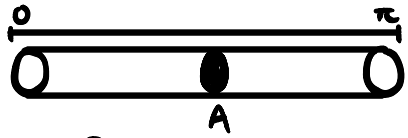
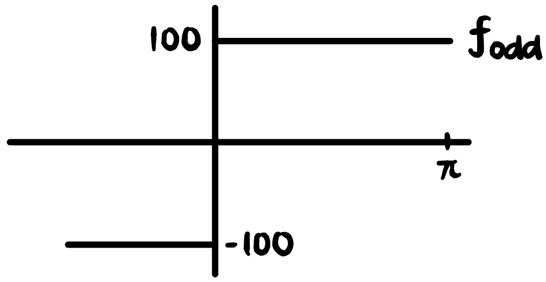
Heat Equation

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2}$$



$w(x,0) = f(x)$ initial temperature

E.g. $f(x) = \begin{cases} 100 & 0<x<\pi \\ 0 & x=0,\pi \end{cases}$



Fourier sine series of f:

$$\frac{400}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1}$$

Then $w(x,t) = \frac{400}{\pi}\sum_{k=1}^{\infty}\frac{e^{-(2k-1)^2a^2t}}{2k-1}\sin(2k-1)x$

Rmk. 1 $t>0$: w is infinitely often differentiable

i.e. temp. smoothens immediately

2 As $t\to\infty$, $w(x,t)$ approaches the solution of the steady-state eqn:

$$0 = \frac{\partial w}{\partial t} = a^2\frac{\partial^2 w}{\partial x^2} \quad \text{i.e.} \quad \frac{\partial^2 w}{\partial x^2} = 0$$

i.e. w is linear, since $w(0) = w(\pi) = 0$

$\longrightarrow w(x) = 0$

if $w(0) = w_1$, $w(\pi) = w_2$, (*)

then $w(x) = w_1 + \frac{1}{\pi}(w_2 - w_1)x$

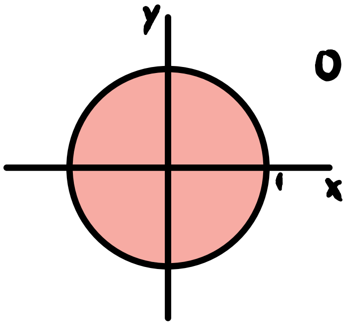
3 If the rod has more dimensions (l, w, h) then heat eqn:

$$\frac{\partial w}{\partial t} = a^2\left(\frac{\partial^2 w}{\partial x^2} \quad \frac{\partial^2 w}{\partial y^2} \quad \frac{\partial^2 w}{\partial z^2}\right)$$

The Dirichlet problem for the disk

Solve Laplace eqn

$$0 = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \Delta w$$



on the unit disk $B_1(0) = \{(x,y) \mid x^2+y^2<1\}$

with boundary conditions

$w(x,y) = f(x,y)$ on $S^1 = \{(x,y) \mid x^2+y^2=1\}$

Rmk. 1 Solution to $\Delta w = 0$ are called harmonic functions

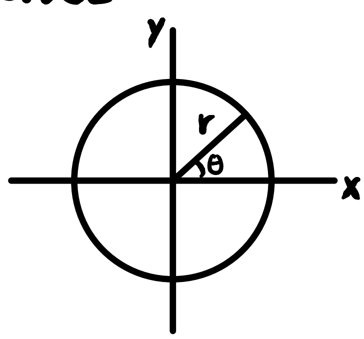
2 The general Dirichlet problem asks for the solutions to $\Delta w = 0$ on a domain $D \subseteq \mathbb{R}^2$ with prescribed values $f(x,y)$ on boundary.

Change to polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

and consider $w(r,\theta)$



Boundary condition: $w(1,\theta) = f(\theta)$

f must be 2π-periodic

Laplace's equation in polar coordinates:

$$\boxed{0 = \Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}}$$

Rmk. This relies on chain rule

e.g. $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$
$= \frac{\partial w}{\partial x}\cdot\cos\theta + \frac{\partial w}{\partial y}\cdot\sin\theta$

Similarly compute $\frac{\partial^2 w}{\partial r^2}, \frac{\partial^2 w}{\partial \theta^2}$

Separation of variables:

$w(r,\theta) = u(r)\cdot v(\theta)$

$\rightarrow 0 = u''v + \frac{1}{r}u'v + \frac{1}{r^2}uv''$

$$\rightarrow \underbrace{-\frac{v''}{v}}_{\text{depends on }\theta} = \frac{u''+\frac{1}{r}u'}{u} = \underbrace{\frac{r^2u''+ru'}{u}}_{\text{depends on }r} = \text{const.} = \lambda$$

$\Rightarrow v'' + \lambda v = 0 \qquad (1)$

$r^2u'' + ru' - \lambda u = 0 \qquad (2)$

recall: $v(\theta)$ 2π-periodic

$v(0) = v(2\pi)$

Note

1. If $\lambda<0$, $a\cosh(\sqrt{|\lambda|}\theta) + b\sinh(\sqrt{|\lambda|}\theta)$
   not 2π-periodic (unless $a, b = 0$)
2. If $\lambda = 0$, $v(\theta) = \text{const} \in \mathbb{R}$
3. If $\lambda > 0$, $a\cos(\sqrt{\lambda}\theta) + b\sin(\sqrt{\lambda}\theta)$
   2π-periodic only if $\sqrt{\lambda}\theta = k\theta$
   for some $k \in \mathbb{N}$

i.e. we have $\lambda = \lambda_k = k^2$

and $v_k(\theta) = a_k\cos(k\theta) + b_k\sin(k\theta)$

Eqn. (2) is $r^2u'' + ru' - k^2u = 0$

require $u(0)$ finite

Solutions are $u_k(r) = \text{const}_k \cdot r^k \quad k \geq 0$

So in total:

$$\begin{aligned} w_k(r,\theta) &= u_k(r) v_k(\theta) \\ &= [a_k \cos(k\theta) + b_k \sin(k\theta)]\, r^k, \quad k \geq 1 \end{aligned}$$

Superpositioning:

$$f(\theta) = w(r,\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} [a_k \cos(k\theta) + b_k \sin(k\theta)]\, r^k$$

must be the Fourier series of f (this determines a_k, b_k uniquely).