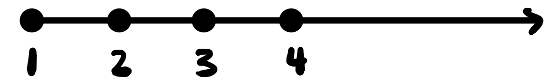
Induction:

$\mathbb{N}$:



If $P(n)$ is a statement about the number n, and both

- $P(1)$ is true (base case)
- for all $n \in \mathbb{N}$, if $P(n)$ is true, then $P(n+1)$ is true (induction step)

First induction:

$$\underbrace{1+2+\ldots+n=\frac{n(n+1)}{2}}_{P(n)} \quad \text{for all } n \in \mathbb{N}$$

Base case : $n=1$

$$P(1): \frac{1(1+1)}{2}=1$$

Induction step: want to prove that if $P(n)$ is true, then $P(n+1)$ is true.

IH (induction hypothesis):

Assume n is a natural number such that $P(n)$ is true.

We assume: $1+2+\ldots+n=\frac{n(n+1)}{2}$

Want to show: $1+2+\ldots+n+n+1=\frac{(n+1)(n+1+1)}{2}$

$$
\begin{aligned}
1+2+\ldots+n+n+1 &= [1+2+\ldots+n]+(n+1) \\
&= \frac{n(n+1)}{2}+(n+1) \\
&= \frac{n(n+1)+2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}
$$

By induction, we conclude that for all n, $1+2+\ldots+n=\frac{n(n+1)}{2}$ $\square$

Ex. For all natural numbers n,

$$n! \geq 2^{n-1}.$$

$$n! = n(n-1)(n-2)\ldots(1)$$

$n=1$ base case:

$$1=1! \geq 2^{1-1}=1 \quad \checkmark$$

Induction step:

Assume $k \in \mathbb{N}$ such that $k! \geq 2^{k-1}$

$\longrightarrow (k+1)! \geq (k+1)2^{k-1}$

$\geq 2 \cdot 2^{k-1}$

$= 2^k$

want to show: $(k+1)! \geq 2^k$

$$
\begin{aligned}
(k+1)! &= (k+1)k! \\
&\geq (k+1)2^{k-1} \\
&\geq 2 \cdot 2^{k-1} \quad (\text{since } k \geq 1) \\
&= 2^k
\end{aligned}
$$

By induction, we conclude that $n! \geq 2^{n-1}$ $\square$

Goal: prove a statement of the form "for all $n \in \mathbb{N}$ $P(n)$ is true"

e.g. $P(n)$: $1 + \ldots + n = \frac{n(n+1)}{2}$

- Base case: prove $P(1)$

  e.g. $1 = \frac{1(1+1)}{2}$

- Induction step: assume $\underbrace{P(n) \text{ is true}}_{\text{inductive hypothesis}}$, then $P(n+1)$ is true, for any n.

Ex. Prove that for real numbers a and r,

$$a + ar + ar^2 + \ldots + ar^n = \frac{a(r^{n+1} - 1)}{r-1}.$$

Fix a and r.

Base case $n=1$:

$$a + ar = \frac{a(r^2-1)}{r-1} = a(r+1) \quad \checkmark$$

Induction step: assume $k \in \mathbb{N}$ such that

$$a + ar + \ldots + ar^k = \frac{a(r^{k+1}-1)}{r-1}$$

Consider $k+1$:

$$\begin{aligned} a + ar + ar^2 + \ldots ar^{k+1} &= \frac{a(r^{k+1}-1)}{r-1} + ar^{k+1} \\ &= \frac{a(r^{k+1}-1)}{r-1} + \frac{ar^{k+2} - ar^{k+1}}{r-1} \\ &= \frac{a(r^{k+2}-1)}{r-1} \end{aligned}$$

Set Theory

Def. A set is a collection of objects.

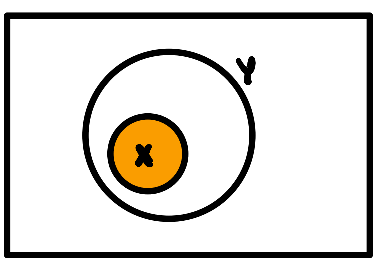
e.g. $A = \{1, 2, 3\}$
$= \{1, 2, 2, 3\}$
$= \{3, 2, 1\}$

Def. Two sets are equal if they contain the same elements.

Notation $B = \{ x \mid x \text{ satisfies the property } P \}$

Ex. $R = \{x \mid x \text{ is prime}\}$
$= \{2, 3, 5, 7, \dots\}$
$R = \{x \mid x \text{ is an even prime}\}$
$= \{2\}$

Def. A set X is a subset of Y if every element in X is also in Y.



Note: $X = Y \iff X \subseteq Y \text{ and } Y \subseteq X$

Cardinality

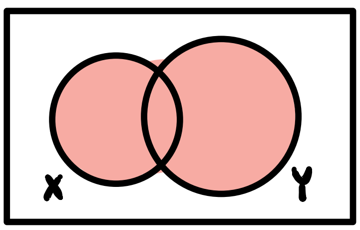
Def. If X is a set, the cardinality of X, denoted $|X|$, is the number of elements contained in X.

Ex. $|\{1,2,3\}| = 3$

$|\{1,2,2,3\}| = 3$

Operations in sets

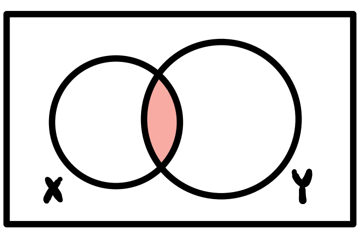
Union: $X \cup Y$ denotes the set whose elements are either elements of X or elements of Y.



$x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$

inclusive

Intersection: $X \cap Y$ denotes the set whose elements are elements of X and Y.

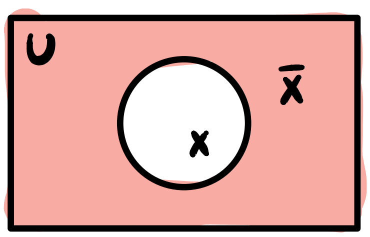


$x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$

U: "universe"; the set in which all elements we are considering are contained — determined by context

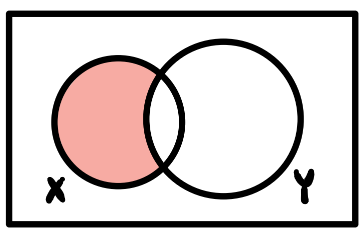
Ex. $\{x \in \mathbb{N} \mid x \text{ is prime}\}$

Complement: The complement $\overline{X}$ of X is the set of all $x \in U$ st. $x \notin X$



$\overline{X} = \{x \in U \mid x \notin X\}$

X-Y: $X \cap (\overline{Y})$



DeMorgan's Law

1. $\overline{(X \cap Y)} = \overline{X} \cup \overline{Y}$
2. $\overline{(X \cup Y)} = \overline{X} \cap \overline{Y}$

Cartesian Product

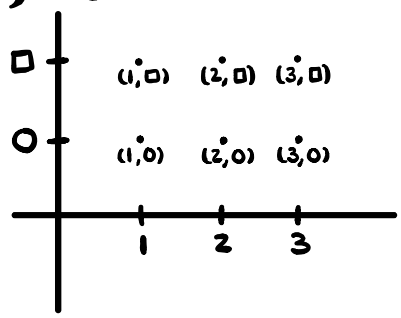
Def. Suppose X, Y are sets, then the **cartesian product** $X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\}$.

Ex. 1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$

2. $A = \{1, 2, 3\}$

$B = \{\square, 0\}$

$A \times B =$



$$|A \times B| = |A| \cdot |B|$$

Functions

Def. Suppose X, Y are sets.

A **function** from X to Y is a set Z of pairs in $X \times Y$ such that for every $x \in X$, there exists $y \in Y$ such that $(x, y) \in Z$.

$f: X \to Y \qquad Z \subseteq X \times Y$

$f(x) = y \Leftrightarrow (x, y) \in Z$

i.e. $Z = \{(x, f(x)) \mid x \in X\}$

If $f: X \to Y$ is a function from X to Y,

we say X is the domain of f,

Y is the codomain of f.

Range of f: $\{y \in Y \mid \exists\, x \in X \text{ such that } f(x) = y\}$

Image

dom(f), ran(f), cod(f)

Def. $f: X \to Y$

- one-to-one (injective)

  For all $y \in \text{ran}(f)$, $\exists!\, x \in X$ such that $f(x) = y$.

  there exists unique

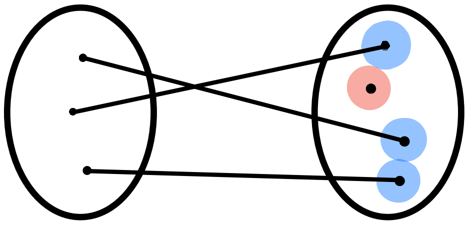
- onto (surjective)

  For every $y \in Y$, there exists $x \in X$ such that $f(x) = y$, i.e. $\text{cod}(f) = \text{ran}(f)$

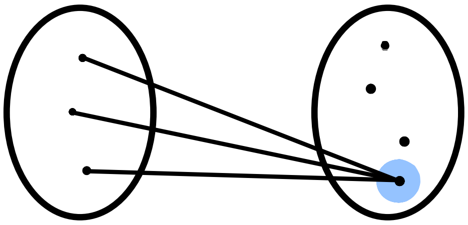
- bijective (one-to-one correspodence)

  Both one-to-one and onto.

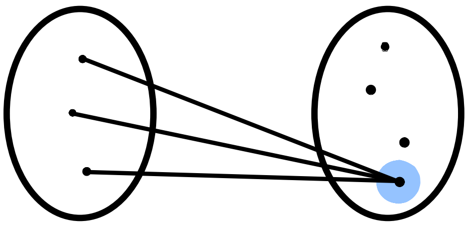
injective



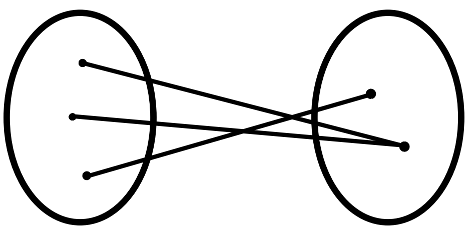
not injective



not a function



surjective



(not injective)

Every function f can be considered as a function from domain to codomain.

$$f : X \to \mathrm{ran}(f)$$

then, f is surjective viewed this way.

Sequences

Def. A sequence is a function whose domain is a set of consecutive integers.

can be a finite set
→ sequence is finite
otherwise, sequence is infinite

If s is a sequence, we write s_n for $s(n)$; n = index of s_n

Def.
- Sequence is increasing if whenever $n<m$, then $s_n < s_m$, $n,m \in D(f)$.
- Sequence is nondecreasing if whenever $n<m$, then $s_n \le s_m$, $n,m \in D(f)$.
- Sequence is decreasing if whenever $n>m$, then $s_n > s_m$, $n,m \in D(f)$.
- Sequence is nonincreasing if whenever $n>m$, then $s_n \ge s_m$, $n,m \in D(f)$.

Notation If s is a sequence with domain $\{n \in \mathbb{N} : n \ge k\}$, then we can write:

$$\{s_n\}_{n=k}^{\infty}$$