Picard Iteration

Observation For a continuous function f,

y solves $\dot{y}(t) = f(t, y(t))$, $y(t_0) = y_0$

$\Leftrightarrow$ y solves $y(t) = y_* + \int_{t_0}^{t} f(s, y(s))\,ds$

because, by the FTC:

$$y(t) - y(t_0) = \int_{t_0}^{t} \dot{y}(t)\,dt$$
$$= \int_{t_0}^{t} f(s, y(s))\,ds$$

Consider $\underbrace{T(y)}_{} = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$

is a function in t
for any continuos f in y

Aim: Find y such that $T(y) = y$.

Then y solves $y = f(t, y)$, $y(t_0) = y_0$

Recipe to find y : Picard Iteration

Set $y_0(t) = y_*$

$y_1(t) = T(y_0) = T(y_*)$

$y_2(t) = T(y_1) = T^2(y_*)$

$\vdots$

$$y_n(t) = T(y_{n-1}(t)) = \underbrace{T^n(y_*)}_{\text{apply } T \text{ n-times}}$$

If f is Lipschitz in y, this process will converge (uniformly) for $t \in [t_0 - \delta, t_0 + \delta]$ to a function $y(t) = \lim_{n \to \infty} y_n(t)$ which solves $T(y) = y$, i.e. our ODE $\dot{y} = f(t, y)$.

Rmk. $$y = \lim_{n\to\infty} y_n$$
$$= \lim_{n\to\infty} T(y_{n-1})$$
$$= T(\lim_{n\to\infty} y_{n-1}) \qquad \text{exchange of integration, taking limit}$$
$$= T(y)$$

i.e. we have a chance of finding y with Picard Iteration

Ex. $\dot{y} = y \qquad y(0) = y_*$

$y_0(t) = y_*$

$f(t, y) = y$

$$y_1(t) = T(y_0(t)) = T(y_*)$$
$$= y_* + \int_0^t f(s, y_0(s))ds$$
$$= y_* + \int_0^t y_* \, ds$$
$$= y_* + y_* \cdot t$$
$$= y_*(1+t)$$

$$y_2(t) = T(y_1(t)) = T(y_*(1+t))$$
$$= y_* + \int_0^t f(s, y_1(s))\, ds$$
$$= y_* + \int_0^t y_*(1+s)ds$$
$$= y_* + y_*(t + \tfrac{1}{2}t^2)$$
$$= y_*(1 + t + \tfrac{1}{2}t^2)$$
$$\vdots$$

$$y_n(t) = y_*(1 + t + \tfrac{1}{2}t^2 + \dots + \frac{t^n}{n!})$$
$$= y_* \sum_{k=0}^{n} \frac{t^k}{k!}$$
$$\lim_{n\to\infty} y_* \sum_{k=0}^{n} \frac{t^k}{k!} = y_* e^t$$

indeed: $y(t) = y_* e^t$, $y(0) = y_*$

Q: Why does Picard Iteration converge?

In general:

$$y_n(t) = y_0 + (y_1 - y_0) + (y_2 - y_1) + \dots + (y_n - y_{n-1})$$
$$= y_0 + \sum_{k=0}^{n} (y_{k+1} - y_k)$$

this series converges if $(y_{k+1} - y_k) \to 0$ fast enough

note: $y_{k+1} - y_k = T(y_k) - T(y_{k-1})$

Claim: If $|f(t,y) - f(t,\tilde{y})| \le L|y - \tilde{y}|$ for all $t \in [t_0 - \delta, t_0 + \delta]$, $y, \tilde{y} \in D$, then

$$\max_{[t_0-\delta, t_0+\delta]} \underbrace{|T(y) - T(\tilde{y})|}_{\text{function in } t} \le L \cdot \delta \max_{[t_0-\delta, t_0+\delta]} |y(t) - \tilde{y}(t)|$$

for every function y, $\tilde{y}$.

i.e. if $q = L \cdot \delta < 1$, then T decreases distances between (continuous) functions, and $q < 1$ is sufficient for convergence of Picard Iteration

note: $\max|y_{k+1} - y_k| = \max|T(y_k) - T(y_{k-1})|$

$$\le q \cdot \max|y_k - y_{k-1}|$$
$$= q \cdot \max|T(y_{k-1}) - T(y_{k-2})|$$
$$\le q^2 \cdot \max|y_{k-1} - y_{k-2}|$$
$$\vdots$$

$\leq q^k |y_1 - y_0|$

Analogy: $a_n = a_0 + \sum_{k=0}^{n} a_{k+1} + a_k$

a_k sequence of numbers st

$|a_{k+1} - a_k| < q^k |a_1 - a_0|$ with $q<1$

Then $|a_n| \leq |a_0| + \sum_{k=0}^{n} |a_{k+1} - a_k|$

$\leq |a_0| + \sum_{k=0}^{n} q^k |a_1 - a_0|$

$= |a_0| + |a_1 - a_0| \sum_{k=0}^{n} q^k$

$\leq |a_0| + |a_1 - a_0| \sum_{k=0}^{\infty} q^k$

$= |a_0| + |a_1 - a_0| \frac{1}{1-q}$

$< \infty$

By Weierstrass M-test: a_n converges.

[more abstract way of proving Picard's Thm. is Barnard's fixed point Thm. $(T(y)=y)$]

Q: Why is the claim true?

Note: $|T(y) - T(\tilde{y})|$

$= y_* + \int_0^t f(s, y(s))\,ds - (y_* + \int_{t_0}^t f(s, \tilde{y}(s))\,ds)$

$\leq \int_{t_0}^t |f(s, y(s)) - f(s, \tilde{y}(s))|\,ds$

F Lipschitz $\rightarrow \leq \int_{t_0}^t L |y(s) - \tilde{y}(s)|\,ds$

$\leq L \cdot \max_{t \in [t_0 - \delta, t_0 + \delta]} |y(s) - \tilde{y}(s)| \cdot \int_{t_0}^t ds$

$\leq L \cdot \max |y(s) - \tilde{y}(s)| \cdot |t + \delta - t|$

$= L \cdot \delta \max |y(s) - \tilde{y}(s)|$

This implies the claim.

Q: Why do we get uniqueness?

Suppose that y_1, y_2 solve

$$\dot{y} = f(t, y) \quad , \quad y(t_0) = y_*$$

Then $\max\limits_{t \in [t_0 - \delta, t_0 + \delta]} |y_1 - y_2| = \max |T(y_1) - T(y_2)|$

$\le \underbrace{q}_{<1} \cdot \max |y_1 - y_2|$

$T(y_i) = y_i$ since y_i solves ODE

This is a contradiction $(1 \le q < 1)$ unless $\max|y_1 - y_2| = 0$, i.e. $y_1 = y_2$.