Theorem Convergence of Bisection

Rmk. This theorem can be used to estimate error bound.

Ex. Determine the # of iterations needed in Bisection Method to solve

$$f(x) = x^3 + 4x^2 - 10 = 0$$

with accuracy 10^{-3} in $[1,2]$.

Sol. By Conv. of Bisection Thm., we have:

$$|P_N - P| \le \frac{1}{2^N}(b-a) < 10^{-3}$$

$$\Rightarrow N > \frac{3}{\log(2)} \approx 9.96$$

$\Rightarrow$ at least 10 iterations needed to achieve accuracy of 10^{-3}

2.2 Fixed Point Iteration

Two related / equivalent functions:

① roots: given a function $f(x)$, find p such that $f(p)=0$

② fixed points: given a function $g(x)$, find p such that $g(p)=p$

Ex. If $g(x) = x - f(x)$ or $g(x) = x + 3f(x)$

then $g(p) = p \iff f(p) = 0$

Ex. Find fixed points of $g(x) = x^2 - 2$

Sol. $x = g(x) \Rightarrow x = x^2 - 2 \Rightarrow x^2 - x - 2 = 0$

?

Thm. Existence and Uniqueness of Fixed Points

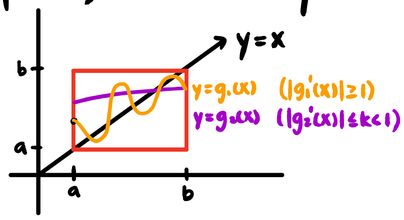
Existence: If $g(x) \in C([a,b])$ and $g(x) \in [a,b]$ for any $x \in [a,b]$,

then there exists $p \in [a,b]$ such that $g(p) = p$.

Uniqueness: If in addition, $g \in C'([a,b])$ and there exists $k \in (0,1)$ such

that $|g'(x)| \leq k < 1$ for any $x \in (a,b)$, then the fixed point

$p \in [a,b]$ is unique.



Pf. **Existence:** Case 1: $g(a) = a$ or $g(b) = b$ $\Rightarrow$ true

Case 2: we have $g(a) > a$ and $g(b) < b$

Let $h(x) = g(x) - x$ $\rightarrow$ $h(a) > 0$ and $h(b) < 0$

Since $h(x) \in C([a,b])$, by IVT $(h(a) \cdot h(b) < 0)$ there exists

$p \in (a,b)$ such that $h(p) = 0$ $\Rightarrow$ p is a fixed pt

Recall MVT: If $f \in C'[a,b]$, $C[a,b]$, then there exists $\xi \in [a,b]$

such that $f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$

Uniqueness: Proof by contradiction

Assume $p, q \in (a,b)$, $p \neq q$

and $g(p) = p$, $g(q) = q$.

By MVT, we can find $\xi \in [a,b]$ such that

$$\frac{g(p) - g(q)}{p - q} = g'(\xi).$$

$$\Rightarrow |p-q| = |g(p) - g(q)| = |(p-q)g'(\xi)| \leq k|p-q| < |p-q|$$

contradiction! $\Rightarrow$ $p = q$

Rmk. ① We can still have $\overset{\text{broader}}{|g'(x)| < 1}$, so $|g'(x)| \overset{\text{stronger}}{\leq k < 1}$ is sufficient condition but not necessary. We will use $|g'(x)| \le k$ later for algorithms.

② This theorem gives sufficient but NOT necessary conditions.

Ex. Let $g(x) = 3^{-x}$ on $[0, 1]$. Discuss existence and uniqueness of f.p of g.

Sol. Existence: $g(1) = \frac{1}{3}$, $g(0) = 1$.

$\Rightarrow g(x) \in [0, 1]$ for all $x \in [0, 1]$.

$\Rightarrow$ there exists a solution

Uniqueness: $g'(x) = -3^{-x} \ln(3)$.

$g'(0) = -\ln(3) \Rightarrow |g'(0)| > 1$

$\Rightarrow$ cannot use uniqueness from thm.

Moreover, fixed point is actually unique

because $g'(x) < 0$ for all $x \in (0, 1)$ (decreasing).