Convergence of Fourier Series

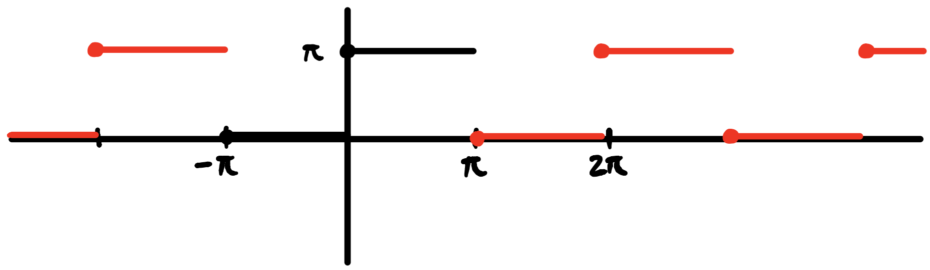
$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx))$

Def. Given $f: [-\pi, \pi) \to \mathbb{R}$ (or $f: (-\pi, \pi] \to \mathbb{R}$),

its extension $\tilde{f}: \mathbb{R} \to \mathbb{R}$ is given by

$$f(x+2\pi k) = f(x) \text{ for all } x \in [-\pi, \pi),\ k \in \mathbb{Z}$$

e.g. $f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ \pi & 0 \le x < \pi \end{cases}$



Def. $f: [a,b] \to \mathbb{R}$ has bounded variation if

$f(x) = u(x) - v(x)$ for non-increasing

functions $u, v: [a,b] \to \mathbb{R}$

Ex. $f \in C^1[a,b] \to f$ has bounded variation

Def. $f(x_0^+) = \lim_{\substack{x \to x_0 \\ x > x_0}} f(x)$

$f(x_0^-) = \lim_{\substack{x \to x_0 \\ x < x_0}} f(x)$

Ex. if $f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ \pi & 0 \le x < \pi \end{cases}$,

$f(0^+) = \pi,\ f(0^-) = 0.$

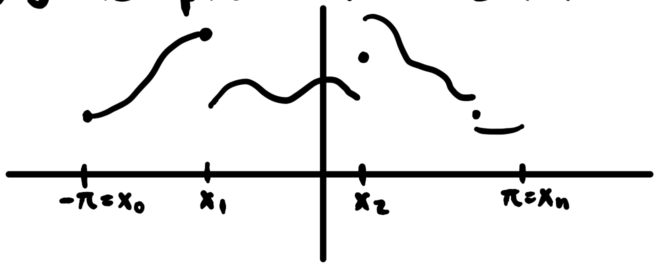
Ex. Suppose $f: [-\pi, \pi) \to \mathbb{R}$

a) f bounded, i.e. $|f(x)| \le M$ for some constant

M for all $x \in [-\pi, \pi)$

b) f has only finitely many minima and maxima

c) f is piecewise continuous



f is continuous on $f|_{(x_i, x_{i+1})}$

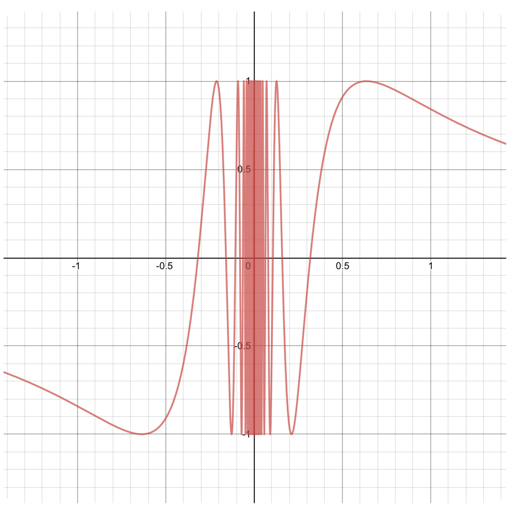
and all limits of $f(x_i^{\pm})$ exist and are finite

($\Leftrightarrow$ f has finitely many jumps

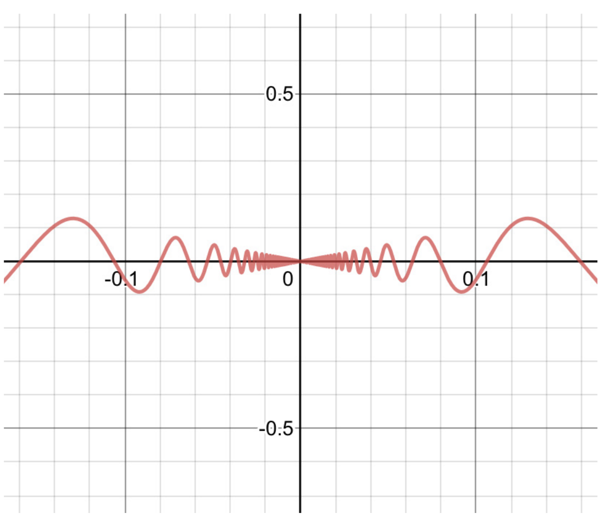
and is otherwise continuous)

If (a)-(c) hold, then f has bounded variation.

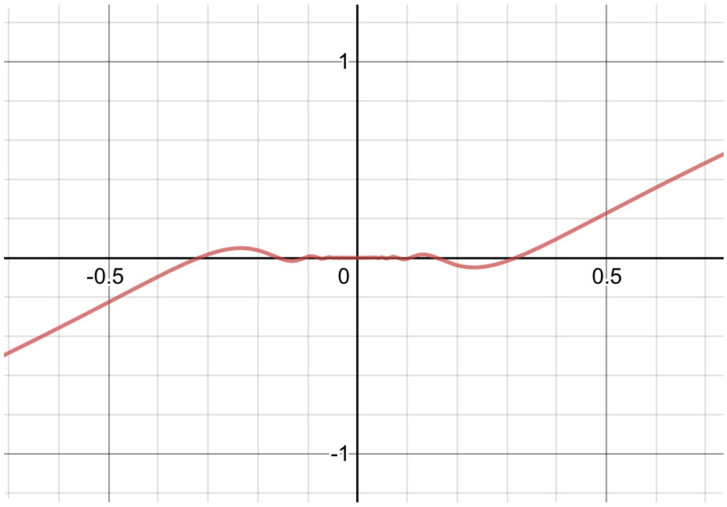
Ex. $f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$



- f is not continuous at $x=0$
- f does not have bounded variation
- f oscillates wildly around $x=0$



- $xf(x)$ is continuous but does not have bounded variation



- $x^2 f(x)$ is even C^n, hence has bounded variation

Thm. (Dirichlet)

Let $f : [-\pi, \pi) \to \mathbb{R}$ be integrable, $\tilde{f}$ its extension to $\mathbb{R}$.

Suppose that f has bounded variation.

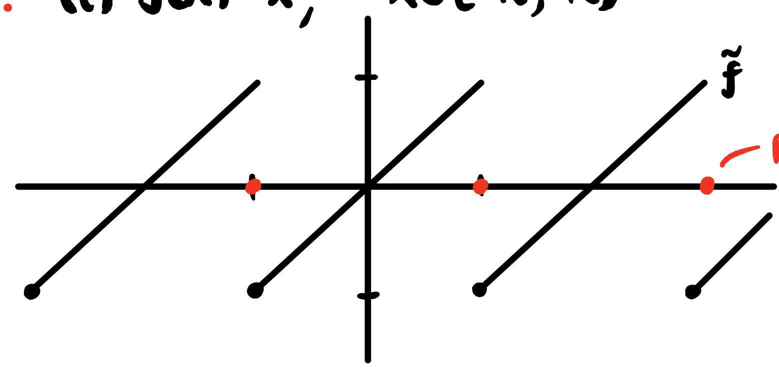
Then $f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$

$$= \frac{1}{2}(\tilde{f}(x^+) + \tilde{f}(x^-))$$

for all $x \in \mathbb{R}$.

In particular, if f is in addition continuous, then $\frac{1}{2}(\tilde{f}(x^+) + \tilde{f}(x^-)) = f(x)$.

Ex. (i) $f(x) = x$, $x \in [-\pi, \pi)$



Last lecture:

$$f(x) \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(kx)$$

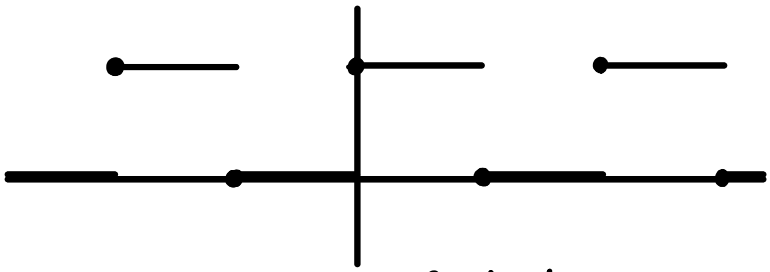
f has bounded variation

$$= \begin{cases} \tilde{f}(x) & \text{if } x \neq 2\pi k+\pi, \quad k\in\mathbb{Z} \\ 0 & \text{if } x = 2\pi k+\pi, \quad k\in\mathbb{Z} \end{cases}$$

$$f((\pi+2\pi k)^+) = \pi$$

$$f((\pi+2\pi k)^-) = -\pi$$

(ii) $f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ \pi & 0 \leq x < \pi \end{cases}$



$$f(x) \sim \frac{\pi}{2} + 2\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

$$= \begin{cases} \tilde{f}(x) & \text{if } x \neq \pi k \quad k\in\mathbb{Z} \\ \frac{\pi}{2} & \text{if } x = \pi k \quad k\in\mathbb{Z} \end{cases}$$

since $\frac{1}{2}(f((k\pi)^-) + f((k\pi)^+))$

$$= \frac{1}{2}(0+\pi) = \frac{1}{2}(\pi+0) = \frac{\pi}{2}$$

Cor. If $f:\mathbb{R}\to\mathbb{R}$ is 2π-periodic and piecewise C^1, i.e. $f|_{(x_i, x_{i+1})} \in C^1$ and all limits $f(x_i^{\pm})$, $f'(x_i^{\pm})$ exist and are finite, then

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k\cos(kx) + b_k\sin(kx))$$

$$= \frac{1}{2}(f(x^+) + f(x^-)).$$