Def. (X, d) m.s., $(Y, \|\cdot\|)$ normed space

$f_n : X \to Y \qquad f_n \in B(X, Y)$

1. $\sum_{n=1}^{\infty} f_n$ converges pointwise if $\forall\ x \in X$, $\lim_{k\to\infty} \sum_{n=1}^{k} f_n(x)$ exists in (Y, d_Y).
2. $\sum_{n=1}^{\infty} f_n$ converges uniformly if $\lim_{k\to\infty} \sum_{n=1}^{k} f_n(x)$ exists in $(B(X, Y), d_\infty)$
3. $\sum_{n=1}^{\infty} f_n$ converges pointwise absolutely if $\forall\ x \in X$, $\sum_{n=1}^{\infty} \|f_n(x)\| < \infty$.
4. $\sum_{n=1}^{\infty} f_n$ converges uniformly absolutely if $\sum_{n=1}^{\infty} \|f\|_\infty < \infty$.

$(4) \overset{\not\Leftarrow}{\Rightarrow} (3) \qquad (4) \Rightarrow (2)$

$(2) \overset{\not\Leftarrow}{\Rightarrow} (1) \qquad (3) \Rightarrow (1)$

Ex. $f_n : (0, 1) \to \mathbb{R}$

$x \mapsto x^n$

(1) $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ pointwise convergence

(3) $\sum_{n=1}^{\infty} |x|^n = \frac{|x|}{1-|x|}$ pointwise absolute convergence

(2) $\lim_{k\to\infty} \overset{S_k}{\sum_{n=1}^{k} x^n} = \overset{S}{\frac{x}{1-x}}$ in $(B(X,Y), d_\infty)$

$\Updownarrow$

$\lim_{k\to\infty} d_\infty(S_k, S) = 0$

S_k converges uniformly to S.

(4) $\sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$?

$\forall$ n, $\|f_n\|_\infty = \sup_{x\in(0,1)} |f_n(x)|$

$= \sup_{x\in(0,1)} |x|^n$

$= (\sup_{x\in(0,1)} |x|)^n$

$= 1$

$\sum_{n=1}^{\infty} \|f_n\|_\infty = \infty$

Thm. (Weierstrass M-test)

(X, d) m.s., $(Y, \|\cdot\|)$ normed vector space,

$f_n \in B(X,Y)$.

Assume Y is complete.

Then (4) $\Rightarrow$ (2).

Pf. (4) $\Rightarrow \sum_{n=1}^{\infty} \|f_n\|_\infty < \infty$.

Define $g_k := \sum_{n=1}^{k} f_n$.

WTS: $\lim_{k\to\infty} \sum_{n=1}^{k} f_n(x)$ exists in $(B(X,Y), d_\infty)$

It is enough to show (S_k) is a Cauchy sequence in $(B(X,Y), d_\infty)$.

Observe that if $N > M$ then

$$g_N - g_M = \sum_{n=1}^{N} f_n - \sum_{n=1}^{M} f_n$$
$$= \sum_{n=M+1}^{N} f_n$$

$$d_\infty(g_N, g_M) = \|g_N - g_M\|_\infty$$
$$= \left\|\sum_{n=M+1}^{N} f_n\right\|_\infty$$
$$\leq \sum_{n=M+1}^{N} \|f_n\|_\infty \qquad (\Delta \text{ ineq.})$$

Define $S_k := \sum_{n=1}^{k} \|f_n\|_\infty$ and $S := \sum_{n=1}^{\infty} \|f_n\|_\infty$.

$$\lim_{k\to\infty} S_k = S$$
$$\lim_{k\to\infty} S_k - S = 0$$
$$\lim_{k\to\infty} \sum_{n=k+1}^{\infty} \|f_n\|_\infty$$
$$\Rightarrow \quad \forall\ \varepsilon > 0 \quad \exists\ \bar{N} \in \mathbb{N} \quad \text{s.t.} \quad \sum_{n=L}^{\infty} \|f_n\|_\infty < \varepsilon \quad \forall\ L \geq \bar{N}.$$

Uniform convergence and integrals

Thm. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of Riemann-integrable functions.

Suppose that $f_n \to f$ uniformly on $[a, b]$.

Then f is Riemann-integrable, and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Recall: given a partition P of $[a, b]$

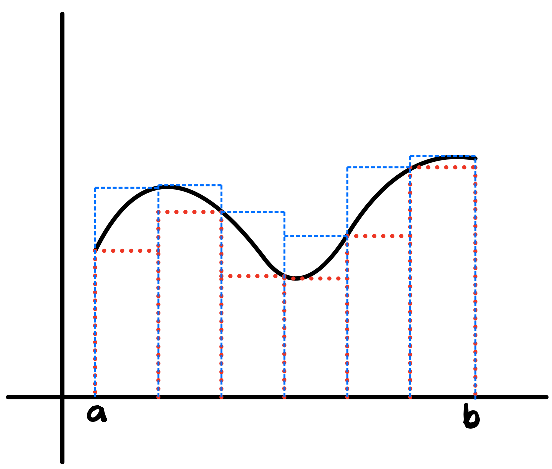
$$a = t_0 < t_1 < \ldots < t_k = b$$

We define the upper sum of f relative to P:

$$U(f, P) = \sum_{i=1}^{k} \sup_{x\in[t_{i-1},t_i]} f(x)\,(t_i - t_{i-1})$$

and the lower sum of f relative to P:

$$L(f, P) = \sum_{i=1}^{k} \inf_{x\in[t_{i-1},t_i]} f(x)\,(t_i - t_{i-1}) .$$



Then, given P, P', $L(f, P) \le U(f, P')$

We define the upper and lower (Darboux) integrals of f:

$$\overline{\int_a^b} f(x)\,dx := \inf U(f, P)$$

$$\underline{\int_a^b} f(x)\,dx := \sup L(f, P)$$

Observe that $\overline{\int_a^b} f(x)\,dx \ge \underline{\int_a^b} f(x)\,dx$ by (*).

If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, we say that f is integrable. (Riemann-Darboux)

Pf. Now suppose (f_n) is a sequence of Riemann-integrable functions, and $f_n \to f$ uniformly.

$$\Rightarrow \|f_n - f\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Let $\varepsilon > 0$. $\exists\ N \in \mathbb{N}$ $\forall\ n \ge N$ $\forall\ x \in [a, b]$

$$|f_n(x) - f(x)| < \varepsilon. \qquad (*)$$

Then, given any partition $P = \{a = t_0 < t_1 < \ldots < t_k = b\}$, observe that

$$\begin{aligned} -\varepsilon + \inf_{x\in[t_{i-1},t_i]} f_n(x) &\le \inf_{x\in[t_{i-1},t_i]} f_n(x) \\ &\le \sup_{x\in[t_{i-1},t_i]} f_n(x) \\ &\le \sup_{x\in[t_{i-1},t_i]} f_n(x) + \varepsilon \end{aligned}$$

Multiply these by $(t_i - t_{i-1})$ and add over i from 1 to k:

$$\sum_{i=1}^{k}\left(-\varepsilon(t_i - t_{i-1}) + \inf_{x\in[t_{i-1},t_i]} f_n(x)\,(t_i - t_{i-1})\right) \le \sum_{i=1}^{k} \inf_{x\in[t_{i-1},t_i]} f_n(x)\,(t_i - t_{i-1})$$

$$\begin{aligned}-\varepsilon(b-a) + L(f_n, P) &\le L(f, P)\\ &\le U(f, P)\\ &\le U(f_n, P) + \varepsilon\end{aligned}$$

Take sup in LHS and inf in RHS:

$$\begin{aligned}-\varepsilon(b-a) + \underline{\int_a^b} f_n(x)\,dx &\le \underline{\int_a^b} f_n(x)\,dx\\ &\le \overline{\int_a^b} f_n(x)\,dx\\ &\le \overline{\int_a^b} f_n(x)\,dx + \varepsilon(b-a)\end{aligned}$$

$$\Rightarrow \left|\overline{\int_a^b} f - \underline{\int_a^b} f\right| \le 2\varepsilon(b-a)$$

$$\varepsilon \to 0: \quad \overline{\int_a^b} f = \underline{\int_a^b} f$$

$\Rightarrow$ f is Riemann-integrable.

Moreover, $\left|\int_a^b f - \int_a^b f_n\right| \le 2\varepsilon(b-a) \quad \forall\ n \ge N$.

$\therefore \quad \lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.