We saw: if $\{\phi_k\}$ ON

and $f \in V$ and $\phi = \sum_{k=1}^{n} \lambda_k \phi_k$

$$\|f-\phi\|^2 = \|f\|^2 - \sum_{k=1}^{n}(f,\phi_k)^2 + \sum_{k=1}^{n}(\lambda_k - (f,\phi_k))^2$$
$$\geq \|f\|^2 - \sum_{k=1}^{n}(f,\phi_k)^2$$

and for $\phi = P(f) = \sum_{k=1}^{n}(f,\phi_k)\phi_k$

we get equality i.e.

$$0 \leq \|f - P(f)\|^2 = \|f\|^2 - \underbrace{\sum_{k=1}^{n}(f,\phi_k)^2}_{\|P(f)\|^2}$$

$$\rightarrow \sum_{k=1}^{n}(f,\phi_k)^2 \leq \|f\|^2$$

and taking $n \to \infty$ we get Bessel's inequality.

Q: If $f \in L^2$ and $\phi_k = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}} \right\}$,

why do we get $\|f - \underbrace{\sum_{k=1}^{n}(f_k,\phi_k)\phi_k}_{\text{Fourier series for } n=\infty}\|_{L^2} \to 0$

as $n \to \infty$?

(i) ϕ_k ON

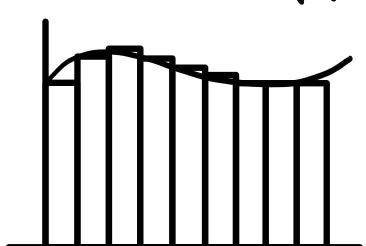
(ii) Every $f \in L^2$ can be approximated

by Fourier series:

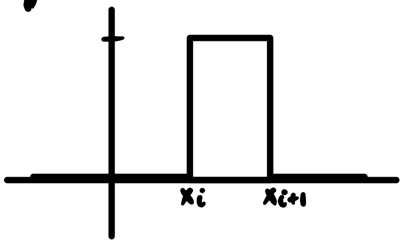
Intuition:

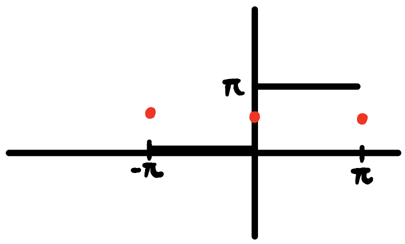
f integrable, e.g. L^2

$\Leftrightarrow$ f can be approximated by step function



Every step function can be approximated by Fourier series:



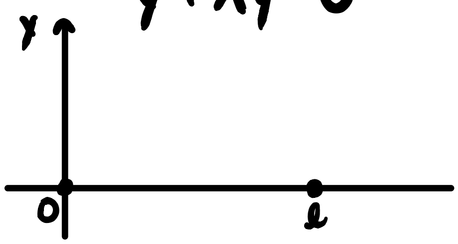


Boundary Value Problems

Ex. For $\lambda \in \mathbb{R}$ consider

$$\ddot{y} + \lambda y = 0 \qquad y(0) = y(\ell) = 0$$

$\ell \in \mathbb{R}$



(i) $\lambda = 0$: $y(t) = c_1 t + c_2 \rightarrow y(t) = 0 \quad \forall t$

(ii) $\lambda < 0$: $y(t) = c_1 \cosh(\sqrt{|\lambda|}t) + c_2 \sinh(\sqrt{|\lambda|}t)$

$0 = y(0) = c_1 + 0 = c_1$

$0 = y(\ell) = c_2 \sinh(\underbrace{\sqrt{|\lambda|}t}_{>0}) \rightarrow c_2 = 0$

$y(t) = 0 \quad \forall t$

(iii) $\lambda > 0$: $y(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t)$

$0 = y(0) = c_1$

$0 = y(\ell) = c_2 \sin(\sqrt{\lambda}t)$

$c_2 = 0 \rightarrow y(t) = 0 \quad \forall t$

$c_2 \neq 0 \rightarrow \sqrt{\lambda} \cdot \ell = \pi k$ for some $k \in \mathbb{N}$

i.e. only for $\lambda^2 = \frac{\pi^2 k^2}{\ell^2}$ with $k \in \mathbb{N}$

we have nonzero solutions

$$y_k(t) = c_k \cdot \sin\left(\frac{\pi k}{\ell} \cdot t\right) \qquad c_k \in \mathbb{R},\ \neq 0$$

$\rightarrow$ space of sol'ns is 1-D $\Leftrightarrow$ $c_k \in \mathbb{R}$

Note If $L(y) = -\ddot{y}$ then $L(y_k) = \lambda_k y_k$

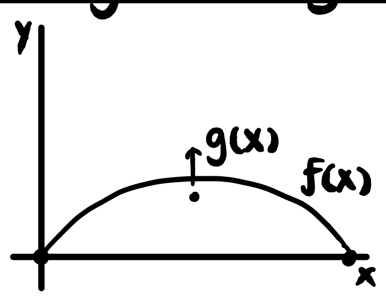
$(-\ddot{y}_k = \lambda_k y_k)$

λ_k : eigenvalues

y_k : eigenfunctions

(eigenspace is 1-D)

Vibrating Strings : 1-D wave equation



mass density $\rho(x)$

(mass of a piece of length dx is $\int \rho(x)dx$)

vibration $y(x,t)$ time t, pos. x

model: 1-D wave eqn

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho(x)} \cdot \frac{\partial^2 y}{\partial x^2}$$

Rmk. This is a linear PDE.

boundary conditions

$y(0,t) = y(\ell,t) = 0$

$y(x,0) = f(x)$ with $f(0) = f(\ell) = 0$

$\frac{\partial y}{\partial t}(x,0) = g(x)$ ($g(x) = 0 \to$ string at rest)

Separation of variables:

$$y(x,t) = u(x) \cdot v(t)$$

Then $\frac{\partial^2 y}{\partial t^2} = u(x)\ddot{v}(t)$

$\frac{\partial^2 y}{\partial x^2} = u''(x)v(t)$

wave eqn: $\frac{\partial^2 y}{\partial t^2} = \frac{1}{\rho(x)} \frac{\partial^2 y}{\partial x^2}$

$u(x)\ddot{v}(t) = \frac{1}{\rho(x)} \cdot u''(x)v(t)$

$$\Rightarrow \underbrace{\frac{\ddot{v}(t)}{v(t)}}_{\text{only dep. on } t} = \underbrace{\frac{1}{\rho(x)} \cdot \frac{u''(x)}{u(x)}}_{\text{only dep. on } x} = \text{const.} = -\lambda a^2$$

$$\Rightarrow \ddot{v}(t) + \lambda a^2 v(t) = 0 \qquad (1)$$

$$u''(x) + \lambda a^2 \cdot \rho(x) \cdot u(x) = 0$$

Suppose from now on: $\rho(x) = \text{const.} = \frac{1}{a^2}$

(variable $\rho(x) \to$ Sturm Liouville)

Then $u''(x) + \lambda u(x) = 0.$ (2)

Since $y(0,t) = y(\ell,t) = 0$: $u(0) = u(\ell) = 0$

Thus, by Ex., (2) only has non-trivial solutions

if $\lambda = \lambda_k = \frac{\pi^2}{\ell^2}k^2$, $k \in \mathbb{N}$

and then $u_k = c_k \cdot \sin\left(\frac{\pi k}{\ell}x\right)$, $c_k \in \mathbb{R}$

are the corresponding solutions.

The general solution of (1) is:

$$V_k(t) = a_k \cos\left(a\frac{\pi k}{\ell} \cdot t\right) + b_k \sin\left(a\frac{\pi k}{\ell} \cdot t\right)$$

Hence : $y_k(x,t) = U_k(t) \cdot V_k(t)$ are solutions of the 1-D equation.