Lenny Wu

5.9 Higher-Order Equations and Systems of Differential Equations

More realistic applications often require solving systems of differential equations (of possibly higher order)

Systems of First-Order IVPs

Def. An n^{th} order system of first-order IVPs has the form

$$(*) \quad \begin{cases} u_j'(t) = f_j(t, u_1, u_2, \dots u_m) & j = 1, 2, \dots m \qquad t \in [a, b] \\ u_j(a) = \alpha_j \end{cases}$$

By introducing m-dimensional vector function

$$\vec{u}(t) = [u_1(t), u_2(t), \dots u_m(t)]^T : [a, b] \to \mathbb{R}^m,$$

we can write the IVP in matrix form as

$$\vec{u}'(t) = F(t, \vec{u}) \qquad t \in [a, b]$$

$$\vec{u}(a) = \alpha$$

where $F(t, \vec{u}) = \begin{bmatrix} f_1(t, \vec{u}) \\ f_2(t, \vec{u}) \\ \vdots \\ f_m(t, \vec{u}) \end{bmatrix} : [a, b] \times \mathbb{R}^m \to \mathbb{R}^m$

The system is called

- linear if $F(t, \vec{u}) = A(t)\vec{u} + B(t)$
    - homogeneous if $B(t) = 0$
    - inhomogeneous if $B(t) \neq 0$
- nonlinear otherwise

Def. The function $f(t, u_1, u_2, \dots u_m)$ defined on the set

$$\Omega = [a,b] \times \mathbb{R}^m = \{(t, u_1, u_2, \dots u_m) \mid a \le t \le b, -\infty < u_j < \infty, j = 1, 2, \dots m\}$$

is said to be Lipschitz in $(u_1, u_2, \dots u_m)$ on Ω

if there exists constant $L > 0$ s.t.

$$|f(t, u_1, u_2, \dots u_m) - f(t, z_1, z_2, \dots z_m)| \le L \sum_{j=1}^{m} |u_i - z_i| = L \|\vec{u} - \vec{z}\|$$

for all $(t, u_1, u_2, \dots u_m)$ and $(t, z_1, z_2, \dots z_m)$ in Ω.

Rmk. By MVT, we can show:

if 1) partial derivatives $\frac{\partial t}{\partial u_j}$ continuous in Ω

2) $\left|\frac{\partial f(t, u_1, \dots u_m)}{\partial u_j}\right| \le L$

for each $j = 1, 2, \dots m$ and all $(t, u_1, u_2, \dots u_m)$ in Ω,

then f is Lipschitz in $(u_1, u_2, \dots u_m)$ on Ω

with constant L.

Thm. 5.12 (Well-posedness of First-Order IVPs)

Let $\Omega = [a,b] \times \mathbb{R}^m$ and $f_j(t, u_1, u_2, \dots u_m)$ be continuous

and Lipschitz continuous in $\vec{u}$ on Ω.

Then the IVP (*) has a unique solution $u_1, u_2, \dots u_m$

for $a \le t \le b$.

Rmk. The hypothesis about Lipschitz continuity, i.e.

$$|f_j(t,\vec{u}) - f_j(t,\vec{z})| \le L_j \|\vec{u}-\vec{z}\|, \quad j=1,2,\dots m$$

implies that

$$\|F(t,\vec{u}) - F(t,\vec{z})\| \le (m \cdot \max_j L_j) \cdot \|\vec{u}-\vec{z}\|_1.$$

Moreover, the norm $\|\cdot\|_1$ can be replaced by any norm $\|\cdot\|$ in $\mathbb{R}^m$ so that

$$\|F(t,\vec{u}) - F(t,\vec{z})\| \le L\|\vec{u}-\vec{z}\|.$$

Rmk. All numerical methods we've seen thus far for solving a single IVP, e.g. one-step methods, can be generalized to solve (*) by replacing the scalar w with the vector $\vec{w}$.

Ex. Let $h=0.5$. Apply Euler's method to solve

$$\begin{cases} u_1' = u_2, \quad u_2' = -u_1 & 0 \le t \le 1 \\ u_1(0) = 1, \quad u_2(0) = 0 \end{cases}$$

Sol. Let $\vec{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$, and $F(t,\vec{u}) = \begin{bmatrix} u_2 \\ -u_1 \end{bmatrix}$.

Then $\vec{w}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{w}_1 \quad w_0 + hF(t,\vec{w}_0)$$

$$\vdots$$

$$\vec{w}_{k+1} = w_k + hF(t,\vec{w}_k)$$

High-Order DEs

Def. A general m^{th}-order IVP

(**) $$\begin{cases} y^{m}(t) = f(t, y, y', \dots y^{(m-1)}) & a \le t \le b \\ y(a) = \alpha_1, \quad y(a) = \alpha_2, \quad \dots \quad y^{(m-1)}(a) = \alpha_{m-1} \end{cases}$$

Idea: can turn into system of first-order IVPs

Let $u_j(t) = y^{(j-1)}(t)$, $j = 1, 2, \dots m$

Then we can convert (**) into

$$\underbrace{\begin{bmatrix} u_1' = u_2 \\ u_2' = u_3 \\ \vdots \\ u_{m-1}' = u_m \\ u_m'(t) = f(t, u_1, u_2, \dots u_{m-1}) \end{bmatrix}}_{u' = F(t, \vec{u})} \quad \text{s.t.} \quad \vec{u}(a) = \vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix}$$

OR : can write (**) as

$$u_j'(t) = y^{(j)}(t) = u_{j+1}(t), \quad j = 1, \dots m-1$$

$$u_m'(t) = y^{(m)}(t) = f(t, y, y', \dots y^{(m-1)})$$

with initial condition $u_j(a) = \alpha_j$, $j = 1, \dots m-1$