1.3 Algorithms and Convergence

Algorithm: procedure that unambiguously describes a finite sequence of of steps in a specified order (can typically be written as pseudocode ⇒ no specific language required)

- **stable:** small changes to input → small changes to output
- **conditionally stable:** stable for some input
- **unstable:** not stable for any input

Let $E_0 > 0$ be error at initial step

E_n be error at n^{th} step

Algorithm has:

- **Linear error growth:** if $E_n \approx c \cdot n \cdot E_0$, $c > 0$ constant
- **Exponential error growth:** if $E_n \approx c^n \cdot E_0$, $c > 1$ constant

Rmk. Linear error growth → stable

Exponential error growth → unstable

Convergence Rate of Sequences

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence such that $\alpha_n \to \alpha$ as $n \to \infty$.

Q. How quickly is α_n approaching α?

Use a second known sequence $\{\beta_n\}$ to describe convergence behavior of $\{\alpha_n\}$.

Def. Let $\alpha_n \to \alpha$ and $\beta_n \to \beta$ as $n \to \infty$.

If there exists $K > 0$ and integer n_0 such that $|\alpha_n - \alpha| \leq K|\beta_n|$ for all $n \geq n_0$, then α_n converges to α with rate/order of $O(\beta_n)$, written as $\alpha_n = \alpha + O(\beta_n)$

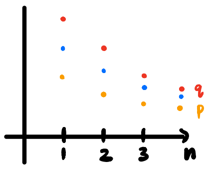
Rmk. 1: β_n is usually chosen as n^{-p}, $p>0$

(generally interested in largest possible p st $\alpha_n = \alpha + O(n^{-p})$)

Rmk. 2: If $0<q<p$ and $\alpha_n = \alpha + O(n^{-p})$, then $\alpha_n = \alpha + O(n^{-q})$

e.g. $|\alpha_n - \alpha| \le K\left|\frac{1}{n^5}\right| \le K\left|\frac{1}{n^2}\right|$

$\uparrow p=5 \qquad q=2 \uparrow$



Ex. Let $n \ge 1$.

$\alpha_n = \frac{n+1}{n^2} \qquad (\alpha = 0)$

$$|\alpha_n - 0| = \left|\frac{n+1}{n^2}\right| = \left|\frac{1}{n} + \frac{1}{n^2}\right| \le \underbrace{2\left|\frac{1}{n}\right|}_{K|\beta_n|}$$

$\rightarrow \alpha_n = 0 + O(\frac{1}{n})$

$\hat{\alpha}_n = \frac{n+1}{n^3} \qquad (\alpha = 0)$

$$|\hat{\alpha}_n - 0| = \left|\frac{n+1}{n^3}\right| = \left|\frac{1}{n^2} + \frac{1}{n^3}\right| \le 2\left|\frac{1}{n^2}\right|$$

$\rightarrow \hat{\alpha}_n = 0 + O(\frac{1}{n^2})$

$\{\hat{\alpha}_n\}$ converges faster!

Similarly for functions:

Let $\lim_{h\to 0} F(h) = L$

Q. How fast is F approaching L? (as $h \to 0$)

Use a known function $G(h)$, where $\lim_{h\to\infty} G(h) = 0$.

Def. Let $\lim_{h\to 0} F(h) = L$, $\lim_{h\to 0} G(h) = 0$. If there exists $K>0$, $h_0>0$ such that $|F(h) - L| \le K|G(h)|$ for $h<h_0$, then we write $F(h) = L + O(G(h))$.

Rmk. $G(h)$ is usually chosen as h^p, $p>0$

(generally interested in $\max\{p : F(h) = L + O(h^p)\}$)

Ex. Analyze the convergence rate of $F(h) = \sin(h) - h\cos(h)$ as $h \to 0$ $(L=0)$

Note: by Taylor's Thm,

$$\sin(h) = h - \frac{h^3}{6}\cos(\xi) \quad \text{where } 0 \le \xi \le h$$

$$\cos(h) = 1 - \frac{h^2}{2}\cos(\eta) \quad \text{where } 0 \le \eta \le h$$

$$\begin{aligned} |\sin(h) - h\cos(h)| &= \left| h - \frac{h^3}{6}\cos(\xi) - h + \frac{h^3}{2}\cos(\eta) \right| \\ &\le \left| \frac{h^3}{6}\underbrace{\cos(\xi)}_{\le 1} \right| + \left| \frac{h^2}{2}\underbrace{\cos(\eta)}_{\le 1} \right| \\ &\le \left(\frac{1}{6} + \frac{1}{2}\right)|h^3| \end{aligned}$$

$\rightarrow \sin(h) - h\cos(h) = 0 + O(h^3)$