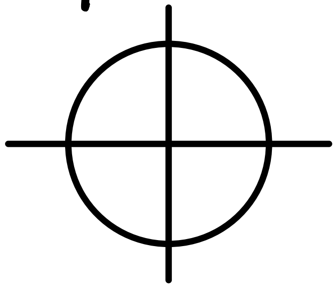
Laplace's Equation



$\Delta w = 0$

$w(1, \theta) = f(\theta)$

We saw: $w(r, \theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k\cos(k\theta) + b_k\sin(k\theta))$

Fourier series of f

Rmk. One can show:

$$w(r, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} f(\phi)\,d\phi$$

Poisson Integral

Note

1. Can compute values of $w(r, \theta)$, in particular, $r<1$, by the values of f only, i.e. the values of w for $r=1$
2. $w(0, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi$ mean value
3. For $r<1$, $w(r, \theta)$ is infinitely often differentiable.

Sturm Liouville Problems

Recall Separation of variables for the heat equation with non-const mass density led to:

$$y''(x) + \lambda \rho(x) y(x) = 0$$

Let $Ly = -P(x)y'' - P'(x)y' + R(x)y$
$= -(P(x)y')' + R(x)y$

$L(\text{function}) = \ \ldots\ \text{function}$

Aim: Given $\rho(x) > 0$, study the eigenvalue problem

$$Ly = \lambda \rho(x) y, \quad \lambda \in \mathbb{R}$$

(e.g. $\rho = 1$, $R = 0$, $P = 1$: $\quad -y''(x) = \lambda y(x)$)

with the boundary conditions

$$(*) \begin{cases} c_1 y(a) + c_2 y'(a) = 0 & c_1, c_2 \neq 0 \\ d_1 y(b) + d_2 y'(b) = 0 & d_1, d_2 \neq 0 \end{cases}$$

Ex. $y''(x) + \lambda y(x) = 0$

i) $y(0) = y(\pi) = 0$

introduction $\Longrightarrow$ nonzero solutions if $\lambda = k^2$, $k \in \mathbb{N}$

$y_k(x) = \text{const} \cdot \sin(kx)$

ii) $y(0)=0,\ y'(\pi)=0$

nonzero solutions iff $\lambda>0$ (check!)

e.g. $\lambda=0$: $y=ax+b$

$0=y(0)=b$

$0=y'(\pi)=a \longrightarrow y(x)=0$

$\lambda>0$: $y(x)=a\cos(\sqrt{\lambda}x)+b\sin(\sqrt{\lambda}x)$

$0=y(0)=a$

$0=y'(\pi)=b\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

For a nonzero solution, must have $\cos(\sqrt{\lambda}\pi)=0$, i.e. $\sqrt{\lambda}\cdot\pi=\frac{\pi}{2}+\pi k$

$\longrightarrow \sqrt{\lambda}=\frac{1}{2}+k,\quad k=0,1,2,\dots$

and $y_k=\text{const}\cdot\sin((\frac{1}{2}+k)x)$

for $k=0,1,2,\dots$

Prop. $Ly=-(Py')'+Ry$ and y_1, y_2 satisfy the boundary conditions (*).

Then $(Ly_1, y_2)_{L^2[a,b]}=(y_1, Ly_2)_{L^2[a,b]}$.

Let's calculate:

$$(Ly_1, y_2)_{L^2[a,b]}=\int_a^b (Ly_1)\cdot y_2$$

$$=-\int_a^b (Py_1')'y_2+\int_a^b Ry_1y_2$$

integration by parts $\quad = Py_1'y_2\Big|_a^b-\int_a^b Py_1'y_2'+\int_a^b Ry_1y_2$

integration by parts (antiderivative of y_1') $\quad = Py_1'y_2\Big|_a^b-\int_a^b y_1(Py_2')'+Py_1y_2'\Big|_a^b+\int_a^b Ry_1y_2$

$$= -Py_1'y_2\Big|_a^b + Py_1y_2'\Big|_a^b + (y_1, Ly_2)_{L^2}$$

$$= \underbrace{P(-y_1'y_2 + y_1y_2')\Big|_a^b}_{=0 \text{ at } x=a,b} + (y_1, Ly_2)_{L^2}$$

Say $c_1 \neq 0$: $-c_1y_1'y_2 + c_1y_1y_2' = 0$ at $x=a$

because boundary condition $c_1y(a) + c_2y'(a) = 0$

yields: $-c_1y_2 = c_2y_2'$, $c_1y_1 = -c_2y_1'$.

Hence $-c_1y_1'y_2 + c_1y_1y_2' = c_2y_1'y_2' - c_2y_1'y_2' = 0$

Cor. Suppose y_1, y_2 with $(*)$ and $Ly_i = \lambda\rho(x)y_i$,

then $\int_a^b y_1(x)y_2(x)\rho(x)dx = 0$

i.e. y_1, y_2 are orthogonal wrt the ρ-weighted L^2 inner product.

Ex. $y''(x) + \lambda y(x) = 0$

$y(0) = y(\pi) = 0$: $y_k(x) = \sin(kx)$

By corollary: $\int_0^\pi \sin(kx)\cdot\sin(\ell x)dx = 0$ if $k \neq \ell$

Why is the corollary true?

$$(\lambda_1 - \lambda_2)\int_a^b y_1(x)y_2(x)\rho(x)dx$$

$$= \int_a^b \lambda_1\rho(x)y_1(x)\cdot y_2(x)dx - \int_a^b y_1\lambda_2\rho(x)y_2(x)dx$$

$$= (Ly_1, y_2) - (y_1, Ly_2) = 0$$