Recap: we're considering $y''+ay'+by=f(x)$, $\quad y(0)=y'(0)=0$

We saw $y(x)=(h*f)(x)=(f*h)(x)$,

where h satisfies $L[h]=\frac{1}{p^2+ap+b}$.

Rmk. (i) h satisfies $h''+ah'+bh=0$, $\quad h(0)=0$, $h'(0)=1$

(check with explicit formula, or take Laplace transform)

$$-\underbrace{h'(0)}_{=1}+(p^2+ap+b)L[h]=0$$

We get $L[h]=\frac{1}{p^2+ap+b}$ as required.

(ii) $A(x)=\int_0^x h(t)dt$

$$A'(x)=h(x)=\int_0^x h'(t)dt \quad (\text{since } h(0)=0)$$

$$A''(x)=h'(x)=h'(x)\underbrace{-h(0)+1}_{=0}$$

$$=\int_0^x h''(t)dt+1$$

$$A''+aA'+bA=\int_0^x \underbrace{(h''(t)+ah'(t)+bh(t))}_{=0 \text{ by ODE for } L}dt+1$$

i.e. $A''+aA'+bA=1$

$A(0)=0$, $A'(0)=0$

Def. $u(x)=\begin{cases}1 & x\geq 0\\ 0 & x<0\end{cases}$

unit step function / Heaviside function

Hence: A satisfies the ODE

$$(1)\begin{cases} y''+ay'+by=f(x)\\ y(0)=0,\quad y'(0)=0\end{cases}$$

for $f(x)=u(x)$ [constant signal $=1$ for $x\geq 0$, no signal for $x<0$]

A is called "indicial response"

$$y = A' * f = A(x)f(0) + (A * f')(x)$$

Q: Is there a function $f(x)$ such that $h(x)$ solves (1)?

A: No; would need $L[f] = (p^2+ap+b)L[h] = 1$

which is impossible:

For every function with a Laplace transform,

$L[f] = F(p) \to 0$ as $p \to \infty$.

Dirac's Delta "Function" $\delta(x)$ is a replacement.

It is not a function; it is a generalized function,

a distribution, implicitly defined by $\int_0^\infty f(x) \cdot \delta(x)\,dx = f(0)$

for every function $f: [0, \infty) \to \mathbb{R}$

$f(x) = 0$ for all large x

Q: How do we think about $\delta(x)$?

Let $\delta_\varepsilon(x) = \begin{cases} 1/\varepsilon & x>0 \\ 0 & x<0 \end{cases}$

$\int_0^\infty \delta_\varepsilon(x) = \int_0^\varepsilon \frac{1}{\varepsilon}\,dx = 1$ for every $\varepsilon > 0$

Then $\int_0^\infty f(x)\delta_\varepsilon(x)\,dx = \int_0^\varepsilon \frac{f(x)}{\varepsilon}\,dx$

$= \frac{1}{\varepsilon}\int_0^\varepsilon f(x)\,dx$

$\to f(0)$ as $\varepsilon \to 0$

Note Let $g(\varepsilon) = \int_0^\varepsilon f(x)\,dx$

$f(0) = g'(0) = \lim_{h\to 0} \frac{g(h)-g(0)}{h} = \frac{\int_0^h f(x)\,dx}{h}$

i.e. $\delta_\varepsilon(x) \longrightarrow \delta(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 \end{cases}$

such that $\int_0^\infty \delta(x)\,dx = 1$

Note $L[\delta_\varepsilon(x)] = \int_0^\infty \delta_\varepsilon(x) e^{-px}\,dx$

$$= \frac{1}{\varepsilon}\int_0^\varepsilon e^{-px}\,dx$$

$$= \frac{1-e^{-p\varepsilon}}{\varepsilon p}$$

(L'Hospital) $\to 1$ as $\varepsilon \to 0$

So $L[\delta(x)] = 1$.

Hence, in the sense of distributions, h solves

$$(1)\ \begin{cases} y'' + ay' + by = 0 \\ y(0) = y'(0) = 0 \end{cases}$$

Why? Take Laplace transform:

$$L[y](p^2 + ap + b) = L[\delta] = 1$$

$$\to L[y] = \frac{1}{p^2+ap+b} \quad \to y = h$$

h is called "impulsive response"
(also fundamental solution)

i.e. solution to (1) with signal that is a Dirac-Delta peak at $x=0$.

$$y = h * f = f * h$$

Rmk. (§51, Ex. 3b) Find a solution to

$$xy'' + (2x+3)y' + (x+3)y = 3e^{-x}, \quad y(0) = 0$$

$\to L[y] = \frac{1}{(p+1)^3} + c\cdot(p+1)$ for a constant $c \in \mathbb{R}$

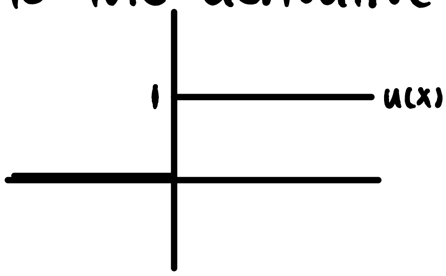
if $c \neq 0$, there is no function $y(x)$ with Laplace transform $c(p+1)$

[must have $L[f] = F(p) \to 0$ as $p \to \infty$]

Hence $c = 0$ is our only option to obtain a solution. Then $L[y] = \frac{1}{(p+1)^3} \to y(x) = xe^{-x}$.

easy check: $y(x)$ indeed solves our ODE.

Rmk. In the sense of distributions, the Delta function is the derivative of the unit step function $u(x)$



"jump at $x=0$ produces δ-peak when taking the derivative"