Laplace Transform

Def. $L[f(x)] = \int_0^\infty f(x) e^{-px}\, dx = F(p)$

is the Laplace Transform of $f(x)$, provided the integral exists.

In particular, $\int_0^\infty f(x) e^{-px}\, dx = \lim_{b\to\infty} \int_0^b f(x) e^{-px}\, dx$.

Ex. Let $p > 0$.

(i) $f(x) = 1 \longrightarrow F(p) = \int_0^\infty e^{-px}\, dx = -\frac{1}{p} e^{-px} \Big|_0^\infty = \frac{1}{p}$

(ii) $f(x) = x \longrightarrow F(p) = \int_0^\infty x e^{-px}\, dx =$

$$= -\frac{1}{p} e^{-px} \cdot x \Big|_0^\infty + \frac{1}{p} \int_0^\infty e^{-px}\, dx$$

$$= 0 + \frac{1}{p} \cdot \frac{1}{p} = \frac{1}{p^2}$$

(iii) $f(x) = x^n \longrightarrow F(p) = \frac{n!}{p^{n+1}}$

(iv) $f(x) = e^{ax} \longrightarrow F(p) = \int_0^\infty e^{(a-p)x}\, dx$

$$= \frac{1}{p-a} \quad \text{(if } p > a\text{; otherwise diverges)}$$

(v) $f(x) = \sin(ax) \longrightarrow F(p) = \frac{a}{a^2+p^2}$

(trick: integrate by parts twice: $F(p) = \frac{a}{p^2} - \frac{a^2}{p^2} \cdot F(p)$)

(iv) $f(x) = \cos(ax) \longrightarrow F(p) = \frac{p}{p^2+a^2}$

Recall Laplace transform is linear: $L[af(x) + bg(x)] = a \cdot L[f(x)] + b \cdot L[g(x)]$

Ex. $L[2x+3] = 2 \cdot L[x] + 3 \cdot L[1] = \frac{2}{p^2} + \frac{3}{p}$

Prop. (Shifting formula)

$$L[e^{ax} f(x)] = \int_0^\infty e^{ax} f(x) e^{-px}\, dx = \int_0^\infty f(x) e^{(a-p)x}\, dx = \int_0^\infty f(x) e^{-(p-a)}\, dx$$

$$= F(p-a)$$

Ex. $L[e^{ax} \sin(bx)] = \frac{b}{(p-a)^2 + b^2}$

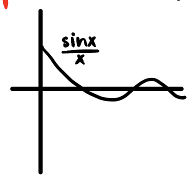
§49 Theoretical Remarks

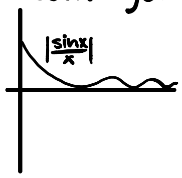
Def. (i) $\int_0^\infty f(x)dx$ **converges** if $\int_0^\infty f(x)dx = \lim_{b\to\infty}\int_0^b f(x)dx$ exists (and is finite)

(and similarly $\lim_{\substack{a\to 0\\ a>0}}\int_a^1 f(x)dx$ exists)

(ii) $\int_0^\infty f(x)dx$ **converges absolutely** if $\int_0^\infty |f(x)|dx$ converges

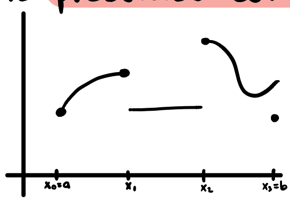
Famous example: $\int_0^\infty \frac{\sin x}{x}dx$ converges but not absolutely.





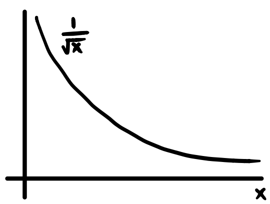
Rmk. requires that f is integrable on intervals $[a,b]$, $0\le a\le b<\infty$,

e.g. $f(x)$ is **piecewise continuous.**



in particular, all limits $\lim_{\substack{x\to x_i\\ (x>x_i \text{ or } x<x_i)}} f(x)$ exist (and are finite)

Ex. $f(x)=\begin{cases}\frac{1}{\sqrt{x}} & x>0\\ 0 & x=0\end{cases}$



f is not piecewise continuous

However, $\int_a^b \frac{1}{\sqrt{x}}dx = 2(\sqrt{b}-\sqrt{a})$

$\longrightarrow 2\sqrt{b}$ as $\underset{a>0}{a\to 0}$

hence $f(x)$ is integrable on all intervals $[0,b]$, $0<b<\infty$.

(but $\int_0^\infty f(x)dx$ diverges to infinity).

We'll see f has a Laplace transform.

Q. Suppose f is integrable on all intervals $[0,b]$, $0<b<\infty$.

How can we ensure that $L[f(x)] = F(p) = \int_0^\infty f(x)e^{-px}\,dx$ exists?

Common assumption: f is of exponential order, i.e. there are constants

$c>0$, $M>0$ such that $|f(x)| \le M\cdot e^{cx}$ for all $x>0$.

Ex. (i) e^{ax}, polynomials, sin/cos

(ii) sums and products of functions of exponential order are again of exponential order

(iii) e^{x^2} is NOT of exponential order

Prop. If f is integrable on all intervals $[0,b]$, $0<b<\infty$, $|f(x)| \le Me^{cx}$, then

$F(p) = L[f(x)]$ exists for all $p>c$ and $F(p)\to 0$ as $p\to\infty$.

Pf. $0 \le \left|\int_0^\infty f(x)e^{-px}dx\right|$

$$\le \int_0^\infty |f(x)|e^{-px}dx$$

$$\le M\int_0^\infty e^{cx}e^{-px}dx$$

$$= M\int_0^\infty e^{(c-p)x}dx$$

$$= M\cdot \lim_{b\to\infty} \frac{1}{c-p} e^{(c-p)x}\Big|_0^b$$

$$= \frac{M}{p-c} < \infty \text{ since } p>c$$

and $\to 0$ as $p\to\infty$

Ex. $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & x>0 \\ 0 & x=0 \end{cases}$

f is integrable on all $[a,b]$, $0<b<\infty$, and since $f(x)\to 0$ as $x\to\infty$,

f is already of exponential order. So f has a Laplace transform:

$p>0$: $L[f(x)] = \int_0^\infty \frac{1}{\sqrt{x}}e^{-px}dx$

$$\begin{aligned}
&\begin{matrix} t=px \\ dt=p\,dx \end{matrix} \quad &&= \int_0^\infty t^{-1/2} p^{1/2} e^{-t} p^{-1} dt \\
&\begin{matrix} t=s^2 \\ dt=2s\,ds \end{matrix} \quad &&= 2p^{-1/2} \int_0^\infty s^{-1} e^{-s^2} s\,ds \\
& &&= 2p^{-1/2} \underbrace{\int_0^\infty e^{-s^2} ds}_{=\frac{\sqrt{\pi}}{2}}
\end{aligned}$$

$$L\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{\pi}{p}}$$