Method of Undetermined Coefficients

(IH) $\quad y''+py'+qy=R(x) \qquad p,q\in\mathbb{R}$

Suppose y_1, y_2 solve (IH).

: defined

Set $y:=y_1-y_2$. Then $y''+py'+qy = y_1''-y_2''+p(y_1'-y_2')+q(y_1-y_2)$

$$= y_1''+py_1'+qy_1-y_2''-py_2'-qy_2$$

$$= R(x)-R(x)=0$$

i.e. y solves the homogeneous equation $y''+py'+q=0$ (H)

Every solution of (H) can be written in terms of the general solution y_g of (H).

Thus, $y_1-y_2 \text{ "="} \, y_g$.

for a specific version of the general solution

Q How do we find a solution to (IH)?

Consider $y''+py'+qy=R(x)$:

(i) $R(x)=e^{ax}$, $\quad a\in\mathbb{R}$

"undetermined coefficient"

Guess: $y_p=Ae^{ax}$ for some constant $A\in\mathbb{R}$

ODE (IH): $\quad Aa^2e^{ax}+pAae^{ax}+qAe^{x}=e^{ax}$

$$A(a^2+pa+q)=1$$

If $a^2+pa+q\neq 0$: $\quad A=\frac{1}{a^2+pa+q}$, and y_p is indeed a solution.

Ex $\quad y''=e^x \quad$ (IH)

particular solution $y_p(x)=e^x$

general solution $y''=0 \quad$ (H)

$y_g=a+bx$ is the general solution of (H)

$\longrightarrow$ general solution of (IH): $\quad y=e^x+a+bx$

- If $a^2+pa+q=0$, try $y_p=A\cdot x\cdot e^{ax}$:

If $a \neq \frac{-p}{2}$, then $A = \frac{1}{2a+p}$ works.

If also $a = \frac{-p}{2}$, then $\frac{1}{2}x^2 e^{-\frac{p}{2}x}$ works.

(ii) $R(x) = \sin(bx)$

Guess: $y_p = A\sin(bx) + B\cos(bx)$

If y_p solves homogeneous equation, then consider $y_p = x(A\sin(bx) + B\cos(bx))$.

Ex. $y'' + y = \sin x$

general solution of homogeneous eqn: $y'' + y = 0$

$$y_g = c_1 \sin x + c_2 \cos x$$

try $y_p = x(A\sin x + B\cos x)$

$\rightarrow y_p' = A\sin x + B\cos x + x(A\cos x - B\sin x)$

$\rightarrow y_p'' = 2A\cos x - 2B\sin x + x(-A\sin x - B\cos x)$

ODE: $2A\cos x - 2B\sin x + 0 = \sin x$

$$B = \frac{-1}{2},\ A = 0$$

i.e. $y_p = \frac{-1}{2}x\cos x$

$\rightarrow$ general solution: $y = -\frac{1}{2}x\cos x + c_1\cos x + c_2 \sin x$.

(iii) $R(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

$= y'' + py' + qy$

- If $q \neq 0$, guess: $y_p = A_0 + A_1x + \ldots + A_nx^n$
- If $q = 0$, guess: $y_p = A_0 + A_1x + \ldots + A_{n+1}x^{n+1}$
- If $p, q = 0$, guess: $y_p = A_0 + A_1x + \ldots + A_{n+2}x^{n+2}$

$\Rightarrow y'' = a_0 + a_1x + \ldots + a_nx^n$, so direct integration also works.

Laplace Transforms: transformations of functions

Ex. (i) $T_1 : f(x) \mapsto f'(x)$ derivative

(ii) $T_2 : f(x) \mapsto \int_0^x f(t)\,dt$ integral

$T_3 : f(x) \mapsto \underbrace{\int_a^b k(p,x) f(x)\,dx}_{\text{function of } p}$

$k(p,x)$ is called kernel

Rmk. These are examples of linear transformations,

i.e. $T[af+bg] = aT[f] + bT[g]$ for functions f, g and constants a, b.

Def. $$L[f(x)] = \int_0^\infty e^{-px} f(x)\,dx = F(p)$$

is the Laplace transform of f, provided the integral exists.