Fourier Analysis

Consider $f: \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) 2π-periodic

i.e. $f(2\pi + x) = f(x)$ for all $x \in \mathbb{R}$

or $f: [-\pi, \pi] \to \mathbb{R}$

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx))$$

approximate f by trig function / expand f wrt frequencies

series might not converge

if it converges, might not converge to $f(x)$

Note $\frac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$

$$= \sum_{k=1}^{n} c_k e^{ikx}$$

where $c_k = \begin{cases} \frac{1}{2}(a_k - ib_k) & k>0 \\ \frac{1}{2}a_0 & k=0 \\ \frac{1}{2}(a_k + ib_k) & k<0 \end{cases}$

and $a_k = c_k + c_{-k}$ $\quad k \geq 0$

$b_k = i(c_k - c_{-k})$ $\quad k \geq 1$

[check w/ Euler's formula $e^{ikx} = \cos(kx) + i\sin(kx)$]

Prop. $\frac{1}{2\pi}\int_{-\pi}^{\pi} (e^{inx} - e^{imx})\,dx = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$

(the functions e^{inx} are orthonormal)

Pf. $\int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = \begin{cases} 2\pi & m=n \\ \left.\frac{e^{i(n-m)x}}{i(n-m)}\right|_{-\pi}^{\pi} = 0 & m \neq n \end{cases}$

If $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ (uniformly on $[-\pi, \pi]$)

then $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx = \sum_{k=-\infty}^{\infty} \frac{c_k}{2\pi} \underbrace{\int_{-\infty}^{\infty} e^{ikx} e^{-imx}\,dx}_{= \begin{cases} 2\pi & \text{if } k=m \\ 0 & \text{otherwise} \end{cases}}$

$= c_m$

f integrable on $[-\pi, \pi]$:

$$f(x) \sim \underbrace{\sum_{k=-\infty}^{\infty} c_k e^{ikx}} = \frac{1}{2}a_0 + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right)$$

Fourier series associated to f

a_k, b_k, c_k Fourier coefficients

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \qquad k \geq 0$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \qquad k \geq 1$$

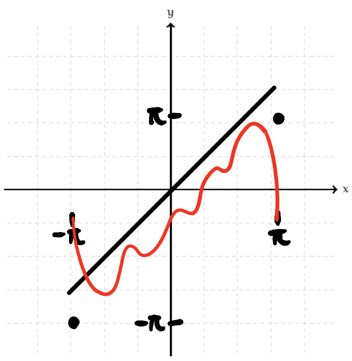
Notation: $c_k = \hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx$

Ex. (i) $f(x) : [-\pi, \pi] \to \mathbb{R}$, $\quad f(x) = x$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx \\ &= \frac{-1}{\pi k}\cos(kx)\cdot x \Big|_{-\pi}^{\pi} + \frac{1}{\pi k}\int_{-\pi}^{\pi}\cos(kx)\,dx \\ &= \frac{-2}{k}\cos(k\pi) + \frac{1}{\pi k^2}\sin(kx)\Big|_{-\pi}^{\pi} \\ &= \frac{-2}{k}(-1)^k + 0 = \frac{-2}{k}(-1)^{k+1} \end{aligned}$$

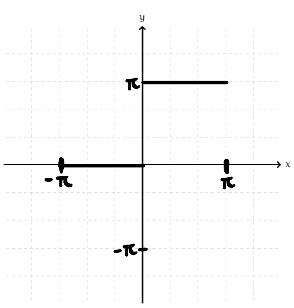
($a_k = 0$ for $k \geq 0$ with similar computation)

$$x \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k}\sin(kx)$$
$$= 2\left(\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \ldots\right)$$



$2\left(\sin(x) - \frac{\sin(2x)}{2} + \ldots + \frac{\sin(5x)}{5}\right)$

(ii) $f(x) : [-\pi, \pi] \to \mathbb{R}$



$$f(x) = \begin{cases} 0 & x \in [-\pi, 0] \\ \pi & x \in [0, \pi] \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} \pi\,dx = 0 + \pi = \pi$$

$$a_k = \frac{1}{\pi}\int_{0}^{\pi} \pi\cos(kx)\,dx$$

$$= \frac{1}{k}\sin(kx)\Big|_0^{\pi} = 0 \quad \text{for } k \geq 1$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} \pi \sin(kx)\,dx$$

$$= \frac{1}{k}\cos(kx)\Big|_0^{\pi}$$

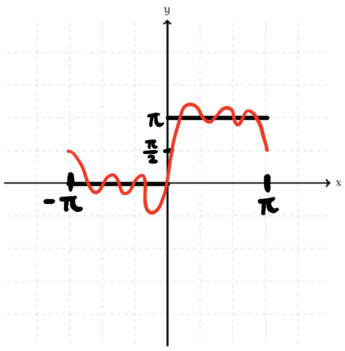
$$= \frac{1}{k}(1-\cos(k\pi)) = \frac{1}{k}(1-(-1)^k)$$

$$= \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{2}{k} & \text{if } k \text{ is odd} \end{cases}$$

$$f(x) \sim \frac{\pi}{2} + \sum_{k=1}^{\infty} b_{2k-1}\sin((2k-1)x)$$

$$= \frac{\pi}{2} + 2\sum_{k=1}^{\infty} \frac{1}{2k+1}\sin((2k-1)x)$$

$$= \frac{\pi}{2} + 2\left(\sin(x) + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \dots\right)$$



$$\frac{\pi}{2} + 2\left(\sin(x) + \frac{\sin(3x)}{3} + \dots + \frac{\sin(7x)}{7}\right)$$

(iii) calculation of fourier coefficients is linear:

if $f(x) = c \cdot g(x) + h(x)$

e.g. $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}(c\cdot g(x) + h(x))\cos(kx)\,dx$

$$= \frac{c}{\pi}\int_{-\pi}^{\pi} g(x)\cos(kx)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos(kx)\,dx$$

Fourier coefficients a_k, b_k, c_k of f

$= c\cdot$(Fourier coefficients a_k, b_k, c_k of g)

$+$ (Fourier coefficients a_k, b_k, c_k of h)

Prop. (i) if f is odd, i.e. $f(-x) = -f(x)$,

then $a_k = 0$, $b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx$

(ii) if f is even, i.e. $f(-x) = f(x)$, then $a_k = \frac{2}{\pi}\int_0^\pi f(x)\cos x\,dx$, $b_k = 0$

Note $\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$

$$= \int_0^\pi f(x)\cos(kx)\,dx + \int_{-\pi}^0 f(x)\cos(kx)\,dx$$

$$= \int_0^\pi f(-x)\cos(-kx)\,dx$$

$$= \int_0^\pi f(-x)\underbrace{\cos(k(-x))}_{=\cos(kx)}\,dx$$

$$= \int_0^\pi (f(-x)+f(x))\cos(kx)\,dx$$

$$= \begin{cases} 0 & \text{if } f \text{ odd} \\ 2\int_0^\pi f(x)\cos(kx)\,dx & \text{if } f \text{ even} \end{cases}$$