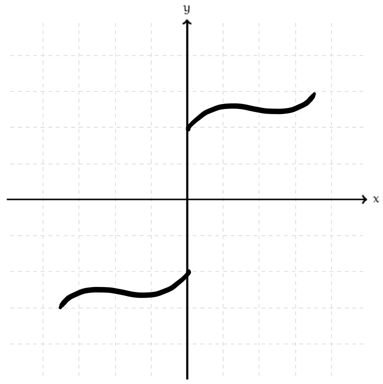
Fourier Sine Series

Consider $f : [0, \pi] \to \mathbb{R}$



$f(x) = -f(-x)$

at $x=0$: $f(0)=0$

Then $\tilde{f}_{odd}(x) = \begin{cases} f(x) & x>0 \\ 0 & x=0 \\ -f(-x) & x<0 \end{cases}$

$\tilde{f}_{odd} : [-\pi, \pi] \to \mathbb{R}$ odd,

$\tilde{f}_{odd}(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx)$, $\quad b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx$

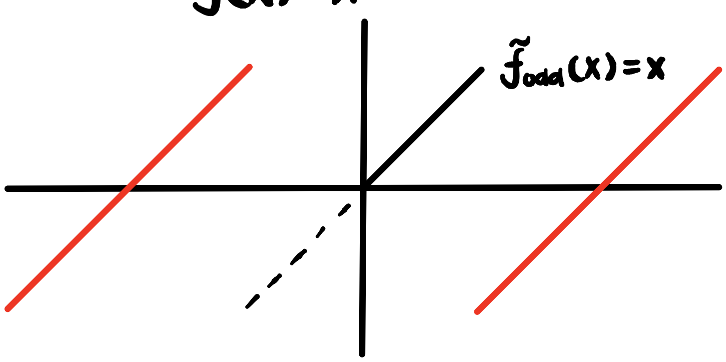
and if f has bounded variation, then

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin(kx) = \frac{1}{2}(\tilde{f}_{odd}(x^-) + \tilde{f}_{odd}(x^+))$$
$$= \frac{1}{2}(f(x^-) + f(x^+)) \quad \text{for } x \in [0, \pi]$$

This is the Fourier sine series associated to f.

Ex. $f : [0, \pi] \to \mathbb{R}$

$f(x) = x$



$$\tilde{f}_{odd}(x) = x \sim 2\sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin(kx)}{k}$$
$$= \begin{cases} x & x \in (-\pi, \pi) \\ 0 & x = -\pi, \pi \end{cases}$$

Orthogonal Functions

Motivation: On $V = \mathbb{R}^n$ we have the inner product

$$(\vec{v}, \vec{w}) = (\vec{v}, \vec{w})_{Eucl} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \sum_{i=1}^{n} v_i w_i$$

The vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$ are orthonormal:

$$(e_k, e_\ell) = \underbrace{\delta_{k\ell}}_{\text{Kronecker Delta}} = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases}$$

and for every $v \in \mathbb{R}^n$, $v = \sum_{k=1}^{n} a_k e_k$ with

$$(v_1, e_1) = \sum_{k=1}^{n} a_k (e_1, e_k) = \sum a_k \delta_{k\ell} = a_\ell,$$

i.e. $v = \sum_{k=1}^{n} (v_1 e_k) e_k$.

Aim: decompose functions in a similar way.

Recall

$$\int_{-\pi}^{\pi} e^{inx} \overline{e^{imx}} \, dx = \int_{-\pi}^{\pi} e^{inx} e^{imx} \, dx = \begin{cases} 2\pi & n = m \\ 0 & n \neq m \end{cases}$$

i.e. the functions $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \mid n \in \mathbb{Z} \right\}$ are ON wrt the inner product $(f, g)_{L^2} = \int_{-\pi}^{\pi} f(x) \underbrace{\overline{g(x)}}_{\text{complex conj. of } g(x)} dx$

Def. An inner product on a vector space V, e.g. $V = \mathbb{R}^n$ or $V = C([a,b], \mathbb{R})$, is a map

$$(\cdot\,, \cdot) : V \times V \to \mathbb{C}$$

i) $(f_1 + c \cdot f_2, g) = (f_1, g) + c(f_2, g)$

ii) $(f, g) = \overline{(g, f)}$

$= (g, f)$ if $f, g : [a,b] \to \mathbb{R}$

iii) $(f, f) \geq 0$, and $(f,f)=0 \Leftrightarrow f=0$

Ex. On $C([a,b], \mathbb{R})$ we have the L^2-inner product

$$(f,g)_{L^2} = \int_a^b f(x)g(x)dx$$

i) $$\begin{aligned}(f_1 + c_2 f_1, g) &= \int_a^b (f_1 g + c f_2) g \\ &= \int_a^b (f_1 g + c f_2 g) \\ &= \int_a^b f_1 g + c\int_a^b f_2 g \\ &= (f_1, g) + c(f_2, g)\end{aligned}$$

ii) $(f,g) = \int_a^b fg = \int_a^b gf = (g,f)$

iii) $(f,f) = \int_a^b \underbrace{f^2}_{\geq 0} \geq 0$

$(f,f)=0$, if $f(x_0) > 0$, then by continuity, $f(x) > 0$ on $(x_0 - \delta, x_0 + \delta)$, and $\int_a^b f^2 \geq \int_{x_0-\delta}^{x_0+\delta} f^2 > 0$

$\Rightarrow f(x_0) > 0 \Rightarrow (f,f) > 0$

equivalently $(f,f)=0 \Rightarrow f=0$

Ex. For $n, m \in \mathbb{N}$, the functions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos(n\pi)}{\sqrt{\pi}}, \quad \frac{\sin(n\pi)}{\sqrt{\pi}}$$

are orthonormal wrt the L^2-inner product on $[a,b] = [-\pi, \pi]$.

(**Rmk.** They are NOT orthonormal on $[0, \pi]$.)

Note $$\cos(nx) = \frac{e^{inx} + e^{imx}}{2}$$

$$\sin(nx) = \frac{e^{inx} - e^{imx}}{2i}$$

$$(\cos(nx), \sin(nx)) = \frac{1}{4i}\int_{-\pi}^{\pi} (e^{inx} + e^{imx})(e^{inx} - e^{imx}) dx$$

$$= \frac{1}{4i}\int_{-\pi}^{\pi} (e^{inx}e^{imx} - e^{inx}e^{-imx} + e^{-inx}e^{imx} - e^{-inx}e^{-imx}) dx$$

$$= \frac{2\pi}{4i}\left(\delta_{n,m} - \underbrace{\delta_{n,-m}}_{0} + \underbrace{\delta_{-n,m}}_{0} - \underbrace{\delta_{-n,-m}}_{=\delta_{n,m}}\right)$$

$n, m \geq 1$

$$= \frac{2\pi}{4i}(\delta_{nm} - \delta_{nm}) = 0$$

Rmk. If $\{\theta_n\}$ are orthogonal,
i.e. $(\theta_n, \theta_m) = 0$ if $n \neq m$,
and $(\theta_n, \theta_n) = c_n > 0$, then
$\{\phi_n = \frac{\theta_n}{\sqrt{c_n}}\}$ are ON.

Ex. Let $g \in C([a,b], \mathbb{R})$ with $g(x) > 0$ for all $x \in [a,b]$.
Then $(f, g)_g = \int_a^b f(x) g(x) \cdot g(x)\,dx$
is an inner product on $C([a,b], \mathbb{R})$