Fixed Point Iteration

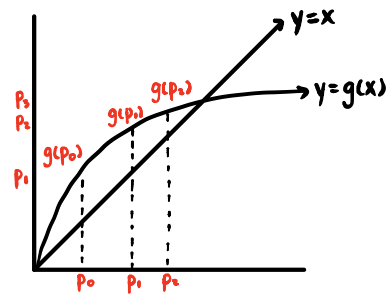
Algorithm goes as follows:

1) Choose initial guess p_0

2) Generate $\{p_n\}_{n=1}^{\infty}$ by setting: $n = g(p_{n-1})$, $n \geq 1$

Note If $p_n \to p$ and g continuous, $p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} g(p_{n-1}) = g(\lim_{n\to\infty} p_{n-1}) = g(p)$

Geometrically:



Thm Fixed Point Theorem

Let $g \in C([a,b])$ with $g(x) \in [a,b]$ for all $x \in [a,b]$

$g \in C'([a,b])$ such that $|g'(x)| \leq k$ for all $x \in [a,b]$

Then for any $p_0 \in [a,b]$, the sequence $\{p_n\}_{n=1}^{\infty}$ defined by $p_n = g(p_{n-1})$, $n \geq 1$ converges to the unique fixed point of g in $[a,b]$ with rate $O(k^n)$

Pf. By Existence & Uniqueness Theorem, there exists $p \in [a,b]$ such that $g(p) = p$.

Since $g(x)$ maps $[a,b]$ to itself, $\{p_n\}$ is well-defined ($g(p_n) \in [a,b]$ for all $n \geq 1$), and $p_n \in [a,b]$ for all $n \geq 1$.

By MVT, there exists $\xi \in (a,b)$ such that

$$
\begin{aligned}
0 \leq |p_n - p| &= g(p_{n-1}) - g(p) \\
&= |g'(\xi_n)| \cdot |p_{n-1} - p| \quad \text{by MVT} \\
&\leq k \cdot |p_{n-1} - p| \\
&\leq k \cdot k |p_{n-2} - p|
\end{aligned}
$$

$$\le \dots \le k^n|p_0-p|$$

Since $k\in(0,1)$, we have $\lim\limits_{n\to\infty}k^n=0$.

$$\Rightarrow \lim_{n\to\infty} k^n|p_0-p|=0$$

$$\lim_{n\to\infty}|p_n-p| \le \lim_{n\to\infty}k^n|p_0-p|=0$$

(abs. value)

$$\lim_{n\to\infty}|p_n-p|=0 \quad \text{by Squeeze Theorem}$$

Thus, $|p_n-p|\le k^n|p_0-p|$, p_n converges to p with rate k^n $(0<k<1)$

i.e. $p_n=p+O(k^n)$.

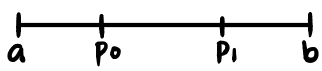
Rmk. Bound $|p_n-p|\le k^n|p_0-p|$ is not useful since we do not have $|p_0-p|$.

Corollary Error bounds for p_n in fixed point iteration can be given by

$$|p_n-p|\le k^n\cdot\max\{|p_0-a|,|p_0-b|\} \quad (1)$$

and $$|p_n-p|\le \frac{k^n}{1-k}|p_1-p_0| \quad (2)$$

Note $|p_0-p|\le\max\{|p_0-a|,|p_0-b|\}$



Rmk. ① If n_1, n_2 are minimum number of iterations required to achieve accuracy ε for (1) and (2), respectively, then take $n=\min\{n_1,n_2\}$

② Convergence rate depends on k (upper bound for $g'(x)$)

So $k\approx 0 \to$ fast convergence

$k\approx 1 \to$ slow convergence

Ex. (a) Show that $g(x)=2^{-x}$ has unique solution in $[\frac{1}{3},1]$

(b) Estimate # of iterations to achieve accuracy $\varepsilon = 10^{-4}$

Sol. (a) g continuous on $[0, 1]$

$g(x) \in [\frac{1}{2}, \frac{1}{\sqrt[3]{2}}] \subset [\frac{1}{3}, 1] \Rightarrow$ solution exists

$g'(x) = -\ln(2) \cdot 2^{-x}$, $|g'(x)| \in [|\frac{\ln 2}{2}|, |\frac{\ln 2}{\sqrt[3]{2}}|]$

$\approx [0.347, 0.552]$

$\Rightarrow |g'(x)| \le k = \frac{\ln 2}{\sqrt[3]{2}}$, so g has a solution in $[\frac{1}{3}, 1]$ by E/U

(b) First bound: since $p_n \in [\frac{1}{3}, 1]$ and $\max\{|p_0 - a|, |p_0 - b|\} \le \frac{2}{3}$

$\Rightarrow n_1 \ge 14.7347 \longrightarrow n_1 \ge 15$

Second bound:

$|p_n - p| \le \frac{k^n}{1-k} |p_1 - p_0| \le \frac{k^n}{1-k} |b - a| \Rightarrow n_2 \ge 16.07$

need at least 15 iterations