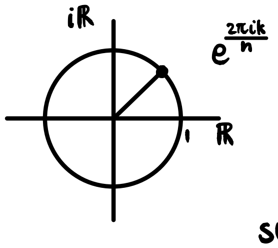
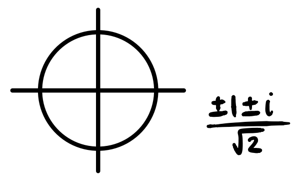
Rmk. i) n^{th} roots of unity: solutions to $z^n=1$



$$\left(e^{\frac{2\pi i k}{n}}\right)^n = e^{2\pi i k\cdot\frac{n}{n}} = 1$$

solutions to $z^n=+1$ are $z_k = e^{\frac{2\pi i k}{n}}$, $k=1,\dots n$

solutions to $z^4=-1$:



ii) if $\omega = re^{i\varphi}$

$$z^n=\omega \quad : \quad z_k = \sqrt[n]{r}\, e^{\frac{i(\varphi+2\pi k)}{n}}$$

Def. The convolution of $f,g:\mathbb{R}\to\mathbb{R}$ is $\underbrace{(f*g)}_{\text{"f convolved with g"}}(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt$

Rmk. $f*g = g*f$ (substitution rule)

If $f,g:[0,\infty)\to\mathbb{R}$ (as always for Laplace transform),

set $f(x)=g(x)=0$ for $x<0$.

It follows that $(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)\,dt = \int_0^x f(x-t)g(t)\,dt$

Thm. Convolution Theorem

$$\begin{aligned} L[(f*g)(x)] &= L\left[\int_0^x f(x-t)g(t)\,dt\right] \\ &= L[f(x)]\cdot L[g(x)] \\ &= F(p)\cdot G(p) \end{aligned}$$

Proof relies on Fubini's theorem.

Ex. $L^{-1}\left[\frac{1}{p^2}\cdot\frac{1}{p^2+1}\right]$

$= L^{-1}\left[L[x]\cdot L[\sin(x)]\right]$

$$= L^{-1}\left[L\left[\int_0^x (x-t)\sin t\,dt\right]\right]$$

$$= \int_0^x (x-t)\sin t\,dt$$

$$= -x\cos t\,\Big|_0^x + t\cos t\,\Big|_0^x - \int_0^x \cos t\,dt$$

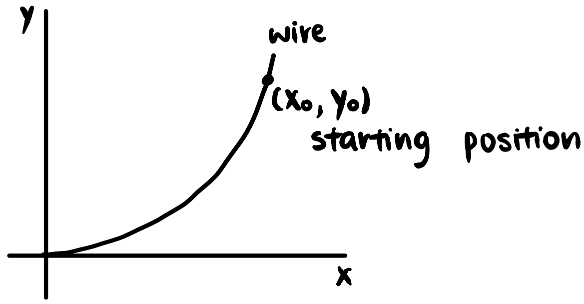
$$= -x\cos x + x + x\cos x - 0 - \sin x + 0$$

$$= x - \sin x$$

alternative: partial fractions

$$\rightarrow \frac{1}{p^2(p^2+1)} = \frac{1}{p^2} + \frac{-1}{p^2+1}$$

Abel's Mechanical Problem



$$\text{transformation: } \underset{\substack{\text{wire}}}{f: x \mapsto f(x)} \longrightarrow \underset{\text{time of descent}}{T: y_0 \mapsto T(y_0)}$$

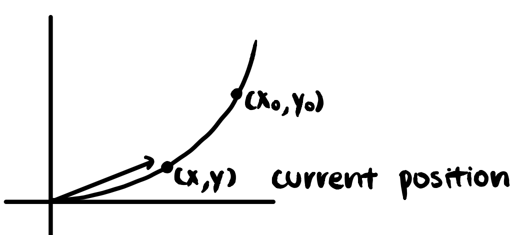
[time that a bead of mass m (starting at rest) takes to slide along the wire from (x_0, y_0)]

Aim: Given $T(y_0)$, find $f(x)$

tautochrone problem: $T(y_0) = T_0$ constant

Conservation of Energy

$$\underbrace{\frac{1}{2m}\left(\frac{ds}{dt}\right)^2}_{\text{kinetic}} = \underbrace{mg(y_0 - y)}_{\text{potential}}$$



s = length of the curve from (0,0) to (x,y)

$$\Rightarrow dt = \frac{-1}{\sqrt{2g(y_0-y)}}\,ds \,,\quad t = t(s)$$

$$T(y_0) = \int_0^{T(y_0)} dt = -\int_{s(y_0)}^{0} \frac{1}{\sqrt{2g(y_0-y)}}\,ds$$

$s=s(y)$, $ds = s'(y)dy$ $\rightarrow$ $$= \int_{-y_0}^{0} \frac{s'(y)}{\sqrt{2g(y_0-y)}}\,dy = \int_0^{y_0} \frac{s'(y)}{\sqrt{2g(y_0-y)}}\,dy$$

$$= \frac{1}{\sqrt{2g}}\left(s' * \frac{1}{\sqrt{y}}\right)(y_0)$$

$\int_0^{y_0} f(y-y_0)\,g(y)\,dy$

take $g(y) = s'(y)$

$f(y) = \frac{1}{\sqrt{y}}$

$$L[T(y_0)] = \frac{1}{\sqrt{2g}} L\left[s'(y) * \frac{1}{\sqrt{y}}\right]$$

$$= \frac{1}{\sqrt{2g}} L[s'(y)] \cdot \underbrace{L\left[\frac{1}{\sqrt{y}}\right]}_{L[y^{-1/2}] = \sqrt{\frac{\pi}{p}}}$$

$$\Rightarrow L[s'(y)] = \sqrt{\frac{2g}{\pi}} \cdot \sqrt{p} \cdot L[T(y_0)]$$

$T(y_0) = T_0$ (tautochrone) $\rightarrow$ $$= \sqrt{\frac{2g}{\pi}} \cdot \sqrt{p} \cdot T_0 \cdot \frac{1}{p}$$

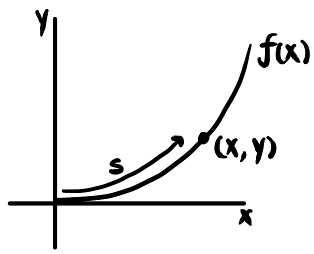
$$= \sqrt{\frac{2g}{\pi}} \cdot T_0 \cdot \frac{1}{\sqrt{p}}$$

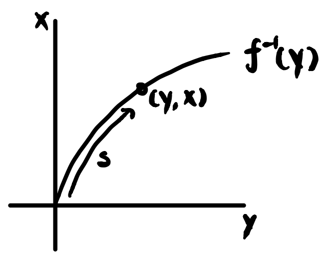
take L^{-1}: $$s'(y) = \sqrt{\frac{2g}{\pi}}\, T_0 \cdot \underbrace{L^{-1}[p^{-1/2}]}_{=\frac{1}{\sqrt{\pi \cdot y}}}$$

(*) $$= \sqrt{2g} \cdot \frac{T_0}{\pi} \cdot \frac{1}{\sqrt{y}}$$

How do we find f(x)?

Sketch:





$$s(y) = \int_0^y \sqrt{1 + (f^{-1\prime}(t))^2}\,dt$$

$$s'(y) = \sqrt{1 + (f^{-1\prime}(y))^2}$$

$$= \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2}$$

(*) $$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \underbrace{2g\left(\frac{T_0}{\pi}\right)}_{=b} \cdot \frac{1}{y} = \frac{b}{y}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{b}{y} - 1} = \sqrt{\frac{b-y}{y}}$$

$$x = \int \sqrt{\frac{y}{b-y}}\,dy = \overset{\text{book}}{\dots} = \frac{b}{2}(2\phi + \sin 2\phi)$$

$y = b\sin^2\theta$

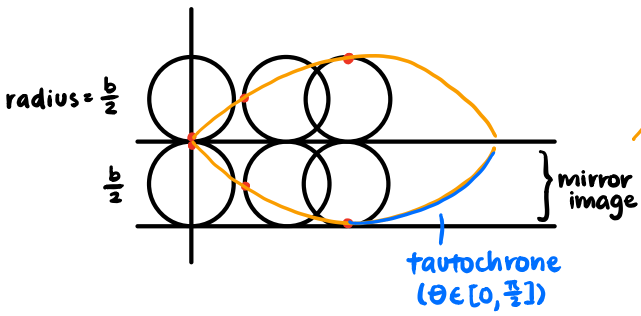
$dy = 2b\sin\theta\cos\theta\,d\theta$

Set $\theta = 2\phi$

$$x = \frac{b}{2}(\theta + \sin\theta) = \frac{b}{2}\left(\theta + \cos\left(\theta - \frac{\pi}{2}\right)\right)$$

$$y = \frac{b}{2} 2\sin^2\left(\frac{\theta}{2}\right) = \frac{b}{2}(1 - \cos\theta)$$

$$= \frac{b}{2}\left(1 - \sin\left(\theta - \frac{\pi}{2}\right)\right)$$



= curve traced by point on circle ($r = 1/2$) moving to the right