Fourier Series

Def. Let $L > 0$. A function $f : \mathbb{R} \to \mathbb{C}$ is L-periodic if $\forall\ x \in \mathbb{R}$: $f(x+L) = f(x)$.

Ex. $\cos x$, $\sin x$: $L = 2\pi$

$e^{ix} = \cos x + i \sin x$

$e^{i\pi} = \cos \pi + i \sin \pi = -1 \implies e^{i\pi} + 1 = 0$

$e^{i(2\pi)} = 1$

Obs. If f is L-periodic, then if $x \in \mathbb{R}$:

$$\begin{aligned} f(x) &= f(x+L) \\ &= \dots \\ &= f(x+kL). \end{aligned}$$

A 1-periodic function is also $\mathbb{Z}$-periodic.

$f(x+k) = f(x) \quad \forall\ k \in \mathbb{Z}$.

Ex. $\cos(2\pi n x)$, $\sin(2\pi n x)$:

$\cos(2\pi n(x+1)) = \cos(2\pi n x)$

$e^{2\pi i n x}$

$C(\mathbb{R}/\mathbb{Z}, \mathbb{C}) := \{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ continuous and } \mathbb{Z}\text{-periodic} \}$

- We can make $C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ a metric space by restricting the d_∞-metric from $C(\mathbb{R}, \mathbb{C})$ to $C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$.

  $d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$

$$|a+bi| = |(a+bi)(a-bi)|^{1/2}$$
$$= (a^2+b^2)^{1/2}$$

Lemma The map sending f to $f|_{[0,1]}$ is a bijection between $C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and $\{ f : [0,1] \to \mathbb{C} \mid f \text{ continuous and } f(0) = f(1) \}$.

Pf. Suppose $f, g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ satisfy $f|_{[0,1]} = g|_{[0,1]}$.

$$\begin{aligned} f(a.b) &= f(0.b) \\ &= f|_{[0,1]}(0.b) \\ &= g|_{[0,1]}(0.b) \\ &= g(a.b) \end{aligned}$$

$$\begin{aligned} f(x) &= f(x - \lfloor x \rfloor) \\ &= g(x - \lfloor x \rfloor) \\ &= g(x) \end{aligned}$$

$\Rightarrow f = g$

$\Rightarrow$ The map is injective.

Let $f \in \{ f : [0,1] \to \mathbb{C} \mid f$ continuous and $f(0) = f(1) \}$.

Define $g : \mathbb{R} \to \mathbb{C}$

$$g(x) = f(x - \lfloor x \rfloor).$$

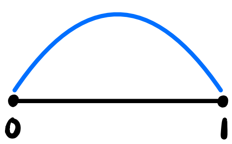
g is 1-periodic.

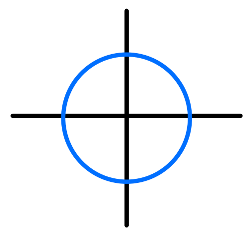
$g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$

$g|_{[0,1]} = f$.

$\Rightarrow$ The map is surjective.

Rmk. This implies we can identify $C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ with continuous functions on $[0,1]$ with endpoints 0 and 1 "identical".





$$\varphi : [0,1] \to \mathbb{C}$$
$$x \mapsto e^{2\pi i x}$$

Def. If $f, g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, we define the inner product by $\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx$.

Rmk. A complex-valued function f is of the form $f(x) = g(x) + ih(x)$, where g, h are real-valued functions.

$$\int_0^1 f(x)dx = \int_0^1 g(x)dx + i\int_0^1 h(x)dx$$

g, h bounded and continuous $\Rightarrow$ integrable

Lemma $\forall\ f, g, h \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, $c \in \mathbb{C}$:

(a) $\langle g, f\rangle = \overline{\langle f, g\rangle}$

(b) $\langle f, f\rangle \geq 0$ and $\langle f, f\rangle = 0 \Leftrightarrow f = 0$

(c) $c\langle f+g, h\rangle = c\langle f, h\rangle + c\langle g, h\rangle$

(d) $\langle f, cg+h\rangle = \langle f, cg\rangle + \langle f, h\rangle$

Ex. $f(x) = 1$ $\quad g(x) = e^{2\pi i x}$

$$\begin{aligned}\langle f, g\rangle &= \int_0^1 e^{-2\pi i x}dx \\ &= \frac{1}{2\pi i}e^{-2\pi i x}\Big|_0^1 \\ &= \frac{1}{2\pi i}(e^{-2\pi i} - 1) \\ &= 0\end{aligned}$$

$\Rightarrow$ f and g are orthogonal.

$\mathbb{R}$: $|x| = \sqrt{x^2}$

$\mathbb{R}^2$: $|(x,y)| = \sqrt{x^2+y^2}$

$= \sqrt{(x,y)\cdot(x,y)}$

$\|f\|_2 = \sqrt{\langle f, f\rangle}$ $\quad$ L^2-norm

(a) $\|f\|_2 \geq 0$ and $\|f\|_2 = 0 \iff f = 0$

(b) $\|cf\|_2 = |c|\,\|f\|_2$

(c) $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$

(d) $|\langle f, g\rangle| \leq \|f\|_2 \cdot \|g\|_2$ $\quad$ C-S ineq.

(e) $\langle f, g\rangle = 0 \implies \|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$ $\quad$ Pythagorean

Rmk. A map $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{C}$ on a vector space V is an inner product if it satisfies the properties above.

Defining $\|v\| := \sqrt{\langle v, v\rangle}$

$d(x,y) := \|x-y\|$ $\quad x, y \in V$

inner product $\implies$ normed $\implies$ metric

If $f, g \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, then $d_2(f,g) = \|f-g\|_2$

$$= \left[\int_0^1 (f(x)-g(x))^2\,dx\right]^{1/2}.$$

$$\begin{aligned}\|f\|_2 &= \langle f, f\rangle^{1/2}\\ &= \left(\int_0^1 f(x)\overline{f(x)}\,dx\right)^{1/2}\\ &= \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}\end{aligned}$$

Note $L^2, d_2 \neq L^\infty, d_\infty$

Ex. $\|f\|_2 \leq \|f\|_\infty \quad \forall \; f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$

$$\int_0^1 |g(x)|\,dx \leq \sup_{x\in[0,1]} |g(x)|$$

Fact uniform convergence $\Rightarrow$ L^2 convergence

$$\begin{aligned} f_n \xrightarrow[d_\infty]{} f \quad &\Rightarrow \quad \|f_n - f\|_\infty \to 0\\ &\Rightarrow \quad \|f_n - f\|_2 \to 0\\ &\Rightarrow \quad f_n \xrightarrow[d_2]{} f\end{aligned}$$

Fact $(C(\mathbb{R}/\mathbb{Z}, \mathbb{C}), d_\infty)$ is complete.

Trigonometric polynomials

write $e_n := e^{2\pi inx} \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$

Def. A function $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ is a **trigonometric polynomial** if we can write

$$f := \sum_{n=-N}^{N} c_n e_n \quad N \in \mathbb{N}, \quad c_n \in \mathbb{C} \quad \forall\, n$$

Ex. $\cos(2\pi nx) = \frac{1}{2}(e_n + e_{-n})$

$\sin(2\pi nx) = \frac{1}{2i}(e_n - e_{-n})$

Thm. The $e^{2\pi inx}$ form an orthonormal system.

$$\langle e_n, e_m \rangle = \langle e^{2\pi inx}, e^{2\pi imx} \rangle = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Pf. $\langle e_n, e_m \rangle = \int_0^1 e^{2\pi inx} \cdot e^{-2\pi imx}\, dx$

if $n = m$: $= \int_0^1 e^0\, dx = 1$

else: $= \int_0^1 e^{2\pi i(n-m)x}\, dx = \dfrac{e^{2\pi i(n-m)x}}{2\pi i(n-m)} \Big|_0^1 = 0$

Cor. If $f = \sum_{n=-N}^{N} c_n e_n$, then $\langle f, e_k \rangle = \langle \sum_{n=-N}^{N} c_n e_n, e_k \rangle$

$$= \sum_{n=-N}^{N} c_n \langle e_n, e_k \rangle$$

$$= c_k$$

$c_n = \langle f, e_n \rangle \quad \forall \; n$

- $\|f\|_2^2 = \langle f, f \rangle$
$= \sum_{n=-N}^{N} |c_n|^2$

Def. Let $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, $n \in \mathbb{Z}$.

The **n^{th} Fourier coefficient** of f is

$$\hat{f}(n) = \langle f, e_n \rangle$$

Fourier transform of f

$$= \int_0^1 f(x) e^{-2\pi i n x} \, dx$$

Thm. (Fourier)

For any $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, the series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ converges to f in the L^2-norm:

$$\lim_{N \to \infty} \|f - \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n\|_2 = 0.$$

Thm. (Parseval)

For any $f \in C(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$