§17 2^{nd} Order ODEs with Constant Coefficients

(1) $y'' + py' + qy = 0 \qquad p, q \in \mathbb{R}$

Note (1) is linear: if y_1, y_2 solve (1), then $a \cdot y_1 + b \cdot y_2$ also solves (1).

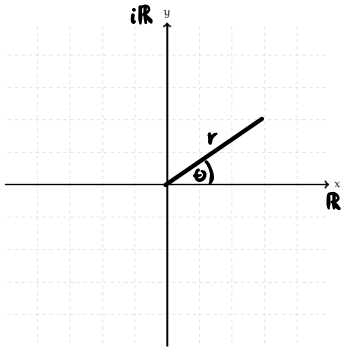
Reminder complex numbers

$z = x + iy \qquad x, y \in \mathbb{R}$

$i^2 = -1$

real part: $\text{Re}\, z = x$

imaginary part: $\text{Im}\, z = y$



Euler's formula:

$e^{i\theta} = \cos\theta + i\sin\theta$

$z = r\cos\theta + ir\sin\theta = x + iy$

$x = r\cos\theta$
$y = r\sin\theta$

(1) $y'' + py' + qy = 0 \qquad p, q \in \mathbb{R}$

consider $y(x) = e^{mx}, \ m \in \mathbb{R}$

for y to be a solution of (1):

need $m^2 e^{mx} + pme^{mx} + qe^{mx} = 0$

$$= (m^2+pm+q)e^{mx} = 0$$

$\hookrightarrow$ $m^2+pm+q=0$ for all x

$$\Rightarrow m_{1,2} = \frac{-p\pm\sqrt{p^2-4q}}{2}$$

$$= \frac{1}{2}\left(-p \pm \sqrt{p^2-4q}\right)$$

Cases: ① $p^2-4q>0 \longrightarrow m_1, m_2 \in \mathbb{R}$

$\longrightarrow$ $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ is the general solution to (1).

Rmk. The general solution of (1) always consists of two real-valued, linearly independent solutions, here: $e^{m_1 x}, e^{m_2 x}$

② $p^2-4q<0 \Rightarrow m_1 \neq m_2$ are complex numbers

Let $m_1 = a+ib$, $m_2 = a-ib$

Both $e^{m_1 x}, e^{m_2 x}$ are complex-valued solutions to (1).

$$e^{m_1 x} = e^{(a+bi)x} = e^{ax}\cdot e^{bix}$$

$$= e^{ax}(\cos bx + i\sin bx)$$

$$e^{m_2 x} = e^{(a-bi)x} = e^{ax}\cdot e^{-bix}$$

$$= e^{ax}(\cos bx - i\sin bx)$$

By linearity, $y = e^{ax}(c_1\cdot\sin bx + c_2\cdot\cos bx)$ is the general, real-valued solution.

③ $p^2-4q=0 \longrightarrow m_1 = m_2 = \frac{-p}{2} \in \mathbb{R}$

general solution is $y = c_1 e^{-\frac{p}{2}x} + c_2 x e^{-\frac{p}{2}x}$

Idea If y is a solution, find $v(x) \neq$ constant such that $v(x)\cdot y(x)$ is a solution of (1). Here this leads to $v''(x)=0$, so $v(x) = a+bx$.

§18. Method of Undetermined Coefficients

(IH) $y''(x)+py'(x)+qy(x)=R(x)$

corresponding homogeneous eqn:

(H) $y''(x)+py'(x)+qy(x)=0$

Recall: (H) is linear; there are 2 L.I. real-valued solutions.

Thm. If y_p is a particular solution of (IH), and y_g is the general solution of (H), then y_p+y_g is the general solution of (IH).

Rmk. Suppose y_1 solves $y''+py'+qy=R_1(x)$

y_2 solves $y''+py'+qy=R_2(x)$

then y_1+y_2 solves $y''+py'+qy=R_1(x)+R_2(x$

Rmk. Note that the difference of two solutions of (IH) solves (H).

[Take $R_1=R_2=R$ in the remark, then y_1-y_2 solves (H). Thus $y_1-y_2=y_g$.]