Changing Intervals

Suppose we have a quadrature rule for $[c, d]$.

Q: What is the rule for $[a, b]$?

- use change of variables formula:

$$\int_{a=g(c)}^{b=g(d)} f(x)dx = \int_c^d f(g(t)) \cdot g'(t)dt$$

where $g(t) = x$

- we want $g(t)$ to satisfy:

$$g(c) = a, \quad g(b) = d$$

use interpolation:

$$g(t) = \frac{t-d}{c-d}a + \frac{t-c}{d-c}b$$

$$g'(t) = \frac{a}{c-d} + \frac{b}{d-c} = \frac{b-a}{d-c}$$

$$\therefore \int_{g(c)}^{g(d)} f(x)dx = \int_c^d f(g(t)) \cdot \frac{b-a}{d-c}dt$$

$$= \frac{b-a}{d-c}\int_c^d f(g(t))dt$$

$$\approx \frac{b-a}{d-c}\sum_{i=0}^{n} w_i f(g(t_i))$$

$$\Rightarrow \int_{a=g(c)}^{b=g(d)} f(x)dx \approx \frac{b-a}{d-c}\sum_{i=0}^{n} w_i f(g(t_i))$$

Ex. Suppose we have:

$$\int_{-1}^{1} f(x)dx \approx \frac{4}{3}f(-\tfrac{1}{2}) - \frac{2}{3}f(0) + \frac{4}{3}f(\tfrac{1}{2})$$

Then $\int_a^b f(x)dx = \frac{b-a}{d-c}\int_c^d f(g(t))dt$

where $c=-1$, $d=1$, $g(t) = \frac{t-1}{-2}a + \frac{t+1}{2}b$

$$= \tfrac{1}{2}(b-a)t + \tfrac{1}{2}(b+a)$$

$$\Rightarrow \int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\tfrac{1}{2}(b-a)t + \tfrac{1}{2}(b+a)\right)dt$$

$$\approx \frac{b-a}{2}\left[\frac{4}{3}f(g(-\tfrac{1}{2})) - \frac{2}{3}f(g(0)) + \frac{4}{3}f(g(\tfrac{1}{2}))\right]$$

$$= \frac{b-a}{2}\Big[\frac{4}{3} f\big(\tfrac{1}{2}(b-a)\tfrac{-1}{2}+\tfrac{1}{2}(b+a)\big)$$
$$- \frac{2}{3} f\big(\tfrac{1}{2}(b-a)\tfrac{0}{2}+\tfrac{1}{2}(b+a)\big)$$
$$+ \frac{4}{3} f\big(\tfrac{1}{2}(b-a)\tfrac{1}{2}+\tfrac{1}{2}(b+a)\big)\Big]$$
$$= \frac{b-a}{2}\Big[\frac{4}{3} f\big(\tfrac{b+3a}{4}\big) - \frac{2}{3} f\big(\tfrac{a+b}{2}\big) + \frac{4}{3} f\big(\tfrac{a+3b}{4}\big)\Big]$$

A more systematic way to determine w_i, x_i:

Orthogonal polynomials on $[-1, 1]$

Def. The functions $f, g \in L^2[-1, 1]$
(i.e. $\int_{-1}^{1} (f(x))^2 dx < \infty$ and $\int_{-1}^{1} (g(x))^2 dx < \infty$)
are orthogonal if $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx = 0$.

Ex. $\langle \sin(\pi x), \cos(\pi x)\rangle = \int_{-1}^{1} \sin(\pi x)\cos(\pi x)dx$
$= \frac{1}{2\pi}\sin^2(\pi x)\Big|_{-1}^{1} = 0$

$\langle 1, x\rangle = \int_{-1}^{1} x\,dx = 0$

$\langle 1, x^2\rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3} \neq 0$

$\langle 1, x^3\rangle = \int_{-1}^{1} x^3 dx = 0$

Rmk. If $f(x)$ is even and $g(x)$ is odd on $[-1, 1]$, then f and g are orthogonal.

Def. Starting from $1, x, x^2, \dots x^n, \dots$
the Gram-Schmidt process w/o

normalization generates a set of orthogonal polynomials
$\{P_0(x), P_1(x), \dots P_n(x)\}$
called **Legendre polynomials**

Here, $P_0(x) = 1$

$$P_1(x) = x - \frac{\langle x, P_0\rangle}{\langle P_0, P_0\rangle} P_0 = x$$

$$P_2(x) = x^2 - \frac{\langle x^2, P_0\rangle}{\langle P_0, P_0\rangle} P_0 - \frac{\langle x, P_1\rangle}{\langle P_1, P_1\rangle} P_1 = x^2 - \frac{1}{3}$$

$$P_3(x) = x^3 - \frac{3}{5}x,$$

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

$$\vdots$$

Rmk. 1 The Legendre polynomials $P_0(x), P_1(x), \dots P_n(x)$ satisfy:

a) for each n, P_n is a monic polynomial of degree n (leading coeff. $= 1$)

b) $\int_{-1}^{1} P(x) \cdot P_n(x)\,dx = 0$ whenever the degree of $P < n$

2 Roots of $P_n(x)$ are distinct on $[-1, 1]$ and symmetric w.r.t. origin, and are **exactly the Gaussian Quadrature nodes!**