(x,\theta), y'(x,\theta)) \frac{\partial y'}{\partial \theta}$$

$$a \le x \le b, \quad \frac{\partial y}{\partial \theta}(a,\theta) = 0, \quad \frac{\partial y'}{\partial \theta}(a,\theta) = 1$$

Let $z(x,\theta) = \frac{\partial y}{\partial \theta}(x,\theta)$.

Want $\frac{\partial y}{\partial \theta}(b,\theta) = z(b,\theta)$ for any given θ.

$$(3) \begin{cases} z''(x,\theta) = \frac{\partial f}{\partial y} \cdot z + \frac{\partial f}{\partial y'} \cdot z' & a \le x \le b \\ z(a,\theta) = 0, \quad z'(a,\theta) = 1 \end{cases}$$

Rmk. 1. Use (3) with $\theta = \theta_{k-1}$ to obtain

$$z(b, \theta_{k-1}) = \frac{\partial y}{\partial \theta}(b, \theta_{k-1})$$ in Newton's method

$\Rightarrow$ Newton's method requires solving 2 IVPs per iteration:

$$\underbrace{y(b, \theta_{k-1})}_{\text{from (2)}} \qquad \underbrace{z(b, \theta_{k-1})}_{\text{from (3)}}$$

2. Another way to solve BVP: discretize derivatives using, e.g., finite difference schemes, e.g.

$$y'(x_n) = \frac{y(x_{k+1}) - y(x_k)}{h}$$

This leads to a linear system, whose solution yields $y(x)$ at $x_0, x_1, \ldots x_n$.

For nonlinear BVP, this requires solving root-finding problem in n dimensions.

3. Solving the BVP is related to solving an optimal control problem.