Cauchy Sequences and Completeness

Def. Let (x_n) be a sequence in a m.s. (X, d).

We say (x_n) is a Cauchy sequence if

$\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that $\forall\ n, m \geq N$: $d(x_n, x_m) < \varepsilon$.

$\subset (X,d)$

Lemma If (x_n) converges, then it is a Cauchy sequence.

Pf. $\exists\ x \in X$ s.t. $\lim\limits_{n \to \infty} x_n = x$.

$\Rightarrow\ \forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}$ s.t. $d(x, x_n) < \frac{\varepsilon}{2}\quad \forall\ n \geq N$.

$\Rightarrow\ \forall\ n, m \geq N$, we have that

$$\begin{aligned} d(x_n, x_m) &\leq d(x, x_n) + d(x, x_m) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Thus, (x_n) is a Cauchy sequence.

In general, convergent $\substack{\not\leftarrow \\ \rightarrow}$ Cauchy.

Def. Let (x_n) be a sequence in a m.s. (X, d) and $n_1 < n_2 < n_3 < \ldots$ a strictly increasing sequence in $\mathbb{N}$. Then (x_{n_k}) is called a subsequence of (x_n).

Lemma Let (x_n) be a Cauchy sequence in a m.s. (X, d). Suppose there is a subsequence (x_{n_k}) which converges to x in X. Then, also $\lim\limits_{n \to \infty} x_n = x$.

Def. Let (x_n) be a sequence in a m.s. (X, d), and let $x \in X$. We say x is a limit point of (x_n) if $\forall\ \varepsilon > 0\ \forall\ N \in \mathbb{N}\ \exists\ n \geq N$ s.t. $d(x_n, x) < \varepsilon$.

Lemma x is a limit point of (x_n) $\Leftrightarrow$ there is a subsequence (x_{n_k}) s.t. $\lim_{n_k \to \infty} x_{n_k} = x$.

Pf. $\Rightarrow$ If $\varepsilon = \frac{1}{k}$ then by assumption $\exists\ N_j \in \mathbb{N}$ s.t. $d(x_n, x) < \frac{1}{j}$.

(x_{N_j}) is a subsequence of (x_n) and $\lim_{N_j \to \infty} x_{N_j} = x$.

$\Leftarrow$ Let $\varepsilon > 0$, $N \in \mathbb{N}$. Since $\lim_{n_j \to \infty} x_{n_j} = x$, $\exists\ N_0 \in \mathbb{N}$ s.t. $d(x_{n_j}, x) < \varepsilon\ \forall\ n_j \geq N_0$.

If $n_j \geq \max\{N_0, N\}$, then x_{n_j} satisfies $d(x_{n_j}, x) < \varepsilon$.

Def. A m.s. (X, d) is complete if every Cauchy sequence converges.

Ex.
- $(\mathbb{R}, |\cdot|)$ is complete
- $((0,1), |\cdot|)$ is not complete $\quad [0,1] = \overline{(0,1)}$
- $(\mathbb{Q}, |\cdot|)$ is not complete

  $\sqrt{2} \notin \mathbb{Q}$

  $\mathbb{Q}$ is dense in $\mathbb{R}$

  $\Rightarrow \exists\ (r_n) \subset \mathbb{R}$ s.t. $\lim_{n \to \infty} r_n = \sqrt{2}$ in $\mathbb{R}$.

  (r_n) is a Cauchy sequence in $\mathbb{Q}$.

Prop. Let (X,d) be a complete m.s., $E \subseteq X$.
Then (E, d_E) is complete $\Leftrightarrow$ E is closed.

restriction of d to E

Pf. $\Leftarrow$ Let (x_n) be a Cauchy sequence in E.

$\Rightarrow$ (x_n) is also a Cauchy sequence in X.

$\Rightarrow$ (x_n) is a convergent sequence in X since X is complete.

$\Rightarrow$ $\exists\ x \in X$ s.t. $\lim_{n\to\infty} x_n = x$.

$\Rightarrow$ $x \in \bar{E}$.

$\Rightarrow$ $x \in E$.

$\Rightarrow$ If $x \in \bar{E}$, $\exists\ (x_n) \subset E$ s.t. $\lim_{n\to\infty} x_n = x$.

(x_n) converges to x.

$\Rightarrow$ (x_n) is a Cauchy sequence in E.

$\Rightarrow$ (x_n) converges to some point in E since E is complete.

$\Rightarrow$ $\lim_{n\to\infty} x_n = x$ is a point in E.

$\Rightarrow$ $\bar{E} \subseteq E$.

$E \subseteq \bar{E}$, so E is closed.

Ex. $X = \mathbb{R}$, $d = |\cdot|$

$E = (0,1)$: not complete $\qquad \bar{E} = [0,1]$: complete

$E = \mathbb{Q}$: not complete $\qquad \bar{E} = \mathbb{R}$: complete

Fact: Any (incomplete) m.s. (X, d) admits a completion, i.e. some m.s. (Y, d') s.t.:

- $X \subseteq Y$, $d'|_X = d$
- (Y, d') is complete
- $\overline{X} = Y$

Compactness

Recall: In $(\mathbb{R}, |\cdot|)$: every bounded sequence has a convergent subsequence

Def. A m.s. (X, d) is **compact** if every sequence in (X, d) has a convergent subsequence.

Ex.
- $X = \mathbb{R}$, $d = |\cdot|$: $(\mathbb{R}, |\cdot|)$ is not compact.
  $x_n = n$ has not a convergent subsequence.
  ($\mathbb{R}$ is unbounded)
- $X = (0, 1)$, $d = |\cdot|$: $((0,1), |\cdot|)$ is not compact.
  $x_n = \frac{1}{n}$ has not a convergent subsequence.
  ($(0,1)$ is open.)

Ex. In $(\mathbb{R}, |\cdot|)$, any bounded and closed set E is compact.

Pf. If $(x_n) \subset E$, then (x_n) is bounded.

But $\exists$ $(x_{n_k}) \subset (x_n)$ and $x \in \mathbb{R}$ s.t. $x_{n_k} \to x$.

$\Rightarrow x \in \bar{E}$.

$\Rightarrow x \in E$.

$\Rightarrow E$ is compact.

In $(\mathbb{R}, |\cdot|)$, compact $\Leftrightarrow$ closed + bounded.

In a general m.s. (X, d), compact $\not\Leftrightarrow$ closed + bounded.

Lemma Any finite m.s. $(X = \{a_1, a_2, \dots a_m\}, d)$ is compact.

Pf. Let (x_n) be a sequence in X.

Write $I_k = \{n \in \mathbb{N} : x_n = a_k \ \forall \ k \in \{1, 2, \dots m\}\}$

$\bigcup_{k=1}^{m} I_k = \mathbb{N}$.

$\mathbb{N}$ is infinite.

Any finite $\cup$ of finite sets is finite.

Then at least one I_{k_0} is infinite.

Let $I_{k_0} = \{n_1, n_2, \dots\}$. Then $x_{n_j} = a_{k_0} \ \forall \ j \geq 1$.

(x_{n_j}) is a constant subsequence of (x_n).

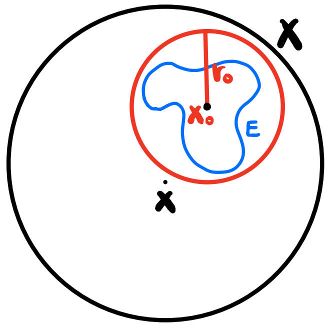
(x_{n_j}) converges to a_{k_0}.

$\Rightarrow (X, d)$ is compact.

Lemma (X, d) m.s., $E \subseteq X$

If E is compact, then E is bounded.

$\exists\ x_0 \in X,\ r_0 > 0$ s.t. $E \subseteq B(x_0, r_0)$



$r = d(x, x_0) + r_0$

note: E is bounded $\Leftrightarrow$ $\forall\ x \in X,\ \exists\ r > 0$ s.t. $E \subseteq B(x, r)$

Pf. E is not bounded $\Rightarrow$ $\exists\ x \in X\ \forall\ r > 0\ \ E \nsubseteq B(x, r)$.

In particular, $\forall\ n \in \mathbb{N},\ \exists\ x_n \in E \setminus B(x, n)$

$\Rightarrow$ by construction $x_n \notin B(x, n)$

$d(x, x_n) > n$

Then $(x_n) \subseteq E$. If E is compact then $\exists$ subsequence (x_{n_k}) of (x_n) s.t. $\lim\limits_{n_k \to \infty} x_{n_k} = y$ for some $y \in X$.

$$\lim_{n_k \to \infty} d(x, x_{n_k}) \geq \lim_{n_k \to \infty} n_k = \infty$$

Moreover, $0 \leq |\underbrace{d(x, x_{n_k}) - d(x, y)}_{\downarrow\, 0}| \leq \underbrace{d(x_{n_k}, y)}_{\downarrow\, 0}$

$$\lim_{n_k \to \infty} d(x, x_{n_k}) = d(x, y) < \infty \quad \lightning$$

Then compact $\Rightarrow$ bounded. $\square$

Lemma (X, d) m.s., $E \subseteq X$

If E is compact, then E is closed.

Pf. If E is not closed, then $E \neq \overline{E} \Rightarrow \exists\ x \in \overline{E} \setminus E$.

Since $x \in \overline{E}$, $\exists\ (x_n) \subset E$ s.t. $\lim_{n \to \infty} x_n = x$. $\notin E$

Any subsequence of (x_n) converges to x.

$\Rightarrow$ (x_n) has not convergent subsequence in E.

$\Rightarrow$ E is not compact. $\boxtimes$

Lemma (X, d) m.s., $E \subseteq X$

If E is compact, then E is complete.

Pf. If (x_n) is a Cauchy sequence in E, then

$\exists\ (x_{n_k})$ a subsequence of (x_n) and $y \in E$ s.t.

$\lim_{n \to \infty} x_{n_k} = y$ since E is compact.

So $\lim_{n \to \infty} x_n = y$.

$\Rightarrow$ E is complete.

Ex. $(\mathbb{Z}, d_{disc})$ $\qquad$ $B_{d_{disc}}(0,2) = \{n \in \mathbb{Z} : d_{disc}(0,n) < 2\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \mathbb{Z}$

- bounded, closed, complete
- NOT compact

$x_n := n$, then $d_{disc}(x_n, x_m) = 1$ if $n \neq m$

$\Rightarrow$ (x_n) has not Cauchy subsequence

$\Rightarrow$ (x_n) has not convergent subsequence

$\Rightarrow$ $(\mathbb{Z}, d_{disc})$ not compact

Def. E is totally bounded if $\forall\, r > 0\ \exists\, N \in \mathbb{N}$

$\exists\, x_1, x_2, \dots x_n \in E$ s. t. $E \subseteq \bigcup_{i=1}^{n} B(x_i, r)$.

(exercise) totally bounded $\Rightarrow$ bounded

In $(\mathbb{R}, |\cdot|)$: compact $\Leftrightarrow$ closed + bounded

In (X, d): compact $\Leftrightarrow$ complete + totally bounded

Lemma (X, d) m.s.

X is compact $\Rightarrow$ X is totally bounded.

Pf. Suppose (X, d) is not totally bounded.

$\exists\ r>0$ s.t. $\forall\ n\in\mathbb{N}$ $\forall$ $x_1, x_2, \dots x_n \in X$,

$$X \nsubseteq \bigcup_{i=1}^{n} B(x_i, r)$$

$$\Rightarrow X \setminus \bigcup_{i=1}^{n} B(x_i, r) \neq \emptyset.$$

Let $x_1 \in X$. Given $x_1, x_2, \dots x_n$, pick $x_{n+1} \in X \setminus \bigcup_{i=1}^{n} B(x_i, r)$.

$\Rightarrow$ We get a sequence (x_n) s.t. $d(x_{n+1}, x_i) \geq r$ $\forall$ $1 \leq i \leq n$

$\Rightarrow$ $d(x_n, x_m) \geq r$ $\forall$ $n \neq m$

$\Rightarrow$ (x_n) has not Cauchy subsequence

$\Rightarrow$ (x_n) has not convergent subsequence

$\Rightarrow$ (X, d) is not compact.

Thm. A m.s. (X, d) is compact $\Leftrightarrow$

it is complete and totally bounded.