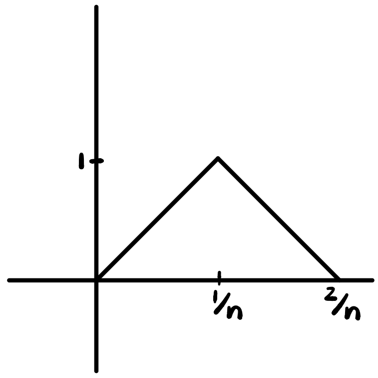
uniform convergence $\overset{\Rightarrow}{\nLeftarrow}$ pointwise convergence

Ex. compact

$([0,1], |\cdot|)$

$f_n : [0,1] \to [0,1]$

bounded
continuous




$f_n(0) = 0$

$f_n(\frac{1}{n}) = 1$

$f_n(\frac{2}{n}) = 0$

Let $x \in [0,1]$.

$\lim_{n\to\infty} f_n(x) = 0$.

For a fixed point $x \in [0,1]$, we have that

$\frac{2}{n} < x \quad \forall \quad n > \frac{2}{x}$

$\Rightarrow f_n(x) = 0$

$\Rightarrow \lim_{n\to\infty} f_n(x) = 0$

$([0,1], |\cdot|)$

$f : [0,1] \to [0,1]$

$x \mapsto 0$

$f_n$ converges to $f$ pointwise.

$f_n$ does not converge to $f$ uniformly

$(X, d_X)$ $(Y, d_Y)$

**Thm.** If $f_n \to f$ uniformly and $x_n \xrightarrow[d_X]{} x$,

then $f_n(x_n) \xrightarrow[d_Y]{} f(x)$.

**(above Ex.)** If $x_n = \frac{1}{n}$, then $x_n \to 0 =: x$

Moreover, $f_n(x_n) = f_n(\frac{1}{n}) = 1$.

$$1 = \lim_{n\to\infty} f_n(x_n) \neq f(x) = 0$$

**Thm.** (Dini's Thm.)

Let $(X, d_X)$ be m.s., $(f_n)$ be a sequence of functions $f_n : X \to \mathbb{R}$. Assume that:

pointwise convergence

(a) $f_n \to f$ pointwise

compactness

(b) $X$ compact

continuity

(c) $f_n$ continuous $\forall$ $n$, $f$ continuous

monotonicity

(d) $\forall$ $n \in \mathbb{N}$, $\forall$ $x \in X$, $f_{n+1}(x) \leq f_n(x)$

Then $f_n \to f$ uniformly on $X$.

**Pf.** $f_n \to f \iff f_n - f \to 0$

W.l.o.g. we can assume that $f = 0$.

Let $\varepsilon > 0$. Define $A_n := \{x \in X : f_n(x) < \varepsilon\}$.

Observe that, by (d), $f_n(x) \geq f_{n+1}(x) \geq \dots$, and $f_n \to 0$ pointwise.

Then $A_n := \{x \in X : f_n(x) < \varepsilon\}$

$$= \underset{\text{continuous}}{f_n^{-1}}(\underset{\text{open}}{(-\infty, \varepsilon)})$$

$\Rightarrow A_n$ is open $\forall\ n \in \mathbb{N}$.

Claim: $X = \bigcup_{n\in\mathbb{N}} A_n$

$\supseteq$: ✓

$\subseteq$: If $x \in X$ and $x \notin \bigcup_{n\in\mathbb{N}} A_n$, then $x \notin A_n\ \forall\ n \in \mathbb{N}$.

Then $f_n(x) \geq \varepsilon\ \ \forall\ n \in \mathbb{N}$.

Since $f_n(x) \to 0$ as $n \to \infty$, we have a contradiction.

Since $X$ is compact, there is a finite subcover $\exists\ N \in \mathbb{N}\quad \bigcup_{k=1}^{N} A_k = X$.

By monotonicity we have $f_{n+1}(x) \leq f_n(x) < \varepsilon$.

If $\underset{x \in A_n}{\underset{\Updownarrow}{f_n(x) < \varepsilon}}$, then $\underset{x \in A_{n+1}}{\underset{\Updownarrow}{f_{n+1}(x) < \varepsilon}}$.

Then $A_n \subseteq A_{n+1}\ \ \forall\ n \in \mathbb{N}$.

$X = \bigcup_{k=0}^{N} A_k = A_N$.

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \underset{X}{\underset{\|}{A_N}} \subseteq A_{N+1}$$

$\therefore\ A_n = X\ \ \forall\ n \geq N$.

$\Rightarrow\ 0 \leq f_n(x) < \varepsilon\ \ \forall\ n \geq N\ \ \forall\ x \in X$

$\Rightarrow\ 0 \leq f_n(x) - f(x) < \varepsilon\ \ \forall\ n \geq N\ \ \forall\ x \in X$

$\Rightarrow\ f_n \to f$ uniformly on $X$.

# Metric of uniform convergence

**Lemma** Let $(X, d_X)$, $(Y, d_Y)$ be m.s.,
$f_n : X \to Y$, $f : X \to Y$.
$f_n$ converges uniformly to $f$

$$\iff \lim_{n\to\infty} \sup_{x\in X} d_Y(f_n(x), f(x)) = 0$$

**Pf.** $\lim_{n\to\infty} \sup_{x\in X} d_Y(f_n(x), f(x)) = 0$

$\iff \forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$,
$\sup_{x\in X} d_Y(f_n(x), f(x)) < \varepsilon$.

$\iff \forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ s.t. $\forall\ n \geq N$,
$d_Y(f_n(x), f(x)) < \varepsilon$.

$\iff f_n$ converges uniformly to $f$.

**Def.** Let $B(X, Y) = \{ f : X \to Y \mid f \text{ is bounded} \}$.

$d_\infty(f, g) := \sup_{x\in X} d_Y(f(x), g(x)) \quad \forall\ f, g \in B(X, Y)$

**Prop.** $d_\infty(f, g)$ defines a metric on $B(X, Y)$.

**Pf.** ① Observe that $d_\infty(f, g) \geq 0 \quad \forall\ f, g \in B(X, Y)$.
Also, $\forall\ f, g \in B(X, Y)$,

$f \in B(X,Y)$

$\Rightarrow$ $f(X)$ is a bounded subset of $Y$.

$\Rightarrow$ $\exists$ $y_1 \in Y$ and $r_1 > 0$ s.t. $f(X) \subseteq B(y_2, r_2)$

Similarly, $\exists$ $y_2 \in Y$, $r_2 > 0$ s.t. $g(X) \subseteq B(y_2, r_2)$.

WTS: $d_\infty(f,g) < \infty$
$$d_\infty(f,g) = \sup_{x \in X} d_Y(f(x), g(x))$$

Assume w.l.o.g. that $r_2 \geq r_1$.

Consider the ball $B(y_1, d_Y(y_1, y_2) + r_2)$.

$$B_{d_Y}(y_2, r_2) \subseteq B_{d_Y}(y_1, r_1) \quad \text{by } \Delta \text{ inequality}$$

and, since $r \geq r_2 \geq r_1$,

$$B_{d_Y}(y_1, r_1) \subseteq B_{d_Y}(y_1, r).$$

$\therefore$ $d_Y(f(x), g(x)) \leq 2r$ $\forall$ $x \in X$.

Moreover, since $d_Y(f(x), g(x)) = d_Y(g(x), f(x))$ $\forall$ $x \in X$,

$d_\infty(f,g) = d_\infty(g,f)$.

② $d_\infty(f,g) = 0 \Rightarrow \sup_{x \in X} d_Y(f(x), g(x)) = 0$

$\Rightarrow 0 \leq d_Y(f(x), g(x)) \leq 0$

$\Rightarrow f(x) = g(x) \quad \forall \ x \in X$

$\Rightarrow f = g$

③ To conclude, we need to verify the $\Delta$ inequality:

$$\begin{aligned} d_\infty(f,h) &= \sup_{x \in X} d_Y(f(x), h(x)) \\ &\leq \sup_{x \in X} \left(d_Y(f(x), g(x)) + d_Y(g(x), h(x))\right) \\ &\leq \sup_{x \in X} d_Y(f(x), g(x)) + \sup_{x \in X} d_Y(g(x), h(x)) \\ &= d_\infty(f,g) + d_\infty(g,h) \qquad \forall \ f, g, h \in B(X,Y). \ \square \end{aligned}$$

**Def.** Let $C(X, Y) = \{ f : X \to Y \mid f \text{ is continuous} \}$.

Let $C_b(X, Y) = \{ f : X \to Y \mid f \text{ is continuous and bounded} \}$.

Clearly, $C_b(X, Y) \subseteq B(X, Y)$.

$(B(X, Y), d_\infty)$ is m.s.

**Lemma** $C_b(X, Y) \subseteq B(X, Y)$ is closed.

**Pf.** If $(f_j) \subseteq C_b(X, Y)$ and $f_j \xrightarrow[d_\infty]{} f$, then

$\lim_{j \to \infty} d_\infty(f_j, f) = 0$.

$\Rightarrow \underbrace{f_j}_{\text{continuous } \forall j} \to f$ uniformly

$\Rightarrow f \in C_b(X, Y)$ (uniform limit at continuous (bounded) pt. is continuous) (bounded)

**Prop.** If $(Y, d_Y)$ is complete, then $(B(X, Y), d_\infty)$ is complete, and $(C_b(X, Y), d_\infty)$ is complete.

Pf. Suppose $(f_n)$ is a Cauchy sequence in $(B(X,Y), d_\infty)$.

$\forall \ \varepsilon > 0, \ \exists \ N \in \mathbb{N}$ s.t.

$\forall \ n, m \geq N, \ d_\infty(f_n, f_m) < \varepsilon.$

$$\sup_{x \in X} d_Y(f_n(x), f_m(x)) < \varepsilon \qquad (*)$$

For every $x \in X$, we have that

$$d_Y(f_n(x), f_m(x)) < \varepsilon \quad \forall \ n, m \geq N$$

Then with the $N$ corresponding to $\varepsilon$ as in $(*)$, we obtain:

$\forall \ n \geq N, \ \forall \ x \in X,$

$$d_Y(f_n(x), f(x)) = \lim_{\substack{m \to \infty \\ m \geq N}} d_Y(f_n(x), f_m(x)) \leq \varepsilon$$

$\Rightarrow f_n \to f$ uniformly

$\Rightarrow \lim_{n \to \infty} d_\infty(f_n, f) = 0.$ ◻

$$\sup_{x \in X} f(x) \leq \underbrace{\sup_{x \in X} d_Y(f_N(x), f(x))}_{\leq \varepsilon} + \sup_{x \in X} f_N(x)$$

**Def.** (Norm)

Given a vector space $V$ over $\mathbb{R}$, a function $\|\cdot\| : V \to \mathbb{R}$ is a **norm** if it satisfies:

(1) $\forall\ v \in V: \ \|v\| \geq 0$

(2) $\forall\ v \in V: \ \|v\| = 0 \iff v = \vec{0}$.

(3) $\forall\ v \in V,\ \lambda \in \mathbb{R}: \ \|\lambda v\| = |\lambda|\,\|v\|$

(4) $\forall\ u, v \in V: \ \|u+v\| \leq \|u\| + \|v\|$

$(V, \|\cdot\|)$ is a **normed space**.

**Lemma** If $(V, \|\cdot\|)$ is a normed space, then $d(x,y) := |x-y|$.

inner product v.s. → normed v.s. · length · distance

· length
· distance
· angles

→ m.s. · distance

→ topological space

**Ex.** · $(\mathbb{R}, |\cdot|)$, $(\mathbb{C}, |\cdot|)$

· $(\mathbb{R}^2, \ \|(x,y)\|_2 = \sqrt{x^2+y^2})$

$\|(x,y)\| = |x| + |y|$

$\|(x,y)\|_\infty = \max\{|x|, |y|\}$

$= \sup\{|x|, |y|\}$

**Def.** For $f \in B(X, Y)$, we let

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

$$\|f-g\|_\infty = \sup_{x \in X} |f(x) - g(x)|$$

$$= d_\infty(f, g)$$

**Rmk.** $f_n \to f$ uniformly

$\Leftrightarrow \quad d_\infty(f_n, f) \to 0 \quad$ as $\quad n \to \infty$

$\Leftrightarrow \quad \|f_n - f\|_\infty \to 0 \quad$ as $\quad n \to \infty$