Q: Why different bases?

- In some cases, better to solve $M\underline{a} = \underline{f}$
- In some cases, M is better conditioned
  $\Rightarrow$ solving $M\underline{a} = \underline{f}$ may produce less errors in $\underline{a}$

MATLAB: $p = \text{polyfit}(\underline{x}_{\text{nodes}}, \underline{f}_{\text{nodes}}, n)$

$\text{polyval}(p, x)$

Error in poly. interpolation

Given $\{(x_i, f_i)\}_{i=1}^{n}$, $f_i = f(x_i)$

Let $p_n(x)$ be the poly of deg. n with $p_n(x_i) = f_i$.

Recall Thm:

Let $[a,b]$ be the interval containing $x_1, \dots x_i$ and $f \in C^{n+1}(a,b)$. Then

$$e_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \underbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}_{W_{n+1}(x)}$$

ξ between $x_0, x_1, \dots x_n$

Rmk. 1 At interpolating points $x_0, \dots x_n$, $e_n(x_i) = a$.

2 We may have little information or control over the size of $\left|\frac{p^{(n+1)}(\xi)}{(n+1)!}\right|$

$\Rightarrow$ we might investigate how to make small: $|W_{n+1}(x)| = |(x-x_0)\cdots(x-x_n)|$

- with equally spaced pts $x_0, \dots x_n$, $W_{n+1}(x)$ tends to oscillate with growing amplitude near endpoints

- a diff. choice of pts can reduce this: a good choice is usually Chebyshev pts, e.g.

$$x_i = \frac{b+a}{2} - \frac{b-a}{2}\cos\left(\frac{2i+1}{2n+2}\pi\right) \qquad i = 0, 1, \ldots n$$

on interval $[a, b]$

Problems with Polynomial Interpolation

- High deg. poly tend to oscillate → large errors can occur
- Low deg. poly do not oscillate as much, but are poorer approximations to functions that do oscillate

Splines (piece-polynomial interpolation)

- partition interval into pieces
- construct low-degree poly approx. on each subinterval
- connect poly pieces together

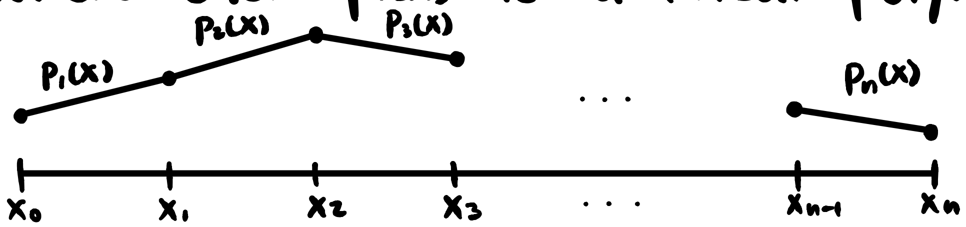
Linear Splines

Given $\{(x_i, f_i)\}_{i=0}^{n}$, define $s(x)$ as

$\Big\{ p_1(x)$ if $x \in [x_0, x_1]$

$$S(x) = \begin{cases} p_2(x) & \text{if } x \in [x_1, x_2] \\ \vdots & \\ p_n(x) & \text{if } x \in [x_{n-1}, x_n] \end{cases}$$

where each $p_i(x)$ is a linear polynomial (line)



How to find $p_i(x)$?

Recall: Given two points (x_{i-1}, f_{i-1}), (x_i, f_i)

point-slope form of the line

$$y - f_i = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}(x - x_i)$$

$$\longrightarrow p_i(x) = f_i + \frac{f_i - f_{i-1}}{x_i - x_{i-1}}(x - x_i)$$

We will use notation: $a_i = f_i$, $b_i = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$

So $p_i(x) = a_i + b_i(x - x_i)$

A linear spline for data $\{(x_i, f_i)\}_{i=0}^{n}$ can be represented as:

$$S(x) = \begin{cases} a_1 + b_1(x - x_1) & \text{if } x_0 \le x < x_1 \\ a_2 + b_2(x - x_2) & \text{if } x_1 \le x < x_2 \\ \vdots & \\ a_n + b_n(x - x_n) & \text{if } x_{n-1} \le x < x_n \end{cases}$$

where $a_i = f_i$, $b_i = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$

Ex. Construct a linear spline for data

x_i	-1	0	1
f_i	0	1	3

$a_1 = 1$, $a_2 = 3$

$b_1 = \frac{1-0}{0-(-1)} = 1$, $b_2 = 2$

$$S(x) = \begin{cases} 1+(x-0) = 1+x & -1 \le x < 0 \\ 3+2(x-1) = 1+2x & 0 \le x \le 1 \end{cases}$$

Error for Linear Splines

Recall:
- For linear interpolation, we use two points: (x_{i-1}, f_{i-1}), (x_i, f_i)
- From then, the error is:

$$|f(x) - p_i(x)| = \left|\frac{f''(\xi_i)}{2!}\right| \cdot |(x-x_{i-1})(x-x_i)|, \quad \xi_i \in [x_{i-1}, x_i]$$

- Let $M_i = \max_{x_{i-1} \le x \le x_i} |f''(x)|$.

  Then $|f(x) - p_i(x)| = \frac{M_i}{2} |(x-x_{i-1})(x-x_i)|$.

Q: Can we find an upper bound on $|(x-x_{i-1})(x-x_i)|$?

Let $W_i(x) = (x-x_{i-1})(x-x_i) = x^2 - (x_{i-1}+x_i)x + x_{i-1}x_i$

$$W_i'(x) = 2x - x_{i-1} - x_i = 0 \rightarrow x = \frac{x_i + x_{i-1}}{2}$$

- Notice that $|W_i(x)|$ is largest either at crit. pt. $x = \frac{x_i + x_{i-1}}{2}$, or at one of the endpts x_{i-1}, x_i.

$$|W_i(x_{i-1})| = 0, \quad |W_i(x_i)| = 0$$

$$\left|W_i\left(\frac{x_{i-1}+x_i}{2}\right)\right| = \left|\left(\frac{x_{i-1}+x_i}{2} - x_{i-1}\right)\left(\frac{x_{i-1}+x_i}{2} - x_i\right)\right|$$

$$= \frac{(x_{i-1}-x_i)^2}{4} \leftarrow \text{largest}$$

$\Rightarrow$ For linear interpolation, the error for $x \in [x_{i-1}, x_i]$ is

$$|f(x) - p_i(x)| \le \frac{1}{2} M_i |W_i(x)| \le \frac{1}{2} \cdot \frac{1}{4} M_i (x_i - x_{i-1})^2$$

$$= \frac{1}{8} M_i h_i^2$$

where $h_i = x_i - x_{i-1}$

Now consider linear splines

$$S(x) = \begin{cases} p_1(x) & x \in [x_0, x_1] \\ p_2(x) & x \in [x_1, x_2] \\ \vdots & \\ p_n(x) & x \in [x_{n-1}, x_n] \end{cases}$$

Let $|e_n(x)| = |f(x) - S(x)| \leq \max_i \frac{M_i}{8} \cdot h_i^2 = \frac{M}{8} h^2$

where $M = \max_i M_i, \quad h = \max_i h_i$