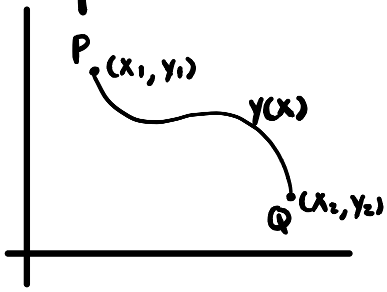
Calculus of Variations

Typical question



(i) What is the shortest path from P to Q?

task: minimize $\int_{x_1}^{x_2} \sqrt{1+y'(x)}\, dx$ over all "nice curves" from P to Q

("admissible curves")

(ii) What is the shape of a wire from P to Q st a bead of mass, driven by its own weight, takes the least time from P to Q?

(Brachistochrone)

Minimize $\int f(x, y, y')dx$

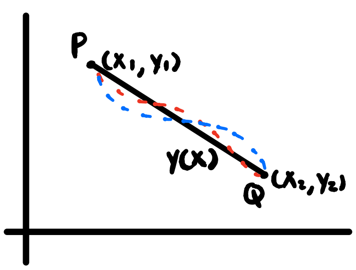
Rmk. in case (ii) $\int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{2g}\cdot\sqrt{y}}\, dx$

How do we find the minimizing curve?

Suppose that $y(x)$ is the minimizing curve.

Then $y(x_1) = y_1$, $y(x_2) = y_2$.

Suppose $\eta : [x_1, x_2] \to \mathbb{R}$ satisfies $\eta(x_1) = \eta(x_2) = 0$.



Then for all small $\alpha \in \mathbb{R}$:

$\bar{y}(x) = y(x) + \alpha\eta(x)$ is also an admissible curve from P to Q and $I(\alpha) = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')dx$

has a minimum at $\alpha = 0$.

Hence $\frac{d}{d\alpha}\big|_{\alpha=0} I(\alpha) = 0$

$$0 = \frac{d}{d\alpha}\Big|_{\alpha=0} \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\,dx$$

$$0 = \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial x} \cdot \underbrace{\frac{\partial x}{\partial \alpha}}_{0} + \frac{\partial f}{\partial \bar{y}} \cdot \underbrace{\frac{\partial \bar{y}}{\partial \alpha}}_{y} + \frac{\partial f}{\partial \bar{y}'} \cdot \underbrace{\frac{\partial \bar{y}'}{\partial \alpha}}_{y'} \Big)\Big|_{\alpha=0} dx$$

$$= \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial y} y + \frac{\partial f}{\partial y'} y' \Big) dx$$

$\int$ by parts: $$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y} y\,dx + \underbrace{\frac{\partial f}{\partial y'} y \Big|_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2} \frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\Big) y\,dx$$

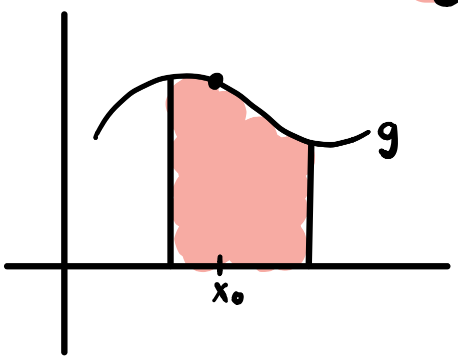
=0 because $y(x_1) = y(x_2) = 0$

$$= \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\Big)\Big) y\,dx$$

for all functions y with $y(x_1) = y(x_2) = 0$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\Big) = 0$$

Euler-Lagrange Eqn



Caution A solution to $\frac{\partial f}{\partial y} - \frac{d}{dx}\Big(\frac{\partial f}{\partial y'}\Big) = 0$ might NOT minimize $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$.
It passes the first derivative test, but we did not check the second derivative.
Hence the solution $y(x)$ is called a stationary point or stationary

function / curve (or extrema if there are no boundary conditions)

Ex. Find E-L equation for

$$I = \int_{x_1}^{x_2} \underbrace{\sqrt{1+(y')^2}}_{f(x,\,y,\,y') = f(y')}\,dx$$

$$0 = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$= 0 - \frac{d}{dx}\left(\frac{2y'}{2\sqrt{1+(y')^2}}\right)$$

caution: $\frac{d}{dx}$ is a total derivative,

hence $\frac{d}{dx}y' = y''(x)$

(instead $\frac{\partial}{\partial y'}y' = 1$,

$\frac{\partial}{\partial x}y' = 0$)

To find y, it's easier to integrate:

$$\frac{y'}{\sqrt{1+(y')^2}} = \text{const.} = c$$

$$(y')^2 = c^2(1+(y')^2)$$

$$0 = c^2 + (y')^2(c^2-1)$$

$$(y')^2 = \frac{c^2}{1-c^2} = \text{const.}$$

$\Rightarrow$ $y''(x) = 0$

$y(x) = ax + b$ is linear

and $y(x_1) = y_1$, $y(x_2) = y_2$

$$\Rightarrow y(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

(ii) Brachistochrone problem:

$$\int_{x_1}^{x_2} \underbrace{\frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}}_{f(x,\,y,\,y') = f(y,\,y')} dx$$

If f does not explicitly depend on x:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}y' - f\right)$$

$$= \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial f}{\partial y'}y'' - \frac{d}{dx}f$$

$$= \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)y' + \frac{\partial f}{\partial y'}y'' - \left(\underbrace{\frac{\partial f}{\partial x}}_{0}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}\right)$$

$$= \left(\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right)y' \underset{\text{E-L}}{=} 0$$