Notes 02/21

Today: - Tools to analyze 2D dynamics

- Change of coordinates
- Stability
- Energy

(Stability, Energy) → Lotka-voterra

Ex. $\begin{cases} \dot{x} = y + a(x^2+y^2)x \\ \dot{y} = -x + a(x^2+y^2)y \end{cases}$

$r^2 = x^2 + y^2 \xrightarrow{\text{diff.}} 2r\cdot\dot{r} = 2x\cdot\dot{x} + 2y\cdot\dot{y} \longrightarrow \dot{r} = \frac{x\dot{x}+y\dot{y}}{r}$

$\tan\theta = \frac{y}{x} \xrightarrow{\text{diff.}} \sec^2\theta\cdot\dot{\theta} = \frac{x\dot{y}-\dot{x}y}{x^2} \longrightarrow \dot{\theta} = \frac{x\dot{y}-\dot{x}y}{x^2(1+\tan^2\theta)} = \frac{x\dot{y}-\dot{x}y}{x^2+y^2} = \frac{x\dot{y}-\dot{x}y}{r^2}$

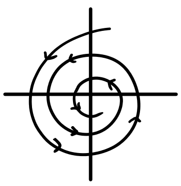
$\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{cases}$

$\dot{\theta} = \frac{x(-x+a(x^2+y^2)y) - y(y+a(x^2+y^2)x)}{r^2} = \frac{-r^2}{r^2} = 1$

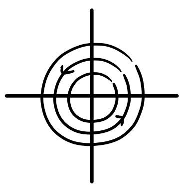
$\dot{r} = \frac{x(y+a(x^2+y^2)x) + y(-x+a(x^2+y^2)y)}{r} = ar^3$

Linearization: $\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$ $\quad DF(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

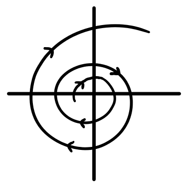
$a<0$: $\dot{r} = ar^3 < 0$

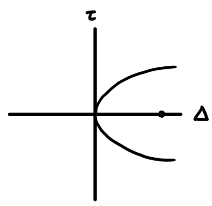


$a=0$: $\dot{r} = 0$



$a>0$: $\dot{r} = ar^3 > 0$





Lotka-Volterra models:

① Competitive population model

x = rabbits   y = sheep

Dynamical system: $\begin{cases} \dot{x} = r_x x\left(1 - \frac{x+\alpha_{xy}y}{k_x}\right) \\ \dot{y} = \underbrace{r_y y}_{\text{exp. growth}}\left(1 - \frac{y+\alpha_{yx}x}{k_y}\right) \end{cases}$

- r_x, r_y growth rates
- k_x, k_y carrying capacities
- α_{xy}, α_{yx} interspecies interaction

$r_x = r_y = 1$

$k_x = k_y = 1$ $\quad \begin{cases} \dot{x} = x(1-x-\frac{1}{2}y) \\ \dot{y} = y(1-y-\frac{1}{2}x) \end{cases}$

$\alpha_{xy} = \alpha_{yx} = \frac{1}{2}$

① Fixed points: $(0,0)$ $(0,1)$ $(1,0)$ $(\frac{2}{3}, \frac{2}{3})$

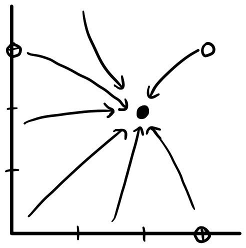
$$DF = \begin{pmatrix} 1-2x-\frac{1}{2}y & -\frac{1}{2}x \\ -\frac{1}{2}y & 1-2y-\frac{1}{2}x \end{pmatrix}$$

$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ $\lambda = 1, 1 \to$ repelling

$DF(1,0) = \begin{pmatrix} -1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ $\quad$ $\lambda = -1, \frac{1}{2} \to$ saddle

$DF(0,1) = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{pmatrix}$ $\quad$ $\lambda = \frac{1}{2}, -1 \longrightarrow$ saddle

$DF(\frac{2}{3}, \frac{2}{3}) = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{pmatrix}$ $\quad$ $\tau^2 > 4\Delta \longrightarrow$ attracting

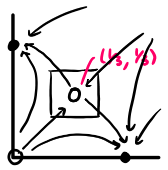


Def. Globally attracting: all initial values converge

Def. Heteroclinic orbit: trajectory / orbit connects two fixed points

Def. Basin of attraction: B a region, space that B is attracted to given fixed pts

$\alpha_{xy} = \alpha_{yx} = 2 \longrightarrow$ "a lot of interaction"

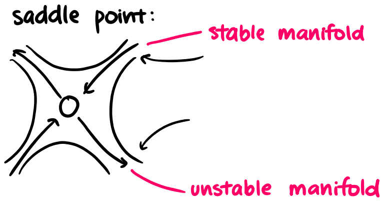


no coexistence

no global attractions

Rmk. In plane: boundaries are trajectories

Zoom at saddle point:



Thm. Stable Manifold Theorem

This picture is accurate at saddle point.

every saddle point (x^*, y^*)

is connected to:

"start at (x^*, y^*)" $\begin{cases} \text{A) trajectory repelling away} \\ \text{B) trajectory attracted to it} \end{cases}$