Rmk. 1 If $f(p)=0$ and $f'(p)\neq 0$, then for any p_0 sufficiently close to p (st $g\in C'([a,b])$, $g(x)\in[a,b]$, $|g'(x)|\leq k\leq 1$), Newton's Method will converge at least quadratically.

Rmk. 2 If $f(p)=0$, then for p_0 close to p, secant method converges to p with order $\frac{\sqrt{5}+1}{2}\approx 1.618$.

Method	Order α
Bisection	1
FP Iteration	1 if $g'(p)\neq 0$ ≥ 2 if $g'(p)=0$
Newton	≥ 2 if $f'(p)\neq 0$ ≤ 1 if $f'(p)=0$
Secant	≈ 1.618 if $f'(p)\neq 0$ ≤ 1 if $f'(p)=0$

2.4 Multiple Roots

Def. Let $f(x)=(x-p)^m\cdot q(x)$ and $\lim_{x\to p} q(x)\neq 0$.

Then p is a root of multiplicity m (multiple root) of f.

$m=1 \rightarrow p$ is a simple root of f.

Thm. 1 A function $f\in C'([a,b])$ has a simple root at p if and only if $f(p)=0$ and $f'(p)\neq 0$.

Pf. $\Rightarrow$ If f has a simple root at p in $[a,b]$,

then 1) $f(p)=0$

2) $f(x)=(x-p)\,q(x)$ where $\lim_{x\to p} q(x)\neq 0$.

Since $f \in C^1[a,b]$,

$$f'(p) = \lim_{x\to p} f'(x) = \lim_{x\to p} q(x) + \overset{\swarrow 0}{(x-p)} q'(x)$$

$$= \lim_{x\to p} q(x) \neq 0.$$

$\Leftarrow$

If $f(p)=0$ and $f'(p)\neq 0$, then

Taylor's Thm

$$f(x) = f(p) + f'(\xi(x))(x-p)$$

$$= f'(\xi(x))(x-p)$$

(ξ between x and p)

Then $\lim_{x\to p} f'(\xi(x)) = f'(p) \neq 0$

Let $g(x) = f'(\xi(x))$. Then p is a simple root of f.

Generalization

Thm. 2 $f \in C^m(a,b)$ has a root of multiplicity m at $p \in (a,b)$ if and only if

$$f(p) = f'(p) = f''(p) = \dots = f^{(m-1)}(p) = 0$$

$$f^{(m)}(p) \neq 0$$

Ex. Let $f(x) = e^x - x - 1$

(a) Show that $x=0$ is a root of mult. 2 of $f(x)$.

(b) Show Newton's method does NOT converge quadratically.

Sol. a) Compute $f(0)$, $f'(0)$, $f''(0)$:

$f(x) = e^x - x - 1$ $\quad$ $f(0)=0$

$f'(x) = e^x - 1$ $\quad$ $f'(0) = 0$

$f''(x) = e^x$ $\quad$ $f''(0) = 1 \neq 0$

By Thm. 2, $p=0$ is a root of multiplicity 2.

b) Since $f'(p)=0$, Newton's method doesn't converge quadratically.

2.4 Modified Newton's Method

Recall: We lose quadratic convergence when $f'(p)=0$, i.e.

when multiplicity of p is $m>1$.

Let $\mu(x) = \dfrac{f(x)}{f'(x)} = \dfrac{(x-p)^m q(x)}{(x-p)^{m-1}q(x)\cdot m + (x-p)^m q'(x)}$

$$= \frac{(x-p)q(x)}{mq(x)+(x-p)q'(x)} \cdot \frac{(x-p)^{m-1}}{(x-p)^{m-1}}$$

Note that p is a simple root of $\mu(x)$ since

$$\lim_{x\to p} \frac{q(x)}{mq(x)+(x-p)q'(x)} = \frac{1}{m} \neq 0$$

Idea: Apply Newton's Method to $\mu(x)$ rather than $f(x)$ if $f(p)\neq 0$.

Since p is simple root $\Rightarrow$ quadratic convergence to p.

Modified Newton: $p_{n+1} = p_n - \dfrac{\mu(p_n)}{\mu'(p_n)}$

$$= p_n - \frac{f(p_n)\cdot f'(p_n)}{[f'(p_n)]^2 - f(p_n)f''(p_n)}$$

Here, we set $g(x) = x - \dfrac{f(x)f'(x)}{(f'(x))^2 - f(x)f''(x)}$.

Rmk. 1 Modified Newton converges quadratically regardless of multiplicity of p.

Rmk. 2 Requires second derivative information (expensive)!