Lenny Wu

Ch.10 Nonlinear Systems of Equations

10.1 Functions of Several Variables

Def. A system of nonlinear equations of the form

$$F(x) = \begin{cases} f_1(x_1, x_2, \dots x_n) = 0 \\ f_2(x_1, x_2, \dots x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \dots x_n) = 0 \end{cases}$$

can be written in matrix form $f(\vec{x}) = \vec{0}$, where $F: \mathbb{R}^n \to \mathbb{R}^n$, $f_i: \mathbb{R}^n \to \mathbb{R}^n$ are called coordinate functions of f.

Def. Let $F = [f_1, f_2, \dots f_n]^T$ be defined from $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^n$. The function F is called continuous, denoted by $F \in C(\Omega)$, if each coordinate function f_i is continuous, i.e.

$$\lim_{\vec{x} \to \vec{x}_0} f_i(\vec{x}) = f_i(\vec{x}_0) \text{ for all } x \in \Omega$$

More precisely, for all $\varepsilon > 0$, there is $\delta > 0$ such that $\|\vec{x} - \vec{x}_0\| < \delta \Rightarrow |f_i(\vec{x}) - f_i(\vec{x}_0)| < \varepsilon$, where $\|\cdot\|$ can be any norm.

Rmk. Boundedness of partial derivatives guarantees continuity of multivariable functions.

Fixed Point Method (Functional Iteration)

Def. Let $G: \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. The point $\vec{P} \in \Omega$ is a fixed point of G if $G(\vec{P}) = \vec{P}$.

Thm. 10.6 (Existence of Fixed Points)

Let $\Omega = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n] \in \mathbb{R}^n$.

If G is:
- continuous on Ω
- $G(\vec{x}) \in \Omega$ for any $\vec{x} \in \Omega$

then G has a fixed point in Ω.

Fixed Point Method

$$\vec{x}^{(k)} = G(\vec{x}^{(k-1)}) \quad \text{with} \quad \vec{x}^{(0)} \in \Omega$$

or $\vec{x}_j^{(k)} \xleftarrow{g_j} x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}, \dots x_n^{(k-1)} \qquad j = 1, \dots, n$

One way to accelerate convergence is the Gauss-Seidel Method

$$\vec{x}_j^{(k)} \xleftarrow{g_j} x_1^{(k-1)}, x_2^{(k)}, x_3^{(k)}, \dots x_{j-1}^{(k)}, x_j^{(k-1)}, \dots x_n^{(k-1)} \qquad j = 1, \dots, n$$

Rmk. 1. This may not always guarantee convergence.
2. Not as parallelizable.

Newton's Method

Thm. 10.7 Let $\vec{p}$ be a fixed point of $G(x)$.

If there exists $\delta > 0$ s.t.

- $\frac{\partial g_i}{\partial x_j}$ are continuous on $N_\delta = \{\vec{x} \mid \|\vec{x} - \vec{p}\| < \delta\}$ for all $i, j = 1, \dots, n$
- $\frac{\partial^2 g_i}{\partial x_j \partial x_k}$ are continuous and bounded $\left|\frac{\partial^2 g_i}{\partial x_j \partial x_k}\right| \le M$ for all $\vec{x} \in N_\delta$
- $\frac{\partial g_i}{\partial x_j}(\vec{p}) = 0$

Then the sequence $\{\vec{x}^{(k)}\}$ generated by the fixed point iteration converges to $\vec{p}$ for any $\vec{x}^{(0)}$ such that $\|\vec{x}^{0} - \vec{p}\| < \tilde{\delta} < \delta$.

Moreover, $\|\vec{x}^{0} - \vec{p}\|_\infty \le \frac{n^2 M}{2} \|\vec{x}^{(k-1)} - \vec{p}\|_\infty^2$.

Quadratic convergence

Newton's Method : apply fixed point method to

$$G(\vec{x}) = \vec{x} - J_F^{-1}(\vec{x}) \cdot F(\vec{x})$$

where $J_F = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$

$$\Rightarrow \quad \vec{x}^{(k)} = \vec{x}^{(k-1)} - J_P(\vec{x}^{(k-1)}) \cdot F(\vec{x}^{(k-1)})$$

Rmk. 1. One can verify that $G(\vec{x})$ satisfies all hypotheses in Thm. 10.7. Thus, Newton's Method converges quadratically, but requires partial derivatives and a good initial condition.

2. J^{-1} is not trivial to compute!

3. At each step, must solve a linear system $O(n^3)$, unless matrix has special structure.

10.3 Quasi-Newton Methods - Broyden's Method

Goal: Generalize Secant method (Quasi-Newton) by approximating J.

These methods typically achieve superlinear convergence:

$$\lim_{k\to\infty} \frac{\|\vec{x}^{(k)}-\vec{p}\|}{\|\vec{x}^{(k-1)}-\vec{p}\|} = 0, \qquad F(\vec{p}) = \vec{0}$$

Broyden's Method

Recall Secant method for solving $f(x)=0$:

$$x^{(k+1)} = x^{(k)} - \frac{x^{(k)}-x^{(k-1)}}{f(x^{(k)})-f(x^{(k-1)})} \cdot f(\vec{x}^{(k)})$$

which replaces $f'(\vec{x}^{(k)})$ in Newton by

$$\frac{f(x^{(k)})-f(x^{(k-1)})}{x^{(k)}-x^{(k-1)}}$$

(note: cannot / by vector)

Generalizing to high dimensions:

$$J(\vec{x}^{(k)})\underbrace{(\vec{x}^{(k)}-\vec{x}^{(k-1)})}_{\vec{s}^{(k)}} = \underbrace{F(\vec{x}^{(k)}) - F(\vec{x}^{(k-1)})}_{\vec{y}^{(k)}}$$

Next, we approximate $J(\vec{x}^{(k)})$ with a matrix A_k.

Build A_k as follows:

Want: $A_k \cdot \vec{s}^{(k)} = \vec{y}^{(k)}$ (not enough to specify A_k. uniquely)

Note that any nonzero vector $\vec{v}$ can be written as a linear combination of $\vec{s}^{(k)}$ and $\vec{z}^{(k)}$, where $\vec{z}^{(k)}$ is the orthogonal complement of $\vec{s}^{(k)}$,

i.e. $\vec{s}^{(k)} \cdot \vec{z}^{(k)} = 0$

To uniquely define A_k, also need to specify how it acts on the orthogonal complement.

In our case, we specify:

$$A_k \cdot \vec{z} = J(\vec{z}^{(k-1)}) \cdot \vec{z} \qquad \text{where } \vec{z}^T \vec{s}^{(k)} = 0$$

Thus, we should have that A_k satisfies

$$\begin{cases} A_k \vec{s}^{(k)} = \vec{y}^{(k)} \\ A_k \vec{z} = J(\vec{x}^{(k-1)})\vec{z}, \quad \text{where } \vec{z}^T \vec{s}^{(k)} = 0 \end{cases}$$

$$\Longleftrightarrow \quad A_k = J(\vec{x}^{(k-1)}) + \frac{[y^{(k)} - J(\vec{x}^{(k-1)})\vec{s}^{(k)}] \cdot \vec{s}^{(k)T}}{\|\vec{s}^{(k)}\|_2^2}$$

can check algebraically

Broyden's Method consists of 2 steps at each iteration:

$$\begin{cases} A_k = A_{k-1} + \dfrac{\vec{y}^{(k)} - A_{k-1}\vec{s}^{(k)}}{\|\vec{s}^{(k)}\|_2^2}\,\vec{s}^{(k)T} & \text{(A-update)} \\ \vec{x}^{(k+1)} = \vec{x}^{(k)} - A_k^{-1} F(\vec{x}^{(k)}) = \vec{x}^{(k)} + \vec{s}^{(k+1)} & \text{(x-update)} \end{cases}$$

Note: still $O(n^3)$ per iteration to compute $A_k^{-1} F(\vec{x}^{(k)})$

or to solve $A_k^{-1} = \left(A_{k-1} + \dfrac{\vec{y}^{(k)} - A_{k-1}\vec{s}^{(k)}}{\|\vec{s}^{(k)}\|_2^2}\,\vec{s}^{(k)T}\right)^{-1}$

Thm. 10.8 (Sherman-Morrison Formula)

Suppose A is a nonsingular matrix and $\vec{x}, \vec{y} \in \mathbb{R}^n$ satisfy $\vec{y}^T A^{-1} \vec{x} \neq -1$.

Then $A + \vec{x}\vec{y}^T$ is nonsingular and

$$(A + \vec{x}\vec{y}^T) = A^{-1} - \frac{A^{-1}\vec{x}\vec{y}^T A^{-1}}{1 + \vec{y}^T A^{-1} \vec{x}}$$

Pf. By straightforward computation,

$$(A + \vec{x}\vec{y}^T)\left[A^{-1} - \frac{A^{-1}\vec{x}\vec{y}^T A^{-1}}{1 + \vec{y}^T A^{-1}\vec{x}}\right]$$

$$= I + \vec{x}\vec{y}^T A^{-1} - \frac{\vec{x}\vec{y}^T A^{-1} + \vec{x}\overbrace{(\vec{y}^T A^{-1}\vec{x})}^{\text{scalar}}\vec{y}^T A^{-1}}{1 + \vec{y}^T A^{-1}\vec{x}}$$

$$= I + \vec{x}\vec{y}^T A^{-1} - (\vec{x}\vec{y}^T A^{-1}) \cdot \left(\frac{1 + \vec{x}\vec{y}^T A^{-1}}{1 + \vec{y}^T A^{-1}\vec{x}}\right)$$

$$= I \quad \square$$

Applying S-M Formula, we get

$$A_{k+1}^{-1} = \left[A_k + \frac{\vec{y}^{(k+1)} - A_k\vec{s}^{(k+1)}}{\|\vec{s}^{(k+1)}\|_2^2}\vec{s}^{(k+1)T}\right]^{-1}$$

$$= A_k^{-1} - \frac{A_k^{-1}\left(\frac{\vec{y}^{(k+1)} - A_k\vec{s}^{(k+1)}}{\|\vec{s}^{(k+1)}\|_2^2}\vec{s}^{(k+1)T}\right)A_k^{-1}}{1 + \vec{s}^{(k+1)T}A_k^{-1}\left(\frac{\vec{y}^{(k+1)} - A_k\vec{s}^{(k+1)}}{\|\vec{s}^{(k+1)}\|_2^2}\right)}$$

$$= A_k^{-1} - \frac{(A_k^{-1}\vec{y}^{(k+1)} - \vec{s}^{(k+1)})\vec{s}^{(k+1)T}A_k^{-1}}{\|\vec{s}^{(k+1)}\|_2^2 + \vec{s}^{(k+1)T}A_k^{-1}\vec{y}^{(k+1)} - \|\vec{s}^{(k+1)}\|_2^2}$$

$$= A_k^{-1} - \frac{(A_k^{-1}\vec{y}^{(k+1)} - \vec{s}^{(k+1)})\vec{s}^{(k+1)T}A_k^{-1}}{\vec{s}^{(k+1)T}A_k^{-1}(\vec{y}^{(k+1)})} \quad (*)$$

Note: A_k^{-1} needs A_{k-1}^{-1} $\Rightarrow$ only need to compute A_0^{-1}!

Algorithm for Improved Broyden's Method

Given $\vec{x}^{(0)}$, set $A_0 = J(\vec{x}^{(0)})$ and $B_0 = A_0^{-1}$.

For $k = 0, 1, 2, \ldots$

$$\vec{s}^{(k+1)} = -B_k F(\vec{x}^{(k)})$$

$$\vec{x}^{(k+1)} = \vec{x}_k + \vec{s}^{(k+1)}$$

$$\vec{y}^{(k+1)} = F(\vec{x}^{(k+1)}) - F(\vec{x}^{(k)})$$

$$B_{k+1} = B_k + \frac{(\vec{s}^{(k+1)} - B_k\vec{y}^{(k+1)})\vec{s}^{(k+1)T}B_k}{\vec{s}^{(k+1)}B_k\vec{y}^{(k+1)}} \quad \text{by } (*)$$