E & U

Perko: Differential Equations and Dynamical Systems (Springer): a reference

Notation

For a vector $x \in \mathbb{R}^n$ (typically for us $n=1$ or 2),

let $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

be its Euclidean length.

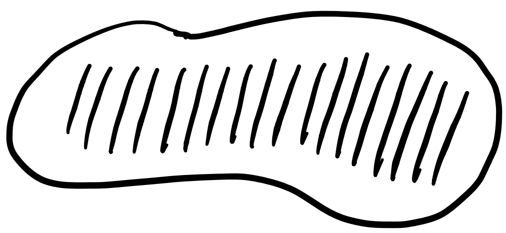
Let $D \subseteq \mathbb{R}^n$ be an open set,

e.g. take $D = \mathbb{R}^n$

$D = B_r(x_0) = \{x \in \mathbb{R}^n \mid |x_0 - x| < r\}$

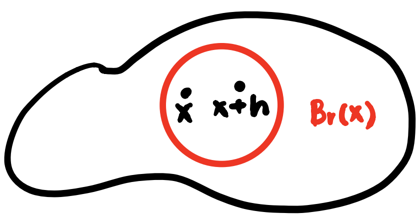
ball of radius $r > 0$ around $x_0 \in \mathbb{R}^n$

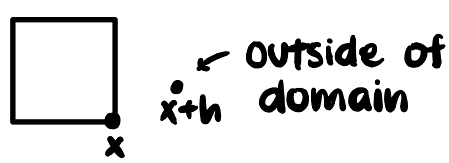
[Technically $D \subseteq \mathbb{R}^n$ is open if for every point in D, there is $r > 0$ such that $B_r(x) \subseteq D$.]



interior is an open set;
the boundary is excluded

Why care? $f'(x) = \lim_{x \to h} \frac{f(x+h) - f(x)}{h}$





D = domain of f

$\frac{1}{x}$ has domain $(-\infty, 0) \cup (0, \infty)$

Notation

i) $C(D) = C(D, \mathbb{R}^n)$

$= \{f : D \to \mathbb{R}^n \mid f \text{ is continuous}\}$

ii) $C'(D) = \{f : D \to \mathbb{R}^n \mid f \text{ is differentiable and } \frac{\partial f}{\partial x_i} \text{ are all continuous}\}$

Rmk. if all 2^{nd} derivatives exist, then $f \in C'$.

iii) $C^k(D) = \{f : D \to \mathbb{R}^n \mid f \text{ is k-times differentiable and } k^{th} \text{ derivative is continuous}\}$

Lipschitz Functions

Def. A continuous $f : D \to \mathbb{R}^n$ is Lipschitz if there is $L > 0$ such that $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$ for all $x_1, x_2 \in D$.
L is called Lipschitz constant.

Prop. Let $I \subseteq \mathbb{R}$ be an open interval, $I = (c, d)$.
boundaries excluded
Suppose $[a, b] \subseteq I$.
boundaries included

If $f \in C'(I)$, then f is Lipschitz on $[a, b]$.

Ex. Every polynomial is Lipschitz on every interval $[a, b]$.

Pf. Let $x_1, x_2 \in [a, b]$, $x_1 < x_2$.

By MVT, $|f(x_1) - f(x_2)| = |f'(z) \cdot (x_1 - x_2)|$

for some $z \in (x_1, x_2)$

$\leq \max\limits_{a \leq z \leq b} |f'(z)| \cdot |x_1 - x_2|$

$= L.$

More generally, if $f \in C^1(D)$, then f is Lipschitz on every closed ball $\overline{B_r(x_0)} = \{x \in \mathbb{R}^n \mid |x_0 - x| \le r\} \subseteq D$

Ex. i) The function $f(x) = x^2$ is Lipschitz on every interval $[a,b]$ (since $f'(x) = 2x$ is continuous $\Rightarrow$ proposition applies)

but NOT Lipschitz on $\mathbb{R}$:

$$\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|} = \frac{|x_1^2 - x_2^2|}{|x_1-x_2|} = \frac{|x_1-x_2||x_1+x_2|}{|x_1-x_2|} = |x_1+x_2|$$

$\rightarrow \infty$ if e.g. $x_1 = 0$, $x_2 \to \infty$

ii) The function $f(x) = \sqrt{x}$ is NOT Lipschitz on $[0,1]$ because

$$\frac{\sqrt{x_1}-\sqrt{x_2}}{|x_1-x_2|} = \frac{(\sqrt{x_1}-\sqrt{x_2})(\sqrt{x_1}+\sqrt{x_2})}{|x_1-x_2|(\sqrt{x_1}+\sqrt{x_2})} = \frac{|x_1-x_2|}{\underbrace{|x_1-x_2|}_{=1}\underbrace{(\sqrt{x_1}+\sqrt{x_2})}_{\to\infty}}$$

if $x_1, x_2 \to 0$

So there is no constant $L > 0$ such that $\frac{\sqrt{x_1}-\sqrt{x_2}}{|x_1-x_2|} \le L$, and f is not Lipschitz on $[0,1]$

note: $f'(x) = \frac{1}{2\sqrt{x}} \to \infty$ as $x \to 0$.

Higher-order ODEs as 1st order systems

Ex. Consider

$$(1) \begin{cases} y'' + ay' + by = g(x) \\ y(0) = y_0, \quad y'(0) = y_0' \end{cases}$$

This is a second order ODE because y'' is the highest derivative.

Set $y_1 = y$

$\quad\;\; y_2 = y'$

Then
$$\begin{cases} y_1' = y_2 \\ y_2' = y'' = g(x) - ay' - by \\ \qquad = g(x) - ay_2 - by_1 \\ y_1(0) = y_0, \quad y_2(0) = y_0' \end{cases}$$

is a 1st order ODE system equivalent to (1).

Note that the vector $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ satisfies

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = f(x, y_1, y_2) = \begin{pmatrix} y_2 \\ g(x) - ay_1 - by_2 \end{pmatrix}$$

A similar trick (introducing a new variable for every derivative) works for any n^{th} order ODE, thus it satisfies to study 1st order systems.

$y' = f(x, y)$