Thm. Picard-Lindelöf $D \subseteq \mathbb{R}^n$

f continuous on $\overset{\text{time}}{[a,b]} \times \overset{\text{space}}{D}$, $(t_0, y_0) \in [a,b] \times D$

(i) If $D = \mathbb{R}^n$: $\dot{y} = f(t,y)$, $y(t_0) = y_0$ has a unique solution $y : [a,b] \to \mathbb{R}^n$

(ii) If $D \neq \mathbb{R}^n$: $\dot{y} = f(t,y)$, $y(t_0) = y_0$ has a unique solution $y : [a,b] \cap [t_0 - \delta, t_0 + \delta] \to \mathbb{R}^n$ for some $\delta > 0$

Previously: $y'' + ay' + by = f(t)$

Ex. Let $P, Q, R \in C([a,b], \mathbb{R})$

$t_0 \in [a,b]$, $y_0, y_0' \in \mathbb{R}$. Then the IVP

$$(1) \quad \begin{cases} \ddot{y} + P\dot{y} + Qy = R(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$

has a unique solution, $y : [a,b] \to \mathbb{R}$.

Pf. Set $y_1 = y$, $y_2 = y'$. Then (1) is equivalent to

$$\begin{cases} y_1' = y_2 \\ y_2' = R - Py' - Qy = R - Py_2 - Qy_1 \\ y_1(t_0) = y_0, \quad y_2(t_0) = y_0' \end{cases}$$

We can also write this as:

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix}$$

$$= f(t, y_1, y_2)$$

Check Lipschitz condition:

$$|f(t, y_1, y_2) - f(t, y_1', y_2')|^2$$

$$= \left| \begin{pmatrix} y_2 - y_2' \\ (R - Py_2 - Qy_1) - (R - Py_2' - Qy_1') \end{pmatrix} \right|^2$$

$$= \left| \begin{pmatrix} y_2 - y_2 \\ P(y_2' - y_2) + Q(y_1' - y_1) \end{pmatrix} \right|^2$$

$$= (y_2 - y_2')^2 + (P(y_2' - y_2) + Q(y_1' - y_1))^2$$

$0 \le (a-b)^2$
$(a\cdot b) \le \frac{1}{2}(a^2+b^2)$

$$
\begin{aligned}
&= (y_2-y_2')^2 + P^2(y_2'-y_2)^2 + 2PQ(y_2'-y_2)(y_1'-y_1) + Q^2(y_1'-y_1)^2 \\
&\le (y_2-y_2')^2 + P^2(y_2'-y_2)^2 + Q^2(y_1'-y_1)^2 \\
&\qquad + \tfrac{1}{2}(P^2+Q^2)\left((y_2'-y_2)^2+(y_1'-y_1)^2\right) \\
&= \left(\frac{P^2}{2}+\frac{3}{2}Q^2\right)(y_1'-y_1)^2 + \left(1+\frac{3}{2}P^2+\frac{Q^2}{2}\right)(y_2'-y_2)^2 \\
&\le \left(1+\tfrac{3}{2}(P^2+Q^2)\right)\left((y_2'-y_2)^2+(y_1'-y_1)^2\right) \\
&\le \underbrace{\max_{t\in[a,b]} \underbrace{\left(1+\tfrac{3}{2}(P^2+Q^2)\right)}_{\text{function of } t}}_{L} \cdot \left|\begin{pmatrix} y_1'-y_1 \\ y_2'-y_2 \end{pmatrix}\right|^2
\end{aligned}
$$

All in all: $|f(t,y_1,y_2) - f(t,y_1',y_2')|^2 \le L\left|\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}\right|^2$

$\forall\, t\in[a,b],\ \forall \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \in \mathbb{R}^2$

hence f is Lipschitz in y.

Recall: $f(t,y) = \begin{pmatrix} 0 & 1 \\ -Q & -P \end{pmatrix} y + \begin{pmatrix} 0 \\ R \end{pmatrix}$

$$f : [a,b] \times \underbrace{\mathbb{R}^2}_{=D} \to \mathbb{R}^2$$
$$t \in [a,b], \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$$

So Picard-Lindelöf part (i) applies, and we get $y : [a,b] \to \mathbb{R}^2$.

E.g. $\ddot{y} + t\dot{y} + 3y = e^t$

We saw: the solution is actually defined on $\mathbb{R}$.

Rmk. if $P, Q, R \in C(\mathbb{R})$

$P, Q, R : \mathbb{R} \to \mathbb{R}$

then we find a solution

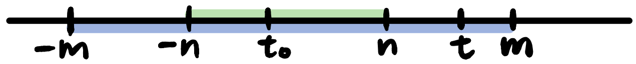
$y : \underline{\underline{\mathbb{R}}} \to \mathbb{R}$ (for all t, not just $t\in[a,b]$)

to $\ddot{y} + P\dot{y} + Qy = R$, $y(t_0) = y_0$, $y'(t_0) = y_0'$

How do we find y?

Let y_n be the solution on $[-n,n]$

(where $n \in \mathbb{N}$ large enough such that $t_0 \in [-n,n]$)



y_n, y_m agree on $[-n,n] \cap [-m,m]$ because they both
$(y_n = y_m)$
are the unique solutions of the same ODE.

Set $y(t) = y_n(t)$ where n is chosen large enough so that $t \in [-n,n]$

Upsot: Solutions to linear ODE are defined for all t

$$\frac{d}{dt} y = \underbrace{M(t)}_{\text{matrix depending continuously on time } (t\in\mathbb{R})} y + \underbrace{R(t)}_{\text{vector in } \mathbb{R}^n \text{ depending continuously on time } (t \in \mathbb{R})}$$

then solutions are defined on $\mathbb{R}$: $y : \mathbb{R} \to \mathbb{R}^n$

Key point: "growth of $\dot{y}$ is (at most) linear in y"

Rmk. (i) $\dot{y} = f(y) = y^2$ is quadratic in y

Recall: Solution to $\dot{y} = y^2$, $y(0) = \frac{1}{y_0} > 0$

is $y(t) = \frac{1}{\frac{1}{y_0} - t}$, and $y(t) \to \infty$ as $t \to \frac{1}{y_0}$

So y only exists up to $\frac{1}{y_0}$.

$y : (-\infty, \frac{1}{y_0}) \to \mathbb{R}$

(ii) For $y(t)$ to be defined for all $t \in \mathbb{R}$

we need $P, Q, R \in C(\mathbb{R})$, and functions like $\frac{1}{t-1}$ (not defined at $t=1$) are not allowed.