$\{s_n\}$ a sequence

$t_n = \sum_{k=1}^{n} s_k = s_1 + s_2 + \ldots + s_n$

$t_N = \prod_{i=1}^{N} s_i = s_1 \cdot s_2 \cdot \ldots \cdot s_N$

Def. A subsequence of a sequence is a sequence t such that

(1) $\mathrm{dom}(t) \subseteq \mathrm{dom}(s)$

(2) for all $n \in \mathrm{dom}(t)$, $t_n = s_n$

$$\{s_n\} = \begin{cases} s_1 = 1 \\ s_2 = 2 \\ s_3 = 4 \\ s_4 = 8 \end{cases} \qquad \mathrm{dom}(s) = \{1, 2, 3, 4\}$$

$$t = \begin{cases} t_3 = 4 \\ t_4 = 8 \end{cases} \qquad \mathrm{dom}(t) = \{3, 4\}$$

Strings

X - "alphabet"

Def. A string over X is a sequence whose values are in X, i.e. a function from a set of consecutive integers to X.

Write strings in a particular way, as words in the alphabet X.

Ex. $X = \{a, b, c\}$

Write
$$\left.\begin{array}{l} s_1 = a \\ s_2 = a \\ s_3 = b \\ s_4 = c \end{array}\right\} \begin{array}{c} aabc \\ \text{or} \\ a^2bc \end{array}$$

- If we are given a string aabc, we understand it is implied that the domain is $\{1, 2, 3, 4\}$.
  - $aabc \neq baac$

$$\{a, a, b, c\} = \{a, b, c\}$$
$$= \{b, a, a, c\}$$

Concatenation

Ex. anti, dis } prefixes ism, arian } suffixes

anti + dis + establishment + arian + ism

antidisestablishmentarianism

Def. The **concatenation** of the strings α and β is the string $\alpha\beta$.

Ex. $\alpha = aab$ $\beta = bcb$

$\alpha\beta = aabbcb$

Def. α is a **substring** of β if there are strings δ and γ such that

$\beta = \delta\alpha\gamma$.

- There is an **empty string** ϕ so that

$\alpha = \phi\alpha\phi$

So α is a substring of itself.

Relations

Def. A **relation from X to Y** is a subset of $X \times Y$.

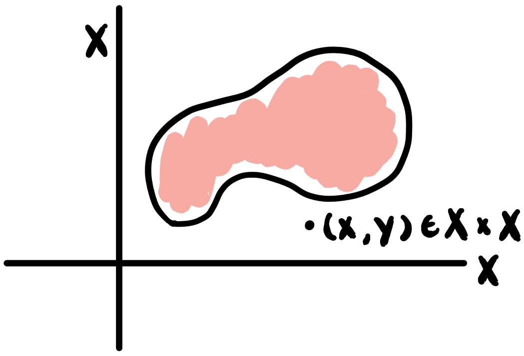
A **relation on X** is a subset of $X \times X$.

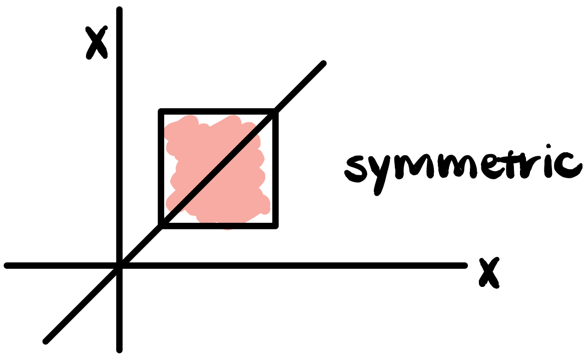
Ex.
- A function from X to Y is a relation from X to Y.
- $X = \mathbb{N}$

  We can identify "$\leq$" with the set
  $\{(n,m) \in \mathbb{N} \times \mathbb{N} \mid n \leq m\}$
- $X = \mathbb{N}$

  We can identify the relation of divisibility $x \mid y$ with
  $\{(n,m) \in \mathbb{N} \times \mathbb{N} \mid n \text{ divides } m\}$

Properties of relations





1. Symmetry

- A relation $R \subseteq X \times X$ is symmetric if whenever $(x,y) \in R$ then $(y,x) \in R$.

Ex. $\cdot \leq$ NOT symmetric $\qquad$ $\cdot \mid$ NOT symmetric

- "$=$" $= \{(n,n) \in X \times X \mid n \in X\}$ symmetric!

2. Antisymmetry

- If $(x,y)\in R$ and $(y,x)\in R$, then $x=y$

Rmk. Consider the relation R on X defined by

$R=\{(x,x)\mid x\in X\}$ "="

symmetric AND antisymmetric

Ex. Suppose R is the relation "$\leq$" on $\mathbb{N}$,

i.e. $R=\{(n,m)\in\mathbb{N}\times\mathbb{N}\mid n\leq m\}$

- R is antisymmetric:

  If $(n,m)\in R$ and $(m,n)\in R$

  $n\leq m$ and $m\leq n$ then $m=n$.

- R is not symmetric:

  $(1,2)\in R$ but $(2,1)\notin R$, i.e.

  $1\leq 2$ but $2\nleq 1$.

Ex. $R'=\{(n,m)\in\mathbb{N}\times\mathbb{N}\mid n<m\}$: not symmetric

! • there are no pairs $(n,m),(m,n)\in R'$!

! $\Rightarrow$ R' is antisymmetric!

(since no such pairs exist to begin with)

Ex. $R = \{(1,3), (3,1), (1,2)\}$

- not symmetric: $(2,1) \notin R$
- not antisymmetric: $1 \neq 3$

3. **Reflexive**:

R is reflexive if for all $x \in X$, $(x,x) \in R$

$$\{(x,x) \mid x \in X\} \subseteq R$$

4. **Transitive**:

R is transitive if whenever $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$

Ex. "$=$" is transitive

"$\leq$" is transitive

"$<$" is transitive

$$\equiv_2 = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid 2 \text{ divides } (x-y)\}$$

$x \equiv_2 y$ and $y \equiv_2 z$:

- 2 divides $(x-y)$
- 2 divides $(y-z)$

$\Rightarrow$ $2n = (x-y)$ for some $n \in \mathbb{Z}$
$2m = (y-z)$ for some $m \in \mathbb{Z}$

$$2(n+m) = (x-y) + (y-z) = x-z$$

So $(x-z)$ is divisible by 2, i.e. $x \equiv_2 z$.
Hence $\equiv_2$ is transitive.

Note: $\equiv_n$ is transitive.

5. **Equivalence relation**:

- R is an equivalence relation on X if R is a symmetric, transitive, and reflexive relation on X.

Ex. "=" reflexive ✓
symmetric ✓
transitive ✓

$\equiv_2$ ✓

6. **Partial Order**:

- A relation R on X is a partial order if R is reflexive, antisymmetric, and transitive.

Ex. "$\leq$" on $\mathbb{N}$: reflexive ✓
antisymmetric ✓
transitive ✓

"=": reflexive ✓
antisymmetric ✓
transitive ✓
(symmetric ✓)

S: set of finite strings in the alphabet $\{0, 1\}$
e.g. $\emptyset$, 0, 1, 00, 01, 10, 11, 001, ...

Define a relation $\trianglelefteq$ so that strings s, t satisfy $s \trianglelefteq t$ iff t extends s, i.e. there is string t' such that $t = st'$. (t' can be empty)

$s = 00$ $\quad t = 0010$

7. **Irreflexive**:
For all $x \in X$, $(x, x) \notin R$.
R

R is antireflexive means for all $x \in X$, $(x,x) \notin R$.

R is reflexive means for all $x \in X$, $(x,x) \in R$.

A relation that is not reflexive is not necessarily antireflexive.

Consider $X = \{1,2,3\}$, and let $R = \{(1,1), (1,2)\}$.

R is not reflexive since $(2,2), (3,3) \notin R$.

R is not antireflexive since $(1,1) \in R$.

Counting

· **Multiplication principle**:

Suppose you are making a solution in two steps and there are n options for the first step and m options for the second step.
Then there are $n \cdot m$ options in total.

Ex.

Option 1	Option 2
32 Gb	old
64 Gb	new
128 Gb	

6 total!

really the same fact as
$$|X_1 \times X_2| = |X_1| \cdot |X_2|$$

· **Addition principle**:

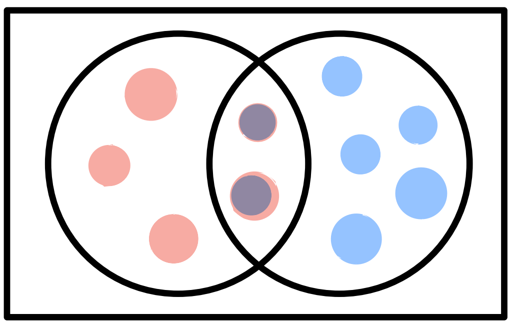
Suppose you are picking one thing out of two separate containers A, B. A has n things and B has m things.
Then there are $n+m$ options in total.

- i.e. if X and Y are disjoint sets, $(X \cap Y = \emptyset)$, then $|X \cup Y| = |X| + |Y|$.

· **Inclusion - Exclusion**

Suppose you are picking one thing from the collections A and B.

$|A \cup B| = |A| + |B| - |A \cap B|$



$|A| = 5$, $|B| = 7$, $|A \cap B| = 2$, $|A \cup B| = 10$

$10 = 5 + 7 - 2$ ✓

· **Permutations**

A permutation on a set $\{a_1, \dots, a_n\}$ is an ordering $a_{i_1} < a_{i_2} < \dots < a_{i_n}$.

A permutation of $\{a_1, \dots, a_n\}$ is a bijection from $\{a_1, \dots, a_n\} \to \{a_1, \dots, a_n\}$

order → bijection

$$\begin{array}{cccccc} a_1, & a_2, & a_3, & \dots & a_n & \\ \downarrow & \downarrow & \downarrow & & \downarrow & \text{bij.} \\ a_{i_1} & a_{i_2} & a_{i_3} & & a_{i_4} & \end{array}$$

Given a bijection of

$$\begin{array}{ccccccc} a_1 & & a_2 & & a_3 & \dots & a_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ f(a_1) & < & f(a_2) & < & f(a_3) & & < f(a_n) \end{array}$$

$\{a_1, a_2, a_3\}$

$a_{\underline{2}} < a_{\underline{3}} < a_{\underline{1}}$ i_n

$i_1 = 2,\ i_2 = 3,\ i_3 = 1$

Q: Suppose $A = \{a_1, a_2, \dots a_n\}$

How many permutations are there?

n choices for 1^{st} element

$n-1$ choices for 2^{nd} element

$n-2$ choices for 3^{rd} element

⋮

$n!$ choices