Thm. Closed-NC Error Thm.

Suppose $R(f)=\sum_{i=1}^{m} w_i f(x_i)$ denotes the $(n+1)$-closed NC rule with $x_0=a$, $x_n=b$, $h=\frac{b-a}{n}$.

Then there exists $\xi\in(a,b)$ such that:

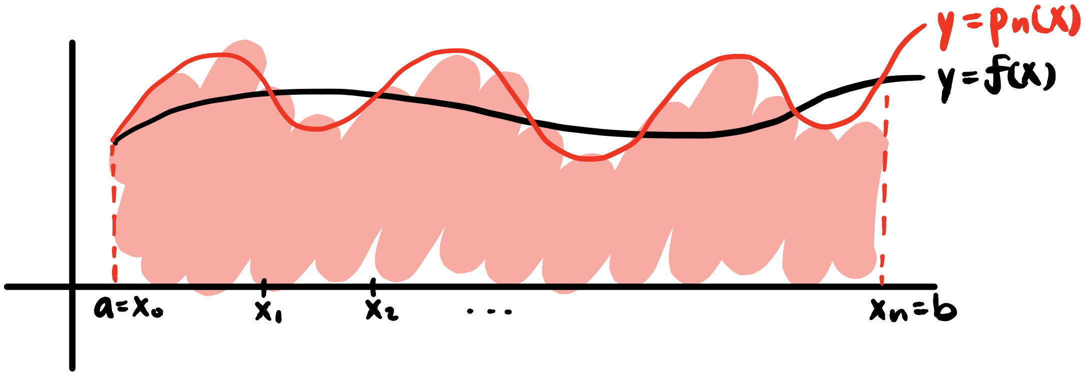
- when n is even and $f\in C^{(n+2)}(a,b)$

$$\int_a^b f(x)dx = R(f)+\frac{h^{n+3}f^{(n+2)}(\xi)}{(n+2)!}\int_0^n t^2(t-1)\cdots(t-n)dt$$

- when n is odd and $f\in C^{(n+1)}[a,b]$

$$\int_a^b f(x)dx = R(f)+\frac{h^{n+2}f^{(n+1)}(\xi)}{(n+1)!}\int_0^n t(t-1)\cdots(t-n)dt$$

Rmk.
- n even $\Rightarrow$ error is $O(h^{n+3})$
- n odd $\Rightarrow$ error is $O(h^{n+2})$



From thm. above, we can derive error bounds for common closed-NC rules.

Ex. Compute error bound for trap. rule

$$R_T(f)=\frac{h}{2}(f(x_0)+f(x_1))$$

Sol. $n=1$:

$$E_T(f)=\frac{h^3\cdot f^{(2)}(\xi)}{2}\cdot\int_0^1 t(t-1)dt$$

$$=\frac{h^3\cdot f''(\xi)}{2}\cdot\left(-\frac{1}{6}\right)$$

$$=\frac{-h^3\cdot f''(\xi)}{12} \qquad x_0<\xi<x_1$$

In the same manner, we can derive the following bounds:

n	rule	formula	error	DOP
1	trapezoid	$\frac{h}{2}(f(x_0)+f(x_1)) - \frac{h^3 \cdot f''(\xi)}{12}$	$O(h^3)$	1
2	Simpson's 1/3	$\frac{h}{3}(f(x_0)+4f(x_1)+f(x_2)) - \frac{h^5}{90} f^4(\xi)$	$O(h^5)$	3
3	Simpson's 3/8	$\frac{3h}{8}(f(x_0)+3f(x_1)+3f(x_2)+f(x_3)) - \frac{3h^5}{80} f^4(\xi)$	$O(h^5)$	3

Rmk. 1 n even $\Rightarrow$ error term depends on $f^{(n+2)}(\xi)$

$\Rightarrow$ exactly integrate functions whose $n+2$-derivative is 0 (poly of deg. $n+1$)

$\Rightarrow$ DOP $= n+1$

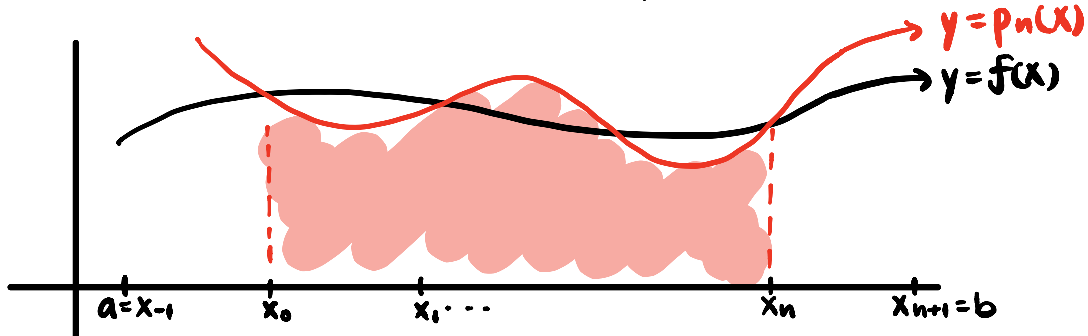
2 This is NOT the case for n odd!

e.g. trap rule w/ $n=1$: DOP $=1$

Similar case holds for open NC formulas.

Let $a = x_{-1} < x_0 < \ldots < x_n < x_{n+1} = b$ and $h = \frac{b-a}{n+2}$.

Then $x_i = x_0 + ih$ for $i = -1, 0, 1, \ldots n+1$



Thm. Open-NC Error Thm.

Suppose $R(f)=\sum_{i=1}^{m} w_i f(x_i)$ denotes the $(n+1)$-open NC rule with $x_{-1}=a$, $x_{n+1}=b$, $h=\frac{b-a}{n+2}$.

Then there exists $\xi\in(a,b)$ such that:

- when n is even and $f\in C^{(n+2)}(a,b)$
$$\int_a^b f(x)\,dx = R(f)+\frac{h^{n+3}f^{(n+2)}(\xi)}{(n+2)!}\int_{-1}^{n+1} t^2(t-1)\cdots(t-n)\,dt$$
- when n is odd and $f\in C^{(n+1)}[a,b]$
$$\int_a^b f(x)\,dx = R(f)+\frac{h^{n+2}f^{(n+1)}(\xi)}{(n+1)!}\int_{-1}^{n+1} t(t-1)\cdots(t-n)\,dt$$

From thm. above, we can derive error bounds for common open-NC rules.

n	rule	formula	error	DOP
0	midpoint	$2hf(x_0)+\frac{h^3}{3}f''(\xi)$	$O(h^3)$	1
1	open trap.	$\frac{3h}{2}[f(x_0)+f(x_1)]+\frac{3h^3}{4}f''(\xi)$	$O(h^3)$	1
2	Milne's	$\frac{4h}{3}[2f(x_0)-f(x_1)+2f(x_2)]+\frac{14h^5}{45}f^{(4)}(\xi)$	$O(h^5)$	3

Rmk. 1. Both NC rules are not suitable when integrating over large intervals using high-deg. polys due to oscillations.

- One remedy: use piecewise-polynomial interpolation, i.e. composite numerical interpolation