Convolution and Delta Functions

Consider $y''+ay'+by=f(x)$, $y(0)=y'(0)=0$

$$\Rightarrow L[y''+ay'+by]=L[f(x)]$$

$$=p^2L[y]-py(0)-y'(0)+apL[y]-ay(0)+bL[y]$$

$$=(p^2+ap+b)L[y]=L[f(x)]$$

$$\Rightarrow L[y]=L[f(x)]\cdot\frac{1}{p^2+ap+b}$$

Suppose that $L[h]=\frac{1}{p^2+ap+b}$.

Then $L[y]=L[f(x)]\cdot L[h(x)]$

$$=L[(f*h)(x)]$$

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$$\rightarrow y=h*f=f*h$$

Prop. Suppose that $p^2+ap+b=(p-m_1)(p-m_2)$

Then $h(x)=\begin{cases} xe^{mx} & \text{if } m_1=m_2=m \\ \frac{1}{\beta}e^{\alpha x}\sin(\beta x) & \text{if } m_{1,2}=\alpha\pm i\beta \\ \frac{e^{m_2x}-e^{m_1x}}{m_2-m_1} & \text{if } m_1\neq m_2,\ m_1,m_2\in\mathbb{R} \end{cases}$

Pf. $L[h]=\frac{1}{p^2+ap+b}=\frac{1}{(p-m_1)(p-m_2)}$

$$=L[e^{m_1x}]\cdot L[e^{m_2x}]$$

$$=L[e^{m_1x}*e^{m_2x}]$$

$$\Rightarrow h(x)=\int_0^x e^{m_1(x-t)}e^{m_2t}\,dt$$

$$=\int_0^x e^{(m_2-m_1)t}\,dt\cdot e^{m_1x}$$

$$=\left[\frac{e^{(m_2-m_1)x}}{m_2-m_1}\Big|_0^x\right]e^{m_1x}$$

$$=\frac{e^{m_2x}-e^{m_1x}}{m_2-m_1}$$

Recall: $(f*g)(x) = \int_0^x f(x-t)g(t)dt$

Ex. $y''+y'-6y = 2e^{3x} = f(x)$

$\to p^2+p-6 = (p+3)(p-2)$

and $h(x) = \dfrac{e^{-3x}-e^{2x}}{-3-2} = \frac{1}{5}(e^{2x}-e^{-3x})$

$$y = h*f = \int_0^x h(x-t)f(t)dt$$
$$= \frac{2}{5}\int_0^x \left(e^{2(x-t)} - e^{-3(x-t)}\right)e^{3t}dt$$
$$= \dots$$
$$= \frac{1}{3}e^{3x} + \frac{1}{15}e^{-3x} - \frac{2}{5}e^{2x}$$

Rmk. $h*f = f*h$

$$h*f = \int_0^x h(x-t)f(t)dt = -\int_x^0 h(x-t)f(t)dt$$

$s=x-t$
$ds=-dt$
$$= \int_0^x f(x-s)h(s)ds$$
$$= f*h$$

Set $A(x) = \int_0^x h(t)dt$. It follows that:

$$y(x) = (h*f)(x) = \int_0^x h(x-t)f(t)dt$$
$$= \int_0^x A'(t)f(x-t)dt = A'*f$$

($\int$ by parts)
$$= A(t)f(x-t)\Big|_0^x - \int_0^x A(t)f'(x-t)(-1)dt$$
$$= A(x)f(0) - \underbrace{A(0)}_{0}f(x) + \int_0^x A(t)f'(x-t)dt$$
$$= A(x)f(0) + (A*f')(x)$$

Rmk. (i) $L[A(x)] = L[\int_0^x h(t)dt]$
$$= \frac{L[h]}{p} = \frac{1}{p(p^2+ap+b)}$$

(ii) A satisfies the ODE
$$\begin{cases} A''+aA'+bA = 1 \\ A(0) = A'(0) = 0 \end{cases}$$

i.e. A solves the ODE for y if we put

$f(x)=1$ (for all $x>0$)

Ex. $y''+y'-6y=2e^{3x}$, $y'(0)=y(0)=0$

$$L[A]=\frac{1}{p(p^2+p-6)}=\frac{1}{p(p+3)(p-2)}$$

$$=-\frac{1}{6p}+\frac{1}{15(p+3)}+\frac{1}{10(p-2)}$$

$$\Rightarrow A(x)=-\frac{1}{6}+\frac{e^{-3x}}{15}+\frac{e^{2x}}{10}$$

$$f(x)=2e^{3x} \longrightarrow f'(x)=6e^{3x}$$

$$\Rightarrow \quad y(x)=A(x)f(0)+(A*f')(x)$$

$$=2A(x)+\int_0^x\left(-\frac{1}{6}+\frac{e^{-3x}}{15}+\frac{e^{2x}}{10}\right)\cdot 6e^{3t}\,dt$$

$$=\ldots$$

$$=\frac{1}{3}e^{-3x}+\frac{1}{15}e^{-3x}-\frac{2}{5}e^{2x}$$

as before.

Ex. Recall $L[x^n]=\dfrac{n!}{p^{n+1}}$

$x^\alpha \quad \alpha\in\mathbb{R}$

Def. $\gamma(z)=\int_0^\infty x^{z-1}e^{-x}\,dx$

Gamma functions

note: $\gamma(n+1)=\int_0^\infty x^n e^{-x}\,dx$
$n\in\mathbb{N}$

$$=L[x^n]\big|_{p=1}=n!$$

"γ interpolates factorials"

$$L[x^\alpha]=\int_0^\infty x^\alpha e^{-px}\,dx$$

$t=px,\ dt=pdx$
$\rightarrow x=\frac{t}{p}$

$$=\int_0^\infty\left(\frac{t}{p}\right)^\alpha e^{-t}\cdot\frac{1}{p}\,dt$$

$$=\frac{1}{p^{\alpha+1}}\int_0^\infty t^\alpha e^{-t}\,dt$$

$$=\frac{\gamma(\alpha+1)}{p^{\alpha+1}}$$

Rmk. (i) $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$

$$L[\sinh(ax)] = \frac{a}{p^2 - a^2}$$

(ii) $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$$L[\cosh(ax)] = \frac{p}{p^2 - a^2}$$