Sturm Liouville Problems

Consider $L(y) = Ly = -Py'' + Qy' + Ry$

Suppose y satisfies:

$$(*) \begin{cases} c_1 y(a) + c_2 y'(a) = 0 \\ d_1 y(b) + d_2 y'(b) = 0 \end{cases}$$

ODEs

Def. Ly is self-adjoint if $P' = -Q$ since then, $Ly = (-Py')' + Ry$, and $(Ly_1, y_2)_{L^2} = (y_1, Ly_2)_{L^2}$ if y_i satisfy $(*)$.

Linear Algebra

Def. $L(v) = Av$, A $n \times n$

L is self-adjoint if $A = A^T$ (i.e. A is symmetric), since then,

$(L(v), w)_{Eucl} = (v, L(w))_{Eucl}$

$$\begin{aligned} (Av)^T w &= v^T A^T w \\ &= v^T A w \\ &= v^T (Aw) \end{aligned}$$

Eigenvalue Problem

$Ly = \lambda g(x) y$

λ: eigenvalue

$y = y(x)$: eigenfunc., $y \neq 0$

Prop. L self-adjoint

$Ly_i = \lambda_i g(x) y_i$, y_i with $(*)$

$\lambda_1 \neq \lambda_2$, then

$(y_1, y_2)_g = \int_a^b y_1 y_2 g = 0$

$L(v) = Av = \lambda v$

for some $v \neq 0$

λ: eigenvalue

v: eigenvectors

Prop. A symmetric

$Av_i = \lambda_i v_i$ with $\lambda_1 \neq \lambda_2$

$\Longrightarrow v_1 \perp v_2$

i.e. $(v_1, v_2) = 0$

check:

$(\lambda_1-\lambda_2)(v_1, v_2)$

$= (\lambda_1 v_1, v_2) - (v_1, \lambda_2 v_2)$

$= (Av_1, v_2) - (v_1, Av_2)$

$= 0$

Thm. Let $Ly = -(Py')' + Ry$.
Suppose $P \in C^1[a,b]$, $P>0$
$R, \rho \in C[a,b]$, $\rho > 0$
Then $Ly = \lambda \rho(x) y$, where y satisfies $(*)$, has infinitely many eigenvalues λ_k, $k \in \mathbb{N}$, with $\lambda_k \to \infty$ as $k \to \infty$
There is precisely one eigenfunction ϕ_k st $L\phi_k = \lambda_k \rho(x)\phi_k$ and $\|\phi_k\|_{L^2} = 1$.
The eigenfunctions ϕ_k are ON and complete, i.e. every $f \in C^2[a,b]$ with $(*)$ can be written as

$$f = \sum_{k=1}^{\infty} (f, \phi_k)\phi_k$$

Thm. (Symmetric matrices are diagonalizable)
There are ON eigenvectors $v_1, \dots v_n$ of $A = A^T$ st any $v \in \mathbb{R}^n$ can be written as $v = \sum_{i=1}^{\infty} (v, v_i) v_i$
and hence

$$Av = \sum_{i=1}^{\infty} (v, v_i) Av_i$$
$$= \sum_{i=1}^{n} (v, v_i) \lambda_i v_i$$

Ex. $Ly = -y''$, take $\rho = 1$
i.e. $y'' + \lambda y = 0$ $\quad y(0) = y(\pi) = 0$
We know $\lambda_k = k^2$, $k = 1, 2, \dots$

$\phi_k(x) = \frac{\sin(kx)}{\sqrt{\pi/2}}$ ON eigenfunction on $[0, \pi]$ (wrt L^2 inner product)

and $f \in C^2[0,\pi]$, $f(0) = f(\pi) = 0$.

Then $f(x) = \sum_{k=1}^{\infty} b_k \sin(kx)$ Fourier sine series

$$= \sum_{k=1}^{\infty} \left(f, \frac{\sin(kx)}{\sqrt{\pi/2}}\right)_{L^2} \frac{\sin(kx)}{\sqrt{\pi/2}}$$

Ex./Rmk. **Legendre eqn**

$$Ly = -((1-x^2)y')' \quad \text{for } x \in [-1, 1]$$

Look at: $Ly = \lambda y$, i.e. $g = 1$

Fact: $\lambda_k = k(k+1)$, $P_k(x) = \frac{1}{2^k k!} \cdot \frac{d^k}{d^k x}(x^2-1)^k$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, ...

are ON: $\int_{-1}^{1} P_k(x) P_l(x)\,dx = 0$ if $k \neq l$

The Legendre polynomial / eqn. arise in solution of $\Delta w = 0$ in 3-D when using separation of variables in spherical coordinates.

Ex. (periodic boundary conditions)

$$y''(x) + \lambda y(x) = 0$$

(*) $y(-\pi) = y(\pi)$, $y'(-\pi) = y'(\pi)$

if $\lambda < 0$: $y(x) = 0$ (check!)

if $\lambda = 0$: $y(x) = ax + b$

$\longrightarrow y(x) = \text{const} = \frac{1}{2}a_0$

if $\omega^2 = \lambda > 0$: $y(x) = a\cos(\omega x) + b\sin(\omega x)$

$a\cos(\omega\pi) - b\sin(\omega\pi) = y(-\pi) = y(\pi) = a\cos(\omega\pi) + b\sin(\omega\pi)$

$a\omega\cos(\omega\pi) + b\omega\sin(\omega\pi) = y'(-\pi) = y'(\pi) = -a\omega\cos(\omega\pi) + b\omega\sin(\omega\pi)$

get: $-2b\sin(\omega\pi) = 0$ or $2a\sin(\omega\pi) = 0$

$a, b \neq 0 \Rightarrow \omega\pi = k\pi, \quad k \in \mathbb{N}$

$\Rightarrow y_k(x) = a_k \cos(kx) + b_k \sin(kx)$ with $\lambda_k = k^2$