$$R(x)=\frac{36x}{(x+3)^2(x^2+9)}=\frac{-6}{(x+3)^2}+\frac{i}{x+3i}+\frac{-i}{x-3i}$$

$$=\frac{-6}{(x+3)^2}+\frac{i(x-3i)-i(i+3i)}{(x+3i)(x-3i)}$$

$$=\frac{-6}{(x+3)^2}+\frac{6}{x^2+9}$$

$$L^{-1}[R(p)]=-6L^{-1}\left[\frac{1}{(p+3)^2}\right]+2L^{-1}\left[\frac{3}{p^2+9}\right]$$

$$=-6e^{-3x}\cdot x+2\sin 3x$$

Applications to Diff. Eqns

Consider $y''+ay'+by=f(x)$, $y(0)=y_0$, $y'(0)=y_0'$

Aim: Apply Laplace transform on both sides

Assumptions: f continuous $\Rightarrow$ y, y', y'' continuous

f is of exponential order

Grönwall Lemma: y, y', y'' all of exp. order

Prop. $L[y']=pL[y]-y(0)$

$L[y'']=p^2L[y]-py(0)-y'(0)$

Pf. i. $L[y']=\int_0^\infty y'(x)e^{-px}dx$

$$=y(x)e^{-px}\Big|_0^\infty+p\int_0^\infty y(x)e^{-px}dx$$

$$=-y(0)+pL[y]$$

y is of exp. order, p large

ii. $L[y'']=L[(y')']$

$$=pL[y']-y'(0)$$

$$=p^2L[y]-py(0)-y'(0)$$

Going back to $y''+ay'+by=f(x)$ $y(0)=y_0$, $y'(0)=y_0'$

$$L[f(x)]=L[y''+ay'+by]$$

$$= L[y''] + aL[y'] + bL[y]$$

$$= (p^2+ap+b)L[y] - (y_0 p + ay_0 - y_0')$$

$$\to L[y] = \frac{L[f(x)] + y_0 p + ay_0 - y_0'}{p^2+ap+b}$$

Applying the inverse Laplace on both sides gives y.

Ex. $y''+4y=4x \quad y(0)=1, \quad y'(0)=5$

$$\frac{4}{p^2} = L[4x] = p^2L[y] - py(0) - y'(0) + 4L[y]$$

$$= p^2L[y] - p - 5 + 4L[y]$$

$$= (p^2+4)L[y] - p - 5$$

$$\to L[y] = \frac{p+5}{p^2+4} + \frac{4}{p^2(p^2+4)}$$

$$= \frac{A}{p} + \frac{B}{p^2} + \frac{Cp+D}{p^2+4}$$

$$\vdots$$

$$= \frac{p+4}{p^2+4} + \frac{1}{p^2}$$

$$\to y = L^{-1}\left[\frac{p+4}{p^2+4} + \frac{1}{p^2}\right]$$

$$= L^{-1}\left[\frac{p}{p^2+4}\right] + 2L^{-1}\left[\frac{2}{p^2+4}\right] + L^{-1}\left[\frac{1}{p^2}\right]$$

$$= \cos 2x + 2\sin 2x + x$$

Rmk. ① previous approach:

$$y = \underbrace{c_1\cos 2x + c_2 \sin 2x}_{y_g} + \underbrace{x}_{y_p}$$

$$y(0)=1, \quad y'(0)=5 \longrightarrow c_1 = 1, \quad c_2 = 2$$

② The solution to $y''+ay'+by=f(x)$, $y(0)=y_0 \quad y'(0)=y_0'$ is unique provided f is continuous.

(This will follow from Picard's Theorem.)

The ODE $y' = \sqrt{y}$, $y(0)=0$ has many solutions, e.g.

e.g. $y(x)=0$, $y(x)=\frac{2}{3}x^{3/2}$.

Recall: $L[y'] = pL[y] - y(0)$

§51 Derivatives and Integrals of Laplace Transforms

Prop. $F^{(n)}(p) = \frac{d^n}{dp^n} F(p)$

$= n^{th}$ derivative of $F(p) = L[f(x)]$

$= L[(-x)^n f(x)]$

In particular, $F'(p) = L[-x \cdot f(x)]$

Note $F'(p) = \frac{d}{dp}\int_0^\infty f(x) e^{-px} dx$

$= \int_0^\infty f(x) \cdot (-x) e^{-px} dx$

$= L[-x \cdot f(x)]$

$F''(p) = \frac{d}{dp} F'(p) = \frac{d}{dp}\int_0^\infty f(x) \cdot (-x) e^{-px} dx$

$= \int_0^\infty f(x)(-x)^2 e^{-px} dx$

$\vdots$

Similarly, $F^{(n)}(p) = \int_0^\infty f(x)(-x)^n e^{-px} dx$

Ex. $L[\sqrt{x}] = L[x \cdot x^{-1/2}]$

$= \frac{-d}{dp} L[x^{-1/2}]$

$= \frac{-d}{dp}\sqrt{\frac{\pi}{p}} = \frac{1}{2p}\sqrt{\frac{\pi}{p}}$