Composite Integration

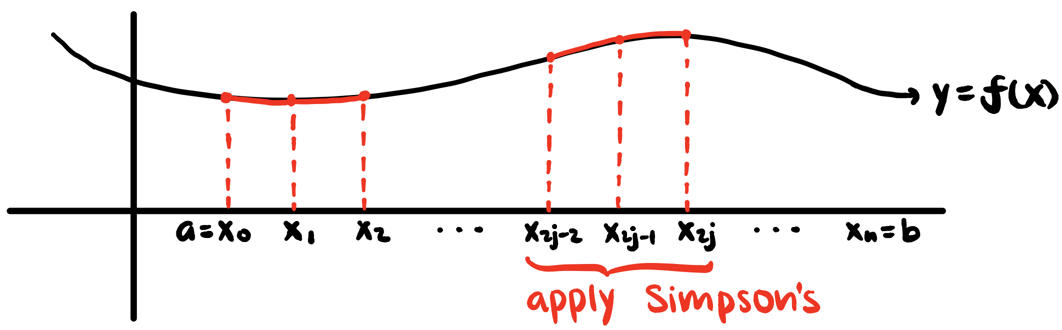
Idea: we consider $I(f) = \int_a^b f(x)\,dx$

- partition $[a,b]$ into n subintervals
- use low-order NC formula at each subinterval (midpt, trap, Simpson's)

Ex. Composite Simpson's Rule

Let $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$, n even,
and consider $x_{2j-2}, x_{2j-1}, x_{2j}$ for $j = 1, \ldots, \frac{n}{2}$
(need 3 pts for Simpson's)

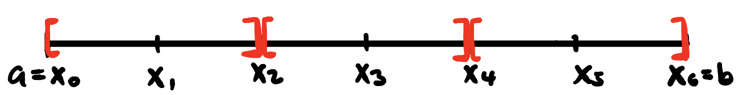
Goal: partition $[a,b]$ into $\frac{n}{2}$ subintervals and apply Simpson's rule on $[x_{2j-2}, x_{2j}]$



$$\int_a^b f(x)\,dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)\,dx$$

$$= \sum_{j=1}^{n/2} \left[\frac{h}{3}\left(f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})\right) - \frac{h^5}{90} f^{(4)}(\xi_j) \right] \qquad x_{2j-2} \le \xi_j \le x_{2j}$$

$$= \frac{h}{3}\left[f(x_0) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(x_n) \right] - \underbrace{\frac{h^5}{90} \sum_{j=1}^{n/2} f^{(4)}(\xi_j)]}_{E_s^c(f)}$$



Note (*) $\min_{x \in [a,b]} f^{(4)}(x) \le f^{(4)}(\xi_j) \le \max_{x \in [a,b]} f^{(4)}(x)$

$$\Rightarrow \min_{x\in[a,b]} f^{(4)}(x) \le \frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j(x)) \le \max_{x\in[a,b]} f^{(4)}(x)$$

the average of $\frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j(x))$ must also satisfy (*)

Note $\frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j(x))$ continuous on $[a,b]$

By IVT, $\exists\ \xi\in(a,b)$ st

$$\frac{2}{n}\sum_{j=1}^{n/2} f^{(4)}(\xi_j(x)) = f^{(4)}(\xi)$$

$$\Rightarrow E_s^c(f) = \frac{-h^5}{90}\sum_{j=1}^{n/2} f^{(4)}(\xi_j) = \frac{-h^5 n}{180} f^{(4)}(\xi), \quad a<\xi<b$$

$$h=\frac{b-a}{n} \Rightarrow E_s^c(f) = \frac{-(b-a)}{180}\cdot h^4 f^{(4)}(\xi)$$

Thm. (Composite Simpson)

Let $f\in C^4(a,b)$, n be even, $h=\frac{b-a}{n}$,

$x_j = a_j + j\cdot h$ for each $j=0,1,\dots n$.

There exists a $\mu\in(a,b)$ s.t. Composite Simpson's rule for n subintervals can be written with error term as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{(b-a)}{180} h^4 f^{(4)}(\mu)$$

Rmk. 1 Composite Simpson's Quadrature rule is given by

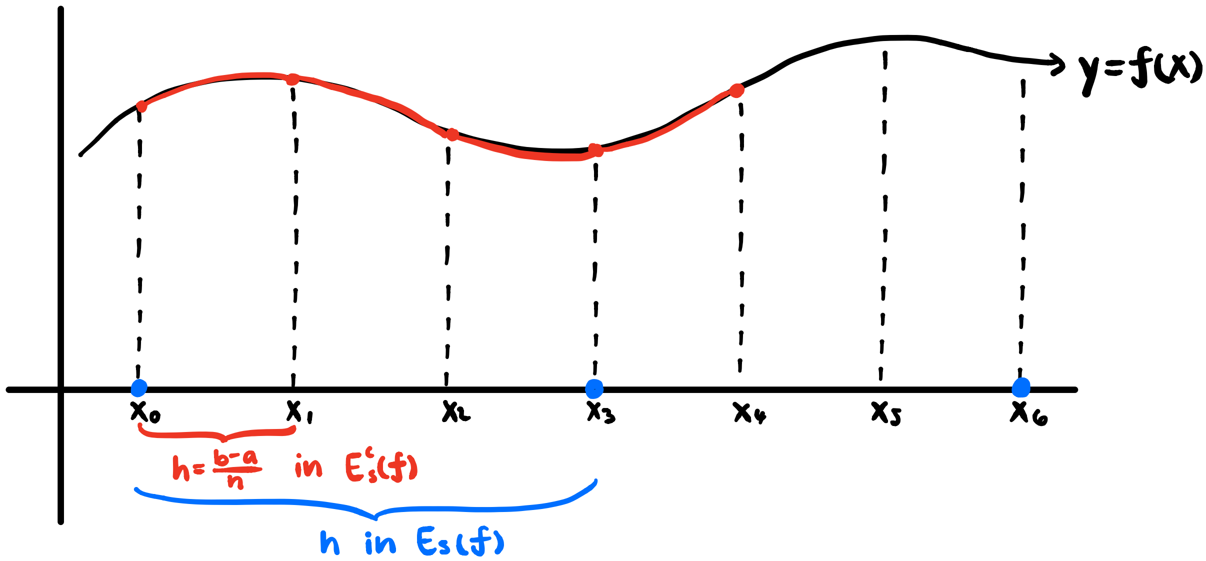
$$R_s^c = f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)$$

2 Composite Standard Simpson's rule error:

$$E_s = -\frac{(b-a)}{180} h^4 f^{(4)}(\mu) \quad a<\mu<b$$

Q: Is Composite Simpson ($O(h^4)$) worse than Standard Simpson ($O(h^5)$)?

A: No! In Standard Simpson, $h=\frac{b-a}{2}$; in Composite Simpson, $h=\frac{b-a}{n}$



Similarly, we can obtain Composite Trapezoidal rule:

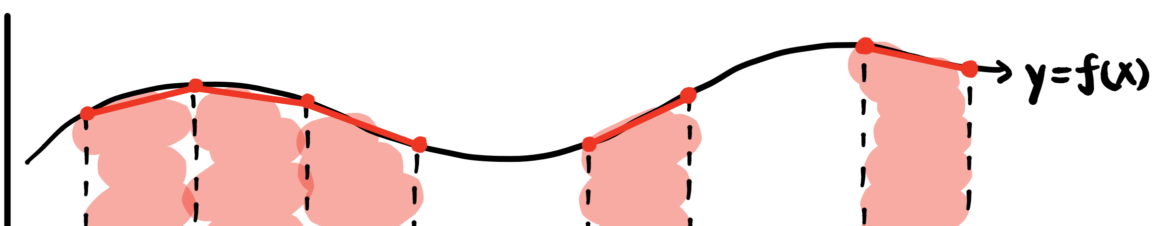
Thm. (Composite Trapezoidal)

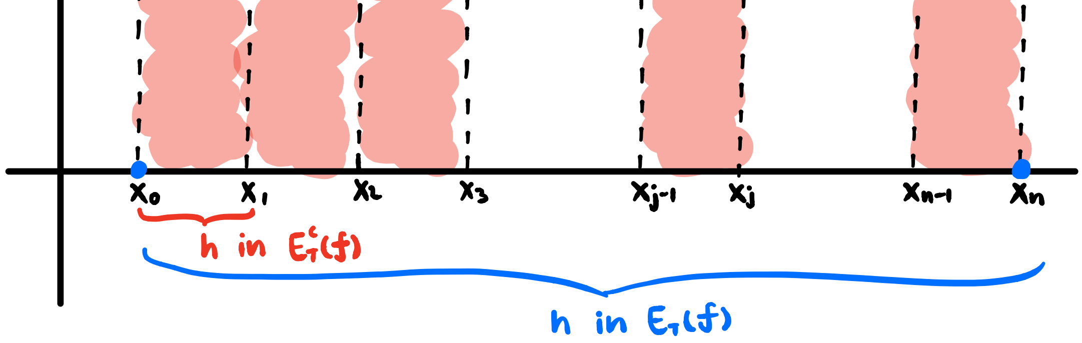
Let $f\in C^2(a,b)$, $h=\frac{b-a}{n}$, (no need to be even

$x_j = a_j + j\cdot h$ for each $j=0,1,\dots n$.

There exists a $\mu\in(a,b)$ s.t. Composite Trapezoidal rule for n subintervals can be written with error term as

$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x)+2\sum_{j=1}^{n-1}f(x_j)+f(b)\right]-\frac{b-a}{12}h^2f''(\mu)$$





Rmk. h in $R_T^c(f)$: $\frac{b-a}{n}$

h in $R_T(f)$: $b-a$

Finally, for the composite midterm rule:

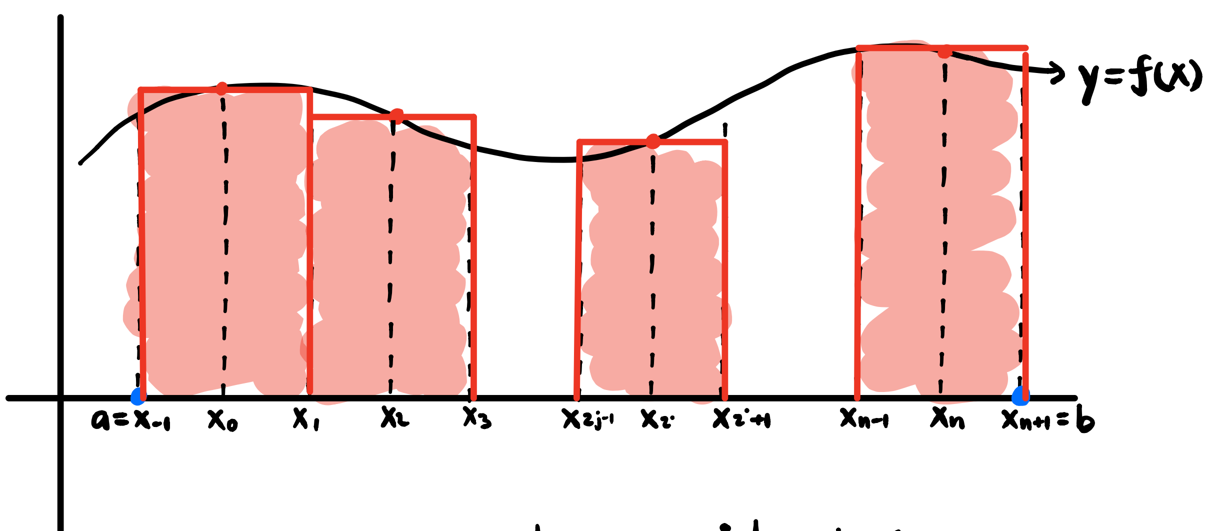
Thm. (Composite Midpoint)

Let $f \in C^2(a,b)$, n be even, $h = \frac{b-a}{n+2}$,

$x_j = a_j + (j+1)h$ for each $j = -1, 0, 1, \ldots n, n+1$

There exists a $\mu \in (a,b)$ s.t. Composite Midpoint rule for $n+2$ subintervals can be written with error term as

$$\int_a^b f(x)dx = 2h \sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6} h^2 f''(\mu)$$



every even node = midpoint

Rmk. h in $R_M^c(f)$: $\frac{b-a}{n+2}$

h in $R_M(f)$: $\frac{b-a}{2}$

Composite Integration Formulae

Rule	Formula	Error
C-Midpt, n even, $h=\frac{b-a}{n+2}$	$R_M^c(f) = 2h\sum_{j=0}^{n/2} f(x_{2j})$	$E_M^c(f) = \frac{b-a}{6}h^2 f''(\mu)$
C-Trap., $h=\frac{b-a}{n}$	$R_T^c(f) = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right]$	$E_T^c(f) = -\frac{b-a}{12}h^2 f''(\mu)$
C-Simp., n even, $h=\frac{b-a}{n}$	$R_S^c(f) = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right]$	$E_S^c(f) = -\frac{b-a}{180}h^4 f^{(4)}(\mu)$

Rmk. Composite rules $\Longrightarrow$ n can be very large whereas standard trap. $(n=1)$, midpt. $(n=0)$, Simpson's $(n=2)$ have fixed n.

Rmk. Can use error to control accuracy of $\int$.

Ex. Determine the value of h that will ensure approx. error $< 2\times10^{-5}$ when approximating $\int_0^\pi \sin x\,dx$ w/ Composite Trapezoidal rule:

Sol. $|E_T^c(f)| = \left|\frac{b-a}{12}h^2 f''(\mu)\right|$

$$= \left|\frac{\pi}{12}h^2(-\sin(\mu))\right|$$

$$= \frac{\pi}{12}h^2 \underbrace{|\sin\mu|}_{\leq 1}$$

$$\leq \frac{\pi}{12}h^2$$

$$\rightarrow \quad h^2 = \frac{24}{\pi}\cdot 10^{-5}$$

$$h = \sqrt{\frac{24}{\pi}\cdot 10^{-5}}$$

↳ use to find n