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Topics

- Numerical solution of ODEs
- Iterative solutions of linear systems of eqns
- Solution of over- and under-determined systems of eqns
- Discrete Fourier transforms, fast Fourier transforms

5.1 Initial Value Problems (IVPs)

IVP: find a function $y(t)$ such that the ODE

$$\begin{cases} y'(t) = f(t, y(t)) & a \le t \le b \\ y(a) = \alpha & \text{initial condition} \end{cases}$$

is satisfied.

Rmk. The IVP arises in many applications, e.g. optimal control, machine learning.

Ex. Given IVP $\begin{cases} y'(t) = \lambda(y - \frac{1}{3}) & 0 \le t \le b \\ y(0) = \frac{1}{3} \end{cases}$

Find the explicit form of the solution.

Sol. This is an autonomous ODE:

$$y(t) = \frac{dy}{dt} = \lambda\left(y - \frac{1}{3}\right)$$

$$\int \frac{dy}{y - 1/3} = \int \lambda \, dt$$

$$\rightarrow \ln\left(y - \frac{1}{3}\right) = \lambda t + C_1$$

$$y(t) = \frac{1}{3} + Ce^{\lambda t}$$

$$y(0) = \frac{1}{3} : \quad C = 0$$

$$\rightarrow y(t) = \frac{1}{3}$$

Review of basic ODE theory:

Def. A function $f(t, y)$ is said to be **Lipschitz continuous** in the variable y on a set $D \subset \mathbb{R}^2$ if

$\forall$ (t, y_1) and (t, y_2), $\exists$ $L > 0$ s.t.

$$|f(t, y_1) - f(t, y_2)| \leq L|y_2 - y_1|$$

a **Lipschitz constant** for f

Rmk. Lipschitz constant for f is not unique, but we care about its existence.

Ex. Let $f(t,y)=t|y|$ and $\Omega=\{(t,y) \mid 1\le t\le 2, -3\le y\le 4\}$

Then
$$
\begin{aligned}
|f(t,y_1)-f(t,y_2)| &= |t|y_1|-t|y_2|| \\
&= t||y_1|-|y_2|| \\
\text{(reverse } \Delta \text{ ineq.)} \quad &\le t|y_1-y_2| \\
&\le 2|y_1-y_2|
\end{aligned}
$$

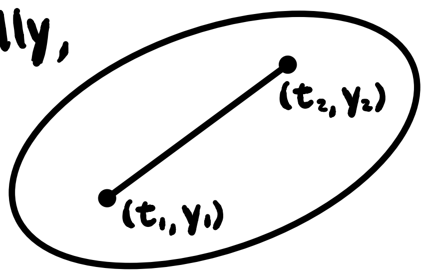
$\rightarrow$ f is Lipschitz continuous in y with Lipschitz constant $L=2$.

Def. A set $D\subset\mathbb{R}^2$ is called **convex** if for all (y_1,y_2) and (t_2,y_2) in D,

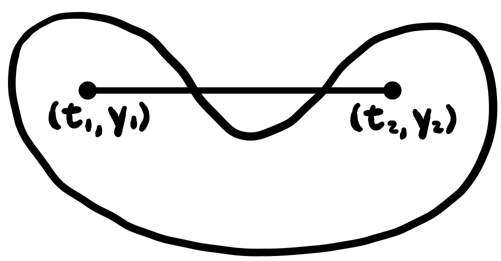
$$
\begin{bmatrix} (1-\lambda)t_1+\lambda t_2 \\ (1-\lambda)y_1+\lambda y_2 \end{bmatrix} = [(1-\lambda)t_1+\lambda t_2,\ (1-\lambda)y_1+\lambda y_2]
$$

is also in D for all $\lambda\in[0,1]$

Geometrically,



convex



not convex

Rmk. Convexity is usually used to show uniqueness of solutions.

Thm. 1 Let $f(t,y)$ be defined on a convex set $D \subset \mathbb{R}^2$.

If there exists $L>0$ such that

$$\left|\frac{\partial t}{\partial y}(t,y)\right| \le L \quad \text{for all } (t,y)\in D$$

then f is Lipschitz continuous in y on D with Lipschitz constant L.

Def. (Well-posedness of IVP)

The IVP

$$\begin{cases} y'(t) = f(t, y(t)) & a \le t \le b \\ y(a) = \alpha \end{cases} \qquad (*)$$

is well-posed if:

① A unique solution, $y(t)$, to $(*)$ exists

② There exists $\varepsilon_0 > 0$ and $k>0$ such that

for all
- $\varepsilon \in (0, \varepsilon_0)$
- $\delta(t)$ continuous with $|\delta(t)| < \varepsilon$ for all $\varepsilon \in [a,b]$
- δ_0 with $|\delta_0| < \varepsilon$

The perturbed problem

$$z'(t) = f(t, z(t)) + \delta(t), \quad a \le t \le b$$

$$z(a) = \alpha + \delta_0$$

has a unique solution $z(t)$ satisfying

$$|z(t) - y(t)| < k\varepsilon \quad \text{for all } t \in [a,b]$$

Rmk. - ② implies that small perturbations to (*) lead to nearby solutions

- if the given IVP is well-posed, then round-off errors do not lead to huge errors in solution $y(t)$.

Thm. 2 (Well-posedness)

Let $D = [a,b] \times \mathbb{R}$ and $f(t,y)$ be overall continuous and Lipschitz continuous in y on the set D. Then the IVP

$$\begin{cases} y'(t) = f(t, y(t)) & a \le t \le b \\ y(a) = \alpha \end{cases}$$

is well-posed with

$$|z(t) - y(t)| \le e^{L|t-a|}(1 + |b-a|\varepsilon)$$

Rmk. For an IVP to be well-posed, only need Lipschitz continuity in y! (and cont. in t / convexity of D)

Ex. Show that the IVP

$$\begin{cases} y'(t) = y - t^2 + 1 & 0 \le t \le 2 \\ y(0) = 0.5 \end{cases}$$

is well-posed on $D = [0,2] \times \mathbb{R}$

Sol. Since $f(t,y) = y - t^2 + 1$, we have

$$\left|\frac{\partial t}{\partial y}(t,y)\right| = 1$$

$$= L$$

$\rightarrow$ f is Lipschitz continuous in y.

Moreover, f is continuous on $D = [0,2] \times \mathbb{R}$.

$\rightarrow$ By Thm. 2, IVP is well-posed.

5.2 Euler's Method

In Calculus / Diff. Eqns, we learn to solve an ODE analytically. But in real life applications, this is not possible.

$\therefore$ We must solve numerically.

Outline of Numerical Methods to Solve IVPs

Given an IVP, the general procedure of Euler's method or other related methods is:

(a) approximate the solution $y(t)$ at mesh points $\{t\}_{i=0}^{N}$ (equidistant pts on $[a,b]$),

- $t_i = a + i \cdot h, \quad i = 0, 1, \dots N,$ where

  $h = t_{i+1} - t_i = \frac{b-a}{n}$ is the step size.

(b) use interpolation to find $y(t)$ for any $t \in [a,b]$.

Euler's Method

$$\left.\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h \cdot f(t_i, w_i) \end{aligned}\right\} \text{difference eqn}$$

Derivation: apply Taylor's Thm.

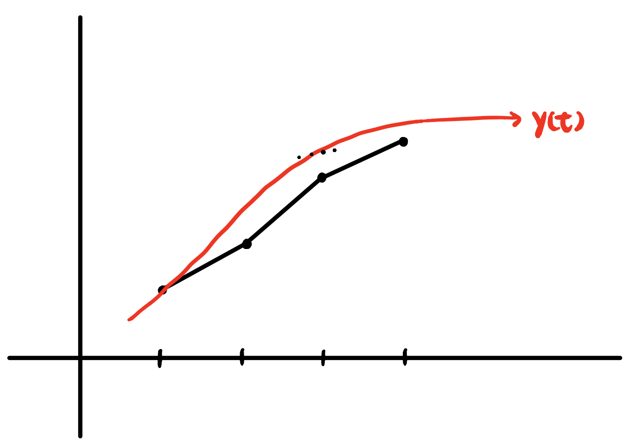
$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)\, y'(t_i) + \frac{1}{2} h^2 y''(\xi_i)$$

$$\longrightarrow y'(t_i) = \frac{y(t_{i+1}) - y(t_i)}{h} + O(h)$$

$$\approx \frac{w_{i+1} - w_i}{h}$$

Note: $w_0 = y(t_0)$, but $w_i \neq y(t_i)$!

Geometric idea: move along tangent line with slope $f(t_i, w_i)$



Note: error propagates along time t (each step incurs error $O(h)$)

Ex. Use Euler's Method to solve the IVP

$$\begin{cases} y' = y - t^2 + 1 \\ y(0) = 0.5 \end{cases} \qquad 0 \le t \le 2 \qquad \text{with } N = 10.$$

Sol. Step size: $h = \frac{2-0}{10} = 0.2$

and $w_0 = y(0) = 0.5$

By Euler's Method, we get:

$$w_{i+1} = w_i + h(w_i + t_i^2 - 1) = w_i + 0.2(w_i - (0.2i)^2 + 1)$$

Here, the actual solution is $y(t) = (t+1)^2 - \frac{1}{2}e^t$.

Can check error:

$|y(0.2) - w_1| \approx 0.0293$ and $|y(0.4) - w_2| \approx 0.0621$

Q: Can we quantify error produced / propagated by Euler's Method?

Lemma 5.8

If
- s, t are ⊕ real numbers
- $\{a_i\}_{i=0}^{k}$ is a sequence such that $a_0 > \frac{-t}{s}$,

  and $a_{i+1} \le (1+s)_i + t$ for $i = 0, 1, \ldots k-1$

Then $a_{i+1} \le e^{(i+1)s}(a_0 + \frac{t}{s}) - \frac{t}{s}$.

Pf. For fixed integer i, we have:

$$
\begin{aligned}
a_{i+1} &\le (1+s)a_i + t \\
&\le (1+s)^1[(1+s)a_{i-1}+t]+t = (1+s)^2 a_{i-1} + [1+(1+s)t] \\
&\le (1+s)^3 a_{i-2} + [1+(1+s)+(1+s)^2]t \\
&\;\;\vdots \\
&\le (1+s)^{i+1} a_0 + [\underbrace{1+(1+s)+(1+s)^2+\ldots+(1+s)^i}_{\sum_{j=0}^{i}(1+s)^j \,:\, \text{geometric series with ratio } (1+s)}]t
\end{aligned}
$$

Note: $\sum_{j=0}^{i}(1+s)^j = \frac{1-(1+s)^{j+1}}{1-(1+s)} = \frac{1}{s}[(1+s)^{j+1}-1]$

$$a_{i+1} \le (1+s)^{i+1}a_0 + \frac{(1+s)^{i+1}-1}{s}t$$

Now, using Taylor expansion for e^x:

$$e^x = 1 + x + \frac{x^2}{2}e^{\xi} \qquad 0<\xi<x$$

$$\rightarrow 1+x \le e^x$$

$$(1+x)^{i+1} \le e^{(i+1)x}$$

$$
\begin{aligned}
a_{i+1} &\le e^{(i+1)s}a_0 + e^{(i+1)s}\cdot\frac{t}{s} - \frac{t}{s} \\
&= e^{(i+1)s}(a_0 + \frac{t}{s}) - \frac{t}{s}. \quad \blacksquare
\end{aligned}
$$

Thm. 5.9 (Error Analysis of Euler's Method)

Suppose
- f is continuous on $D=[a,b]\times\mathbb{R}$
- f is Lipschitz continuous in y with constant L
- $|y''(t)|\le M$ for all $t\in[a,b]$, and
- $y(t)$ is unique sol. to IVP

$$\begin{cases} y'(t)=f(t,y(t)) & a\le t\le b \\ y(a)=\alpha \end{cases}$$

Then the sequence $\{w_i\}_{i=0}^{N}$ generated by Euler's Method satisfies

$$|y(t_i)-w_i|\le \frac{h\cdot M}{2L}\left(e^{L(t_i-a)}-1\right)$$

Pf. For $i=0$, $w_i=y(t_0)=\alpha$.

For $i\ge 1$, Taylor's Thm. satisfies:

$$y(t_{i+1})=y(t_i)+h\cdot f(t_i,y_i)+\frac{h^2}{2}y''(\xi_i),$$

$$t_i\le\xi_i,t_{i+1},\quad i=0,1,\dots N-1$$

By Euler's Method, we have:

$$w_{i+1}=w_i+hf(t_i,w_i)$$

Denoting $y_i=y(t_i)$, we have:

$$y_{i+1}-w_{i+1}=y_i-w_i+h[f(t_i,y_i)-f(t_i,w_i)]+\frac{h^2}{2}y''(\xi_i)$$

$$\rightarrow |y_{i+1}-w_{i+1}|\le|y_i-w_i|+h\underbrace{|f(t_i,y_i)-f(t_i,w_i)|}_{\le L|y_i-w_i|}+\frac{Mh^2}{2}$$

$$\rightarrow |y_{i+1}-w_{i+1}|\le(1+hL)|y_i-w_i|+\frac{Mh^2}{2}$$

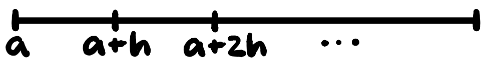
Let $d_i = |y_i - w_i|$, $s = hL$, $t = \frac{Mh^2}{2}$:

$$d_{i+1} \le (1+s)d_i + t$$
$$\le e^{(i+1)s}\left(d_0 + \frac{t}{s}\right) - \frac{t}{s} \quad \text{by Lemma 5.8}$$

Note: $d_0 = 0 > -\frac{t}{s}$

$$|y_{i+1} - w_{i+1}| e^{(i+1)hL}\left[|y_0 - w_0| + \frac{h^{\cancel{2}}M}{2\cancel{h}L}\right] - \frac{hM}{2L}$$

Since $(i+1)h = t_{i+1} - t_0 = t_{i+1} - a$



Rmk. 1. This theorem requires an upper bound M for 2nd order derivative, which is NOT always available.

Instead, consider:

$$y''(t) = \frac{d}{dt}y'(t) = \frac{d}{dt}(f(t,y))$$
$$= \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y) \cdot \frac{dy}{dt} \quad f$$

assuming f is differentiable.

2. Smaller step size $h \Rightarrow$ smaller error, $O(h)$.

3. A similar error bound can be found for the perturbed IVP (see Thm. 5.10).

Ex. Find error bounds for Euler's Method with $h=0.2$ for the IVP

$$\begin{cases} y'(t) = y(t) - t^2 + 1 & 0 \le t \le 2 \\ y(0) = 0.5 \end{cases} \quad (*)$$

Sol. Recall from previous example that sol. of $(*)$ is:

$$y(t) = (t-1)^2 - \frac{1}{2}e^t \quad \text{and} \quad \frac{\partial y}{\partial t} = 1 \rightarrow L = 1$$

$$|y''(t)| = |2 - \frac{1}{2}e^t| \le \frac{1}{2}e^2 - 2 = M \quad t \in [0,2]$$

The estimated error is thus

$$|y_i - w_i| \le 0.1\left(\frac{1}{2}e^2 - 2\right)\left(e^{t_i} - 1\right)$$

(by Thm. 5.9)