We have seen: $L[y'] = pL[y] - y(0)$

$L[y''] = p^2 L[y] - py(0) - y'(0)$

<u>Integrals of Laplace transforms:</u>

Prop. i) $L\left[\int_0^x f(t)\,dt\right] = \frac{F(p)}{p}$

ii) $\int_p^\infty F(q)\,dq = L\left[\frac{f(x)}{x}\right]$

Pf. i) Let $y(x) = \int_0^x f(t)\,dt$. Then $y(0) = 0$ and $y'(0) = f(x)$.

$\Rightarrow L[f(x)] = L[y'] = pL[y] - y(0)$

$= pL\left[\int_0^x f(t)\,dt\right]$

$\rightarrow L\left[\int_0^x f(t)\,dt\right] = \frac{F(p)}{p}$

ii) Set $g(x) = \frac{f(x)}{x}$, $G(p) = L[g(x)]$.

$G'(p) = L[-x \cdot g(x)] = L\left[-x \cdot \frac{f(x)}{x}\right]$

$= L[-f(x)] = -F(p)$

Integrate from a to p:

$G(p) - G(a) = \int_a^p G'(x)\,dx$

$= -\int_a^p F(q)\,dq$

$= \int_p^a F(q)\,dq$

as $a \to \infty$, $G(a) \to 0$, thus $G(p) = \int_p^\infty F(q)\,dq$.

Hence $\int_p^\infty F(q)\,dq = L\left[\frac{f(x)}{x}\right](p)$

$= \int_0^\infty \frac{f(x)}{x} e^{-px}\,dx$

for $p=0$: $\int_0^\infty F(q)\,dq = \int_0^\infty \frac{f(x)}{x}\,dx$

Ex. $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty L[\sin x](p)\,dp$ function of p

$= \int_0^\infty \frac{1}{1+p^2}\,dp$

$= \arctan(p)\Big|_0^\infty$

$= \frac{\pi}{2}$

Bessel's Equation: $x^2y''(x)+xy'(x)+(x^2-\alpha^2)y(x)=0$

$\alpha\in\mathbb{C}$ is called the order of the equation or the Bessel function $J_\alpha(x)$

$\alpha=0$: $xy''(x)+y'(x)+xy(x)=0$

$y(0)=1$

Apply Laplace transform: Recall $F'(p)=L[-xf(x)]$

$$\begin{aligned}
0&=L[xy''(x)+y'(x)+xy(x)]\\
&=L[xy''(x)]+L[y'(x)]+L[xy(x)]\\
&=-\tfrac{d}{dp}L[y''(x)]+L[y'(x)]-\tfrac{d}{dp}L[y(x)]\\
&=-\tfrac{d}{dp}(p^2L[y]-py(0)-y'(0))+pL[y]-y(0)-\tfrac{d}{dp}L[y]\\
&=-2pL[y]-p^2\tfrac{d}{dp}L[y]+y(0)+pL[y]-y(0)-\tfrac{d}{dp}L[y]\\
&=-(p^2+1)\tfrac{d}{dp}L[y]-pL[y]
\end{aligned}$$

$$\rightarrow \frac{d}{dp}L[y]=\frac{-p}{1+p^2}L[y]$$

$$\begin{aligned}
\rightarrow \ln L[y]&=-\int\frac{p}{1+p^2}dp\\
&=\frac{-1}{2}\ln(1+p^2)+C_1\\
&=\ln(1+p^2)^{-1/2}+C_1
\end{aligned}$$

$$\begin{aligned}
L[y]&=C_2\cdot\frac{1}{\sqrt{1+p^2}}\\
&=C_2\cdot\frac{1}{p}\cdot\frac{1}{(\frac{1}{p^2}+1)^{1/2}}\\
&=\frac{C_2}{p}\left(\frac{1}{p^2}+1\right)^{-1/2}
\end{aligned}$$

$$(1+x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n \qquad |x|<1$$

binomial series

for series to converge

(take $x=\frac{1}{p^2}=p^{-2}$ $\qquad \binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{n!}$

$$\begin{aligned}
L[y]&=\frac{C_2}{p}\left(1+\frac{1}{p^2}\right)^{-1/2}\\
&=\frac{C_2}{p}\sum_{n=0}^{\infty}\binom{-1/2}{n}p^{-2n}
\end{aligned}$$

$$= \frac{C_2}{p} \sum_{n=0}^{\infty} \frac{(-1)^n 1\cdot3\cdot5\cdot\ldots(2n-1)}{2^n \cdot n!}$$

Since $1\cdot3\cdot5\cdot\ldots(2n-1) = \frac{2n!}{2^n \cdot n!}$

Compute Laplace transform:

$$y(x) = C_2 \sum_{n=0}^{\infty} (-1)^n \frac{1}{2^{2n}(n!)^2} L\left[\frac{(2n!)}{p^{2n+1}}\right]$$

$$= C_2 \cdot \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n} \cdot (n!)^2}$$

note: $y(0) = 1 \rightarrow C_2 = 1$

$y^{(k)} = J_0(x)$ Bessel function (of first kind) for $\alpha = 0$

Property: $L[J_0(x)] = \frac{1}{1+p^2}$