3.1 Interpolation

Q: Are polynomials a "good" set of functions to approximate/ interpolate arbitrary function f?

Thm. 1 Weierstrass Approximation Thm.

Suppose $f \in C([a, b])$

For any $\varepsilon > 0$, there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \varepsilon$ for all $x \in [a, b]$

Rmk. Derivative and integral of polynomials are easy to compute $\Rightarrow$ often used to approximate <u>continuous</u> functions

Ex. Let $f(x) = e^x$. Taylor's expansion about $x_0 = 0$ yields

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \underbrace{\frac{e^{\xi} x^{n+1}}{(n+1)!}}_{R(x)} \quad \xi \text{ betw. } x \text{ and } 0$$

$\Rightarrow$ can use $P_n(x)$ to approximate $f(x)$ with error $R(x)$

Thm. 2 Given $(x_0, f(x_0)), (x_1, f(x_1)), \dots (x_n, f(x_n))$ with distinct x_k, $k = 0, 1, \dots n$.

Then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k) \text{ for each } k = 0, 1, \dots n.$$

Q: Given $(x_0, f(x_0)), (x_1, f(x_1)), \dots (x_n, f(x_n))$

How to construct $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

to interpolate $f(x)$?

Power Series Approach: Let $p(x)$ have the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

$$\left.\begin{aligned} p(x_0) &= f_0 \\ p(x_1) &= f_1 \\ &\vdots \\ p(x_{n+1}) &= f_{n+1} \end{aligned}\right\} \quad \begin{aligned} a_n x_0^n + a_{n-1} x_0^{n-1} + \ldots + a_1 x_0 + a_0 &= f_0 \\ a_n x_1^n + a_{n-1} x_1^{n-1} + \ldots + a_1 x_1 + a_0 &= f_1 \\ &\vdots \\ a_n x_{n+1}^n + a_{n-1} x_{n+1}^{n-1} + \ldots + a_1 x_{n+1} + a_0 &= f_{n+1} \end{aligned} \qquad (*)$$

x_n's distinct $\rightarrow$ $n+1$ equations

Want to find coefficients a_i, $i = 0, 1, \ldots n$.

This can be done by solving the system $(*)$ consisting of $n+1$ equations and $n+1$ variables (a_i)

We can write $(*)$ in matrix-vector form

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$V \underline{a} = \underline{f}$

(Vandermonde matrix)

in MATLAB: $\underline{a} = V \backslash \underline{f}$

Rmk. 1 V is generally ill-conditioned

2 Intuitive to build, difficult to solve

3 General cost to invert ?

4. V nonsingula

A more popular approach to construct $p(x)$:

Lagrange Form

$$p(x) = f(x_0)L_{n,0}(x) + f(x_1)L_{n,1}(x) + \dots + f(x_n)L_{n,n}(x)$$

Recall that we want $p(x_k) = f(x_k)$, $k = 0, 1, \dots n$

$\Rightarrow$ want $L_{n,k} = \begin{cases} 1 & \text{if } x = x_k \\ 0 & \text{else} \end{cases}$ (**)

We can thus construct L as follows:

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

(note this satisfies (**))

$L_{n,k}$ is called a **Lagrange interpolating polynomial.**

Ex. (a) Let $x_0 = 2$, $x_1 = 2.75$, $x_2 = 4$. Find $P_2(x)$ for $f(x) = \frac{1}{x}$.

(b) Use $P_2(x)$ to approximate $f(3) = \frac{1}{3}$.

Sol. Note $f_0 = \frac{1}{2}$, $f_1 = \frac{1}{2.75}$, $f_2 = \frac{1}{4}$

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2.75)(x-4)}{(-0.75)(-2)}$$

$$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-16}{15}(x-2)(x-4)$$

$$L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{2}{5}(x-2)(x-2.75)$$

$$\Rightarrow P_2(x) = \sum_{i=0}^{2} L_{2,i}(x) \cdot f(x_i)$$