Notes 02/24

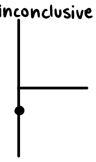
Today: - Energy / Lyapunov Function

- Lotka-Volterra: Predator-Prey
- Symmetry

Ex. $\begin{cases} \dot{x} = -xy^2 \\ \dot{y} = 3x^2y - y \end{cases}$ $\quad DF = \begin{pmatrix} -y^2 & -2xy \\ 6xy & -1 \end{pmatrix}$

fixed point: $(0,0)$ $\quad DF(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$ inconclusive!

$\tau = -1, \ \Delta = 0$



Tool: Lyapunov function $\supset$ Energy functions

Def. Lyapunov function

$V(x,y)$ Functions of state

Thm. Hosch-Smale Theorem

If (x^*, y^*) is a fixed point of a system and V is a Lyapunov function, $V(x^*, y^*) = 0$:

① $\dot{V} < 0$ & $V(x,y) \neq 0$ for all $(x,y) \neq (x^*, y^*)$

$\longrightarrow (x^*, y^*)$ globally attracting

② $\dot{V} = 0$ & $V(x,y) \neq 0$ for all $(x,y) \neq (x^*, y^*)$

$\longrightarrow$ DF gives a circle

③ $\dot{V} > 0$ & $V(x,y) \neq 0$ for all $(x,y) \neq (x^*, y^*)$

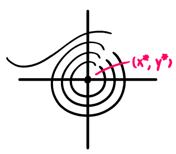
$\longrightarrow (x^*, y^*)$ unstable and repelling

To be more precise, consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^*$. Suppose that we can find a ***Liapunov function***, i.e., a continuously differentiable, real-valued function $V(\mathbf{x})$ with the following properties:

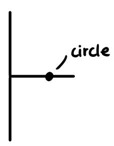
1. $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, and $V(\mathbf{x}^*) = 0$. (We say that V is *positive definite.*)
2. $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. (All trajectories flow "downhill" toward $\mathbf{x}^*$.)

Then $\mathbf{x}^*$ is globally asymptotically stable: for all initial conditions, $\mathbf{x}(t) \rightarrow \mathbf{x}^*$ as $t \rightarrow \infty$. In particular the system has no closed orbits. (For a proof, see Jordan and Smith 1987.)

Guaranteed to have closed cycles



Rmk. DF gives a circle if DF has 2 complex eigenvalues and $\tau = 0$.



Pf. Idea:

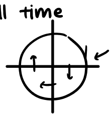
① V is decreasing close by (x^*, y^*)

but assumptions for a min (x^*, y^*)

$\dot{V} < 0 \longrightarrow$ forces convergence to (x^*, y^*)

② DF gives a circle for all time

$\begin{cases} x' = y + O(x^2, y^2) \\ y' = x + O(x^2, y^2) \end{cases}$



preserved energy forces loop to close

approximately close to (x^*, y^*) / linearization

How to find energy functions?

$$\begin{cases} \dot{x} = y \\ \dot{y} = 2x - 4x^3 \end{cases}$$

Want: E such that $\dot{E} = 0$

$\partial_x E \dot{x} + \partial_y E \dot{y} = 0$

$\hookrightarrow \begin{cases} \frac{\dot{x}}{y} = 1 \\ \frac{\dot{y}}{2x-4x^3} = 1 \end{cases}$ $\Rightarrow \frac{\dot{x}}{y} - \frac{\dot{y}}{2x-4x^2} = 0$ $\cdot y$ $\cdot 2x - 4x^3$

$\downarrow$ separate variables

$$\begin{cases} \partial_x E = u(x,y) \\ \partial_y E = v(x,y) \end{cases}$$

Solve: $\begin{cases} \partial_x E = 2x - 4x^3 \\ \partial_y E = -y \end{cases}$ $\qquad \underbrace{(2x-4x^3)}_{\partial_x E}\dot{x} - \underbrace{y}_{\partial_y E} \cdot \dot{y} = 0$

Integrate in x and y $\longrightarrow E(x,y) = x^2 - x^4 - \frac{1}{2}y^2$

Show that *a conservative system cannot have any attracting fixed points.*

Solution: Suppose $\mathbf{x}^*$ were an attracting fixed point. Then all points in its basin of attraction would have to be at the same energy $E(\mathbf{x}^*)$ (because energy is constant on trajectories and all trajectories in the basin flow to $\mathbf{x}^*$). Hence $E(\mathbf{x})$ must be a *constant function* for $\mathbf{x}$ in the basin. But this contradicts our definition of a conservative system, in which we required that $E(\mathbf{x})$ be nonconstant on all open sets. ■

If attracting fixed points can't occur, then what kind of fixed points *can* occur? One generally finds saddles and centers, as in the next example.

$E(x,y) = \frac{1}{2}y^2 - x^2 + x^4$ (switch sign)

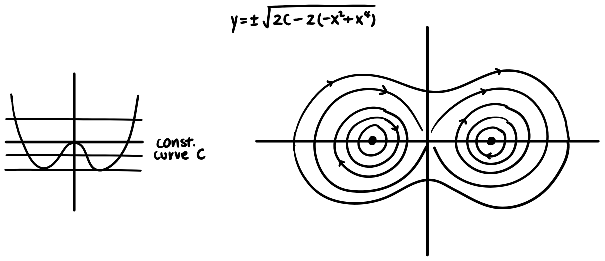
constant along solutions $(x(t), y(t))$

$\Rightarrow (x(t), y(t))$ solves $E(x(t), y(t)) = C$ ↖constant-dependent trajectory $\subset$ level curves

Def. Level curves: curves where $E(x,y) = C$

Plot level curves of $E = \frac{1}{2}y^2 - x^2 + x^4 = C$

$y = \pm\sqrt{2C - 2(-x^2 + x^4)}$



Lotka-Volterra predator-prey

x = rabbits, y = wolves ; wolves eat rabbits

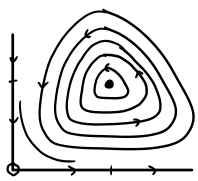
$$\begin{cases} \dot{x} = \underbrace{r_x x}_{\text{grow exponentially}} - \overbrace{\alpha xy}^{\text{get eaten}} \\ \dot{y} = \underbrace{\beta xy}_{\text{incr. by eating}} - \delta y \leftarrow \text{rate of decay} \end{cases}$$

Ex. $\begin{cases} \dot{x} = x - xy \\ \dot{y} = xy - y \end{cases}$ $\qquad \begin{matrix} x(1-y) = 0 \\ y(1-x) = 0 \end{matrix}$ $\Rightarrow$ fixed points: $(0,0)$ $(1,1)$

Stability: $DF = \begin{pmatrix} 1-y & -x \\ y & x-1 \end{pmatrix}$

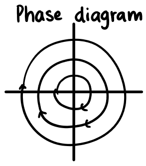
$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow$ saddle

$DF(1,1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\qquad \tau = 0, \Delta = 1 \rightarrow$ center, inconclusive



Symmetry

Ex. Phase diagram: consider



$$\begin{cases} \dot{x} = y - y^3 \\ \dot{y} = -x - x^3 \end{cases}$$

How to see this symmetry:

Def. Reversible system: one that remains the same after $t \to -\tau$, $y \mapsto -y$

if $t \to \tau$

$x(t) \mapsto x(\tau)$

$$\begin{cases} \frac{dx}{dt} = \frac{dx}{-d\tau} = y - y^3 \\ \frac{dy}{dt} = \frac{dy}{-d\tau} = -x - x^3 \end{cases} \Longrightarrow \begin{matrix} \frac{dx}{d\tau} = y^3 - y \\ \frac{dy}{d\tau} = x + x^3 \end{matrix} \xRightarrow{y \mapsto -Y} \begin{cases} \frac{dX}{d\tau} = Y - Y^3 \\ \frac{dY}{d\tau} = -x - x^3 \end{cases}$$

$x(t) = \sin(t)$

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{dx}{-d\tau} = y - y^3 \\ \dfrac{dy}{dt} = \dfrac{dy}{-d\tau} = -x - x^3 \end{cases} \implies \begin{matrix} \dfrac{dx}{d\tau} = y^3 - y \\ \dfrac{dy}{d\tau} = x + x^3 \end{matrix} \xRightarrow{y \mapsto -Y} \begin{cases} \dfrac{dX}{d\tau} = Y - Y^3 \\ \dfrac{dY}{d\tau} = -x - x^3 \end{cases}$$

$x(t) = \sin(t)$

$x(\tau) = \sin(-\tau)$