Chapter 8 - NP-Completeness

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May 2021

NP-Completeness

How do you compare the difficulties of problems?

Given problems X and Y, we say $X \leq_P Y$ if Y can be solved in polynomial time implies X can be solved in polynomial time.

Polynomial time reduction: X can be polynomial time reduced to Y if any instance of X can be solved by:

- polynomial time computations
- + polynomial numbers of calls to solver of Y

Example:

instance of problem X
$\downarrow$ computation
create instances of problem Y
$\downarrow$ solver for Y
use the results to get solution of X

Independent set:

A. Given $G = (V, E)$, find the maximum size of node subset $S \subseteq V$ such that no two nodes in S are adjacent.
B. Given $G = (V, E)$ and input k, is there any independent set of size at least k? (decision version of the independent set)

```
// solver of B:
// given G, k for instance of B
  pass G to solver of A --> obtain k*
  if k <= k*:
      return yes
  if k > k*:
      return no

// solver of A (binary search):
// given G, independent set size [1, n]
  l = 1, r = n
  k = (l + r) / 2
  if B(G, k) == yes:
      l = k
  else:
      r = k
```

$B \leq_P A$.
$A \leq_P B$.
$A \equiv_P B$.

Vertex Cover

Given $G = (V, E)$, is there a subset of $\leq k$ nodes such that each edge is incident to at least one node in the set?

Lemma. *Node set S is an independent set if and only if $V - S$ is a vertex cover.*

Proof.
($\Rightarrow$)
For every edge $(u, v) \in E$, S is an independent set implies at least one of u, v is not in S.
Then at least one of u, v is in $V - S$.
Thus, $V - S$ is a vertex cover.

($\Leftarrow$)
For any $(u, v) \in E$, S $V - S$ is a vertex cover implies at least one of u, v is in $V - S$.
Then at least one of u, v is not in S.
Thus, S is an independent set. $\square$

Vertex cover $\leq k \Longleftrightarrow$ Independent set $\geq n - k$.
Vertex over $\leq_P$ independent set.
Independent set $\leq_P$ vertex set.

Set Cover

Given a set of elements U and m subsets $S_1, ..., S_m \subseteq U$, can we find $\leq k$ subsets to cover all the elements in U?

Vertex cover $\leq_P$ set cover.

Proof.
Given a vertex cover problem $G = (V, E), k$, create a set cover problem:

$U = \{e \in E\}$

For each node $v \in V$, create a subset $S_v = \{e : e \text{ incident to } v\}$.

G has vertex cover $\leq k \iff V, S$ has set cover $\leq k$. $\square$

Independent set $\equiv_P$ vertex cover $\leq_P$ set cover.

3-Satisfiability (3-SAT)

$\Phi = (\overline{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_4)$.

Determine whether there exists true / false assignment of $x_1, \ldots, x_n$ such that $\Phi =$ true.

$\Phi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee x_3)$.

Theorem. *3-SAT $\leq_P$ independent set.*

Proof.
Given an instance of 3-SAT problem $\Phi = C_1 \wedge \ldots... \wedge C_m$, construct an independent set problem:

For each C_i, construct 3 nodes.
Connect all edges within each C_i.
Connect x_i with $\overline{x}_i$ for all i. □

Φ is satisfiable $\Longleftrightarrow G$ has an independent set of size m.

Proof.
($\Leftarrow$)
S is an independent set of size m such that:

- one node in each triangle,
- no $(x_i, \overline{x}_i$ selected together.

Set the true / false values according to the selected nodes.
Then, $\Phi =$ true.

($\Rightarrow$)
Suppose Φ is satisfiable.
There exists a satisfying true / false assignment of $x_1, \ldots, x_n$.
There is at least one true in each triangle.
C_i selects a true literal (node) in each triangle.
S with m nodes has no conflict, so there is no link between those nodes. □

3-SAT $\leq_P$ independent set $\equiv_P$ vertex cover $\equiv_P$ set cover.