We saw it suffices to study 1st order ODE systems

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, y_1, \dots y_n) \\ \vdots \\ f_n(x_1, y_1, \dots y_n) \end{pmatrix}$$

in short: $y' = f(x, y)$

Here, x, y are not interchangeable.

However, typically they are both variables of space.

Instead: use derivative w.r.t. time:

$$\frac{d}{dt} y = \dot{y} = f(t, y)$$

$$y = y(t) \qquad t = \text{time}$$

Ex. $\frac{d}{dt} y = \dot{y} = y^2, \quad y(0) = y_0 = 0$

Here: $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f(t, y) = y^2$

Separation of variables:

$$\dot{y} = \frac{dy}{dt} = y^2 \rightarrow \quad dt = \frac{dy}{y^2}$$

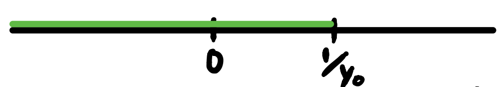
$$\rightarrow t = \int dt = \int \frac{1}{y^2} dy = -\frac{1}{y} + C$$

initial condition: $0 = \frac{-1}{y_0} + C$

$$C = \frac{1}{y_0}$$

$$t = \frac{1}{y_0} - \frac{1}{y} \rightarrow y(t) = \frac{1}{\frac{1}{y_0} - t}$$

$y_0 > 0$: maximal domain of definition $y: (-\infty, \frac{1}{y_0}) \to \mathbb{R}$



$y(t) \to \infty$ as $t \to \frac{1}{y_0}$

Recall: $f(y) = y^2$ is not Lipschitz on $\mathbb{R}$

Thm. Picard-Lindelöf (Picard's Theorem)

Let $f \in C([a,b] \times D)$, $(t_0, y_0) \in D$. $(D \subseteq \mathbb{R}^n$ open$)$

Suppose that f is Lipschitz in y, i.e.

$|f(t_1, y_1) - f(t_1, y_2)| \leq L |y_1 - y_2|$ for all $y_1, y_2 \in D$.

i) if $D = \mathbb{R}^n$, then the initial value problem (IVP)

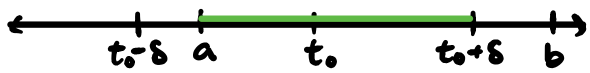
$\dot{y} = f(t, y)$, $y(t_0) = y_0$ (i.e. $\frac{d}{dt}\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, y_1, \dots y_n) \\ \vdots \\ f_n(x_1, y_1, \dots y_n) \end{pmatrix}$, $y(t_0) = y_0$)

has a unique solution $y : [a,b] \to \mathbb{R}^n$.

ii) if $D \neq \mathbb{R}^n$, there is $\delta > 0$ such that the IVP

$\dot{y} = f(t, y)$, $y(t_0) = y_0$ has a unique solution

$y : [a,b] \cap [t_0 - \delta, t_0 + \delta] \to \mathbb{R}^n$.



(i.e. the IVP has a unique local solution)

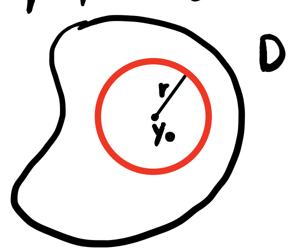
Rmk. i) How large can $\delta > 0$ be?

$\delta > 0$ is at least:

Pick $r > 0$:

$\overline{B_r(y_0)} = \{ y \in \mathbb{R}^n \mid |y - y_0| \leq r \}$

$\subseteq D$



and set $\delta = r \cdot \dfrac{1}{\max\limits_{t \in [a,b],\ y \in \overline{B_r(y_0)}} |f(t,y)|}$

ii) for $n=1$, this is Thm. B in §70 (Simmons)

Corollary Let $f \in C([a,b] \times D)$, $D \in \mathbb{R}^n$ such that all partial derivatives $\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}$ exist and are continuous.

Then there is $\delta > 0$ such that the IVP $\dot{y} = f(t,y)$, $y(t_0) = y_0$ has a unique solution $y : [t_0 - \delta, t_0 + \delta] \cap [a,b] \to \mathbb{R}^n$.

($\frac{\partial f}{\partial y_i}$ continuous $\to$ f Lipschitz in y)

Rmk. This is Thm. A in §70 (Simmons)

Ex. The ODE $\dot{y} = \sqrt{y}$, $y(0) = 0$ has many solutions,

e.g. $y_1(t) = 0$ for all t

$y_2(t) = \frac{1}{4}t^2$: $\dot{y}_2(t) = \frac{1}{2}t$

In particular, $f(t,y) = \sqrt{y}$ cannot be Lipschitz in y close to $y = 0$.

Rmk. Existence (but not Uniqueness) of solutions to $\dot{y} = f(t,y)$, $y(t_0) = y_0$ for continuous functions f follows from Peano's Thm.

Ex. The Riccati equation

$$\dot{y}(t) = t^2 + y^2, \quad y(\underbrace{0}_{t_0 = 0}) = y_0$$

has a unique solution $y : (-\delta, \delta) \to \mathbb{R}$ for some $\delta > 0$.

Here: $f(t,y) = t^2 + y^2$

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

f is differentiable; in particular, $\frac{\partial f}{\partial y} = 2y$ is continuous. So the corollary applies and we get a unique solution $y : (-\delta, \delta) \to \mathbb{R}$, $t_0 = 0$ (take any $\underset{a<0<b}{[a, b]} \subseteq \mathbb{R}$ in the corollary).

Alternatively, check f is Lipschitz in y:

$$\frac{|f(t, y_1) - f(t, y_2)|}{|y_1 - y_2|} = \frac{|t^2 + y_1^2 - t^2 - y_2^2|}{|y_1 - y_2|}$$

$$= \frac{|y_1^2 - y_2^2|}{|y_1 - y_2|}$$

$$= \frac{|y_1 + y_2||y_1 - y_2|}{|y_1 - y_2|}$$

$$= |y_1 + y_2| \leq L$$

$$\leq 2r$$

if $y_1, y_2 \in [y_0 - r, y_0 + r]$

$$= \overline{B_r(y_0)} \in \{y \in \mathbb{R} \mid |y_0 - y| \leq r\}$$

so by P-L part (ii), with $D = (y_0 - r, y_0 + r)$, we get a solution $y : (-\delta, \delta) \to \mathbb{R}$.

Note Part (i) of Picard's Theorem does NOT apply because $g(y) = y^2$ is NOT Lipschitz on $\mathbb{R}$.