Gaussian Quadrature

Basic Idea: choose weights w_i and $x_i \in [a, b]$ (optimally) st $R(f) = \sum_{i=0}^{n} w_i f(x_i)$

have DOF $2n+1$ (n : # intervals)

Basic Properties:

- all $w_i > 0$
- all pts $x_i \in (-1, 1)$
- weights satisfy symmetry condition:

  n odd $\Rightarrow$ $x_0 = -x_n, \quad x_1 = -x_{n-1}, \quad x_2 = -x_{n-2}, \ \ldots$

  n even $\Rightarrow$ $x_0 = -x_n, \quad x_1 = -x_{n-1}, \quad \ldots, \quad x_{\frac{n}{2}} = 0$

Ex. Find a "1-pt" Gauss rule for $\int_{-1}^{1} f(x)dx$

($DOP = 2n+1 = 2 \cdot 0 + 1 = 1$)

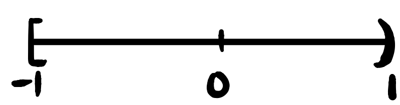
Sol. $R(f) = w_0 f(x_0)$

- symmetry: $x_0 = 0$
- DOP: $2 \cdot 0 + 1 = 1$

$\Rightarrow R(f) = \int_{-1}^{1} f(x)dx$ for $f(x) = 1, x$

$f(x) = 1 \qquad \int_{-1}^{1} 1 dx = 2 = w_0$

$\Rightarrow$ "1-pt" Gauss-rule is: $R(f) = 2f(x_0)$



Rmk. For $[-1, 1]$ this is simply midpt rule!

Ex. Find a "2-pt" Gauss rule for $\int_{-1}^{1} f(x)dx$

i.e. $R(f) = w_0 f(x_0) + w_1 f(x_1)$

Sol. · symmetry : $x_0 = -x_1$, $w_0 = w_1$

· DOP $= 2\cdot 1 + 1 = 3$

$\Rightarrow R(f) = \int_{-1}^{1} f(x)dx$ for $f(x) = 1, x, x^2, x^3$

$\int_{-1}^{1} 1dx = 2 = w_0 + w_1$

$\int_{-1}^{1} xdx = 0 = w_0x_0 + w_1x_1$

$\int_{-1}^{1} x^2dx = \frac{2}{3} = w_0x_0^2 + w_1x_1^2$

$\int_{-1}^{1} x^3dx = 0 = w_0x_0^3 + w_1x_1^3$

(symm.) $2w_0 = 2 \longrightarrow w_0 = w_1 = 1$

(symm.) $x_0 = -x_1 \longrightarrow x_0^2 = x_1^2$

$\frac{2}{3} = 2x_0^2 \rightarrow x_0 = \pm\frac{1}{\sqrt{3}}$

$x_0 = \frac{-\sqrt{3}}{3}$, $x_1 = \frac{\sqrt{3}}{3}$

$\Rightarrow R(f) = f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$