We saw:

$$y_k(x,t) = v_k(t)\,u_k(x)$$
$$= \left(a_k \cos\left(\frac{\pi k}{\ell} t\right) + b_k \sin\left(\frac{\pi k}{\ell} t\right)\right) \cdot \sin\left(\frac{\pi k}{\ell} x\right)$$

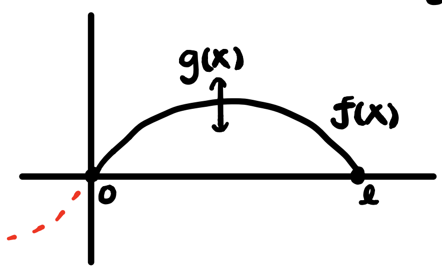
solve 1-D wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

with boundary condition $y(0,t) = y(\ell,t) = 0$

Also want: $y(x,0) = f(x)$

$\frac{\partial y}{\partial t}(x,0) = g(x)$



Aim Superposition the solutions $\frac{1}{k}$ to satisfy boundary conditions.

Note If $y(x,t)$, $z(x,t)$ solve $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$, so does $c \cdot y(x,t) + z(x,t)$.

Look at $y(x,t) = \sum_{k=1}^{\infty} y_k(x,t)$

For simplicity: assume $\ell = \pi$

$$y(x,t) = \sum_{k=1}^{\infty} (a_k \cos(akt) + b_k \sin(akt)) \sin(kx)$$

boundary cond.:

$$f(x) = y(x,0) = \sum_{k=1}^{\infty} a_k \sin(kx)$$

we choose a_k so that $\sum_{k=1}^{\infty} a_k \sin(kx)$ is the Fourier sine series $\tilde{f}_s$ of f.

$$g(x) = \frac{\partial y}{\partial t}(x,0) = \sum_{k=1}^{\infty} \underbrace{a \cdot k \cdot b_k \sin(kx)}_{\text{has to be Fourier sine series of } g(x)}$$

Prop. If $f \in C^2[0,\pi]$, $f(0) = f(\pi) = 0$, $f''(0) = f''(\pi) = 0$ and $g \in C^1[0,\pi]$ with $g(0) = g(\pi) = 0$ then the above procedure works (i.e. the series converges to a solution $y \in C^2(\mathbb{R}^2)$ of the wave equation.)

Rmk. 1 C^2 solutions to the BVP of the wave equation are unique.

2 Every solution can be written as
$$y(x,t) = F(x+at) + G(x-at)$$
i.e. it consists of two waves, one moving right and one moving left.

Indeed: $\cos(akt)\sin(kx)$

$$\frac{1}{2}\sin(k(x+at)) + \frac{1}{2}\sin(k(x-at))$$

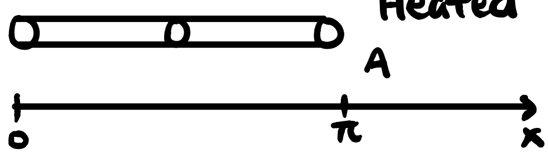
and if, e.g. $g(x) = 0$, then
$$y(x,t) = \sum_{k=1}^{\infty} a_k \cos(akt) \sin(kx)$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} a_k \sin(k(x+at)) + \frac{1}{2}\sum_{k=1}^{\infty} a_k \sin(k(x-at))$$

$$= \frac{1}{2}\left(\underbrace{\tilde{f}_s}_{\text{F.s.s. of } f}(x+at) + \tilde{f}_s(x-at)\right)$$

Heat Equation

Heated rod



heat is evenly distributed across the section A (i.e. only dep. on x, t)

Heat flow modeled by

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2}$$

where $a^2 = \frac{k}{c\rho}$ thermal diffusion

where k = thermal conductivity

c = specific heat capacity

ρ = mass density

Boundary conditions:

$w(x,0) = f(x)$ initial temp.

$w(0,t) = w(\pi,t) = 0$ i.e. $f(0) = f(\pi) = 0$

Separation of variables

$w(x,t) = u(x) \cdot v(t)$

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2}$$

$$\frac{\partial w}{\partial t} = u(x)\cdot \dot{v}(t) \qquad \frac{\partial^2 w}{\partial x^2} = u''(x)\cdot v(t)$$

$$u(x)\dot{v}(t) = a^2 u''(x) v(t)$$

$$\underbrace{\frac{u''(x)}{u(x)}}_{\text{only dep. on } x} = \frac{1}{a^2}\underbrace{\frac{\dot{v}(t)}{v(t)}}_{\text{only dep. on } t} = \text{const} = -\lambda$$

$$\Rightarrow \quad u''(x) + \lambda u(x) = 0 \qquad (1)$$

$$\dot{v}(t) + a^2\lambda v(t) = 0 \qquad (2)$$

Since $w(0,t) = w(\pi,t) = 0$ for all t (and $u(t) \neq 0$ for some t), we get $u(0) = u(\pi) = 0$.

By the examples in the intro, (1) has a nonzero solution if $\lambda_k = k^2$, $k \in \mathbb{N}$, namely

$$u_k(x) = c_k \sin(kx), \quad c_k \in \mathbb{R}, \quad k \in \mathbb{N}$$

With $\lambda_k = k^2$, (2) becomes $\dot{v}(t) + (ak)^2 v(t) = 0$ and the general solution is

$$v_k(t) = \tilde{c}_k \cdot e^{-a^2k^2t}, \quad \tilde{c}_k \in \mathbb{R}$$

So $w_k(x,t) = e^{-a^2k^2t}\sin(kx)$ are solutions to the heat equation.

To satisfy the initial condition $w(x,0) = f(x)$, consider $w(x,t) = \sum_{k=1}^{\infty} b_k e^{-a^2k^2t}\sin(kx)$

$$f(x) = w(x,0) = \sum_{k=1}^{\infty} b_k \sin(kx)$$

Take $b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx$.

Then $\sum_{k=1}^{\infty} b_k \sin(kx)$ is the Fourier sine series of $f(x)$. This determines the b_k's uniquely.

If $f(x)$ is continuous for $0<x<\pi$ and $f(0)=f(\pi)=0$, this yields the unique solution of the 1D heat equation.