Note: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

If $\sin(h) = h \cdot \cos(\xi) \quad 0 \le \xi \le h$

$h \cdot \cos(h) = h \cdot \cos(\eta) \quad 0 \le \eta \le h$

$\Rightarrow |\sin h - h\cosh| \le h \cdot [\cos(\xi) - \cos(\eta)]$

$\Rightarrow F(h) = 0 + O(h)$

If $\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!}\cos(\xi)$

$h \cdot \cos(h) = h - \frac{h^3}{2!} + \frac{h^5}{4!}\cos(\eta)$

$$\Rightarrow |\sin h - h\cosh| = \left| -\frac{h^3}{3!} + \frac{h^3}{2!} + \frac{h^5}{5!}\underbrace{\cos(\xi)}_{\le 1} - \frac{h^5}{4!}\underbrace{\cos(\eta)}_{\le 1} \right|$$

$$\le \left(\frac{1}{3!} + \frac{1}{2!}\right)|h|^3 + \left(\frac{1}{5!} - \frac{1}{4!}\right)\underbrace{|h|^5}_{\le h^3}$$

$$\le Kh^3$$

$$\Rightarrow F(h) = 0 + O(h^3)$$

Recall Taylor's Theorem

$$f(x) = \underbrace{f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + \dots + \frac{f^{(n)}(x^*)}{n!}(x - x^*)^n}_{P_n(x)}$$

$$+ \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x^*)^{n+1}}_{R_n(x)} \quad \text{where } \xi \text{ between } x \text{ and } x^*$$

Truncation error: $R_n(x) = f(x) - P_n(x)$

2.1 Bisection Method

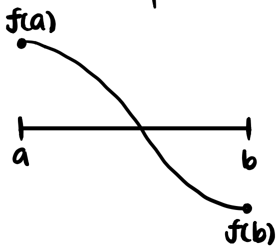
Goal: Given $f(x) \in C([a,b])$, want to find root $p \in [a,b]$ such that $p(c) = 0$

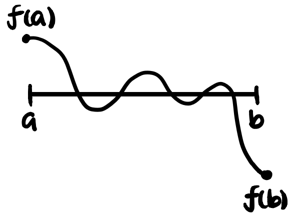
Q1: Is there a root? (existence)

Thm. **Intermediate Value Theorem**

If $f \in C([a,b])$ and K between $f(a)$ and $f(b)$, then there exists $p \in [a,b]$ such that $f(p) = K$.

Corollary If $f \in C([a,b])$ and $f(a) \cdot f(b) < 0$, then there exists $p \in [a,b]$ such that $f(p) = 0$.





<u>**Bisection:**</u> Find interval $[a_1, b_1]$ such that $f(a_1) \cdot f(b_1) < 0$

Let $p_1 = \frac{a_1 + b_1}{2}$ be the midpoint.

3 possibilities:

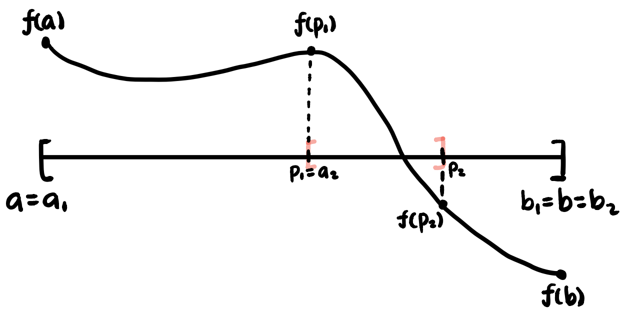
① $f(p_1) = 0$. Then $p = p_1$. Done!

② If p_1 has the same sign as $f(a_1)$, then set $a_2 = p_1$ and $b_2 = b_1$.

Consider new interval $[a_2, b_2] = [p_1, b_1]$.

③ If p_1 has the same sign as $f(b_1)$, then set $b_2 = p_1$ and $a_2 = a_1$.

Consider new interval $[a_2, b_2] = [a_1, p_1]$.



Bisection generates $p_1, p_2, \ldots p_n, \ldots \to p$

Rmk. ① Each halved interval $[a_{n+1}, b_{n+1}]$ contains a root since it satisfies $f(a_{n+1}) \cdot f(b_{n+1}) < 0$.

② For stopping criterion, choose:

$$\left.\begin{array}{l} - \; |p_n - p_{n-1}| < \varepsilon \\ - \; |f(p_n)| < \varepsilon \end{array}\right\} \text{can combine these two}$$

ε chosen by user

- max # of iterations reached

③ To avoid over/underflow when computing $f(a)\cdot f(b)$, compute

$$\operatorname{sgn}(f(a_n))\cdot \operatorname{sgn}(f(b_n))$$

$$\operatorname{sgn}(x)=\begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \end{cases}$$

Thm. Convergence of Bisection

Suppose that $f\in C[a,b]$ and $f(a)\cdot f(b)<0$.

Then the sequence $\{p_n\}_{n=1}^{\infty}$ generated by Bisection Method approximates a zero p of $f(x)$ with rate $|p_n-p|\le \frac{b-a}{2^n}$, $n\ge 1$.

That is, $p_n=p+O(\frac{1}{2^n})$

Note Since $a_1=a$ and $b_1=b$, $b_2-a_2=\frac{1}{2}(b_1-a_1)$

By induction, we have:

$$|b_2-a_2|=\frac{1}{2}|b-a|$$

$$|b_3-a_3|=\frac{1}{2}|b_2-a_2|=\frac{1}{4}|b-a|$$

$$\vdots$$

$$|b_n-a_n|=\frac{1}{2^{n-1}}|b-a|$$

By construction, $p_n=\frac{1}{2}(a_n+b_n)$ and $p\in(a_n,b_n)$

$$\Rightarrow |p_n-p|\le\frac{1}{2}(b_n-a_n)=\frac{1}{2^n}(b-a)$$

Thus, $p_n=p+O(\frac{1}{2^n})$.