Def / Prop. Let $f \in (V, (\cdot,\cdot))$ and consider the subspace

$$W = \text{span}\{\phi_1, \dots \phi_k\} = \left\{\sum_{i=1}^{n} \lambda_i \phi_i \;\middle|\; \lambda_i \in \mathbb{R}\right\}$$

The orthogonal projection of f onto W is $P_W(f) = \sum_{k=1}^{n} \underbrace{(f, \phi_k)}_{\text{generalized Fourier coeffs.}} \phi_k$

$P_W(f)$ is the unique vector in W which minimizes

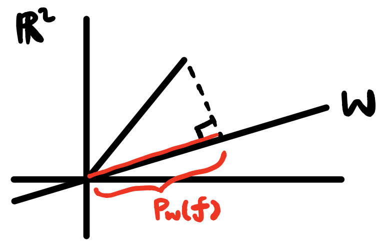
$$\|f-\phi\|^2 = \int_{-\pi}^{\pi} |f(x) - \phi(x)|^2 dx$$

among all $\phi \in W$, that is

$$\min_{\phi \in W} \|f-\phi\|^2 = \|f - P_W(f)\|^2 = \|f\|_{L^2}^2 - \sum_{i=1}^{n} (f_i \phi_i)^2$$

Furthermore, $P_W(f)$ is the unique vector in W such that $f - P_W(f) \perp W$.

$\uparrow$ orthogonal



Pf. Suppose $\phi = \sum_{k=1}^{n} \lambda_k \phi_k$:

$$0 \le \|f-\phi\|^2 = (f-\phi, f-\phi)$$
$$= (f, f-\phi) - (\phi, f-\phi)$$
$$= (f, f) - (f, \phi) - (\phi, f) + (\phi, \phi)$$
$$= \|f\|^2 - 2(f, \phi) + (\phi, \phi)$$
$$= \|f\|^2 - 2(f, \sum \lambda_k \phi_k) + (\sum \lambda_k \phi_k, \sum \lambda_\ell \phi_\ell)$$
$$= \|f\|^2 - 2\sum \lambda_k (f, \phi_k) + \sum \lambda_k \phi_k \underbrace{(\phi_k, \phi_\ell)}_{\delta_{k\ell} \text{ since } \{\phi_k\} \text{ ON}}$$
$$= \|f\|^2 - 2\sum \lambda_k (f, \phi_k) + \sum \lambda_k^2$$

$$= \|f\|^2 + \sum(\lambda_k - (f,\phi_k))^2 - \sum(f,\phi_k)^2$$
$$\geq \|f\|^2 - \sum(f,\phi_k)^2$$

with equality if and only if $\lambda_k = (f,\phi_k)$

i.e. $\phi = \sum_{k=1}^{n}(f,\phi_k)\phi_k = P_W(f)$.

Hence $0 \leq \|f - P_W(f)\|^2 = \|f\|^2 - \sum_{k=1}^{n}(f_k,\phi_k)^2$

Taking $k\to\infty$, we get:

Cor. (Bessel's Inequality)

Suppose that $\{\phi_k \mid k\in\mathbb{N}\}$ are ON.

Then $\sum_{k=1}^{\infty}(f,\phi_k)^2 \leq \|f\|^2$

and thus $(f,\phi_k)\to 0$ as $k\to\infty$.

Ex. Specifically, we have:

$\phi_0 = \frac{1}{\sqrt{2\pi}}$ $\quad (f,\phi_0)_{L^2} = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f\cdot 1\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)\,dx$

$\phi_{1k} = \frac{\cos(kx)}{\sqrt{\pi}}$ $\quad (f,\phi_{1k}) = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$

$\phi_{2k} = \frac{\sin(kx)}{\sqrt{\pi}}$ $\quad (f,\phi_{2k}) = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$

For $W = \{\phi_0, \phi_{1k}, \dots \phi_{1n}, \phi_{2k}, \dots \phi_{2n}\}$ we have:

$$P_W(f) = \sum_{k=1}^{n}(f,\phi_k)_{L^2}\phi_k$$
$$= \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx))$$

with usual Fourier coefficients

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

and as $k\to\infty$, we recover the Fourier series.

Furthermore, in $L^2[-\pi,\pi]$ we have equality,

i.e. $f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k\cos(kx) + b_k\sin(kx))$ in L^2

i.e. $\|f - P_W(f)\|_{L^2} \to 0$ as $k \to \infty$

Thm. The functions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos(kx)}{\sqrt{\pi}}, \quad \frac{\sin(kx)}{\sqrt{\pi}}, \quad k \in \mathbb{N}$$

form a complete sequence of ON functions in $(L^2[-\pi,\pi], (\cdot,\cdot)_{L^2})$,

i.e. for $f \in L^2[-\pi,\pi]$ we have

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx)) \text{ in } L^2$$

Furthermore, we have Parseval's equality:

$$\|f\|_{L^2}^2 = \int_{-\pi}^{\pi}|f(x)|^2dx$$
$$= \pi \cdot \left(\frac{1}{2}a_0^2 + \sum_{k=1}^{\infty}(a_k^2 + b_k^2)\right)$$

Bessel: $\|f\|^2 \geq \sum (f,\phi_k)^2$

$$\underbrace{\|f - \sum_{k=1}^{n}(f,\phi_k)\phi_k\|^2}_{\to 0 \text{ as } n \to \infty} = \|f\|^2 - \sum_{k=1}^{n}(f,\phi_k)^2$$

$$\Longleftrightarrow \|f\|^2 = \sum_{k=1}^{\infty}(f,\phi_k)^2.$$

For Sturm Liouville problems we'll use

$$(f,g)_\rho = \int_{-\pi}^{\pi} f(x)\cdot g(x)\cdot \rho(x)dx$$

More generally:

Thm. For a vector space with an inner product $(V,(\cdot,\cdot))$ and an ON sequence

$\{\phi_k\}$, we have:

(i) For every $f \in V$,

$$f = \sum_{k=1}^{\infty} (f, \phi_k)\phi_k \quad \text{in } (V, (\cdot,\cdot))$$

i.e. $\|f - \sum_{k=1}^{n} (f,\phi_k)\phi_k\| \to 0$ as $n \to \infty$

if and only if

(ii) $\|f\|^2 = \sum_{k=1}^{\infty} (\phi_k, f)^2$ (Parseval's Equality)

In case (i) (or equivalently (ii)) is true, $\{\phi_k\}$ is called complete.

Previously, if $f : [-\pi, \pi] \to \mathbb{R}$ is integrable and of bounded variation, then

$$P_{W_k}(f)(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx).$$

$$n \to \infty : \quad \longrightarrow \frac{1}{2}(\tilde{f}(x^+) + \tilde{f}(x^-))$$

for every $x \in \mathbb{R}$ individually (i.e. pointwise)