Lenny Wu

5.3 Higher-Order Taylor

Consider again the IVP

$$\begin{cases} y'(t) = f(t, y(t)) & a \le t \le b \\ y(a) = \alpha \end{cases}$$

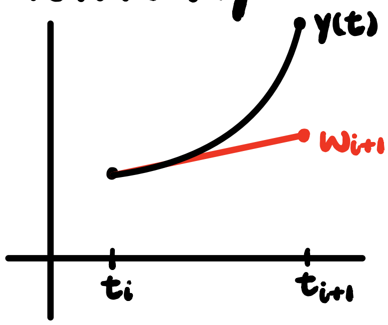
and its corresponding difference equation

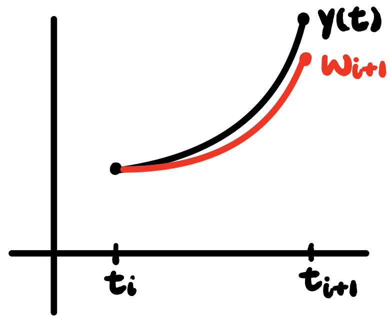
$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\phi(t_i, w_i, h) \end{cases}$$

Recall that Euler's Method uses a linear approximation of $y(t)$ at $t = t_{i+1}$ to obtain $w_{i+1} \approx y(t_{i+1})$. (Error: $O(h)$)

Idea Use a higher order approximation of $y(t)$ to create next step.

Geometrically:





Algebraically:

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \dots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i)$$

$$= y(t_i) + hf(t_i, y_i) + \frac{h^2}{2}\frac{df}{dt}(t_i, y_i) + \dots$$

$$+ \frac{h^n}{n!}\frac{d^{n-1}}{(dt)^{n-1}}f(t_i, y_i) + \frac{h^{n+1}}{(n+1)!}\frac{d^n}{dt^n}f(\xi_i)$$

$$= y(t_i) + h\Big[\underbrace{f(t_i, y_i) + \frac{h}{2}\frac{df}{dt}(t_i, y_i) + \dots + \frac{h^{n-1}}{n!}\frac{d^{n-1}}{(dt)^{n-1}}f(t_i, y_i) + \frac{h^n}{(n+1)!}\frac{d^n}{dt^n}f(\xi_i)}_{\phi(t_i, w_i, h)}\Big]$$

Ex. Let $f(t,y) = y - t^2 + 1$. Find ϕ corresponding to Taylor's Method of order 2.

Sol. For order 2, we need $y''(t) = f'(t,y)$

$$= y' - 2t$$

$$= y - t^2 - 2t + 1$$

$$\rightarrow \phi(w, t, h) = f(t, w) + \frac{h}{2}f'(t, w)$$

$$= w - t^2 + 1 + \frac{h}{2}(w - t^2 - 2t + 1)$$

$$= \left(1 + \frac{h}{2}\right)(w - t^2 + 1) - ht$$

Q: How to compare different methods?
We look at truncation errors.

2 types of truncation errors:

- local truncation error τ_{i+1} at $i+1^{th}$ step (time t_{i+1}) is defined as:

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i, h))}{h}$$
$$= \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i, h)$$

Note: called "local" error b/c it measures accuracy at specific step, and assumes that the method was exact at previous steps.

(hence y_i and not w_i)

- global truncation error e_{i+1} at $i+1^{th}$ step (time t_{i+1}) is defined as:

$$e_{i+1} = |y_{i+1} - w_{i+1}|$$
$$= |y_{i+1} - \underbrace{(w_0 + h\phi(t_0, w_0) + h\phi(t_1, w_1) + \ldots + h\phi(t_i, w_i))|}_{\text{contains errors propagated over time}}$$

Def. A one-step difference equation with local truncation error $\tau_i(h)$ and global error $e_i(h)$ at i^{th} step is called

- **consistent** w.r.t. IVP if
$$\lim_{h\to 0} \max_{1\le i\le N} |\tau_i(h)| = 0$$
- **convergent** w.r.t. IVP if
$$\lim_{h\to 0} \max_{1\le i\le N} \underbrace{|w_i - y_i|}_{e_i(h)} = 0$$

Rmk.
- consistency : difference eqn $\Rightarrow$ differential eqn as $h\to 0$
- convergence : sol. of difference eqn $\Rightarrow$ sol. of differential eqn as $h\to 0$

Def. A numerical method to solve IVP is **of order p** if $\max_{1\le i\le N} \tau_i(h) = O(h^p)$

Ex. Show that Euler's Method is of order 1 and consistent.

Pf. Euler's $\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h \cdot f(t_i, w_i) \end{cases}$

Then $\tau_{i+1} = \dfrac{y_{i+1} - y_i - hf(t_i, y_i)}{h} = \dfrac{\frac{1}{2}h^2 y''(\xi_i)}{h}$

Since $|y''(\xi_i)| \le M$, $\tau_{i+1}(h) = O(h)$

Hence, $\max\limits_{1 \le i \le N} \tau_i(h) = O(h)$.

$\Rightarrow$ Euler's Method is of order 1 and consistent since $\lim\limits_{h \to 0} O(h) = 0$

Rmk. Similar to derivation of Euler's, Taylor's Method of order p and consistent

5.10 Stability of One-Step Methods

Def. A numerical method to solve the IVP is called stable if there is constant K and step size $h_0>0$ such that the difference between the two solutions w_i and $\tilde{w}_i$ to the IVP, with initial values y_0 and $\tilde{y}_0$, respectively, satisfies

$$|w_i-\tilde{w}_i| \le K|y_0-\tilde{y}_0|, \quad i=0,1,\dots N$$

whenever $h<h_0$ and $Nh \le |b-a|$.

Rmk. Roughly speaking, in a stable method, small perturbations in the initial data

$\Rightarrow$ small changes in subsequent approximated solutions.

(Analogous to ODE being well-posed!)

Thm. 5.20

Suppose the IVP $y'=f(t,y)$, $a\le t\le b$, $y(a)=\alpha$ is approximated by a one-step difference method of the form

$$\begin{cases} w_0=\alpha \\ w_{i+1}=w_i+h\phi(w_i,t_i,h) \end{cases}$$

Suppose that ϕ is continuous, Lipschitz continuous in w with constant L on the set

$$\Omega = \{(t,w,h) \mid a \le t \le b,\ -\infty < w < \infty,\ 0 \le h \le h_0\}$$

for some h_0. Then the one-step method is

i) stable

ii) convergent iff it is consistent

i.e. $\phi(t,y,0) = f(t,y)$

iii) if there exists $\tau(h)$ such that

$|\tau_i(h)| \le \tau(h)$ for $i = 0, 1, \dots N$ and $0 \le h \le h_0$,

then $|e_i| = |y_i - w_i| \le \frac{\tau(h)}{L} e^{L(t_i - a)}$

Rmk. 1. ϕ continuous

$$\lim_{(t,y,h) \to (t_0,y_0,h_0)} \phi(t,w,h) = \phi(t_0,w_0,h_0)$$

2. (ii) under assumptions, convergence is equivalent to consistency.

3. (iii) $\Rightarrow$ we can control global error by controlling local error.

5.4 Runge-Kutta Methods (RK)

Recall that for Taylor's methods, to obtain high order accuracy, say $O(h^p)$ with $p>1$, need to compute up to $p-1$ derivatives of f.

Goal of Runge-Kutta Methods:

achieve high order accuracy without evaluating derivatives of f.

Thm. 5.13 (Taylor's Theorem in 2 Variables)

Suppose that $f(t,y)$ and all its derivatives of order $\le n+1$ $\left(\frac{\partial^{i+j} f}{\partial t^i \partial y^j}(t,y) \quad 1 \le i+j \le n+1\right)$ are continuous on $\Omega = [a,b] \times [c,d]$, and let $(t_0, y_0) \in \Omega$.

For every $(t,y) \in \Omega$, there exists ξ between t_0 and t, and μ between y_0 and y s.t.

$$f(t,y) = P_n(t,y) + R_n(t,y), \text{ where}$$

$$\begin{aligned} P_n(t,y) = f(t_0,y_0) &+ \Big[(t-t_0)\frac{\partial f}{\partial t}(t_0,y_0) \\ &+ (y-y_0)\frac{\partial f}{\partial y}(t_0,y_0) + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(t_0,y_0)\Big] \\ &+ \dots + \Big[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(t-t_0)^{n-j}(y-y_0)^j \frac{\partial^n f}{\partial t^{n-j}\partial y^j}(t_0,y_0)\Big] \end{aligned}$$

and $R_n(t,y) = \frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(t-t_0)^{n+1-j}(y-y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j}\partial y^j}(\xi,\mu)$.

Rmk. $\binom{n}{j} = \frac{n!}{j!(n-j)!}$

Ex. Let $f(t,y) = e^{t-y}$. Find $P_2(t,y)$ about $(0,0)$.

Sol. Since $f(t,y) = e^t e^{-y}$:

$$\frac{\partial^n}{\partial t^{n-j}\partial y^j} f(t,y) = e^t (-1)^j e^{-y}$$

$$\rightarrow P_2(t,y) = f(0,0) + \left[t\frac{\partial f}{\partial t}(0,0) + y\frac{\partial f}{\partial y}(0,0)\right]$$
$$+ \frac{1}{2!}\left[\binom{2}{0}t^2\frac{\partial^2 f}{\partial t^2}(0,0) + \binom{2}{1}ty\frac{\partial^2 f}{\partial t\partial y}(0,0) + \binom{2}{2}y^2\frac{\partial^2 f}{\partial y^2}(0,0)\right]$$

$$\rightarrow P_2(t,y) = 1 - t - y + \frac{1}{2}(t^2 - 2ty + y^2)$$

RK Methods of Order 2

Recall: Taylor methods of order 2 required 1st order Taylor expansion:

$$\phi(t,y,h) = T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\underbrace{f'(t,y)}_{\frac{\partial f}{\partial t}}$$

By chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}$$
$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f$$

Main idea: approximate $T^{(2)}(t,y)$ without evaluating derivatives of f.

That is, find a_1, α_1, β_1, such that

$$a_1 \cdot f(t+\alpha_1, y+\beta_1) \approx T^{(2)}(t,y)$$

Note: 1. no derivative eval for $a_1 f(t+\alpha_1, y+\beta_1)$

2. will obtain a_1, α_1, β_1, by using Taylor's Thm. in 2D.

So we have:

$$T^{(2)}(t,y) = f(t,y) + \frac{h}{2}\frac{\partial f}{\partial t}(t,y) + \frac{h}{2}\frac{\partial f}{\partial y}(t,y)\cdot f(t,y) + O(h^2)$$

$$\underbrace{a_1 f(t+\alpha_1, y+\beta_1)}_{\phi(t,y)} = a_1\left[f(t,y) + \alpha_1\frac{\partial f}{\partial t}(t,y) + \beta_1\frac{\partial f}{\partial y}(t,y)\right] + O(h^2)$$

By matching coefficients, we have:

$$\begin{cases} a_1 = 1 \\ a_1\cdot\alpha_1 = \frac{h}{2} \\ a_1\cdot\beta_1 = \frac{h}{2}f(t,y) \end{cases} \Rightarrow \begin{cases} a_1 = 1 \\ \alpha_1 = \frac{h}{2} \\ \beta_1 = \frac{h}{2}f(t,y) \end{cases}$$

Also,

$$\begin{aligned} R_1(t+\alpha_1, y+\beta_1) &= \frac{\alpha_1^2}{2}\frac{\partial^2 f}{\partial t^2}(\xi,\eta) + \alpha_1\beta_1\frac{\partial^2 f}{\partial t\partial y}(\xi,\eta) + \frac{\beta_1^2}{2}\frac{\partial^2 f}{\partial y^2}(\xi,\eta) \\ &= O(h^2) \text{ since } \alpha_1 = \frac{h}{2} \end{aligned}$$

Thus, we achieve the midpoint method:

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)\right) \end{cases}$$

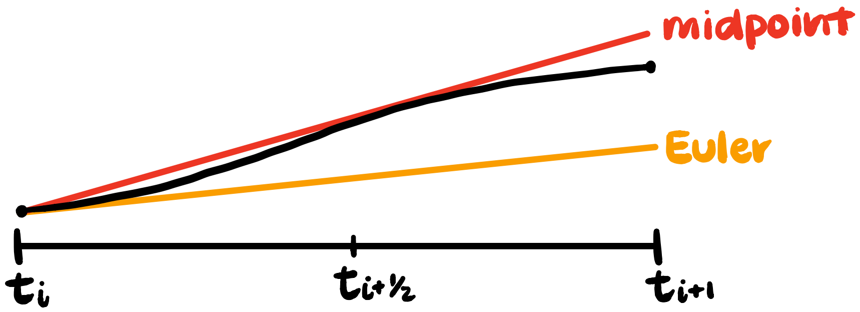
which has local truncation error $O(h^2)$

Rmk. 1. Called "midpoint method" b/c we are evaluating f at midpoint

$$\underbrace{\left(t_i+\tfrac{h}{2}\right.}_{t_{i+\frac{1}{2}}},\ \underbrace{\left.w_i+\tfrac{h}{2}f(t_i,w_i)\right)}_{\approx\, w_{i+\frac{1}{2}}}$$

↑ Euler step to midpoint

2. Instead of moving along tangent line at (t_i, w_i), move along tangent line at $\left(t_i+\frac{h}{2}, w_i+\frac{h}{2}f(t_i,w_i)\right)$



3. Note we had 3 parameters, a_1, α_1, β_1, and we were able to match all partial derivatives of $O(h)$, leading error $O(h^2)$.

Modified Euler's Method

Main idea: approximate $T^{(3)}(t,y)$ with

$$\phi = a_1 f(t,y) + a_2 f(t+\alpha_2,\ y+\delta_2 f(t,y))$$

$$\begin{aligned} T^3(t,y) &= f(t,y) + \frac{h}{2}f'(t,y) + \frac{h^2}{6}f''(t,y) \\ &= f(t,y) + \frac{h}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}f\right) + \frac{h^2}{6}\left(\frac{\partial^2 f}{\partial t^2} + 2\frac{\partial^2 f}{\partial t \partial y} + \frac{\partial^2 f}{\partial y^2}f^2 \right. \\ &\qquad \left. + \frac{\partial f}{\partial y}\frac{\partial f}{\partial t} + \frac{\partial^2 f}{\partial y^2}f\right) \end{aligned}$$

And applying the 2D Taylor's Thm to Φ:

$$a_1 f(t,y) + a_2 f(t+\alpha_2,\ y+\delta_2 f(t,y))$$
$$= (a_1+a_2) f(t,y) + a_2\alpha_2 \frac{\partial f}{\partial t} + a_2\delta_2 f \frac{\partial f}{\partial y}$$
$$+ a^2 \frac{\alpha_2^2}{2} \frac{\partial^2 f}{\partial t^2} + a_2\alpha_2\delta_2 f \frac{\partial^2 f}{\partial t \partial y} + a_2 \frac{\delta^2}{2} f^2 \frac{\partial^2 f}{\partial y^2}$$

Here, cannot match all coefficients of partial derivatives!

i.e. coefficients of both polynomials $a_1, a_2, \alpha_2, \delta_2$ cannot be uniquely determined due to lack of conditions.

Thus, we are left with an $O(h^2)$ approximation by choosing $a_1 = a_2 = \frac{1}{2}$, $\delta_1 = \alpha_2 = h$.

We get the Modified Euler's Method:

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \frac{h}{2}(f(t_i, w_i) + f(t_i+h,\ w_i + hf(t_i, w_i)) \end{cases}$$