3.3 Divided Differences and Newton Form

Q: Given data (x_i, y_i) for $i = 0, 1, \dots n$, how to find coefficients a_i's such that the polynomial

(Newton Form) $$p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)\cdots(x-x_{n-1})$$

interpolates (x_i, f_i)?

Forward Divided Differences

- 0^{th} divided difference of f wrt x_i:
$$f[x_i] = f(x_i)$$
- 1^{st} divided difference of f wrt x_i, x_{i+1}:
$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$
- 2^{nd} divided difference of f wrt x_i, x_{i+1}, x_{i+2}:
$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_i, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$
- k^{th} divided difference of f wrt $x_i, x_{i+1}, \dots x_{i+k}$:
$$f[x_i, \dots x_{i+k}] = \frac{f[x_{i+1}, \dots x_{i+k}] - f[x_i, \dots x_{i+k-1}]}{x_{i+k} - x_i}$$

In particular, $$f[x_0, \dots x_n] = \frac{f[x_1, \dots x_n] - f[x_0, \dots x_{n-1}]}{x_n - x_0} \quad (1)$$

Thm. Let P_n be the n^{th} order polynomial st $P_n(x_i) = f(x_i)$ for $i = 0, 1, \dots n$. Then

$$(*) \quad P(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \dots + f[x_0, \dots x_n](x-x_0)\cdots(x-x_n)$$
$$= f[x_0] + \sum_{j=1}^{n}\left[f[x_0, \dots x_j]\left(\prod_{i=0}^{j-1}(x-x_i)\right)\right]$$

Rmk. 1 (*) referred to as Newton form of polynomial

2 obtain $f[x_0, \dots x_j]$ recursively using (1)

Another (equivalent) way to obtain Newton coefficients:

Recall $P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)\cdots(x-x_{n-1})$

$$\left.\begin{array}{l} P_n(x_0) = f(x_0) \\ P_n(x_1) = f(x_1) \\ \quad \vdots \\ P_n(x_n) = f(x_n) \end{array}\right\} \begin{array}{l} a_0 \\ a_0 + a_1(x-x_0) \\ \quad \vdots \\ a_0 + a_1(x-x_0) + \dots + a_n(x-x_0)\cdots(x-x_{n-1}) \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & (x_1-x_0) & 0 & & 0 \\ 1 & (x_2-x_0) & (x_2-x_0)(x_2-x_1) & \ddots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & (x_n-x_0) & (x_n-x_0)(x_n-x_1) & & (x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1}) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$L\underline{a} = \underline{f}$$

Rmk. 1 This can be easily solved by forward substitution

2 $\text{Det}(L) = 1[(x-x_0)][(x_2-x_0)(x_2-x_1)]\cdots[(x_n-x_0)\cdots(x_n-x_{n-1})]$

$\rightarrow$ if all x_i distinct, L is nonsingular

$\Rightarrow L\underline{a} = \underline{f}$ has unique solution

Ex. Given $(0,0)$, $(\frac{\pi}{2}, 1)$, $(\pi, 0)$, find the

Newton form of polynomial.

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$
$$= a_0 + a_1 x + a_2(x)(x - \tfrac{\pi}{2})$$

$p(0) = 0 \rightarrow a_0 = 0$

$p(\frac{\pi}{2}) = 1 \rightarrow 0 + a_1 \frac{\pi}{2} = 1 \rightarrow a_1 = \frac{2}{\pi}$

$p(\pi) = 0 \rightarrow 0 + \frac{2}{\pi}(\pi) + a_2 \pi(\pi - \frac{\pi}{2}) \longrightarrow a_2 = \frac{-4}{\pi^2}$

$\Rightarrow \ p(x) = 0 + \frac{2}{\pi}x - \frac{4}{\pi^2}x(x - \frac{\pi}{2})$

Recap: We have seen 3 forms of $P(x)$:

- power series (Vandermonde)
- Lagrange
- Newton

Although these may look diff, they each give the SAME $p(x)$.

A general form for interpolation:

- given data $\{(x_i, f_i)\}_{i=0}^{n}$ and a basis of polynomials of deg. n $\{\phi_0(x), \phi_1(x), \dots \phi_n(x)\}$:
  - Find coeffs $a_0, a_1, \dots a_n$ such that $p(x) = a_0\phi_0(x) + a_1\phi_1(x) + \dots + a_n\phi_n(x)$, and $p(x_i) = f_i$.
  - To find $a_0, \dots a_i$, we need to solve a linear system:

$$\left.\begin{aligned} p(x_0) &= f_0 \\ p(x_1) &= f_1 \\ &\vdots \\ p(x_n) &= f_n \end{aligned}\right\} \begin{bmatrix} \phi_0(x_0) & \cdots & \phi_n(x_0) \\ \phi_0(x_1) & & \phi_n(x_1) \\ \vdots & & \vdots \\ \phi_0(x_n) & & \phi_n(x_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$M\underline{a} = \underline{f}$$

Ex. (a) Power Series basis : $\{1, x, \ldots, x^n\}$

$\longrightarrow M = V =$ Vandermonde matrix $O(n^3)$

(b) Newton basis : $\{1, x-x_0, \ldots, (x-x_0)\cdots(x-x_{n-1})\}$

$\longrightarrow M = L =$ Lower triangular matrix $O(n^2)$

(c) Lagrange basis : $\{L_{n,0}(x), \ldots L_{n,n}(x)\}$

where $L_{n,j} = \dfrac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$

$\longrightarrow M = I =$ identity matrix 0

(d) Chebyshev basis : $\{T_0(x), \ldots T_n(x)\}$

where $T_j(x) = \cos(j \cdot \arccos(x))$, $-1 \leq x \leq 1$

$\rightarrow M$: no special structure