Last time: Modified Newton improved linearly convergent scheme to quadratically convergent scheme, but required higher derivatives (expensive)!

Q: Given a linearly convergent sequence, how to modify it to achieve faster convergence?

Def. Assume $\{p\}_{n=0}^{\infty}$ converges and $\lim\limits_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lambda$.

a) If $\lambda = 0$, $\{p\}_{n=1}^{\infty}$ conver esg **superlinearly** to p.

b) If $0<\lambda<1$ $\{p\}_{n=1}^{\infty}$ converges **linearly** to p.

c) If $\lambda = 1$, $\{p\}_{n=1}^{\infty}$ converges **sublinearly** to p.

Ex. The sequence $p_n = \frac{1}{n+1}$ converges sublinearly to 0.

Pf. $\lim\limits_{n\to\infty} \frac{|p_{n+1}-0|}{|p_n-0|} = \lim\limits_{n\to\infty} \frac{\left|\frac{1}{n+2}\right|}{\left|\frac{1}{n+1}\right|} = \lim\limits_{n\to\infty} \frac{n+1}{n+2} = 1$

Aitken's Δ^2 Method

Idea: speed up convergence of linearly conv. sequence.

Let $\{p_n\}_{n=0}^{\infty}$ converge linearly to p, i.e. $\lim\limits_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lambda^{\in(0,1)}$.

Assume that $p_n - p$, $p_{n+1}-p$, $p_{n+2}-p$ have the same sign.

Then $\underbrace{\frac{p_{n+2}-p}{p_{n+1}-p}}_{\approx\lambda} \approx \underbrace{\frac{p_{n+1}-p}{p_n-p}}_{\approx\lambda}$ for sufficiently large n.

Isolating p:

$$p \approx \frac{p_{n+2} - 2p_n - p_{n+1}^2}{p_{n+2}-2p_{n+1}+p_n}$$

$$= \frac{p_n(p_{n+2}-2p_{n+1}-p_n) - (p_{n+1}^2 - 2p_np_{n+1}+p_n^2)}{p_{n+2}-2p_{n+1}+p_n}$$

$$= p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$$

Define new sequence **(Aitken's Method)**:

$$\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$$

Rmk. Need to compute 2 sequences:

$$\begin{array}{ccccc} p_0 & p_1 & p_2 & \longrightarrow & \hat{p}_0 \\ & & p_3 & \longrightarrow & \hat{p}_1 \\ & & \vdots & & \end{array}$$

Def. Given $\{p_n\}_{n=0}^{\infty}$, the 1st-order forward difference is defined as:

$$\Delta p_n = p_{n+1} - p_n$$

2nd-order forward difference:

$$\Delta^2(p_n) = \Delta(\Delta p_n) = \Delta p_{n+1} - \Delta p_n$$
$$= p_{n+2} - 2p_{n+1} + p_n$$

k^{th}-order forward difference:

$$\Delta^k(p_n) = \Delta(\Delta^{k-1} p_n), \quad k \geq 2$$

Thm. Aitken's Method

$$\hat{p}_n = p_n - \frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n}$$

Assume $\{p_n\}_{n=0}^{\infty}$ converges to p linearly, and $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} < 1$.

Then Aitken's Δ^2 sequence $\{\hat{p}_n\}_{n=0}^{\infty}$ converges to p faster than $\{p_n\}$ in the sense that

$$\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = 0$$

$$\hat{p}_n = p_n - \frac{(\Delta p_n)^2}{\Delta^2(p_n)}, \quad n \geq 0$$

Pf. Let $\lim_{n\to\infty} \frac{p_{n+1}-p}{p_n-p} = \lambda$, $\delta = \frac{p_{n+1}-p}{p_n-p}$

Then we have:

$$p_{n+1} - p_n = (p_{n+1}-p) - (p_n - p) = (\delta - 1)(p_n - p)$$

$$p_{n+2} - 2p_{n+1} - p_n = (p_{n+2} - p) - 2(p_{n+1} - p) + (p_n - p)$$

$$= (\delta_{n+1}\delta_n - 2\delta_n + 1)(p_n - p)$$

and
$$\lim_{n\to\infty} \frac{\hat{p}_n - p}{p_n - p} = \lim_{n\to\infty} \frac{1}{p_n - p}\left(p_n - \frac{(p_{n+1}-p)^2}{p_{n+2} - 2p_{n+1} + p_n} - p\right)$$
$$= \lim_{n\to\infty} \frac{1}{p_n - p}(p_n - p) - \frac{(p_{n+1}-p)^2}{(p_n - p)(p_{n+2} - 2p_{n+1} + p_n)}$$
$$= \lim_{n\to\infty} 1 - \frac{(\delta_n - 1)^2}{\delta_{n+1}\delta_n - 2\delta_n + 1}$$
$$= 1 - \frac{(\lambda-1)^2}{\lambda^2 - 2\lambda + 1} = 1 - 1 = 0$$

Rmk. 1 Aitken's may not accelerate convergence if $\{p_n\}_{n=0}^{\infty}$ converges quadratically.

2 Require same signs for $p_{n+1} - p$, $p_n - p$, $p_{n+2} - p$

Aitken's applied to F.P. Iteration

Assume $\{p_n\}_{n=0}^{\infty}$ is generated by FPI method.

Aitken's Δ^2:

p_0, $p_1 = g(p_0)$, $p_2 = g(p_1)$ $\quad\quad$ $\hat{p}_0 = \{\Delta^2\}(p_0) = p_0 - \frac{(p_1 - p_0)^2}{p_2 - 2p_1 + p_0}$

$p_3 = g(p_2)$ $\quad\quad$ $\hat{p}_1 = \{\Delta^2\}(p_1) = p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1}$

$\vdots$ $\quad\quad$ $\vdots$