Thm. (X, d) m.s. is compact $\Leftrightarrow$ it is complete and totally bounded.

Pf. $\Leftarrow$ (X, d) totally bounded

$\forall\ r>0\ \exists\ c_1, c_2, \dots c_n$ s.t. $X \subseteq \bigcup_{i=1}^{n} B(c_i, r)$

In particular, this holds for $r=1$.

Let (x_n) be a sequence in X.

WTS: has convergent subsequence

$\exists\ c_1^{(1)}, c_2^{(1)}, \dots, c_{m_1}^{(1)} \in X$ s.t. $X \subseteq \bigcup_{i=1}^{n} B(c_i^{(1)}, 1)$.

$\exists\ j \in \{1, 2, \dots m_1\}$ s.t. $B(c_j^{(1)}, 1)$ contains a subsequence of (x_n): $(x_n^{(1)})$.

Observe that, the distance between any two elements of $(x_n^{(1)})$ is <2.

Since (X, d) is totally bounded, choosing $r=\frac{1}{2}$, we obtain:

$\exists\ c_1^{(2)}, c_2^{(2)}, \dots, c_{m_2}^{(2)} \in X$ s.t. $X \subseteq \bigcup_{i=1} B(c_i^{(2)}, \frac{1}{2})$

$\exists\ j \in \{1, 2, \dots m_2\}$ s.t. $B(c_i^{(2)}, \frac{1}{2})$ contains a subsequence of $(x_n^{(1)})$: $(x_n^{(2)})$.

Then, the distance between any two elements of $(x_n^{(2)})$ is <1.

$x_1, x_2, x_3, \ldots$

$x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots$

$x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots$

$\vdots$

k^{th} step: $x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots$

Observe that:

$d(x_1^{(1)}, x_2^{(2)}) \leq 2$ $\qquad$ $d(x_1^{(1)}, x_3^{(3)}) \leq 2$

$d(x_2^{(2)}, x_3^{(3)}) \leq 1$

$d(x_3^{(3)}, x_4^{(4)}) \leq \frac{1}{2}$

$\vdots$

In general, if $n > m$ are 2 natural numbers,

then $d(x_n^{(n)}, x_m^{(m)}) < \frac{1}{2^{m-2}}$.

m level: $x_1^{(m)}, x_2^{(m)}, \ldots$

n level: $x_1^{(n)}, x_2^{(n)}, \ldots$

Then $(x_n^{(n)})$ is a Cauchy subsequence of (x_n)

Since X is complete, $(x_n^{(n)})$ converges.

$\Rightarrow (X, d)$ is compact. $\boxtimes$

Thm. (X, d) m.s., $E \subseteq X$. T.F.A.E.:

1. E is compact.
2. Every open cover has a finite subcover, i.e. if A_i, $i \in I$ are open sets and $E \subseteq \bigcup_{i \in I} A_i$, then $\exists$ $F \subset I$ finite s.t. $E \subseteq \bigcup_{i \in F} A_i$

Pf. $2 \Rightarrow 1$ WTS: E not compact $\Rightarrow$ (2) does not hold

Suppose E is not compact.

$\Rightarrow \exists$ $(x_n) \subset E$ w/o convergent subsequence.

Note: a sequence (x_n) converges to x $\Longleftrightarrow$ $\forall$ $r > 0$ $\exists$ infinitely many n s.t. $x_n \in B(x, r)$.

$x_{n_k} \to x$: $\exists$ N s.t. $x_{n_k} \in B(x, r)$ $\forall$ $n_k > N$

Since (x_n) has no convergent subsequence, $\forall$ $x \in E$, $\exists$ $r_x > 0$ s.t. there are only finitely many x_n in $B(x, r_x)$.

Observe that:

$$E = \bigcup_{x \in E} \{x\} \subseteq \bigcup_{x \in E} B(x, r_x)$$

For any $F \subseteq E$ finite, we have:

$\bigcup_{x \in F} B(x, r_x)$ contains only finitely many x_n.

Therefore, there is not finite subcover.

$1 \Rightarrow 2$

Suppose E is compact.

Suppose $V_\alpha, \alpha \in I$ is an open cover of E s.t. there is no finite $F \subset I$ w/ $E \subseteq \bigcup_{\alpha \in F} V_\alpha$

By assumption, $x \in E \subseteq \bigcup_{\alpha \in I} V_\alpha$.

$\forall$ $x \in E$, $\exists$ $\alpha \in I$ s.t. $x \in V_\alpha$.

$\Rightarrow$ $\exists$ $r > 0$ s.t. $B(x, r) \subseteq V_\alpha$ (since V_α is open).

$\forall$ $y \in E$, $\exists$ $r > 0$ s.t. $B(y, r) \subseteq V_\alpha$ for some $\alpha \in I$.

Let $r(y) := \sup\{r > 0 : B(y, r) \subseteq V_\alpha$ for some $\alpha \in I\}$.

$r_0 = \inf\{r(y) \mid y \in E\}$

Case 1: $r_0 = 0$

$\forall$ n, $\exists$ $y_n \in E$ s.t. $r(y_n) < \frac{1}{n}$.

Since E is compact, $\exists$ a subsequence (y_{n_j}) of (y_n) s.t. $\lim_{n_j \to \infty} y_{n_j} = y$ for some $y \in E$.

$r(y) > 0$

$\exists$ $N \in \mathbb{N}$ s.t. $d(y_{n_j}, y) < \frac{r(y)}{2}$ $\forall$ $n_j \geq N$.

$B(y_{n_j}, \frac{r(y)}{2}) \subset B(y, r(y))$ by the Δ inequality.

$\Rightarrow B(y_{n_j}, \frac{r(y)}{2}) \subseteq V_\alpha$ for some $\alpha \in I$.

$\Rightarrow r(y_{n_j}) \geq \frac{r(y)}{2}$

$\Rightarrow \lim_{n_j \to \infty} r(y_{n_j}) \geq \frac{r(y)}{2} > 0$ $\forall$ $n_j \geq N$

On the other hand, $0 < r(y_{n_j}) < \frac{1}{n_j}$ $\forall$ $j \in \mathbb{N}$.

So $\lim_{n_j \to \infty} r(y_{n_j}) = 0$. $\lightning$

Case 2 : $r_0 > 0$

$\forall\ y \in E,\quad r(y) > \frac{r_0}{2}$

$\Rightarrow\ \exists\ \alpha \in I\ \text{ s.t. } B(y_{n_j}, \frac{r_0}{2}) \subseteq V_\alpha$.

Pick $y_1 \in E$.

$\Rightarrow B(y_1, \frac{r_0}{2}) \subseteq V_{\alpha_1}$ for some $\alpha_1 \in I$.

V_{α_1} doesn't cover E.

Pick $y_2 \in E \setminus V_{\alpha_1}$.

$\Rightarrow\ y_2 \in E \setminus B(y_1, \frac{r_0}{2})$.

$\Rightarrow\ \exists\ \alpha_2 \in I\ \text{ s.t. } B(y_2, \frac{r_0}{2}) \subseteq V_{\alpha_2}$.

$E \nsubseteq V_{\alpha_1} \cup V_{\alpha_2}$.

Pick $y_3 \in E \setminus (V_{\alpha_1} \cup V_{\alpha_2})$.

$\Rightarrow\ y_3 \notin B(y_1, \frac{r_0}{2}),\ y_3 \notin B(y_2, \frac{r_0}{2})$

$\Rightarrow\ \exists\ \alpha_3 \in I\ \text{ s.t. } B(y_3, \frac{r_0}{2}) \subseteq V_{\alpha_3}$.

$E \nsubseteq V_{\alpha_1} \cup V_{\alpha_2} \cup V_{\alpha_3}$.

$\vdots$

$\Rightarrow\ (y_n) \subseteq E$.

$d(y_n, y_m) \geq \frac{r_0}{2} \quad \forall\ n \neq m$

$\Rightarrow\ (y_n)$ has not Cauchy subsequence. $\lightning$