Ex. $f_n : [0, 1] \to \mathbb{R}$

$$f_n(x) = \begin{cases} 2n & \text{if } \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

$f(x) := 0 \quad \forall \quad x \in [0, 1]$

$f_n \to f$ pointwise.

$$\int_0^1 f_n(x)\,dx = \int_{1/2n}^{1/n} f_n(x)\,dx$$
$$= 2n\left(\frac{1}{n} - \frac{1}{2n}\right)$$
$$= 1$$

$$\int_0^1 f(x)\,dx = 0$$

Ex. $Q \cap [0, 1] = \{q_1, q_2, \dots\}$

$$f_n(x) = \begin{cases} 1 & \text{if } x = q_i,\ i \le n \\ 0 & \text{otherwise} \end{cases}$$

$f_n$ is R.I. $\forall$ $n \in \mathbb{N}$.

$$\int_0^1 f_n(x)\,dx = 0$$

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

**Cor.** Suppose $f_n : [a,b] \to \mathbb{R}$ are Riemann-integrable and $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a,b]$.
Then $\sum_{n=1}^{\infty} f_n$ is Riemann-integrable and
$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx$.

**Pf.** $S_N := \sum_{n=1}^{N} f_n$

$S := \sum_{n=1}^{\infty} f_n$

$S_N$ is Riemann-integrable $\forall\ N \in \mathbb{N}$.

$S_N \to S$ uniformly.

$$\begin{aligned}
\lim_{N\to\infty} \int_a^b S_N &= \int_a^b S \\
&= \int_a^b \left(\sum_{n=1}^{\infty} f_n\right) \\
&= \lim_{N\to\infty} \int_a^b \left(\sum_{n=1}^{N} f_n\right) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \left(\int_a^b f_n\right) \\
&= \sum_{n=1}^{\infty} \int_a^b f_n
\end{aligned}$$

## Uniform convergence and derivatives

Ex. $f_n : [0, 2\pi] \to \mathbb{R}$ $\qquad$ $f : [0, 2\pi] \to \mathbb{R}$

$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ $\qquad$ $f(x) = 0$

$\forall \ x \in [0, 2\pi], \ \lim_{n\to\infty} f_n(x) = 0$

$f_n \to f$ pointwise.

$|f_n(x) - f(x)| = \frac{|\sin(nx)|}{\sqrt{n}} \leq \frac{1}{\sqrt{n}} \to 0$

$f_n \to f$ uniformly.

$f_n'(x) = \sqrt{n} \cos(nx)$

$f_n'(0) = \sqrt{n}$

$f'(0) = 0$

$\Rightarrow \ \lim_{n\to\infty} f_n'(x) = f'(x)$ at $x = 0$

Ex. $f_n : [-1, 1] \to \mathbb{R}$ $\qquad$ $f : [-1, 1] \to \mathbb{R}$

$f_n(x) = \sqrt{x^2 + 1/n^2}$ $\qquad$ $\sqrt{x^2} = |x|$ $\qquad$ $f(x) = |x|$

$\sqrt{x^2 + 1/n^2} - \sqrt{x^2} \leq \sqrt{1/n^2} = \frac{1}{n}$

$f_n \to f$ uniformly.

$f_n'(0) = 0$

$f'(0)$ does not exist.

**Thm.** Suppose that $f_n : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ are functions s.t.:

1. $\forall\ n \in \mathbb{N} :\ \forall\ x \in (a,b) :\ f_n'(x)$ exists.
2. $f_n \to g$ uniformly on $(a,b)$
3. $\exists\ x_0 \in (a,b) :\ \lim_{n\to\infty} f_n'(x_0)$ exists.

Then $\forall\ x \in [a,b] :\ f(x) := \lim_{n\to\infty} f_n(x)$ exists and $f$ is differentiable on $(a,b)$ with $f' = g$.

Moreover, $f_n \to f$ uniformly.

**Pf.** By Fundamental Theorem of Calculus, $\forall\ x \in [a,b]$:

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt$$

$$f(x) := f(x_0) + \int_{x_0}^{x} g(t)\,dt \qquad \forall\ x \in [a,b] \qquad (*)$$

$(*)$: $f$ is differentiable and $f' = g$.

- $\forall\ \varepsilon > 0\ \exists\ N_1 \in \mathbb{N}$ s.t. $|f(x_0) - f_n(x_0)| < \varepsilon/2\ \ \forall\ n \geq N_1$
- $\forall\ \varepsilon > 0\ \exists\ N_2 \in \mathbb{N}$ s.t. $|g(t) - f_n'(t)| < \frac{\varepsilon}{2(b-a)}$

  $\forall\ n \geq N_2\ \ \forall\ t \in [a,b]$

$$\begin{aligned}
|f(x) - f_n(x)| &= \left| f(x_0) + \int_a^b g(t)\,dt - f_n(x_0) - \int_a^b f_n'(t)\,dt \right| \\
&\leq |f(x_0) - f_n(x_0)| + \left| \int_a^b (g(t) - f_n'(t))\,dt \right| \\
&< \varepsilon/2 + \int_a^b \frac{\varepsilon}{2(b-a)}\,dt \\
&= \varepsilon \qquad \forall\ n \geq \max\{N_1, N_2\} \quad \forall\ t \in [a,b]
\end{aligned}$$

**Cor.** Suppose $(f_n)$ is a sequence of differentiable functions on $(a, b)$.

Assume:

(i) $\exists\ x_0 \in [a, b]$ s.t. $\sum_{n=1}^{\infty} f_n(x_0)$ converges

(ii) $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $(a, b)$

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$, is differentiable, and $\left(\sum_{n=1}^{\infty} f_n\right)'(x) = \sum_{n=1}^{\infty} f_n'(x)$.

**Pf.** $S_N := \sum_{n=1}^{N} f_n$

(i) $\Rightarrow \exists\ x_0 \in [a, b]$ s.t. $(S_N(x_0))$ converges

(ii) $\Rightarrow (S_N')$ converges uniformly.

**Def.** $(X, d)$ m.s., $E \subseteq X$.

We say $E$ is **dense** if one of the following is satisfied:

1. $\overline{E} \subseteq X$.
2. $\forall\ x \in X\ \ \exists\ (x_n) \subset E$ s.t. $\lim_{n\to\infty} x_n = x$.
3. For any $A \subseteq X$ open, $A \cap E \neq \phi$.
4. For any open ball $B \subseteq X$, $B \cap E \neq \phi$.
5. $\forall\ x, y \in X : \exists\ a \in E$ s.t. $d(x, a) < d(x, y)$

**Thm.** (Stone - Weierstrass)

$(X, d)$ compact m.s., $A \subseteq C(X, \mathbb{R})$.

Assume:

1. The constant function $\mathbb{1} \in A$.
2. $\forall$ $f, g \in A$, $\forall$ $\lambda \in \mathbb{R}$: $\lambda f + g \in A$.
3. $A$ separates points:

   $\forall$ $x, y \in X$ : $\exists$ $f \in A$ s.t. $f(x) \neq f(y)$

Then $A$ is dense in $(C(X, \mathbb{R}), d_\infty)$.

**Thm.** (Weierstrass Approximation Thm.)

If $f \in C([a, b], \mathbb{R})$, there exists a sequence of polynomials $(p_n)$ s.t. $p_n \to f$ uniformly on $[a, b]$.

In other words, $\forall$ $\varepsilon > 0$ $\exists$ a polynomial $p$ s.t. $\|f - p\|_\infty = \sup_{x \in [a,b]} |f(x) - p(x)| < \varepsilon$.

**Prop.** If $f \in C([a, b], \mathbb{R})$ and $\int_a^b f(x) x^n \, dx = 0$ $\forall$ $n \in \mathbb{N}$, then $f(x) = 0$ on $[a, b]$.

**Thm.** (Contraction Mapping)

Suppose $(X, d)$ is a complete m.s., and $f: X \to X$ is a continuous function which is a contraction: $\exists\ 0 \le c < 1$ s.t.

$$d(f(x), f(y)) \le c \cdot d(x, y) \quad \forall\ x, y \in X.$$

Then $\exists!\ x \in X$ s.t. $f(x) = x$.

**Pf.** Choose $x_0 \in X$, and define inductively

$$x_1 = f(x_0)$$

$$x_2 = f(x_1) = f(f(x_1))$$

$$\vdots$$

$$x_n = f^{(n)}(x_0)$$

$$d(x_1, x_2) = d(f(x_1), f(x_0)) \le c \cdot d(x_1, x_0)$$

$$d(x_3, x_2) \le c \cdot d(x_2, x_1) \le c^2 \cdot d(x_1, x_0)$$

$$\vdots$$

In general, $d(x_{n+1}, x_n) \le c^n \cdot d(x_1, x_0)$.

Since $f$ is a contraction, if $n \ge m$:

$$\begin{aligned} d(x_n, x_m) &= d(f^{(n)}(x_0), f^{(m)}(x_0)) \\ &\le c^m\, d(f^{(n-m)}(x_0), x_0). \\ &\le c^m \big[ d(f^{(n-m)}(x_0), f^{(n-m-1)}(x_0)) + d(f^{(n-m-1)}(x_0), f^{(n-m-2)}(x_0)) \\ &\qquad + \dots + d(f(x_0), x_0) \big] \end{aligned}$$

$$\le c^m \left[ c^{n-m-1} \cdot d(x_1, x_0) + c^{n-m-1} \cdot d(x_1, x_0) + \ldots + c \cdot d(x_1, x_0) + d(x_1, x_0) \right]$$

$$\le c^m \cdot d(x_1, x_0) \, (1 + c + c^2 + \ldots)$$

$$= \frac{c^m}{1-c} \cdot d(x_1, x_0)$$

Since $\frac{c^m}{1-c} \to 0$ as $m \to \infty$:

$(x_n)$ is Cauchy sequence.

$\Rightarrow$ $x_n \to x$ for some $x \in X$.

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$$

$$= \lim_{n\to\infty} x_{n+1}$$

$$= x$$

Suppose also $f(y) = y$. Then

$$d(x, y) \le d(f(x), f(y)) \le c \cdot d(x, y)$$

$$\Rightarrow d(x, y) = 0$$

$$\Rightarrow x = y.$$

# Power Series

**Def.** A power series centered at $a$ is any series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n \qquad c_n \in \mathbb{R}$$

**Def.** Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a formal power series. The radius of convergence of that series is:

$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$$

**Lemma** (Interval of convergence)

Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a formal power series and $R$ its radius of convergence.

1. The series converges absolutely $\forall \; x \in (a-R, a+R)$.
2. The series diverges for $x \notin [a-R, a+R]$.
3. For any $r \in (0, R)$, the series converges absolutely uniformly on $[a-r, a+r]$.
4. The limit function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $x \in (a-R, a+R)$ is continuous on its domain.

Pf. (1) and (2) follow from the root test.

$$\rho = \limsup_{n\to\infty} |c_n(x-a)^n|^{\frac{1}{n}}$$
$$= |x-a| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$
$$= \frac{|x-a|}{R}$$

If $\rho < 1$, series converges absolutely.

If $\rho > 1$, series diverges.

(3): $r \in (0, R)$

Choose $\bar{r} \in (r, R)$ : $\frac{1}{R} < \frac{1}{\bar{r}} < \frac{1}{r}$

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

$\exists \; N \in \mathbb{N} \;\; \forall \; n \geq N$ : $|c_n|^{\frac{1}{n}} \leq \frac{1}{\bar{r}}$

$$|c_n| \leq \frac{1}{\bar{r}^n} \quad \forall \; n \geq N$$

Then
$$\sum_{n=0}^{\infty} |c_n(x-a)^n| = \sum_{n=0}^{N-1} |c_n(x-a)^n| + \sum_{n=N}^{\infty} |c_n(x-a)^n|$$
$$\leq \sum_{n=0}^{N-1} |c_n| r^n + \sum_{n=N}^{\infty} |c_n(x-a)^n|$$
$$\leq \sum_{n=0}^{N-1} |c_n| r^n + \sum_{n=N}^{\infty} \left(\frac{r}{\bar{r}}\right)^N$$

(4): $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$

$$= \lim_{n\to\infty} \sum_{n=0}^{N} c_n(x-a)^n$$

$\forall \; x \in (a-R, a+R)$ : $\exists \; r \in (0, R)$ s.t. $x \in [a-r, a+r]$

Ex. $\cdot \sum_{n=1}^{\infty} \frac{x^n}{n}$ $\quad R=1 \quad [-1, 1)$

$\cdot \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ $\quad R=1 \quad [-1, 1]$

**Lemma** Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on the interval of convergence $(a-R, a+R)$.

Then for any $y, z \in (a-R, a+R)$ w/ $y<z$:

$[y, z] \subseteq (a-R, a+R)$.

$$\begin{aligned}
\int_y^z f(x)dx &= \int_y^z \sum_{n=0}^{\infty} c_n(x-a)^n dx \\
&= \sum_{n=0}^{\infty} \int_y^z c_n(x-a)^n dx \\
&= \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \Big|_y^z \\
&= \sum_{n=0}^{\infty} \frac{c_n}{n+1}\left[(z-a)^{n+1} - (y-a)^{n+1}\right]
\end{aligned}$$

**Lemma** (Infinite differentiability)

Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ on the interval of convergence $(a-R, a+R)$.

$f \in C((a-R, a+R), \mathbb{R})$.

Then $f$ is infinitely differentiable on $(a-R, a+R)$.

Moreover, $\forall\ k \in \mathbb{N}\ \ \forall\ x \in (a-R, a+R)$:

$$\frac{d^k}{dx^k} f(x) = \sum_{n=0}^{\infty} c_n \frac{n!}{(n-k)!} (x-a)^{n-k}$$

where RHS has radius of convergence $R$.