For linear splines, $|e_n(x)| \leq \frac{M}{8}h^2$

Note This result says:

If nodes $x_0, x_1, \dots x_n$ are very close, then h is small, and error can be small.

Problems with Linear Splines:

- They are NOT smooth. That is, although $f(x)$ is continuous for all x, $s'(x)$ is not continuous at break points. (think about sols to ODEs / PDEs)

High-Degree splines — properties of deg. m spline

1. Domain is a closed interval $[\alpha, \beta]$
2. $s(x), s'(x), \dots s^{(m-1)}(x)$ are continuous on $[a,b]$
3. $[\alpha, \beta]$ is partitioned st $\alpha = x_0 < x_1 < x_2 < \dots < x_n = \beta$ where $s(x)$ is a polynomial of degree at most m on $[x_{i-1}, x_i]$.

Terminology

- knots: break points. points
- nodes: points where spline interpolates data often, knots = nodes

Cubic splines: used often in applications

Basic Idea:

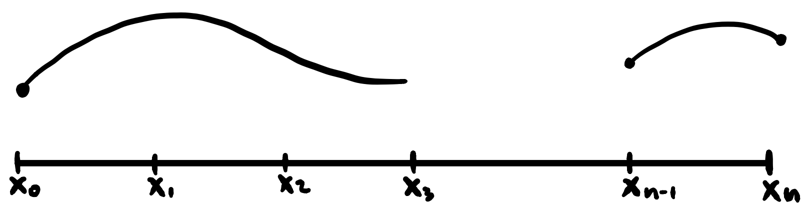
- Suppose we are given interpolation data $\{(x_i, f_i)\}_{i=0}^{n}$
- Find $s(x)$ satisfying

$$s(x) = \begin{cases} p_1(x) & x_0 \le x \le x_1 \\ p_2(x) & x_1 < x \le x_2 \\ \vdots & \vdots \\ p_n(x) & x_{n-1} < x \le x_n \end{cases}$$

where $p_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$

To find $s(x)$, we need to find a_i, b_i, c_i, d_i, $i = 1, \dots n$

We have $4n$ unknowns $\rightarrow$ need $4n$ equations



Conditions we need satisfied:

1. Interpolation: $s(x_i) = f_i$, $\quad i = 0, 1, \dots n$

$$\left.\begin{aligned} p_1(x_0) &= f_0 \\ p_2(x_1) &= f_1 \\ &\vdots \\ p_{n-1}(x_{n-1}) &= f_{n-1} \\ p_n(x_n) &= f_n \end{aligned}\right\}$$

2. **Continuity of $s(x)$**: $p_{i+1}(x_i) = p_i(x_i) \quad i = 1, 2, \ldots n-1$

$$\left.\begin{array}{l} p_2(x_1) = p_1(x_1) \\ p_3(x_2) = p_2(x_2) \\ \vdots \\ p_n(x_{n-1}) = p_{n-1}(x_{n-1}) \end{array}\right\} \quad \begin{array}{l} a_2 + b_2(x_1 - x_2) + c_2(x_1 - x_2)^2 + d_3(x_1 - x_2)^3 = a_1 \\ a_3 + b_3(x_2 - x_3) + c_2(x_2 - x_3)^2 + d_3(x_2 - x_3)^3 = a_2 \\ \vdots \\ a_n + b_n(x_{n-1} - x_n) + c_n(x_{n-1} - x_n)^2 + d_n(x_{n-1} - x_n)^3 = a_{n-1} \end{array}$$

This gives $n-1$ equations

3. **Continuity of $s'(x)$**: $P'_{i+1}(x_i) = P'_i(x_i), \quad i = 1, 2, \ldots n-1$

Note $P'_i(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$

$$\left.\begin{array}{l} P'_2(x_1) = P'_1(x_1) \\ P'_3(x_2) = P'_2(x_2) \\ \vdots \\ P'_n(x_{n-1}) = P'_{n-1}(x_{n-1}) \end{array}\right\} \quad \begin{array}{l} b_2 + 2c_2(x_1 - x_2) + 3d_2(x_1 - x_2)^2 = b_1 \\ b_3 + 2c_3(x_2 - x_3) + 3d_3(x_2 - x_3)^2 = b_2 \\ \\ b_n + 2c_n(x_{n-1} - x_n) + 3d_n(x_{n-1} - x_n)^2 = b_n \end{array}$$

This gives $n-1$ equations

4. **Continuity of $s''(x)$**: $P''_{i+1}(x_i) = P''_i(x_i), \quad i = 1, 2, \ldots n-1$

Note $P''_i(x) = 2c_i + 6d_i(x - x_i)$

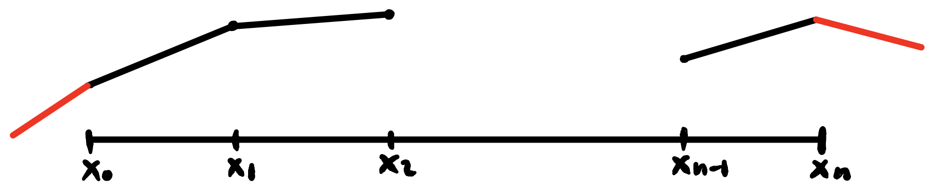
$$\left.\begin{array}{l} P''_2(x_1) = P''_1(x_1) \\ P''_3(x_2) = P''_2(x_2) \\ \vdots \\ P'_n(x_{n-1}) = P'_{n-1}(x_{n-1}) \end{array}\right\} \quad \begin{array}{l} 2c_2 + 6d_2(x_1 - x_2) = 2c_1 \\ 2c_3 + 6d_3(x_2 - x_3) = 2c_2 \\ \vdots \\ 2c_n + 6d_n(x_{n-1} - x_n) = 2c_{n-1} \end{array}$$

This gives $n-1$ equations

5. **End conditions** (several options)

(a) Natural (or free) boundary conditions

Assume $s(x)$ is a linear polynomial (line) for $x < x_0$ and $x > x_n$



This means:

$s''(x_0) = 0 \quad \Rightarrow \quad p_1''(x_0) = 0$

$s''(x_n) = 0 \quad \Rightarrow \quad p_n''(x_n) = 0$

Recall: $p_i''(x) = 2c_i + 6d_i(x - x_i)$

$\rightarrow$ $2c_1 + 6d_1(x_0 - x_1) = 0$

$2c_n = 0$

(b) Clamped end conditions (1st deriv. cond.)

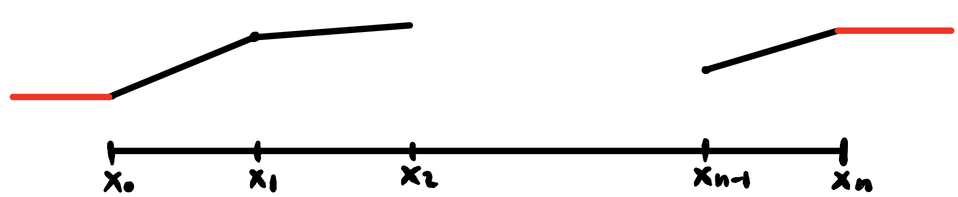
Here, we specify slope at endpoints.
We suppose we want the slope at x_0 to be δ_0 and slope at x_n to be δ_n.

Then: $s'(x_0) = \delta_0$, $s'(x_n) = \delta_n$ $\Rightarrow$ $p_1'(x_0) = \delta_0$, $p_n'(x_n) = \delta_n$

$b_1 + 2c_1(x_0 - x_1) + 3d_1(x_0 - x_1)^2 = \delta_0$

$b_n = \delta_n$

E.g. if we choose $\delta_0 = \delta_n = 0$, our spline may look like:



(c) Can similarly clamp 2^{nd} derivative