Norms and distances associated to an inner product:

Ex. $(v,w)=\sum_{i=1}^{n} v_i w_i$ on $\mathbb{R}^n$

Then $|v| = \left(\sum_{i=1}^{n} v_i^2\right)^{1/2} = \sqrt{(v,v)}$

Euclidean length: $|v-w|$ = dist. betw. v, w

Def. If $(\ ,\ )$ is an inner product on V, then

$\|v\| = \sqrt{(v,v)}$ (norm / length of v)

$\|v-w\|$ is the distance between v and w.

Ex. $f, g \in C([a,b], \mathbb{R})$

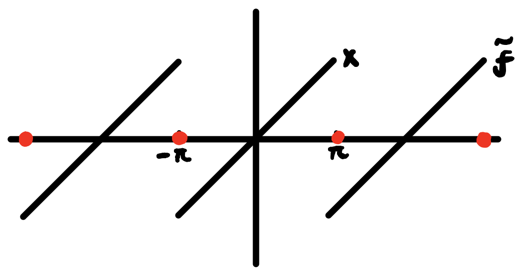
$(f,g)_{L^2} = \int_a^b f(x)g(x)dx$

$\|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \left(\int_a^b |f(x)|^2 dx\right)^{1/2}$

L^2-norm of f

$\|f-g\|_{L^2} = \left(\int_a^b |f(x)-g(x)|^2 dx\right)^{1/2}$

mean square distance / error



$|\tilde{f} - \text{Fourier series}|^2(x)$

$= 0$ if $x \neq -\pi, \pi$

but $\|\tilde{f} - \text{Fourier series}\|_{L^2} = 0$

Prop. If $(\ ,\ )$ is an inner product on V, then

(a) $\|v\| = \sqrt{(v,v)}$ is a norm i.e.

- $\|c \cdot v\| = |c| \cdot \|v\|$
- $\|v+w\| \leq \|v\| + \|w\|$ triangle inequality
- $\|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$

(b) Cauchy-Schwarz inequality

$$|(v,w)| \le \|v\| \, \|w\|$$
$$(v,w) = \|v\| \|w\| \cos \measuredangle (v,w)$$

Ex. For $(,)_{L^2}$, we have

(i) $\left(\int_a^b |f(x)+g(x)|^2 dx\right)^{1/2} \le \left(\int_a^b |f(x)|^2 dx\right)^{1/2} + \left(\int_a^b |g(x)|^2 dx\right)^{1/2}$

Minkowski inequality

(ii) $\left|\int_a^b f(x)g(x)dx\right| \le \left(\int_a^b |f(x)|^2 dx\right)^{1/2} \left(\int_a^b |g(x)|^2 dx\right)^{1/2}$

Cauchy - Schwarz inequality

The space L^2 of square integrable functions

Recall: $(f,g)_{L^2} = \int_a^b f(x)g(x)dx$

is an inner product on $C([a,b],\mathbb{R})$.

On the other hand, by Cauchy - Schwarz, $(f,g)_{L^2}$ is finite already if $\|f\|_{L^2} < \infty$, $\|g\|_{L^2} < \infty$.

Def. $$L^2([a,b]) = \left\{ f:[a,b] \to \mathbb{R} \;\middle|\; \int_a^b |f(x)|^2 dx < \infty \right\} = \left\{ f:[a,b]\to\mathbb{R} \;\middle|\; \|f\|_{L^2} < \infty \right\}$$

space of square integrable $f(x)$

H. Lebesgue

Rmk. On L^2, $(\cdot,\cdot)_{L^2}$ satisfies all properties of an inner product except: $\|f\| = 0 \Leftrightarrow f = 0$

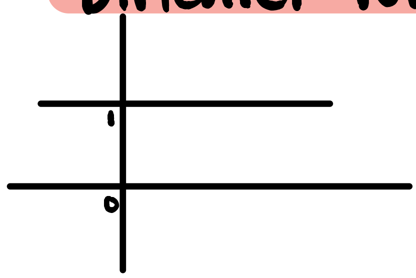
For example, if $f(x) = 0$ except at finitely many points $x_0, \dots x_n$, then $\|f\|_{L^2} = 0$

but f is NOT identically zero.

Def. A function $f \in L^2$ with $\|f\|_{L^2} = 0$ is called a **null function**.

Rmk. $D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Dirichlet function



$\|D\|_{L^2} = 0$

Equality in L^2: $f = g$ in $L^2 \iff \|f - g\|_{L^2} = 0$

$\iff f - g$ is a null function

(not necessarily $f = g$ for all x)

Convergence in L^2: $f_n \to f$ as $n \to \infty$ in L^2

$\iff \|f_n - f\|_{L^2} = \left(\int_a^b |f_n(x) - f(x)|^2 dx \right)^{1/2}$

$\to 0$ as $n \to \infty$

and in this case we write:

$\lim_{n \to \infty} f_n = f$ <u>in L^2</u>

Approximation of functions in L^2

Rmk. The following works for any (possibly ∞-dim.) vector space V with an (arbitrary) inner product $(\, , \,)$ and any sequence $\{\phi_n\}$ of ON functions.

We'll always think about

$$(V, (\, , \,)) = \left(L^2[-\pi,\pi], (\, , \,)_{L^2}\right)$$

and the ON functions

$$\{\phi_n\} = \{\phi_0, \phi_{1k}, \phi_{2k}\}$$
$$= \left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}}\right\}$$