Notes 02/26

Today: - Visual: Lyapunov functions

- How to use symmetry?
- Topology : what is special about 2D?

· $\dot V<0$ local min of $V \longrightarrow$ attracting

· $\dot V=0$ linear center $\longrightarrow$ nonlinear cycles /

· $\dot V>0$ local max of $V \longrightarrow$ repelling

Q: If you have a quantity V such that $\dot V<0$, can $\begin{cases}\dot x=f\\ \dot y=g\end{cases}$ have a cycle?

Physical systems

$$\begin{cases} x'=y \\ y'=-V'(x,y) \end{cases}$$

$V(x)$ potential energy

$$x''=\underbrace{-V'(x)}_{F}$$

Always have conserved quantities

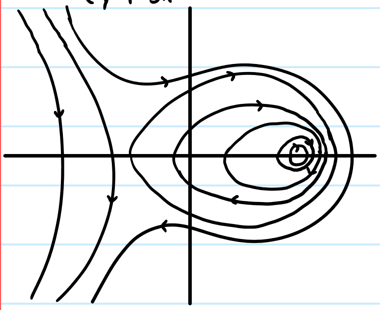
$E=\frac{1}{2}y^2+V(x)$

Ex. Compute $\dot E=0$.

E conserved $\rightarrow$ trajectories are level curves.

$V(x)=-x+x^3$

$$\begin{cases} \dot x=y \\ \dot y=1-3x^2 \end{cases}$$



Same function

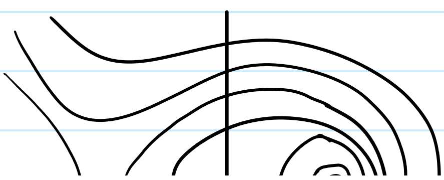
$E(x,y)=\frac{1}{2}y^2+(-x+x^3)$

What about $\dot E<0$?

Systems with friction / damping

$x''=V'(x)\underbrace{-bx'}_{\text{damping}}$ $(b>0)$

$$\begin{cases} \dot x=y \\ \dots \end{cases}$$



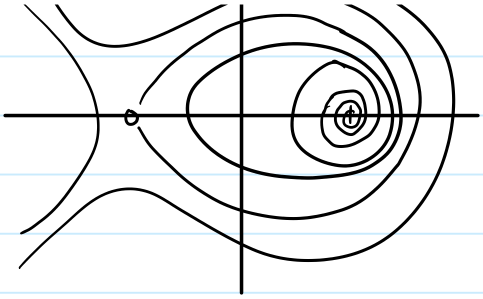
$\dot E<0$: spiral inwards

$$\begin{cases} \dot{x} = y \\ \dot{y} = 1 - 3x^2 - by \end{cases}$$

Non-conservative $(b>0)$

fixed point: $(\dot{x}=0,\ \dot{y}=0)$

$x = \pm \frac{\sqrt{3}}{3}$



$\dot{E}<0$: spiral inwards

Symmetry

$$\begin{cases} \dot{x} = y - y^3 \\ \dot{y} = -x - y^2 \end{cases} \qquad DF = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix}$$

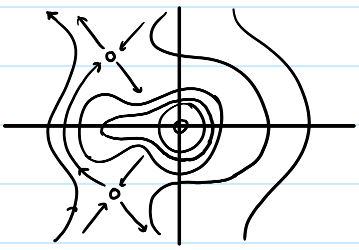
reversible: system invariant

$t \to -T$

$y \mapsto -Y$

· symmetry forces linear centers to be nonlinear centers

fixed points	$(0,0)$	$(-1,-1)$	$(-1,1)$
	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix}$
	center	saddle	saddle



"Topology": "facts that involve being in $\mathbb{R}^2$" (vs. on sphere / $\mathbb{R}^3$)

Why special? A) trajectories don't cross

B) closed curve / trajectory divides $\mathbb{R}^2$ / plane into 2 parts